# MPHY0030 Coursework - Part 2 Report

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## 1 Question 1

The polynomial part is not required as the Gaussian spline is always positive definite [1], simplifying the system of linear equations that will need to be computed.

## 2 Question 2

The system of linear equations representing the spline fitting problem are as follows

$$\begin{pmatrix} K_1 + \lambda I \\ K_2 + \lambda I \\ K_3 + \lambda I \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \tag{1}$$

where $K_i$ is the Gaussian kernel, $I$ is the identity matrix of the same size of $K$, $\alpha_i$ are the respective coefficients, and $q_i$ are the the target landmarks. The subscript number represents each coordinate dimension i.e. $x$, $y$, and $z$.

## 3 Question 3

These equations form a least-squares problem. Therefore, to solve them, I utilised the Moore-Penrose Inverse on the matrix given by $K_i + \lambda I$, as this gives the minimised norm solution of $\alpha_i$. For example, given $Ax = b$, where $A$ is a matrix, and $x$ & $b$ are vectors, the Moore-Penrose inverse of $A$, $A^\dagger$, can be used as such: $x = A^\dagger b$.

## 4 Question 4

The control points here are the source landmarks used in [1], which control the locality of the transformation. These points need to be the same as the source points used to calculate the $\alpha$ coefficients, as these control points are used consistently to calculate the Gaussian radial basis functions.

## 5 Question 5

The parameter $\lambda$ is needed at this stage. $\lambda$ helps control the transform and keeps it local to the control points. If $\lambda$ is too large, the transformation approaches an affine transformation, with all the points being transformed in a similar manner.

## 6 Question 6

The elements of Gaussian kernel are given by $K_{ij} = R_G(x_i - p_j)$. Therefore, the $x$ points were given as a column vector and the $p$ values as a row vector. Then when $x - p$ was performed, it resulted in a matrix of the correct size and form as needed for $K$. I simply then needed to perform the Gaussian RBF equation $R_G$ with the whole matrix of $x - p$.

## 7 Question 7

The Gaussian parameter $\sigma$ determines a type of strength for the transformation. As the Gaussian spline RBF is given by

$$R_G(r) = \exp(-r^2/2\sigma^2) \tag{2}$$

a smaller $\sigma$ will result in a larger $(r^2/2\sigma^2)$. Therefore, the negative exponential will be small, giving a small $R_G$. And vice versa, a larger $\sigma$ will give a smaller $(r^2/2\sigma^2)$, and thus larger $R_G$.

## 8 Question 8

To randomly displace the control points, I employed the use of a normal distribution. The amount of displacement is randomly pulled from this distribution. This helps to prevent large displacement that may not be biophysically reasonable. Therefore, the displacements are focussed instead around smaller positive and negative amounts. Constraints have been added to ensure the displaced control points remain within the boundaries of the image itself. This is an issue for control points at/very near to the edges of the image, and the random displacement could move them out of bounds. Instead, if that happens, they are instead constrained to the edge instead.

## 9 Question 9

As the random displacement of the control points is kept small, and the displacements are prevented from moving out of the domain of the image, the resultant coordinate transformation will also be small and constrained within the image. As such, the individual local transformations could be seen as biophysically plausible due to it avoiding extreme transformations that could either very unlikely, or impossible, in a real biological scenario. However, due to the transformation being based upon a number of displaced control points, and therefore it could be seen as a number of different local transformations, the overall transformation may not be seen as biophysically plausible.

## 10 Question 10

To computer a warped image, the `warp_image` function takes in an object from each of the Image3D, FreeFormDeformation, and RBFSpline classes, as well as the Gaussian parameter $\sigma$. The Image3D object provides the coordinates of the image to be transformed that were found via the size of the image and the voxel dimensions. The coordinates are provided in meshgrid format, as a list of points is inefficient to compute due to the number of individual coordinate points. The FreeFormDeformation object provides the control points that the transformation is based around (i.e the points $p$ in [1]), which have been obtained from the image itself. By specifying the number of control points in each direction, the coordinates of the control points are found using the dimensions of the image, and creating a equally spaced grid of the points given by the number. Lastly, the RBFSpline object provides the $\alpha$ coefficients required for the spline-based transformation. The $\alpha$ coefficients are found using source and target points. The source points are the same as the control points. The target points are generated by randomly displacing the control points.

