

MEM52220 - Applied Econometrics

Nicolas Reigl

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Chapter 1

Introduction

Welcome to MEM5220 - Applied Econometrics. This handout was originally (and currently) designed for the use with MEM5220 - Applied Econometrics at Tallinn University of Technology. Note that this workbook is still under heavy development!

1.1 Prerequisites

A basic knowledge of the R (Team, 2013) programming language is required.

1.2 Resources

Our primary resource is Heiss (2016)¹. For theoretical concepts see Wooldridge (2015).

1.3 Acknowledgements

I thank Kadri Männasoo and Juan Carlos Cuestas for proofreading and their useful comments.

¹Heiss (2016) builds on the popular Introductory Econometrics by Wooldridge (2016) and demonstrates how to replicate the applications discussed therein using R.

Chapter 2

Linear Regression

The general form in which we specify regression models in R:

```
## response ~ terms
##
## y ~ age + sex          # age + sex main effects
## y ~ age + sex + age:sex # add second-order interaction
## y ~ age*sex            # second-order interaction +
##                        # all main effects
## y ~ (age + sex + pressure)^2
##                        # age+sex+pressure+age:sex+age:pressure...
## y ~ (age + sex + pressure)^2 - sex:pressure
##                        # all main effects and all 2nd order
##                        # interactions except sex:pressure
## y ~ (age + race)*sex    # age+race+sex+age:sex+race:sex
## y ~ treatment*(age*race + age*sex) # no interact. with race,sex
## sqrt(y) ~ sex*sqrt(age) + race
## # functions, with dummy variables generated if
## # race is an R factor (classification) variable
## y ~ sex + poly(age,2)   # poly generates orthogonal polynomials
## race.sex <- interaction(race,sex)
## y ~ age + race.sex      # for when you want dummy variables for
##                        # all combinations of the factors
```

2.1 Simple Linear Regression

We start off with a simple OLS Regression. We will work with multiple data sources:

- Data from Wooldridge (2015) : Introductory Econometrics: A Modern Approach.
- More datasources in the future

To load the dataset and necessary functions:

```
# This function 1. checks if the packages are installed. 2. It installs the packages if they were not i
PACKAGES<-c("wooldridge", # Wooldrige Datasets
            "tidyverse",  # for data manipulation and ggplots
            "broom",      # Tidy regression output
            "ggpubr",      # Multiple ggplots on a page. Note that, the installation of ggpubr will automat
            "ggfortify")  # Simple ggplot recipe for lm objects)
```

```
inst<-match(PACKAGES, .packages(all=TRUE))
need<-which(is.na(inst))
if (length(need)>0) install.packages(PACKAGES[need])
lapply(PACKAGES, require, character.only=T)
```

Classic examples of quantities modeled with simple linear regression:

- College GPA SAT scores $\beta > 0$
- Change in GDP change in unemployment $\beta < 0$
- House price number of bedrooms $\beta > 0$
- Species heart weight species body weight $\beta > 0$
- Fatalities per year speed limit $\beta < 0$

Notice that these simple linear regressions are simplifications of more complex relationships between the variables in question.

In this exercise we use the dataset *ceosal1*. Let us analyse the dataset first

```
data("ceosal1")
help("ceosal1")
?ceosal1
```

As we see from the R documentation the *ceosal1* dataset contain of a random sample of data reported in the May 6, 1991 issue of Businessweek.

To get a first look at the data you can use the `View()` function inside R Studio.

```
View(ceosal1)
```

We could also take a look at the variable names, the dimension of the data frame, and some sample observations with `str()`.

```
str(ceosal1)

## 'data.frame': 209 obs. of 12 variables:
## $ salary : int 1095 1001 1122 578 1368 1145 1078 1094 1237 833 ...
## $ pcsalary: int 20 32 9 -9 7 5 10 7 16 5 ...
## $ sales : num 27595 9958 6126 16246 21783 ...
## $ roe : num 14.1 10.9 23.5 5.9 13.8 ...
## $ pcroe : num 106.4 -30.6 -16.3 -25.7 -3 ...
## $ ros : int 191 13 14 -21 56 55 62 44 37 37 ...
## $ indus : int 1 1 1 1 1 1 1 1 1 1 ...
## $ finance : int 0 0 0 0 0 0 0 0 0 0 ...
## $ consprod: int 0 0 0 0 0 0 0 0 0 0 ...
## $ utility : int 0 0 0 0 0 0 0 0 0 0 ...
## $ lsalary : num 7 6.91 7.02 6.36 7.22 ...
## $ lsales : num 10.23 9.21 8.72 9.7 9.99 ...
## - attr(*, "time.stamp")= chr "25 Jun 2011 23:03"
```

As we have seen before in the general R tutorial, there are a number of additional functions to access some of this information directly.

```
dim(ceosal1)
```

```
## [1] 209 12
```

```
nrow(ceosal1)
```

```
## [1] 209
```



```
ncol(ceosal1)
```

```
## [1] 12
```

```
summary(ceosal1)
```

```
##      salary      pcsalary      sales      roe
##  Min.   : 223    Min.   : -61.00   Min.   : 175.2   Min.   : 0.50
## 1st Qu.: 736    1st Qu.:  -1.00   1st Qu.: 2210.3   1st Qu.:12.40
## Median : 1039   Median :   9.00   Median : 3705.2   Median :15.50
## Mean   : 1281   Mean   :  13.28   Mean   : 6923.8   Mean   :17.18
## 3rd Qu.: 1407   3rd Qu.:  20.00   3rd Qu.: 7177.0   3rd Qu.:20.00
## Max.   :14822   Max.   :212.00   Max.   :97649.9   Max.   :56.30
##      pcroe      ros      indus      finance
##  Min.   : -98.9   Min.   : -58.0   Min.   :0.0000   Min.   :0.0000
## 1st Qu.: -21.2   1st Qu.:  21.0   1st Qu.:0.0000   1st Qu.:0.0000
## Median :  -3.0   Median :  52.0   Median :0.0000   Median :0.0000
## Mean   :  10.8   Mean   :  61.8   Mean   :0.3206   Mean   :0.2201
## 3rd Qu.:  19.5   3rd Qu.:  81.0   3rd Qu.:1.0000   3rd Qu.:0.0000
## Max.   : 977.0   Max.   :418.0   Max.   :1.0000   Max.   :1.0000
##      consprod      utility      lsalary      lsales
##  Min.   :0.0000   Min.   :0.0000   Min.   :5.407   Min.   : 5.166
## 1st Qu.:0.0000   1st Qu.:0.0000   1st Qu.:6.601   1st Qu.: 7.701
## Median :0.0000   Median :0.0000   Median :6.946   Median : 8.217
## Mean   :0.2871   Mean   :0.1722   Mean   :6.950   Mean   : 8.292
## 3rd Qu.:1.0000   3rd Qu.:0.0000   3rd Qu.:7.249   3rd Qu.: 8.879
## Max.   :1.0000   Max.   :1.0000   Max.   :9.604   Max.   :11.489
```

The interesting task here is to determine how far a high the CEO salary is, for a given return on equity.

Your turn

What sign would be expect of β (the slope)?

A: Without seeing the data **my** prior is that $\beta > 0$.

Note

A simple linear model as assumes that the mean of each y_i conditioned on x_i is a linear function of x_i . But notice that simple linear regressions are simplifications of more complex relationships between the variables in question.

```
# Use ggplot style
ggplot(ceosal1, aes(x = roe, y = salary)) +
  geom_point()
```

Consider a simple regression model

$$\text{salary} = \beta_0 + \beta_1 \text{roe} + u$$

We are concerned with the population parameter β_0 and β_1 . The general form of the model.

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \quad (2.1)$$

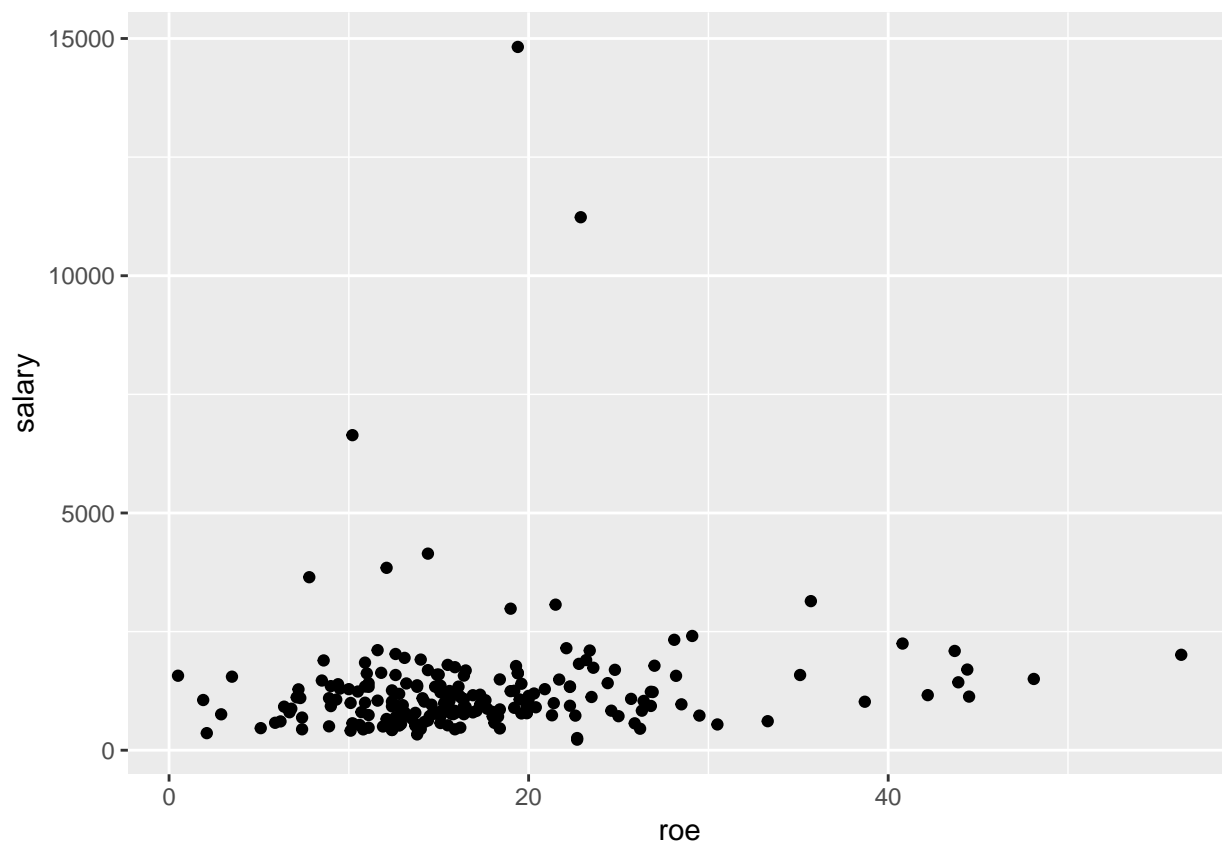


Figure 2.1: Relationship between ROE and Salary

The ordinary least squares (OLS) estimators are

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \quad (2.2)$$

Ingredients for the OLS formulas

```
attach(ceosal1)

## The following objects are masked from ceosal1 (pos = 6):
##
##   consprod, finance, indus, lsalary, lsales, pcroe, pcsalary,
##   roe, ros, salary, sales, utility
cov(roe, salary)

## [1] 1342.538
var(roe)

## [1] 72.56499
mean(salary)

## [1] 1281.12
```

