Applications of Bessel Functions to Physical Systems

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February 6, 2024

1 Introduction

When discussing the applications of various mathematical techniques found in the field of Physics, it would be amiss if we were to not explore how the Bessel Functions we have learned can be found and applied to various situations that one can find within the realm of Physics, so the following paper aims to do just that. Beginning with the electrostatic analysis of a cylindrical cavity, we will demonstrate how a given partial differential equation in cylindrical coordinates can be made simple through the use of the Bessel Function, as well as how we can derive and solve a differential equation to explain the physics behind the tension in an elastic strong that possesses a non-uniform mass density. Through the analysis of these two situations, the reader will be walked through the process of how to approach various situations (pulling inspiration from both classical mechanics and electromagnetism, two of the main fields students will have dealt with in their studies) and how to apply their knowledge of the Bessel Function to better understand these instances and trivialize what would have been difficult and tedious mechanisms to analyse.

2 Discussion

2.1 Electrostatic Analysis of a Cylindrical Cavity

A cylindrical cavity, of radius a and height h, has conducting walls. Its circular ends, at $z = 0$ and $z = h$, are insulated from the cylindrical sleeve and are held at zero potential. The sleeve is held at a constant non-zero potential V_0. Show that the electrostatic potential within the cavity, $V(\rho, \varphi, z)$, which obeys Laplace's equation, is

$$V(\rho, \varphi, z) = \frac{4V_0}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \frac{I_0\left((2n+1) + \pi \frac{\rho}{h}\right)}{I_0\left((2n+1) + \pi \frac{a}{h}\right)} \sin\left((2n+1)\pi \frac{z}{h}\right)$$

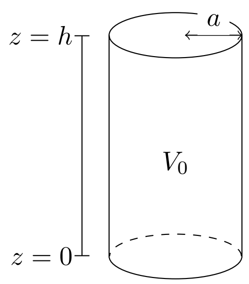

Figure 1: Cylindrical Cavity

To find the potential, we note that the electric potential satisfies Laplace's equation: $\nabla^2 V = 0$. From the geometry of the problem, we see that we should switch to cylindrical coordinates. Laplace's equation in cylindrical coordinates is [1]:

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{2.1}$$

Performing the usual separation of variables, let $V = R(r)\,\Phi(\phi)\,Z(z)$. Substituting this in for V and dividing, we obtain:

$$R\frac{1}{\rho}\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho}\frac{1}{R}\frac{\partial R}{\partial R} + \frac{1}{\Phi}\frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0 \tag{2.2}$$

Suppose that

$$\frac{1}{Z}\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} = -k^2 \tag{2.3}$$

Putting this into Equation 2.2,

$$\frac{1}{R}\frac{\partial^2 R}{\partial \rho^2} + \frac{1}{\rho}\frac{1}{R}\frac{\partial R}{\partial \rho} + \frac{1}{\Phi}\frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} - k^2 = 0$$

Solving for Equation 2.3 gives the usual periodic solution $Z\left(z\right) = A\cos(kz) + B\sin(kz)$. Then multiplying Equation 2.4 by ρ^2 yields

$$\frac{\rho^2}{R}\frac{\partial^2 R}{\partial \rho^2} + \frac{\rho}{R}\frac{\partial R}{\partial \rho} + \frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \phi^2} - k^2\rho^2 = 0$$

This allows us to separate out an equation for Φ, by supposing that

$$\frac{1}{\Phi}\frac{\mathrm{d}^2 \Phi}{\mathrm{d}\phi^2} = -m^2 \tag{2.4}$$

This has periodic solutions as well; solving for Φ gives $\Phi\left(\phi\right) = C\cos(m\phi) + D\sin(m\phi)$. This sets our final equation for the R as:

$$\frac{\rho^2}{R}\frac{\partial^2 R}{\partial \rho^2} + \frac{\rho}{R}\frac{\partial R}{\partial \rho} - m^2 - k^2\rho^2 = 0$$

When multiplying both sides by R, the equation whose solutions are modified Bessel functions [1] appears:

$$\rho^2\frac{\partial^2 R}{\partial \rho^2} + \rho\frac{\partial R}{\partial \rho} - \left(m^2 + k^2\rho^2\right)R = 0 \tag{2.5}$$

The solutions to which are the modified Bessel functions $I_m\left(k\rho\right)$. We can now identify the main equation as:

$$V(\rho,\phi,z) = R\left(r\right)\Phi\left(\phi\right)Z\left(z\right) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} I_m(k\rho)\left[A\cos(kz) + B\sin(kz)\right]\left[C\cos(m\phi) + D\sin(m\phi)\right] \tag{2.6}$$