Using the Gaussian spline, the coefficients are found by solving the equations set out in Question 2. The `evaluate` function from the RBFSpline class is called using the image coordinates, the control points, $\alpha$ and $\sigma$. This returns a meshgrid of new, transformed coordinates interpolated using the the control points, $\alpha$, and $\sigma$ with the Gaussian RBF. Interpolation is then utilised to interpolated new image intensity values for the new coordinates, based off of the original coordinates and intensity values.

## 11 Question 11

Based upon the images from my code, I can't concretely say whether the deformations are biophysically plausible. This is due to my final images not correctly displaying warping. However, based upon the theory set out, I would imagine that in many causes the deformation from the displaced control points would be biophysically plausible. This is because the control point displacement is prevented from being too large, and is constrained within the boundaries of the image itself. Small displacements would lead to small deformations, which could be easily seen a real-world scenario.

## 12 Question 12

Some of the parameters within this work could be changed one at a time to visualise how each one individually affects the final image. However, due to not being able to get my code to be fully functional, with my images not warping properly, they are not indicative of the actually effects. Therefore, for this question, I will explain how my images have changed, but also how proper images should be affected by the parameter changes.

A base original image is shown in figure 1 with parameters as follows: strength of randomiser - 1, number of control points - [4, 4, 4], and Gaussian parameter - $\sigma = 2$. In each case, only one of these parameters is changed from the 'base' value so see how it alone affects the image.

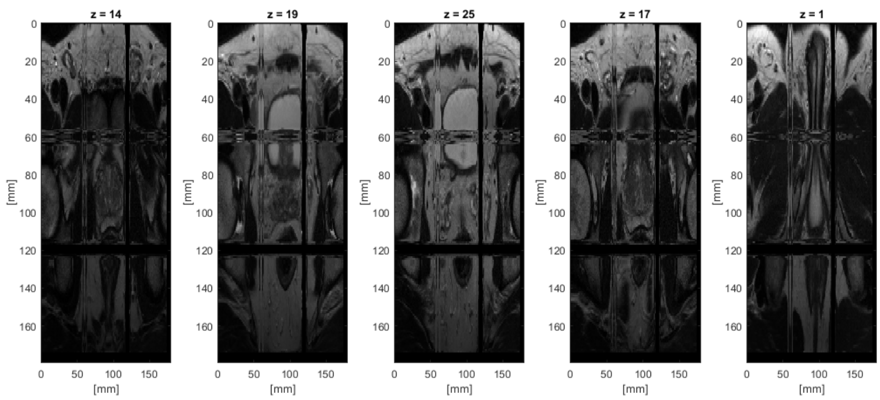


Figure 1: Base image with strength of randomiser - 1, number of control points - [4, 4, 4], and Gaussian parameter - $\sigma = 2$

To begin with, the strength of the randomiser, which controls the strength of displacement of the control points, was reduced to 0.2. This image is shown in figure 2. From my example, there is no apparent change between the images when the randomisation strength is changed. However, what should happen, is a lower amount of deformation when going from a higher to lower strength. As the randomisation strength controls the amount of displacement of the control points, a higher strength will results in greater displacements. As the transformation is trying to match these displacements, the amount of deformation/warping occur will increase. Likewise, a small strength will give small displacements, and small deformations. So for my case, going from a strength of 1 to 0.2 should result in a significant decrease in the amount of deformation.

