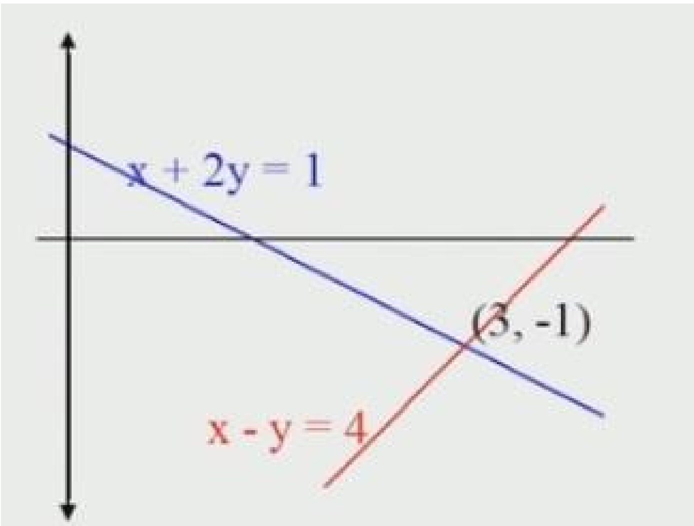
Numerical Solution of System of Linear Algebric Equations:

Rank & Solution of system of algebric equations:

Case-1:

$$x + 2y = 1$$
$$x - y = 4$$



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Equivalent Matrix formulation of the set of equation: $\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

We can Write this as, AX=B; where A= $\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$, B= $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ & X= $\begin{bmatrix} x \\ y \end{bmatrix}$

Therefore X can be solved: X= $A^{-1}B$, It is clear from the graph that there is unique solution exist for these set of linear equations.

This conclusion can also be draw by checking the Rank of the matrix A

If rank(A)=number of rows/columns of the square matrix A, then unique solution will exist.

```
A_1=[1 2;1 -1];
B_1=[1;4];
[row,columns]=size(A_1)
```

```
row = 2
columns = 2
```

```
rank_A_1=rank(A_1)
```

```
rank_A_1 = 2
```

In this case rows=rank of A, so unique solution exist.

To get the solution

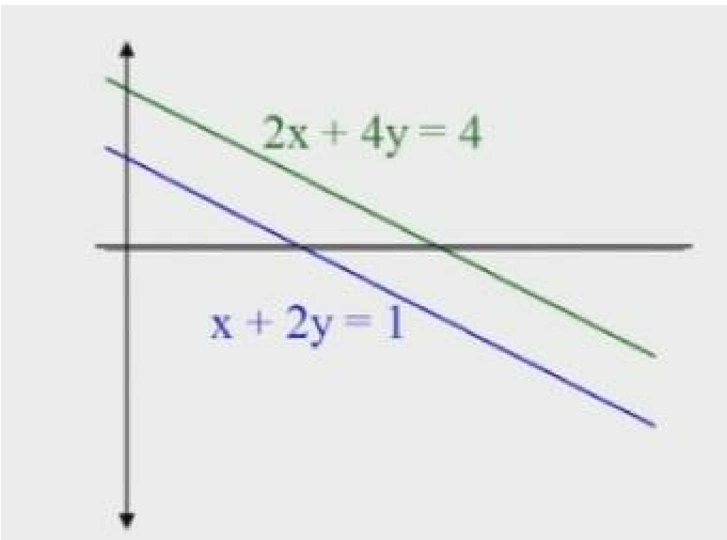
```
X=A_1\B_1  % you can also use X=inv(A)*B;
```

```
X =

     3
    -1
```

Case-2:

$$x + 2y = 1$$
$$2x + 4y = 4$$



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Equivalent Matrix formulation of the set of equation: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

It is clear from the graph there is no solution for this set of equatons.

This can be shown in the following way

```
A_2=[1 2;2 4];
B_2=[1;4];
[row,columns]=size(A_2)
```

```
row = 2
columns = 2
```

```
rank_A_2=rank(A_2)
```

```
rank_A_2 = 1
```

As Rank(A)<number of rows so there will not be unique solution.

