Ramsey Theory

The Computation of Ramsey Graphs

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Abstract

Preface

The work on the project took place from the 1. of September to the 20. of December 2023.

Sources are stated at the start of each chapter, or section if the sources used in that particular section differ from the sources, which are generally used within the chapter.

Definitions, algorithms, theorems, propositions, lemmas, corollaries, examples, and remarks are numbered according to each chapter and consecutively. Equations are also numbered according to each chapter but separately and likewise with figures and tables. The conclusions of proofs and examples are marked with ■ and □ respectively.

A table of notation and shorthands is given after the list of contents. Please note that some symbols may be used differently in different chapters.

Finally, I wish to extend a thank you to my supervisor, Matteo Bonini, for his wonderfull guidance throughout the project.

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Notation and Shorthands

1 Ramsey Numbers

Before we can get started we will need some basic notions from graph theory.

Definition 1.1. Let $G = (V, E)$ be a graph, G is called *complete* if $v, u \in V$ such that $v \neq u$ implies $\{v, u\} \in E$. A *subgraph* of G is a graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. Finally a complete subgraph is called a *clique*.

We often denote a complete graph with n vertices as K_n.

Theorem 1.2 (Generalized Pigeonhole Principle)**.** *Let A be a set such that $|A| > m \cdot r$ if $A = \bigcup_{i=1}^{r} A_i$, then there exists a A_j such that $|A_j| > m$.*

Proof. Assume for the sake of contradiction that $|A_j| < m$ for $j = 1, 2, \ldots, r$, then:

$$|A| \leq \sum_{j=1}^{r} |A_i| < r \cdot m$$

clearly a contradiction. ∎

Definition 1.3. Let A be a set, a r*-coloring* on A is a function $\chi : A \to C$ where C is the set of *colors* and $|C| = r$. The subset $A' \subseteq A$ is called *monochromatic* if $\chi(a) = c$ for all $a \in A$.

Remark 1.4. The concrete choice of color set C, in our r-coloring $\chi : A \to C$, is not really a concern, since given any bijection ϕ between C and another color set C' one would obtain a new r-coloring $\chi' = \phi \circ \chi$. Hence for the sake of simplicity we pick $C = [r]$, unless $r = 2$ or $r = 3$ in which cases we usually let $C = \{red, blue\}$ or $C = \{red, blue, green\}$ respectively. Additionally we note that an r-coloring χ on A, can be thought of as an partitioning of A into the sets $A_i = \{a \in A | \chi(a) = i\}$.

We will primarily be interested in the case where A is the set of edges E of the complete graph K_n, in which case we will refer to χ as an edge coloring on K_n.

Definition 1.5. Let n be a positive natural number we will write $n \to (\ell_1, \ell_2, \ldots, \ell_r)$ if for every r-edge coloring $\chi : E_n \to \{c_1, c_2, \ldots, c_r\}$ on K_n, there exists an $i \in \{1, 2, \ldots, r\}$ such that χ emits a c_i-monochromatic clique of order k_i.

Remark 1.6. Clearly $k_i \leq k_i'$ and $n \to (k_1, k_2, \ldots, k_r)$ implies that $n \to (k_1', k_2', \ldots, k_r')$

Theorem 1.7 (Ramsey's Theorem)**.** *Let $\ell_1, \ell_2 \geq 2$, then there exists a non-zero $n \in \mathbb{N}$ such that $n \to (\ell_1, \ell_2)$.*

Proof. Clearly $\ell_1 \to (\ell_1, 2)$ and $\ell_2 \to (2, \ell_2)$ (after all either K_{ℓ_i} is monochromatic or there exist a monochromatic clique of order 2). We will proceed using induction on $\ell_1 + \ell_2$, we may assume that $\ell_1 + \ell_2 \geq 6$, with $\ell_1, \ell_2 \geq 3$. Additionally by our induction hypothesis we may assume the existence of non-zero $n_1, n_2 \in \mathbb{N}$ such that $n_1 \to (\ell_1, \ell_2 - 1)$ and $n_2 \to (\ell_1 - 1, \ell_2)$. Next let $n := n_1 + n_2$ we will show that $n \to (\ell_1, \ell_2)$. Fix an arbitrary 2-edge coloring χ on K_n and let v be a vertex in K_n, then v is adjcent to $n - 1$ other vertices in K_n, hence:

$$n_1 + n_2 - 1 = n - 1 = |N_\chi(v; 1)| + |N_\chi(v; 2)|$$

meaning either $|N_\chi(v; 1)| \geq n_2$ or $|N_\chi(v; 2)| \geq n_1$. Without loss of generality assume that the second inequality holds, namely $|N_\chi(v; 2)| \geq n_1$. By our inductive hypothesis, the complete graph $G = (N_\chi(v; 2), N_\chi(v; 2) \times N_\chi(v; 2))$ contains either a 1-monochromatic clique of order ℓ_1 (in which case we are done) or a 2-monochromatic clique of order $\ell_2 - 1$, in which case we note that v is connected to the vertices in $N_\chi(v; 2)$ via edges that χ assigns the color 2, hence complete graph on the vertex set $N_\chi(v; 2) \cup \{v\}$ forms a 2-monochromatic clique of order ℓ_2. ∎