Manual calculation of the OLS coefficients

```
b1hat <- cov(roe,salary)/var(roe)
b0hat <- mean(salary) - b1hat * mean(roe)
```

Or use the `lm()` function

```
lm(salary ~ roe, data=ceosal1)

##
## Call:
## lm(formula = salary ~ roe, data = ceosal1)
##
## Coefficients:
## (Intercept)          roe
##      963.2         18.5
lm1_ceosal1 <- lm(salary ~ roe, data=ceosal1)

unique(ceosal1$roe)

## [1] 14.1 10.9 23.5  5.9 13.8 20.0 16.4 16.3 10.5 26.3 25.9 26.8 14.8 22.3
## [15] 56.3 12.6 20.4  1.9 19.9 15.4 38.7 24.4 15.6 14.4 19.0 16.1 12.1 16.2
## [29] 18.4 14.2 14.9 12.4 17.1 16.9 18.1 19.3 18.3 13.7 12.7 15.1 16.5 10.2
## [43] 19.6 12.8 15.9 17.3  8.5 19.5 19.2 28.1 25.0 15.0 20.3 22.7 13.2 10.3
## [57] 17.7 10.0  6.8 13.1 15.8 15.3  0.5 13.0 11.1  8.9 17.5  9.3  9.5 15.5
## [71]  8.6 24.6  7.2 11.6 26.4 21.4  9.0  9.4  3.5 22.1 33.3 22.8 20.9  6.7
## [85]  7.1 11.8 14.0 10.1  6.4 17.6 23.6 35.7 23.2 44.4  2.1 23.4 25.7 27.0
## [99] 43.7 24.8 26.2 44.5 35.1 11.0 19.4 28.5 43.9 15.7 28.2 42.2 21.5 29.5
## [113] 22.6 22.9  7.8 48.1 18.0 21.7 21.3 26.9 30.5 29.1 40.8 10.8  5.1 12.3
## [127]  7.4  6.2 10.6  2.9 13.5 10.7 11.9 12.9  7.3 14.6 14.5 14.7
```

Plot the linear regression fit the *base r* way.

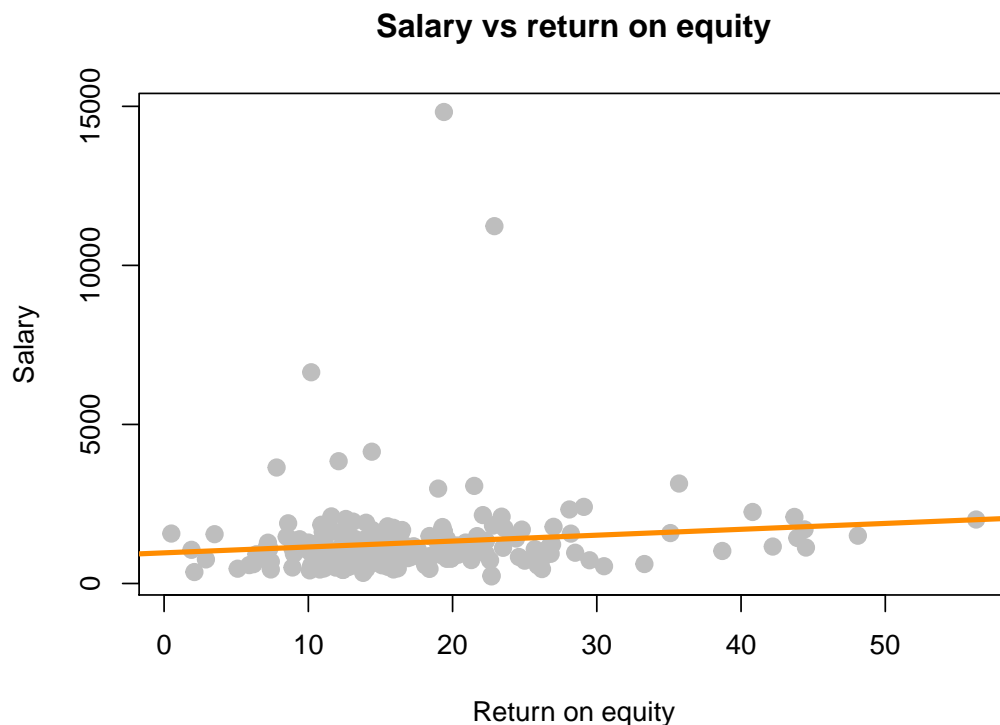


Figure 2.2: OLS regression base Rstyle

```
plot(salary~ roe, data = ceosal1,
     xlab = "Return on equity",
     ylab = "Salary",
     main = "Salary vs return on equity",
     pch = 20,
     cex = 2,
     col = "grey")
abline(lm1_ceosal1, lwd = 3, col = "darkorange")
```

Or use ggplot

```
ggplot(ceosal1, aes(x = roe, y = salary)) +
  geom_point() +
  stat_smooth(method = "lm", col = "red")
```

Determine the names of the elements of the list using the `names()` command.

```
names(lm1_ceosal1)
```

```
## [1] "coefficients" "residuals"    "effects"      "rank"
## [5] "fitted.values" "assign"       "qr"          "df.residual"
## [9] "xlevels"      "call"        "terms"       "model"
```

Extract one element, for example the residuals from the list object

```
head(lm1_ceosal1$residuals) # head() just prints out the first 6 residual values
```

```
##      1      2      3      4      5      6
## -129.0581 -163.8543 -275.9692 -494.3483  149.4923 -188.2151
```

Another way to access stored information in `lm1_ceosal1` are the `coef()`, `resid()`, and `fitted()` functions.

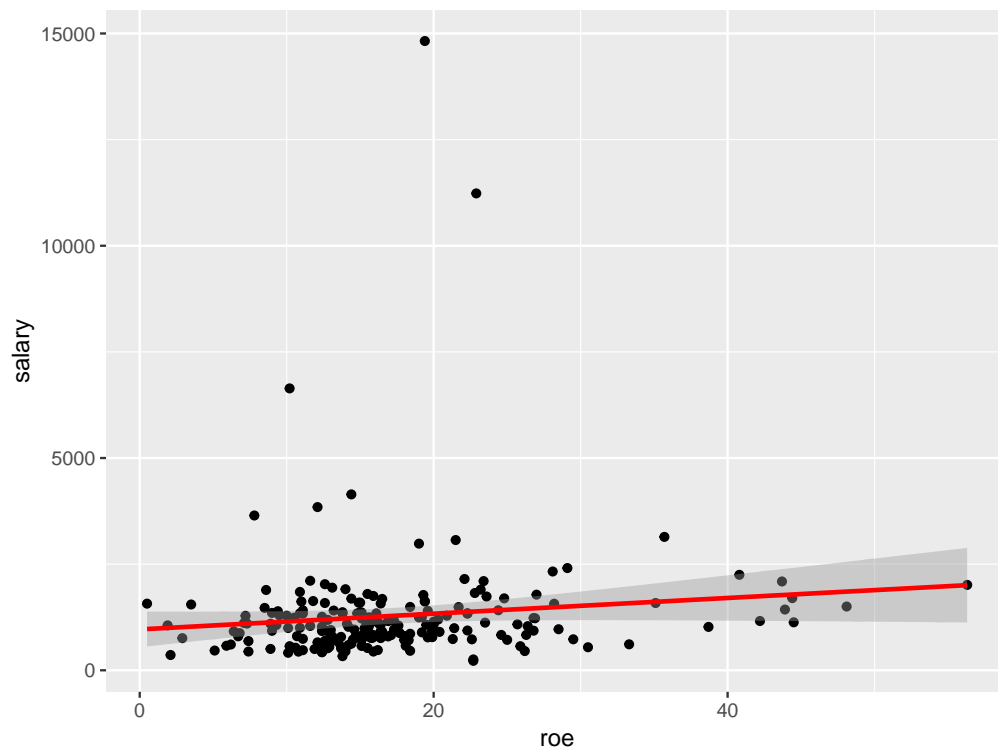


Figure 2.3: OLS regression ggplot2 style

These return the coefficients, residuals, and fitted values, respectively.

```
coef(lm1_ceosal1)
```

```
## (Intercept)      roe
##   963.19134   18.50119
```

The function `summary()` is useful in many situations. We see that when it is called on our model, it returns a good deal of information.

```
summary(lm1_ceosal1)
```

```
##
## Call:
## lm(formula = salary ~ roe, data = ceosal1)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1160.2  -526.0  -254.0   138.8 13499.9
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)   963.19     213.24   4.517 1.05e-05 ***
## roe           18.50      11.12   1.663  0.0978 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1367 on 207 degrees of freedom
## Multiple R-squared:  0.01319,    Adjusted R-squared:  0.008421
```

```
## F-statistic: 2.767 on 1 and 207 DF, p-value: 0.09777
```

The `summary()` command also returns a list, and we can again use `names()` to learn what about the elements of this list.

```
names(summary(lm1_ceosal1))
```

```
## [1] "call"          "terms"          "residuals"      "coefficients"
## [5] "aliased"        "sigma"          "df"             "r.squared"
## [9] "adj.r.squared" "fstatistic"     "cov.unscaled"
```

So, for example, if we wanted to directly access the value of R^2 , instead of copy and pasting it out of the printed statement from `summary()`, we could do so.

```
summary(lm1_ceosal1)$r.squared
```

```
## [1] 0.01318862
```

Your turn

Recall that the residual sum of squares (SSR) is

$$R^2 = \frac{\text{Var}(\hat{y})}{\text{Var}(y)} = 1 - \frac{\text{Var}(\hat{u})}{\text{Var}(y)} \quad (2.3)$$

Calculate R^2 manually:

```
var(fitted(lm1_ceosal1))/var(ceosal1$salary)
```

```
## [1] 0.01318862
```

```
1 - var(residuals(lm1_ceosal1))/var(ceosal1$salary)
```

```
## [1] 0.01318862
```

Another useful function is the `predict()` function.

```
set.seed(123)
```

```
roe_sample <- sample(ceosal1$roe, 1)
```

Let's make a prediction for salary when the return on equity is 20.2999992.

```
b0hat_sample <- mean(salary) - b1hat * roe_sample
```

We are not restricted to observed values of the explanatory variable. Instead we can supply also our own predictor values

```
predict(lm1_ceosal1, newdata = data.frame(roe = 30))
```

```
##          1
```

```
## 1518.227
```

The above code reads “predict the salary when the return on equity is 30 using the *lm1_ceosal1* model.”