The first set of boundary conditions are that $V(\rho,\phi,0) = 0$ and $V(\rho,\phi,h) = 0$. This gives us that $A = 0$ and $\sin(kh) = 0$ which means $k = \pi n/h$. The remaining boundary condition gives us that the sleeve is held at a constant potential V_0. Applying this these to Equation 2.6 above,

$$V_0 = V(a,\phi,z) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} I_m(\frac{\pi n}{h}a)\left[\sin(\frac{\pi n}{h}z)\right]\left[C\cos(m\phi) + D\sin(m\phi)\right]$$

Where B has been absorbed into the C and D constants. To get a value for the C and D constants we can use the orthogonality by multiplying both sides of the equation by $\cos(m'\phi)$ and $\sin(\pi n' z/h)$:

$$\int_0^{2\pi}\int_0^h V_0\cos(m'\phi)\sin(\frac{\pi n'}{h}z)\mathrm{d}\phi\mathrm{d}z = \int_0^{2\pi}\int_0^h \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} I_m(\frac{\pi n}{h}a)\left[\sin(\frac{\pi n}{h}z)\sin(\frac{\pi n'}{h}z)\right]\left[C\cos(m\phi)\cos(m'\phi) + D\sin(m\phi)\cos(m'\phi)\right]\mathrm{d}\phi\mathrm{d}z$$

For the left hand side, we find that

$$\int_0^{2\pi}\int_0^h V_0\cos(m'\phi)\sin(\frac{\pi n'}{h}z)\mathrm{d}\phi\mathrm{d}z = \frac{2hV_0}{n'}(1-(-1)^{n'})$$

On the right side, the orthogonality rule makes the cosines under the integral into $\delta_{nn'}$. Due to the absence of any angular dependence on the boundary condition, $m = m' = 0$. The integral then equates to: $I_0(ka)\pi nC$. These conditions allow us to solve for C as:

$$C = \frac{2V_0}{\pi n I_0\left((n\pi\frac{a}{h}\right)}(1-(-1)^n)$$

It should be noted that $(1-(-1)^n)$ is zero for any even n and 2 for any odd n; thus a change of variable $n \rightarrow 2n+1$ can be used to re-write the term $(1-(-1)^n)$ as 2. This allows for the constant C to be re-written as:

$$C = \frac{4V_0}{\pi\left(2n+1\right)}\frac{1}{I_0\left(\left(2n+1\right)\pi\frac{a}{h}\right)}$$

Since we got that $m = 0$ due to the absence of any angular dependence, the $D\sin(m\phi)$ term drops out. Putting everything back together, we obtain the final solution as

$$\begin{aligned} V(\rho, \phi, z) &= \sum_{n=0}^{\infty} CI_0\left((2n+1)\pi\frac{\rho}{h}\right)\sin\left((2n+1)\pi\frac{z}{h}\right) \\ &= \sum_{n=0}^{\infty} \frac{4V_0}{\pi(2n+1)} \frac{I_0\left((2n+1)\pi\frac{\rho}{h}\right)}{I_0\left((2n+1)\pi\frac{a}{h}\right)}\sin\left((2n+1)\pi\frac{z}{h}\right) \end{aligned}$$

2.2 Tension in an Elastic String with Non-Uniform Mass Density

An elastic string whose mass per unit length is $\mu_0(1+\alpha x)$ where x is the distance from one end of the string, is stretched under tension T between two points a distance l apart. Find the equation defining the natural frequencies of the string.

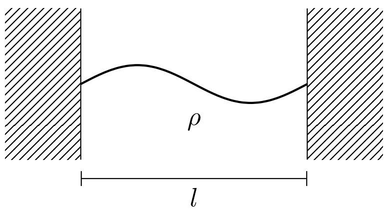

Figure 2: Elastic String of Length l and Density ρ

Suppose that we zoom in and take a look at a small segment of the string with tension τ:

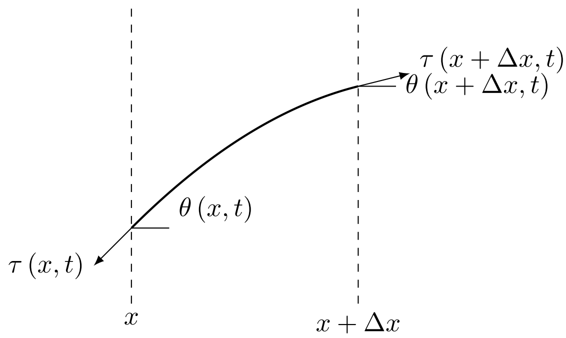

Figure 3: Stretching of a Segment of Elastic String

The slope of the string may be represented as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan(\theta(x,t)) = \frac{\partial u}{\partial x}$$

