Here, I am making an attempt to fill in the initial data for a scattering diagram in the surfaces case. I'm trying to follow the table of data in the "Scattering fans" manuscript for easy comparison.—Nathan

I fix $(\mathbf{S}, \mathcal{M})$, a tagged triangulation T and a multi-lamination $\mathbf{L}$. 1

1. maybe it matches better with other papers to call the individual laminations L, not Λ

TABLE 1. Initial data and preliminary definitions for scattering diagrams from surfaces

Notation	Description/requirements
N	$N_{\mathrm{uf}} \oplus N_{\mathrm{fr}}$ N_{uf} is formal sums of tagged arcs of T with integer coefficients. N_{fr} is formal sums of laminations in $\mathbf{L}$ with integer coefficients.
$M = \mathrm{Hom}(N, \mathbb{Z})$	$M_{\mathrm{uf}} \oplus M_{\mathrm{fr}}$ M_{uf} nonnegative integer-weighted quasi-laminations. M_{fr} formal sums of coefficients $(y_\Lambda : \Lambda \in \mathbf{L})$ with integer coefficients.
V	formal sums of tagged arcs of T with real coefficients.
I	$T \cup \mathbf{L}$
$I_{\mathrm{uf}} \subseteq I$	T
$I_{\mathrm{fr}} \subseteq I$	$\mathbf{L}$
$\{\cdot, \cdot\} : N \times N \to \mathbb{Q}$	$\{\alpha, \gamma\}$ is signed adjacency b_{α_γ} with the usual convention. $\{\alpha, \Lambda\}$ is shear coordinate $b_\alpha(T, \Lambda)$ with the usual convention giving positive coordinate when a curve sees α on its right when entering the rectangle. $\{\Lambda, \alpha\}$ is $-b_\alpha(T, \Lambda)$ But we need to think carefully about this convention: This is set up so that the shear coordinates give coefficients in the **wide** matrices sense. I think that's right because it matches what GHKK do, but I might be wrong, or we might have the freedom to choose either way. Tall matrices would be nicer for universal coefficients, at least psychologically.
$N^\circ \subseteq N$	$N^\circ = N$, so we're requiring that $\{\cdot, \cdot\}$ actually maps $N \times N$ to the integers.
$M^\circ = \mathrm{Hom}(N^\circ, \mathbb{Z})$	$M^\circ = M$
d_i	$d_i = 1$ for all i
$\mathbf{s} = (e_i : i \in I)$	$T \cup \mathbf{L}$
$N^+ = N^+_{\mathbf{s}}$	nonzero formal sums of tagged arcs of T with **nonnegative** integer coefficients

Table 1. (continued)

Notation	Description/requirements
$[\,\cdot\,,\,\cdot\,]_{\mathbf{s}} : N \times N \to \mathbb{Q}$	$[e_i, e_j]_{\mathbf{s}} = \{e_i, e_j\}$
ε_{ij}	$\{e_i, e_j\}$, entries of the B-matrix, extended in both directions, but we never need to care about entries indexed by two laminations
$\{e_i^* : i \in I\}$	$\{$elementary laminations $L_\alpha : \alpha \in T\} \cup \{y_\Lambda : \Lambda \in \mathbf{L}\}$
$\{f_i : i \in I\}$	$f_i = e_i$
V^*	"real quasi-laminations" = closure of set of real-weighted quasi-laminations. Alternatively, we could just take V^* to be formal sums of elementary laminations with integer coefficients. In some contexts below (e.g. p^*), that makes more sense anyway. (Maybe it makes sense to have both realizations of V^*. The only real point is that every formal rational sum of elementary laminations is actually a lamination.)
$\langle\,\cdot\,,\,\cdot\,\rangle : V^* \times V \to \mathbb{R}$	sum of shear coordinates (or limit of such sums) Alternatively, weighted sum of shear coordinates of elementary laminations.
$\langle\,\cdot\,,\,\cdot\,\rangle : M^\circ \times N \to \mathbb{Q}$	sum of shear coordinates on $M_{\mathrm{uf}} \times N_{\mathrm{uf}}$, also $\langle y_\Lambda, \Pi\rangle = \delta_{\Lambda\Pi}$ and zero otherwise.
$p^* : N_{\mathrm{uf}} \to M^\circ$	$p^*(\alpha) = \sum_{\beta \in T} b_{\alpha\beta} L_\beta + \sum_{\Lambda \in \mathbf{L}} b_\alpha(T, \Lambda) y_\Lambda$ A good reason to keep the definition of $\{\,\cdot\,,\,\cdot\,\}$ as it is above: If I switch it, I get a minus sign between the two sums defining $p^*(\alpha)$, which doesn't seem nice. $(p^*(e_i) : i \in I_{\mathrm{uf}})$ required to be linearly independent
$(v_i : i \in I)$	$v_i = p^*(e_i)$ (as above) for $i \in I_{\mathrm{uf}}$ and $v_i \in M^\circ$ for $i \in I_{\mathrm{fr}}$ chosen to make $(v_i : i \in I)$ linearly independent For definiteness, we can always take the v_i for $i \in I_{\mathrm{fr}}$ to be in $\{e_i^* : i \in I\} = \{L_\alpha : \alpha \in T\} \cup \{y_\Lambda : \Lambda \in \mathbf{L}\}$. If we do principal coefficients, we can the v_i for $i \in I_{\mathrm{fr}}$ to be $\{L_\alpha : \alpha \in T\}$

