Sequential Auctions of Identical Items with Budget-Constrained Bidders

Zhiyi Huang* Nikhil R. Devanur† David Malec‡

Abstract

In this paper, we study sequential auctions with two budget constrained bidders and any number of identical items. All prior results on such auctions consider only two items. We construct a canonical outcome of the auction that is the only natural equilibrium and is unique under a refinement of subgame perfect equilibria. We show certain interesting properties of this equilibrium; for instance, we show that the prices decrease as the auction progresses. This phenomenon has been observed in many experiments and previous theoretic work attributed it to features such as uncertainty in the supply or risk averse bidders. We show that such features are not needed for this phenomenon and that it arises purely from the most essential features: budget constraints and the sequential nature of the auction. A little surprisingly we also show that in this equilibrium one agent wins all his items in the beginning and then the other agent wins the rest. The major difficulty in analyzing such sequential auctions has been in understanding how the selling prices of the first few rounds affect the utilities of the agents in the later rounds. We tackle this difficulty by identifying certain key properties of the auction and the proof is via a joint induction on all of them.

*University of Pennsylvania. Supported in part by an ONR MURI Grant N000140710907. Part of this work was done while the author was an intern at Microsoft Research, Redmond. Email: `hzhiyi@cis.upenn.edu`.

†Microsoft Research, Redmond. Email: `nikdev@microsoft.com`.

‡University of Wisconsin-Madison. Part of this work was done while the author was an intern at Microsoft Research, Redmond. Email: `dmalec@cs.wisc.edu`.

1 Introduction

Currently, there is a rich and well developed theory of auctions, most of it focused on single-round auctions, where all the items are auctioned off at once. Yet, many real world instances of auctions for multiple items are sequential in nature and the theory behind such auctions is still in a nascent stage. Another commonly occurring real-world aspect of auctions is budget constraints and incorporating budget constraints into traditional auction theory has often been quite challenging. The combination of sequential auctions and budget constraints is very natural and occurs commonly. This has been well recognized by the economics community. A modern-day instance of this combination is the much studied ad-auction; there have been many theoretical results capturing different aspects of ad-auctions. However there has been little to no theoretical results on the sequential nature of these auctions, even though there have been empirical studies that suggest bidders in these auctions do strategize to exhaust their competitors' budgets and take advantage of an emptier playing field Zhou and Lukose (2007). The goal of this paper is to characterize and extend our understanding of sequential auctions with budget constrained bidders.

Our results are based on a simple yet rich model that preserves the most essential features we wish to understand: the sequential nature combined with budget constraints. The model is simple because the items are identical, the number of items is fixed and known and agents have complete information.[1] Yet, the model is rich enough that the equilibrium outcomes display a rather complicated pattern (See Tables 1 and 2). Our results need substantial work and the difficulties we face and the techniques we employ are summarized in Section 1.2.

Our model is as follows: there are multiple identical items and two agents interested in obtaining them. Each agent wants to acquire as many items as possible but is subject to a *budget* constraint. The items are auctioned off *sequentially*, that is, the items are auctioned one after the other, each auction starting after the previous one is completed. The agents have *complete information*, that is each agent knows the budget of the other and the total number of items. This scenario can be thought of as a 2-player extensive game. The question we seek to answer in this paper is, what are the subgame perfect equilibria of this game.

The above question is one of the most basic that can be asked about sequential auctions with budget constrained bidders. The intuition for what should happen in the above game is also fairly straight forward: each agent tries to exhaust the budget of the other as soon as possible, so that he can have the rest of the items for himself at a low price. This intuition as well as the importance of this question was recognized as early as Pitchik and Schotter (1986). However so far it has not been possible to analyze such games with more than two items and these games have come to be regarded as quite difficult to analyze. The difficulty arises from the fact that as the number of rounds increases, the effects of an outcome in the first round on those in the later rounds become extremely complex. At least part of this difficulty arises due to the budget constraint as opposed to, say, a unit-demand constraint. (We present a detailed discussion of these aspects in Section 1.2).

In each round we consider a first-price sealed-bid auction. One could also consider alternatives such as a second-price auction (See Section 3.4 for a comparison) or an ascending price auction, but for technical reasons a first-price auction is most appropriate. One issue with subgame perfect equilibria as such is that there are many unnatural equilibria, for instance, even for a single-item auction. We need to consider a stronger notion of equilibrium in order to rule out these unnatural equilibria, and the one we use is a variant of the trembling hand perfect equilibrium. (See Section 2 for a precise definition.) Under this refinement, we show that there is a unique subgame perfect equilibrium of this game, *for any number of items*. (All earlier results only consider two items.) This unique equilibrium is indeed the natural equilibrium of the game and we refer to this as the canonical equilibrium. We show several properties of this equilibrium,

[1] Understanding idealized models such as one with complete information case provides a benchmark with which we can compare more realistic settings.

some surprising and some expected.

- The number of items won by an agent is approximately proportional to his budget.
- The prices are monotonically non-increasing as the auction progresses.
- One agent wins all of his items in the beginning and then the other agent wins the remaining items.

We now discuss each of these properties in more detail.

Number of Items Won This is perhaps the least surprising of all the properties. It is natural to expect that the number of items won is approximately proportional to the budget. However, an interesting contrast surfaces when we compare the number of items won here with the number of items won in the adaptive clinching auction of Dobzinski et al. (2011). It turns out that the number of items won in the sequential auction is more equitable, and closer to the proportion of the budget than in the adaptive clinching auction. A detailed comparison is presented in Section 5.

Monotonicity of Prices Pitchik and Schotter (1988) ran lab experiments of sequential auctions that showed that prices decrease as the auction progresses. Since then such experiments have been repeated by others and the monotonic decrease of prices has been reaffirmed Ashenfelter (1989). Almost all subsequent theoretical results have tried to capture this phenomenon. Actually the first among these, e.g., Weber (1981) and Milgrom and Weber (1982) showed the opposite, that prices increase. Later, different results attributed the decreasing price phenomenon to different features of the model: Jeitschko (1999) showed that decreasing prices could occur due to uncertainty in the number of items, Black and Meza (1992) attributed it to decreasing marginal utilities and McAfee and Vincent (1993) to risk aversion. The game we consider does not have any of these features: the number of items is fixed and known, the marginal utilities are constant and the agents are risk neutral. Our result shows that the decreasing prices phenomenon occurs purely from the most essential properties: the budget constraint and the sequential nature of the game.

The Order of Sale As one would expect, the number of items won by an agent is approximately proportional to his budget. The real surprise is in the order in which the items are won. Although the intuition we mentioned earlier says that an agent tries to exhaust the other agent's budget, it is conceivable that they win alternately to maintain the ratio of items won along the way. But the equilibrium outcome is at the other extreme. The ratio of the budgets not only determines the number of items each agent wins, but also determines which agent can force the other one to win all of his items in the beginning and then win the rest of the items. Suppose that the budget of agent 1 is fixed and the budget of agent 2 is monotonically increasing. The outcome changes as follows. Say agent 2 is winning the first few items initially. At some point his budget becomes large enough that he can force agent 1 to win the items first, while keeping the *number* of items won the same. As his budget increases further, there is once again a transition point where agent 2 can win one more item, but now has to move back to winning items first. The pattern then continues until he can win all the items. There may be some discontinuity at the transition points since tie breaking rules come into effect.

1.1 Related Work

Although many variants of sequential auctions with budget constrained bidders have been considered, they are all limited to just two rounds. The seminal paper of Pitchik and Schotter (1986) considered sequential auctions with two (non-identical) items and two budgeted constrained bidders with additive valuation over the items. They characterize the equilibria of this game in the complete information setting and prove certain properties. They followed it up with an experimental validation of the decreasing prices phenomenon in Pitchik and Schotter (1988). Additional empirical evidence of decreasing prices was provided in Ashenfelter (1989). In Benoît and Krishna (2001) the agents have a common value for 2 non identical items with combinatorial valuations. Rodriguez (2009) constructs equilibrium with declining prices for sequential auction with multiple agents under the assumption of decreasing marginal utility. An isolated example of sequential auctions in the context of ad-auctions is Iyengar et al. (2006). They also consider two rounds and their results are dominating strategies for bidders participating in such auctions. Some examples of sequential auctions in settings other than budget constrained bidders are Black and Meza (1992), Hassidim et al. (2011), Jeitschko (1999), McAfee and Vincent (1993), Paes Leme et al. (2012), Weber (1981). Budget constraints have also been considered in the setting of simultaneous (vs. sequential) auction design by Bhattacharya et al. (2010), Borgs et al. (2005), Che and Gale (1998, 2000), Dobzinski et al. (2011).

Our model is also similar to some of the literature on contests such as Robson (2005), but there are crucial differences. For instance, the problem considered by Robson (2005) is as follows: for one item, if the two bidders bid x and y, then they *both pay their respective bids* and the result is that bidder 1 wins with probability $x^r/(x^r + y^r)$ where $0 < r \leq 1$ is some parameter, and bidder 2 wins with the remaining probability. Plus there is no *sequentiality* in his model. These make a huge difference; in his model the equilibrium can be easily solved for and a closed form expression for equilibrium bids is derived. In fact, if the items are identical (as in our model), then his solution is simply to divide the budgets equally among all items. This is very different from the equilibrium behavior in our model as can be seen from Tables 1 and 2.

1.2 Techniques and Difficulties

The simplest of all the properties we prove is that the number of items won is approximately proportional to the budgets. Let the total number of rounds be k and the budgets of the agents be B_1 and B_2. Suppose agent 1 bids B_2/k_2 in every round, for some integer $k_2 \in [0, k]$. Agent 2 can afford to win at most k_2 items with this strategy. Therefore agent 1 could ensure $k_1 := \min\{k - k_2, B_1 k_2/B_2\}$ items. Optimizing the choice of k_1, agent 1 can ensure he wins k_1 items for the highest integer k_1 for which $k_1 + k_2 = k$ and $B_1/k_1 \geq B_2/k_2$.

The major difficulty comes (naturally) from understanding the interplay between individual auctions resulting from the shared budget across them. The central question is what the outcome of the current auction means to an agent in terms of later rounds. Intuitively, it seems sensible that a winning agent always prefers to pay as little as possible, while a losing agent prefers the winner's payment to be as high as possible. Thus, in order to understand the outcome of our auctions, we need to understand how the utility from a sequence of k auctions behaves as a function of the remaining budgets. Unfortunately the structure of the utility function is rather complicated. While an agent's utility from the current round is still linear in the payment for the current round, there is no guarantee that the role the payment plays in reducing the budget for future rounds is also linear – and in fact, analyzing even cases with few items makes it clear that it is not. Analyzing small cases, i.e. $k = 1, 2, 3$ shows that while it does start out simple, adding more items quickly makes the function very complicated. This can be seen in Table 1 and Table 2.

It is interesting to compare sequential auctions with unit-demand bidders and budget constrained bidders. Unit-demand bidders only want one copy of the item, as long as the price is less than their valuation of the item. In one round of a sequential auction with unit-demand bidders, only the identity of the winner in that round matters to the remaining bidders. The subgame only depends on the remaining bidders. This can be interpreted as bidders having externalities on each other Paes Leme et al. (2012). In contrast, in a sequential auction with budget constrained bidders, not only the identity of the winner but also the price paid by her matters to the other bidder. The subgame and in particular the utility function depends on the residual budgets.

First of all, in order to even talk about *the* utility, you need that the equilibrium is unique, which seems circular. We get around this by first defining a *canonical outcome* and consider the utilities of the agents in this outcome. We identify certain key properties of the canonical outcome such as the monotonicity of prices, the sequence of wins and the monotonicity and continuity of the utility function that help us show that this outcome is indeed an equilibrium. We finally argue that this equilibrium is essentially unique. The proof that the canonical outcome is an equilibrium is by a joint induction on all the properties mentioned. The proof of each of these properties depends intricately on the others, inductively. This is summarized in Section 3.

The essence of all this is that any one round looks like a single item auction in the sense that for each agent there is a critical price above which he prefers to lose that item and below which he prefers to win it. (At the critical price she could either have a strict preference for winning over losing or be indifferent.) Monotonicity of prices as the auction progresses is shown by supposing for the sake of contradiction that the price of the first item is less than that of the second[2] and showing that this leads to a profitable deviation for the loser of the first item. This leads to a contradiction since (inductively) we have shown that the outcome must be an equilibrium. The construction of such a profitable deviation itself relies inductively on other properties. The sequence of wins, that is that one agent wins all the items in the beginning followed by the other agent winning the rest of the items, is also proved by contradiction via a profitable deviation. The utility function is discontinuous at the points of transition where an agent can afford to win one extra item. We need to show that this discontinuity propagates in a controlled manner, all of which makes the whole proof quite technically challenging.

Organization In Section 2 we give formal definitions of the game, the notions of equilibria we consider and analyze the cases with a few items. We give the construction of the canonical outcome, the properties on which we do the joint induction to prove that this outcome is an equilibrium and their dependence on each other in Section 3. In Section 4 we give proof sketches of some of the important steps in the induction. Section 5 contains a comparison of the outcome of the sequential auction and that of the adaptive clinching auction. Section 6 discusses extensions and future work. The full proof is in Appendix A.

2 Preliminaries

Problem definition Suppose a central principal wants to auction k identical items to two budget constrained agents by running k first-price sealed-bid auctions in a sequential manner. We seek to understand the equilibrium allocation and prices of this sequential auction. Here, we define an agent's utility to be the valuation of the items she gets *plus her remaining budget* at the end of the auction. If the budget is exceeded then the utility is negative infinity. Note that this is not the standard definition of quasi-linear utilities. Using this definition of utilities is essential for obtaining the natural monotonicity of utilities with respect to the budgets, which will play an important role in our proofs.

Further, we let B_i denote the budget of agent i and let v_i denote the value of agent i for winning each item. We will assume that the values are the dominant factor in the sense that the agents' primary goal is to maximize the number of items that they get in the sequential auction. Minimizing the total price paid for the items is only a secondary goal, conditioned on getting the same number of items. More precisely, we will enforce this assumption by assuming $v_1, v_2 > 2^k B_1, 2^k B_2$. In this case, any change in the prices will be overcompensated by even a tiny chance of getting an extra item. We consider the complete information case, that is, we assume that each agent knows all the valuations and the budgets.

Formally, this is an instance of an *extensive game with complete information and simultaneous moves*. (See Mas-Colell et al. (1995) or Osborne and Rubinstein (1994) for a formal definition.) In each round, both the agents move simultaneously and submit a bid. The outcome is a first-price auction: an item is awarded to

[2] This is the only case that we have to consider, by induction.

the agent with the higher bid and the amount of his bid is deducted from his budget. For technical reason, we will allow each agent i to bid b and $b+$ for each $0 \leq b < B_i$, where bidding $b+$ means bidding infinitesimally larger than b. If one agent bids b and the other agent bids $b+$ then the agent bidding $b+$ wins the item and is charged b. This treatment is essential for the existence of Nash equilibrium in the first price auction and therefore essential for the sequential auction. See Paes Leme et al. (2012), for example, for other use of this treatment in the literature. Finally if both agents bid the same, then the item is awarded to an agent uniformly at random.

The natural solution concept for such games is a *subgame perfect equilibrium*. Given any partial history of an extensive game, the remainder of the game is yet another extensive game, called a subgame of the original game. A subgame perfect equilibrium is a Nash equilibrium of the extensive game such that for any partial history of the game, the induced strategies of the Nash equilibrium is a Nash equilibrium of the induced subgame. Intuitively, this requires that given what has already occurred, each player acts rationally with respect to the remainder of the game. This rules out agents playing threats that are not credible.

Equilibrium refinement The sequential auction we consider has unnatural subgame perfect equilibria. So we consider refinements of the solution concept that rule out such equilibria. For instance, in a first-price sealed-bid auction of a single item where the agents' values are $v_1 > v_2$, agent 1 bidding $b+$ and agent 2 bidding b is an equilibrium for any $v_1 > b \geq v_2$. However, we want to rule out the unnatural equilibria where $b > v_2$ because in these equilibria agent 2 bids above her value.

A widely-used refinement for finite games is the *trembling-hand-perfection* originally proposed in Selten (1975), which is defined as follows:

Definition 1 (Trembling-Hand-Perfection)**.** *An equilibrium strategy profile (σ, τ) of a two-player finite game G is* trembling-hand-perfect *if there exists a sequence $\{(\sigma_j, \tau_j)\}_j$ of completely mixed strategy profiles such that (σ_j, τ_j) converges (σ, τ) as j goes to infinity, and σ is a best reply to every τ_j, and τ is a best reply to every σ_j in the sequence.*

Here, a completely mixed strategy profile is one in which every strategy of every player is played with some positive probability. The positive, and presumably tiny, probability of playing strategies that are not in the equilibrium profile corresponds to the fact that players make small mistakes from time to time (trembling hand). So what trembling-hand-perfection indicates is that the equilibrium strategy profile is robust to such small mistakes in the sense that using the equilibrium strategy maximizes the agent's utility even if we assume the other agent makes small mistakes.

Note that this is not the standard definition of trembling-hand-perfection in the literature. We choose this definition because it is more convenient for our discussion. See Selten (1975) for the standard definition of trembling-hand-perfection and the equivalence of the different definitions.

Despite the many appealing properties of trembling-hand-perfection in finite games, there is no well-accepted definition of trembling-hand-perfection in the literature for games with continuous strategy space such as the sequential auctions in this paper. Moreover, if we use the definition of finite games directly, then *trembling-hand-perfect equilibria may not exist*. Indeed, a trembling-hand-perfect equilibrium may not exists even for the first-price auction with a single item and two agents. Suppose the values are $v_1 > v_2$. Then, consider the equilibrium agent 1 bidding v_2+ and agent 2 bidding v_2 (it is easy to rule out the other equilibria). But bidding v_2 is dominated by bidding $v_2 - \epsilon$ for agent 2: if both cases lose, then they yield the same utility; if bidding v_2 wins and bidding $v_2 - \epsilon$ loses, then winning at v_2 is still dominated because it yield utility zero; finally if both cases win, then bidding $v_2 - \epsilon$ is strictly better because it pays less. Therefore, this cannot be a trembling-hand-perfect equilibrium because folklore result asserts the strategy of each agent in a trembling-hand-perfect equilibrium must be non-dominated.

A commonly used remedy to the absence of a theory for trembling-hand-perfection in continuous games is to consider a sequence of discretized version of the continuous game that converges to it, and then analyze the limit of the trembling-hand-perfect equilibria of the discretized games (e.g. Bagnoli and Lipman (1989), Broecker (1990)). This approach, however, is very difficult to apply to games with complicated structures such as the sequential auction with arbitrary number of items.

In order to settle this problem, we will use a slightly weaker equilibrium refinement, which we refer to as the *semi-trembling-hand-perfect equilibrium*.

Definition 2 (Semi-Trembling-Hand-Perfection)**.** *An equilibrium strategy profile (σ, τ) of a two-player game G is* semi-trembling-hand-perfect *if there exists a sequence $\{(\sigma_j, \tau_j)\}_j$ of completely mixed strategies such that (σ_j, τ_j) converges to (σ, τ), and the best reply to σ_j converges to τ, and the best reply to τ_j converges to σ as j goes to infinity.*

As we can see by comparing Definition 2 and Definition 1, the notion of semi-trembling-hand-perfection is still trying to model the robustness of the equilibrium strategies with respect to small mistakes made by the agents, but in a weaker sense: when the other agent makes small mistakes, we no longer require the best reply to be exactly the equilibrium strategy; however, the best reply has to converge to the equilibrium strategy as the other agent makes fewer and fewer mistakes.

