Synthesis of Logical Operators for Quantum Computers using Stabilizer Codes

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Introduction

- Quantum computers solve some problems more efficiently than classical computers, e.g., Shor's algorithm for prime factorization is *exponentially* faster than classical ones [NC10].
- But the components of a quantum circuit, e.g., gates, measurements, wires, are faulty.
- An insightful idea is to *encode* the qubits using a **quantum error-correcting code (QECC)** and try to address faults via error-correction procedures [NC10].
- **Fault-tolerance**: If a quantum operation is performed on an encoded block of qubits, and a single component of the circuit fails, then the number of errors in the output state should be within the error-correcting capacity of the code.
- Goals of fault-tolerant quantum computation:
  - Find codes that efficiently encode information (logical qubits) into the code states (physical qubits).
  - The codes must also have sufficiently high error-correcting capacity.
  - For a chosen code, determine the circuits that realize non-trivial operations on the logical qubits. These physical circuits are called the logical operators for the code.
  - Determine fault-tolerant implementations for a *universal* set of logical operators for the code. This guarantees reliable and *arbitrary* quantum computation.

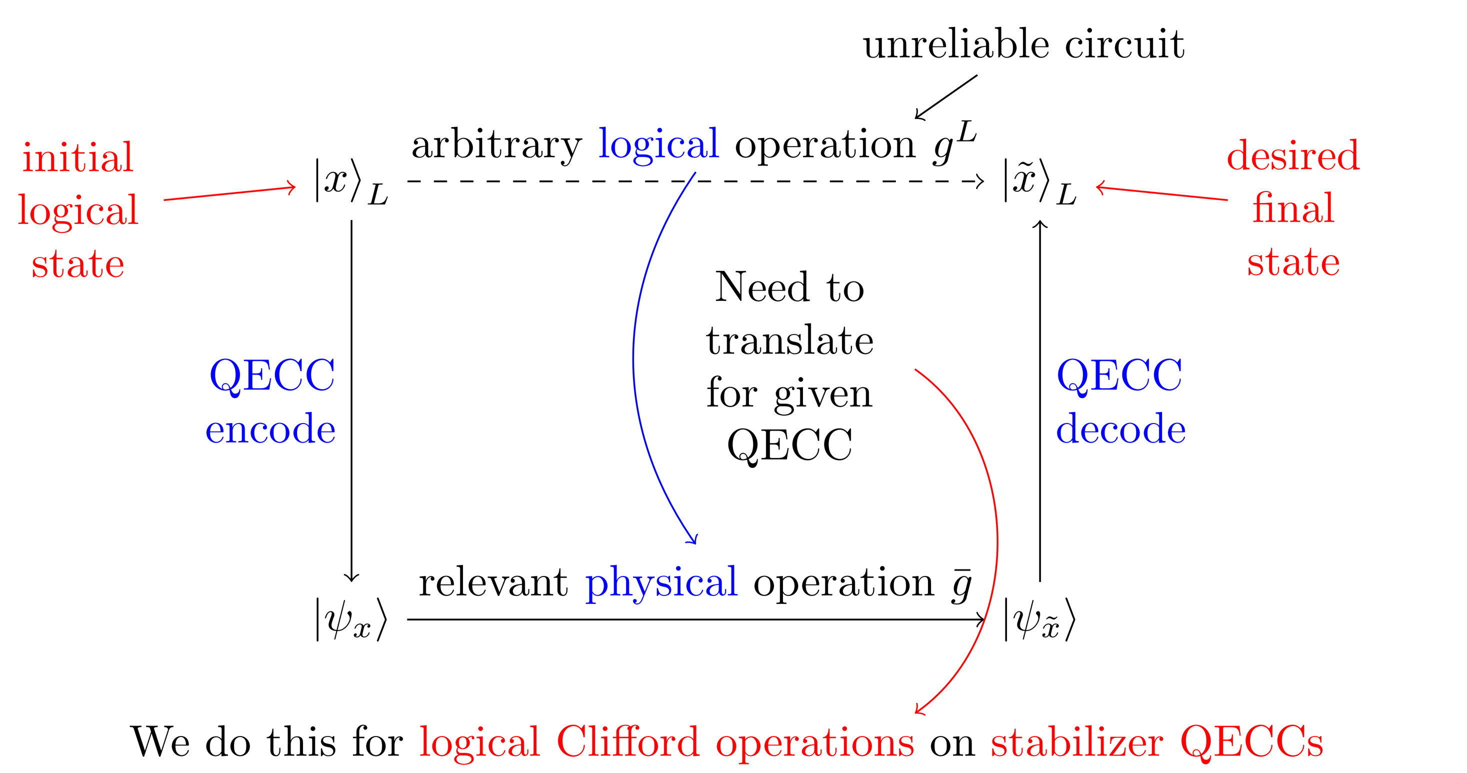


Figure 1: Problem: Quantum Operations on Encoded (Protected) Qubits

The Heisenberg-Weyl Group $HW_N (N = 2^m)$

- *Qubit*: Mathematically, a 2-dimensional Hilbert space over $\mathbb{C}$.
- *Pure state*: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, with $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.
- Example (2 qubits): $|0\rangle \otimes |1\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} = |01\rangle$, a basis vector for $\mathbb{C}^4$.