From Newton's law in the vertical direction, the equation of motion $F_y = ma_y$ states that

$$\tau(x+\Delta x, t)\sin(\theta(x+\Delta x, t)) - \tau(x,t)\sin(\theta(x+\Delta x, t)) + \rho\Delta x Q(x,t) = \rho\Delta x\frac{\partial^2 u}{\partial t^2}$$

Where $Q(x,t)$ is the force per unit mass depending on any external forces. Assuming that the string is light enough such that outside forces (such as gravity) are negligible, $Q(x,t) = 0$. Then taking the limit as $\Delta x \to 0$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tau(x,t)\sin(\theta(x,t))) = \rho\frac{\partial^2 u}{\partial x^2}$$

Taking θ to be small, we can use the small angle approximation so that

$$\frac{\partial u}{\partial x} = \tan\theta \approx \sin\theta$$

For a perfectly elastic string, the magnitude of the tensile force T only depends on the local stretching [2]; and for small angles, which we have assumed, it is essentially a constant $\tau = T_0$. Thus we obtain the form of the wave equation that governs the shape of the string:

$$\rho\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} \tag{2.7}$$

For this specific case, we have that $\rho = \mu_0 (1 + \alpha x)$ and so we must solve

$$\mu_0 (1 + \alpha x) \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2}$$

Performing the usual separation of variables, suppose that $u(x,t) = X(x) T(t)$. Then

$$\frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = \frac{T_0}{X \mu_0 (1 + \alpha x)} \frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = -k^2$$

The solution for the time component can be found as:

$$T(t) = A \cos(kt) + B \sin(kt)$$

For the spatial part, we have

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \frac{k^2}{T_0} \mu_0 (1 + \alpha x) X = 0 \tag{2.8}$$

This is clearly not Bessel's equation, and it is not immediately obvious that the solutions to this differential equation are Bessel functions. Referring to Handy Hint #13 from the Tutorial[1], Bessel's equation can be written in the from

$$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} + (1 - 2A) x \frac{\mathrm{d}y}{\mathrm{d}x} + \left[A^2 + B^2 \left(C^2 x^{2B} - m^2\right)\right] y = 0 \tag{2.9}$$

which gives three parameters A, B, and C that can be used to transform our differential equation to look like Bessel's equation. To achieve this, the term $1 + \alpha x$ needs to be shifted first. So let $v = 1 + \alpha x$. Then Equation 2.8 reads:

$$v^2 \frac{\mathrm{d}^2 X}{\mathrm{d}v^2} + \frac{k^2}{T_0 \alpha^2} v^3 \mu_0 X = 0$$

Comparing this with the form given in Equation 2.9, we can see that the term with the first derivative must vanish: $1 - 2A = 0$ and so $A = 1/2$. the only power of v that multiplies the solution X is v^3. Comparing this with the x^{2B} gives us that $B = 3/2$. We also see that there is no term that looks like X times some constant, and so

$$A^2 - B^2 m^2 = 0$$
$$m = \pm \frac{1}{3}$$

The remaining term coefficient in front of $v^3 X$, which by inspection gives us the following relation:

$$B^2 C^2 = \frac{k^2 \mu_0}{T_0 \alpha^2}$$
$$C^2 = \left(\frac{2k}{3\alpha}\right)^2 \frac{\mu_0}{T_0}$$

For which we will take the positive value for C in order to keep the argument of the Bessel function positive. From Handy Hint #13, the solution can be written as a linear combination of functions taking the form $x^A J_m \left(C x^B\right)$. As such, the solution is

$$X(v) = E\sqrt{v} J_{1/3} \left(\frac{2\alpha k}{3} \sqrt{\frac{\mu_0}{T_0}} v^{\frac{3}{2}}\right) + F\sqrt{v} J_{-1/3} \left(\frac{2\alpha k}{3} \sqrt{\frac{\mu_0}{T_0}} v^{\frac{3}{2}}\right)$$
$$X(x) = E\sqrt{1 + \alpha x} J_{1/3} \left(\frac{2\alpha k}{3} \sqrt{\frac{\mu_0}{T_0}} (1 + \alpha x)^{\frac{3}{2}}\right) + F\sqrt{1 + \alpha x} J_{-1/3} \left(\frac{2\alpha k}{3} \sqrt{\frac{\mu_0}{T_0}} (1 + \alpha x)^{\frac{3}{2}}\right)$$

