Draft Auctions

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Abstract

We introduce draft auctions, which is a sequential auction format where at each iteration players bid for the right to buy items at a fixed price. We show that draft auctions offer an exponential improvement in social welfare at equilibrium over sequential item auctions where predetermined items are auctioned at each time step. Specifically, we show that for any subadditive valuation the social welfare at equilibrium is an $O(\log^2(m))$-approximation to the optimal social welfare, where m is the number of items. We also provide tighter approximation results for several subclasses. Our welfare guarantees hold for Bayes-Nash equilibria and for no-regret learning outcomes, via the smooth-mechanism framework. Of independent interest, our techniques show that in a combinatorial auction setting, efficiency guarantees of a mechanism via smoothness for a very restricted class of *cardinality* valuations, extend with a small degradation, to subadditive valuations, the largest complement-free class of valuations. Variants of draft auctions have been used in practice and have been experimentally shown to outperform other auctions. Our results provide a theoretical justification.

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1 Introduction

Consider the scenario where several indivisible items are to be auctioned off to bidders with *combinatorial* valuations, i.e., valuations that depend on the entire set of items obtained. In practice, simple auctions such as *sequential item auctions* are commonly used for such purposes. As a motivating example, the *Indian Premiere League*[1] conducts an "IPL player auction" annually, an auction where the teams in the league recruit players [thehindu.com, youtube.com, cricinfo.com]. The format is a sequential auction: the players are considered one after the other in some order. When a player is up for auction the teams participate in an ascending price auction and the highest bidding team "wins" the player. The winning bid is the salary of the player for a given period.[2] The process is repeated with the next player.

It is known that sequential item auctions could lead to a highly inefficient allocation of items, as measured by the *price of anarchy*, even for very simple combinatorial valuations. We introduce a natural and simple alternative called a *draft auction* which has a much (exponentially) better price of anarchy for the very general class of subadditive valuation functions. We discuss why this makes a strong case for the following message of the paper: *if you are running a sequential item auction, then replace it with a draft auction.*

We now define the model formally: an instance of a *combinatorial auction* consists of m *items* that are to be auctioned off, n *bidders* wishing to obtain these items, and a *valuation* function $v_i : 2^{[m]} \to \mathbb{R}_+$ for each bidder i. (We identify the set of items and the set of bidders with $[m]$ and $[n]$ respectively.) We assume that the v_is are monotone and non-decreasing. The *result* of an auction is an allocation of items to bidders and payments of bidders: bidder i gets a set $S_i \subseteq [m]$ of items, and makes a payment P_i, with the S_is forming a partition of $[m]$. Bidders are selfish and try to maximize their utility from the auction, which is assumed to be *quasi-linear*, i.e., $v_i(S_i) - P_i$. The valuation $v_i(S)$ can then be interpreted as how much the set of items S is worth to i, in terms of the *numeraire* in which the payments are made. Suppose that the objective of the auction designer is to maximize the *social welfare* of the resulting allocation, which is defined as $SW := \sum_{i \in [n]} v_i(S_i)$.

An example of such an auction that is commonly seen in practice is what is called a *sequential item auction*: items are auctioned off one after the other (in some arbitrary order), using a simple auction such as an ascending price auction or a sealed bid first or second price auction. To be precise, consider a sequential, sealed-bid first price auction which is formally defined as follows. There are m rounds, and in each round $j \in [m]$ each bidder $i \in [n]$ submits a bid b_{ij}. Item j is sold to the highest bidder $i^* = \arg\max_{i \in [n]}\{b_{ij}\}$, at the price equal to her bid, b_{i^*j}, breaking ties arbitrarily. The winner's identity i^* and the winning bid b_{i^*j} are publicly revealed before proceeding to the next round.

Notice that the allocation of items in this auction is a function of the bids, and each bidder strategizes to maximize her own utility. The bid of a bidder in any round could be a function of her own valuation, the information the bidder has about other bidders' valuations, and the observed history until that time, which includes the winners and their bids in all previous rounds. In general there is no single utility-maximizing strategy for a bidder since her utility also depends on other bidders' strategies, thus setting up a *game* among the bidders.

Rational players are assumed to play *equilibrium* strategies, where each bidder's strategy is a

[1] A professional league for the sport of cricket

[2] The salaries for the most demanded players are in the range of a few million dollars, for playing about 6 weeks a year for 3 years. This is substantially higher than the player incomes prior to the auction.

"best response" to the strategies of all the other bidders. There are many equilibrium definitions in the same spirit as above but differing in technical details; see Section 3 for precise definitions.

Bounding the inefficiency at equilibrium via the price of anarchy Equilibria of certain auctions lead to allocations that are not welfare optimal. It is standard practice to analyze this inefficiency by bounding the ratio of welfares of the optimal allocation and the welfare-minimizing equilibrium of the auction. Such a bound is called the **Price of Anarchy** (PoA). The Price of Anarchy provides a quantitative scale with which we can measure such auctions;[3] a smaller price of anarchy is more desirable. To be precise, for a given valuation profile $\mathbf{v}$, let $SW(\textsc{Opt}(\mathbf{v}))$ be the optimal social welfare, which is the highest social welfare obtainable over all possible allocations of items to bidders. $SW(\textsc{Opt}(\mathbf{v})) := \max\left\{\sum_{i\in[n]} v_i(S_i) : (S_i)_{i\in[n]} \text{ is a partition of } [m]\right\}$. Let T denote a particular set of equilibria, s an equilibrium in T and $SW(s)$ the social welfare at this equilibrium. Then

$$PoA(T) := \max_{s\in T} \frac{SW(\textsc{Opt}(\mathbf{v}))}{SW(s)}.$$

The price of anarchy defined above is for a given instance; it can be generalized to a *Bayesian* setting, which formalizes the notion that bidders have probabilistic beliefs about each others valuations:[4] each v_i is drawn independently from a probability distribution $\mathcal{D}_i$ for all $i \in [n]$. The $\mathcal{D}_i$s are public knowledge, but v_i is bidder i's private information. $\mathcal{D}_i$ represents the belief about bidder i's valuation based on publicly available information. The price of anarchy is then defined as a ratio of expectations, expectation of $SW(\textsc{Opt})$ and expectation of $SW(s)$. The expectations are taken over the draws of v_i from $\mathcal{D}_i$ for each i. The *complete information* setting where all bidders know all valuations is a special case of the Bayesian setting.

The price of anarchy of an auction can crucially depend on the structure of the valuation functions; therefore, we consider special classes of valuation functions and study the worst-case (maximum) price of anarchy over all instances with valuation functions belonging to each class. Among the simplest valuation function classes are *additive* valuations, which are of the form $v_i(S_i) = \sum_{j\in S_i} v_{ij}$ and *unit-demand* valuations which are of the form $v_i(S_i) = \max_{j\in S_i}\{v_{ij}\}$. That is, a unit-demand bidder values a bundle only according to his most-valued item in the bundle.

It was recently shown by Feldman et al. [2013] that for sequential first price auctions, when bidders may have *either* additive or unit-demand valuations, **the price of anarchy could be** $\Omega(m)$ for the set of pure Nash equilibria in the complete information setting.[5] Since the class of additive/unit-demand valuations are among the simplest valuations and the set of pure Nash equilibria in the complete information setting is among the smallest set of equilibria, the price of anarchy for this case should be among the lowest. Yet the lower bound of $\Omega(m)$ is nearly as bad as it gets since it is easy to show an upper bound of $O(m)$ for a much more general class of valuations (subadditive valuations) and a much bigger set of equilibria.