- The single qubit *Pauli* operators are given by
$$I_2 \triangleq \begin{bmatrix}1&0\\0&1\end{bmatrix},\ X \triangleq \begin{bmatrix}0&1\\1&0\end{bmatrix},\ Z \triangleq \begin{bmatrix}1&0\\0&-1\end{bmatrix},\ Y \triangleq \imath \cdot XZ = \begin{bmatrix}0&-\imath\\ \imath&0\end{bmatrix};\ \imath \triangleq \sqrt{-1}.$$
- Bit-flip ($X|v\rangle = |v \oplus 1\rangle$) and phase-flip ($Z|v\rangle = (-1)^v|v\rangle$) anti-commute: $XZ = -ZX$.

m-qubit Heisenberg-Weyl Group $HW_N (N = 2^m)$: Operators $\imath^\kappa D(a,b)$, where
$$D(a,b) \triangleq X^{a_1}Z^{b_1} \otimes X^{a_2}Z^{b_2} \otimes \cdots \otimes X^{a_m}Z^{b_m} \in \mathbb{U}_{2^m},$$
$a = (a_1, \ldots, a_m), b = (b_1, \ldots, b_m) \in \mathbb{F}_2^m$, $\kappa \in \{0,1,2,3\}$ and $\mathbb{U}_N$ is the unitary group.

- Example: $D(a,b)|v\rangle = (-1)^{vb^T}|v+a\rangle \Rightarrow D(11010, 10110)|10101\rangle = |01111\rangle$.
$(XZ \otimes X \otimes Z \otimes XZ \otimes I_2)|10101\rangle = XZ|1\rangle \otimes X|0\rangle \otimes Z|1\rangle \otimes XZ|0\rangle \otimes I_2|1\rangle = |01111\rangle$.
- Symplectic Inner Product: For vectors $[a,b], [a',b'] \in \mathbb{F}_2^{2m}$, define
$$\langle [a,b],[a',b'] \rangle_s \triangleq a'b^T + b'a^T = [a,b]\ \Omega\ [a',b']^T \pmod 2,$$
where $\Omega = \begin{bmatrix}0 & I_m\\ I_m & 0\end{bmatrix}$ is the symplectic form in $\mathbb{F}_2^{2m}$.
- $D(a,b)D(a',b') = (-1)^{\langle [a,b],[a',b'] \rangle_s} D(a',b')D(a,b) \Rightarrow$ commute iff $\langle [a,b],[a',b'] \rangle_s = 0$.

Isomorphism γ: $HW_N / \langle \imath^\kappa I_N \rangle \to \mathbb{F}_2^{2m}$ defined as $\gamma(D(a,b)) \triangleq [a,b]$.

Clifford Group and Symplectic Matrices

$\mathsf{Cliff}_N \triangleq \mathcal{N}_{\mathbb{U}_N}(HW_N)$: all $g \in \mathbb{U}_N$ s.t. $gHW_Ng^\dagger = HW_N$ (normalizer of HW_N in $\mathbb{U}_N$).
$$\mathsf{Cliff}_N = \langle HW_N, H, P, \mathsf{CNOT} \text{ or } \mathsf{CZ} \rangle.$$