We will construct a subgame-perfect equilibrium that is unique after the refinement of semi-trembling-hand-perfection.

2.1 Warm-up: Few-item cases

Let us first examine the cases with only 1, 2 or 3 items in order to build our intuition for the problem.

$k = 1$ First of all, let us consider the simplest case of a single item. In this case, the problem becomes the classic first price auction. Therefore, suppose the agent values are $B_1 < B_2$, then the unique Nash equilibrium that survives the iterated elimination of dominated strategies is where agent 1 bids B_1 and agent 2 bids B_1^+ and wins the item. In the case of $B_1 = B_2$, both agent will bid their budgets and we have a tie.

$k = 2$ Next, let us move on to the more interesting case of a two-item sequential auction. In order to examine how the allocation of items changes as the ratio between the budgets changes, let us fix $B_2 = 1$ and gradually increase B_1 starting from 0.

If $B_1 < \frac{1}{2}$, then agent 2 has enough budget to get both items. Since the value of a single item dominates any changes in the payment, agent 2 wins both items by bidding B_1^+ in both rounds.

If $\frac{1}{2} < B_1 < 1$, then agent 1 can guarantee herself an item by bidding B_1 in both rounds. So the best strategy for agent 2 is to let agent 1 win the first item and pay the highest possible price. There are two types of credible threats that agent 2 could use to set the price for the first item. The first threat is to offer agent 1 the first item at price $B_2 - B_1$, threatening that if agent 2 wins the first item at this price then she has sufficient budget remaining to win the second item as well. The second threat is to offer the first item at price $\frac{B_1}{2}$, threatening that if agent 1 do not take the first item at this price then she would need to pay a higher price in order to win the second item.

Agent 2 uses the larger of these two threats to set the price in round 1. If $\frac{1}{2} < B_1 < \frac{2}{3}$, then agent 2 uses the first threat. Agent 1 wins the first item at price $B_2 - B_1$ and agent 2 wins the second item paying agent 1's remaining budget, $2B_1 - B_2$. If $\frac{2}{3} < B_1 < 1$, then agent 2 exploits the second threat. Agent 1 wins the first item at price $\frac{1}{2}B_1$ and agent 2 wins the second item also at price $\frac{1}{2}B_1$.

By symmetry the case of $B_1 > 1$ is identical with the roles of the agents swapped. The equilibrium allocations in various cases are summarized in Table 1.

Table 1: Equilibrium strategies for $k = 2$

Phase	Budget ratio (B_1/B_2)	Round 1	Round 2
1	$(0, \frac{1}{2})$	2 wins at B_1	2 wins at B_1
2	$(\frac{1}{2}, \frac{2}{3})$	1 wins at $B_2 - B_1$	2 wins at $2B_1 - B_2$
3	$(\frac{2}{3}, 1)$	1 wins at $\frac{1}{2}B_1$	2 wins at $\frac{1}{2}B_1$

$k = 3$ The case with three items is much more complicated in the sense that there are more possibilities of allocation sequences. Here we briefly demonstrate these different allocation sequences and the intuition behind them. Again, we fix the budget of agent 2 to be $B_2 = 1$ and gradually increase agent 1's budget starting from 0.

The first phase is when $B_1 < \frac{1}{3}$, where agent 2 has enough budget to win all three items by bidding B_1^+ in all three rounds.

The second phase is when $\frac{1}{3} < B_1 < \frac{3}{8}$, where agent 1 has enough budget to obtain one item. In this phase, agent 2 forces agent 1 to get the first item at price $B_1 - 2B_2$ and win the next two items cheaply paying agent 1's remaining budget. The threat used by agent 2 is that she could win the remaining two items as well if agent 1 does not accept this offer. If agent 2 gets the first item at this price, then the induced subgame falls into phase 1 of the two-item case.

The third phase is when $\frac{3}{8} < B_1 < \frac{1}{2}$, where agent 2 forces agent 1 to get the first item at price $\frac{1}{4}B_2$ via a different threat: if agent 2 wins the first item at this price, then the induced subgame falls into phase 2 of the two-item case. In fact the threshold at which the induced subgame falls into phase 2 of the two-item case is $B_2 - \frac{3}{2}B_1$. However for this to be a credible threat by agent 2, she must weakly prefer winning the item at this price to losing it. On the one hand, the utility of agent 2 for winning the first item at some price p, assuming that the induced subgame falls into phase 2 of the two-item case, is $2v_2 + B_2 - p - (2B_1 - (B_2 - p)) = 2v_2 + 2B_2 - 2B_1 - 2p$. On the other hand, the utility of agent 2 for losing the first item at price p is $2v_2 + B_2 - 2(B_1 - p) = 2v_2 + B_2 - 2B_1 + 2p$. The price $p = \frac{1}{4}B_2$ is obtained by equating the two and solving $2v_2 + 2B_2 - 2B_1 - 2p = 2v_2 + B_2 - 2B_1 + 2p$.

The rest of the phases are similar in spirit to the above. In the fourth phase, agent 2 forces agent 1 to get the first item at price $\frac{1}{2}B_1$, threatening that if she wins the item at this price then the induced subgame falls into phase 3 of the two-item case. In the fifth to the seventh phases, it is still the case that agent 1 gets one item and agent 2 gets two. However, agent 1 now has enough budget to be in the dominant position in the price competition. So agent 1 forces agent 2 to pay higher prices for the first two items and then wins the last item, paying agent 2's remaining budget. Depending on the budget of agent 1, the prices she can set are determined by the phases in the two-item subgame they would end up in, if she wins the first item at those prices. We do not discuss further details of these threats but summarize the outcomes in Table 2.

Observations We end this section with a few observations regarding the equilibrium outcomes discussed above.

1. First of all, agent 1 and agent 2 get k_1 and k_2 items respectively ($k_1 + k_2 = k$) if and only if the budgets satisfy $\frac{k_1}{k_2+1} < \frac{B_1}{B_2} < \frac{k_1+1}{k_2}$.[3] Intuitively, these conditions can be interpreted as follows: if the average price an agent, say, agent 1, can afford for k_1 items is strictly greater than the average price agent 2 can afford for $k_2 + 1$ items, then agent 1 can guarantee winning k_1 items. For instance, agent 1 could keep bidding $\frac{B_1}{k_1}$.

[3]We omit the boundary cases $\frac{B_1}{B_2} = \frac{k_1}{k_2+1}$ here. As we will see in the next section, in these boundary cases the agents keep making the same bids until one of them runs out of budget. We handle these cases separately.

Table 2: Equilibrium strategies for $k = 3$

Budget ratio (B_1/B_2)	Round 1	Round 2	Round 3
$(0, \frac{1}{3})$	2 wins at B_1	2 wins at B_1	2 wins at B_1
$(\frac{1}{3}, \frac{3}{8})$	1 wins at $B_2 - 2B_1$	2 wins at $3B_1 - B_2$	2 wins at $3B_1 - B_2$
$(\frac{3}{8}, \frac{1}{2})$	1 wins at $\frac{1}{4}B_2$	2 wins at $B_1 - \frac{1}{4}B_2$	2 wins at $B_1 - \frac{1}{4}B_2$
$(\frac{1}{2}, \frac{2}{3})$[1]	1 wins at $\frac{1}{2}B_1$	2 wins at $\frac{1}{2}B_1$	2 wins at $\frac{1}{2}B_1$
$(\frac{2}{3}, \frac{5}{6})$	2 wins at $\frac{1}{3}B_2$	2 wins at $\frac{1}{3}B_2$	1 wins at $\frac{1}{3}B_2$
$(\frac{5}{6}, \frac{9}{10})$	2 wins at $B_1 - \frac{1}{2}B_2$	2 wins at $\frac{3}{4}B_2 - \frac{1}{2}B_1$	1 wins at $\frac{3}{4}B_2 - \frac{1}{2}B_1$
$(\frac{9}{10}, 1)$	2 wins at $B_1 - \frac{1}{2}B_2$	2 wins at $2B_1 - \frac{3}{2}B_2$	1 wins at $3B_2 - 3B_1$

[1] In this case, due to a tie in the first round, it could also be that agent 2 wins the first item at price $\frac{1}{2}B_1$, agent 1 wins the second item at price $\frac{1}{2}B_1$, and agent 2 wins the last item at price $\frac{1}{2}B_1$. Since the agents get the same number of items at the same prices in both the outcomes, we will break ties in some particular way in order to get more consistent structures, which are explained in more details in the next section.

2. It is not difficult to verify that in each of the phases of different allocation sequences, the prices of the items are non-increasing in the number of rounds.

3. Intriguingly, it is always the case that one of the agents wins her share of the items and then the other agent wins the rest of the items. In other words, there are no interleaving of winners as one might imagine.

4. Finally, we can easily examine that the utility of an agent in the equilibrium is monotonically increasing in the agent's budget and non-increasing in the budget of the other agent. Although this observation seems intuitive, an example in Benoît and Krishna (2001) indicates that the utilities might not be monotone in the agents' budgets. However, in Benoît and Krishna (2001) the utility is defined as the total value of the items an agent gets minus the total price that she pays. Indeed, with this definition, an agent's utility could counter-intuitively decrease as her budget increases. For instance, suppose we fix agent 2's budget to be 1 and gradually increases agent 1's budget from $\frac{2}{3}$ to 1 in the two-item case. Agent 1 always gets one item but agent 2 forces agent 1 to pay higher and higher price for the item as agent 1's budget increases. In this paper, we use a different notion, an agent's utility is defined to be the total value of the items she gets plus the remaining budget that she has at the end. With this definition, we can show the desired monotonicity, which plays a crucial role in our analysis.

3 General Case

In this section, we consider the general case of arbitrary number of items and outline the proof structure and the high level ideas. The easiest part of the whole proof is showing how the number of items won by each agent in any equilibrium depends on the budgets. The intuition is that the number of items an agent wins should be approximately proportional to his budget. Consider $\frac{k_1}{k-k_1+1}$ as a function of k_1 and observe that it is monotonically increasing. So there must exists a unique $k_1 \in \mathbb{Z}_{\geq 0}$ such that $\frac{k_1}{k-k_1+1} \leq \frac{B_1}{B_2} < \frac{k_1+1}{k-k_1}$. If we let $k_2 = k - k_1$, then either $\frac{k_1}{k_2+1} < \frac{B_1}{B_2} < \frac{k_1+1}{k_2}$, or $\frac{k_1}{k_2+1} = \frac{B_1}{B_2}$. We will handle these two cases separately as follows.

Proposition 3.1. *Suppose $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ satisfy $k_1 + k_2 = k$ and $\frac{k_1}{k_2+1} < \frac{B_1}{B_2} < \frac{k_1+1}{k_2}$. Then, in any subgame perfect equilibrium of the sequential auction, agent 1 gets k_1 items and agent 2 gets k_2 items.*

Proof. Regardless of the strategy of agent 2, agent 1 can guarantee k_1 items for herself by bidding $\frac{B_1}{k_1}$ until her budget is exhausted. This is because at this price, agent 2 can afford to win at most k_2 items after which her remaining budget is strictly less than $\frac{B_1}{k_1}$. Therefore any outcome in which agent 1 gets less than k_1 items cannot be an equilibrium regardless of the price paid, since we assumed that $v_1 > B_1$. By symmetry agent 2 gets at least k_2 items in any equilibrium. Since $k_1 + k_2 = k$, it must be the case that agent 1 gets exactly k_1 items and agent 2 gets exactly k_2 items. $\square$

Proposition 3.2. *Suppose $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ satisfy $k_1 + k_2 = k$ and $\frac{k_1}{k_2+1} = \frac{B_1}{B_2}$. Then, in any equilibrium, agent 1 gets at least $k_1 - 1$ items and agent 2 gets at least k_2 items. (The tie-breaking comes into effect for the extra item.)*

The proof of this proposition is essentially the same as that of Proposition 3.1.

3.1 Canonical Outcome

We now construct a canonical outcome of the game. The heart of the proof is showing that this is a subgame perfect equilibrium. This involves exploiting several other properties of this outcome. We then show that under the refinement of semi-trembling-hand-perfection this is the only equilibrium.

The canonical outcome is defined recursively. We start with the single-item case and assume w.l.o.g. that $B_1 \geq B_2$. In this case, the sequential auction is simply the first-price sealed-bid auction.

> *Canonical Outcome (base case):* The canonical outcome for a single-item is agent 1 bidding B_2+ and agent 2 bidding B_2 if $B_1 > B_2$, and both agents bidding $B_1 = B_2$ otherwise.

Suppose we have defined the canonical outcome for $k-1$ or fewer items. In order to recursively define the canonical outcome for k items, we need some definitions. We use i and $-i$ to denote agents. If i is 1 then $-i$ is 2 and vice versa.

Definition 3 (Utility). *Let $U_i^{(k)}(B_i, B_{-i})$ denote the utility (expected utility in case tie-breaking comes into effect) of agent i in the canonical outcome when the budgets are B_i and B_{-i}.*

Definition 4 (Winning Utility). *Let $W_i^{(k)}(B_i, B_{-i}, p) \stackrel{\text{def}}{=} v_i + U_i^{(k-1)}(B_i - p, B_{-i})$ denote the* winning utility *of agent i, that is, the utility when agent i wins the first item at price p and both agents follow the canonical outcome in the remaining sequential auction with $k-1$ items.*

Definition 5 (Losing Utility). *Let $L_i^{(k)}(B_i, B_{-i}, p) \stackrel{\text{def}}{=} U^{(k-1)}(B_i, B_{-i} - p)$ denote the* losing utility *of agent i, that is, the utility when agent i wins the first item at price p and both agents follow the canonical outcome in the remaining sequential auction with $k-1$ items.*

We now state a monotonicity property of these functions which is used in the recursive definition of the canonical outcome. This property is proved along with other properties later.

Proposition 3.3. *$U_i^{(k)}(B_i, B_{-i})$ is increasing in B_i, and non-increasing in B_{-i}.*

Proposition 3.4. *$W_i^{(k)}(B_i, B_{-i}, p)$ is decreasing in p while $L_i^{(k)}(B_i, B_{-i}, p)$ is non-decreasing in p, and both are increasing in B_i and non-increasing in B_{-i}.*

It is easy to see that Proposition 3.3 implies Proposition 3.4. Given the monotonicity in Proposition 3.4, we define the *critical prices* of the agents for the first round.

Definition 6 (Critical Prices). *There exists a unique price $p_i^{(k)}$ s.t. for any $p > p_i^{(k)}$, $W_i^{(k)}(B_i, B_{-i}, p) < L_i^{(k)}(B_i, B_{-i}, p)$, and for any $p < p_i^{(k)}$, $W_i^{(k)}(B_i, B_{-i}, p) > L_i^{(k)}(B_i, B_{-i}, p)$. We will refer to $p_i^{(k)}$ as the* critical price *of agent i.* [4]

The definition of critical prices is identical to the definition of *critical values* in Pitchik and Schotter (1986) for the two-item case. Here, we extend the definition to arbitrary number of items. The critical price $p_i^{(k)}$ is similar, although not identical, to an agent's valuation in the single-item auction in the following sense: agent i is willing to get the first item at any price lower than the critical price $p_i^{(k)}$, while she has no interest in winning the first item at a price higher than $p_i^{(k)}$. Note that one or both of $W_i^{(k)}$ and $L_i^{(k)}$ could be discontinuous at $p_i^{(k)}$. So the monotonicity in Proposition 3.4 does not have further indication on whether agent i prefers winning or losing the first item at price $p_i^{(k)}$. In the later sections, we do show that an agent weakly prefers winning to losing at the critical price. In order to show this, we will need to utilize a few subtle structures of the canonical outcome.

We are now ready to complete the construction of the canonical outcome.

> *Canonical Outcome (recursive step):* Suppose that the canonical outcome is defined for $k-1$ items and we have computed the critical prices. Assume w.l.o.g that $p_i^{(k)} \geq p_{-i}^{(k)}$. Then, the canonical outcome of the k-item case is agent i bidding $p_{-i}^{(k)}+$ and agent $-i$ bidding $p_{-i}^{(k)}$ and both agents following the canonical outcome in the subgame of $k-1$ items, if $p_i^{(k)} > p_{-i}^{(k)}$. Otherwise, the canonical outcome is both agents bidding $p_i^{(k)} = p_{-i}^{(k)}$ and following the canonical outcome in the resulting subgame of $k-1$ items.

3.2 Properties of the Canonical Equilibrium

The main structural theorem in this paper is the following.

Theorem 3.5. *The canonical outcome is a subgame perfect equilibrium of the sequential auction with k items.*

The proof of this theorem is by a joint induction on Proposition 3.3 and several other properties of canonical outcomes. We now detail these properties and specify how the proof of each of these properties inductively depends on the others.

The next two properties characterize which agent wins which round in the canonical outcome of the sequential auctions and the monotonicity of the prices paid in the canonical outcome.

Proposition 3.6. *Suppose $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ satisfy $k_1 + k_2 = k$ and $\frac{k_1}{k_2+1} \leq \frac{B_1}{B_2} < \frac{k_1+1}{k_2}$. Based on the different cases, the following happens in the canonical outcome:*

Case 1 (Type I tie-breaking) $\frac{k_1}{k_2+1} = \frac{B_1}{B_2}$. *In this case, both agents will keep bidding $p^* \stackrel{\text{def}}{=} \frac{B_1}{k_1} = \frac{B_2}{k_2+1}$ until one of the agents runs out of her budget, and then the other agent will win the rest of the items for free.*

Case 2 $\frac{k_1}{k_2+1} < \frac{B_1}{B_2} < \frac{k_1+1}{k_2+1}$. *In this case, agent 1 wins the first k_1 items and then agent 2 wins the rest.* [1]

[4] We note that it might be the case that even if p achieves its maximum value B_i, we still have $W_i^{(k)}(B_i, B_{-i}, p) > L_i^{(k)}(B_i, B_{-i}, p)$. But it is not difficult to prove this can only happen when $B_i < \frac{B_{-i}}{k}$, in which case it is clear agent $-i$ will win all the items paying B_i per item. We will omit this trivial case in our discussion and assume p_i always exists.

Case 3 (Type II-A tie-breaking) $\frac{B_1}{B_2} = \frac{k_1+1}{k_2+1}$. *In this case, both agents will keep bidding* $p^* \stackrel{\text{def}}{=} \frac{B_1}{k_1+1} = \frac{B_2}{k_2+1}$ *until one of the agent's budget becomes* p^**, and then the other agent will get the remaining items by bidding* p^*+. [2]

Case 4 $\frac{k_1+1}{k_2+1} < \frac{B_1}{B_2} < \frac{k_1+1}{k_2}$. *In this case agent* 2 *wins the first* k_2 *items and then agent* 1 *wins the rest.* [1]

[1] *When* $\frac{B_1}{B_2}$ *is sufficiently close to* $\frac{k_1+1}{k_2+1}$*, we may end up in another type tie-breaking case. However, in this tie-breaking case, we can assume w.l.o.g. the allocation is as asserted in Proposition 3.6 without changing the agents' utilities. We will explain in details in Appendix A. Such treatment will be convenient for our proofs.*

[2] *In this tie-breaking case, we can assume either agent* 1 *gets the first* k_1 *items and then agent* 2 *gets the remaining* k_2 *items, or the other way around, without changing the utilities of the agents. Such treatment will be convenient for our proofs.*

Proposition 3.7. *The prices paid in each round of the canonical outcome is non-increasing as the auction proceeds.*

Finally, we will also show the following property about the continuity of the utility functions in the agents' budgets.