Our Contributions

We propose a natural and simple variant of the sequential item auction which we call a **draft auction**. Draft auctions also proceed in rounds: each round is a sealed-bid first price auction. The

[3] Analogous to an approximation factor for approximation algorithms or a competitive ratio for online algorithms.
[4] What we call the *Bayesian* setting here is also called the *incomplete information* setting.
[5] See Section 3 for formal definitions of equilibria and the complete information setting.

difference is that there is no designated item in any round; instead, the winner decides which items she wishes to purchase in that round, paying her bid for *each* such item. Formally, a draft auction is as follows.

1. Initialize, for all $i \in [n]$, $S_i = \emptyset, P_i = 0$. The set of remaining items $I = [m]$.
2. While $I \neq \emptyset$,
3. Each bidder $i \in [n]$ submits a sealed bid b_i and a set $X_i \subseteq I$.
4. Allocate set X_{i^*} to $i^* = \arg\max_{i \in [n]}\{b_i\}$, i.e., $S_{i^*} = S_{i^*} \cup X_{i^*}$. Break ties arbitrarily.
5. Bidder i^* pays her bid for each item in X_{i^*}, i.e., $P_{i^*} = P_{i^*} + b_{i^*}|X_{i^*}|$.
6. The winner i^*, winning bid b_{i^*} and allocated bundle X_{i^*} is announced.
7. End While.

We show that draft auctions have a much better price of anarchy than sequential item auctions, for the very general class of subadditive valuation functions. **Subadditive** valuations are those v that satisfy the property $v(S \cup T) \leq v(S) + v(T)$ for all $S, T \subseteq [m]$. The class of subadditive valuations, which are also called complement-free valuations, contains other well-studied classes of valuations such as submodular, gross substitutes (see Appendix A for formal definitions), additive and unit-demand valuations. We show the following price of anarchy bound for draft auctions for subadditive valuations.

Theorem 1.1. *The price of anarchy for draft auctions for subadditive valuations with respect to Nash equilibria (Definition 3.1) in the Bayesian setting or correlated equilibria (Definition B.1) in the complete information setting is $O(\log^2 m)$.*

We show a slightly better bound for the class of XOS valuations, which is the class of valuations that are representable as a maximum of linear functions, i.e., valuations of the form

$$v(S) = \max\left\{\sum_{j \in S} v_{1j}, \ldots, \sum_{j \in S} v_{kj}\right\}.$$

Theorem 1.2. *The price of anarchy for draft auctions for* XOS *valuations with respect to Nash equilibria in the Bayesian setting or correlated equilibria in the complete information setting is $O(\log m)$.*

The relations between these classes of valuations are given below.

unit-demand $\cup$ additive $\subseteq$ gross substitutes $\subseteq$ submodular $\subseteq$ XOS $\subseteq$ subadditive.

When compared to the $\Omega(m)$ lower bound on the price of anarchy for sequential item auctions for the class unit-demand $\cup$ additive [Feldman et al., 2013], our results above give an exponential improvement.

We also show constant factor upper and lower bounds for the price of anarchy for unit-demand valuations as well as for symmetric concave valuations (where the valuation is a concave function of only the *number* of items; see Section 4.2 for a precise definition).

Theorem 1.3. *The price of anarchy for draft auctions for unit demand bidders with respect to Nash equilibria in the Bayesian setting or correlated equilibria in the complete information setting is at most 4, and w.r.t. pure Nash equilibria in the complete information setting is at most 2.*

Theorem 1.4. *The price of anarchy for draft auctions for unit demand bidders w.r.t. pure Nash equilibria in the complete information setting is at least 1.22. Further there are instances where no equilibrium achieves a welfare within $1 + \epsilon$ of the optimum, for some small universal constant $\epsilon > 0$.*

Theorem 1.5. *The price of anarchy for draft auctions for bidders with symmetric concave valuations with respect to Nash equilibria in the Bayesian setting or correlated equilibria in the complete information setting is at most 8.*

The price of anarchy bounds we show are exponentially better than those for sequential item auctions. In fact, it is possible that draft auctions have a constant price of anarchy for subadditive valuations. We use this contrast to advocate the use of draft auctions in place of sequential item auctions in practice.

To prove our upper bounds, we use the smoothness approach introduced by Roughgarden [2009] and extended to auctions by Syrgkanis and Tardos [2013]. It boils down to the following main technique: for every equilibrium, construct a deviating strategy for each player which gets at least some fraction of her value in the social-welfare maximizing allocation, while paying at most a small multiple of the revenue in equilibrium. The deviations we construct are more involved than those for sequential item auctions and the technical difficulties involved are detailed in Section 4. On a separate note, we show that efficiency bounds proven via the smoothness approach for a very special class of valuations directly extend with only a polylogarithmic degradation to the whole class of subadditive valuations and with no degradation to the class of symmetric concave valuations. Specifically, we show that it suffices to analyze settings where the value of a player is simply proportional to the number of items he acquired from a specific interest set of items. Then we show that smoothness for these simple constrained, cardinality valuations directly implies smoothness for concave symmetric valuations (i.e. identical items) with no loss, for submodular valuations with only a $\log(m)$ loss and for subadditive valuations with a $\log^2(m)$ loss. Our approach may have potential applications to the analysis of other simple mechanisms for combinatorial auction settings.

2 Illustrative Example and Related Issues

In this section we give an illustrative example, discuss several related issues and questions, and survey related work.

Illustrative example To illustrate the advantages of draft auctions over sequential item auctions, we revisit an instance introduced by Paes Leme et al. [2012], that shows that inefficiency is bound to arise at the unique subgame perfect equilibrium in undominated strategies of sequential item auctions with unit-demand bidders: Consider an instance with 4 bidders, a, b, c, d and 3 items A, B, C. Bidder a has value $v_a = \epsilon$ only for item A, bidder b has value α for either A or B, bidder c has value α for either B or C and bidder d has value $\alpha - \epsilon$ for C. It is shown by Paes Leme et al. [2012] that assuming that auctions occur in order A, C, B then in the unique equilibrium, bidder b

will let the ϵ-valued bidder a, win the auction, so that he gets the last auction for item B for free. The reasoning being that bidder c will go for item C and will not bid in the last auction. This yields a price of anarchy of $3/2$.

However, observe that the latter behavior is very much tied to the ordering of the item auctions. If the auctioneer were to run a draft auction in the same setting then it is easy to see that the optimal allocation can arise at equilibrium: bidders b, c, d all bid ϵ^+ at every iteration until they get allocated. If bidder b wins he gets item A, if bidder c wins then he gets item B and if bidder d wins he gets item C. It is easy to see that no bidder has an incentive to deviate.

Right to choose auctions A simpler variant of the draft auction is obtained by restricting each bidder to only pick one item when she wins a round. This auction format has been studied and used previously, under the names of "right to choose" (RTC) auctions or "pooled auctions". Intuitively, the two formats should not differ much; if a bidder wins a round at a certain price in an RTC auction, then she should be able to win subsequent rounds at the same price too, thus simulating a draft auction. The reason that our results don't readily extend to this format is that the deviations we construct in our proofs need the ability to win multiple items at once. The same deviation for RTC auctions would occur over multiple rounds and necessarily involve reasoning about "off-equilibrium" paths, which is perhaps the biggest technical hurdle in proving price of anarchy bounds for sequential settings. In fact, we believe that the draft vs. RTC auctions might prove to be a good training ground where this technical hurdle could be crossed.

Instances of sequential item auctions Here we identify natural candidates for implementing a draft auction in place of a sequential item auction. Sequential item auctions are used by auction houses such as Sotheby's and Christie's, which auction off art, jewelry, wine, etc. The United States government auctions off a whole bunch of seized and surplus merchandise ranging from electronics and automobiles to industrial equipment and real estate [USA.gov]. We believe that in many of these cases switching to a draft auction would be easy and beneficial. RTC auctions have already been used in some instances and have been found to give a higher revenue, for example condo sales in Miami [Ashenfelter and Genesove, 1992] and selling water rights in Chile [Alevy et al., 2010].