Gate	Unitary Matrix	Action on Paulis
Hadamard	$H \triangleq \frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}$	$HXH^\dagger = Z$ $HZH^\dagger = X$
Phase	$P \triangleq \begin{bmatrix}1&0\\0&\imath\end{bmatrix}$	$PXP^\dagger = Y$ $PZP^\dagger = Z$
Controlled-NOT	$\mathsf{CNOT}_{1\to2} \triangleq \begin{bmatrix}I_2&0\\0&X\end{bmatrix}$	$\mathsf{CNOT}_{1\to2}(X \otimes I_2)\mathsf{CNOT}_{1\to2}^\dagger = X \otimes X = X_1X_2$
Controlled-Z	$\mathsf{CZ}_{12} \triangleq \begin{bmatrix}I_2&0\\0&Z\end{bmatrix}$	$\mathsf{CZ}_{12}(X \otimes I_2)\mathsf{CZ}_{12}^\dagger = X \otimes Z = X_1Z_2$

Symplectic Representation: Define $E(a,b) \triangleq \imath^{ab^T}D(a,b)$. If $g \in \mathsf{Cliff}_N$ then
$$gE(a,b)g^\dagger = \pm E\left([a,b]F_g\right), \text{ where } F_g = \begin{bmatrix}A_g & B_g\\ C_g & D_g\end{bmatrix} \text{ is symplectic,}$$
i.e., $F_g \Omega F_g^T = \Omega$, and hence preserves inner products: $\langle [a,b],[a',b'] \rangle_s = \langle [a,b]F_g, [a',b']F_g \rangle_s$.

$$g = \mathsf{CZ}_{12}, F_g = \begin{bmatrix}1&0&0&1\\0&1&1&0\\&&1&0\\&&0&1\end{bmatrix} : g(X \otimes I_2)g^\dagger = gE(10,00)g^\dagger = E\left([10,00]F_g\right) = E(10,01) = X_1Z_2.$$

Homomorphism ϕ: $\mathsf{Cliff}_N \to \mathsf{Sp}(2m, \mathbb{F}_2)$ defined as $\phi(g) \triangleq F_g$, where $\mathsf{Sp}(2m,\mathbb{F}_2)$ is the binary symplectic group. Note that for $g \in HW_N$ we have $F_g = I_{2m}$.

Stabilizer Codes and Logical Pauli Operators

- k-dimensional Stabilizer: commutative subgroup $S \subset HW_N$ generated by linearly independent Hermitian operators $E(a_j, b_j) \triangleq \imath^{a b^T} D(a_j, b_j)$, $j = 1, \ldots, k$.
- $[\![m, m-k, d]\!]$ Stabilizer Code: The 2^{m-k} dimensional subspace $V(S)$ jointly fixed by all elements of the stabilizer S, i.e., $V(S) \triangleq \left\{ |\psi\rangle \in \mathbb{C}^N \mid g|\psi\rangle = |\psi\rangle\ \forall\ g \in S \right\}$.
- The $[\![6,4,2]\!]$ Code: $S \triangleq \langle \boldsymbol{g}^X \triangleq X^{\otimes 6} = E(r,0),\ \boldsymbol{g}^Z \triangleq Z^{\otimes 6} = E(0,r) \rangle$, $r = [111111]$.
$\boldsymbol{g}^X = X_1X_2X_3X_4X_5X_6 = X^{\otimes 6}$, $\boldsymbol{g}^Z = Z_1Z_2Z_3Z_4Z_5Z_6 = Z^{\otimes 6}$.
- CSS Construction: Let $\mathcal{C}$ be the $[6,5,2]$ single-parity check code with $m = 6, k = 1$. The dual $\mathcal{C}^\perp \subset \mathcal{C}$ is the $[6,1,6]$ repetition code. Two possible generator matrices for $\mathcal{C}/\mathcal{C}^\perp$:
$$G^X_{\mathcal{C}/\mathcal{C}^\perp} = \begin{bmatrix}1&1&0&0&0&0\\1&0&1&0&0&0\\1&0&0&1&0&0\\1&0&0&0&1&0\end{bmatrix} =: \begin{bmatrix}h_1\\h_2\\h_3\\h_4\end{bmatrix} \quad \text{or} \quad G^Z_{\mathcal{C}/\mathcal{C}^\perp} = \begin{bmatrix}0&1&0&0&0&1\\0&0&1&0&0&1\\0&0&0&1&0&1\\0&0&0&0&1&1\end{bmatrix} =: \begin{bmatrix}h'_1\\h'_2\\h'_3\\h'_4\end{bmatrix}.$$
- So if we have an $(m-2k)$-qubit state $|x\rangle_L$ then the CSS code will encode this into
$$|\psi_x\rangle \equiv \left|v + \mathcal{C}^\perp\right\rangle \triangleq \frac{1}{\sqrt{|\mathcal{C}^\perp|}} \sum_{c \in \mathcal{C}^\perp} \left|c + x \cdot G^X_{\mathcal{C}/\mathcal{C}^\perp}\right\rangle = \frac{1}{\sqrt{|\mathcal{C}^\perp|}} \sum_{c \in \mathcal{C}^\perp} \left|c + \sum_{j=1}^{m-2k} x_j h_j\right\rangle.$$

