Fisher Markets and Convex Programs

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1 Introduction

Convex programming duality is usually stated in its most general form, with convex objective functions and convex constraints. (The book by Boyd and Vandenberghe is an excellent reference [2].) At this level of generality the process of constructing a dual of a convex program is not so simple, in contrast to LP duality where there is a simple set of rules one can use for the same purpose. In this article, we consider a special class of convex programs: with convex objective functions and *linear* constraints, and derive a simple set of rules to construct the dual, analogous to LPs. We also show applications of this: the main application in this article will be to derive convex programs for Fisher markets.

2 Convex programming duality

Conjugate

We now define the *conjugate* of a function, and note some of the properties. This will be the key *new* ingredient to extend the simple set of rules for LP duality to convex programs. Suppose that $f : \mathbf{R}^n \to \mathbf{R}$ is a function. The conjugate of f is $f^* : \mathbf{R}^n \to \mathbf{R}$ and is defined as follows:

$$f^*(\mu) := \sup_x \{\mu^T x - f(x)\}.$$

Although the conjugate is defined for any function f, for the rest of the article, we will assume that f is *strictly convex and differentiable*, since this is the case that is most interesting to the applications we discuss.

Properties of f^*:

- f^* is strictly convex and differentiable. (This property holds even if f is not strictly convex and differentiable.)
- $f^{**} = f$. (Here we use the assumption that f is strictly convex and differentiable.)
- If f is separable, that is $f(x) = \sum_i f_i(x_i)$ then $f^*(\mu) = \sum_i f_i^*(\mu_i)$.
- If $g(x) = cf(x)$ for some constant c, then $g^*(\mu) = cf^*(\mu/c)$.
- If $g(x) = f(cx)$ for some constant c, then $g^*(\mu) = f^*(\mu/c)$.
- If $g(x) = f(x + a)$ for some constant a, then $g^*(\mu) = f^*(\mu) - \mu^T a$.
- If μ and x are such that $f(x) + f^*(\mu) = \mu^T x$ then $\nabla f(x) = \mu$ and $\nabla f^*(\mu) = x$.
- Vice versa, if $\nabla f(x) = \mu$ then $\nabla f^*(\mu) = x$ and $f(x) + f^*(\mu) = \mu^T x$.

We say that (x, μ) form a complementary pair wrt f if they satisfy one of the last two conditions stated above. We now calculate the conjugates of some simple strictly convex and differentiable functions. These will be useful later.

- If $f(x) = \frac{1}{2}x^2$, then $\nabla f(x) = x$. Thus $f^*(\mu)$ is obtained by letting $\mu = x$ in $\mu^T x - f(x)$, which is then equal to $\frac{1}{2}\mu^2$.
- If $f(x) = -\log(x)$, then $\nabla f(x) = -1/x$. Set $\mu = -1/x$ to get $f^*(\mu) = -1 + \log(x) = -1 - \log(-\mu)$.
- Suppose $f(x) = x \log x$. Then $\nabla f(x) = \log x + 1 = \mu$. So $x = e^{\mu-1}$. $f^*(\mu) = \mu x - f(x) = x(\log x + 1) - x \log x = x = e^{\mu-1}$. That is, $f^*(\mu) = e^{\mu-1}$.

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Convex programs with linear constraints

Consider the following (primal) optimization problem.

$$\max \sum_i c_i x_i - f(x) \text{ s.t.}$$

$$\forall\ j, \sum_i a_{ij} x_i \le b_j.$$

We will derive a minimization problem that is the *dual* of this, using Lagrangian duality. This is usually a long calculation. The goal of this exercise is to identify a shortcut for the same.