Why first price? The auction in each round of the draft auction is a sealed bid first price auction. Our results continue to hold for second price auctions under an extra assumption of "no overbidding". Although overbidding (i.e., bidding above one's valuation) seems unnatural and unhelpful, one cannot easily rule out such strategies in a second price auction. This makes analyzing second price auctions much harder, and the no overbidding assumption has become a common way around this difficulty . One exception is the unit-demand case where we can show that overbidding is a dominated strategy.

Ascending price auctions hold additional difficulties since in this case, each round is itself a sequential game. Due to this, we cannot hide our deviations until the very end and win a whole bunch of items once our deviation is apparent to others, like we do now. Even within a round, as soon as it becomes clear that we are deviating from the equilibrium, other players may change their behaviour before we can win the round. Nonetheless, we believe that our bounds should hold "in principle" for these auctions as well, and resolving whether they do is an interesting open question.

Why social welfare? We picked social welfare as the objective in this paper. Social welfare is probably the most common objective in the study of combinatorial auctions, and is well motivated when the auctioning authority is something like the government. Folklore has it that social welfare is also the "right" objective in the absence of a monopoly, that is if similar items can be obtained by other sellers as well.

Another natural objective is the revenue from the auction. As mentioned in the section on related work, experimental results indicate that the revenue from draft auctions is higher than other formats such as sequential item auctions on real world instances. Theoretical analysis of revenue seems more difficult as is evidenced by the dearth of such results. One difficulty is, unlike social welfare which only depends on the allocation, the revenue depends on the payments as well and therefore there is no clear benchmark for revenue as an objective. We can answer simple questions about revenue, such as "Is the revenue from one auction instance-wise better than the other?" The answer is, no, for the complete information case. Resolving this question for the Bayesian case for reasonable distributions such as regular or monotone hazard rate distributions and analysing the revenue of these auctions in general is also an important direction for future research.

Sequential vs. simultaneous auctions Another simple auction is a simultaneous item auction, where bidders submit sealed bids for all the items simultaneously and each item is sold to the highest bidder. Feldman et al. [2013] have shown a constant price of anarchy for simultaneous auctions for subadditive valuations, which indicates that simultaneous auctions are better than sequential auctions. In practice, as evidenced by the examples we cited earlier, sequential item auctions still seem to be quite commonly preferred over simultaneous item auctions. Yet another variant is the simultaneous ascending price auction, which is commonly used in FCC spectrum auctions. While there has been a lot of research into the design of these auctions, they become quite difficult to participate in as the number of items becomes very large. Thus, there are reasons other than price of anarchy bounds that might affect whether one prefers a sequential or a simultaneous auction for each scenario. A better theoretical understanding of the advantages of each and direct comparisons between the two would be very valuable.

Computational issues An important issue with all price of anarchy results is whether the players actually play at equilibrium. One justification for this issue is that often the price of anarchy bounds also hold for outcomes of no-regret learning which are computationally efficient, provided the same game is repeated many times. A nicer solution would be to change the rules of the game so that computing an equilibrium actually becomes easier for the players. Another solution is to give an algorithm that the players could use to compute a *quasi-equilibrium:* a strategy that is resistant to a reasonable subclass of deviations, and show price of anarchy bounds for these outcomes. We see resolving these issues as among the most important future directions.

Other open problems Our work raises several open questions, the most intriguing one being whether the price of anarchy for subadditive valuations is at most a constant. A constant upper bound on subclasses such as gross substitutes or submodular valuations would also be very interesting. Can we show any upper bounds for classes beyond subadditive valuations? A natural candidate is the class of valuations with restricted complements, introduced by Abraham et al. [2012].

Related work A predominant approach to combinatorial auctions is the design of "truthful mechanisms". Although the VCG mechanism is truthful and gives the socially optimal allocation, it is not computationally efficient. There has been a long line of research into designing truthful mechanisms that run in polynomial time and approximate the social welfare for various classes of valuations: see Blumrosen and Nisan ['07].

More recently, an alternate approach has been to analyze simple auctions that are commonly used in practice, by quantifying the inefficiency of equilibria via the price of anarchy [Christodoulou et al., 2008, Bhawalkar and Roughgarden, 2011, Hassidim et al., 2011, Feldman et al., 2013, Lucier and Borodin, 2010, Paes Leme and Tardos, 2010, Lucier and Paes Leme, 2011, Caragiannis et al., 2011]. Our work is most closely related to recent results on sequential item auctions: Paes Leme et al. [2012] showed a price of anarchy of 2 for unit-demand valuations in the complete information case, and that for submodular bidders the price of anarchy can grow linearly with the number of items. point. They are all closely related, so I just left it like this The positive results were later extended to the incomplete-information setting by Syrgkanis and Tardos [2012]. A dominating theme here has been the emergence of a "smoothness" framework that captures many of the price of anarchy bounds, and allows these bounds to be extended to larger classes of equilibria: Roughgarden [2009] to outcomes of learning algorithms and Roughgarden [2012] and Syrgkanis [2012] to games of incomplete information. Syrgkanis and Tardos [2013] give a specialized smoothness framework for auctions with quasi-linear preferences, which we also use. In fact, we provide a way to extend the smoothness for a very simple class of valuations to smoothness for subadditive valuations with only a polylogarithmic loss. This potentially has applicability in the analysis of other simple mechanisms for subadditive valuations. On the negative side, Feldman et al. [2013] showed that even when some valuations are unit-demand and some are additive, the price of anarchy of sequential item auctions can grow linearly with the number of items. Our work shows that this inefficiency can be largely alleviated by switching to the draft auction, thereby portraying that it was not the sequentiality that caused the inefficiency but rather the specific ordering of the items being auctioned.

In the economics community the literature on right to choose (RTC) auctions is the closest to our work. Most of this work is empirical, some in the field and others in the lab, and shows that the revenue of RTC auctions is higher than that of other auctions. Among field experiments Ashenfelter and Genesove [1992] studied the result of RTC auctions in condominium sales in Miami, which indicated empirically[6] that the revenue of RTC auctions could be higher than other formats.

Alevy et al. [2010] studied RTC auctions for water rights sales in Chile and found higher revenue than in the analogous sequential item auction. Laboratory experiments by Eliaz et al. [2008], Goeree et al. [2004] and Salmon and Iachini [2007] all find evidence of higher revenue in RTC auctions under various settings.

Most theoretical work on RTC focuses on very special cases. Harstad [2010] finds that revenue equivalence holds between RTC and sequential item auctions, for 2 superadditive bidders. Gale and Hausch [1994] has shown that all Bayes-Nash equilibria yield socially optimal allocations for 2 unit-demand bidders. [Burguet, 2007] shows that RTC generates more revenue than sequential item auctions, when there are 2 items and many single-minded, risk-averse bidders, each equally likely to prefer either item, whose valuations are drawn i.i.d from a continuous distribution.Yet, it is not clear if RTC auctions always generate a higher revenue than other auctions for a general setting.

The economics literature on sequential item auctions is focused on exact characterizations,

[6]We find the results inconclusive, due to reasons we cannot go into here.

once again for very special cases [Weber, 1981, Milgrom and Weber, 1982]. These become exceedingly difficult as we go beyond a few items. Finally, there is a substantial literature on simultaneous ascending price auctions, which is the choice of auction for FCC spectrum auctions. Many variants of these auctions have been designed, most notably by Gul and Stacchetti [2000], Ausubel and Milgrom [2002], Ausubel [2004], with the main focus of showing that truthful bidding is an equilibrium and that it results in an efficient (welfare optimal) outcome, for gross substitute valuations. However in most cases other inefficient equilibria could exist and there is no bound on how bad these equilibria could be, so these results don't imply a price of anarchy bound.