In that case we ahve to form the augmented matrix Aug=[A B]

Then we have to find out rank of augmented matrix.

If rank(Aug)$\neq$rank(A) then No solution of the system of equations

If rank(Aug)=rank(A) then infinitely many solutions will exist

Check it for case-2

```
Aug_2=[A_2 B_2];
rank_Aug_2=rank(Aug_2)
```

```
rank_Aug_2 = 2
```

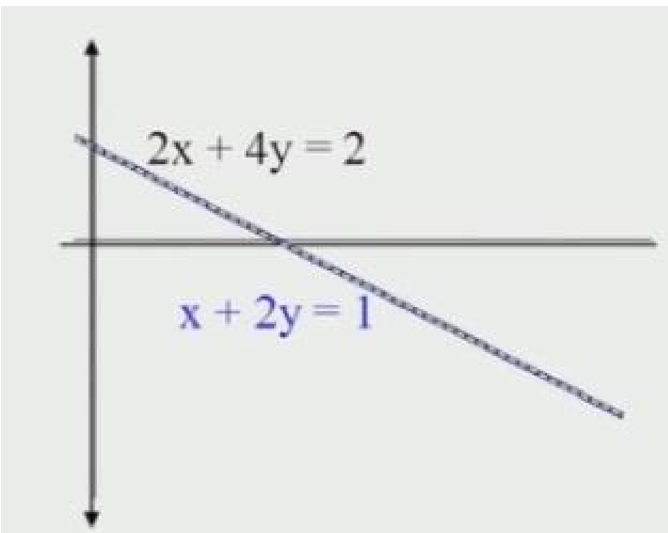
```
rank_A_2=rank(A_2)
```

```
rank_A_2 = 1
```

In this case rank(Aug)$\neq$rank(A) therefore, there is no solution in this system of equations.

Case-3:

$$x + 2y = 1$$
$$2x + 4y = 2$$



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Equivalent Matrix formulation of the set of equation: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

It is clear from the graph that there are infinitely many solutions exists

This can be shown in the following way

Compare number of rows & rank(A)

```
A_3=[1 2;2 4];
B_3=[1;2];
rank(A_3)
```

```
ans = 1
```

Clear that rank(A)<number of rows.So no unique solution exist.

Compare rank of augmented matrix & rank of matrix A

```
Aug_3=[A_3 B_3];
rank(Aug_3)
```

```
ans = 1
```

```
rank(A_3)
```

```
ans = 1
```

Since rank of augmented matrix is equal to rank of A ,

rank(Aug_3)=rank(A_3)=1

Therefore there are infinitely many solution exist for this case.

Gauss Elimination (Algorithm)

Take one example:

$$x_1 + x_2 + x_3 = 4$$
$$2x_1 + x_2 + 3x_3 = 7$$
$$3x_1 + 4x_2 - 2x_3 = 9$$

To solve this set of equations by Gauss Elimination

Form the coefficient matrix A, B matrix & augmented matrix Aug

```
A=[1 1 1;2 1 3;3 4 -2];
B=[4;7;9];
Aug=[A B]
```

```
Aug =

     1     1     1     4
     2     1     3     7
     3     4    -2     9
```

In each step A(i,i) is the pivot element

Use pivot element to make zeros in pivot column

. $R_j = R_j - \alpha_{ij} R_i$, where $\alpha_{ij} = \dfrac{A(j,i)}{A(i,i)}$

Formation of upper triangular matrix

```
%A(1,1) as pivot element
alpha=Aug(2,1)/Aug(1,1);
Aug(2,:)=Aug(2,:)-alpha*Aug(1,:); % R2=R2-alpha*R1
alpha=Aug(3,1)/Aug(1,1);
Aug(3,:)=Aug(3,:)-alpha*Aug(1,:); % R3=R3-alpha*R1
Aug
```

```
Aug =

     1     1     1     4
     0    -1     1    -1
     0     1    -5    -3
```

```
%A(2,2) as pivot element
alpha=Aug(3,2)/Aug(2,2);
Aug(3,:)=Aug(3,:)-alpha*Aug(2,:); % R3=R3-alpha*R2
Aug
```

```
Aug =

     1     1     1     4
     0    -1     1    -1
     0     0    -4    -4
```