Define the Lagrangian function

$$L(x,\lambda) := \sum_i c_i x_i - f(x) + \sum_j \lambda_j (b_j - \sum_i a_{ij} x_i).$$

We say that x is feasible if it satisfies all the constraints of the primal problem. Note that for all $\lambda \ge 0$ and x feasible, $L(x,\lambda) \ge \sum_i c_i x_i - f(x)$. Define the dual function

$$g(\lambda) = \max_x L(x,\lambda).$$

So for all λ, x, $g(\lambda) \ge L(x,\lambda)$. Thus $\min_{\lambda \ge 0} g(\lambda) \ge$ the optimum value for the primal program. The dual program is essentially $\min_{\lambda\ge 0} g(\lambda)$. We further simplify it as follows. Rewriting the expression for L,

$$L = \sum_i \mu_i x_i - f(x) + \sum_j b_j \lambda_j$$

$$\text{where } \mu_i = c_i - \sum_j a_{ij}\lambda_j.$$

Now note that $g(\lambda) = \max_x L(x,\lambda) = \max_x \{\sum_i \mu_i x_i - f(x)\} + \sum_j b_j \lambda_j = f^*(\mu) + \sum_j b_j \lambda_j$. Thus we get the dual optimization problem:

$$\min \sum_j b_j \lambda_j + f^*(\mu) \text{ s.t.}$$

$$\forall\ i, \sum_j a_{ij}\lambda_j = c_i - \mu_i,$$

$$\forall\ j, \lambda_j \ge 0.$$

Note the similarity to LP duality. The differences are as follows. Suppose the concave part of the primal objective is $-f(x)$. There is an extra variable μ_i for every variable x_i that occurs in f. In the constraint corresponding to x_i, $-\mu_i$ appears on the RHS along with the constant term. Finally the dual objective has $f^*(\mu)$ in addition to the linear terms. In other words, we *relax* the constraint corresponding to x_i by allowing a slack of μ_i, and *charge* $f^*(\mu)$ to the objective function.

Similarly, suppose we start with the primal problem

$$\max \sum_i c_i x_i - f(x) \text{ s.t.}$$

$$\forall\ j, \sum_i a_{ij} x_i \le b_j,$$

$$\forall\ i, x_i \ge 0.$$

Then the dual problem is

$$\min \sum_j b_j \lambda_j + f^*(\mu) \text{ s.t.}$$

$$\forall\ i, \sum_j a_{ij}\lambda_j \ge c_i - \mu_i,$$

$$\forall\ j, \lambda_j \ge 0.$$

As we saw, the optimum for the primal program is lower than the optimum for the dual program (weak duality). In fact, if the primal constraints are strictly feasible, that is there exist x_i such that for all j $\sum_i a_{ij} x_{ij} < b_j$, then the two optima are the same (strong duality) and the following generalized complementary slackness conditions characterize them:

- $x_i > 0 \Rightarrow \sum_j a_{ij}\lambda_j = c_i - \mu_i$,
- $\lambda_j > 0 \Rightarrow \sum_i a_{ij} x_i = b_i$ and
- x and μ form a complementary pair wrt f, that is, $\mu = \nabla f(x), x = \nabla f^*(\mu)$ and $f(x) + f^*(\mu) = \mu^T x$.

Similarly suppose we start from a minimization problem of the form

$$\min \sum_i c_i x_i + f(x) \text{ s.t.}$$

$$\forall\ j, \sum_i a_{ij} x_i \ge b_j,$$

$$\forall\ i, x_i \ge 0.$$

Then the dual of this is

$$\max \sum_j b_j \lambda_j - f^*(\mu) \text{ s.t.}$$

$$\forall\ i, \sum_j a_{ij}\lambda_j \le c_i + \mu_i,$$

$$\forall\ j, \lambda_j \ge 0.$$

Infeasibility and Unboundedness

When an LP is infeasible, the dual becomes unbounded. The same happens with these convex programs as well. We now give the proof for some special cases. Suppose first that the set of linear constraints is itself infeasible, that is, there is no solution to the set of inequalities

$$\forall\ j, \sum_i a_{ij} x_i \leq b_j. \tag{1}$$

Then by Farkas' lemma, we know that there exists numbers $\lambda_j \geq 0$ for all j such that

$$\forall\ i, \sum_j a_{ij}\lambda_j = 0, \text{and} \sum_j \lambda_j b_j < 0.$$

Now $g(\lambda) = f^*(c) + \sum_j \lambda_j b_j$, and by multiplying all the λ_j by a large positive number, g can be made arbitrarily small.