Proposition 3.8. *For* $i = 1, 2$, $U_i^{(k)}$ *is continuous in both* B_i *and* B_{-i}*, except when* $\frac{B_i}{B_{-i}} = \frac{k_i}{k_{-i}+1}$ *for* $k_i, k_{-i} \in \mathbb{Z}_{\geq 0}$ *such that* $k_i + k_{-i} = k$*. Moreover,*

$$\lim_{B_{-i} \to \frac{k_{-i}+1}{k_i} B_i^-} U_i^{(k)}(B_i, B_{-i}) = k_i v_i \ , \quad \lim_{B_{-i} \to \frac{k_{-i}+1}{k_i} B_i^+} U_i^{(k)}(B_i, B_{-i}) = (k_i - 1)v_i + B_i$$

$$\lim_{B_i \to \frac{k_i}{k_{-i}+1} B_{-i}^-} U_i^{(k)}(B_i, B_{-i}) = (k_i - 1)v_i + \frac{k_i}{k_{-i}+1} B_{-i} \ , \quad \lim_{B_i \to \frac{k_i}{k_{-i}+1} B_{-i}^-} U_i^{(k)}(B_i, B_{-i}) = k_i v_i$$

If $\frac{B_i}{B_{-i}} = \frac{k_i}{k_{-i}+1}$, then the tie-breaking comes into effect and the utilities are random variables. We can calculate the expected utilities, and these are summarized later.

Dependencies For the sake of presentation, we will let $\mathcal{P}_1^{(k)}$ denote Theorem 3.5 for k items. Let $\mathcal{P}_2^{(k)}$ denote Proposition 3.6 for k items. Let $\mathcal{P}_3^{(k)}$ denote Proposition 3.7 for k items. Let $\mathcal{P}_4^{(k)}$ and $\mathcal{P}_5^{(k)}$ denote Proposition 3.3 and Proposition 3.8 respectively. The joint induction proceeds by assuming each of the properties $\mathcal{P}_1^{(k-1)}$ to $\mathcal{P}_5^{(k-1)}$ and proving properties $\mathcal{P}_1^{(k)}$ to $\mathcal{P}_5^{(k)}$. The dependencies are summarized below:

$\mathcal{P}_1^{(k)}$ depends on $\mathcal{P}_1^{(k-1)}$, $\mathcal{P}_4^{(k-1)}$, and $\mathcal{P}_5^{(k-1)}$.

$\mathcal{P}_2^{(k)}$ depends on $\mathcal{P}_1^{(k)}$, $\mathcal{P}_4^{(k-1)}$, and $\mathcal{P}_5^{(k-1)}$.

$\mathcal{P}_3^{(k)}$ depends on $\mathcal{P}_1^{(k)}$, $\mathcal{P}_2^{(\leq k)}$, $\mathcal{P}_4^{(k-1)}$, and $\mathcal{P}_5^{(k-1)}$;

$\mathcal{P}_4^{(k)}$ and $\mathcal{P}_5^{(k)}$ depend on $\mathcal{P}_4^{(k-1)}$, $\mathcal{P}_5^{(k-1)}$, and $\mathcal{P}_2^{(k)}$.

3.3 Uniqueness of equilibrium

Note that when $p_i^{(k)} > p_{-i}^{(k)}$ (or symmetrically the other way around), we face the problem of equilibrium section. We can argue that agent i bidding $p_{-i}^{(k)}+$ and agent $-i$ bidding $p_{-i}^{(k)}$ is the only "stable" equilibrium in the sense that it is the unique semi-trembling-hand-perfect equilibrium.

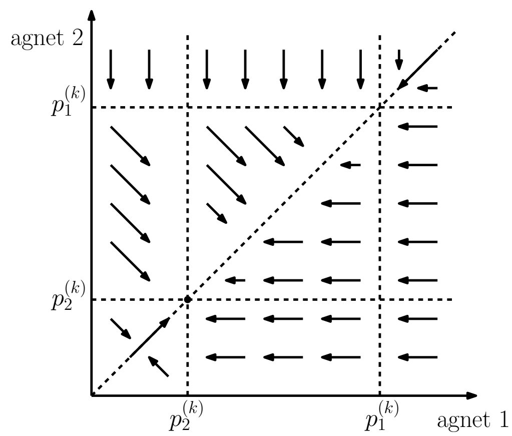

Figure 1: Proof sketch of uniqueness of the canonical equilibrium by analyzing the dynamic of the game.

Proposition 3.9. *The canonical equilibrium is the unique semi-trembling-hand-perfect and subgame-perfect equilibrium.*

The analysis is very similar to the equilibrium refinement in a first price auction and therefore we will defer the proof to Appendix B. Here we will give an informal explanation by presenting the dynamic of the game in Figure 1. Suppose w.l.o.g. that $p_1^{(k)} > p_2^{(k)}$, and agent 1 bids b_1 and agent 2 bids b_2. If $b_1 > b_2$ and $b_1 > p_2^{(k)}$, then agent 1 wants to decease her bid while agent 2 has no incentive of changing her strategy. If $b_1 < b_2$ and $b_2 > p_1^{(k)}$, then agent 1 has no incentive to change her bid while agent 2 wants to decrease her bid to lose the item. If $b_1 < b_2 < p_1^{(k)}$, then agent 1 wants to increase her bid to wins the item while agent 2 wants to decrease her bid to lower the price. Finally, if $b_2 < b_1 < p_2^{(k)}$, then agent 1 wants to decrease her bid to lower the price, while agent 2 wants to increase her bid to win the item. Summarizing these cases, it is easy to see the only "stable" point is agent 1 bidding $p_2^{(k)}+$ and agent 2 bidding $p_2^{(k)}$.

3.4 First-price vs. second-price

We can also consider the second-price version of the sequential auction, where the winner and the price are chosen using the second-price rule in each round: the agent with the higher bid wins and pays the other agent's bid.

In the second-price version, the canonical outcome we construct in this paper is still a subgame-perfect equilibrium, via almost identical arguments. However, it is no longer the stable one because underbidding is weakly-dominated under the second-price rule. In the second-price auction of a single item, truthful bidding is a stable equilibrium as it is a dominant-strategy equilibrium. We can use the critical-price methodology in this paper to analyze the case of multiple items. Roughly speaking, when it comes to 2 or more items, the agent with the lower critical price may have incentive to overbid (unlike the single-item case) in order to deplete the other agent's budget. In such case, the first-round bids in the equilibria must be the winner bidding $b+$ while the loser bidding b for some b between the critical prices. Further, since underbidding is weakly-dominated in the second-price rule, in the stable equilibrium b must equal the higher one of the critical prices. In other words, it is conceivable that the stable equilibrium is both agent bidding the higher one of the agents' critical prices in each round, rather than the lower one as in the first-price version. This observation is implicitly proved for the value-dominant case of two items in Pitchik and Schotter (1986). Also, see Pitchik and Schotter (1988) for experimental results comparing the first-price and second-price sequential auctions. We shall not discuss any further the second-price version but leave it as an interesting open question to study the equilibrium of the second-price sequential auction.

4 Proof Sketches

In this section, we will sketch the proofs of some of the important implications outlined in Section 3. The complete joint induction of these claims is in Appendix A.

4.1 Two-Phase Winner Sequence

Consider the two-phase winner sequence described in Proposition 3.6. The simplest case is the type I tie-breaking, which we restate and prove as follows.

Proposition 4.1 (Type I tie-breaking)**.** *Suppose $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ satisfy $k_1 + k_2 = k$ and $\frac{k_1}{k_2+1} = \frac{B_1}{B_2}$. Then, in the canonical outcome, both agents keep bidding $p^* \stackrel{\text{def}}{=} \frac{B_1}{k_1} = \frac{B_2}{k_2+1}$ until one agent is out of budget, and then the other agent wins the rest for free.*

Proof. Intuitively, this is the case where the budget ratio lies at the intersection of the region where agent 1 gets k_1 items and agent 2 gets k_2 items and the region where agent 1 gets $k_1 - 1$ items and agent 2 gets $k_2 + 1$ items. In other words, agent 1 can guarantee $k_1 - 1$ items and agent 2 can guarantee k_2 items for sure, and both agents are competing for the extra item by keep bidding the highest rational bid they have.

Now we formally prove the claim. We first prove that $p_1^{(k)} = p_2^{(k)} = p^*$. Consider any price $p^* + \epsilon$ for sufficiently small $\epsilon > 0$. It is easy to verify that $\frac{k_1-2}{k_2+2} < \frac{B_1-p-\epsilon}{B_2} < \frac{k_1-1}{k_2+1}$ and $\frac{k_1}{k_2} < \frac{B_1}{B_2-p-\epsilon} < \frac{k_1+1}{k_2-1}$ when ϵ is sufficiently small. So by Proposition 3.6, if agent 1 wins the first item at price $p^* + \epsilon$, then in the subgame she only gets $k_1 - 2$ items and thus $k_1 - 1$ items in total; on the other hand, if agent 1 loses the first item at price $p^* + \epsilon$, then in the subgame she gets k_1 items. Therefore, we have $W_1^{(k)}(B_1, B_2, p^* + \epsilon) < L_1^{(k)}(B_1, B_2, p^* + \epsilon)$ since $v_1 \gg B_1$. So by the definition of $p_1^{(k)}$ we have $p_1^{(k)} \leq p^*$. Similarly, we can show $p_1^{(k)} \geq p^*$. So we have $p_1^{(k)} = p^*$. Via an almost identical proof we can show that $p_2^{(k)} = p^*$.

Now note that $\frac{B_1-p^*}{B_2} = \frac{k_1-1}{k_2+1}$ and $\frac{B_1}{B_2-p^*} = \frac{k_1}{k_2}$. So we can recursively apply the same argument in the subgames to finish the proof of Proposition 4.1. □

Next, we sketch the proof of the other three cases. We first clarify the tie-breaking case described in the footnote of Proposition 3.6.

Proposition 4.2. *Suppose $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ satisfy $k_1 + k_2 = k$ and $\frac{k_i}{k_{-i}+1} < \frac{B_i}{B_{-i}} \leq \frac{k_i+1}{k_{-i}+1}$. Then, in the canonical outcome it is always the case that agent i wins the first k_i items, and then agent $-i$ gets the remaining k_{-i} items paying agent i's remaining budget for each item. Here, there are two tie-breaking caveats:*

Type II-A Tie-breaking *If $\frac{B_1}{B_2} = \frac{k_1+1}{k_2+1}$, then in the canonical outcome both agents keep bidding $p^* \stackrel{\text{def}}{=} \frac{B_1}{k_1+1} = \frac{B_2}{k_2+1}$ until one agent's budget becomes p^*, and then the other agent gets the remaining item by bidding p^*+.*

In this tie-breaking case, we can assume either agent 1 gets the first k_1 items and then agent 2 gets the remaining k_2 items, or the other way around, without changing the utilities of the agents.

Type II-B Tie-breaking *If $\frac{B_i}{B_{-i}}$ is smaller than and sufficiently close to $\frac{k_i+1}{k_{-i}+1}$, then in the canonical outcome both agents keep bidding $p^* \stackrel{\text{def}}{=} \frac{B_i}{k_i+1}$ until either agent i gets k_i items or agent $-i$ gets $k_{-i} - 1$ items. In the former case, agent $-i$ then wins the remaining items paying agent i's remaining budget p^*. In the latter case, agent i then keeps bidding p^*+ until she gets k_i items, and then agent i wins the remaining items paying agent i's remaining budget p^*.*

In this tie-breaking case, we can assume agent i gets k_i items and then agent i gets the remaining k_{-i} items, without changing the utilities of the agents.

Sketch of Proposition 4.2. Here we only show that one of the agent gets her share of the items first and then the other agent gets the rest. Once we have shown this, it is easy to deduce who gets the items first.

Suppose for contradiction that agent 1 gets the first item paying $p_2^{(k)}$, and then agent 2 gets the next k_2 items, and finally agent 1 gets the rest. Briefly speaking, we first show that in this case, the first item must be sold at a price that is lower than the average price that agent 1 pays. Then, we argue that since the first item is so cheap, agent 1 must have a profitable deviation by winning the first item.

There will be several cases we need to consider in the complete proof. Here we describe one of the cases. We assume the following: (1) no tie in the first round: $p_1^{(k)} > p_2^{(k)}$; (2) agent 2 is indifferent between winning and losing the first item at $p_2^{(k)}$; (3) if agent 2 wins the first item paying $p_2^{(k)}$, then in the subgame agent 2 wins the next $k_2 - 1$ items and then agent 2 wins the rest.

By the second assumption, agent 2 has the same remaining budget regardless of whether she wins the first item or not. If agent 2 wins the first item, agent 1 gets k_1 items at price equal to agent 2's remaining budget. If agent 1 wins the first item, then agent 1 gets $k_1 - 1$ items at a price equal to agent 2's remaining budget, and one item at price $p_2^{(k)}$. Further, agent 1 must strictly prefer winning the first item. Therefore, $p_2^{(k)}$ must be strictly smaller than agent 2's remaining budget. Now by the price monotonicity (Proposition 3.7) of $k-1$ items, $p_2^{(k)}$ is strictly smaller than any other price in the assumed equilibrium. Now we have a contradiction by constructing the following equilibrium. Agent 2 could win the first item paying a price greater than $p_2^{(k)}$ but lower than the prices of the other items in the assumed equilibrium. Then, agent 2 could keep bidding $p_2^{(k)}$ until agent 1 gets at least one item, say item j^*. Finally, agent 2 could follow the canonical outcome thereafter. In the first j^* rounds of this deviation, agent 1 gets one item at price $p_2^{(k)}$, and agent 2 gets $j^* - 1$ items paying roughly $p_2^{(k)}$ per item. In the first j^* rounds of the assumed equilibrium, however, agent 1 also gets one item at price $p_2^{(k)}$, and agent 2 gets $j^* - 1$ items paying strictly greater than $p_2^{(k)}$ per item. Therefore, agent 2 is strictly better off in the deviation by the monotonicity of an agent's utility in her budget. $\square$

4.2 Monotonicity of Utilities

The complete proof of Proposition 3.3 requires a more detailed case analysis. Here we sketch the proof by describing the analysis for one of the cases.

For the sake of the discussion here, we assume the following: (1) agent 1 wins the first item at agent 2's critical price in the canonical outcome; (2) agent 2 is indifferent between winning and losing the first item at price $p_2^{(k)}$; (3) the above two assumptions still hold when agent 2's budget increases from B_2 to $B_2 + \epsilon$ or when agent 1's budget increases from B_1 to $B_1 + \epsilon$ for sufficiently small $\epsilon > 0$; (4) we have inductively proved Proposition 3.3 for the case of $k-1$ items.

First, consider increasing agent 2's budget from B_2 to $B_2 + \epsilon$. Note that agent 2's utility when her budget is B_2 is $U_2^{(k)}(B_2, B_1) = L_2^{(k)}(B_2, B_1, p_2^{(k)}) = W_2^{(k)}(B_2, B_1, p_2^{(k)})$ as she is indifferent between winning and losing at price $p_2^{(k)}$. Moreover, we have $L_2^{(k)}(B_2 + \epsilon, B_1, p_2^{(k)}) > L_2^{(k)}(B_2, B_1, p_2^{(k)}) = U_2^{(k)}(B_2, B_1)$ where we get the inequality by comparing the budgets after the first round when agent 2's budget is B_2 and $B_2 + \epsilon$. Similarly, we have $W_2^{(k)}(B_2 + \epsilon, B_1, p_2^{(k)}) > W_2^{(k)}(B_2, B_1, p_2^{(k)}) = U_2^{(k)}(B_2, B_1)$. So by bidding $p_2^{(k)}$ agent 2 could guarantee strictly more utility than $U_2^{(k)}(B_2, B_1)$ when her budget is $B_2 + \epsilon$. Thus, in the canonical outcome agent 2's utility will strictly increase when her budget increases.

Next, consider increasing agent 1's budget from B_1 to $B_1 + \epsilon$. We will let $p'(\epsilon)$ denote the critical price of agent 2 when agent 1's budget is $B_1 + \epsilon$ instead of B_1. We will show that $p'(\epsilon) < p_2^{(k)} + \epsilon$. Note that $L_2^{(k)}(B_2, B_1 + \epsilon, p_2^{(k)} + \epsilon) = L_2^{(k)}(B_2, B_1, p_2^{(k)})$ because we will end up in the same subgame after the first round in both cases. On the other hand, we have $W_2^{(k)}(B_2, B_1 + \epsilon, p_2^{(k)} + \epsilon) < W_2^{(k)}(B_2, B_1, p_2^{(k)})$ by

Table 3: Clinching auction outcomes for $k = 2$

Budget ratio (B_1/B_2)	Round 1	Round 2
$(0, \frac{2}{3})$	2 wins at $\frac{B_1}{2}$	2 wins at B_1
$(\frac{2}{3}, 1)$	2 wins at $\frac{B_1}{2}$	1 wins at $1 - \frac{B_1}{2}$

comparing the budgets after the first round in the two cases. So by our assumption that agent 2 is indifferent at price $p_2^{(k)}$ when the budgets are B_1 and B_2, we have $L_2^{(k)}(B_2, B_1+\epsilon, p_2^{(k)}+\epsilon) > W_2^{(k)}(B_2, B_1+\epsilon, p_2^{(k)}+\epsilon)$. Since agent 2 is indifferent at price $p'(\epsilon)$ when agent 1's budget is $B_1 + \epsilon$, we have $p'(\epsilon) < p_2^{(k)} + \epsilon$. Hence, we get that

$$U_2^{(k)}(B_2, B_1 + \epsilon) = U_2^{(k-1)}(B_2, B_1 + \epsilon - p'(\epsilon)) \leq U_2^{(k-1)}(B_2, B_1 - p_2^{(p)}) = U_2^{(k)}(B_2, B_1) \ .$$

Therefore, we have proved the desired monotonicity for this case. The analysis of the other cases are similar in spirit to the above arguments. But some special treatments are needed on a case-by-case basis. The full proof is in Appendix A.

5 Comparison with the clinching auction

In this section we compare the outcome of the sequential auction with that of the clinching auction of Ausubel (2004), as modified by Dobzinski et al. (2011) to accommodate budgets. At a high level, the clinching auction proceeds as follows. It starts with a per-unit price of 0, and slowly raises this until some agent becomes critical for keeping overall demand above supply, i.e. the total demand from *other* agents drops below the supply. When this happens, we allocate to this agent at the current price, until the supply is reduced back to below other agents' total demand (this process is the "clinching" from the auction's name). We then return to slowly raising the per-unit price. This process continues until the supply has been exhausted.

When analyzing sequential auctions, we focus on the case with just 2 bidders, both of whom have values exceeding their budgets. In this case, the clinching auction simplifies to the following process: items are allocated one-by-one, and at a given round, if there are k items left, and the remaining budgets of agents 1 and 2 are B_1 and B_2, respectively, an item is allocated to the agent with the higher budget at a price of $\min(B_1, B_2)/k$.

We begin by considering cases with small k, as we did with sequential auctions, and then present some observations on the outcomes of the clinching auction for general k. For $k = 1$, we can see from the description above that if $B_1 < B_2$ (without loss of generality), then agent 2 will simply win the single item at a price of B_1. Consider the case where $k = 2$, again assuming that $B_1 < B_2$. Then in the first round, agent 2 wins the item at a price of $B_2/2$; the outcome of the second round simply depends on whether or not paying this price depletes agent 2's budget below B_1, i.e. whether or not $B_2 > 3B_1/2$. Note that this induction is much simpler than the one for sequential auctions – once we know the budgets of the agents, we may immediately infer the winner and payment in the current round, which tells us the budgets for the next round. As such, we omit further details, and simply present the outcomes and prices for $k = 2$ and $k = 3$ in Table 3 and Table 4, respectively.

