Notes on Mixed Integer Linear Programming

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October 26, 2025

1 Introduction

The term *Mixed Integer Linear Programming (MILP)* arises from the field of mathematical optimization; in general, a *Linear Programming (LP)* problem is an optimization problem of the form:

$$\begin{aligned} \min_x \ & z = c^T x \\ \text{s.t. } & Ax \leq b \\ & x \geq 0 \end{aligned}$$

where $x, c \in \mathbb{R}^n, b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$. If the additional constraint that some or all of the entries in x only take on integer values, that is values in $\mathbb{N}$, is imposed then the LP problem transforms into a MILP problem.

Remark. The tool Xion, which these notes partly document is a tool for MILP problems. However it should be noted that xion was created purely for educational purposes.

Definition 1.1. A MILP is in *standard form* if its of the form:

$$\begin{aligned} \min_x \ & z = c^T x \\ \text{s.t. } & Ax = b \\ & x \geq 0 \end{aligned} \tag{1}$$

with $A \in \mathbb{R}^{m \times n}$ and $b \geq 0$.

Remark. It is sufficient to consider problems in standard form since, every MILP problem can be converted into a MILP problem in standard form

1.1 Converting a MILP problem to standard form

If the objective is $\max_x z = c^T x$, then since $\max_x = -\min_x(-c^T x)$ we can consider $\bar{c} = -c$.
Converting the inequality constraints to equality constraints is less straight forward however consider the following inequality constraint:

$$a_i^T x \leq b_i \tag{2}$$

Where a_i is the ith row of A and b_i is the ith entry in b. Introducing a slack variable $s_i \geq 0$, allows us to transform the constraint into:

$$\begin{aligned} a_i^T x + s_i &= b_i \\ s_i &\geq 0 \end{aligned}$$

Finally if $b_i < 0$ multiplying both sides of the equation by -1 makes sure that the right hand side is positive. Conversely every constraint of the form:

$$a_i^T x \geq b_i \tag{3}$$

can be written as:

$$\begin{aligned} a_i^T x - s_i &= b_i \\ s_i &\geq 0 \end{aligned}$$

Remark. Under the hood xion only works with MILP problems in standard form.

2 Linear Programming and the Simplex Method

Let C be a convex set an *extreme point* $p \in C$ is a point, where $x\lambda + (1-\lambda)y = p$ implies $x = y = p$ for all $\lambda \in (0;1)$.

Let $P \subseteq \mathbb{R}^n$, then P is called a *polytope* if it is a convex hull of fintely many points in $\mathbb{R}^n$. If there exits a matrix $A \in \mathbb{R}^{n\times m}$ and vector $b \in \mathbb{R}^m$ such that $P = \{x \in \mathbb{R}^n | Ax \leq b\}$, then P is called a *polyhedron*.

Theorem 2.1 (Representation Theorem)**.** *Let $A \in \mathbb{R}^{n\times m}$, $b \in \mathbb{R}^m$ and $Q := \{x \in \mathbb{R}^n | Ax \leq b\}$ and P be the convex hull of the extreme points in Q, and $C = \{x \in \mathbb{R}^n | Ax \leq 0\}$. If* $\operatorname{rank}(A) = n$ *then* $Q = P + C := \{p + c | p \in P, c \in C\}$.

Definition 2.1. Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m_{\geq 0}$ with $\operatorname{rank}(A) = \operatorname{rank}(A, b) = m$ and $n > m$, then a points $\hat{x}$ is called a *basic solution* to (1) if $Ax = b$ and the columns of A corresponding to the non-zero components of $\hat{x}$ is linearly independent. Furthermore if $\hat{x} \geq 0$ then $\hat{x}$ is refered to as a *basic feasible solution (BFS)* to (1). If more than $n-m$ variables of a basic solution $\hat{x}$ is zero, then $\hat{x}$ is said to *degenerate*.

Theorem 2.2. *Suppose $A \in \mathbb{R}^{m\times n}$ with* $\operatorname{rank}(A) = m$ *and $b \in \mathbb{R}^m$, then $\hat{x} \in \{x \in \mathbb{R}^m_{\geq 0} | Ax = b\}$ is an extreme point if and only if $\hat{x}$ is a BFS.*