1. I-beams

In this section we introduce a new combinatorial object on triangulated marked surfaces, the I-beam. Our goal is to construct a scattering diagram $\mathfrak{D}_I(B(\Delta))$ with walls indexed by I-beams in $(\mathbf{S}, \mathcal{M})$ with triangulation Δ, and then show that $\mathfrak{D}_I(B(\Delta)) = \mathrm{Scat}^T(B(\Delta))$, the transposed cluster scattering diagram for $B(\Delta)$.

Roughly speaking, an I-beam can be thought of as corresponding to a collection of facets in the rational quasi-lamination fan which share a common normal vector. Thus, an I-beam either corresponds to a collection of pairs of "exchangeable" allowable curves 2 which each intersect in the same way, or it corresponds to an allowable closed curve which may be completed to a quasi-lamination of co-dimension 1. Such allowable closed curves are referred to as non-shielding.

2. i.e., allowable curves arising as the image, under κ (see [2, Section 5]), of exchangeable tagged arcs) SV

Definition 1.1. An allowable closed curve λ is ***non-shielding*** if every component of $\mathbf{S} \setminus \lambda$ contains at least one marked point.

Example 1.2. The curve λ in the dread torus which is parallel to the boundary is a shielding loop. Triangulations of the dread torus consist of 5 arcs, but quasi-laminations containing λ are only of size 3.

Definition 1.3. Given an (ordinary, untagged) triangulation Δ of $(\mathbf{S}, \mathcal{M})$, an ***I-beam*** I is a non-self-intersecting (branching) curve in $\mathbf{S}$ considered up to isotopy relative to $\mathcal{M}$. We require that I is disjoint from the boundary of $\mathbf{S}$, except possibly at its endpoints, and is either

(1) a non-shielding allowable closed curve, or
(2) a (branching) curve with two branch points 3 in $\mathbf{S} \setminus \Delta$ such that:
   (a) If a branch point lies inside a self-folded triangle, then extend I with a single branch terminating at the puncture on the fold, and tag that end of I either plain or notched.
   (b) If a branch point lies inside a non-self-folded triangle, then I intersected one of its arcs transversely immediately before reaching the branch point. In this case, extend I with two branches, terminating at unmarked points on each of the remaining two arcs.
   (c) If both branch points lie in triangles of the same type (self-folded or non-self-folded), then these triangles must be distinct.

3. Is this a misuse of 'branching'? maybe 'forked' is better? SV

Remark 1.4. Each non-closed I-beam maps naturally to either an ordinary arc (if both branch points lie in non-self-folded triangles), or to a tagged arc with at least one endpoint at a puncture which is not in the initial triangulation Δ. The map is as follows: instead of extending a branch point in a non-self-folded triangle with two branches terminating at unmarked points on adjacent arcs, extend with a single branch terminating at the marked point the two arcs share.

Remark 1.5. There is a choice to be made in whether to define I-beams with respect to a tagged triangulation or an ordinary triangulation. The advantage of the ordinary triangulation, which we use for now, is that it is easier to see how an I-beam encodes different ways of interacting with a self-folded triangle. Tagged triangulations offer the advantage of a more natural definition of the intersection vector (see Definition 1.6), without appealing to the map τ.

To define the wall $(\mathfrak{d}_I, f_{\mathfrak{d}_I})$ associated to an I-beam I, we first define the vector $n_I \in N^+$ which is normal to the codimension-1 rational cone $\mathfrak{d}_I$. 4.

4. May need to check that n_I is primitive SV

Figure 1. Some simple examples

ibeam_examples.jpg

Recall there is a map τ from ordinary arcs to tagged arcs that sends each arc which does not bound a once-punctured monogon to the same curve, with both endpoints tagged plain, and sends any arc which does bound a once-punctured monogon to the arc, in the monogon, tagged notched at the puncture and plain at its other end. (See, e.g., [2, Definition 3.3].)