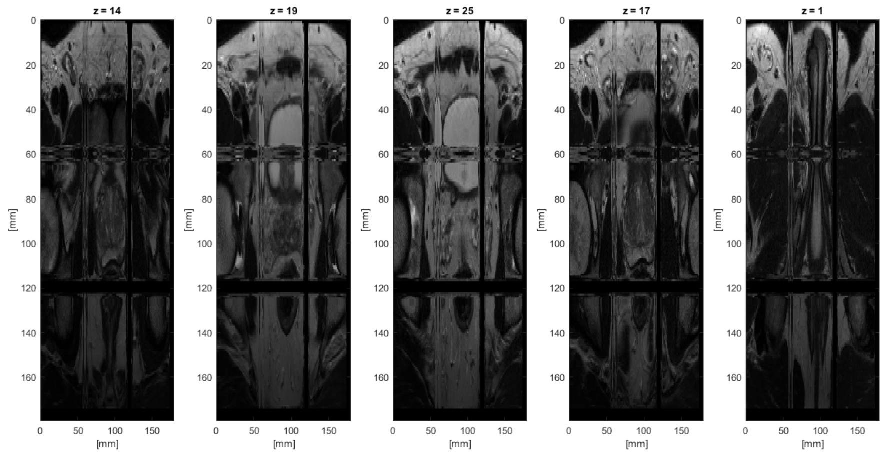


Figure 2: Image with strength of randomiser = 0.2

Next, the number of control points in each of the $x$, $y$ and $z$ direction was changed from [4, 4, 4] to [10, 10, 10]. Again, in my case, this is not being displayed correctly. Here, the images appeared almost ‘cut up‘, with black lines running through the grid of non-displaced control points. Therefore, with the increase of control points, there are more of these line artefacts that break up the image significantly. It causes the actual anatomy to be impossible to discern.

What should happen is that as more control points are added, there will be more areas of local transformation and warping. The RBF tries to retain locality of the transformations around the control points, so points of the image far away from control points should be minimally affected. Therefore, if there are a large number of control points, there are increased areas that are being warped.

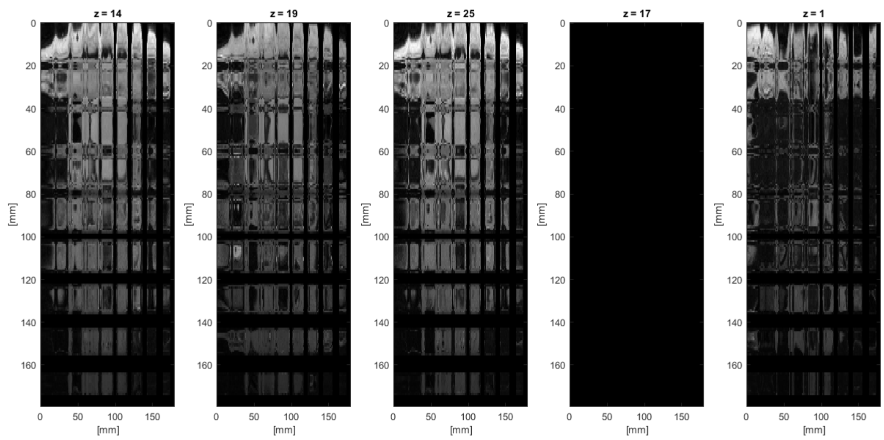


Figure 3: Image with number of control points = [10, 10, 10]

Finally, the value of the Gaussian parameter $\sigma$ was changed to two different values: 0.5 (figure 4), and 5 (figure 5). As mentioned previously in Question 7, the Gaussian parameter $\sigma$ affects the Gaussian RBF. In my images, a smaller $\sigma$ appears to have less of an effect, with the gridded line artefacts actually becoming narrower. With a larger $\sigma$, the lines become thicker, and a greater level of distortion is apparent within the image. In a correctly warped image, the same concept would apply as explained in Question 7. A smaller $\sigma$ gives smaller transformations/deformations, and a larger $\sigma$ gives larger transformations/deformations. Therefore, increasing the $\sigma$ parameter provides stronger warping in the final images.