3 Preliminaries and Notation

Recall that in the **Bayesian setting**, each v_i is drawn independently from a distribution $\mathcal{D}_i$ on a set of possible valuations $\mathcal{V}_i$, all $\mathcal{D}_i$s are public knowledge and v_is are private information. In each round, the winner, the winning set and the winning price are publicly revealed. The **complete information** setting is a special case where each bidder knows the valuation of all the other bidders.[7]

A *strategy* $s_i : \mathcal{V}_i \to \Delta(B_i)$ of bidder i is a function, from her valuation to a distribution over bid plans $b_i \in B_i$. Each bid plan b_i determines the bid b_{it} that a player makes at some round t and the set X_{it} of items he gets conditional on winning, based on the information h_{it} available to her up to that round. For any given valuation profile $\mathbf{v}$, a tuple of strategies $b = \mathbf{s}(\mathbf{v}) = (s_i(v_i))_{i\in[n]}$ determines the outcome of the auction; let $u_i(b; v_i)$ denote the utility, (or expected utility when b is a distribution over bid plans) obtained by bidder i as a function of the bid plans b. Recall that for a deterministic profile the utility is $v_i(S_i(b)) - P_i(b)$ where $S_i(b)$ is the set of items i wins and $P_i(b)$ is her total payment. Additionally, for any bid plan b, we denote with $p_j(b)$ the price that item j was sold at, under bid plan b. Observe that a bid plan actually also contains information about *what might have happened*, i.e., they specify the result of possible deviations from the actual outcome, which becomes important in the definitions of equilibria. We now define the most basic equilibrium concept, that of a Nash equilibrium.

Definition 3.1. *A* **pure (resp. mixed) Bayes-Nash** *equilibrium is a pure (resp. mixed) strategy tuple* $\mathbf{s}$ *such that no player can unilaterally deviate to obtain a better utility. In other words,*

$$\forall\ i \in [n], \forall v_i \in \mathcal{V}_i, \forall\ b_i' \in B_i, \quad \mathbb{E}_{\mathbf{v}_{-i}}[u_i(b_i', \mathbf{s}_{-i}(\mathbf{v}_{-i}); v_i)] \leq \mathbb{E}_{\mathbf{v}_{-i}}[u_i(\mathbf{s}(\mathbf{v}); v_i)],$$

where as is standard, $\mathbf{s}_{-i}(\mathbf{v}_{-i})$ *denotes* $(s_j(v_j))_{j\in[n], j\neq i}$*, the strategy tuple* $\mathbf{s}$ *restricted to players other than* i*, and* $(b_i', \mathbf{s}_{-i}(\mathbf{v}_{-i}))$ *denotes the tuple where* $s_i(v_i)$ *is replaced by* b_i' *in* $\mathbf{s}(\mathbf{v})$*. Similarly* $\mathbf{v}_{-i}$ *denotes the tuple of valuations* $(v_j)_{j\in[n], j\neq i}$*. The expectations are taken over the draw of* $\mathbf{v}_{-i}$*.*

A Nash equilibrium in sequential games allows for *irrational threats*, where an equilibrium strategy of a bidder could be suboptimal beyond a certain round. A standard refinement of the Nash equilibrium for extensive form games is the *subgame perfect equilibrium*, that allows only for strategies that constitute an equilibrium of any subgame, conditional on any possible history of play (see Fudenberg and Tirole [1991] for a formal definition and a more comprehensive treatment.) Our results also extend to complete-information correlated equilibria, as defined in Appendix B.1.

[7] It is the case where D_i is v_i with probability 1.

$$\text{Subgame perfect } \subseteq \text{ Nash } \subseteq \text{Correlated Equilibria}$$

The price of anarchy may be defined w.r.t any of these equilibria; larger classes have higher price of anarchy. In the Bayesian setting the price of anarchy is defined as the worst-case ratio of the expectations, over the random values, of the social welfare at the optimum $\mathbb{E}_{\mathbf{v}}[SW(\textsc{Opt}(\mathbf{v}))]$ and at an equilibrium $\mathbb{E}_{\mathbf{v}}[SW(\mathbf{s}(\mathbf{v}))]$.

To prove our results we will use the following notion of a smooth mechanism and its corresponding implications on the price of anarchy.

Definition 3.2 (Syrgkanis and Tardos [2013]). *A mechanism is (λ, μ)-smooth for a class of valuations $\mathcal{V} = \times_i \mathcal{V}_i$ if for any valuation profile $v \in \mathcal{V}$, there exists a mapping $b_i' : B_i \to B_i$ such that for all $b \in \times_i B_i$:*

$$\sum_i u_i \left(b_i'(b_i), b_{-i}; v_i \right) \geq \lambda SW(\textsc{Opt}(\mathbf{v})) - \mu \sum_i P_i(b) \tag{1}$$

Theorem 3.3 (Syrgkanis and Tardos [2013]). *If a mechanism is (λ, μ)-smooth then the price of anarchy of mixed Bayes-Nash equilibria of the incomplete information setting and of correlated equilibria in the complete information setting is at most $\frac{\max\{1,\mu\}}{\lambda}$*

4 Price of Anarchy Upper Bounds

We will show that draft auctions are smooth mechanisms according to Definition 3.2 and therefore they achieve good social welfare at every correlated equilibrium of the complete information setting and every mixed Bayes-Nash equilibrium of the incomplete information setting.

For expository purposes, we begin by analyzing the case of unit-demand bidders. In this setting, each player is allocated only one item in the optimal allocation. To prove the smoothness property, we need to show that from any current bid profile, every player has a deviating strategy that depends only on his valuation and what he was doing previously, such that either she gets utility that is a constant fraction of his value in the optimal allocation, or his item in the optimal allocation is currently sold at a high price.

One of the technical difficulty is that, unlike sequential item auctions, a player is not aware, without information about other bidders' strategies, at which step his optimal item is going to be allocated, since this is endogenously chosen by one of his opponents. Thus, deviations of the form: "behave exactly as previously until the optimal item arrives and then deviate to acquire it", are not feasible in the case of draft auctions.[8]

Instead, our deviations for the unit-demand case have a player always attempt to get his optimal item, while it is still available, without changing the observed history when she loses. We show a deviation of the following form does just that: *At each time step, as long as your optimal item is still available, bid the maximum of your equilibrium bid and half your value for your optimal item. If you ever win, buy your optimal item.*

Theorem 4.1. *The draft auction for unit-demand bidders is a $(\frac{1}{2}, 2)$-smooth mechanism.*

[8]Even in the complete information setting, the time at which an item sells is defined by the strategies of other players: using this information to construct a deviation would not fit into the smoothness framework. In the case of mixed strategies, or incomplete information, the time an item sells is a random variable, so such a strategy is not even well-defined.