Definition 1.6. Given an (ordinary, untagged) triangulation Δ of $(\mathbf{S}, \mathcal{M})$ and an I-beam I, for each arc $\gamma \in T$ we define a scalar $c_{I\gamma} \in \mathbb{Z}_{\geq 0}$ as follows. If γ is an arc in a self-folded triangle and I has an endpoint at the puncture on the fold, then $c_{I\gamma} = 0$ if the taggings on $\tau(\gamma)$ and I agree and 1 if they do not. Otherwise, $c_{I\gamma}$ is the number of times I and γ intersect transversely. The ***intersection vector*** $\mathbf{n}_I$ is the formal sum

$$\mathbf{n}_I = \sum_{\gamma \in T} c_{I\gamma}\gamma \in N^+ \tag{1.1}$$

Alternatively, $\mathbf{n}_I$ may be thought of as the vector $\mathbf{n}_I = (I(\gamma) : \gamma \in T) \in \mathbb{Z}_{\geq 0}^{|\Delta|}$. Note that the definition of an I-vector ensures that $\mathbf{n}_I$ is nonzero.

Conjecture 1.7. *Each non-closed I-beam can be extended to a pair of allowable curves which intersect exactly once. (Two curves which are compatible except for opposite spiral directions at a shared endpoint are considered to only intersect once.) Further, if λ is an allowable curve which is compatible with both, then its shear coordinate vector $\boldsymbol{b}(T, \lambda)$ is orthogonal to $\mathbf{n}_I$.*

Remark 1.8. The tagging on an I-beam terminating at a puncture in a self-folded triangle determines how it can be extended to two "exchangeable" allowable curves.

Remark 1.9. There may be some subtlety here which we won't be able to tease out until we write the scattering diagram proof. There appear to be some pairs of exchangeable allowable curves which don't correspond to any I-beams as we've defined them. However, the corresponding facet in the lamination fan (consisting of a collection of pairwise compatible allowable curves which form a maximal lamination with either) does have a normal vector given by one of our I-beams.

new_ibeam.jpg

References

[1] M. Gross, P. Hacking, S. Keel, and M. Kontsevich, *Canonical bases for cluster algebras.* Preprint, 2014. `arXiv:1411.1394`

[2] N. Reading, *Universal geometric cluster algebras from surfaces.* Trans. Amer. Math. Soc. **366** no. 12 (2014), 6647–6685.

2. A possible alternative to I-beams and joints

Definition 2.1. Given a triangulation Δ of unpunctured5 $(\mathbf{S}, \mathcal{M})$, a ***barricade*** B in Δ is given by a topological graph embedded in $\mathbf{S}$, such that, when restricted to the neighborhood of a triangle in Δ, the connected components are isotopic to one of the pictures in Figure 2.

5. I haven't attempted punctured surfaces yet. GM

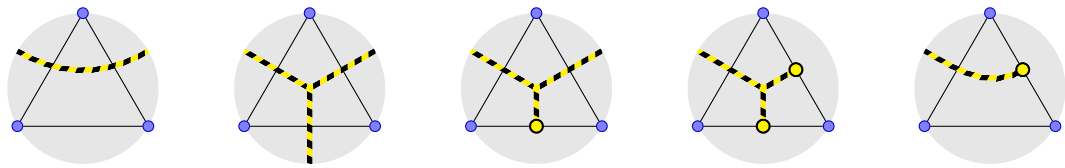

Figure 2. Local pictures of barricades (up to rotation and reflection)

A ***short leaf***6 is a leaf whose adjacent vertex is in the adjacent triangle (e.g. the third and fourth local picture in Figure 2). A ***long leaf*** is a leaf on the boundary of $\mathbf{S}$.

6. I can't decide if we want to allow non-short leaves in the definition. They make the proof of Prop. 2.2 easier, but they don't appear in barricades related to the scattering diagram. GM

Barricades are considered up to isotopy within the set of barricades (so leaves must remain on arcs in Δ). A measured lamination μ is ***compatible*** with a barricade B if there is are equivalent μ' and B' such that μ' intersects Δ minimally and μ' and B' do not intersect.

Proposition 2.2. *Given a barricade B, a measured lamination μ is compatible with B iff the following inequalities hold.*

- *For each path in B which begins with a right turn and ends with a left turn,*
$$S_1(\mu) + S_2(\mu) + ... + S_n(\mu) \geq 0$$
where $S_1, S_2, ..., S_n$ are the shear coordinates of the arcs in Δ crossed by P.
- *For each path in B which begins with a left turn and ends with a right turn,*
$$S_1(\mu) + S_2(\mu) + ... + S_n(\mu) \leq 0$$
where $S_1, S_2, ..., S_n$ are the shear coordinates of the arcs in Δ crossed by P.
- *For each cycle in B,*
$$S_1(\mu) + S_2(\mu) + ... + S_n(\mu) = 0$$
where $S_1, S_2, ..., S_n$ are the shear coordinates of the arcs in Δ crossed by P.