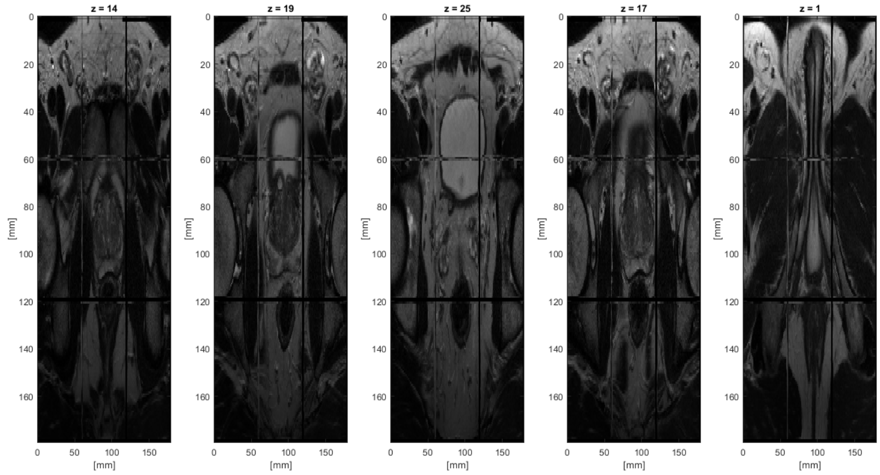


Figure 4: Image with $\sigma = 0.5$

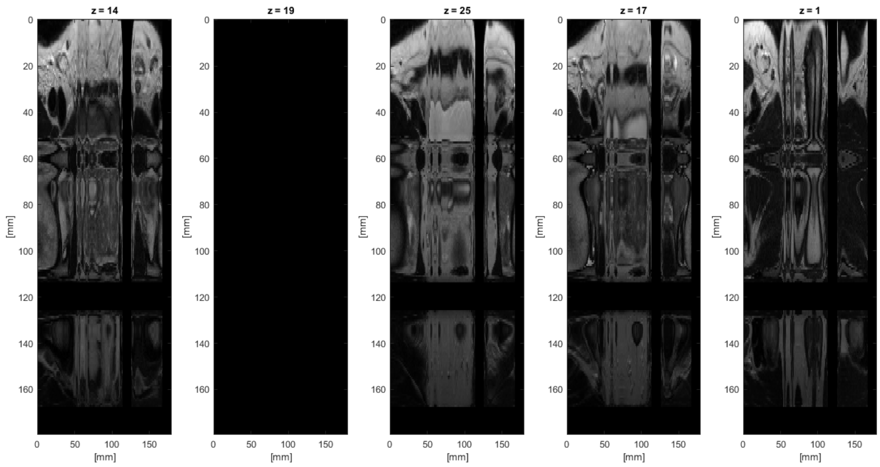


Figure 5: Image with $\sigma = 5$

Having a higher randomisation strength, a large number of control points, and a larger sigma will altogether result in many areas of strong warping, leading to a highly deformed image. For my incorrect case, this is displayed in figure 6, where the image is almost completely lost. A correct image with these high parameters would show large amounts of deformation, spread across a large area of the slices. In a case such a this, it would be less likely to represent a biophysically plausible case.

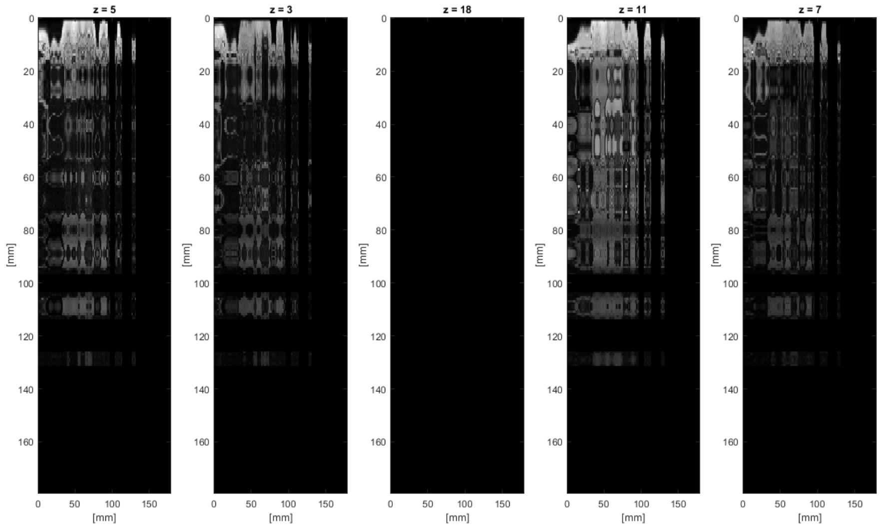


Figure 6: Image with all aforementioned changed applied

## References

[1] Fornefett M, Rohr K, Stiehl H. Radial basis functions with compact support for elastic registration of medical images. Image and Vision Computing. 2001;19(1-2):87-96.