Back substitution:

$$.x_3 = \frac{\mathrm{Aug}(3,4)}{\mathrm{Aug}(3,3)}$$

$$.x_2 = \frac{\mathrm{Aug}(2,4) - \mathrm{Aug}(2,3) * x_3}{\mathrm{Aug}(2,2)}$$

$$.x_1 = \frac{\mathrm{Aug}(1,4) - \mathrm{Aug}(1,2) * x_2 - \mathrm{Aug}(1,3) * x_3}{\mathrm{Aug}(1,1)}$$

```
x=zeros(3,1);
x(3)=Aug(3,4)/Aug(3,3);
x(2)=(Aug(2,4)-Aug(2,3)*x(3))/Aug(2,2);
x(1)=(Aug(1,4)-Aug(1,2)*x(2)-Aug(1,3)*x(3))/Aug(1,1);
x
```

```
x =

     1
     2
     1
```

We can generalize the code in the following way

```
 %Define A, B & Aug=[A B]
A=[1 1 1;2 1 3;3 4 -2];
B=[4;7;9];
Aug=[A B]
```

```
Aug =

     1     1     1     4
     2     1     3     7
     3     4    -2     9
```

```
% Find n  number of unknowns as rows =column in square matrix
[m,n]=size(A);
```

```
n
```

```
n = 3
```

```
% Gauss Elimination
for i=1:n-1
    for j=i+1:n
    alpha=Aug(j,i)/Aug(i,i);
    Aug(j,:)=Aug(j,:)-alpha*Aug(i,:); % Rj=Rj-alpha*Ri
    end
end
% Back Substitution
for i=n:-1:1
    x(i)=(Aug(i,end)-Aug(i,i+1:n)*x(i+1:n))/Aug(i,i);
end
x
```

```
x =

     1
     2
     1
```

Tri Diagonal Matrix Algorithm (TDMA)

Form of the matrices for n=7

$$A = \begin{bmatrix} \mathrm{d}_1 & \mathrm{u}_1 & 0 & 0 & 0 & 0 & 0 \\ \mathrm{l}_1 & \mathrm{d}_2 & \mathrm{u}_2 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{l}_2 & \mathrm{d}_3 & \mathrm{u}_3 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{l}_3 & \mathrm{d}_4 & u_4 & 0 & 0 \\ 0 & 0 & 0 & l_4 & d_5 & u_5 & 0 \\ 0 & 0 & 0 & 0 & l_5 & d_6 & u_6 \\ 0 & 0 & 0 & 0 & 0 & l_6 & \mathrm{d}_7 \end{bmatrix}; B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{bmatrix}$$

Example:

Consider the following set of equations with variables $T_1,\ T_2, \ldots, T_{10}, T_{11}$

$T_1 = 100$

$T_1 - (2 + \alpha)T_2 + T_3 = \beta$

$T_2 - (2 + \alpha)T_3 + T_4 = \beta$

$T_3 - (2 + \alpha)T_4 + T_5 = \beta$.

.

.

