Neutron Scattering and Legendre Polynomials

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(c) Neutrons (mass= 1 in atomic units) are scattered by a nucleus of mass A. In the center of mass system, the scattering is isotropic (the same in all directions). In the lab system, the average of the cosine of the angle of deflection, ψ, is

$$\langle\cos\psi\rangle = \frac{1}{2}\int_0^\pi \frac{A\cos\theta+1}{\left(A^2+2A\cos\theta+1\right)^{1/2}}\sin\theta \mathrm{d}\theta,$$

where θ is the usual polar angle. Show that $\langle\cos\psi\rangle = \frac{2}{3A}$. [Hint: Think about the generating function].

Answer: The generating function for Legendre Polynomials is

$$F(x,h) = \sum_{n=0}^{\infty} P_n(x)\, h^n = \frac{1}{\sqrt{1-2xh+h^2}} \qquad \text{(Tutorial: 19)}$$

It should be noted that the nucleus of mass 1 is the smallest one; (you can't really scatter off a nucleus if there's less than one neutron) and so $A \geq 1$. We can use the substitution $x = \cos\theta$ with no problems. However, for h, naively using $h = -A$ means that $|h|$ gets bigger as A gets bigger; but the generating function only converges for $|x| \leq 1$. As a consequence, we need $|h|$ to get *smaller* as A gets bigger. The easiest way to achieve this is to simply take the reciprocal so then let $h = -1/A$. This is also consistent with the minimum of A being 1, which lies within our convergence radius. So then we can re-write the term with the square root in terms of $-1/A$ as:

$$\frac{A\cos(\theta)+1}{\sqrt{A^2+2A\cos\theta+1}} = \frac{A\cos(\theta)+1}{A\sqrt{\left(-\frac{1}{A}\right)^2 - 2\left(-\frac{1}{A}\right)\cos\theta+1}}$$

We note now, that this is $(\cos\theta + 1/A)$ times the generating function for Legendre polynomials with $x = \cos\theta$ and $h = -1/A$, and so

$$\frac{1}{\sqrt{\left(-\frac{1}{A}\right)^2 - 2\left(-\frac{1}{A}\right)\cos\theta+1}} = \sum_{n=0}^{\infty} P_n(\cos(\theta))\left(-\frac{1}{A}\right)^n$$

Plugging this back into the main integral yields:

$$\begin{aligned}\frac{1}{2}\int_0^\pi \frac{A\cos\theta+1}{\sqrt{A^2+2A\cos\theta+1}}\sin\theta\mathrm{d}\theta &= \frac{1}{2}\int_0^\pi \frac{A\cos(\theta)+1}{A\sqrt{\left(-\frac{1}{A}\right)^2 - 2\left(-\frac{1}{A}\right)\cos\theta+1}}\sin\theta\mathrm{d}\theta \\ &= \frac{1}{2A}\int_0^\pi (A\cos(\theta)+1)\sum_{n=0}^{\infty} P_n(\cos(\theta))\left(-\frac{1}{A}\right)^n \sin\theta\mathrm{d}\theta\end{aligned}$$

The first two Legendre polynomials are $P_0(\cos\theta) = 1$ and $P_1(\cos\theta) = \cos\theta$ and so we can write most of the integral in terms of other Legendre polynomials:

$$\begin{aligned}&\frac{1}{2A}\int_0^\pi (AP_1(\cos(\theta)) + P_0(\cos(\theta)))\sum_{n=0}^{\infty} P_n(\cos(\theta))\left(-\frac{1}{A}\right)^n \sin\theta\mathrm{d}\theta \\ &= \frac{1}{2A}\sum_{n=0}^{\infty}\left(-\frac{1}{A}\right)^n \int_0^\pi (AP_1(\cos(\theta)) + P_0(\cos(\theta)))\, P_n(\cos(\theta))\sin\theta\mathrm{d}\theta\end{aligned}$$

We can now evaluate the rest using orthogonality. From the tutorial,

$$\int_{-1}^{1} P_m(x) P_n(x) \, \mathrm{d}x = \frac{2}{2n+1} \delta_{mn} \tag{Tutorial: 34}$$

Making the variable change $x = \cos\theta$ gives $\mathrm{d}x = \sin\theta \mathrm{d}\theta$ and so

$$\int_{0}^{\pi} P_m(\cos\theta) P_n(\cos\theta) \sin\theta \mathrm{d}\theta = \frac{2}{2n+1} \delta_{mn}$$

As a consequence, the only terms that will survive in the sum are $n = m = 0$ and $n = m = 1$. This basically kills of the entire integral, only leaving

$$\begin{aligned}
\frac{1}{2A} \left(A \left(\frac{2}{2(1)+1} \right) \left(-\frac{1}{A} \right)^{1} + \frac{2}{2(0)+1} \left(-\frac{1}{A} \right)^{0} \right) &= \frac{1}{2A} \left(-\frac{2}{3} + 2 \right) \\
&= \frac{1}{2A} \left(\frac{4}{3} \right) \\
&= \boxed{\frac{2}{3A}}
\end{aligned}$$

Just to re-iterate, this gives that for $A \geq 1$,

$$\langle \cos\psi \rangle = \frac{1}{2} \int_{0}^{\pi} \frac{A\cos\theta + 1}{\left(A^2 + 2A\cos\theta + 1\right)^{1/2}} \sin\theta \mathrm{d}\theta = \frac{2}{3A}$$

References

[1] Jacob, Richard J. *Tutorials in the Mathematical Methods of Physics*. Tempe, 2022.