Now suppose that the feasible region defined by the inequalities (1) and the domain of f defined as $dom(f) = \{x : f(x) < \infty\}$ are disjoint. Further assume for now that $f^*(c) < \infty$ and that there is a strict separation between the two, meaning that for all x feasible and $y \in dom(f)$, $d(x, y) > \epsilon$ for some $\epsilon > 0$. Then once again by Farkas' lemma we have that there exist $\lambda_j \geq 0$ for all j and $\delta > 0$ such that

$$\forall y \in dom(f), \sum_{i,j} a_{ij}\lambda_j y_i > \sum_j \lambda_j b_j (1+\delta).$$

This implies that $g(\lambda) < f^*(c) - \delta \sum_j \lambda_j b_j$, and as before, by multiplying all the λ_j by a large positive number, g can be made arbitrarily small.

3 Convex programs for Fisher markets

In a Fisher market, there are n buyers and m goods. The goods are divisible, and there is a given supply for each of them. Buyer i has a budget of B_i and a utility function U_i. Given prices for the goods, he wants to buy a bundle of goods that maximizes his utility, subject to the constrain that he does not spend more than his budget. The market is at an equilibrium, if each buyer is allocated a utility maximizing bundle of goods and the market clears, that is all the goods are allocated.

An interesting special class of markets is when the buyers utilities are linear, i.e., $U_i = \sum_j u_{ij} x_{ij}$, where x_{ij} is the amount of good j allocated to him. The following is the classic Eisenberg-Gale convex program for Fisher markets with linear utilities. An optimum solution to this program captures equilibrium allocation for the corresponding market.

$$\max \sum_i B_i \log u_i \text{ s.t.}$$

$$\forall\ i, u_i \leq \sum_j u_{ij} x_{ij},$$

$$\forall\ j, \sum_i x_{ij} \leq 1,$$

$$x_{ij} \geq 0.$$

We now use the technology we developed in the previous section to construct the dual of this convex program. We let the dual variable corresponding to the constraint $u_i \leq \sum_j u_{ij} x_{ij}$ be β_i and the dual variable corresponding to the constraint $\sum_i x_{ij} \leq 1$ be p_j (these will correspond to the equilibrium prices, hence the choice of notation). We also need a variable μ_i that corresponds to the variable u_i in the primal program since it appears in the objective in the form of a concave function, $B_i \log u_i$. We now calculate the conjugate of this function. Recall that if $f(x) = -\log x$ then $f^*(\mu) = -1 - \log(-\mu)$, and if $g(x) = cf(x)$ then $g^*(\mu) = cf^*(\mu/c)$. Therefore if $g(x) = -c\log x$ then $g^*(\mu) = -c - c\log(-\mu/c) = c\log c - c - c\log(-\mu)$. In the dual objective, we can ignore the constant terms, $c\log c - c$. We are now ready to write down the dual program which is as follows.

$$\min \sum_j p_j - \sum_i B_i \log(-\mu_i) \text{ s.t}$$

$$\forall\ i, j, p_j \geq u_{ij}\beta_i,$$

$$\forall\ i, \beta_i = -\mu_i.$$

We can easily eliminate μ_i from the above to get the following program.

$$\min \sum_j p_j - \sum_i B_i \log(\beta_i) \text{ s.t} \tag{2}$$

$$\forall\ i, j, p_j \geq u_{ij}\beta_i.$$

The variables p_j's actually correspond to equilibrium prices. In fact, we can even eliminate the

β_i's by observing that in an optimum solution, $\beta_i = \min_j \{p_j/u_{ij}\}$. This gives a convex (but not strictly convex) function of the p_j's that is minimized at equilibrium. Note that this is an unconstrained[1] minimization. The function is as follows

$$\min \sum_j p_j - \sum_i B_i \log(\min_j \{p_j/u_{ij}\}).$$

It would be interesting to give an intuitive explanation for why this function is minimized at equilibrium. Another interesting property of this function is that the (sub)gradient of this function at any price vector corresponds to the (set of) excess supply of the market with the given price vector. This implies that a tattonement style price update, where the price is increased if the excess supply is negative and is decreased if it is positive, is actually equivalent to gradient descent.