$T_9 - (2 + \alpha)T_{10} + T_{11} = \beta$

$T_{11} = 25$

Consider β=-1, α =0.04

If we arrange the equation in AT=B form, then A will be a TDM and T is a column matrix containing variables T1,T2,...

```
n=11
```

```
n = 11
```

```
A=zeros(11,11);
B=zeros(11,1);
alpha=0.04;
bita=-1;
d=-(2+alpha)
```

```
d = -2.0400
```

```
l=1;
u=1;
for i=2:n
    A(i,i)=d;
    A(i,i-1)=l;
    A(i-1,i)=u;
    B(i)=bita;
end
[A B]
```

```
ans =

         0    1.0000         0         0         0         0         0         0
    1.0000   -2.0400    1.0000         0         0         0         0         0
         0    1.0000   -2.0400    1.0000         0         0         0         0
         0         0    1.0000   -2.0400    1.0000         0         0         0
         0         0         0    1.0000   -2.0400    1.0000         0         0
         0         0         0         0    1.0000   -2.0400    1.0000         0
         0         0         0         0         0    1.0000   -2.0400    1.0000
         0         0         0         0         0         0    1.0000   -2.0400
         0         0         0         0         0         0         0    1.0000
         0         0         0         0         0         0         0         0
```

```
A(1,1)=1;
A(11,11)=1;
A(1,2)=0;
A(11,10)=0;
B(1)=100;
B(11)=25;
AB=[A B]
```

```
AB =

    1.0000         0         0         0         0         0         0         0
    1.0000   -2.0400    1.0000         0         0         0         0         0
         0    1.0000   -2.0400    1.0000         0         0         0         0
         0         0    1.0000   -2.0400    1.0000         0         0         0
         0         0         0    1.0000   -2.0400    1.0000         0         0
         0         0         0         0    1.0000   -2.0400    1.0000         0
         0         0         0         0         0    1.0000   -2.0400    1.0000
         0         0         0         0         0         0    1.0000   -2.0400
         0         0         0         0         0         0         0    1.0000
         0         0         0         0         0         0         0         0
```

```
T=A\B
```

```
T =

  100.0000
   85.8589
   74.1521
   64.4114
   56.2471
   49.3327
   43.3917
   38.1863
   33.5084
   29.1708
```

TDMA Algoritm to solve this problem

```
    for i=1:n-1
    %Normalize with respect to pivot element
    A(i,i+1)=A(i,i+1)/A(i,i);
    B(i)=B(i)/A(i,i);
    A(i,i)=1;
    % Elimination using pivot element
    alpha=A(i+1,i);
    A(i+1,i+1)=A(i+1,i+1)-alpha*A(i,i+1);
    B(i+1)=B(i+1)-alpha*B(i);
    A(i+1,i)=0;
end
AB=[A B]
```

```
AB =

    1.0000         0         0         0         0         0         0         0
         0    1.0000   -0.4902         0         0         0         0         0
         0         0    1.0000   -0.6452         0         0         0         0
         0         0         0    1.0000   -0.7170         0         0         0
         0         0         0         0    1.0000   -0.7558         0         0
         0         0         0         0         0    1.0000   -0.7787         0
         0         0         0         0         0         0    1.0000   -0.7928
         0         0         0         0         0         0         0    1.0000
         0         0         0         0         0         0         0         0
         0         0         0         0         0         0         0         0
```

```
B;
% Back Substitution
T=zeros(n,1);
T(n)=B(n)/A(n,n);
for i=n-1:-1:1
    T(i)=B(i)-A(i,i+1)*T(i+1);
end
T
```