Corollary 2.3. *Given a barricade B, the set $C_B \subset \mathbb{R}^\Delta$ of measured laminations which are compatible with B is a closed cone.*

3. Proof of Proposition 2.2

Let us collectively refer to the inequalities in Proposition 2.2 as the ***compatibility inequalities*** for B and μ.

A barricade is ***simple*** if it is connected and has no degree 3 vertices. We start by proving a stronger version of Proposition 2.2 in the case of a simple barricades.

Lemma 3.1. *Let B be a simple barricade in Δ. A measured lamination μ is compatible with B iff it satisfies the compatibility inequalities.*

Proof idea. Induction, I hope. I don't know about loops, though. ☐

Lemma 3.2. *A measured lamination μ is compatible with a barricade B iff μ is compatible with every simple sub-barricade of B.*

Proof idea. If they are incompatible, there should be some unmarked curve that is crossed. □

Proof of Proposition 2.2. Assume μ is compatible with B. For any path or cycle in B, there is a simple sub-barricade B' containing that path or cycle which is also compatible with μ, and so the associated compatibility inequality holds by Lemma 3.1.

Assume the compatibility inequalities hold for μ and B. Then they also hold for any simple sub-barricade of B, and so by Lemma 3.1 μ is compatible with each of them. By Lemma 3.2, μ is compatible with B. □

4. Minimal barricades

Definition 4.1. The ***codimension*** of a barricade is the codimension of (the span of) the compatible cone. A barricade is ***minimal*** if it has no sub-barricades of the same codimension.

Easy observation: all leaves in a minimal barricade are short, since they can be cut off without increasing the codimension.

Conjecture 4.2. *The codimension of a barricade B is*

$$(\#\ \textit{of edges between trivalent vertices}) + (\#\ \textit{of loops without leaves})$$

Corollary 4.3. *The minimal barricades of codimension 1 are:*

- *Trees with 4 leaves, all short.*
- *A cycle with 0 leaves.*

Corollary 4.4. *The minimal barricades of codimension 2 are:*

- *Trees with 5 leaves, all short.*
- *A cycle with 2 leaves, all short.*
- *Disjoint unions of 2 minimal barricades of codimension 1.*

Lemma 4.5. *If B and B' are minimal barricades, then*

$$C_B \cap C_{B'} = \bigcup_i C_{B_i}$$

where each of the B_i are minimal barricades.

5. The scattering diagram of $(\mathbf{S}, \mathcal{M})$

We fix a triangulation Δ of $(\mathbf{S}, \mathcal{M})$, and we use the language of barricades to construct a scattering diagram in $\mathbb{R}^\Delta$. We distinguish two kinds of minimal barricades of codimension 1. Construct a scattering diagram $\mathfrak{D}$ in $\mathbb{R}^\Delta$ as follows.

- A ***I-beam*** is a tree with 4 short leaves. For each I-beam B, $\mathfrak{D}$ has a wall

  $$(C_B, 1 + x^{\mathsf{B}n})$$

  where $n \in \mathbb{N}^\Delta$ counts the transverse crossings of B and the arcs in Δ.

- A ***non-shielding loop*** is a cycle with 0 leaves, such that each component of the complement contains a marked point. For each non-shielding loop B, $\mathfrak{D}$ has a wall
$$\left(C_B, (1 - x^{\mathsf{B}n})^{-2}\right)$$
where $n \in \mathbb{N}^{\Delta}$ counts the transverse crossings of B and the arcs in Δ.

Theorem 5.1. *The scattering diagram $\mathfrak{D}$ is consistent, and is the GHKK scattering diagram of the seed Δ in the cluster algebra associated to $(\mathbf{S}, \mathcal{M})$.*

5.1. **Proof of Theorem 5.1.**

Lemma 5.2. *The joints of $\mathfrak{D}$ are as follows.*

- *If B is a tree with 5 short leaves, then C_B is a joint which is contained in the 3 walls; specifically, the 3 I-beams contained in B.*
- *If B is a cycle with 2 short leaves on the same side, then C_B is a joint which is contained in 2 or 3 walls; specifically, the 2 I-beams contained in B, and the cycle (if it is non-shielding).*
- *If B is a cycle with 2 short leaves on different sides, then C_B is a joint which is contained in infinitely many walls; specifically, the infinitely-many I-beams contained in B, and the cycle.*[1]
- *If B is a disjoint union of 2 I-beams or non-shielding loops, then C_B is a joint contained in 2 walls.*

[1]Note that the cycle in this case must be non-shielding.