Online Load Balancing

Note Title 2/14/2013

- n machines., $i = 1 \dots n$.
- jobs arrive online, $j = 1 \dots m$.

Repeat
- $\forall i$ load / processing time $= l_{ij}$
- assign job to some machine

Minimize Makespan $= \max_i \left\{ \sum_{j \to i} l_{ij} \right\}$

Covering Problem

- Suppose we know OPT $= \Lambda$,

$\boxed{\text{Normalize units so that } \Lambda = 1 \quad \text{Then WMA, } l_{ij} \leq 1.\ \forall i,j}$

Let $x_{ij} = \mathbb{1}(j \to i)$ $\quad L_{i,j} = \sum_{j'=1}^{j} l_{ij} x_{ij}$

Algo Minimizes $\Phi_m = \sum_i (1+\varepsilon)^{L_{i,m}}$

$\forall j = 1, \dots m,$

Repeat
- Let $\Phi_{i,j-1} = (1+\varepsilon)^{L_{i,j-1}}$
- $j \to \arg\min_i \{ \Phi_{i,j-1} l_{ij} \}$

Let x_{ij}^* be the optimum solution.

Then since OPT $= 1$,

$\forall i \quad \sum_{j=1}^{m} l_{ij} x_{ij}^* \leq 1$

– cont'd....

$$\Phi_j = \sum_i (1+\varepsilon)^{L_{i,j}} = \sum_i (1+\varepsilon)^{L_{i,j-1} + l_{ij} x_{ij}}$$

$$= \sum_i \phi_{i,j-1} \cdot (1+\varepsilon)^{l_{ij} x_{ij}}$$

$$\le \sum_i \phi_{i,j-1} \left(1 + \varepsilon l_{ij} x_{ij}\right)$$

$$= \sum_i \phi_{i,j-1} + \varepsilon \sum_i \phi_{i,j-1} l_{ij} x_{ij}$$

$$\le \Phi_{j-1} + \varepsilon \sum_i \phi_{i,j-1} l_{ij} x_{ij}^*$$

$$\therefore \quad \Phi_m \le \Phi_0 + \varepsilon \sum_{i,j} \phi_{i,j-1} l_{ij} x_{ij}^*$$

$$\le \Phi_0 + \varepsilon \sum_i \phi_{i,m} \sum_j l_{ij} x_{ij}^*$$

$$\le \Phi_0 + \varepsilon \sum_i \phi_{i,m} \cdot 1$$

$$= \Phi_0 + \varepsilon \Phi_m$$

$$\therefore \quad \Phi_m (1-\varepsilon) \le \Phi_0 \quad \text{or} \quad \Phi_m \le \Phi_0 / (1-\varepsilon)$$

$$\Phi_m (L_{i1}, \dots L_{im}) \ge (1+\varepsilon)^{\max\{L_{i1}, \dots L_{im}\}} = (1+\varepsilon)^{ALG}$$

$$\Phi_0 = \sum_i (1+\varepsilon)^0 = m$$

$$\therefore \quad (1+\varepsilon)^{ALG} \le m / 1-\varepsilon$$

or $\quad ALG \log(1+\varepsilon) \le \log m - \log(1-\varepsilon)$

$\therefore$ ALG $\leq \dfrac{\log m}{\log(1+\varepsilon)} - \dfrac{\log(1-\varepsilon)}{\log(1+\varepsilon)}$

Can pick ε to be a small constant so that

ALG $\leq O(\log m)$

Generalization to resource allocation problem

- Resource $i = 1..n$. capacity C_i.
- Request $j = 1..m$.
  - set of feasible options F_j
  - $\forall k \in F_j$, resource consumption $a(i,j,k)$

$x_{jk} = \mathbb{1}(j \to k)$ pick option k for request j.

Goal: Minimize $\max_i \left\{ \dfrac{\sum_{j,k} a_{i,j,k} x_{jk}}{C_i} \right\}$

Same approach works.