Note: A convex program that is very similar to (2) was also discovered independently by Garg [4]. However it is not clear how they arrived at it, or if they realise that this is the dual of the Eisenberg-Gale convex program.

Going back to Convex Program (2), we write an equivalent program by taking the logs in each of the constraints.

$$\min \sum_j p_j - \sum_i B_i \log(\beta_i) \text{ s.t}$$

$$\forall\ i, j, \log p_j \geq \log u_{ij} + \log \beta_i.$$

We now think of $q_j = \log p_j$ and $\gamma_i = -\log \beta_i$ as the variables, and get the following convex program.

$$\min \sum_j e^{q_j} + \sum_i B_i \gamma_i \text{ s.t} \tag{3}$$

$$\forall\ i, j, \gamma_i + q_j \geq \log u_{ij}.$$

We now take the dual of this program. Again, we need to calculate the conjugate of the convex function that appears in the objective, namely e^x. We could calculate it from scratch, or derive it from the ones we have already calculated. Recall that if $f(x) = e^{x-1}$, then $f^*(\mu) = \mu \log \mu$, and if $g(x) = f(x+a)$ then $g^*(\mu) = f^*(\mu) - \mu^T a$. Thus if $g(x) = e^x = f(x+1)$ then $g^*(\mu) = f^*(\mu) - \mu = \mu \log \mu - \mu$. The dual variable corresponding to the constraint $\gamma_i + q_j \geq \log u_{ij}$ is b_{ij} and the dual variable corresponding to e^{q_j} is p_j (by abuse of notation, but it turns out that these once again correspond to equilibrium prices). Thus we get the following convex program.

$$\max \sum_{i,j} b_{ij} \log u_{ij} - \sum_j (p_j \log p_j - p_j) \text{ s.t.}$$

$$\forall\ j, \sum_i b_{ij} = p_j,$$

$$\forall\ i, \sum_j b_{ij} = B_i,$$

$$b_{ij} \geq 0 \quad \forall\ i, j.$$

A simple observation shows that $\sum_j p_j = \sum_i B_i$ is a constant and and hence can be dropped from the objective function. Thus we finally get the following convex program, which was introduced by Birnbaum, Devanur and Xiao [1].

$$\max \sum_{i,j} b_{ij} \log u_{ij} - \sum_j p_j \log p_j \text{ s.t.} \tag{4}$$

$$\forall\ j, \sum_i b_{ij} = p_j,$$

$$\forall\ i, \sum_j b_{ij} = B_i,$$

$$b_{ij} \geq 0 \quad \forall\ i, j.$$

3.1 Extensions to other markets

The Eisenberg-Gale convex program can be generalized to capture the equilibrium of many other markets, such as markets with Leontief utilities, or network flow markets. In fact, Jain and Vazirani [5] identify a whole class of such markets whose equilibrium is captured by convex programs similar to that of Eisenberg and Gale (called *EG markets*). We can take the dual of all such programs to get corresponding generalizations for the convex program (2). For instance, a Leontief utility is of the form $U_i = \min_j \{x_{ij}/\phi_{ij}\}$ for some given values ϕ_{ij}. The Eisenberg-Gale-type convex program for Fisher markets with Leontief utilities is as follows.

$$\max \sum_i B_i \log u_i \text{ s.t.}$$

[1] Although with some analysis, one can derive that the optimum solution satisfies that $p_j \geq 0$, and $\sum_j p_j = \sum_i B_i$, the program itself has no constraints.

$$\forall\ i, j, u_i \leq x_{ij}/\phi_{ij},$$
$$\forall\ j, \sum_i x_{ij} \leq 1,$$
$$x_{ij} \geq 0.$$