We finish this section with some observations about the outcomes of the clinching auction for general k.

1. First, we observe that the order of allocation is quite different from that in the sequential auction. While in the sequential auction winners are never interleaved, in the clinching auction winners are

Table 4: Clinching auction outcomes for $k = 3$

Budget ratio (B_1/B_2)	Round 1	Round 2	Round 3
$(0, \frac{6}{11})$	2 wins at $\frac{B_1}{3}$	2 wins at $\frac{B_1}{2}$	2 wins at B_1
$(\frac{6}{11}, \frac{3}{4})$	2 wins at $\frac{B_1}{3}$	2 wins at $\frac{B_1}{2}$	1 wins at $1 - \frac{5B_1}{6}$
$(\frac{3}{4}, 1)$	2 wins at $\frac{B_1}{3}$	1 wins at $\frac{3-B_1}{6}$	2 wins at $\frac{5B_1-3}{6}$

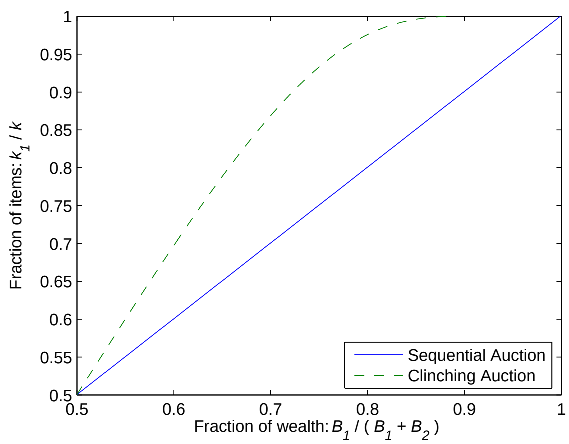

Figure 2: A comparison of how the sequential and clinching auctions compare in terms of the fraction of items an agent can expect to receive as a function of the fraction of the total wealth they control under each.

nearly always interleaved. In particular, it is easy to verify that once there are two consecutive rounds where the winning agent changes, the winner will change in every round thereafter. Thus, we find that in general the clinching auction repeatedly allocates to the agent with the higher budget until the two budgets are "close," and then proceeds to alternate between the two agents.

2. Unlike sequential auctions, where the prices are non-increasing as we proceed to later rounds, the very definition of the clinching auction ensures that prices with only increase as we proceed to later rounds.

3. The fact that the clinching auction generally alternates between the two agents when allocating items might give the impression that it is more "fair" than the sequential auction; considering how the items are divided between the agents, however, gives a much different impression. In particular, the clinching auction increases the number of items it allocates to agent 1 as B_1 increases much more quickly than the sequential auction does. For example, agent 1 cannot win all the items in a sequential auction until $B_1 > k \cdot B_2$, but in the clinching auction does so with only $B_1 > H_k \cdot B_2 \approx \log k \cdot B_2$. More generally, if agent 1 controls a p-fraction of the total wealth (i.e. $p = B_1/(B_1 + B_2)$), the fraction of items agent 1 can expect to win is $k_1/k \approx p$ under the sequential auction, but $k_1/k \approx 1 - \exp(\frac{2p-1}{p-1})/2$ in the clinching auction. See Figure 2 for a graphical comparison of these two quantities.

6 Conclusion and future directions

As mentioned in the introduction, we consider the most basic setting of sequential auctions with budget constrained bidders in order to analyze, for the first time, the case of arbitrary number of items. Now that we understand this basic case, one can hope to extend it to more settings. In particular as "next steps", the following are interesting questions that our work raises.

- Even for the case of two bidders, we assume that the values are much bigger than the budgets, in essence to ensure that the primary goal of the agents is to win as many items as possible and the price paid is only a secondary goal. Can we extend our results to arbitrary valuations?
- A crucial property in our results was the sequence of wins. Even with 3 bidders, it is not clear how this generalizes. How to extend our results to an arbitrary number of bidders is of course the big question.
- Another direction in which we can relax our model is to consider incomplete information. This would become more interesting when the valuations are comparable to the budget.

Finally, we feel we are still far from a thorough understanding of even the two agent case presented here. It would be insightful to further simplify the proof we currently have.

References

O. Ashenfelter. How auctions work for wine and art. *The Journal of Economic Perspectives*, 3(3):23–36, 1989. 2, 3

L.M. Ausubel. An efficient ascending-bid auction for multiple objects. *The American Economic Review*, 94 (5):1452–1475, 2004. 15

M. Bagnoli and B.L. Lipman. Provision of public goods: Fully implementing the core through private contributions. *The Review of Economic Studies*, 56(4):583, 1989. 6

J.P. Benoît and V. Krishna. Multiple-object auctions with budget constrained bidders. *Review of Economic Studies*, 68(1):155–179, 2001. 3, 8

S. Bhattacharya, G. Goel, S. Gollapudi, and K. Munagala. Budget constrained auctions with heterogeneous items. In *Proceedings of the 42nd ACM Symposium on Theory of Computing*, pages 379–388. ACM, 2010. 3

J. Black and D. Meza. Systematic price differences between successive auctionsare no anomaly. *Journal of Economics & Management Strategy*, 1(4):607–628, 1992. 2, 3

C. Borgs, J. Chayes, N. Immorlica, M. Mahdian, and A. Saberi. Multi-unit auctions with budget-constrained bidders. In *Proceedings of the 6th ACM Conference on Electronic Commerce*, pages 44–51. ACM, 2005. 3

T. Broecker. Credit-worthiness tests and interbank competition. *Econometrica*, pages 429–452, 1990. 6

Y.K. Che and I. Gale. Standard auctions with financially constrained bidders. *The Review of Economic Studies*, 65(1):1, 1998. 3

Y.K. Che and I. Gale. The optimal mechanism for selling to a budget-constrained buyer. *Journal of Economic Theory*, 92(2):198–233, 2000. 3

S. Dobzinski, R. Lavi, and N. Nisan. Multi-unit auctions with budget limits. *Games and Economic Behavior*, 2011. 2, 3, 15

A. Hassidim, H. Kaplan, Y. Mansour, and N. Nisan. Non-price equilibria in markets of discrete goods. In *Proceedings of the 12th ACM Conference on Electronic Commerce*, 2011. 3

G. Iyengar, D. Phillips, and Stein C. Bidding strategically with budget-constraints in sequential auctions. *Technical Report*, 2006. 3

T.D. Jeitschko. Equilibrium price paths in sequential auctions with stochastic supply. *Economics Letters*, 64(1):67–72, 1999. 2, 3

A. Mas-Colell, M.D. Whinston, J.R. Green, and Universitat Pompeu Fabra. Facultat de Cičncies Econňmiques i Empresarials. *Microeconomic theory*, volume 1. Oxford University Press New York, 1995. 4

R.P. McAfee and D. Vincent. The declining price anomaly. *Journal of Economic Theory*, 60(1):191–212, 1993. 2, 3

P.R. Milgrom and R.J. Weber. A theory of auctions and competitive bidding. *Econometrica*, pages 1089–1122, 1982. 2

M.J. Osborne and A. Rubinstein. *A course in game theory*. The MIT Press, 1994. 4

R. Paes Leme, V. Syrgkanis, and E. Tardos. Sequential auctions and externalities. In *SODA*, 2012. 3, 5

C. Pitchik and A. Schotter. Budget constrained sequential auctions. *Working Papers*, 1986. 1, 2, 10, 12

C. Pitchik and A. Schotter. Perfect equilibria in budget-constrained sequential auctions: An experimental study. *The Rand Journal of Economics*, pages 363–388, 1988. 2, 3, 12

Alex Robson. Multi-item contests. Anucbe school of economics working papers, Australian National University, College of Business and Economics, School of Economics, 2005. URL http://EconPapers.repec.org/RePEc:acb:cbeeco:2005-446. 3

G.E. Rodriguez. Sequential auctions with multi-unit demands. *The BE Journal of Theoretical Economics*, 9(1):45, 2009. 3

R. Selten. Reexamination of the perfectness concept for equilibrium points in extensive games. *International Journal of Game Theory*, 4(1):25–55, 1975. 5

R.J. Weber. Multiple-object auctions. *Discussion Papers*, 1981. 2, 3

Yunhong Zhou and Rajan M. Lukose. Vindictive bidding in keyword auctions. In *ICEC*, pages 141–146, 2007. 1

A Joint Induction

A.1 Notations

As we will see later in this section, the probability trial of tossing (at most $m+n+1$) coins until either we get $m+1$ head or we get $n+1$ tails will play an important role in the tie-breaking cases of the equilibrium. Therefore, we will define the following notation to denote the probability of getting $n+1$ tails for presentation convenience.

Definition 7. *For any $m, n \in \mathbb{Z}_{\geq 0}$, we let $\phi(m, n)$ denote the probability that we get at least $n+1$ heads if we repeatedly toss a fair coin $m+n+1$ times.*

We will prove two straightforward properties of the probability $\phi(m, n)$.

Lemma A.1. *For any $m, n \in \mathbb{Z}_{\geq 0}$,*

$$\phi(m, n) = \sum_{j=0}^{m} \binom{n+j}{n} \left(\frac{1}{2}\right)^{n+j+1} .$$

Proof. First, the number of different head/tail sequences such that we get n heads and j tails in the first $n+j$ tosses and get another head in the last toss is $\binom{n+j}{n}$. Each of these sequences happens with probability $\left(\frac{1}{2}\right)^{n+j+1}$. So the probability that we get $n+1$ heads and before that get exactly j tails is $\binom{n+j}{n}\left(\frac{1}{2}\right)^{n+j+1}$. Summing up for j from 0 to m proves the lemma. □

Lemma A.2. *For any $m, n \in \mathbb{Z}$ such that $m \geq 1$ and $n \geq 0$ we have*

$$\phi(m, n) - \phi(m-1, n+1) = \binom{n+m+1}{n+1} \left(\frac{1}{2}\right)^{m+n+1} .$$

Proof. By the definition of ϕ, we have that $\phi(m, n) - \phi(m-1, n+1)$ equals the probability that in $m+n+1$ tosses we get exactly $n+1$ heads. Note that there are $\binom{m+n+1}{n+1}$ such head/tail sequences and each happens with probability $\left(\frac{1}{2}\right)^{m+n+1}$. So we have proved the lemma. □

A.2 Main Proof

Now let us present the joint inductive proof of Theorem 3.5, Proposition 4.2 (the elaborated version of Proposition 3.6, Case 2 - 4) , Proposition 3.7, Proposition 3.3, and Proposition 3.8. First, let us restate and clarify these claims as follows:

$\mathcal{P}_1^{(k)}$ (**Theorem 3.5**) The canonical outcome is a subgame perfect equilibrium of the sequential auction with k items.

$\mathcal{P}_2^{(k)}$ (**Proposition 4.2**) Suppose $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ satisfy $k_1 + k_2 = k$ and $\frac{k_i}{k_{-i}+1} < \frac{B_i}{B_{-i}} \leq \frac{k_i+1}{k_{-i}+1}$. Then, in the canonical outcome it is always the case that agent i wins the first k_i items, and then agent $-i$ gets the remaining k_{-i} items paying agent i's remaining budget for each item. Here, there are two tie-breaking caveats:

Type II-A Tie-breaking If $\frac{B_1}{B_2} = \frac{k_1+1}{k_2+1}$, then in the canonical outcome both agents keep bidding $p^* \stackrel{\text{def}}{=} \frac{B_1}{k_1+1} = \frac{B_2}{k_2+1}$ until one agent's budget becomes p^*, and then the other agent gets the remaining item by bidding p^*+.

In this tie-breaking case, we can assume either agent 1 gets the first k_1 items and then agent 2 gets the remaining k_2 items, or the other way around, without changing the utilities of the agents.

Type II-B Tie-breaking If $\frac{B_i}{B_{-i}}$ is smaller than and sufficiently close to $\frac{k_i+1}{k_{-i}+1}$, then in the canonical outcome both agents keep bidding $p^* \overset{\text{def}}{=} \frac{B_i}{k_i+1}$ until either agent i gets k_i items or agent $-i$ gets $k_{-i}-1$ items. In the former case, agent $-i$ then wins the remaining items paying agent i's remaining budget p^*. In the latter case, agent i then keeps bidding p^*+ until she gets k_i items, and then agent i wins the remaining items paying agent i's remaining budget p^*.

In this tie-breaking case, we can assume agent i gets k_i items and then agent i gets the remaining k_{-i} items, without changing the utilities of the agents.

$\mathcal{P}_3^{(k)}$ **(Proposition 3.7)** The prices paid in each round of the canonical outcome is non-increasing as the auction proceeds.

$\mathcal{P}_4^{(k)}$ **(Proposition 3.3)** $U_i^{(k)}(B_i, B_{-i})$ is increasing in B_i, and non-increasing in B_{-i}.

$\mathcal{P}_5^{(k)}$ **(Proposition 3.8)** $U_i^{(k)}$ is continuous in both B_i and B_{-i}, except when $\frac{B_i}{B_{-i}} = \frac{k_i}{k_{-i}+1}$ for $k_i, k_{-i} \in \mathbb{Z}_{\geq 0}$ such that $k_i + k_{-i} = k$. Moreover,

$$\begin{aligned}
\lim_{B_{-i} \to \frac{k_{-i}+1}{k_i} B_i^-} U_i^{(k)}(B_i, B_{-i}) &= k_i v_i \ , \\
\lim_{B_{-i} \to \frac{k_{-i}+1}{k_i} B_i^+} U_i^{(k)}(B_i, B_{-i}) &= (k_i - 1) v_i + B_i \ , \\
\lim_{B_i \to \frac{k_i}{k_{-i}+1} B_{-i}^-} U_i^{(k)}(B_i, B_{-i}) &= (k_i - 1) v_i + \frac{k_i}{k_{-i}+1} B_{-i} \ , \\
\lim_{B_i \to \frac{k_i}{k_{-i}+1} B_{-i}^-} U_i^{(k)}(B_i, B_{-i}) &= k_i v_i \ .
\end{aligned}$$

If $\frac{B_i}{B_{-i}} = \frac{k_i}{k_{-i}+1}$, then

$$U_i^{(k)}(B_i, B_{-i}) = (k_i - 1) v_i + \phi(k_{-i}, k_i - 1) v_i + \phi(k_i - 1, k_{-i}) B_i - \phi(k_i - 2, k_{-i} + 1) B_{-i} \ .$$

Also, recall that the dependencies can be summarized as follows:

$\mathcal{P}_1^{(k)}$ depends on $\mathcal{P}_1^{(k-1)}$, $\mathcal{P}_4^{(k-1)}$, and $\mathcal{P}_5^{(k-1)}$.

$\mathcal{P}_2^{(k)}$ depends on $\mathcal{P}_1^{(k)}$, $\mathcal{P}_4^{(k-1)}$, and $\mathcal{P}_5^{(k-1)}$.

$\mathcal{P}_3^{(k)}$ depends on $\mathcal{P}_1^{(k)}$, $\mathcal{P}_2^{(\leq k)}$, $\mathcal{P}_4^{(k-1)}$, and $\mathcal{P}_5^{(k-1)}$.

$\mathcal{P}_4^{(k)}$ and $\mathcal{P}_5^{(k)}$ depend on $\mathcal{P}_4^{(k-1)}$, $\mathcal{P}_5^{(k-1)}$, and $\mathcal{P}_2^{(k)}$.

Base case These propositions in the case of 1 or 2 items are easy to verify from the discussions in Section 2.1. So we will omit the tedious calculations here.

Inductive step Let us move on to the inductive step. Suppose we have prove the propositions for the cases of $k-1$ or less items. Now let us consider the case of k items. The rest of the section is organized as follows. In Section A.2.1, we will verify the monotonicity and continuity of the winning and losing utilities (Proposition 3.4) from the monotonicity and continuity of the utility functions of $k-1$ items. In Section A.2.2, we will prove the agents weakly prefer winning at the critical prices, which will be useful for the rest of the proof. In Section A.2.3 we will prove that the canonical outcome is indeed a subgame-perfect equilibrium ($\mathcal{P}_1^{(k-1)}, \mathcal{P}_4^{(k-1)}, \mathcal{P}_5^{(k-1)} \Rightarrow \mathcal{P}_1^{(k)}$). In Section A.2.4, we will prove the two-phase winner sequence in the canonical outcome ($\mathcal{P}_2^{(k-1)}, \mathcal{P}_4^{(k-1)}, \mathcal{P}_5^{(k-1)}, \mathcal{P}_1^{(k)} \Rightarrow \mathcal{P}_2^{(k)}$). In Section A.2.5, we will explain why the prices paid for the items weakly declines as the action proceeds ($\mathcal{P}_3^{(k-1)}, \mathcal{P}_4^{(k-1)}, \mathcal{P}_5^{(k-1)}, \mathcal{P}_1^{(k)}, \mathcal{P}_2^{(\leq k)} \Rightarrow \mathcal{P}_3^{(k)}$). Finally, in Section A.2.6, we will analyze the monotonicity and continuity of the utility function in the sequential auction with k items ($\mathcal{P}_4^{(k-1)}, \mathcal{P}_5^{(k-1)}, \mathcal{P}_2^{(k)} \Rightarrow \mathcal{P}_4^{(k)}, \mathcal{P}_5^{(k)}$).

A.2.1 Monotonicity and Continuity of Winning and Losing Utility

In this step, we will establish the monotonicity and continuity of the winning and losing utilities of the k-item sequential auction.

Lemma A.3 (Proposition 3.4 restated)**.** *$W_i^{(k)}(B_i, B_{-i}, p)$ is decreasing in p while $L_i^{(k)}(B_i, B_{-i}, p)$ is non-decreasing in p, and both are increasing in B_i and non-increasing in B_{-i}.*

Lemma A.4. *$W_i^{(k)}$ and $L_{-i}^{(k)}$ are continuous in p except when $p = B_i - \frac{k_i}{k_{-i}} B_{-i}$ for $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ such that $k_1 + k_2 = k$. Moreover, we have*

$$
\begin{aligned}
\lim_{p \to (B_{-i} - \frac{k_{-i}}{k_i} B_i)^-} L_i^{(k)}(B_i, B_{-i}, p) &= (k_i - 1)v_i + B_i \ , \\
\lim_{p \to (B_{-i} - \frac{k_{-i}}{k_i} B_i)^+} L_i^{(k)}(B_i, B_{-i}, p) &= k_i v_i \ , \\
\lim_{p \to (B_i - \frac{k_i}{k_{-i}} B_{-i})^-} W_i^{(k)}(B_i, B_{-i}, p) &= (k_i + 1)v_i \ , \\
\lim_{p \to (B_i - \frac{k_i}{k_{-i}} B_{-i})^+} W_i^{(k)}(B_i, B_{-i}, p) &= k_i v_i + \frac{k_i}{k_{-i}} B_{-i} \ .
\end{aligned}
$$

of Lemma A.3 and Lemma A.4. By definition, we have $L_i^{(k)}(B_i, B_{-i}, p) = U_i^{(k-1)}(B_i, B_{-i} - p)$. Therefore, by Proposition 3.3 and Proposition 3.8 of the $k-1$ item case we get the asserted monotonicity of $L_i^{(k)}$. Also, we get that $L_i^{(k)}$ is continuous in p except when $p = B_{-i} - \frac{k_i}{k_{-i}} B_i$ for $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ s.t. $k_1 + k_2 = k$, and

$$
\begin{aligned}
\lim_{p \to (B_{-i} - \frac{k_{-i}}{k_i} B_i)^-} L_i^{(k)}(B_i, B_{-i}, p) &= \lim_{B \to \frac{k_{-i}}{k_i} B_i^+} U_i^{(k-1)}(B_i, B) = (k_i - 1)v_i + B_i \ , \\
\lim_{p \to (B_{-i} - \frac{k_{-i}}{k_i} B_i)^+} L_i^{(k)}(B_i, B_{-i}, p) &= \lim_{B \to \frac{k_{-i}}{k_i} B_i^-} U_i^{(k-1)}(B_i, B) = k_i v_i \ .
\end{aligned}
$$

Similarly, $W_i^{(k)}(B_i, B_{-i}, p) = v_i + U_i^{(k-1)}(B_i - p, B_{-i})$. So by Proposition 3.3 and Proposition 3.8 we have the desired monotonicity and its continuity in p except when $B_{-i} = \frac{k_{-i}}{k_i}(B_i - p)$ for $k_1, k_2 \in \mathbb{Z}_{\geq 0}$

s.t. $k_1 + k_2 = k$, and

$$\lim_{p \to (B_i - \frac{k_i}{k_{-i}} B_{-i})^-} W_i^{(k)}(B_i, B_{-i}, p) = v_i + \lim_{B \to \frac{k_i}{k_{-i}} B_{-i}^+} U_i^{(k-1)}(B, B_{-i}) = (k_i + 1)v_i \ ,$$

$$\begin{aligned} \lim_{p \to (B_i - \frac{k_i}{k_{-i}} B_{-i})^+} W_i^{(k)}(B_i, B_{-i}, p) &= v_i + \lim_{B \to \frac{k_i}{k_{-i}} B_{-i}^-} U_i^{(k-1)}(B, B_{-i}) \\ &= k_i v_i + \frac{k_{-i}}{k_i} B_{-i} \ . \end{aligned}$$

So we have proved the lemmas. □

A.2.2 Agents Weakly Prefer wining at Critical Prices

In this section, we will prove that the agents always weakly prefer winning the first item at the critical prices. First we need to prove a few lemmas.