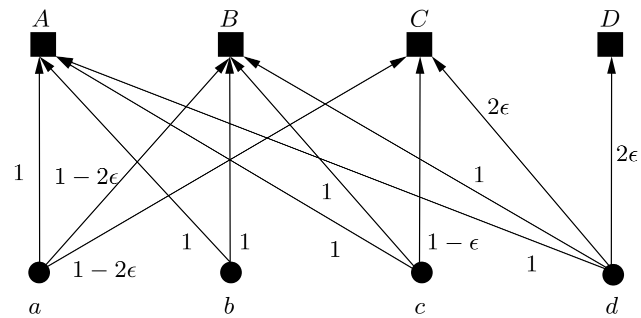

Figure 1: An example of inefficiency in draft auctions for unit-demand bidders

Proof. Consider a unit-demand valuation profile v (i.e. $v_i(S) = \max_{j\in S} v_{ij}$) and let j_i^* be the item assigned to player i in the optimal matching for valuation profile v. We will show that there exists a deviation mapping $b_i' : B_i \to B_i$ for each player i, such that for any bid profile b:

$$u_i(b_i'(b_i), b_{-i}) \geq \frac{1}{2} v_{ij_i^*} - p_{j_i^*}(b) - P_i(b). \tag{2}$$

Consider the following b_i': in every auction t, the player bids the maximum of her previous bid b_{it} (conditional on the history) and $\frac{v_{ij_i^*}}{2}$, until j_i^* gets sold. If she ever wins some auction, she picks j_i^*. Suppose that j_i^* was sold at some auction t under strategy profile b. We consider the following two cases separately, which are exhaustive since i drops out after round t at most.

Case 1: i wins an auction $t' \leq t$ in b_i'. If i wins with bid $b_{it'}$ then there must have been her payment under b_i as well, and $P_i(b) = b_{it'}$. Otherwise it is $b_i^* = \frac{v_{ij_i^*}}{2}$. Therefore her utility is

$$u_i(b_i', b_{-i}) \geq v_{ij_i^*} - \max\left\{\frac{v_{ij_i^*}}{2}, P_i(b)\right\} \geq v_{ij_i^*} - \frac{v_{ij_i^*}}{2} - P_i(b) \geq \frac{1}{2} v_{ij_i^*} - p_{j_i^*}(b) - P_i(b).$$

Case 2: i does not win any auction in b_i'. In this case, it must be that $p_{j_i^*}(b) \geq \frac{1}{2} v_{ij_i^*}$ since otherwise i would have won auction t. Her utility in this case utility is zero. Therefore (2) holds in this case as well.

Thus we have shown that the deviation b_i' always satisfies (2). The smoothness property follows by summing over all players and using the fact that $\sum_i p_{j_i^*}(b) = \sum_{j\in[m]} p_j(b) = \sum_i P_i(b)$. ∎

This implies that the draft auction has Bayes-Nash and correlated price of anarchy of at most 4 (Theorem 1.3). This bound is comparable but not identical to our bound on the pure price of anarchy, which we show to be upper-bounded by 2 in the appendix.

We note here that the Price of Anarchy for unit-demand bidders is at least 1.22 (Lemma C.4). In fact, even in the case where unit-demand bidders have the same ordinal preferences over items, draft auctions may have a pure, sequential price of stability strictly greater than 1, e.g. there are cases where *no equilibrium is optimal.*

The example shown in Figure 1 has no equilibrium where the optimal allocation is given. The optimal allocation is $(a, A), (b, B), (c, C), (d, D)$, with a total weight of $3 + \epsilon$. Since the agents have the same ordering on items, item a will be selected first. So, for the optimal allocation to be an equilibrium, a must win the first round. We show that a will have strictly higher utility when she chooses to lose the first round, assuming the optimal allocation occurs in all subgames. In the subgame where (a, A), is removed, the prices (indexed by time) for this allocation are $(1 - 2\epsilon, 1 - 2\epsilon, 0, 0)$, where player b price-sets in the first round and player d price-sets in the second round[9]. Thus, player a gets utility 2ϵ from winning the first round. On the other hand, in any subgame where a doesn't win the first round, the allocation will be $(a, C), (c, B), (d, D)$ (by assumption of optimality). Then, the prices will be $(1 - 2\epsilon, 1 - 2\epsilon, 0, 0)$; a's utility will be $1 - 4\epsilon$ larger than if she won round 1.

4.1 Smoothness for constraint-homogeneous valuations

As a next step to general subadditive valuations, we analyze smoothness of the draft auction for a simple class of valuations. We subsequently show that this is the key element in proving our efficiency results for all subadditive valuations. Specifically, we construct a deviating strategy for the class of valuations, where each player i is interested in a subset of the items $S \subseteq [m]$, and treats all items in S homogeneously, i.e. their value is a linear in the number of items from the interest set. We will denote such valuations as *constraint-homogeneous valuations.*

Definition 4.2 (Constraint-Homogeneous Valuation)**.** *A valuation on a set of items is* constraint-homogeneous *if it is defined via an interest set S and a per-unit value $\hat{v}$ such that:*

$$\forall T \subseteq [m] : v(T) = \hat{v} \cdot |T \cap S| \tag{3}$$

Unlike the unit-demand case, each player might be allocated several units in the optimal allocation. As before, a good deviating strategy should achieve a constant fraction of a player's valuations for her optimal number of units, or show that the price being paid for those units at equilibrium is high enough. Constructing such a deviating strategy is inherently more difficult than in the unit demand case. The main new technical difficulty here is to construct deviations which buy multiple units, while paying only equilibrium prices. Once a deviation has affected the winning history, the prices in the remaining off-equilibrium subgame are difficult to reason about. Thus, a player should always be trying to acquire her optimal number of units at a good price, whilst at the same time not changing the observed history of play.

The first idea is that the "right price" that a player should bid to acquire her units is half of the per-unit value, and then try to acquire the "right number" of items, which is at least half the number of units in her optimal allocation. However, consider a round where her equilibrium bid is higher than the "right price". If the bidder shades her bids down to the right price, then she may not win that round, which changes the history for all the other players and sets the game down an off-equilibrium path. In order to avoid this, the deviation bids the maximum of the original bid and the right price. If the original bid is higher, she follows the original strategy and picks the same set of items [10]. If the right price is higher, she then buys sufficient number of items to win the "right number" of units, and drops out of subsequent rounds.

The main technical meat of the paper which uses the construction of such a deviation and forms the basis of almost all the smoothness results in the paper is captured in the following lemma.

[9] We can ensure b is the price-setter for player a by slightly increasing the weight on the edges (b, A). That way, b will have a slightly higher incentive to beat a than d has.

[10] If the deviation were for the bidder to buy all the right number of units when she won because of her equilibrium bid, she might pay too much for them.

Lemma 4.3 (Core Deviation Lemma)**.** *Suppose that a player i has a constraint-homogeneous valuation with interest set S and per-unit value $\hat{v}$. Then in a draft auction there exists a deviation mapping $b'_i : B_i \to B_i$ such that, for any strategy profile b:*

$$u_i(b'_i(b_i), b_{-i}; v_i) \geq \frac{1}{2}\frac{\hat{v} \cdot |S|}{2} - \sum_{j \in S} p_j - P_i(b).$$

The lemma is proved using the following deviation which we call the Core Deviation. We refer to the items in S as *units*, and to items not in S as *items*. We denote by k_{it} (resp. $k_{i,<t}$) the number of units that player i obtains in (resp. before) auction t under the original strategy b_i. We use the shorthand notation $s^* := \left\lceil \frac{|S|}{2} \right\rceil$

Definition 4.4 (Core Deviation)**.** *The core deviation b'_i for player i with a constraint-homogeneous valuation with interest set S and per-unit value $\hat{v}$ is defined as follows.*

Let $b^*_i = \frac{\hat{v}}{2}$. In every auction t, she submits $b'_{it} = \max\{b^*_i, b_{it}\}$. If she wins with bid b^*_i, she buys $s^* - k_{i,<t}$ units of S and drops out. If she wins with a bid of b_{it}, she buys what she did under b_i: k_{it} units together with any other items she was buying under strategy profile b_i at auction t. She continues to bid b'_{it} until she acquires s^* units or the number of units remaining are not sufficient for her to complete s^* units.