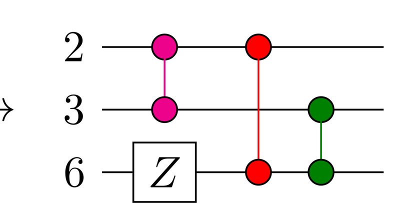
Synthesizing Logical Paulis: Find physical realizations $\bar{X}_j, \bar{Z}_j$ for X_j^L, Z_j^L resp. s.t.
- $\bar{X}_j, \bar{Z}_j \in \mathbb{U}_N$ act on $|\psi_x\rangle$ and realize action of X_j^L, Z_j^L (resp.) on $|x\rangle_L$.
- $\bar{X}_j, \bar{Z}_j$ satisfy the commutation relations: $\bar{X}_i\bar{Z}_j = \begin{cases} -\bar{Z}_j\bar{X}_i & \text{if } i = j,\\ \bar{Z}_j\bar{X}_i & \text{if } i \neq j\end{cases}$.
- $\bar{X}_j, \bar{Z}_j$ normalize the stabilizer S (preserve the code subspace) so that, for $g \in S\ \exists\ g' \in S$ s.t.
$$\bar{X}_j|\psi_x\rangle = \bar{X}_jg|\psi_x\rangle = (\bar{X}_jg\bar{X}_j^\dagger)\bar{X}_j|\psi_x\rangle = g'\bar{X}_j|\psi_x\rangle \text{ (similarly for } \bar{Z}_j).$$

- For the $[\![6,4,2]\!]$ CSS code the logical Pauli operators are given by
$$\begin{array}{l|l} \bar{X}_1 \triangleq D(h_1, 0) = X_1X_2 & \bar{Z}_1 \triangleq D(0, h'_1) = Z_2Z_6\\ \bar{X}_2 \triangleq D(h_2, 0) = X_1X_3 & \bar{Z}_2 \triangleq D(0, h'_2) = Z_3Z_6\\ \bar{X}_3 \triangleq D(h_3, 0) = X_1X_4 & \bar{Z}_3 \triangleq D(0, h'_3) = Z_4Z_6\\ \bar{X}_4 \triangleq D(h_4, 0) = X_1X_5 & \bar{Z}_4 \triangleq D(0, h'_4) = Z_5Z_6 \end{array}.$$

Synthesis of Logical Clifford Operators

- Conditions similar to that for logical Paulis, but commutation constraints $\bar{X}_i\bar{Z}_j$ replaced by conjugation constraints with logical Paulis, i.e., $\bar{g}$ must satisfy $\bar{g}\bar{X}_j\bar{g}^\dagger = \bar{h}$ if $g^L X_j^L (g^L)^\dagger = h^L \in HW_{2^{m-k}}$ and $\bar{g}\bar{Z}_j\bar{g}^\dagger = \bar{h}'$ if $g^L Z_j^L (g^L)^\dagger = (h')^L \in HW_{2^{m-k}}$.
- Synthesizing $g^L = \mathsf{CZ}_{12}^L$ for the $[\![6,4,2]\!]$ code: Find physical operator $\bar{g} = \overline{\mathsf{CZ}}_{12}$ such that
$$\overline{\mathsf{CZ}}_{12}\bar{X}_j\overline{\mathsf{CZ}}_{12}^\dagger \triangleq \begin{cases} \bar{X}_1\bar{Z}_2 & \text{if } j = 1,\\ \bar{Z}_1\bar{X}_2 & \text{if } j = 2,\\ \bar{X}_j & \text{if } j \neq 1,2 \end{cases},$$
$$\overline{\mathsf{CZ}}_{12}\bar{Z}_j\overline{\mathsf{CZ}}_{12}^\dagger \triangleq \bar{Z}_j\ \forall\ j = 1,2,3,4.$$
- Using the symplectic representation translate these into constraints on the desired symplectic matrix for $\overline{\mathsf{CZ}}_{12}$ as follows:
$$\bar{X}_1 = X_1X_2 \xmapsto{\overline{\mathsf{CZ}}_{12}} X_1X_2Z_3Z_6 \overset{\gamma,\phi}{\Longleftrightarrow} [110000, 000000]F_{\overline{\mathsf{CZ}}_{12}} = [110000, 001001]$$
$$\bar{X}_2 = X_1X_3 \xmapsto{\overline{\mathsf{CZ}}_{12}} X_1X_3Z_2Z_6 \overset{\gamma,\phi}{\Longleftrightarrow} [101000, 000000]F_{\overline{\mathsf{CZ}}_{12}} = [101000, 010001]$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$\boldsymbol{g}^X = X^{\otimes 6} \xmapsto{\overline{\mathsf{CZ}}_{12}} X^{\otimes 6} = X_1X_2\cdots X_6 \overset{\gamma,\phi}{\Longleftrightarrow} [111111, 000000]F_{\overline{\mathsf{CZ}}_{12}} = [111111, 000000]$$
$$\boldsymbol{g}^Z = Z^{\otimes 6} \xmapsto{\overline{\mathsf{CZ}}_{12}} Z^{\otimes 6} = Z_1Z_2\cdots Z_6 \overset{\gamma,\phi}{\Longleftrightarrow} [000000, 111111]F_{\overline{\mathsf{CZ}}_{12}} = [000000, 111111].$$