Lemma A.5. *For $i = 1, 2$, suppose $p > 0$ satisfies $p = \frac{B_i}{k_i+1} = \frac{B_{-i}}{k_{-i}}$ for $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ such that $k_1 + k_2 = k$, then $W_i^{(k)}(B_i, B_{-i}, p) \geq k_i v_i + \phi(k_{-i} - 1, k_i - 1)v_i$.*

Proof. If agent i wins the first item at price p, then in the subgame of $k - 1$ items, the remaining budgets satisfy $\frac{B_i - p}{B_{-i}} = \frac{k_i}{k_{-i}}$. So we are in the tie-breaking case of Proposition 4.1. In this case, agent i will get k_i items for sure, and have probability $\phi(k_{-i} - 1, k_i - 1)$ of getting an extra item. Further, the remaining budget is at least zero. So we have $W_i^{(k)}(B_i, B_{-i}, p) \geq k_i v_i + \phi(k_{-i} - 1, k_i - 1)v_i$. □

Lemma A.6. *Suppose $p > 0$ satisfies that $p = \frac{1}{k_i+1} B_i = \frac{1}{k_{-i}} B_{-i}$ for $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ s.t. $k_1 + k_2 = k$, then $k_i v_i + \phi(k_{-i} - 1, k_i) + L_i^{(k)}(B_i, B_{-i}, p) \leq k_i v_i + B_i + \phi(k_{-i} - 1, k_i)v_i$.*

Proof. If agent i loses the first item at price p, then in the subgame of $k - 1$ items, the remaining budges satisfies $\frac{B_i}{B_{-i}} = \frac{k_i+1}{k_{-i}-1}$. So we are in the tie-breaking case of Proposition 4.1. In this case, agent i will get k_i items for sure, and have probability $\phi(k_{-i} - 2, k_i)$ of getting an extra item. Further, the remaining budget is at most B_i and at least zero. So we have the desired inequality. □

Lemma A.7. *Suppose $p > 0$ satisfies that $p = \frac{1}{k_i+1} B_i = \frac{1}{k_{-i}} B_{-i}$ for $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ s.t. $k_1 + k_2 = k$, then $W_i^{(k)}(B_i, B_{-i}, p) > L_i^{(k)}(B_i, B_{-i}, p)$.*

Proof. Let us assume $i = 1$ for the sake of presentation. In this case, we have

$$\begin{aligned} W_1^{(k)}(B_1, B_2, p) &= v_1 + U_1^{(k-1)}(B_1 - p, B_2) \\ &= k_1 v_1 + \phi(k_2 - 1, k_1 - 1)v_1 + \phi(k_1 - 1, k_2 - 1)B_1 - \phi(k_1 - 2, k_2)B_2 \ . \end{aligned}$$

and

$$\begin{aligned} L_1^{(k)}(B_1, B_2, p) &= U_1^{(k-1)}(B_1, B_2 - p) \\ &= k_1 v_1 + \phi(k_2 - 2, k_1)v_1 + \phi(k_1, k_2 - 2)B_1 - \phi(k_1 - 1, k_2 - 1)B_2 \ . \end{aligned}$$

Further, by Lemma A.2 we have $\phi(k_2-1,k_1-1)-\phi(k_2-2,k_1)=\binom{k-1}{k_1}\left(\frac{1}{2}\right)^{k-1}$ and $\phi(k_1,k_2-2)-\phi(k_1-1,k_2-1)=\binom{k-1}{k_2-1}\left(\frac{1}{2}\right)^{k-1}=\binom{k-1}{k_1}\left(\frac{1}{2}\right)^{k-1}$. So we have

$$\begin{aligned} W^{(k)}(B_i,B_{-i},p)-L^{(k)}(B_i,B_{-i},p) &= (\phi(k_2-1,k_1-1)-\phi(k_2-2,k_1))v_1 \\ &\quad -(\phi(k_1,k_2-2)-\phi(k_1-1,k_2-1))B_1 \\ &\quad +(\phi(k_1-1,k_1-1)-\phi(k_2-2,k_1))B_2 \\ &= \binom{k-1}{k_1}\left(\frac{1}{2}\right)^{k-1}(v_1-B_1+B_2)\ >\ 0\ . \end{aligned}$$

So we have proved Lemma A.7. □

Lemma A.8. *Either both $W_i^{(k)}$ and $L_i^{(k)}$ are continuous at $p_i^{(k)}$, in which case agent i is indifferent between winning or losing the first item at $p_i^{(k)}$, or $W_i^{(k)}$ is discussions at $p_i^{(k)}$, in which case agent i strictly prefers winning the first item at price $p_1^{(k)}$.*

Proof. For the sake of presentation, we will assume w.l.o.g. that $i=1$. There are four cases:

Case 1 Both $W_1^{(k)}$ and $L_1^{(k)}$ are continuous at $p_1^{(k)}$. In this case, we can deduce by the definition of $p_1^{(k)}$ that $W_1^{(k)}(B_1,B_2,p_1^{(k)})=L_1^{(k)}(B_1,B_2,p_1^{(k)})$, that is, agent 1 is indifferent between winning and losing the first item at price $p_1^{(k)}$.

Case 2 $L_1^{(k)}$ is continuous at $p_1^{(k)}$ but $W_1^{(k)}$ is not. In this case, by Lemma A.4 we conclude that $p_1^{(k)}=B_1-\frac{k_1}{k_2}B_2$ for $k_1,k_2\in\mathbb{Z}_{\geq 0}$ s.t. $k_1+k_2=k$. By our choice of $p_1^{(k)}$, we have $W_1^{(k)}(B_1,B_2,p)<L_1^{(k)}(B_1,B_2,p)$ for any $p>p_1^{(k)}$, and $W_1^{(k)}(B_1,B_2,p)>L_1^{(k)}(B_1,B_2,p)$ for any $p<p_1^{(k)}$. So we have

$$L_1^{(k)}(B_1,B_2,p_1^{(k)})=\lim_{p\to p_1^{(k)}+}L_1^{(k)}(B_1,B_2,p)\geq\lim_{p\to p_1^{(k)}+}W_1^{(k)}(B_1,B_2,p)=k_1v_1+\frac{k_1}{k_2}B_2$$

and

$$L_1^{(k)}(B_1,B_2,p_1^{(k)})=\lim_{p\to p_1^{(k)}-}L_1^{(k)}(B_1,B_2,p)\leq\lim_{p\to p_1^{(k)}-}W_i^{(k)}(B_1,B_2,p)=(k_1+1)v_1\ .$$

By the value range of $L_1^{(k)}$ in each continuous interval in Lemma A.4, we know that $B_2-\frac{k_2}{k_1}B_1<p_1^{(k)}<B_2+\frac{k_2-1}{k_1+1}B_1$ and $k_1v_1\leq L_1^{(k)}(B_1,B_2,p_1^{(k)})\leq k_1v_1+B_1$. Finally, the utility for winning the first item is

$$\begin{aligned} W_1^{(k)}\left(B_1,B_2,p_1^{(k)}\right) &= v_1+U^{(k-1)}\left(\frac{k_1}{k_2}B_2,B_2\right)\geq k_1v_1+\phi(k_2-1,k_1-1)v_1 \\ &\geq k_1v_1+\frac{1}{2^k}v_1>k_1v_1+B_1\ . \end{aligned}$$

Here, the first inequality holds because agent 1 gets k_1 items for sure and gets an extra item with probability $\phi(k_2-1,k_1-1)$ in this case. So agent 1 strictly prefers winning the first item.

Case 3 $W_1^{(k)}$ is continuous at $p_1^{(k)}$ but $L_1^{(k)}$ is not. We will argue this case is impossible. Suppose for contradiction that this is the case, then $p_1^{(k)} = B_2 - \frac{k_2}{k_1}$. Similar to the analysis in the second case, we have that.

$$W_1^{(k)}(B_1, B_2, p_1^{(k)}) = \lim_{p \to p_1^{(k)}+} W_1^{(k)}(B_1, B_2, p) \le \lim_{p \to p_1^{(k)}+} L_1^{(k)}(B_1, B_2, p) = k_1 v_1$$

and

$$W_1^{(k)}(B_1, B_2, p_1^{(k)}) = \lim_{p \to p_1^{(k)}-} W_1^{(k)}(B_1, B_2, p) \ge \lim_{p \to p_1^{(k)}-} L_1^{(k)}(B_1, B_2, p) = (k_1 - 1)v_1 + B_1 \ .$$

However, by Lemma A.4 we have $B_1 - \frac{k_1-2}{k_2+2} B_2 < p_1^{(k)} < B_1 - \frac{k_1-1}{k_2+1} B_2$. When $p_1^{(k)}$ is in this range, the value of $W_1^{(k)}$ is upper bounded by $W_1^{(k)}(B_1, B_2, p_1^{(k)}) < (k_1 - 1)v_1 + \frac{k_1-1}{k_2+1} B_2 < (k_1 - 1)v_1 + B_1$. So we have a contradiction.

Case 4 Both $W_1^{(k)}$ and $L_1^{(k)}$ are discontinuous at $p_1^{(k)}$. Then, by Lemma A.4 it must be the case that $p_1^{(k)} = B_2 - \frac{k_2-1}{k_1+1} B_1 = B_1 - \frac{k_1}{k_2} B_2$ for $k_1, k_2 \in \mathbb{Z}_{\ge 0}$ s.t. $k_1 + k_2 = k$. So we have that $p_1^{(k)} = \frac{1}{k_1+1} B_1 = \frac{1}{k_2} B_2$. By Lemma A.7, agent 1 strictly prefers winning the first item at price $p_1^{(k)}$ in this case.

Summarizing the four cases, we have proved the lemma. □

A.2.3 Subgame-Perfection of the Canonical Outcome

In this step, we aim to establish the fact that the canonical outcome is a subgame-perfect equilibrium. Let us first prove several lemmas.

Lemma A.9. *$L_i^{(k)}$ is continuous at $p_i^{(k)}$.*

Proof. Let us assume w.l.o.g. that $i = 1$. Suppose for contradiction that $L_1^{(k)}$ is discontinuous at $p_1^{(k)}$. Then, by Lemma A.8, $W_1^{(k)}$ must be discontinuous at $p_1^{(k)}$ as well. By Lemma A.4 it must be the case that $p_1^{(k)} = B_2 - \frac{k_2-1}{k_1+1} B_1 = B_1 - \frac{k_1}{k_2} B_2$ for $k_1, k_2 \in \mathbb{Z}_{\ge 0}$ s.t. $k_1 + k_2 = k$. So we have that $p_1^{(k)} = \frac{1}{k_1+1} B_1 = \frac{1}{k_2} B_2$. So it is easy to verify $W_2^{(k)}$ and $L_2^{(k)}$ are discontinuous at $p_1^{(k)}$ as well, and $p_i^{(k)}$ is also the critical price of agent 2, which contradicts our assumption. □

Lemma A.10. *Suppose $p_1^{(k)} = p_2^{(k)}$. Then, either all of $W_1^{(k)}$, $L_1^{(k)}$, $W_2^{(k)}$, $L_2^{(k)}$ are continuous at $p_1^{(k)} = p_2^{(k)}$, or all of them are discontinuous at $p_1^{(k)} = p_2^{(k)}$.*

Proof. Suppose at least one of $W_1^{(k)}$, $L_1^{(k)}$, $W_2^{(k)}$, $L_2^{(k)}$ is discontinuous at $p_1^{(k)} = p_2^{(k)}$. Note that by $\mathcal{P}^{(k)}$, $W_i^{(k)}$ is discontinuous at a price p if and only if $L_{-i}^{(k)}$ is discontinuous at p. So we can assume without loss of generality that $W_1^{(k)}$ and $L_2^{(k)}$ are discontinuous at $p_1^{(k)} = p_2^{(k)}$. Now by $\mathcal{P}^{(k)}$, $L_2^{(k)}$ is discontinuous at $p_1^{(k)} = p_2^{(k)}$ implies $W_2^{(k)}$ is discontinuous at $p_1^{(k)} = p_2^{(k)}$ as well, which further indicates $L_1^{(k)}$ is discontinuous at $p_1^{(k)} = p_2^{(k)}$. So we have proved the lemma. □

Lemma A.11. *Suppose $p_i^{(k)} > p_{-i}^{(k)}$. Then, agent i bidding $p_{-i}^{(k)}+$ and agent $-i$ bidding $p_{-i}^{(k)}$ and then both agents following the canonical outcome in the subgames of $k - 1$ items is a subgame perfect equilibrium.*

Proof. It suffices to show that neither of the agents has profitable deviations. Let us assume w.l.o.g. that $i = 1$ for the sake of presentation.

First, we will consider the possible deviation of agent 1. We note that agent 1 could not benefit from bidding over $p_2^{(k)}$ because that would only increases her price for getting the first item. Further, agent 1 could not benefit from bidding below $p_2^{(k)}$ because $p_2^{(k)} < p_1^{(k)}$ implies that agent 1 strictly prefers winning the first item at prices $p_2^{(k)}$.

Next, let us consider the possible deviation of agent 2. Note that underbidding has no effect for that agent 1 would still win the first item at $p_2^{(k)}$. Further, by Lemma A.9, $L_2^{(k+1)}$ is continuous at $p_2^{(k)}$. So we get that overbidding has utility at most

$$\lim_{p \to p_2^{(k)}+} W_2^{(k)}(B_2, B_1, p) \le \lim_{p \to p_2^{(k)}+} L_2^{(k)}(B_2, B_1, p) = L_2^{(k)}(B_2, B_1, p_2^{(k)}) \ .$$

Therefore, overbidding could not be a profitable deviation for agent 2 either. $\square$

Lemma A.12. *If $p_1^{(k)} = p_2^{(k)}$, then both agents bidding $p_1^{(k)} = p_2^{(k)}$ and then following the canonical outcomes in the subgames of $k-1$ items is a subgame-perfect equilibrium.*

Proof. We will let $p^* \stackrel{\text{def}}{=} p_1^{(k)} = p_2^{(k)}$ for convenience. By symmetry, it suffices to prove agent 1 has no profitable deviation. By Lemma A.10 we only need to consider the following two cases.

The first case is when all of $W_1^{(k)}$, $L_1^{(k)}$, $W_2^{(k)}$, $L_2^{(k)}$ are continuous at p^*. In this case, both agents are indifferent between winning and losing at price p^*. Therefore, underbidding yield utility $L_1^{(k)}(B_1, B_2, p^*)$, which is the same as the utility of bidding p^*. Overbidding is strictly worse because for any $p > p^*$, we have $W_1^{(k)}(B_1, B_2, p) < W_1^{(k)}(B_1, B_2, p^*)$, which again equals the utility of bidding p^*.

The second case is when all of $W_1^{(k)}$, $L_1^{(k)}$, $W_2^{(k)}$, $L_2^{(k)}$ are discontinuous at p^*. In this case, we have $p^* = B_1 - \frac{k_1}{k_2} B_2 = B_2 - \frac{k_2 - 1}{k_1 + 1} B_1$ and thus $p^* = \frac{1}{k_1+1} B_1 = \frac{1}{k_2} B_2$. By Lemma A.7 agent 1 strictly prefers winning the first item at price p^* than losing it. So agent 1 will not underbid. If agent 1 overbids $p > p^*$, then her utility is

$$\begin{aligned}
W_1^{(k)}(B_1, B_2, p) &= v_1 + U_1^{(k)}(B_1 - p, B_2) \\
&< v_1 + \lim_{B \to (\frac{k_1}{k_2} B_2)^-} U_1^{(k)}(B, B_2) && (B_1 - p < \frac{k_1}{k_2} B_2) \\
&= v_1 + (k_1 - 1) v_1 + \frac{k_1}{k_2} B_2 && (\text{Lemma A.4}) \\
&< k_1 v_1 + B_1 \ . && (\frac{B_1}{B_2} \le \frac{k_1 + 1}{k_2})
\end{aligned}$$

On the other hand, the utility of bidding p^* is

$$\begin{aligned}
U_1^{(k)}(B_1, B_2) &= \frac{1}{2} \left(W_1^{(k)}(B_1, B_2, p^*) + L_1^{(k)}(B_1, B_2, p_1^{(k)}) \right) \\
&\ge k_1 v_1 + \frac{1}{2} B_1 + \frac{1}{2} \left(\phi(k_2 - 1, k_1 - 1) + \phi(k_2 - 1, k_1) \right) v_1 \quad (\text{Lemma A.5, Lemma A.6}) \\
&\ge k_1 v_1 + \frac{1}{2} B_1 + \frac{1}{2^k} v_1 \\
&> k_1 v_1 + B_1 \ .
\end{aligned}$$

So bidding p^* is strictly better. $\square$

By Lemma A.11 and Lemma A.12, we have shown that the canonical outcome is indeed a subgame-perfect equilibrium of the k-item sequential auction.

A.2.4 Two-Phase Winner Sequence

In this section, we will present the inductive proof for the two-phase winner sequence structure in the canonical outcome. Again, we will start with a few technical lemmas.

Proposition A.13. *Suppose we are not in the type I tie-breaking case of Proposition 4.1. In other words, there are $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ such that $k_1 + k_2 = k$ and $\frac{k_1}{k_2+1} < \frac{B_1}{B_2} \leq \frac{k_1+1}{k_2}$. Then, for $i = 1, 2$, agent i gets k_i items with average price at most $\frac{B_{-i}}{k_{-i}+1}$ in the canonical outcome.*

Proof. By symmetry, it suffices to prove it for $i = 1$. Let us consider the following strategy for agent 1: keep bidding $p^* \stackrel{\text{def}}{=} \frac{B_2}{k_2+1}$ until agent 2's remaining budget becomes p^*, and then keep bidding p^*+. It is easy to verify this strategy guarantees winning at least k_1 items regardless of agent 2's strategy and paying p^* per item. So in the canonical outcome, agent 1 pays at most p^* per item on average. □

Lemma A.14. *Suppose the budgets do not fall into the tie-breaking cases of Proposition 4.1, then the budgets after the first round in the canonical outcome do not fall into the tie-breaking cases either.*

Proof. We may assume w.l.o.g. that there exists $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ such that $k_1 + k_2 = k$, and $\frac{B_1}{k_1+1} < \frac{B_2}{k_2+1} < \frac{B_1}{k_1}$.