The crucial observation is this: as long as the player hasn't already acquired s^* units, she has not affected the game path created by strategy b_i in any way. From the perspective of the other bidders, she behaved exactly as under b_i, by winning at her price under b_i and getting the items she would have got under b_i. If she ever wins at a higher price, she acquires all the units needed to reach s^* units in that auction and then drops out. Thus the prices that she faces in all the auctions prior to having won s^* units are the same as the prices under strategy b_i.

The Core Deviation Lemma follows immediately from Lemmas 4.5 and 4.6.

Lemma 4.5. *If player i wins at least s^* units of S under the Core Deviation b'_i then*

$$u_i(b'_i(b_i), b_{-i}; v_i) \geq \frac{1}{2}s^*\hat{v} - P_i(b).$$

Proof. If player i wins at least s^* units of S under b'_i then the valuation for the items she wins is at least $s^*\hat{v}$. For the auctions in which she wins with a bid of b_{it} she pays a total amount of at most $P_i(b)$ and for the (at most one) auction she wins with a bid of b^*_i she pays at most $s^* b^*_i$. So her total payment is at most $s^* b^*_i + P_i(b) = s^* \frac{\hat{v}}{2} + P_i(b)$. ∎

Lemma 4.6. *If player i wins fewer than s^* units of S under the Core Deviation b'_i then*

$$u_i(b'_i(b_i), b_{-i}; v_i) \geq \frac{1}{2}s^*\hat{v} - \sum_{j \in S} p_j - P_i(b).$$

Proof. Consider the auction under the original strategy profile b. Let (by an abuse of notation) $p_1 \leq p_2 \leq \ldots \leq p_{|S|}$ be the prices at which the items in S are sold under b. This is not necessarily the order in which they are sold. We show in Lemma 4.7 that, when bidder i wins fewer than s^* units under b'_i, it must be that $p_{s^*} \geq \frac{\hat{v}}{2}$. Using this we obtain that

$$\sum_{j \in S} p_j \geq \sum_{l=s^*}^{|S|} p_l \geq (|S| - s^* + 1)\, p_{s^*} \geq s^* p_{s^*} \geq \frac{\hat{v}}{2}s^*, \tag{4}$$

where we also used the simple observation that $s^* \leq \frac{|S|+1}{2}$.

The total payment of player i under b_i' in this case where she wins fewer than $|S|/2$ units of S is at most $P_i(b)$, therefore her utility is (trivially) at least $-P_i(b)$. The lemma now follows from adding the inequalities $u_i(b_i'(b_i), b_{-i}; v_i) \geq -P_i(b)$ and $0 \geq \frac{\hat{v}}{2}s^* - \sum_{j\in S} p_j$ (which holds by inequality (4)). ■

Lemma 4.7. *If player i wins fewer than s^* units of S under the Core Deviation b_i' then the s^*-th lowest price of the units in S under b, is at least $\hat{v}/2$.*

Proof. First, observe that if player i was obtaining at least s^* units under b then she is definitely winning s^* units under b_i', since she is always bidding at least as high. So, we can assume that under b player i wins fewer than s^* units.

Recall that $p_1 \leq p_2 \leq \ldots \leq p_{|S|}$ are the prices at which the units in S are sold under b. Let P_t be the price of auction t (under b). Let t^* be the first auction that was won at price $P_{t^*} \leq p_{s^*}$ under b but not by bidder i. We know that such an auction must exist; under b there are s^* units of S that are sold at a price at most p_{s^*}, and since player i wins fewer than s^* of them, some of them are not won by player i.

We now argue that player i is still bidding in auction t^* under b_i'. First of all, she has not won s^* units prior to t^*. The other condition needed for her to be active is that there are at least $s^* - k_{i,<t^*}$ units available for sale in that auction. This follows from the fact that for any auction $t < t^*$ for which $P_t \leq p_{s^*}$, we know that player i was winning under b_i. Thus every unit that was sold prior to t^* at a price of less than or equal to p_{s^*} was sold to player i. There are s^* units sold at a price $\leq p_{s^*}$ and the number of such units sold prior to t^* is at most the number of total units won by bidder i prior to t^*. Thus the number of available units available at t^* is at least: $s^* - k_{i,<t^*}$.

Finally, we argue that $P_{t^*} \geq b_i^*$. Suppose for the sake of contradiction that $P_{t^*} < b_i^*$. Then player i wins auction t^*. Since she was not winning t^* under b_i, it must be that she is winning t^* with a bid of b_i^*. Thus in that auction she will buy every unit needed to reach s^* units. By the analysis in the previous paragraph, we know that there are still enough units available for sale to reach s^*. Thus in this case she will win s^* items, a contradiction with the main assumption of the Lemma. Therefore, $b_i^* \leq P_{t^*}$ and by definition, $P_{t^*} \leq p_{s^*}$ and $b_i^* = \frac{\hat{v}}{2}$. ■

An easy corollary of the above core deviation lemma is that when all players have constraint-homogeneous valuations, the draft auction is a $\left(\frac{1}{4}, 2\right)$-smooth mechanism, and thus has a price of anarchy of at most 8 for these valuations.

Corollary 4.8. *The draft auction is a $(\frac{1}{4}, 2)$-smooth mechanism when bidders have constraint-homogeneous valuations.*

Proof. Consider a constraint-homogeneous valuation profile v and a bid profile b. Let S_i^* be the units allocated to player i in the optimal allocation for profile v. Also let S_i be the interest set of each player and $\hat{v}_i$ his per-unit value. Consider the alternative valuation profile where each player i has a constraint-homogeneous valuation v_i' with interest set $S_i' = S_i \cap S_i^*$ and per unit value $\hat{v}_i' = \hat{v}_i$.

Observe that for any $T \subseteq [m]$, $v_i(T) \geq v_i'(T)$ and $v_i(S_i^*) = v_i'(S_i^*)$. Thus, for any bid profile b: $u_i(b; v_i) \geq u_i(b; v_i')$ and $SW(\textsc{Opt}(v')) \geq SW(\textsc{Opt}(v))$. Invoking Lemma 4.3 on valuations v_i', we get that there exists a deviation mapping $b_i' : B_i \to B_i$ for each player i such that for any strategy profile b:

$$\sum_i u_i(b_i'(b_i), b_{-i}; v_i) \geq \sum_i u_i(b_i'(b_i), b_{-i}; v_i') \geq \frac{1}{4}\textsc{Opt}(v') - 2\sum_i P_i(b) \geq \frac{1}{4}\textsc{Opt}(v) - 2\sum_i P_i(b),$$

where we have once again used the fact that $\sum_i p_{j_i^*}(b) = \sum_{j\in[m]} p_j(b) = \sum_i P_i(b)$. ■

4.2 Extension to more general valuations

We will next show that smoothness for constraint-homogeneous valuations implies smoothness for a much larger class of valuations. We achieve this based on the following re-interpretation of the results in Syrgkanis and Tardos [2013][11].

Definition 4.9 (Pointwise Valuation Approximation)**.** *A valuation class V is pointwise β-approximated by a valuation class V', if for any valuation profile $v \in V$, and for any set $S \subseteq [m]$, there exists a valuation profile $v' \in V'$ such that: $\beta v'(S) \geq v(S)$ and for all $T \subseteq [m]$: $v(T) \geq v'(T)$.*

Note that, importantly, the valuation v' can depend on S. $\beta v'$ only needs to upper bound v at S, while v' needs to lower bound v everywhere else. This is much weaker than the related notion of approximation by a function class, where for every v we ask for a single v' such that v is sandwiched between $\beta v'$ and v' everywhere.