To shorten things up slightly, we can substitute the group of constants in the argument of the Bessel function as

$$\gamma = \frac{2\alpha k}{3} \sqrt{\frac{\mu_0}{T_0}} \tag{2.10}$$

There are homogeneous Dirichlet conditions at the boundary which will allow us to determine F in terms of E (or vice versa) by applying them to the solution:

$$X(0) = E J_{1/3}(\gamma) + F J_{-1/3}(\gamma) = 0$$
$$F = -E \frac{J_{1/3}(\gamma)}{J_{-1/3}(\gamma)}$$

Plugging this back in, we have that the solution so far is

$$\begin{aligned} X(x) &= E\sqrt{1+\alpha x}J_{1/3}\left(\gamma(1+\alpha x)^{\frac{3}{2}}\right) - E\frac{J_{1/3}(\gamma)}{J_{-1/3}(\gamma)}\sqrt{1+\alpha x}J_{-1/3}\left(\gamma(1+\alpha x)^{\frac{3}{2}}\right) \\ &= E\left(\sqrt{1+\alpha x}J_{-1/3}(\gamma)J_{1/3}\left(\gamma(1+\alpha x)^{\frac{3}{2}}\right) - J_{1/3}(\gamma)\sqrt{1+\alpha x}J_{-1/3}\left(\gamma(1+\alpha x)^{\frac{3}{2}}\right)\right) \end{aligned}$$

We must now apply the boundary condition at $x = l$. Unfortunately, this boundary condition does not provide enough information to be able to solve for the constant E, which acts as a sort of amplitude for the string's vibration. It does however, give us a way to find the allowed values for γ:

$$X(l) = E\left(\sqrt{1+\alpha l}J_{-1/3}(\gamma)J_{1/3}\left(\gamma(1-\alpha l)^{\frac{3}{2}}\right) - J_{1/3}(\gamma)\sqrt{1+\alpha l}J_{-1/3}\left(\gamma(1-\alpha l)^{\frac{3}{2}}\right)\right) = 0 \tag{2.11}$$

As such, γ is constrained so that it is the roof of Equation 2.11 above. By definition, we then have a relation between the nth root γ_n of the function $X(l)$ and the allowed frequencies k_n defined by Equation 2.10 as:

$$k_n = \frac{3\gamma_n}{2\alpha}\sqrt{\frac{T_0}{\mu_0}}$$

For example, with the ratio of $\alpha l = 2$ and every other constant set to 1 ($T_0 = \mu_0 = E = 1$),

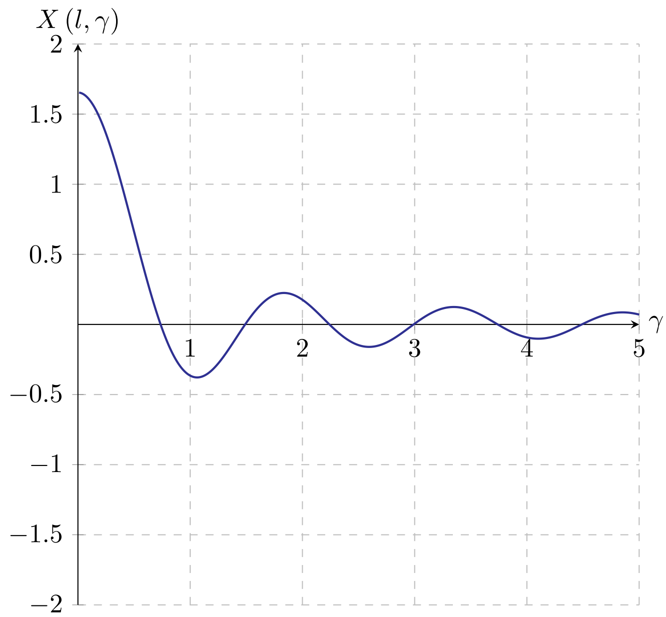

Figure 4: $X(l,\gamma)$ as a function of γ such that $\alpha l = 2$

The first few roots of γ are as follows:

$$\begin{aligned} \gamma_1 &\approx 0.736552 \\ \gamma_2 &\approx 1.489779 \\ \gamma_3 &\approx 2.240607 \\ \gamma_4 &\approx 2.990515 \\ \gamma_5 &\approx 3.739983 \end{aligned}$$

References

[1] Jacob, Richard J. *Tutorials in the Mathematical Methods of Physics*. Tempe, 2022.

[2] Haberman, Richard. *Applied Partial Differential equations with Fourier Series and Boundary Value Problems*. Southern Methodist University, 2013.