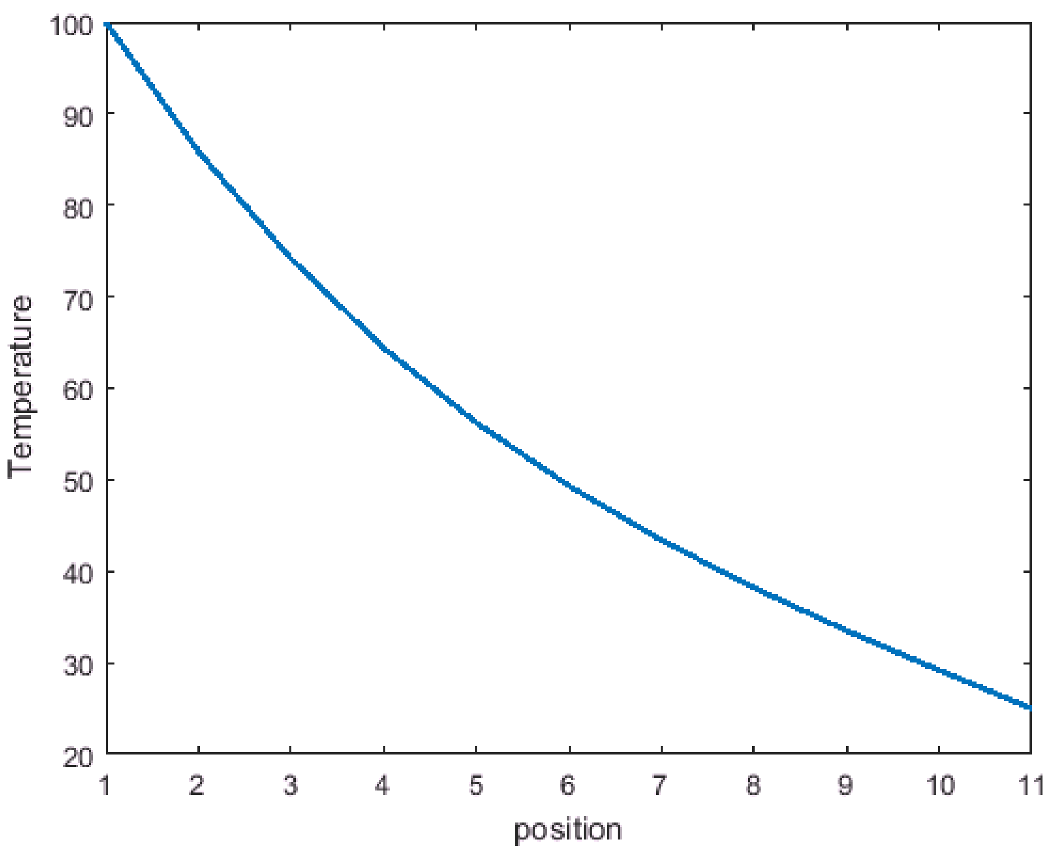
```
T =

  100.0000
   85.8589
   74.1521
   64.4114
   56.2471
   49.3327
   43.3917
   38.1863
   33.5084
   29.1708
```

```
% Plot
plot([1:11],T,'LineWidth',2)
```

```
Warning: MATLAB has disabled some advanced graphics rendering features by switching to
software OpenGL. For more information, click here.
```

```
xlabel('position');
ylabel('Temperature');
```



Iterative Methods of Solving Linear System of Equations of the form Ax=b

Jacobi:

$$x_k^{(i+1)} = \frac{B_k - \left(\sum_{j \neq k} A_{k,j} x_j^{(i)}\right)}{A_{k,k}}$$

Gauss Siedel:

$$x_k^{(i+1)} = \frac{B_k - \left(\sum_{j=1}^{k-1} A_{k,j} x_j^{(i+1)} + \sum_{k+1}^{n} A_{k,j} x_j^{(i)}\right)}{A_{k,k}}$$

Example:

$$x_1 + x_2 + x_3 = 4$$
$$2x_1 + x_2 + 3x_3 = 7$$
$$3x_1 + 4x_2 - 2x_3 = 9$$

Solve this set of equations by Gauss Siedel Method:

Solution:

Order the equation based on diagonal dominance.

$$2x_1 + x_2 + 3x_3 = 7$$

$$3x_1 + 4x_2 - 2x_3 = 9$$

$$x_1 + x_2 + x_3 = 4$$

```
A=[2 1 3;3 4 -2;1 1 1];
B=[7;9;4];
AB=[A B];
n=3;

x=zeros(n,1);
err=zeros(n,1);
iter=25;
Err=zeros(iter,1);

% Gauss Siedel Iteration

for i=1:iter
    for k=1:n
        xold=x(k);
        x(k)=(AB(k,end)-AB(k,1:k-1)*x(1:k-1)-AB(k,k+1:n)*x(k+1:n))/AB(k,k);
        err(k)=abs(xold-x(k));
    end
   disp(['Iteration No. ',num2str(i),':  Error= ',num2str(max(err))]);
   Err(i)=max(err);
end
```

```
Iteration No. 1:  Error= 3.5
Iteration No. 2:  Error= 1.2813
Iteration No. 3:  Error= 0.40625
```