2.1.1 Regression through the Origin and Regression on a Constant

Regression without intercept (through origin)

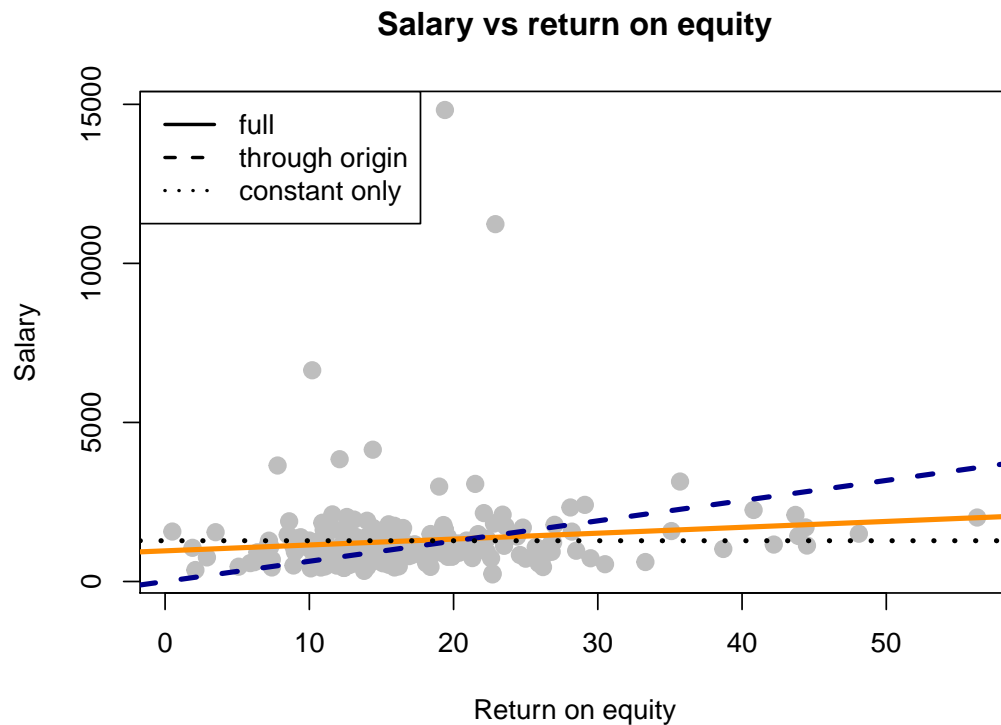


Figure 2.4: Regression through the Origin and on a Constant

```
lm2 <- lm(salary ~ 0 + roe, data = ceosal1)
```

Regression without slope

```
lm3 <- lm(salary ~ 1, data = ceosal1)
```

```
plot(salary ~ roe, data = ceosal1,
     xlab = "Return on equity",
     ylab = "Salary",
     main = "Salary vs return on equity",
     pch = 20,
     cex = 2,
     col = "grey")
abline(lm1_ ceosal1, lwd = 3, lty = 1, col = "darkorange")
abline(lm2, lwd = 3, lty = 2, col = "darkblue")
abline(lm3, lwd = 3, lty = 3, col = "black")
legend("topleft",
      c("full",
        "through origin",
        "constant only"),
      lwd = 2,
      lty = 1:3)
```

2.1.2 Simulating SLR

2.1.2.0.1 Expected Values, Variance, and Standard Errors

The **Gauss–Markov theorem** tells us that when estimating the parameters of the simple linear regression

model β_0 and β_1 , the $\hat{\beta}_0$ and $\hat{\beta}_1$ which we derived are the best linear unbiased estimates, or BLUE for short. (The actual conditions for the Gauss–Markov theorem are more relaxed than the SLR model.)

In short those assumptions are:

- SLR.1 Linear population regression function $y = \beta_0 + \beta_1 \times x + u$
- SLR.2 Random sampling of x and y from the population
- SLR.3 Variation in the sample values: x_1, \dots, x_n
- SLR.4 Zero conditional mean: $\mathbf{E}(u|x) = 0$
- SLR.5 Homoskedasticity: $\text{Var}(u|x) = \sigma^2$

Recall that under **SLR.1 - SLR.4** the OLS parameter estimators are unbiased. Under **SLR.1 - SLR.4** the OLS parameter estimators have a specific sampling variance.

Simulating a model is an important concept. In practice you will almost never have a true model, and you will use data to attempt to recover information about the unknown true model. With simulation, we decide the true model and simulate data from it. Then, we apply a method to the data, in this case least squares. Now, since we know the true model, we can assess how well it did.

Simulation also helps to grasp the concepts of estimators, estimates, unbiasedness, the sampling variance of the estimators, and the consequences of violated assumptions.

Sample size

```
n <- 200
```

True parameters

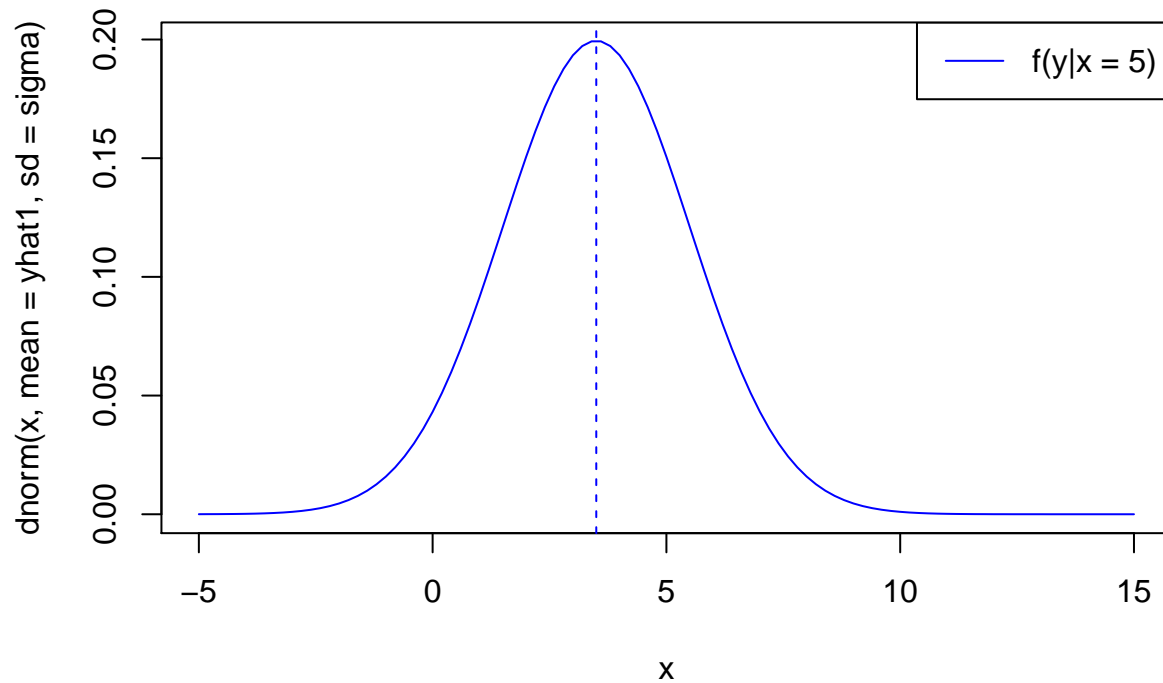
```
b0<- 1
b1 <- 0.5
sigma <- 2 # standard deviation of the error term u
x1 <- 5
```

Determine the distribution of the independent variable

```
yhat1 <- b0 + b1 * x1 # Note that we do not include the error term
```

Plot a Gaussian distribution of the dependent variable based on the parameters

```
curve(dnorm(x, mean = yhat1, sd = sigma), -5, 15, col = "blue")
abline(v = yhat1, col = "blue", lty = 2)
legend("topright", legend = c("f(y|x = 5)"), lty = 1, col = c("blue"))
```

This represent the theoretical (true) probability distribution of y , given x

We can calculate the variance of b_1 and plot the corresponding density function.

$$\text{var}(b_2) = \frac{\sigma^2}{\sum (x_1 - \bar{x})} \quad (2.4)$$

Assume that x_2 represents a second possible predictor of y

```
x2 <- 18

x <- c(rep(x1, n/2), rep(x2, n/2))
xbar <- mean(x)

sumxbar <- sum((x-xbar)^2)
varb <- (sigma^2)/sumxbar
sdb <-sqrt(varb)
leftlim <- b1-3*sdb
rightlim <- b1+3*sdb

curve(dnorm(x, mean = b1, sd = sdb), leftlim, rightlim,)
abline(v = b1, col = "blue", lty = 2)
```

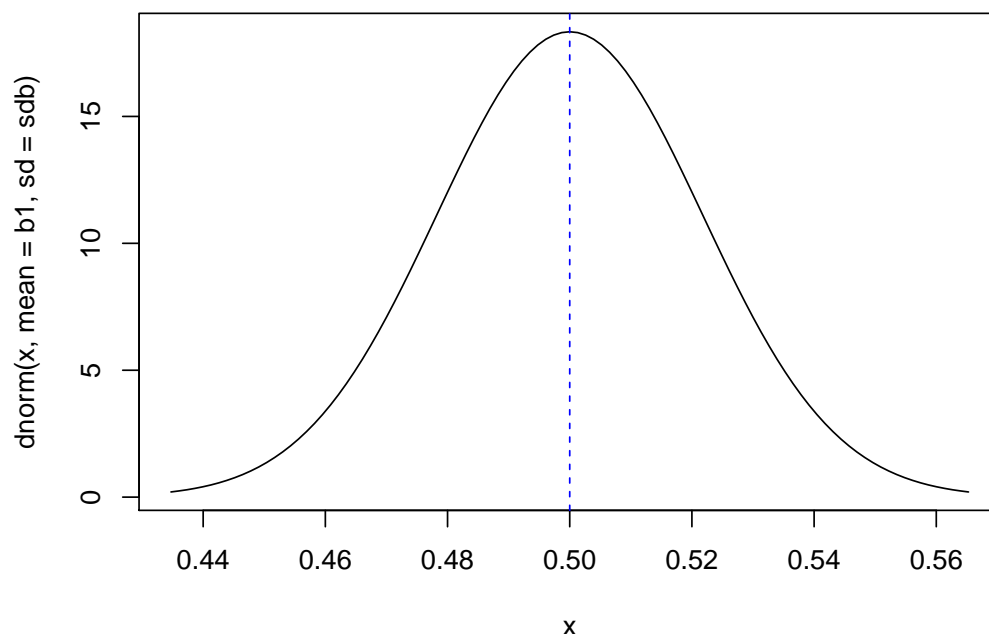
Draw sample of size n

```
x <- rnorm(n, 4, sigma)
# Another way is to assume that the values for x are fixed and know
# x= seq(from = 0, to = 10, length.out = n)

u <- rnorm(n, 0, sigma)

y <- b0 + b1 * x + u
```