Note that $\frac{B_1}{k_1+1} < \frac{B_2}{k_2+1}$ implies $\frac{B_1}{k_1+1} < \frac{B_2}{k_2}$. So by Proposition 3.6 agent 1 gets k_1 items and agent 2 gets k_2 items in the canonical outcome. Assume for contradiction that the equilibrium of the subgame after the first round do fall into the tie-breaking case of Proposition 4.1. Then, by Proposition 4.1 the expected number of items agent 1 gets will be a non-integral number for that in the type I tie-breaking case, there exists some $k_1', k_2' \in \mathbb{Z}_{\geq 0}$, $k_1' + k_2' = 1$, such that agent 1 gets at least k_1', and agent 2 gets at least $k_2' - 1$, and both agents have some non-zero probability of winning the last item. But Proposition 3.6 asserts that the number of items agent 1 gets shall be integral. So we have a contradiction. □

Lemma A.15. *Suppose $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ satisfy that $k_1 + k_2 = k$, $\frac{k_1}{k_2+1} < \frac{B_1}{B_2} < \frac{k_1+1}{k_2}$, and $\frac{B_1}{B_2} \neq \frac{k_1+1}{k_2+1}$. Then, it must be that one of the agent, say, agent i, wins the first k_i items and then the other agent wins the remaining items.*

Proof. By Proposition 3.6, agent 1 gets k_1 items and agent 2 gets k_2 items in the canonical outcome. Suppose for contradiction that the lemma does not hold. Then, by Lemma A.14, and our assumption that Proposition 4.2 and Proposition 3.7 hold for the canonical outcome of $k - 1$ item, we can assume w.l.o.g. that agent 1 wins the first item with price $p_2^{(k)}$ in the equilibrium; and then agent 2 wins the next $k_2 \geq 1$ items with a non-increasing price sequence $q_1 \geq \cdots \geq q_{k_2}$ such that the average price is at least $\frac{B_2}{k_2+1}$; and finally agent 1 wins the remaining $k_1 - 1 \geq 1$ items with prices equal $r = B_2 - \sum_{j=1}^{k_2} q_j \leq q_{k_2}$, the remaining budget of agent 2.

Let us consider the deviation in which agent 2 wins the first item at price $p_2^{(k)}$. By Lemma A.8, either agent 2 is indifferent between winning and losing the first item at price $p_2^{(k)}$, or $W_2^{(k)}(B_2, B_1, p)$ is discontinuous at $p = p_2^{(k)}$.

Case 1: Agent 2 is indifferent In this case, agent 2 gets k_2 items as well in the deviation. So the subgame does not fall into the tie-breaking case of Proposition 4.1. Our proof strategy is to first show that the price of the first item must be very small. In particular, it is smaller than the average price that agent 2 pays in the canonical outcome. Then, we will derive a contradiction by concluding agent 2 has a profitable deviation by wining the first item since it is so cheap. We can further divide the situation into three cases depending on the allocation sequence after the first round.

Case 1a: Agent 1 gets items first after the first round More precisely, after the first round of this deviation, agent 1 will win the first k_1 items with average price at least $\frac{B_1}{k_1+1}$ and then agent 2 wins the remaining $k_2 - 1$ item with prices equal agent 1's remaining budget.

By Proposition 4.2 of $k-1$ items and our assumed allocation sequence after agent 1 gets the first item, such allocation sequence could hold only if the remaining budgets $B_1 - p_2^{(k)}$ and B_2 after the first round satisfy $\frac{B_1 - p_2^{(k)}}{B_2} > \frac{k_1}{k_2+1}$. So we get our first upper bound on the price of the first item: $p_2^{(k)} < B_1 - \frac{k_1}{k_2+1} B_2$. Via similar reasoning, by Proposition 4.2 and the assumed allocation sequence after agent 2 wins the first time, we get that the price of the first item is also upper bounded by $p_2^{(k)} < B_2 - \frac{k_2}{k_1+1} B_1$. Combining these two upper bounds we get $p_2^{(k)} < \frac{B_2}{k_2+1}$.

In this case, we argue agent 2 has a profitable deviation because she could have won the first item by bidding $p' \in (p_2^{(k)}, \frac{B_2}{k_2+1})$; and then keeps bidding $p_2^{(k)}$ until agent 1 wins an item; and finally follows the canonical outcome strategy thereafter. We let j^* denote the first item that agent 1 wins in this (profitable) deviation. On the one hand, agent 2 gets $j^* - 1$ items after j^* rounds in the equilibrium and her remaining budget is $B_2 - \sum_{j=1}^{j^*-1} q_j$ while agent 1's remaining budget is $B_1 - p_2^{(k)}$. On the other hand, agent 2 gets $j^* - 1$ as well after j^* rounds of the deviation and agent 1's remaining budget is also $B_1 - p_2^{(k)}$. However, agent 2's remaining budget after j^* rounds becomes $B_2 - p' - (j^* - 2)p_2^{(k)}$. Recall that $q_1 \geq \cdots \geq q_{k_2}$ and $\frac{1}{k_2} \sum_{j=1}^{k_2} q_j \geq \frac{B_2}{k_2+1}$. So we have $\sum_{j=1}^{j^*-1} q_j \geq (j^* - 1)\frac{B_2}{k_2+1} > p' + (j^* - 2)p_2^{(k)}$. In other words, agent 2's remaining budget in the deviation is strictly larger than that in the equilibrium. By the strict monotonicity of $U_2^{(k-j^*)}$ in agent 2's budget, agent 2 is strictly better off in the deviation. So we have a contradiction.

Case 1b: Agent 2 gets items first after the first round In this case, agent 2 wins the first $k_2 - 1$ items with average price at least $\frac{B_2 - p_2^{(k)}}{k_2}$ after the first round of the deviation, and then agent 1 wins the remaining items with prices equal agent 2's remaining budget. Since agent 2 is indifferent between winning and losing the first item, her remaining budget in the two cases shall be the same. So agent 1 wins the remaining item with prices equal r in the deviation. Further, agent 1 pays a total price $p_2^{(k)} + (k_1 - 1)r$ in the equilibrium and she weakly prefer winning the first item than losing it. So we have

$$p_2^{(k)} \leq r \leq q_{k_2} \leq \cdots \leq q_1 \quad .$$

If at least one of the inequalities is strict, then similar to case 1a, agent 2 has a profitable deviation by winning the first item at price $p' \in (p_2^{(k)}, q_1)$, and then bidding $p_2^{(k)}$ until agent 1 wins an item, and finally following the equilibrium strategy.

If all the above inequalities hold with equality, then we conclude that the prices in the sequential auction are all the same. Let p^* denote this fixed price in the auction. Note that $B_2 = \sum_{i=1}^{k_2} q_i + r = (k_2+1)p^*$. We have $p^* \stackrel{\text{def}}{=} \frac{B_2}{k_2+1}$. Note that by Proposition A.13 agent 2 could guarantee getting k_2 items paying $\frac{B_1}{k_1+1}$ per item. So it must be the case that $\frac{B_1}{k_1+1} \geq \frac{B_2}{k_2+1} = p^*$. Therefore, we have $\frac{B_1 - p^*}{k_1} \geq p^*$. By Proposition 4.2 and Proposition A.13, if agent 1 loses the first item at price p^*, then in the subgame it will be the case that agent 2 gets the next $k_2 - 1$ items with average price at least $\frac{B_2}{k_2+1} = p^*$ and then agent 1 gets the remaining k_1 item paying agent 2's remaining budget which is at most $\frac{B_2}{k_2+1} = p^*$. But agent 1 shall weakly prefer winning than losing the first item at p^*. So we conclude in the deviation where agent 1 loses the first item at p^*, all the prices are exactly p^*. Thus, agent 1 has the same utility for winning and losing the first item at p^* and $p_1^{(k)} = p^*$. Hence, we are in the Type II-B Tie-breaking case of Proposition 4.2. We could assume agent 2 gets the first k_2 items and then agent 1 gets the remaining k_1 items without changing the utilities in the equilibrium.

Case 2: $W_2^{(k)}(B_2, B_1, p)$ is discontinuous at $p = p_2^{(k)}$ In this case, let us consider a deviation where agent 2 wins the first item with price $p_2^{(k)} + \epsilon$ and follows the canonical outcome thereafter. By Proposition 3.4 of the $k-1$ item case, the utility of agent 2 in this deviation approaches $k_2 v_2 + \frac{k_2}{k_1} B_1$ as ϵ goes to zero, while the utility in the equilibrium is $k_2 v_2 + r < k_2 v_2 + \frac{1}{k_2+1} B_2$. Yet by our assumption $\frac{k_2}{k_1} B_1 \geq \frac{1}{k_1} B_1 > \frac{1}{k_2+1} B_2$. So the deviation we considered is profitable.

Summing up all these cases, we either derive contradiction or conclude we are in fact in the Type II-A Tie-breaking case. Thus, we have proved the lemma. □

Now we are ready to present to proof of Proposition 4.2.

of Proposition 4.2. Suppose $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ satisfy that $k_1 + k_2 = k$ and $\frac{k_i}{k_{-i}+1} < \frac{B_i}{B_{-i}} < \frac{k_i+1}{k_{-i}+1}$. For the sake of presentation, let us assume w.l.o.g. that $i = 1$. By Lemma A.15, it must be the case that some agent j wins the first k_j items and then the other agent wins the rest of the items. It remains to prove that $j = 1$. Suppose for contradiction that $j = 2$. By Proposition A.13, agent 2 could guarantee getting the items with prices equal $\frac{B_1}{k_1+1}$. So the remaining budget of agent 2 is at least $B_2 - k_2 \frac{B_1}{k_1+1} > B_2 - \frac{k_2}{k_2+1} B_2 = \frac{B_2}{k_2+1}$. But now agent i must be paying an average price at least $\frac{B_2}{k_2+1}$, contradicting Lemma A.15.

Now it remains to analyze the case of $\frac{B_1}{B_2} = \frac{k_1+1}{k_2+1}$. Since inductively we have assume the Type II-B Tie-breaking of Proposition 4.2 holds in the $k-1$ item case, it suffices to show that the critical prices in the first round are $p_1^{(k)} = p_2^{(k)} = p^* \stackrel{\text{def}}{=} \frac{B_1}{k_1+1} = \frac{B_2}{k_2+1}$. Further, by symmetry it suffices to show $p_1^{(k)} = p^*$.

For any price of the first item p that is less than p^*, if agent 1 wins the first item at p, then in the induced subgame agent 1 has budget strictly greater than $B_1 - p^* = \frac{k_1}{k_2+1} B_2$. So by Proposition A.13, agent 1's budget at the end shall be strictly greater than $B_1 - p^* - (k_1 - 1)\frac{B_2}{k_2+1} > \frac{B_1}{k_1+1}$. If agent 1 loses the item at p^*, on the other hand, then in the induced subgame agent 2's budget will be strictly greater than $B_2 - p^* = \frac{k_2}{k_1+1} B_1$ according to Proposition A.13. So in this case agent 1, at best, could win k_1 items with average price at least than $\frac{B_1}{k_1+1}$ and have remaining budget at most $\frac{B_1}{k_1+1}$ in the end. Therefore, we conclude that for any price that is strictly less than p^*, agent 1 will strictly prefer winning the first item than losing it. Similarly, we could show that for any price that is strictly greater than p^*, agent 1 would strictly prefers losing the first item. In sum, we have $p_1^{(k)} = p_2^{(k)} = p^*$. □

A.2.5 Weakly Declining Prices

In this section, we will prove the prices in the sequential auction is non-increasing in the number of rounds. If we are in the tie-breaking cases of Proposition 4.1 or Proposition 4.2, then clearly the price sequence is non-increasing. So it remains to discuss the case without ties.

Let us assume w.l.o.g. that $\frac{B_1}{k_1+1} < \frac{B_2}{k_2+1} < \frac{B_1}{k_1}$ for $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ s.t. $k_1 + k_2 = k$. Then, by Proposition 4.2 and Proposition A.13 agent 1 will buy the first k_1 items with average price at least $\frac{B_1}{k_1+1}$ and then agent 2 will win the rest of the items with price equals agent one's remaining budget, which is at most $\frac{B_1}{k_1+1}$. If the prices are all the same, then it is clearly non-increasing. So let us further assume the prices are not all the same. So the average price agent 1 pays is strictly greater than $\frac{B_1}{k_1+1}$.

If $k_1 = 1$ then the price sequence is clearly non-increasing. Next, we will assume $k_1 \geq 2$ and agent 1 wins the first k_1 items with prices $q_1, \ldots, q_{k_1}$. Further, we will assume for contradiction that $q_1 < q_2$. By $\mathcal{P}_3^{(k)}$, we have $q_2 \geq \cdots \geq q_{k_1}$.

Note that q_1 equals the critical price of agent 2. So by Lemma A.8, either $W_2^{(k)}(B_2, B_1, p)$ is discontinuous at $p = q_1$, or agent 2 is indifferent between winning the first item and losing it at price q_1.

In the first case, we have $q_1 = B_2 - \frac{k_2}{k_1} B_1$. Note that if agent 2 deviates by winning the first item with price $q_1 + \epsilon$, then her utility approaches $k_2 v_2 + B_2 - q_1$ as ϵ goes to zero. So in the equilibrium, the total price

Table 5: Summary of the allocation sequences in the equilibrium path and in the deviations considered in the proof.

	Round 1	Round 2	$\cdots$	Round j^*	$\cdots$
Equilibrium	1 wins at q_1	1 wins at q_2	$\cdots$	1 wins at q_{j^*}	$\cdots$
Deviation 1	2 wins at q_1	1 wins at q'_2	$\cdots$	1 wins at q'_{j^*}	$\cdots$
Deviation 2	1 wins at q_1	2 wins at q_2	$\cdots$	1 wins at q''_{j^*}	$\cdots$
Profitable Deviation	1 wins at q_1	1 wins at q_2	$\cdots$	2 wins at q_1	$\cdots$

that agent 2 pays is at most q_1. Now let us consider the second round, in which agent 1 has remaining budget $\frac{k_1+k_2}{k_1}B_1 - B_2$. Since $q_2 > q_1$ and that agent 2 pays at most q_1 in total in the equilibrium, we know that agent 2 could not be indifferent between winning and losing the second item at price q_2. By Lemma A.8, we get that q_2 is a discontinuous point of $W_2^{(k-1)}(\frac{k_1+k_2}{k_1}B_1 - B_2, B_2)$. Thus, $q_2 = B_2 - \frac{k_2}{k_1-1}\left(\frac{k_1+k_2}{k_1}B_1 - B_2\right)$. By our assumption that $q_2 > q_1$, we have $B_2 - \frac{k_2}{k_1}B_1 < B_2 - \frac{k_2}{k_1-1}\left(\frac{k_1+k_2}{k_1}B_1 - B_2\right)$, simplifying which we get $\frac{B_2}{k_2+1} > \frac{B_1}{k_1}$. So we have a contradiction to Proposition 3.6.

Let us move on to the next case that agent 2 is indifferent between winning the first item and losing it at price q_1. We let U_1 and U_2 denote the utilities in the equilibrium of agent 1 and agent 2 respectively. We will consider three possible deviation from the equilibrium path that are summarized in Table 5.

The first deviation is when agent 2 wins the first item at price q_1 and both agents follows the unique equilibrium of the subgame thereafter. In this deviation, the allocation sequence after the first round must by agent 1 wins the next k_1 items with a non-increasing price sequence and then agent 2 wins the remaining $k_2 - 1$ items paying agent 1's remaining budget. Otherwise, by Proposition 4.2 and Proposition A.13, agent 1 pays an average price that is at most $\frac{B_1}{k_1+1}$. So agent 1 is strictly better off by losing the first item, contradicting our assumption. We will let $q'_2 \geq \cdots \geq q'_{k_1+1}$ denote the prices at which agent 1 wins the items. Let U'_1 and U'_2 denote the utilities in this deviation of agent 1 and agent 2 respectively. By our assumption, $U_2 = U'_2$ and $U_1 \geq U'_1$.

The second deviation is when agent 1 wins the first item at price q_1 as in the equilibrium, but agent 2 wins the second item at price q_2, and then both agents follows the unique equilibrium of the subgame thereafter. Similar to the previous reasonings, in this subgame it must be the case that agent 1 wins the next $k_1 - 1$ items at some non-increasing price sequence and then agent 2 wins the remaining $k_2 - 1$ items paying agent 1's remaining budget. We will let $q''_3 \geq \cdots \geq q''_{k_1+2}$ denote the prices at which agent 1 wins the next $k_1 - 1$ items starting from round 3, and let U''_1 and U''_2 denote the utilities in this deviation of agent 1 and agent 2 respectively. By Lemma A.8, both agents shall weakly prefer winning the second item at price q_2 after losing the first one at q_1. So we have $U''_2 \geq U_2$ and $U''_1 \leq U_1$.

By comparing the utilities of agent 2 in the unique equilibrium and in these two deviations, we have $U''_2 \geq U_2 = U'_2$. Note that $U''_2 = v_2 + U_2^{(k-2)}(B_1 - q_1, B_2 - q_2)$ and $U'_2 = v_2 + U_2^{(k-2)}(B_1 - q'_2, B_2 - q_1)$. So we have $U_2^{(k-2)}(B_1 - q_1, B_2 - q_2) \geq U_2^{(k-2)}(B_1 - q'_2, B_2 - q_1)$. Further, we have $B_2 - q_2 < B_2 - q_1$ due to our assumption that $q_1 < q_2$. So by the monotonicity of $U_2^{(k-2)}$, we must have $B_1 - q_1 < B_1 - q'_2$, and thus $q_1 > q'_2$.

Further, by $U_1 \geq U'_1$, and by $U_1 = k_1 v_1 + B_1 - \sum_{i=1}^{k_1} q_i$ and $U'_1 = k_1 v_1 + B_1 = \sum_{i=2}^{k_1+1} q'_i$, we have $\sum_{i=1}^{k_1} q_i \leq \sum_{i=2}^{k_1+1} q'_i$. Since $q_1 > q'_2 \geq \cdots \geq q'_{k_1+1}$, we conclude that $\sum_{i=1}^{k_1} q_i < k_1 q_1$. Thus, there exist $3 \leq j \leq k_1$ such that $q_j < q_1$. Let j^* denote the smallest such j.

Now we conclude that agent 2 has a profitable deviation because she could let agent 1 wins the first

$j^* - 1$ items at price $q_1, \dots q_{j^*-1}$, and then wins the next item by bidding $q_1 - \epsilon > q_{j^*}$. The utility of agent 2 in this deviation will be $U_2^{(k-j^*)}(B_1 - \sum_{i=1}^{j^*-1} q_i, B_2 - q_1 + \epsilon) > U_2^{(k-j^*)}(B_1 - \sum_{i=2}^{j^*} q_i', B_2 - q_1) = U_2'$ due to the fact that $\sum_{i=1}^{j^*-1} q_i > (j^* - 1)q_1 > \sum_{i=2}^{j^*} q_i'$. Further, $U_2' = U_2$ by our assumption. So this is a profitable deviation for agent 2 and we have a contradiction.

A.2.6 Monotonicity and Continuity of Utility

Finally, let us analyze the monotonicity and continuity of $U_i^{(k)}$ with respect to the budgets and prove Proposition 3.3 and Proposition 3.8.