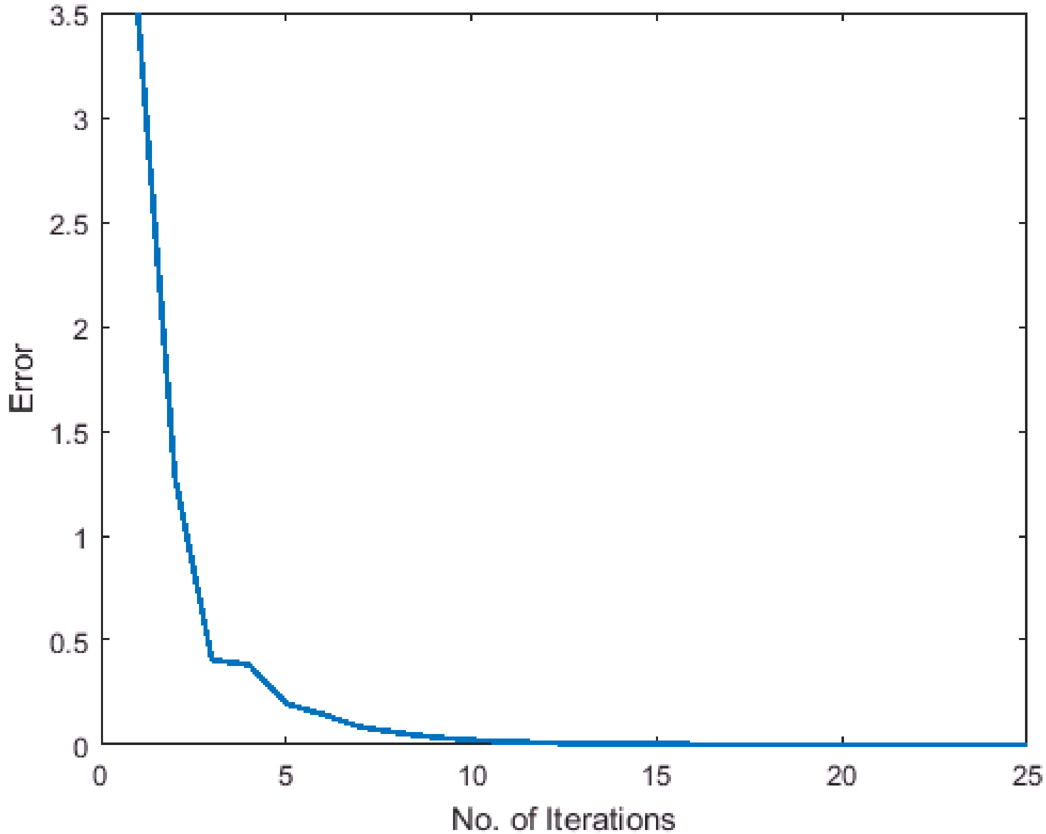
```
Iteration No. 4:  Error= 0.38281
Iteration No. 5:  Error= 0.19727
Iteration No. 6:  Error= 0.14502
Iteration No. 7:  Error= 0.085571
Iteration No. 8:  Error= 0.057648
Iteration No. 9:  Error= 0.035805
Iteration No. 10:  Error= 0.023363
Iteration No. 11:  Error= 0.014792
Iteration No. 12:  Error= 0.0095388
Iteration No. 13:  Error= 0.0060827
Iteration No. 14:  Error= 0.0039054
Iteration No. 15:  Error= 0.002497
Iteration No. 16:  Error= 0.0016006
Iteration No. 17:  Error= 0.0010244
Iteration No. 18:  Error= 0.00065625
Iteration No. 19:  Error= 0.00042016
Iteration No. 20:  Error= 0.0002691
Iteration No. 21:  Error= 0.00017232
Iteration No. 22:  Error= 0.00011035
Iteration No. 23:  Error= 7.0668e-05
Iteration No. 24:  Error= 4.5256e-05
Iteration No. 25:  Error= 2.8981e-05
```

```
x
```

```
x =

    1.0001
    2.0000
    1.0000
```

```
% Plot
plot([1:iter],Err,'LineWidth',2)
xlabel('No. of Iterations');
ylabel('Error');
```



Assignment:

Write MATLAB code for Jacobi Iteration to solve the same set of equations and show the convergence rate of the solution by plotting errors.