Estimate parameter by OLS

Figure 2.5: The theoretical (true) probability density function of b_1

```
olsreg <- lm(y ~ x )
```

```
simulation.df <- data.frame(x,y)
```

```
population.df <- data.frame(b0, b1)
```

```
plot(simulation.df,
     xlab = "x",
     ylab = "y",
     # main = "Simulate least squares regression",
     pch = 20,
     cex = 2,
     col = "grey")
abline(olsreg, lwd = 3, lty = 1, col = "darkorange")
abline(b0, b1, lwd = 3, lty = 2, col = "darkblue")
legend("topleft",
     c("OLS regression function",
       "Population regression function"),
     lwd = 2,
     lty = 1:2)
```

```
lable1 <- "OLS regression function"
```

```
ggplot(simulation.df, aes(x = x, y = y)) +
```

```
  geom_point() +
```

```
  geom_abline(aes(intercept=b0,slope=b1,colour="Population regression function"), linetype = "dashed", sl
```

```
  stat_smooth(aes(colour = "OLS regression function"), method = "lm", se=FALSE, show.legend = TRUE)+
```

```
  labs(colour = "Regression functions"
```

```
    # , title = "Simulate least squares regression"
```

```
  ) +
```

```
  theme_bw()
```

Since the expected values and variances of our estimators are defined over separate random samples from

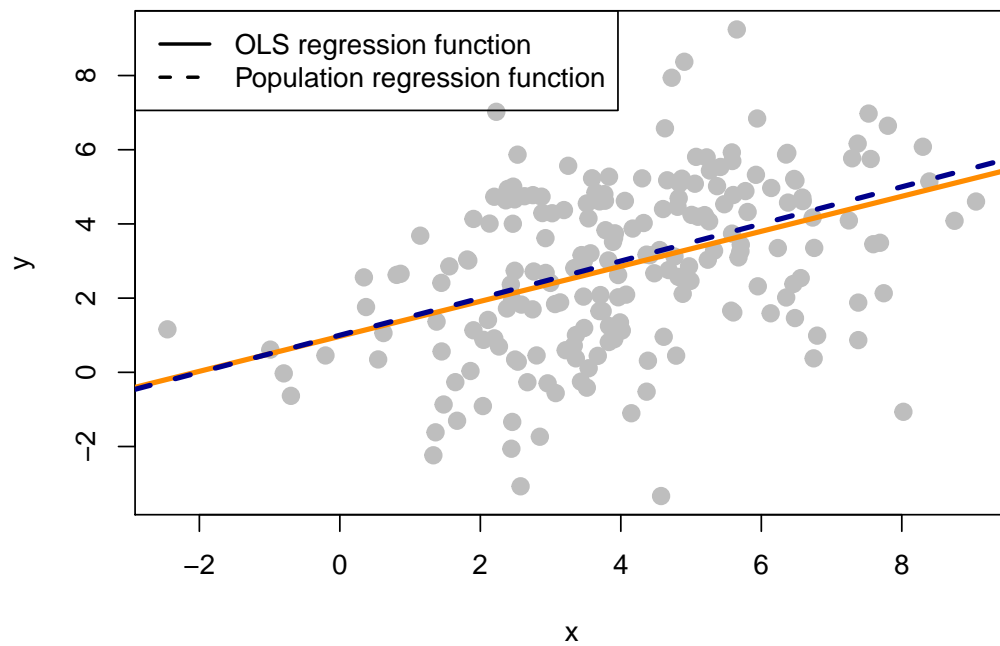


Figure 2.6: Simulated Sample and OLS Regression Line

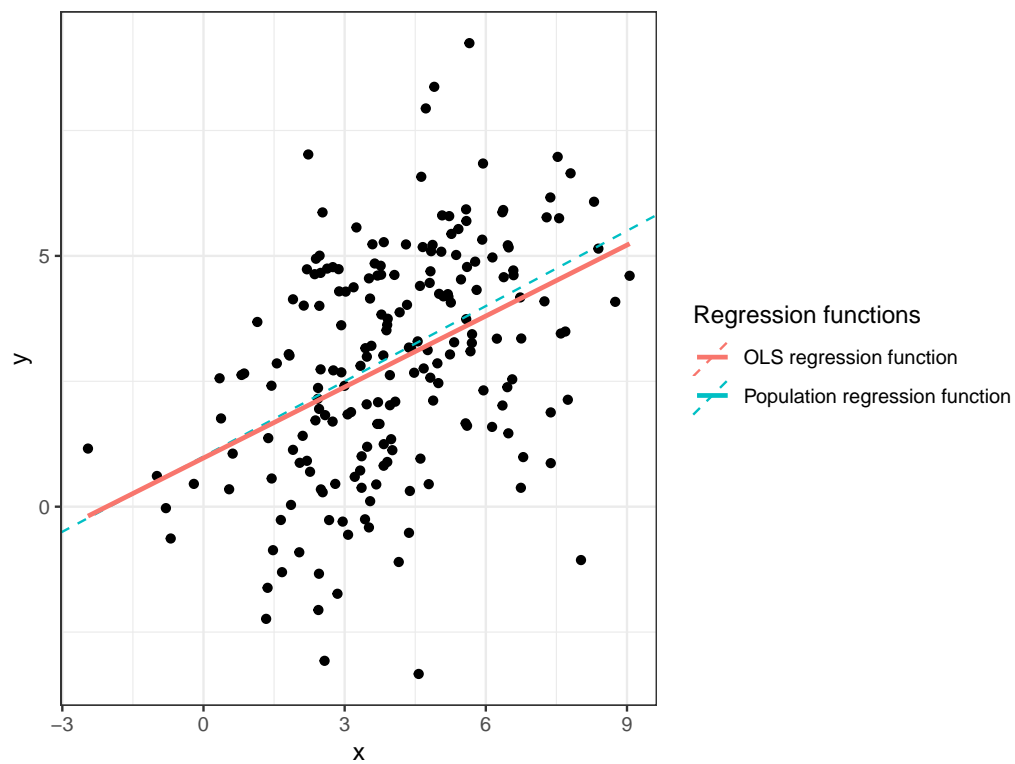


Figure 2.7: Simulated Sample and OLS Regression Line (ggplot Style)

the same population, it makes sense to repeat our simulation exercise over many simulated samples.

```
# Set the random seed
set.seed(1234567)

# set sample size and number of simulations
n<-1000; r<-10000

# set true parameters: betas and sd of u
b0<-1.0; b1<-0.5; sigma<-2

# initialize b0hat and b1hat to store results later:
b0hat <- numeric(r)
b1hat <- numeric(r)

# Draw a sample of x, fixed over replications:
x <- rnorm(n,4,1)

# repeat r times:
for(j in 1:r) {
  # Draw a sample of y:
  u <- rnorm(n,0,sigma)
  y <- b0 + b1*x + u

  # estimate parameters by OLS and store them in the vectors
  bhat <- coefficients( lm(y~x) )
  b0hat[j] <- bhat["(Intercept)"]
  b1hat[j] <- bhat["x"]
}

# MC estimate of the expected values:
mean(b0hat)

## [1] 0.9985388
mean(b1hat)

## [1] 0.5000466
# MC estimate of the variances:
var(b0hat)

## [1] 0.0690833
var(b1hat)

## [1] 0.004069063

# Initialize empty plot
plot( NULL, xlim=c(0,8), ylim=c(0,6), xlab="x", ylab="y")
# add OLS regression lines
for (j in 1:10) abline(b0hat[j],b1hat[j],col="gray")
# add population regression line
abline(b0,b1,lwd=2)
# add legend
legend("topleft",c("Population","OLS regressions"),
      lwd=c(2,1),col=c("black","gray"))
```

Even though the loop solution is transparent, let us take a look at a different, more *modern* approach.

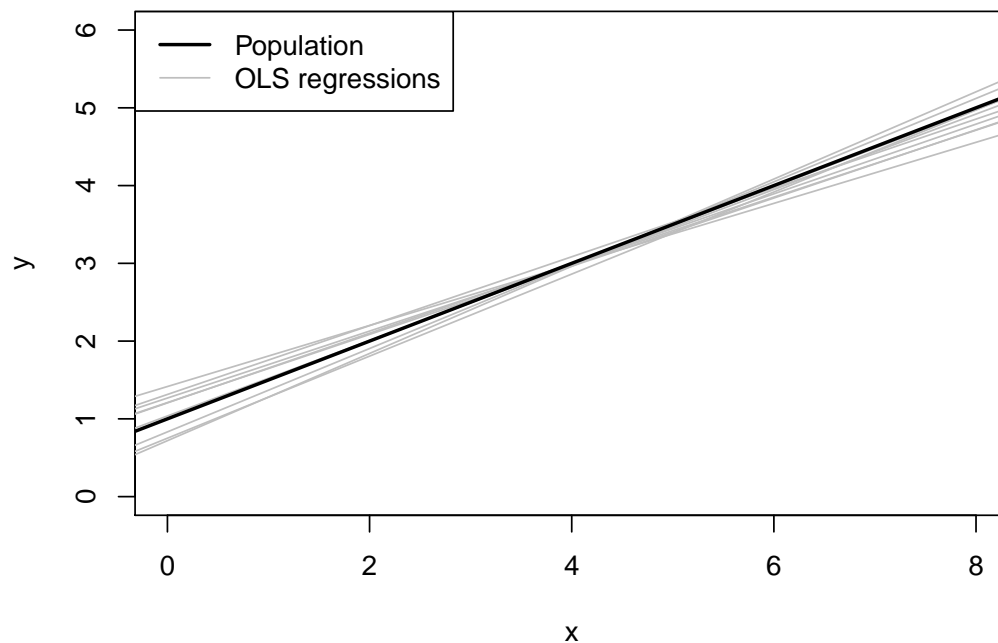


Figure 2.8: Population and Simulated OLS Regression Lines

```
# define a function the returns the alpha -- its point estimate, standard error, etc. -- from the OLS
x <- rnorm(n,4,1) # NOTE 1: Although a normal distribution is usually defined by its mean and variance,
# NOTE 2: We use the same values for x in all samples since we draw them outside of the loop.

iteration <- function() {
  u <- rnorm(n,0,sigma)
  y <- b0 + b1*x + u

  lm(y~x) %>%
    broom::tidy() # %>%
    # filter(term == 'x') # One could only extract the slope
}

# 1000 iterations of the above simulation
MC_coef<- map_df(1:1000, ~iteration())
str(MC_coef)
```

```
## Classes 'tbl_df', 'tbl' and 'data.frame': 2000 obs. of 5 variables:
## $ term : chr "(Intercept)" "x" "(Intercept)" "x" ...
## $ estimate : num 1.577 0.372 1.44 0.387 1.355 ...
## $ std.error: num 0.2672 0.0639 0.2623 0.0628 0.2626 ...
## $ statistic: num 5.9 5.82 5.49 6.17 5.16 ...
## $ p.value : num 4.94e-09 7.91e-09 5.13e-08 9.92e-10 2.99e-07 ...
```