Lemma 4.10 (Extension Lemma)**.** *If a mechanism for a combinatorial auction setting is (λ, μ)-smooth for the class of valuations V' and V is pointwise β-approximated by V', then it is $\left(\frac{\lambda}{\beta}, \mu\right)$-smooth for the class V.*

Proof. Consider a valuation profile v where each valuation comes from valuation class V . For each player i let S_i^* be her optimal allocation under v and let v^* be the valuation profile such that $v_i^* \in V'$ is the valuation that β-dominates v_i for set S_i^*: i.e. $\beta \cdot v_i^*(S_i^*) \geq v_i(S_i)$ and for all $T \subseteq [m]$: $v_i(T) \geq v_i^*(T)$. By the first property we get that $\beta \cdot SW(\textsc{Opt}(v^*)) \geq SW(\textsc{Opt}(v))$. By the second property we get that for all bid profiles b: $u_i(b; v_i) \geq u_i(b; v_i^*)$. Let $b_i' : B_i \to B_i$ be the deviation mapping that is designated by the smoothness property of the mechanism under v^*. Then for any bid profile b:

$$\begin{aligned}\sum_i u_i(b_i'(b_i), b_{-i}; v_i) \geq\ & \sum_i u_i(b_i'(b_i), b_{-i}; v_i^*) \geq \lambda SW(\textsc{Opt}(v^*)) - \mu \sum_i P_i(b) \\ \geq\ & \frac{\lambda}{\beta} SW(\textsc{Opt}(v)) - \mu \sum_i P_i(b)\end{aligned}$$

which implies the mechanism is smooth for the valuation class V. ■

Identical Items and Concave Symmetric Valuations. We first consider the case where all items are identical and players have a valuation that is a concave function of the number of items acquired, i.e., $v_i(S) = f_i(|S|)$ for some non-decreasing concave function $f_i : \mathbb{N} \to \mathbb{R}^+$. We call these as *concave symmetric valuations*. We show that all such valuations can be pointwise 1-approximated by constraint-homogeneous valuations. As a corollary we get that the price of anarchy of draft auctions for this case is at most 8 (Theorem 1.5).

Theorem 4.11. *The class of concave symmetric valuations is pointwise 1-approximated by constraint-homogeneous valuations.*

[11] Hartline [2013] gives a special case of this re-interpretation for the mechanism defined by simultaneous single-item auctions, showing how smoothness for additive valuations implies smoothness for unit-demand (and XOS) valuations

Proof. Consider a valuation profile v as described in the theorem (i.e. $v_i(S) = f_i(|S|)$). Consider a set $S \subseteq [m]$ and let v_i' be the constraint-homogeneous valuation with interest set S and per-unit valuation $\hat{v}_i' = \frac{f_i(|S_i^*|)}{|S_i^*|}$. By concavity of the valuation v_i we have that for any $T \subseteq [m]$:

$$v_i'(T) = \hat{v}_i' \cdot |T \cap S_i^*| = \frac{f_i\left(|S_i^*|\right)}{|S_i^*|} \cdot |T \cap S_i^*| \leq f_i\left(|T \cap S_i^*|\right) \leq v_i(T) \tag{5}$$

Additionally, $v_i'(S_i^*) = f_i(|S_i^*|) = v_i(S_i^*)$. ■

Heterogeneous items. We next turn to simplest class of valuations over heterogeneous items, additive valuations. We show that the Core Deviating Lemma implies a $O(\log(m))$ price of anarchy. The key technical step is showing that any additive valuations can be pointwise approximated within a logarithmic factor by a constraint-homogeneous valuation, via a standard bucketing argument.

Lemma 4.12. *Additive valuations can be pointwise* $2(\log(m-1)+1)$*-approximated by constraint-homogeneous valuations.*

Proof. Consider an additive valuation v, i.e $v(T) = \sum_{j \in T} v_j$. Let S be a set of k items and sort the items in S in decreasing order of value: $v_1 \geq v_2 \geq \ldots \geq v_k$. Consider the partition of items $\mathcal{P}$ where $I_1 = \left\{j : v_j \geq \frac{v_1}{2}\right\}$, and more generally, for any $t \in [2, \log(k-1)]$

$$I_t = \left\{j \mid \frac{v_1}{2^{t-1}} > v_j \geq \frac{v_1}{2^t}\right\}.$$

Let the final set I_f contain all the smallest items, $I_f = \left\{j : v_j < \frac{v_1}{k-1}\right\}$. Notice that the largest-valued item in I_f has value at most $\frac{v_1}{k-1}$ and there are at most $k-1$ items in I_f, thus, $v_i(I_f) < v_1$, and so $v_i(I_1) > v_i(I_f)$. There are $\log(k-1)+1$ sets in $\mathcal{P}$, so the largest valued one has value at least $\frac{v(S)}{\log(k-1)+1}$. It cannot be I_f so it is one of the first $\log(k-1)$ sets. Thus if we denote with $\tau = \arg\max_{t \in [1,\ldots,f-1]} v_i(I_t)$ we get that:

$$v(I_\tau) \geq \frac{v(S)}{\log(k-1)+1} \tag{6}$$

Now consider the constraint-homogeneous valuation v_i' with interest set I_τ and $\hat{v}' = \min_{j \in I_\tau} v_j$. It is obvious that for any set $T \subseteq [m]$: $v(T) \geq v'(T)$, since an element's valuation was either set to zero or decreased under v'. Additionally, since items in I_τ only differ by a factor of 2, we also get that $v'(S) = \hat{v} \cdot |I_\tau| \geq \frac{v(I_\tau)}{2} \geq \frac{v(S)}{2(\log(k-1)+1)}$. ■

Moreover, as we show in the Appendix (Section C.5), the latter approximation is asymptotically tight. Combining the latter lemma with the smoothness of draft auctions for constraint-homogeneous valuations we get the efficiency guarantee for additive valuations.

Corollary 4.13. *The draft auction is a* $(\frac{1}{8(\log(m-1)+1)}, 2)$*-smooth mechanism for additive bidders, implying a price of anarchy of at most* $16(\log(m-1)+1)$.

Additionally, by the definition of XOS valuations, it is easy to see that they are pointwise 1-approximated by additive valuations, in the sense of Definition 4.9. Moreover, it is known (see e.g. Bhawalkar and Roughgarden [2011]) that subadditive valuations can be pointwise H_m-approximated by additive valuations, in the sense of Definition 4.9 and this is tight. This leads to the following two corollaries.

Corollary 4.14. *The draft auction is a* $(\frac{1}{8(\log(m-1)+1)}, 2)$*-smooth mechanism for XOS valuations and it is a* $\left(\frac{1}{8H_m(\log(m-1)+1)}, 2\right)$*-smooth mechanism for subadditive valuations.*

This in turn implies that the price of anarchy of draft auctions is $O(\log(m))$ for XOS valuations (Theorem 1.2) and $O(\log^2(m))$ for subadditive valuations (Theorem 1.1).

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A Valuation Classes

Definition A.1. *A monotone valuation function is* **submodular** *if it exhibits the diminishing marginal value property, which to be precise is that*

$$\forall\ S \subseteq T, \forall\ i \notin T, v(S \cup \{i\}) - v(S) \geq v(T \cup \{i\}) - v(T).$$

Definition A.2. *Valuations with the* **gross substitutes** *property are defined in terms of the corresponding* demand *function. Given prices p_j for all $j \in [m]$, the demand correspondence is*

$$x(p_j)_{j\in[m]} := \arg\max_{S\subseteq[m]} \left\{ v(S) - \sum_{j\in S} p_j \right\}.$$

A demand function satisfies gross-substitutes if increasing the price of one item does not decrease the demand for any other item. If the demand function is a correspondence, then it satisfies the gross-substitute condition when the following holds: if an item j is in some demand set under price $p = (p_1, \ldots, p_m)$, then after increasing the price of item j and keeping the rest of the prices the same, there exists a demand set under the new prices that contains j.