The dual of the above (after some simplification as before) is as follows.

$$\min \sum_j p_j - \sum_i B_i \log(\beta_i) \text{ s.t}$$
$$\forall\ i, \sum_j \phi_{ij} p_j = \beta_i.$$

In general for an EG-type convex program, the dual has the objective function $\sum_j p_j - \sum_i B_i \log(\beta_i)$ where β_i is the minimum cost buyer i has to pay in order to get one unit of utility. For instance, for the network flow market, where the goods are edge capacities in a network and the buyers are source-sink pairs looking to maximize the flow routed through the network, then β_i is the cost of the cheapest path between the source and the sink given the prices on the edges.

However, for some markets, it is not clear how to generalize the Eisenberg-Gale convex program, but the dual generalizes easily. In each of the cases, the optimality conditions can be easily seen to be equivalent to equilibrium conditions. We now show some examples of this.

Quasi-linear utilities

Suppose the utility of buyer i is $\sum_j (u_{ij} - p_j)x_{ij}$. In particular, if all the prices are such that $p_j > u_{ij}$, then the buyer prefers to not be allocated any good and go back with his budget unspent. It is easy to see that the following convex program captures the equilibrium prices for such utilities.

$$\min \sum_j p_j - \sum_i B_i \log(\beta_i) \text{ s.t} \tag{5}$$
$$\forall\ i, j, p_j \geq u_{ij}\beta_i,$$
$$\forall\ i, \beta_i \leq 1.$$

In fact, given this convex program, one could take its dual to get an EG-type convex program as well.

$$\max \sum_i B_i \log u_i - v_i \text{ s.t.}$$
$$\forall\ i, u_i \leq \sum_j u_{ij} x_{ij} + v_i,$$
$$\forall\ j, \sum_i x_{ij} \leq 1,$$
$$x_{ij}, v_i \geq 0.$$

Although this is a small modification of the Eisenberg-Gale convex program, it is not clear how one would arrive at this directly without going through the dual.

Transaction costs

Suppose that we are given, for every pair, buyer i and good j, a transaction cost c_{ij} that the buyer has to pay per unit of the good in addition to the price of the good. Thus the total money spent by buyer i is $\sum_j (p_j + c_{ij})x_{ij}$. Chakraborty, Devanur and Karande [3] show that the following convex program captures the equilibrium prices for such markets.

$$\min \sum_j p_j - \sum_i B_i \log(\beta_i) \text{ s.t} \tag{6}$$
$$\forall\ i, j, p_j + c_{ij} \geq u_{ij}\beta_i,$$
$$\forall\ i, \beta_i \leq 1.$$

Nash Bargaining

Vazirani [6] considers a convex programming that captures the Nash Bargaining solution. The program is as follows. (c_i's are all constants.)

$$\max \sum_i B_i \log(u_i - c_i) \text{ s.t.}$$
$$\forall\ i, u_i \leq \sum_j u_{ij} x_{ij},$$
$$\forall\ j, \sum_i x_{ij} \leq 1,$$
$$x_{ij} \geq 0.$$

The dual of this program is as follows

$$\min \sum_j p_j - \sum_i c_i \beta_i - \sum_i B_i \log(\beta_i) \text{ s.t} \tag{7}$$
$$\forall\ i, j, p_j \geq u_{ij}\beta_i.$$

One could follow the change of variables and try to get a convex program similar to (4) for the same. However the change of variables $q_j = \log p_j$ and $\gamma_i = \log \beta_i$ makes the program non-convex, since now the objective function has the term $-c_i e^{\gamma_i}$.

Spending constraint utilities

[1] also give an extension of Convex program (4) to what is called as the spending constraint model. We refer the reader to [1] for details.

4 Conclusions

We presented a general framework of convex programming duality for a special class of convex programs, namely programs with convex objectives and linear functions. We hope that the simplified form of this duality would lead to a greater adoption of these techniques among the community. We show how these techniques can be used to obtain new and interesting convex programs for several of the Fisher markets.

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