Instead of plotting simulated and true parameter regression lines we can take a look at the kernel density of the simulated parameter estimates

Figure 2.9 shows the simulated distribution of β_0 and β_1 the theoretical one.

```
# plot the results
str(MC_coef)
```

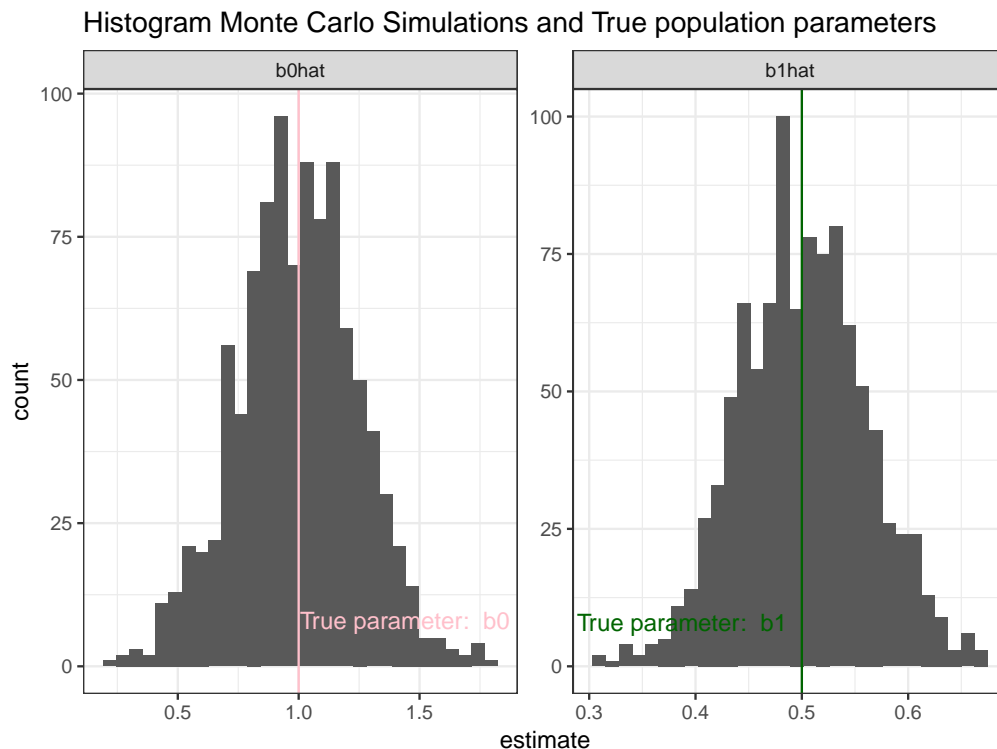


Figure 2.9: Histogram b0 and b1 and true parameter

```
## Classes 'tbl_df', 'tbl' and 'data.frame':    2000 obs. of  5 variables:
## $ term      : chr  "(Intercept)" "x" "(Intercept)" "x" ...
## $ estimate  : num  1.577 0.372 1.44 0.387 1.355 ...
## $ std.error : num  0.2672 0.0639 0.2623 0.0628 0.2626 ...
## $ statistic : num  5.9 5.82 5.49 6.17 5.16 ...
## $ p.value   : num  4.94e-09 7.91e-09 5.13e-08 9.92e-10 2.99e-07 ...
```

```
MC_coef <- MC_coef %>%
  mutate(OLScoeff = ifelse(term == "x", "b1hat", "b0hat")) %>% # rename the x to b1hat and (Intercept) to b0hat
  mutate(Simulated = ifelse(term == "x", "b1", "b0")) # %>%
```

```
ggplot(data= MC_coef, aes(estimate)) +
  geom_histogram() +
  geom_vline(data = filter(MC_coef, OLScoeff == "b0hat"), aes(xintercept=b0), colour="pink") +
  geom_vline(data = filter(MC_coef, OLScoeff == "b1hat"), aes(xintercept=b1), colour="darkgreen") +
  geom_text(data=MC_coef[3,], mapping=aes(x=estimate, y=8, label=paste("True parameter: ", MC_coef[3,7])) +
  geom_text(data=MC_coef[4,], mapping=aes(x=estimate, y=8, label=paste("True parameter: ", MC_coef[4,7])) +
  facet_wrap( ~ OLScoeff, scales = "free") +
  labs(
    title = "Histogram Monte Carlo Simulations and True population parameters") +
  theme_bw()
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```

```
b1_sim <- MC_coef %>%
  filter(Simulated == "b1")

mean(b1_sim$estimate)
```

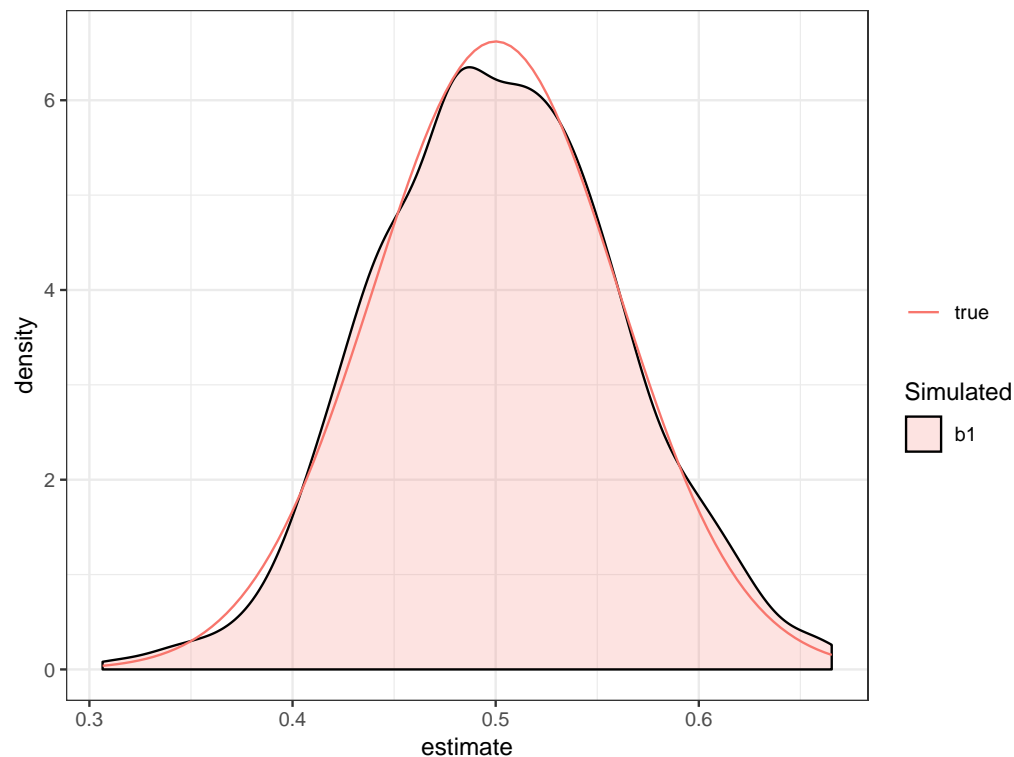


Figure 2.10: Simulated and theoretical distributions of b1

```
## [1] 0.5011414
```

```
var(b1_sim$estimate) == (sd(b1_sim$estimate))^2
```

```
## [1] FALSE
```

```
all.equal(var(b1_sim$estimate) , (sd(b1_sim$estimate))^2) # Floating point arithmetic!
```

```
## [1] TRUE
```

```
ggplot(data= b1_sim, aes(estimate)) +
  geom_density(aes(fill = Simulated), alpha = 0.2) + # computes and draws the kernel density, which is
  # stat_function(fun = dnorm, args = list(mean = mean(b1_sim$estimate), sd = sd(b1_sim$estimate)), aes
  stat_function(fun = dnorm, args = list(mean = 0.5, sd = sd(b1_sim$estimate)), aes(colour = "true")) +
  # labs(
  #   title = "Kernel Density Monte Carlo Simulations vs. True population parameters"
  # ) +
  scale_color_discrete(name="") +
  theme_bw()
```

Rework this section might have mixed up what is simulated and what is biased

2.1.2.0.2 Violation of SLR.4

To implement a violation of **SLR.4** (zero conditional mean) consider a case where in the population u is not mean independent of x , for example

$$\mathbf{E}(u|x) = \frac{x-4}{5}$$

```

# Set the random seed
set.seed(1234567)

# set sample size and number of simulations
n<-1000; r<-10000

# set true parameters: betas and sd of u
b0<-1; b1<-0.5; su<-2

# initialize b0hat and b1hat to store results later:
b0hat <- numeric(r)
b1hat <- numeric(r)

# Draw a sample of x, fixed over replications:
x <- rnorm(n,4,1)

# repeat r times:
for(j in 1:r) {
  # Draw a sample of y:
  u <- rnorm(n, (x-4)/5, su) # this is where manipulate the assumption of zero conditional mean
  y <- b0 + b1*x + u

  # estimate parameters by OLS and store them in the vectors
  bhat <- coefficients( lm(y~x) )
  b0hat[j] <- bhat["(Intercept)"]
  b1hat[j] <- bhat["x"]
}

```

OLS coefficients

```

# MC estimate of the expected values:
mean(b0hat)

```

```
## [1] 0.1985388
```

```
mean(b1hat)
```

```
## [1] 0.7000466
```

```

# MC estimate of the variances:
var(b0hat)

```

```
## [1] 0.0690833
```

```
var(b1hat)
```

```
## [1] 0.004069063
```

The average estimates are far from the population parameters $\beta_0 = 1$ and $\beta_1 = 0.5$!

2.1.2.0.3 Violation of SLR.5

Homoskedasticity is not required for unbiasedness but for it is a requirement for the theorem of sampling variance. Consider the following heteroskedastic behavior of u given x .