One possible solution
$$\Rightarrow F_{\overline{\mathrm{CZ}}_{12}} = \begin{bmatrix}I_6 & B\\ 0 & I_6\end{bmatrix}, B = \begin{bmatrix}0&0&0&0&0&0\\0&0&1&0&0&1\\0&1&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&1&1&0&0&0\end{bmatrix} \longleftrightarrow$$



$$\Downarrow$$
$$\overline{\mathrm{CZ}}_{12} = \mathrm{diag}\left(\imath^{vBv^T}\right)Z_6$$
$$= \mathrm{CZ}_{36}\mathrm{CZ}_{26}\mathrm{CZ}_{23}Z_6$$

Not captured in $F_{\overline{\mathrm{CZ}}_{12}}$ – added to fix signs

- We solve such symplectic systems of linear equations using symplectic transvections.
- Definition: Given a row vector $h \in \mathbb{F}_2^{2m}$, the transvection $Z_h : \mathbb{F}_2^{2m} \to \mathbb{F}_2^{2m}$ is
$$Z_h(x) \triangleq x + \langle x, h \rangle_s h \iff F_h \triangleq I_{2m} + \Omega h^T h \in \mathsf{Sp}(2m, \mathbb{F}_2).$$

Summary of Our Results

- Given a sequence of binary vectors $x_i, y_i,\ i = 1, \ldots, t \leq 2m$ s.t. $\langle x_i, x_j \rangle_s = \langle y_i, y_j \rangle_s$, there exists a symplectic matrix F, expressible as a product of at most $2t$ transvections, s.t. $x_iF = y_i$. We also given an explicit algorithm to compute such a matrix.
- Let $\{(u_a, v_a),\ a \in \{1, \ldots, m\}\}$ be a collection of pairs of binary vectors that form a symplectic basis for $\mathbb{F}_2^{2m}$, where $u_a, v_a \in \mathbb{F}_2^{2m}$. Consider a system of linear equations $u_iF = u'_i, v_jF = v'_j$, where $i \in \mathcal{I} \subseteq \{1, \ldots, m\}, j \in \mathcal{J} \subseteq \{1, \ldots, m\}$ and $F \in \mathsf{Sp}(2m, \mathbb{F}_2)$. Let $\alpha \triangleq |\bar{\mathcal{I}}| + |\bar{\mathcal{J}}|$. Then there are $2^{\alpha(\alpha+1)/2}$ solutions F to the system. We also give an algorithm to efficiently enumerate them.
- For an $[\![m, m-k]\!]$ stabilizer code, the number of symplectic solutions for each logical Clifford operator is $2^{k(k+1)/2}$. We give a complete algorithm to determine all solutions and their circuits.
- For an $[\![m, m-k]\!]$ stabilizer code with stabilizer S, each physical realization of a given logical Clifford operator that normalizes S can be converted into a circuit that centralizes S, i.e., commutes with every element of S, while realizing the same logical operation.

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