Let us first consider the case when $\frac{B_1^*}{B_2^*} \neq \frac{k_1+1}{k_2+1}$. We will assume w.l.o.g. that $\frac{k_1}{k_2+1} < \frac{B_1^*}{B_2^*} < \frac{k_1+1}{k_2+1}$. Then, there exists a neighborhood of (B_1^*, B_2^*) such that for any budget profile (B_1, B_2) in the neighborhood we have $\frac{k_1}{k_2+1} < \frac{B_1}{B_2} < \frac{k_1+1}{k_2+1}$. In other words, it is the case that agent 1 gets the first k_1 items and then agent 2 gets the remaining items in the equilibrium.

We will consider the monotonicity and continuity $U_1^{(k)}$ and $U_2^{(k)}$ as B_1 increases. By Proposition 4.2, agent 1 will get items first in the canonical outcome and thus agent 1 will get the first item paying agent 2's critical price $p_2^{(k)}$. Further, consider any sufficiently small $\epsilon > 0$ such that $\frac{k_1}{k_2+1} < \frac{B_1^*+\epsilon}{B_2^*} < \frac{k_1+1}{k_2+1}$. We let $p_2'(\epsilon)$ denote the critical price of agent 2 when the budgets are $B_1^* + \epsilon$ and B_2^*. We need the following lemmas in our argument.

Lemma A.16. *$L_2^{(k)}(B_2^*, B_1^*, p)$ is continuous at $p = p_2^{(k)}$.*

Proof. Suppose not. Then, $p_2^{(k)}$ is a discontinuous point of $L_2^{(k)}(B_2^*, B_1^*, p)$. By Lemma A.8, we know $W_2^{(k)}(B_2^*, B_1^*, p)$ must be discontinuous at $p_2^{(k)}$ as well. Thus, it must be the case that $p_2^{(k)} = B_1^* - \frac{k_1-1}{k_2+1}B_2^*$ and $p_2^{(k)} = B_2^* - \frac{k_2}{k_1}B_1^*$, which implies $p_2^{(k)} = \frac{1}{k_2+1}B_2^* = \frac{1}{k_1}B_1^*$. So we have $\frac{B_1^*}{B_2^*} = \frac{k_1}{k_2+1}$, contradicting the assumption in the lemma. □

Given the continuity of $L_2^{(k)}$ at $p = p_2^{(k)}$, we can upper bound how much the price of the first item increases as B_1 increases as follows.

Lemma A.17. *$p_2'(\epsilon) < p_2^{(k)} + \epsilon$ for sufficiently small $\epsilon > 0$.*

Proof. By the monotonicity of $W_2^{(k)}$, for nay $p < p_2^{(k)} + \epsilon$ we have $W_2^{(k)}(B_2^*, B_1^* + \epsilon, p_2^{(k)} + \epsilon) < W_2^{(k)}(B_2^*, B_1^*, p)$. So

$$
\begin{aligned}
W_2^{(k)}(B_2^*, B_1^* + \epsilon, p_2^{(k)} + \epsilon) &< \lim_{p \to p_2^{(k)}+} W_2^{(k)}(B_2^*, B_1^*, p) \\
&\leq \lim_{p \to p_2^{(k)}+} L_2^{(k)}(B_2^*, B_1^*, p) && \text{(By } \mathcal{P}^{(k)}\text{)} \\
&= L_2^{(k)}(B_2^*, B_1^*, p_2^{(k)}) && \text{(By Lemma A.16)} \\
&= L_2^{(k)}(B_2^* + \epsilon, B_1^*, p_2^{(k)} + \epsilon) \ .
\end{aligned}
$$

So agent 2 would prefer losing than winning the first item at price $p_2^{(k)} + \epsilon$ when the budgets are $B_1^* + \epsilon$ and B_2^*. So $p_2'(\epsilon) < p_2^{(k)} + \epsilon$. □

Given Lemma A.17, the monotonicity of the agents' utilities in B_1 follows straightforwardly.

Lemma A.18. *$U_1^{(k)}(B_1^*+\epsilon, B_2^*) > U_1^{(k)}(B_1^*, B_2^*)$ and $U_2^{(k)}(B_2^*, B_1^*+\epsilon) \le U_2^{(k)}(B_2^*, B_1^*)$ for sufficiently small $\epsilon > 0$.*

Proof. By Lemma A.17, we have

$$\begin{aligned} U_1^{(k)}(B_1^*+\epsilon, B_2^*) &= W_1^{(k)}(B_1^*+\epsilon, B_2^*, p_2'(\epsilon)) > W_1^{(k)}(B_1^*+\epsilon, B_2^*, p_2^{(k)}+\epsilon) \\ &= W_1^{(k)}(B_1^*, B_2^*, p_2^{(k)}) = U_1^{(k)}(B_1^*, B_2^*) \end{aligned}$$

and

$$\begin{aligned} U_2^{(k)}(B_2^*, B_1^*+\epsilon) &= L_2^{(k)}(B_2^*, B_1^*+\epsilon, p_2'(\epsilon)) \le L_2^{(k)}(B_2^*, B_1^*+\epsilon, p_2^{(k)}+\epsilon) \\ &= L_2^{(k)}(B_2^*, B_1^*, p_2^{(k)}) = U_2^{(k)}(B_2^*, B_1^*) \ . \end{aligned}$$

Therefore, we have deduced the desired monotonicity of the utilities. □

In order to prove continuity, we will first show the continuity of agent 2's critical price as B_1 increases. Since Lemma A.17 already provides us with a lower bound of $p'(\epsilon)$, we only need to come up with a lower bound of $p'(\epsilon)$.

Lemma A.19. *For any sufficiently small $\epsilon > 0$, we have $p_2'(\epsilon) > p_2^{(k)} - \frac{k_2}{k_1}\epsilon$.*

Proof. First, let us consider the case when $W_2^{(k)}(B_2^*, B_1^*, p)$ is continuous at $p = p_2^{(k)}$. In this case, by Lemma A.8 we know that $W_2^{(k)}(B_2^*, B_1^*, p_2^{(k)}) = L_2^{(k)}(B_2^*, B_1^*, p_2^{(k)})$. We let $p_2''(\epsilon) = p_2^{(k)} - \epsilon\frac{(B_2^*-p_2^{(k)})}{B_1^*}$. Then, $p_2''(\epsilon) < p_2^{(k)}$ and it is easy to verify that $B_2^* - p_2''(\epsilon) = \frac{B_1^*+\epsilon}{B_1^*}(B_2^* - p_2^{(k)})$.

On the one hand, we have

$$\begin{aligned} W_2^{(k)}(B_2^*, B_1^*+\epsilon, p_2''(\epsilon)) &= v_2 + U_2^{(k-1)}(B_2^* - p_2''(\epsilon), B_1^*+\epsilon) \\ &= v_2 + U_2^{(k-1)}\left(\frac{B_1^*+\epsilon}{B_1^*}(B_2^* - p_2^{(k)}), \frac{B_1^*+\epsilon}{B_1^*}B_1^*\right) \\ &\ge v_2 + U_2^{(k-1)}(B_2^* - p_2^{(k)}, B_1^*) = W_2^{(k)}(B_2^*, B_1^*, p_2^{(k)}) \ . \end{aligned} \tag{1}$$

Here the last inequality holds because when both agents' budgets are multiply by the same factor, we will have the same allocation sequences and the remaining budget is multiply by the same factor.

On the other hand, by $p_2''(\epsilon) < p_2^{(k)}$ we have $B_1^* - p_2^{(k)} < B_1^* + \epsilon - p_2''(\epsilon)$. So we have

$$\begin{aligned} L_2^{(k)}(B_2^*, B_1^*, p_2^{(k)}) &= U_2^{(k-1)}(B_2^*, B_1^* - p_2^{(k)}) < U_2^{(k-1)}(B_2^*, B_1^*+\epsilon - p_2''(\epsilon)) \\ &= L_2^{(k)}(B_2^*, B_1^*+\epsilon, p_2''(\epsilon)) \ . \end{aligned} \tag{2}$$

By (1), (2), and $L_2^{(k)}(B_2^*, B_1^*, p_2^{(k)}) = W_2^{(k)}(B_2^*, B_1^*, p_2^{(k)})$, we get that $L_2^{(k)}(B_2^*, B_1^*+\epsilon, p_2''(\epsilon)) < W_2^{(k)}(B_2^*, B_1^*, p_2''(\epsilon))$. Thus, $p_2'(\epsilon) \ge p_2''(\epsilon)$.

Note that $W_2^{(k)}(B_2^*, B_1^*, p_2^{(k)}) = L_2^{(k)}(B_2^*, B_1^*, p_2^{(k)}) \in [k_2v_2, (k_2+1)v_2)$ because agent 2 gets k_2 items in the equilibrium. So by $W_2^{(k)}(B_2^*, B_1^*, p_2^{(k)}) = v_2 + U_2^{(k-1)}(B_2^* - p_2^{(k)}, B_1^*)$, we get that $U_2^{(k-1)}(B_2^* - p_2^{(k)}, B_1^*) \in [(k_2-1)v_2, k_2v_2)$. Therefore, agent 1 gets k_1 items and agent 2 gets $k_2 - 1$ items in the induced subgame after agent 2 wins the first item. So by $\mathcal{P}^{(k)}$ we have that $\frac{B_2^*-p_2^{(k)}}{B_1^*} < \frac{k_2}{k_1}$ and therefore $p_2'(\epsilon) \ge p_2''(\epsilon) = p_2^{(k)} - \frac{\epsilon(B_2^*-p_2^{(k)})}{B_1^*} > p_2^{(k)} - \epsilon\frac{k_2}{k_1}$.

Next, we will consider the case when $W_2^{(k)}(B_2^*, B_1^*, p)$ is discontinuous at $p = p_2^{(k)}$. In this case, we know that $\frac{B_2^*-p_2^{(k)}}{B_1^*} = \frac{k_2}{k_1}$ and thus $p_2^{(k)} = B_2^* - \frac{k_2}{k_1}B_1^*$. Since for any $p < B_2^* - \frac{k_2}{k_1}(B_1^*+\epsilon)$ we have $\frac{B_2^*-p}{B_1^*} > \frac{k_2}{k_1}$, we get that $W_2^{(k)}(B_2^*, B_1^*+\epsilon, p) \geq (k_2+1)v_2 > L_2^{(k)}(B_2^*, B_1^*+\epsilon, p)$ for any $p < B_2^* - \frac{k_2}{k_1}(B_1^*+\epsilon)$. Therefore, we have $p_2'(\epsilon) \geq B_2^* - \frac{k_2}{k_1}(B_1^*+\epsilon) = p_2^{(k)} - \epsilon\frac{k_2}{k_1}$. □

Now we are ready to show the continuity of $U_1^{(k)}$ and $U_2^{(k)}$ in B_1.

Lemma A.20. *We have*

$$\lim_{\epsilon\to 0^+} U_1^{(k)}(B_1^*+\epsilon, B_2^*) = U_1^{(k)}(B_1^*, B_2^*) \quad , \quad \lim_{\epsilon\to 0^+} U_2^{(k)}(B_2^*, B_1^*+\epsilon) = U_2^{(k)}(B_2^*, B_1^*) \ .$$

Proof. Note that

$$\lim_{\epsilon\to 0^+} U_1^{(k)}(B_1^*+\epsilon, B_2^*) = \lim_{\epsilon\to 0^+} \left(v_1 + U_1^{(k-1)}(B_1^*+\epsilon-p_2'(\epsilon), B_2^*)\right) \tag{3}$$

$$\lim_{\epsilon\to 0^+} U_2^{(k)}(B_2^*, B_1^*+\epsilon) = \lim_{\epsilon\to 0^+} U_2^{(k-1)}(B_2^*, B_1^*+\epsilon-p_2'(\epsilon)) \tag{4}$$

Note that we have inductively assume that $U_1^{(k-1)}$ and $U_2^{(k-1)}$ are continuous at $B_1 = B_1^* - p_2^{(k)}$. Further, by Lemma A.17 and Lemma A.19, we know that $\lim_{\epsilon\to 0^+}(B_1^*+\epsilon-p_2'(\epsilon)) = B_1^* - p_2^{(k)}$. So from (3) we have $\lim_{\epsilon\to 0^+} U_1^{(k)}(B_1^*+\epsilon, B_2^*) = v_1 + U_1^{(k-1)}(B_1^*-p_2'(\epsilon), B_2^*) = U_1^{(k)}(B_1^*, B_2^*)$ and from (4) we have $\lim_{\epsilon\to 0^+} U_2^{(k)}(B_2^*, B_1^*+\epsilon) = U_2^{(k-1)}(B_2^*, B_1^*-p_2'(\epsilon)) = U_2^{(k)}(B_2^*, B_1^*)$. □

Similarly, we could show the desired monotonicity and continuity of $U_1^{(k)}$ and $U_2^{(k)}$ as B_1 decreases. We will omit the tedious calculation in this paper.

Next, let us move on to how $U_1^{(k)}$ and $U_2^{(k)}$ behaves as B_2 changes. Again, we will only consider the case that B_2 increases as the opposite case is very similar.

Lemma A.21. *Suppose a budget profile (B_1, B_2) satisfies $\frac{k_1'}{k_2'+1} < \frac{B_1}{B_2} < \frac{k_1'+1}{k_2'+1}$ for $k_1'+k_2' = k-1$. Then, for sufficiently small $\epsilon > 0$, we have $U_2^{(k-1)}(B_2+\epsilon, B_1) \geq U_2^{(k-1)}(B_2, B_1)+\epsilon$.*

Proof. Since $\frac{k_1'}{k_2'+1} < \frac{B_1}{B_2} < \frac{k_1'+1}{k_2'+1}$, in the canonical outcome of a sequential auction with $k-1$ item and budget profile (B_1, B_2), agent 1 will get the first k_1' items and agent 2 gets the remaining k_2' items. Further, for sufficiently small ϵ, we have the same allocation sequence in the canonical outcome when the budget profile is $(B_1, B_2+\epsilon)$.

By Proposition 3.3 of the $k-1$ item cases, we have $U_1^{(k-1)}(B_1, B_2) \geq U_1^{(k-1)}(B_1, B_2+\epsilon)$. So the total price that agent 1 pays becomes higher or remains the same as agent 2's budget changes from B_2 to $B_2+\epsilon$. Note that in the equilibrium allocation sequences of both cases, agent 2 pays agent 1's remaining budget for each item. So the total price that agent 2 pays decreases and remains the same as her budget increases from B_2 to $B_2+\epsilon$. Thus, we have $U_2^{(k-1)}(B_2+\epsilon, B_1) \geq U_2^{(k-1)}(B_2, B_1)+\epsilon$. □

Lemma A.22. $\frac{k_1}{k_2} \leq \frac{B_1^*}{B_2^*-p_2^{(k)}} < \frac{k_1+1}{k_2}$.

Proof. By $W_2^{(k)}(B_2^*, B_1^*, p_2^{(k)}) \geq L_2^{(k)}(B_2^*, B_1^*, p_2^{(k)}) = U_2^{(k)}(B_2^*, B_1^*)$ and that agent 2 gets k_2 items in equilibrium, we get that agent 2 gets at least k_2-1 items in the induced subgame after winning the first item at price $p_2^{(k)}$. So we have $\frac{k_1}{k_2} \leq \frac{B_1^*}{B_2^*-p_2^{(k)}} < \frac{k_1+1}{k_2-1}$.

Next, we will show that $\frac{B_1^*}{B_2^*-p_2^{(k)}} < \frac{k_1+1}{k_2}$. Suppose not. Then, by $\mathcal{P}_2^{(k)}$ and $\mathcal{P}^{(k)}$ we have that agent 2's remaining budget at the end of the induced subgame after she wins the first round is at most $\frac{B_1^*}{k_1+1}$. On the other hand, by $\mathcal{P}_2^{(k)}$ we know that in the equilibrium of the k-item sequential auction, agent 2's remaining budget is at least $B_2^* - k_2\frac{B_1^*}{k_1+1} > \frac{k_2+1}{k_1+1}B_1^* - k_2\frac{B_1^*}{k_1+1} = \frac{B_1^*}{k_1+1}$. So we deduce that $W_2^{(k)}(B_2^*, B_1^*, p_2^{(k)}) < L_2^{(k)}(B_2^*, B_1^*, p_2^{(k)})$, contradicting Lemma A.8. □

Further, since that agent 1 gets the first k_1 item in the equilibrium, we know that agent 1 gets the first $k_1 - 1$ in the induced subgame after she wins the first item at price $p_2^{(k)}$. So by Proposition 4.2 of the $k-1$ item cases, we have the following.

Lemma A.23. $\frac{k_1-1}{k_2+1} < \frac{B_1^*-p_2^{(k)}}{B_2^*} < \frac{k_1}{k_2+1}$.

Lemma A.24. *$U_2^{(k)}(B_2^*+\epsilon, B_1^*) \geq U_2^{(k)}(B_2^*, B_1^*) + \epsilon$ for sufficiently small $\epsilon > 0$.*

Proof. If $\frac{k_1}{k_2} = \frac{B_1^*}{B_2^*-p_2^{(k)}}$, then $W_2^{(k)}(B_2^*, B_1^*, p)$ is discontinuous at $p = p_2^{(k)}$. In this case, agent 2 could have bid $p_2^{(k)} + \epsilon$ when her budget is $B_2^* + \epsilon$. If she wins the first round, then by $B_2^* + \epsilon - (p_2^{(k)} + \epsilon) = B_2^* - p_2^{(k)} = \frac{k_2}{k_1}B_1^*$, she has some chance of winning $k_2 + 1$ items. So the utility would greater than $U_2^{(k)}(B_2^*, B_1^*) + \epsilon$, where agent 2 only gets k_2 items. If she loses the first round, then her utility is $U_2^{(k-1)}(B_2^*+\epsilon, B_1^*-p_2^{(k)}-\epsilon)$. By Lemma A.21 and the monotonicity of $U_2^{(k-1)}$ in B_1, this utility is greater or equal to $U_2^{(k-1)}(B_2^*, B_1^*-p_2^{(k)})+\epsilon = U_2^{(k)}(B_2^*, B_1^*)+\epsilon$. If $\frac{k_1}{k_2} \neq \frac{B_1^*}{B_2^*-p_2^{(k)}}$, then by Lemma A.22 we have $\frac{k_1}{k_2} < \frac{B_1^*}{B_2^*-p_2^{(k)}} < \frac{k_1+1}{k_2}$. Moreover, for sufficiently small $\epsilon > 0$, we have $\frac{k_1}{k_2} < \frac{B_1^*}{B_2^*+\epsilon-p_2^{(k)}} < \frac{k_1+1}{k_2}$. Thus, by letting $k_1' = k_1$ and $k_2' = k_2 - 1$ in Lemma A.21, we get that

$$
\begin{aligned}
W_2^{(k)}(B_2^*+\epsilon, B_1^*, p_2^{(k)}) &= v_2 + U_2^{(k-1)}(B_2^*+\epsilon-p_2^{(k)}, B_1^*) && \text{(Definition of } W_2^{(k)}\text{)}\\
&\geq v_2 + U_2^{(k-1)}(B_2^*-p_2^{(k)}, B_1^*)+\epsilon && \text{(Lemma A.21)}\\
&= W_2^{(k)}(B_2^*, B_1^*, p_2^{(k)})+\epsilon && \text{(Definition of } W_2^{(k)}\text{)}\\
&\geq L_2^{(k)}(B_2^*, B_1^*, p_2^{(k)})+\epsilon && (\mathcal{P}^{(k)})\\
&= U_2^{(k)}(B_2^*, B_1^*)+\epsilon\ . && (5)
\end{aligned}
$$

Further, by Lemma A.23 we have $\frac{k_1-1}{k_2+1} < \frac{B_1^*-p_2^{(k)}}{B_2^*} < \frac{k_1}{k_2+1}$. So for sufficiently small $\epsilon > 0$, we have $\frac{k_1-1}{k_2+1} < \frac{B_1^*-p_2^{(k)}}{B_2^*+\epsilon} < \frac{k_1}{k_2+1}$. Therefore, by letting $k_1' = k_1 - 1$ and $k_2' = k_2$ in Lemma A.21, we have that

$$
\begin{aligned}
L_2^{(k)}(B_2^*+\epsilon, B_1^*, p_2^{(k)}) = U_2^{(k-1)}(B_2^*+\epsilon, B_1^*-p_2^{(k)}) &\geq U_2^{(k-1)}(B_2^*, B_1^*-p_2^{(k)})+\epsilon\\
= L_2^{(k)}(B_2^*, B_1^*, p_2^{(k)})+\epsilon &= U_2^{(k)}(B_2^*, B_1^*)+\epsilon\ . \qquad (6)
\end{aligned}
$$

By (5) and (6), agent 2 could have bid $p_2^{(k)}$ and guaranteed at least $U_2^{(k)}(B_2^*, B_1^*)+\epsilon$ utility when her budget is $B_2^*+\epsilon$ for sufficiently small ϵ. Thus, $U_2^{(k)}(B_2^*+\epsilon, B_1^*) \geq U_2^{(k)}(B_2^*, B_1^*)+\epsilon$. □

Lemma A.25. *$U_1^{(k)}(B_1^*, B_2^*+\epsilon) \leq U_1^{(k)}(B_1^*, B_2^*)$ for sufficiently small $\epsilon > 0$.*

Proof. Let us consider the equilibrium allocation sequences when the budget profiles are (B_1^*, B_2^*) and $(B_1^*, B_2^* + \epsilon)$ for sufficiently small $\epsilon > 0$. By our assumption, in both cases agent 1 will gets the first k_1 items and then agent 2 will get the remaining k_2 items paying agent 1's remaining budget for each item.