```

# Set the random seed
set.seed(1234567)

```



```

# set sample size and number of simulations
n<-1000; r<-10000

# set true parameters: betas and sd of u
b0<-1; b1<-0.5; su<-2

# initialize b0hat and b1hat to store results later:
b0hat <- numeric(r)
b1hat <- numeric(r)

# Draw a sample of x, fixed over replications:
x <- rnorm(n,4,1)

# repeat r times:
for(j in 1:r) {
  # Draw a sample of y:
  varu <- 4/exp(4.5) * exp(x)
  u <- rnorm(n, 0, sqrt(varu) )
  y <- b0 + b1*x + u

  # estimate parameters by OLS and store them in the vectors
  lm_heterosced <- lm(y~x)

  bhat <- coefficients( lm(y~x) )
  b0hat[j] <- bhat["(Intercept)"]
  b1hat[j] <- bhat["x"]
}

summary(lm_heterosced) # just the last sample of the MC-simulation

##
## Call:
## lm(formula = y ~ x)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -23.6742  -0.9033   0.0052   1.0012   9.3411
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  1.24088    0.27158   4.569 5.51e-06 ***
## x            0.44561    0.06593   6.759 2.37e-11 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.075 on 998 degrees of freedom
## Multiple R-squared:  0.04377,    Adjusted R-squared:  0.04281
## F-statistic: 45.68 on 1 and 998 DF,  p-value: 2.367e-11

```

Plot the residual against the regressor suspected of creating heteroskedasticity, or more generally, the fitted values of the regression.

```

res <- residuals(lm_heterosced)
yhat <- fitted(lm_heterosced)

```

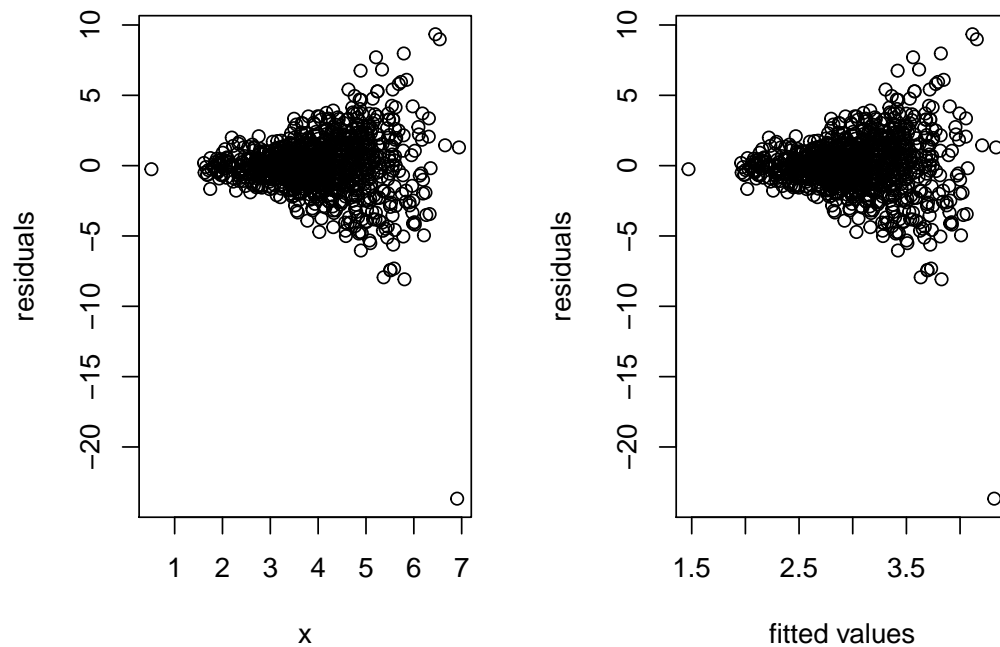


Figure 2.11: Heteroskedasticity in the simulated data

```

par(mfrow = c(1,2))
plot(x, res, ylab = "residuals")
plot(yhat, res, xlab = "fitted values", ylab = "residuals")

# MC estimate of the expected values:
mean(b0hat)

## [1] 1.0019
mean(b1hat)

## [1] 0.4992376
# MC estimate of the variances:
var(b0hat)

## [1] 0.08967037
var(b1hat)

## [1] 0.007264373

```

Unbiasedness is provided but sampling variance is incorrect (compared to the results provided above).

2.1.3 Nonlinearities

Sometimes the scatter plot diagram or some theoretical considerations suggest a non-linear relationship. The most popular non-linear relationships involve logarithms of the dependent or independent variables and polynomial functions.

We will use a new dataset, *wage1*, for this section. A detailed exploratory analysis of the dataset is left to the reader.

```
data("wage1")
attach(wage1)

## The following objects are masked from wage1 (pos = 5):
##
##   clerocc, construc, educ, exper, expersq, female, lwage,
##   married, ndurman, nonwhite, northcen, numdep, profocc,
##   profserv, services, servocc, smsa, south, tenure, tenursq,
##   trade, trcompu, wage, west
```

2.1.3.1 Predicated variable transformation

A common variance stabilizing transformation (VST) when we see increasing variance in a fitted versus residuals plot is $\log(Y)$.

Related, to use the \log of an independent variable is to make its distribution closer to the normal distribution.

```
# wage1$logwage <- log(wage1$wage) # one could also create a new variable
```

```
p1_wagehisto <- ggplot(wage1) +
  geom_histogram(aes(x = wage), fill = "red", alpha = 0.6) +
  theme_bw()
```

```
p2_wagehisto <- ggplot(wage1) +
  geom_histogram(aes(x = wage), fill = "blue", alpha = 0.6) +
  scale_x_continuous(trans='log2', "Log Wage") + # instead of creating a new variable with simply defin
  theme_bw()
```

```
ggarrange(p1_wagehisto, p2_wagehisto,
  labels = c("A", "B"),
  ncol = 2, nrow = 1)
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```

A model with a log transformed response:

$$\log(Y_i) = \beta_0 + \beta_1 \times x_i + \epsilon_i \quad (2.5)$$

```
lm_wage <- lm(wage ~ educ, data = wage1)
lm_wage1 <- lm(log(wage) ~ educ, data = wage1)
summary(lm_wage)
```

```
##
## Call:
## lm(formula = wage ~ educ, data = wage1)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -5.3396 -2.1501 -0.9674  1.1921 16.6085
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.90485    0.68497  -1.321   0.187
```

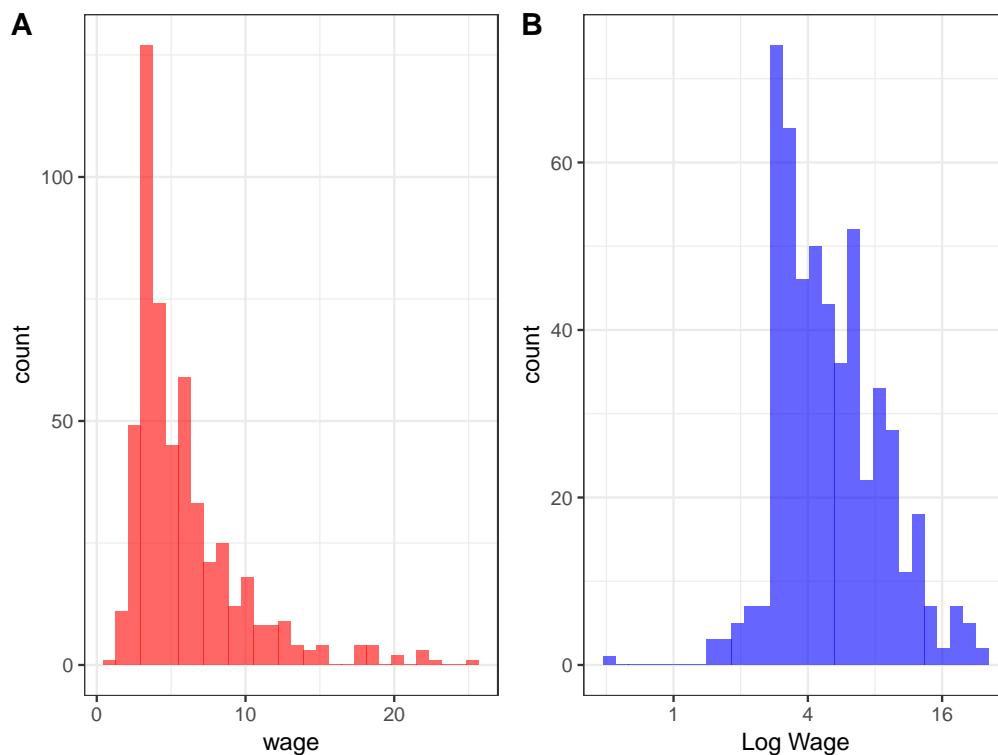


Figure 2.12: Histogram of wage and log(wage)

```
## educ      0.54136    0.05325  10.167  <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.378 on 524 degrees of freedom
## Multiple R-squared:  0.1648, Adjusted R-squared:  0.1632
## F-statistic: 103.4 on 1 and 524 DF,  p-value: < 2.2e-16
summary(lm_wage1)
```

```
##
## Call:
## lm(formula = log(wage) ~ educ, data = wage1)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -2.21158 -0.36393 -0.07263  0.29712  1.52339
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.583773   0.097336   5.998 3.74e-09 ***
## educ         0.082744   0.007567  10.935 < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.4801 on 524 degrees of freedom
## Multiple R-squared:  0.1858, Adjusted R-squared:  0.1843
```