Corollary 1.8. *Let $\ell_1, \ell_2, \ldots, \ell_r \geq 2$, then there exists a non-zero $n \in \mathbb{N}$ such that $n \to (\ell_1, \ell_2, \ldots, \ell_r)$.*

Proof. We proceed using induction on r, the base case $r = 2$ is proven in Theorem 1.7. Next assume that the theorem holds for $r - 1$. From Theorem 1.7, it follows that there exists a $\ell \in \mathbb{N}_{>0}$ such that $\ell \to (\ell_{r-1}, \ell_r)$. By our induction hypothesis we may find a $n \in \mathbb{N}_{>0}$ for which it holds that:

$$n \to (\ell_1, \ell_2, \ldots, \ell_{r-2}, \ell)$$

Now given any r-edge coloring χ on K_n we may obtain a $r - 1$-edge coloring χ' on K_n by defining:

$$\chi'(e) = \begin{cases} \chi(e) & \text{if } \chi(e) < r - 1 \\ r - 1 & \text{otherwise} \end{cases}$$

Hence χ' must either emit a i-monochromatic clique of order ℓ_i, for some $i \in \{1, 2, \ldots, r - 2\}$ (in which case we are done) or a $(r - 1)$-monochromatic clique of order ℓ. Let K_ℓ be this monochromatic clique, then χ associates every edge in G with either $r - 1$ or r. However ℓ was chosen so that $\ell \to (\ell_{r-1}, \ell_r)$, hence G must contain either a $(r - 1)$-monochromatic clique of order ℓ_{r-1} or a r-monochromatic clique of order ℓ_r. ∎

Definition 1.9. The *Ramsey number* $R(k_1, k_2, \ldots, k_r)$, is the minimal $n \in \mathbb{N} \setminus \{0\}$ such that $n \to (k_1, k_2, \ldots, k_r)$, additionally we let $R(k; r)$ denote $R(k_1, k_2, \ldots, k_r)$, with $k_1 = k_2 = \cdots = k_r = k$.

Generally direct computation of Ramsey numbers are extremely difficult,

Example 1.10. In this example we will show that $R(3, 3) = 6$, we start by showing that $R(3, 3) \leq 6$. Consider the edge coloring $\chi : E(K_6) \to \{red, blue\}$ on K_6 and let v be a vertex in K_6, then v has 5 adjacent neighbours, by the generalized pigeonhole principle, Theorem 1.2, there is a color c (either *red* or *blue*) such that $|N_\chi(v; c)| \geq 3$.

Without loss of generality we may assume that $c = red$. Next take pairwise distinct $v_1, v_2, v_3 \in N_\chi(v; red)$, then we must have $\chi(v_i, v_j) = blue$ for pairwise distinct $i, j \in [3]$, since we otherwise would have that $K_6|_{\{v, v_i, v_j\}}$ would form a *red*-monochromatic clique of order 3. However this in turn means that $\chi(v_i, v_j) = blue$ for all pairwise distinct $i, j \in [3]$, hence $K_6|_{\{v_1,v_2,v_3\}}$ forms a *blue*-monochromatic clique of order 3.

On the other hand it is easy to construct a 2-edge coloring on K_5 which emits no monochromatic subclique of order 3, see for instance the graph in Figure 1.1.

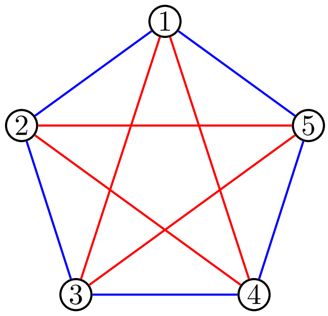

Figure 1.1: *A 2-edge coloring on* K_5 *that emits no monochromatic subclique of order* 3.

Finally since $R(3,3) > 5$ and $R(3,3) \leq 6$, we obtain that $R(3,3) = 6$. $\square$

1.1 Bounds on Ramsey Numbers

1.1.1 The Probabilistic Method

2 Schurs

In this chapter, we will answer the question: "Given a r-coloring on $\mathbb{N}^+$, does there always exist a monochromatic set $\left\{x_1, x_2, \ldots, x_k, \sum_{i=0}^k x_i\right\} \subseteq \mathbb{N}^+$". To get a better intuition for the problem, consider the case where $r = 3$ and $k = 2$ and $\mathbb{N}^+$ is colored by:

$$\chi(n) = \begin{cases} \mathit{red} & \text{if } n \equiv 0 \mod 3 \\ \mathit{blue} & \text{if } n \equiv 1 \mod 3 \\ \mathit{green} & \text{otherwise} \end{cases}$$

A visual representation of χ would be: $1, 2, 3, 4, 5, 6, \ldots$ meaning one example of a *red*-monochromatic subset of the form $\{x, y, x + y\}$, under χ, would be $x = y = 3$, notice that we do not require x and y to be distinct.