B Definition of Correlated Equilibrium

Definition B.1. Correlated equilibrium *A correlated equilibrium is a distribution X over joint strategy profiles such that, for each player i, following the suggestion s_i drawn from the distribution X is a best-response, in expectation over the suggestions s_{-i}, not known to i and assuming everyone else plays according to their suggestion:*

$$\mathbb{E}_{s_{-i},\mathbf{v}}[u_i(\mathbf{s}(\mathbf{v})) \mid s_i] \geq \mathbb{E}_{s_{-i},\mathbf{v}}[u_i\left(s'_i(v_i), \mathbf{s}_{-i}(\mathbf{v}_{-i})\right) \mid s_i]$$

Note that the deviation is allowed to depend on the suggestion (in the event that s'_i is required to be independent of s_i for all i, we call s a coarse correlated equilibrium).

C Omitted Proofs

C.1 Existence of Pure SPE for Single-Item Draft Auctions

Observation C.1. *Single draft auctions always have pure subgame perfect equilibria, where bidders do not use weakly dominated strategies, for bidders who have arbitrary valuations.*

Proof. By Theorem 2.1 of previous work Syrgkanis [2012], every first-price single-item auction with externalities has a pure Nash equilibrium which doesn't use dominated strategies. Single draft auctions can be thought of as single auctions with externalities: bidder i has an associated value in the remaining game as a function of the player j who wins the current auction (since j has a well-defined best item to take once he wins). Thus, one can construct a subgame perfect equilibrium by backwards induction. The final auction has no externalities. As a function of who wins the kth auction and what they take, there is a well-defined value each bidder has for the remaining auctions. Those define the externalities for the kth auction. ■

Observation C.2. *The above proof relies on the single selection process: if a player can select more than one item, the set she chooses (and thus her externalities for winning, and other's externalities for her winning) change as a function of the price at which she wins.*

C.2 The Pure PoA for unit-demand bidders is at most 2

We give a tighter upper bound on the price of anarchy for pure Nash equilibria, which was the second part of Theorem 1.3.

Observation C.3. *The pure-Nash PoA is 2 for unit-demand bidders.*

Proof. Consider an agent i who gets the item $j^*(i)$ in OPT. There are 2 cases: that i wins in a round where $j^*(i)$ has yet to be sold, and where i wins a round after $j^*(i)$ has been sold. In case 1, $v_{i,j(i)} \geq v_{i,j^*(i)}$ (since i has the choice to take $j^*(i)$). In case 2, $v_{i,j(i)} \geq v_{i,j^*(i)} - p(j^*(i))$. Summing up over all players, we have $SW_{EQ} \geq SW_{OPT} - REV_{EQ}$. Thus, since $REV_{EQ} \geq SW_{EQ}$, by individual rationality, $2SW_{EQ} \geq SW_{OPT}$. ■

C.3 Inefficiency: Proof of Theorem 1.4

In this section, we show that draft auctions for unit demand bidders may be inefficient, even when all players agree on the relative ordering of the items by their value. We also show that the pure price of anarchy is between 2 and $209/177 > 1.22$ for unrestricted unit-demand bidders. Here, we present an example of the inefficiency which arises from the competition between agents.

Lemma C.4. *The pure price of anarchy of draft auctions for unit-demand bidders is at least 1.22.*

Proof. Consider the matrix

$$\left(\begin{array}{c|ccc} A & 32 & 31 & 83 \\ B & 9 & 84 & 97 \\ C & 2 & 42 & 93 \end{array}\right)$$

and the strategy profile where B wins first, supported by price $p_1 = 51$ by agent A, then C wins at price $p_2 = 0$, then A wins at price $p_3 = 0$. The allocation will be $(1, 3, 2)$, with social welfare

$97 + 32 + 42 = 171$, while the optimal allocation $(1, 2, 3)$ gives social welfare of $32 + 84 + 93 = 209$. Thus, the PoA is at least $209/171 > 1.22$.

Showing this is a SPE is not difficult: it is necessary, however, for A to be the price supporter in round 1, rather than C (both have equal externality for B winning round 1). if C price-sets, B would rather lose round 1 and the outcome will be efficient. ■

The proof of Theorem 1.4 follows from Lemma **??** and Lemma C.4.

C.4 Non-uniqueness of equilibria and Non-dominance of Revenue

We show that equilibria are non-unique for draft auctions, even with unit-demand bidders. Consider the following valuations, where a, b, c are the bidders, and the items are A^*, B^*. Suppose a has value 1 for A^* and 0 for B^*, and b and c have value 2 for either item. Then, there are two pure equilibria: one where b wins A^* for price 1 and c wins B^* for 0, and the other where c wins A^* for 1 and b wins B^* for 0.

This example also shows that item auctions have multiple equilibria (if A^* is sold first, either b or c winning at price 1 and then the other winning the second round at price 0 is an equilibrium). If we consider the item auction with the reverse order, where B^* is sold and then A^*, the price in each round will be 1, which shows the revenue from the best order is better than the revenue from draft auctions.

C.5 Tightness of pointwise approximation by constraint-homogeneous valuations

Theorem C.5. *There exist additive valuations v for which an $o(\log(m))$-pointwise approximation by a constraint-homogenous valuation does not exist.*

Proof. We construct such a valuation v. Consider an instance with $m = 2^{k+1} - 1$ items divided into $k + 1$ buckets, indexed by $\{0, \ldots, k\}$. Each bucket $t \in \{0, \ldots, k\}$ has 2^t items and each such item has a value of $\frac{1}{2^{t+1}}$. Notice that $m = \sum_{t=0}^{k} 2^t = 2^{k+1} - 1$ and $v([m]) = \frac{k+1}{2}$, since each bucket yields a total value of $1/2$.

Suppose for contradiction there exists some β-pointwise approximation to v. Consider the β-pointwise approximation to v at $[m]$, v'_m. It must be the case that

$$v'_m([m]) \geq \frac{v([m])}{\beta} = \frac{k+1}{2 \cdot \beta} \tag{7}$$

Now, since v'_m is constraint-homogeneous, it must be the case that there is some set S for which

$$v'_m([m]) = v'_m([m] \cap S) = \hat{v}'_m \cdot |m \cap S| = \hat{v}'_m \cdot |S|.$$

Therefore, it must be that $\hat{v}'_m \cdot |S| \geq \frac{k+1}{\beta}$. Moreover, we need to satisfy that for any other set $T \subseteq S$ is not over-valued:

$$v'_m(T) = \hat{v}'_m \cdot |T| \leq v(T).$$

Therefore, if S contains any element from bucket t, then it must be that $\hat{v}'_m \leq \frac{1}{2^{t+1}}$. Otherwise, the value for the singleton set containing only that item, would be over-valued. Let t, be the largest-indexed bucket from which S contains an element. In that case the total number of items contained in S is at most $\sum_{k=0}^{t} 2^k = 2^{t+1} - 1$. Hence, the total value of the constraint homogeneous valuation for the whole set of items is at most:

$$v'_m([m]) = \hat{v}'_m \cdot |S| \leq \frac{1}{2^{t+1}} \cdot (2^{t+1} - 1) = 1 - \frac{1}{2^{t+1}} \leq 1 \tag{8}$$

Thus we get a lower bound on the approximation β that can be achieved:

$$\beta \geq \frac{v([m])}{v'_m([m])} \geq \frac{k+1}{2} \geq \frac{\log(m+1)}{2} \tag{9}$$

Thus it must be that $\beta = \Omega(\log(m))$. ■