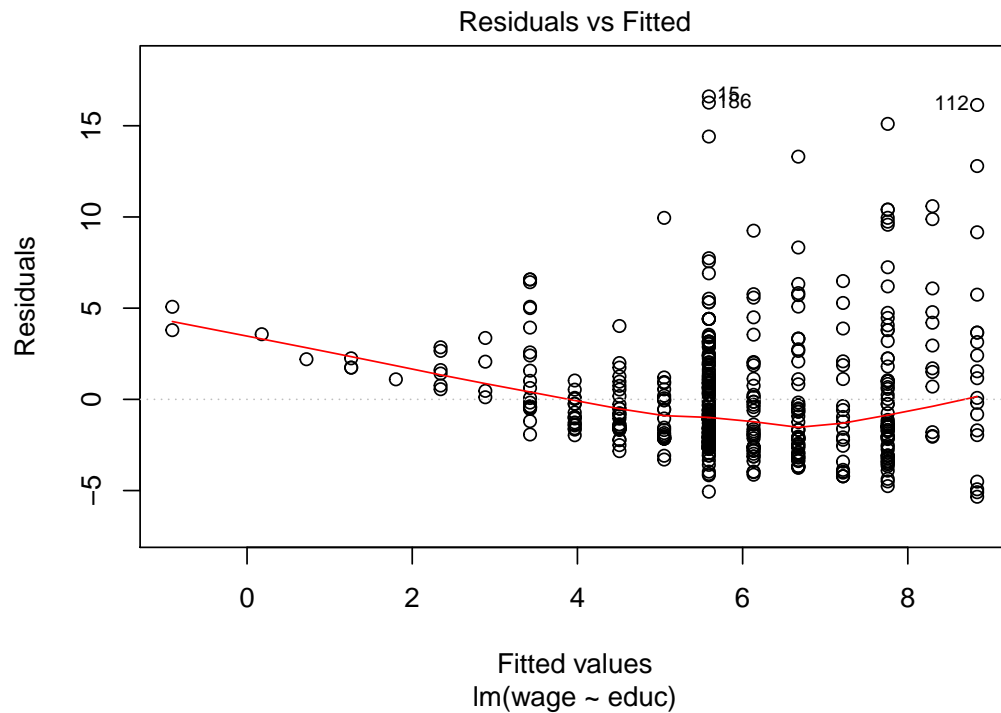


Figure 2.13: Regression diagnostics plot base R - Linear Relationship

```
## F-statistic: 119.6 on 1 and 524 DF, p-value: < 2.2e-16
```

Plotting Diagnostics for Linear Models

```
plot(lm_wage)
```

```
autoplot(lm_wage, which = 1:6, colour = 'dodgerblue3',
  smooth.colour = 'red', smooth.linetype = 'dashed',
  ad.colour = 'blue',
  label = FALSE,
  label.size = 3, label.n = 5, label.colour = 'blue',
  ncol = 3) +
  theme_bw()
```

```
autoplot(lm_wage1, which = 1:6, colour = 'dodgerblue3',
  smooth.colour = 'red', smooth.linetype = 'dashed',
  ad.colour = 'blue',
  label = FALSE,
  label.size = 3, label.n = 5, label.colour = 'blue',
  ncol = 3) +
  theme_bw()
```

```
p1_nonlinearities <- ggplot(wage1, aes(x = educ, y = wage )) +
  geom_point() +
  scale_y_continuous(trans='log2', "Log Wage") +
  stat_smooth(aes(fill="Linear Model"),size=1,method = "lm" ,span =0.3, se=F) +
  guides(fill = guide_legend("Model Type")) +
  theme_bw()
```

Note that if we re-scale the model from a log scale back to the original scale of the data, we now have