Next we will show that the proposition mentioned in the begining of the section does indeed hold.

Theorem 2.1 (Schur's Theorem)**.** *Let r, k be positive integers, then there exists a least positive integer $S(r, k)$ such that any r-coloring $\chi : [1, S(r, k)] \to C$ there exists a monochromatic subset of $[1, S(r, k)]$ of the form $\left\{x_1, x_2, \ldots, x_k, \sum_{i=1}^k x_i\right\}$.*

Proof. We will show that any r-coloring χ of $[1, R(k + 1; r)]$ emits a monochromatic subset of the form $\left\{x_1, x_2, \ldots, x_k, \sum_{i=1}^k x_i\right\}$.

Let G be the complete graph with vertex set $[1, R(k + 1; r) + 1]$. We will define an r-edge coloring on G by defining $\chi' : E(G) \to C$ as $\chi'(\{a, b\}) := \chi(|a - b|)$, by the definition of $R(k+1; r)$, the edge coloring χ' must emit a monochromatic clique G' of order $k+1$. Next if we order the verticies $v_1, v_2, \ldots v_{k+1}$ in G' in increasing order, meaning $v_1 < v_2 < \ldots < v_{k+1}$. Then, since G' is monochromatic, we see that

$$\chi(v_i - v_j) = \chi'(\{v_i, v_j\}) = \chi'(\{v_{i'}, v_{j'}\}) = \chi(v_{i'} - v_{j'})$$

for all $i > j$ and $i' > j'$, since the verticies are ordered in increasing order. The rest follows by setting $x_j := v_{j+1} - v_j$. ∎

Let $\chi_0 : \mathbb{N}^+ \to C$ be an r-coloring then, by Schur's Theorem 2.1, there exists a monochromatic subset $A_0 := \left\{x_1, x_2, \ldots, x_k, \sum_{i=1}^k x_i\right\}$ of $[1, S(r, k)]$. Next we may define a new coloring χ_1, which colors every element in $\mathbb{N}^+ \setminus A_0$ the same color as χ_0, but each element in A_0 a distinct new color, hence χ_1 is at most a $(r + k + 1)$-coloring, which similarly admits a monochromatic subset $A_1 := \left\{y_1, y_2, \ldots, y_k, \sum_{i=1}^k y_i\right\}$ of $[1, S(r + k + 1, k)]$. Additionally we note that $\chi_1(n) = \chi_0(n)$ for each $n \in A_1$ since each element in A_0 is colored a new distinct color, by χ_1. Hence A_1 is also monochromatic under χ_0. Repeating this argument we obtain the following corollary:

Corollary 2.2. *Let r, k be positive integers, any r-coloring $\chi : \mathbb{N}^+ \to \{c_1, c_2, \ldots, c_r\}$ emits infinitely many monochromatic subsets $\mathbb{N}^+$ of the form $\left\{x_1, x_2, \ldots, x_k, \sum_{i=1}^k x_i\right\}$.*

Next we show that statement of Fermats last theorem: "The equation $x^n + y^n = z^n$, with $n \geq 2$, has no solution $x, y, z \in \mathbb{Z}$, such that $xyz \neq 0$." is false if we instead require that x, y, z are elements in some specific family of finite fields.

Theorem 2.3. *Let $n \geq 1$, then there exists a prime p such that for all primes $q \geq p$, the equation $x^n + y^n = z^n$ has a solution $x, y, z \in \mathbb{F}_q$ with $xyz \neq 0$.*

Proof. Let $q > S(n, 2)$ be a prime, we will consider the multiplicative group $\mathbb{F}_q^*$, and the subgroup $G = \{x^n \mid x \in \mathbb{F}_p\}$.

Hence there exists $a_1, a_2, \ldots a_k \in \mathbb{F}_q^*$ such that $\mathbb{F}_q^* = \bigcup_{i=1}^k a_i G$ with $k = \frac{n}{\gcd(n,p-1)}$ **TODO**. Since $\left|\mathbb{F}_q^*\right| = \left|\mathbb{F}_q^*/S\right| |S|$ (lagrange index theorem) and

Next, we define a k-coloring $\chi : \mathbb{F}_q^* \to [k]$ by $\chi(y) = j$ if and only if $y \in a_j G$. Now since $k \leq n$ and $p - 1 \geq S(n, 2)$, there exists a monochromatic triple $\{x, y, z\} \subseteq \mathbb{F}_q^*$ such that $x + y = z$, by Theorem 2.1. Meaning there exists an index $j \in [k]$ such that $a_j x^n, a_j y^n, a_j z^n \in a_j S$ with $a_j x^n + a_j y^n = a_j z^n$, the rest follows by multiplying by a_j^{-1}. ■

Appendices

Bibliography