By Lemma A.24, we have $U_2^{(k)}(B_2^* + \epsilon, B_1^*) \geq U_2^{(k)}(B_2^*, B_1^*) + \epsilon$. Thus, the total price that agent 2 pays when her budget is $B_2^* + \epsilon$ is lower than that when her budget is B_2^*. So we conclude that the remaining budget of agent 1 when agent 2's budget is $B_2^* + \epsilon$ is smaller than that when agent 2's budget is B_2^*. In other words, $U_1^{(k)}(B_1^*, B_2^* + \epsilon) \leq U_1^{(k)}(B_1^*, B_2^*)$. □

By Lemma A.24 and Lemma A.25, we have shown the desired monotonicity of $U_1^{(k)}$ and $U_2^{(k)}$ in B_2. It remains to show the utility functions are continuous in B_2 at point (B_1^*, B_2^*). We will let $p_2'(\epsilon)$ denote the critical price of agent 2 when the budgets are B_1^* and $B_2^* + \epsilon$. In other words, $p_2'(\epsilon)$ is the price that agent 1 pays in the first round.

Lemma A.26. *For sufficiently small $\epsilon > 0$, we have $p_2'(\epsilon) \leq p_2^{(k)} + \epsilon$.*

Proof. Consider a price $p_2^{(k)} + \epsilon + \epsilon'$ for sufficiently small $\epsilon' > 0$. By the definition of $W_2^{(k)}$, we have

$$\begin{aligned} W_2^{(k)}(B_2^* + \epsilon, B_1^*, p_2^{(k)} + \epsilon + \epsilon') &= v_2 + U_2^{(k-1)}(B_2^* - p_2^{(k)} - \epsilon', B_1^*) \\ &= W_2^{(k)}(B_2^*, B_1^*, p_2^{(k)} + \epsilon') \ . \end{aligned} \tag{7}$$

Further, by the monotonicity of $U_2^{(k-1)}$, we have

$$\begin{aligned} L_2^{(k)}(B_2^* + \epsilon, B_1^*, p_2^{(k)} + \epsilon + \epsilon') &= U_2^{(k-1)}(B_2^* + \epsilon, B_1^* - p_2^{(k)} - \epsilon - \epsilon') \\ &> U_2^{(k-1)}(B_2^*, B_1^* - p_2^{(k)} - \epsilon') \\ &= L_2^{(k)}(B_2^*, B_1^*, p_2^{(k)} + \epsilon') \ . \end{aligned} \tag{8}$$

Finally, by definition of $p_2^{(k)}$, for any $p > p_2^{(k)}$, we have $W_2^{(k)}(B_2^*, B_1^*, p) < L_2^{(k)}(B_2^*, B_1^*, p)$. So by (7), (8), we have $W_2^{(k)}(B_2^* + \epsilon, B_1^*, p_2^{(k)} + \epsilon + \epsilon') < L_2^{(k)}(B_2^* + \epsilon, B_1^*, p_2^{(k)} + \epsilon + \epsilon')$. Note that this hold for any $\epsilon' > 0$. So by the definition of critical prices, $p_2'(\epsilon) \leq p_2^{(k)} + \epsilon$. □

Lemma A.27. *We have*

$$\lim_{\epsilon \to 0^+} U_1^{(k)}(B_1^*, B_2^* + \epsilon) = U_1^{(k)}(B_1^*, B_2^*) \quad \textit{and} \quad \lim_{\epsilon \to 0^+} U_2^{(k)}(B_2^* + \epsilon, B_1^*) = U_2^{(k)}(B_2^*, B_1^*) \ .$$

Proof. On the one hand, by Lemma A.24 we have $U_2^{(k)}(B_2^* + \epsilon, B_1^*) \geq U_2^{(k)}(B_2^*, B_1^*) + \epsilon$, which goes to $U_2^{(k)}(B_2^*, B_1^*)$ as ϵ goes to zero. On the other hand, by Lemma A.26, we have $U_2^{(k)}(B_2^* + \epsilon, B_1^*) = U_2^{(k-1)}(B_2^* + \epsilon, B_1^* - p_2'(\epsilon)) \leq U_2^{(k-1)}(B_2^* + \epsilon, B_1^* - p_2^{(k)} - \epsilon)$, which also goes to $U_2^{(k-1)}(B_2^*, B_1^* - p_2^{(k)}) = U_2^{(k)}(B_2^*, B_1^*)$ as ϵ goes to zero due to the continuity of $U_2^{(k)}$. So we have $\lim_{\epsilon \to 0^+} U_2^{(k)}(B_2^* + \epsilon, B_1^*) = U_2^{(k)}(B_2^*, B_1^*)$.

Note that agent 2 will get the last k_2 items paying agent 1's remaining budget for each item. So the continuity of agent 2's utility implies that the remaining budget of agent 1 and and the utility of agent 1 are continuous in B_2 at point (B_1^*, B_2^*). □

Finally, it remains to prove the continuity and monotonicity at $\frac{B_1^*}{B_2^*} = \frac{k_1+1}{k_2+1}$. In fact, we can use the same arguments as above, except that we will assume the allocation sequence is agent 1 gets item first when we consider B_1 approaches B_1^* from below or B_2 approaches B_2^* from above, and assume agent 2 getting items first for the other two cases.

Proof. Since we have proved Lemma A.18, Lemma A.20, Lemma A.24, Lemma A.25, Lemma A.27, it remains to verify the boundary conditions. We will analyze how $U_1^{(k)}(B_1, B_2)$ behaves as B_2 approaches $\frac{k_2}{k_1}B_1$ from below. The proof of the other cases are very similar and we will omit them in this extended abstract.

Suppose $B_2 = \frac{k_2+1}{k_1}B_1 - \epsilon$ for sufficiently small $\epsilon > 0$. Then, by Proposition 4.2, agent 1 will get the first k_1 items in the canonical outcome and then agent 2 will get the remaining k_2 items paying agent 1's remaining budget. In particular, agent 1 will win the first item paying agent 2's critical price $p_2^{(k)}$.

Next, we will let $p^* = \frac{B_1}{k_1}$ for the sake of expedition and argue $p_2^{(k)} \geq p^* - \epsilon$. If agent 2 wins the first item at $p^* - \epsilon$, then her remaining budget becomes $\frac{k_2}{k_1}B_1$. So by Proposition 4.1, agent 2 will have non-zero probability of winning $k_2 + 1$ items. Hence, agent 2 shall strictly prefers winning the first item at $p^* - \epsilon$. So we have $p_2^{(k)} \geq p^* - \epsilon$.

Therefore, we have $U_1^{(k)}(B_1, B_2) = U^{(k-1)}(B_1 - p_2^{(k)}, B_2) \leq U_1^{(k-1)}(B_1 - p^* + \epsilon, B_2)$. If we let ϵ goes to zero, the limit of the right-hand-size goes to $k_1 v_1$ by our inductive hypothesis. So

$$\lim_{B_2 \to \frac{k_2+1}{k_1}B_1^-} U_1^{(k)}(B_1, B_2) \leq k_1 v_1 \ .$$

Further, $U_1^{(k)}(B_1, B_2) \geq k_1 v_1$ for any $\epsilon > 0$. So the above holds with equality.

Finally, let us consider the utility of agent i when $\frac{B_i}{B_{-i}} = \frac{k_i}{k_{-i}+1}$. We will assume w.l.o.g. that $i = 1$ for the sake of presentation. By Proposition 4.1, both agents will keep bidding $p^* \stackrel{\text{def}}{=} \frac{B_1}{k_1} = \frac{B_2}{k_2+1}$ until one of the agent runs out of her budget. So agent 1 will get $k_1 - 1$ items for sure, and with probability $\phi(k_2, k_1 - 1)$, agent 1 will get an extra item. For $n = 1, \ldots, k_1 - 1$, with probability $\phi(n, k_2) - \phi(n-1, k_2) = \binom{k_2+n}{n}\left(\frac{1}{2}\right)^{k_2+n+1}$, agent 1 gets exactly n items before agent 2 gets $k_2 + 1$ items. So the utility for agent i is

$$U_1^{(k)}(B_1, B_2) = (k_1 - 1)v_1 + \phi(k_2, k_1 - 1)v_1 + \sum_{n=1}^{k_1-1} \binom{k_2+n}{n} \left(\frac{1}{2}\right)^{k_2+n+1} (B_1 - np^*) \ .$$

Further,

$$\begin{aligned}
& \sum_{n=1}^{k_1-1} \binom{k_2+n}{n} \left(\frac{1}{2}\right)^{k_2+n+1} (B_1 - np^*) \\
= \ & \phi(k_1 - 1, k_2)B_1 - (k_2 + 1)p^* \sum_{n=1}^{k_1-1} \binom{k_2+n}{n-1} \left(\frac{1}{2}\right)^{k_2+n+1} \\
= \ & \phi(k_1 - 1, k_2)B_1 - (k_2 + 1)p^* \phi(k_1 - 2, k_2 + 1) \ .
\end{aligned}$$

Since, $p^* = \frac{B_2}{k_2+1}$ by our assumption, we have proved the asserted utility. □

B Semi-Trembling-Hand-Perfection of the Canonical Outcome

In this section, we will prove that the canonical outcome is the only stable equilibrium under the refinement of semi-trembling-hand-perfection.

Lemma B.1. *For $i = 1, 2$, suppose $p_i^{(k)} > p_{-i}^{(k)}$. Then, the canonical outcome is the unique semi-trembling-hand-perfect and subgame-perfect equilibrium.*

In the following discussion, let us assume w.l.o.g. that $i = 1$ in Lemma B.1 for the sake of presentation. In order to prove Lemma B.1, we will first show a few lemmas. The first lemma clarifies the potential equilibrium strategies we need to consider.

Lemma B.2. *Suppose $p_1^{(k)} > p_2^{(k)}$ and the agents follow the canonical outcome in the subgames of $k-1$ items. Then, the only candidate equilibrium strategy for the first round is agent 1 bidding $p+$ and agent 2 bidding p for $p_1^{(k)} \geq p \geq p_2^{(k)}$.*

Proof. First, we note that one of the agents bidding strictly greater than the other ($p+$ is not considered greater than p) cannot be an equilibrium because by the monotonicity of $W_1^{(k)}$ and $W_2^{(k)}$, the winner will make a lower bid in order to get the first item with a lower price. Further, both agents bidding strictly greater than $p_1^{(k)}$ cannot be an equilibrium because the winner (in case of tie, both agents) will prefer losing the first item at such a high price. Moreover, both agents bidding strictly smaller then $p_2^{(k)}$ cannot be an equilibrium either, because the loser (in case of tie, both agents) will prefer bid slightly higher and wins the first item. Finally, both agents bidding p for $p_1^{(k)} \geq p \geq p_2^{(k)}$, that is, agent 1 did not use the privilege of bidding $p+$, cannot be an equilibrium, because either agent 1 would strictly prefer bidding $p+$ (if $p < p_1^{(k)}$) or agent 2 would strictly prefer underbids and losing the item (if $p > p_2^{(k)}$). In sum, the only candidate equilibrium strategy for the first round is agent 1 bidding $p+$ and agent 2 bidding p for $p_1^{(k)} \geq p \geq p_2^{(k)}$. □

Lemma B.3. *Then, bidding $p > p_2^{(k)}$ in the first round (and follows the unique equilibrium in the induced subgame) is weakly dominated for agent 2.*

Proof. Let us consider the alternative strategy of bidding $p_2^{(k)} + \epsilon < p$ for sufficiently small $\epsilon > 0$. We will show this strategy weakly dominates bidding p for agent 2.

If agent 1 bids $\hat{p} > p$, then both strategies lose the first item at $\hat{p}$ and therefore yield the same payoff.

If agent 1 bids $\hat{p} = p$ or $p+$, then by $p > p_2^{(k)}$ we have $L_2^{(k)}(B_2, B_1, p) > W_2^{(k)}(B_2, B_1, p)$. So the utility of bidding p is at most $L_2^{(k)}(B_2, B_1, p)$, which equals the utility of bidding $p_2^{(k)} + \epsilon$ and losing the first item.

If agent 1 bids $\hat{p}$ s.t. $p_2^{(k)} + \epsilon \leq \hat{p} < p$, then the utility of bidding p is $W_2^{(k)}(B_2, B_1, p)$. By the monotonicity of $W_2^{(k)}$, this is less than $W_2^{(k)}(B_2, B_1, \hat{p})$. Further, by $\hat{p} > p_2^{(k)} + \epsilon \geq p_2^{(k)}$, we get that $W_2^{(k)}(B_2, B_1, \hat{p}) \leq L_2^{(k)}(B_2, B_1, \hat{p})$. So bidding $p_2^{(k)} + \epsilon$ and losing the first item at $\hat{p}$ is strictly better.

Finally , if agent 1 bids $\hat{p} < p_2^{(k)} + \epsilon$, then the both strategies wins the first item. So bidding $p_2^{(k)}$ is strictly better for that $W_2^{(k)}$ is decreasing as the price increases. □

Lemma B.4. *Suppose $p_1^{(k)} > p_2^{(k)}$ and the agents follow the canonical outcome in the subgames of $k-1$ items. Then, for any p s.t. $p_1^{(k)} \geq p > p_2^{(k)}$, agent 1 bidding $p+$ and agent 2 bidding p is not a semi-trembling-hand-perfect equilibrium.*

Proof. Consider any sequence $\{\sigma_j\}_j$ of completely mixed strategies of agent 1 that converges to bidding $p+$. We will argue bidding strictly greater than $p_2^{(k)}$ is sub-optimal for agent 2 when agent 1 use strategy σ_j for any j because it is weakly dominated. Therefore, the best responses of $\{\sigma_j\}_j$ cannot converges to bidding p since $p > p_2^{(k)}$. So it is not a semi-trembling-hand-perfect equilibrium. □

Lemma B.5. *Suppose $p_1^{(k)} > p_2^{(k)}$ and the agents follow the canonical outcome in the subgames of $k-1$ items. Then, agent 1 bidding $p_2^{(k)}+$ and agent 2 bidding $p_2^{(k)}$ is a semi-trembling-hand-perfect equilibrium.*

Proof. Let us first consider the stability of agent 2's strategy. Consider the following sequence $\{\sigma_j\}_j$ of completely mixed strategies of agent 1 that converges to bidding $p_2^{(k)}+$. Let

$$\alpha_j = W_2^{(k)}(B_2, B_1, p_2^{(k)}(1-2^{-j})) - W_2^{(k)}(B_2, B_1, 0)$$

denote the gain of winning the first item for free instead of $p_2^{(k)}(1-2^{-j-1})$ for agent 2. Let

$$\beta_j = W_2^{(k)}(B_2, B_1, p_2^{(k)}(1-2^{-j-1})) - L_2^{(k)}(B_2, B_1, p_2^{(k)}(1-2^{-j-1}))$$

denote the gain of winning the first item rather than losing it for agent 2 when the price is $p_2^{(k)}(1-2^{-j-1})$. We will let $\gamma_j = \min\{1, \frac{\beta_j}{\alpha_j}\}$ and define σ_j such that the probability density of agent 1 bidding p when she uses σ_j is:

$$f_{\sigma_j}(p) = \begin{cases} \dfrac{1}{2^j p_2} & \text{, if } \left|p - p_2^{(k)}\right| \leq \dfrac{p_2}{2^j} \\ \dfrac{\gamma_j}{2^{2j+2} p_2} & \text{, if } \left|p - p_2^{(k)}\right| > \dfrac{p_2}{2^j} \end{cases},$$

and we will choose the probability of bidding $p_2^{(p)}+$ properly such that the probability sum up to 1. It is easy to verify this sequence of completely mixed strategies converges to bidding $p_2^{(p)}$.

Further, by Lemma B.3 we get that the best response must be bids smaller or equal to $p_2^{(p)}$.

Finally, we claim any bid p that is smaller than $p_2^{(p)}(1-2^{-j})$ is strictly worse off comparing to bidding $p_2^{(p)}(1-2^{-j-1})$. When agent 1 bids above $p_2^{(k)}(1-2^{-j-1})$, both strategy yields the same payoff. When agent 1 bids between $p_2^{(k)}(1-2^{-j})$ and $p_2^{(k)}(1-2^{-j-1})$, which happens with prabability 2^{-2j-1} by our choice of σ_j, bidding $p_2^{(k)}(1-2^{-j-1})$ is better by at least β_j. When agent 1 bids below $p_2^{(k)}(1-2^{-j})$, which happens with probability at most $\gamma_j 2^{-2j-2}$ by our choice of σ_j, bidding p could be better off by at most α_j. So by $\gamma_j \leq \frac{\beta_j}{\alpha_j}$, we get that bidding $p_2^{(k)}(1-2^{-j-1})$ is better for agent 2. Therefore, we get that any best response bid to σ_j must be at least $p_2^{(k)}(1-2^{-j-1})$.

Summing up the above upper and lower bounds on the best response bids of agent 2, we get that the best responses of σ_j converges to bidding $p_2^{(p)}$ as j increases.

The stability of agent 1's strategy can be proved similarly. So the canonical outcome is semi-trembling-hand-perfect. □

Summarizing Lemma B.2, Lemma B.3, Lemma B.4, and Lemma B.5, we have proved Lemma B.1. Via simlar analysis, we can show the canonical outcome is "stable" as well when the critical prices are the same in the first round. We will omit the details here.

Lemma B.6. *Suppose $p_1^{(k)} = p_2^{(k)}$. Then, the canonical outcome is the unique semi-trembling-hand-perfect and subgame-perfect equilibrium.*