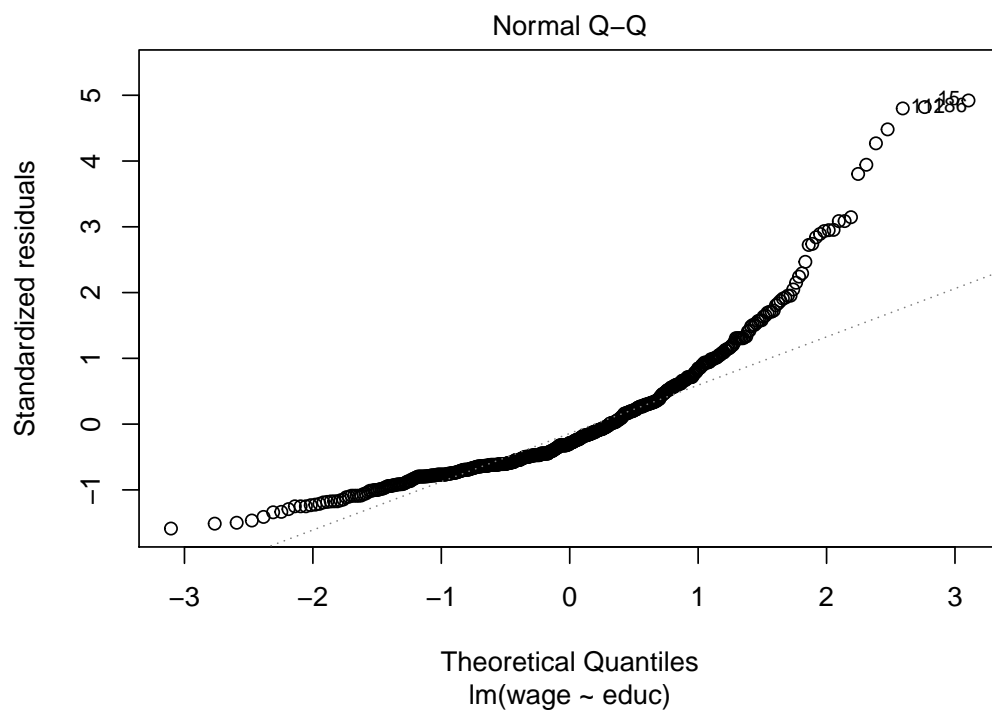


Figure 2.14: Regression diagnostics plot base R - Linear Relationship

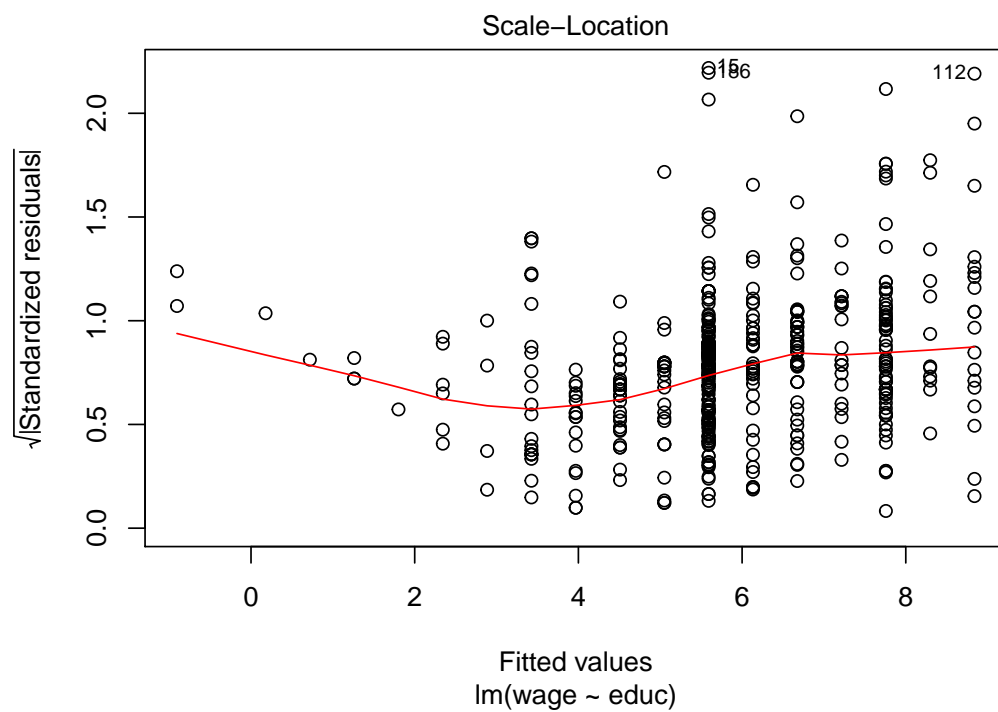


Figure 2.15: Regression diagnostics plot base R - Linear Relationship

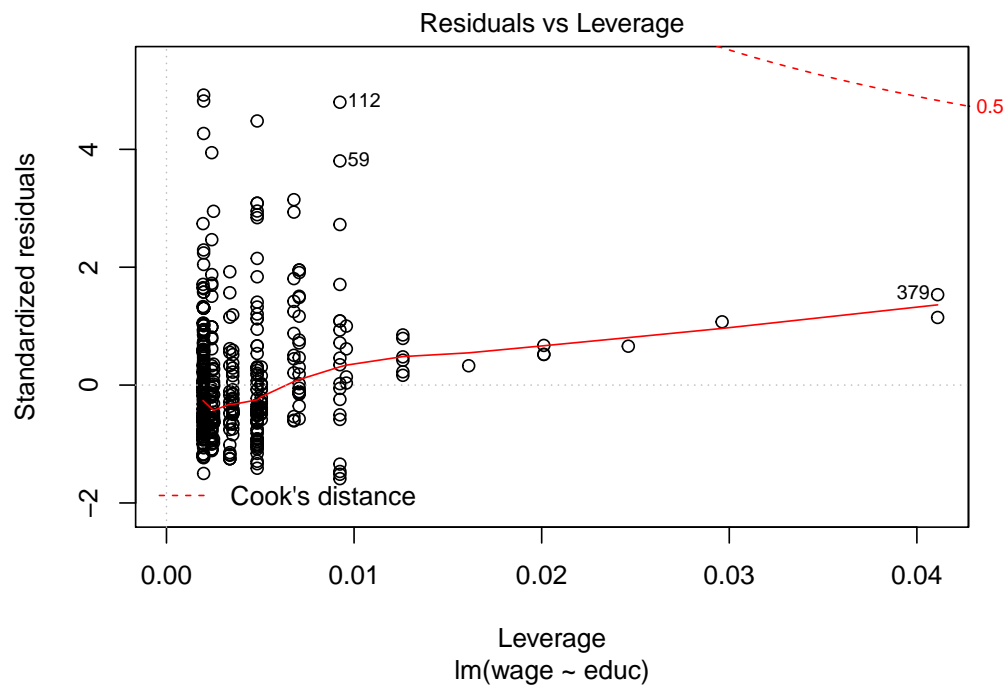


Figure 2.16: Regression diagnostics plot base R - Linear Relationship

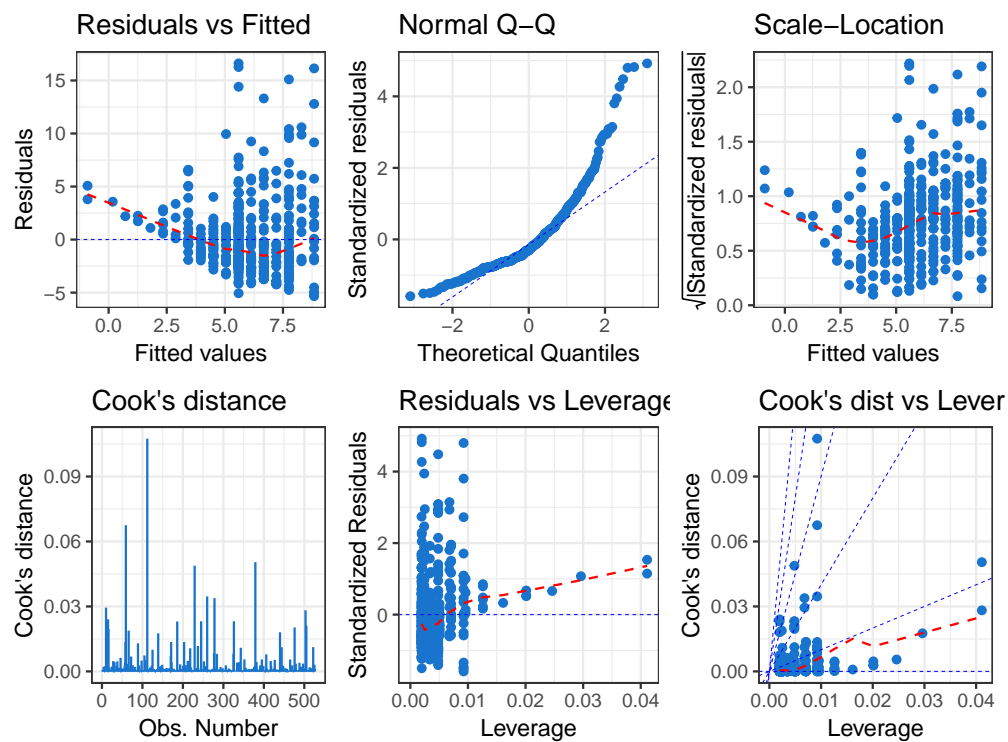


Figure 2.17: Regression diagnostics autoplot(ggplot) - Linear Relationship

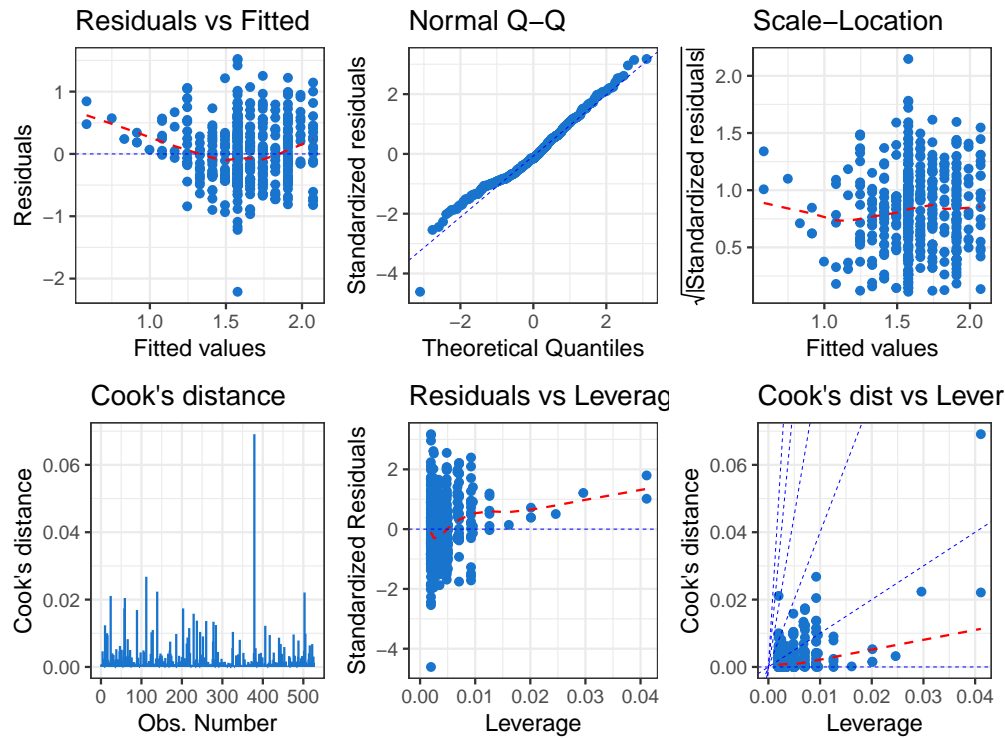


Figure 2.18: Regression diagnostics - Non-Linear Relationship

$$Y_i = \exp(\beta_0 + \beta_1 \times x_i) \times \exp(\epsilon_i) \quad (2.6)$$

which has errors entering in a multiplicative fashion.

```
log.model.df <- data.frame(x = wage1$educ,
                           y = exp(fitted(lm_wage1))) # This is essentially exp(b0_wage1 + b1_wage1 * w

p2_nonlinearities <- ggplot(wage1, aes(x = educ, y = wage)) +
  geom_point() +
  geom_line(data = log.model.df, aes(x, y, color = "Log Model"), size = 1, linetype = 2) +
  guides(color = guide_legend("Model Type")) +
  theme_bw()

ggarrange(p1_nonlinearities, p2_nonlinearities,
          labels = c("A", "B"),
          ncol = 2, nrow = 1)
```

A: Plotting the data on the transformed log scale and adding the fitted line, the relationship again appears linear, and the variation about the fitted line looks more constant.

B: By plotting the data on the original scale, and adding the fitted regression, we see an exponential relationship. However, this is still a *linear* model, since the new transformed response, $\log(Y_i)$, is still a *linear* combination of the predictors. In other words, only β needs to be linear, not the x values.

NOTE:

The example comes from the Wooldridge book but the variable `educ` looks more like count data. A Poisson GLM might seem like a better choice.

Quadratic Model

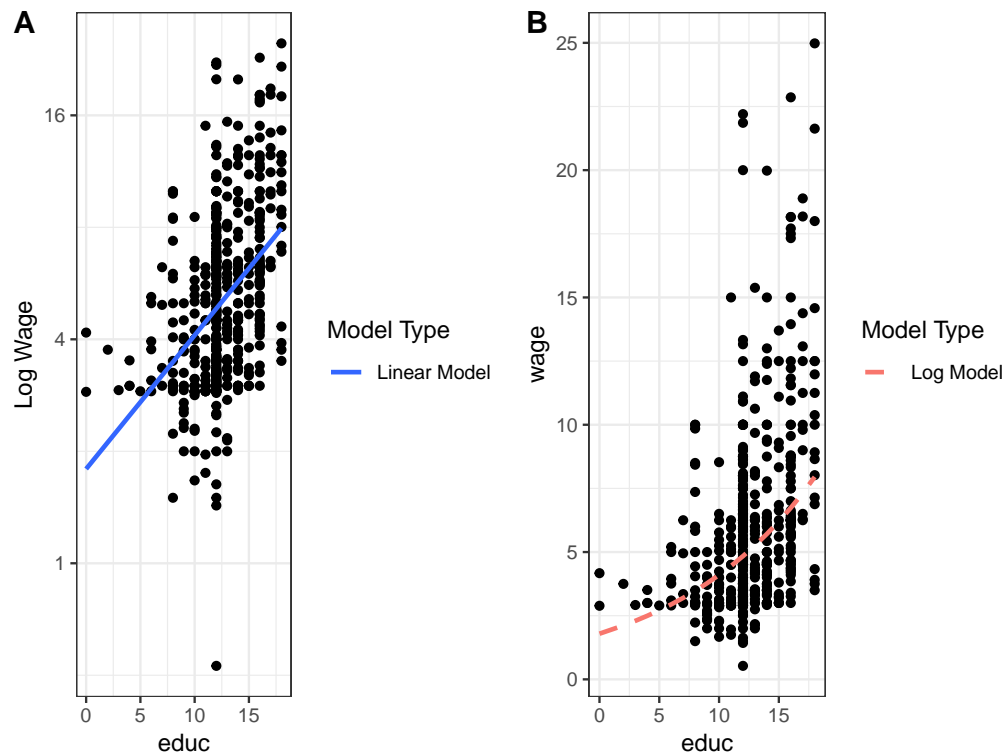


Figure 2.19: Wages by Education - Different transformations

$$Y_i = \beta_0 + \beta_1 \times x_i^2 \times \epsilon_i \quad (2.7)$$

New dataset from Wooldrige: Collected from the real estate pages of the Boston Globe during 1990. These are homes that sold in the Boston, MA area.

```
data("hprice1")
attach(hprice1)
```

```
## The following objects are masked from hprice1 (pos = 5):
##
##   assess, bdrms, colonial, lassess, llotsize, lotsize, lprice,
##   lsqrft, price, sqrft
```

In R, independent variables involving mathematical operators can be included in regression equation with the function `I()`

```
lm_hprice <- lm(price ~ sqrft, data = hprice1)
lm_hprice1 <- lm(price ~ sqrft + I(sqrft^2), data = hprice1)
```

Alternatively use the `poly()` function. Be careful of the additional argument `raw`.

```
lm_hprice2 <- lm(price ~ poly(sqrft, degree = 2), data = hprice1)
lm_hprice3 <- lm(price ~ poly(sqrft, degree = 2, raw = TRUE), data = hprice1) # if true, use raw and
unnname(coef(lm_hprice1))
```

```
## [1] 1.849453e+02 -1.710855e-02 3.262809e-05
```

```

unnname(coef(lm_hprice2))

## [1] 293.5460 754.8517 135.6051
unnname(coef(lm_hprice3))

## [1] 1.849453e+02 -1.710855e-02 3.262809e-05
all.equal(unnname(coef(lm_hprice1)), unnname(coef(lm_hprice2)))

## [1] "Mean relative difference: 5.401501"
all.equal(unnname(coef(lm_hprice1)), unnname(coef(lm_hprice3)))

## [1] TRUE
all.equal(fitted(lm_hprice1), fitted(lm_hprice2))

## [1] TRUE
all.equal(fitted(lm_hprice1), fitted(lm_hprice3))

## [1] TRUE

```

2.1.4 Inference for Simple Linear Regression

“There are three types of lies: lies, damn lies, and statistics” *Benjamin Disraeli*

2.2 Multiple Linear Regression

Note

A **(general) linear model** is similar to the simple variant, but with a multivariate $x \in \mathbb{R}^p$ and a mean given by a hyperplane in place of a single line.

- General principles are the same as the simple case
- Math is more difficult because we need to use matrices
- Interpretation is more difficult because the β_j are effects conditional on the other variables

Many would retain the same signs as the simple linear regression, but the magnitudes would be smaller. In some cases, it is possible for the relationship to flip directions when a second (highly correlated) variable is added.

The file was created using R version 3.5.1.

Chapter 3

Binary limited dependent variable models

In this section we discuss probit and logit models

Bibliography

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