On Optimality of CSS Codes for Transversal T

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Overview

1. Motivation and Related Work
2. Logical Clifford Synthesis (LCS) for Stabilizer Codes
3. Quadratic Form Diagonal (QFD) Gates
4. Stabilizer Codes Matched to QFD Gates

Goal: Logical Operations from Physical T Gates

QECC: Quantum Error Correcting Code

$$
\begin{tikzcd}
|x\rangle_L \arrow[rr, "\text{logical operation}"] \arrow[d, "\text{encode}"', "{[\![n,k,d]\!]}\ \text{QECC}"'] & & |\tilde{x}\rangle_L \\
|\psi_x\rangle \arrow[rr, "\text{relevant physical operation}"] & & |\psi_{\tilde{x}}\rangle \arrow[u, "{[\![n,k,d]\!]}\ \text{QECC decode}"']
\end{tikzcd}
$$

Need to translate for the $[\![n, k, d]\!]$ QECC

What stabilizer structure is required so that
the physical application of T gates preserves the code subspace?

Literature related to Magic State Distillation

- [GC99]: Universal computation via quantum teleportation
- [BK05]: Ideal Clifford gates and noisy ancillas
- [BH12]: Distillation with low overhead, triorthogonal codes
- [KB15]: Transversal gates on color codes
- [CH17]: Quasitransversality
- [HH17]: Generalized triorthogonality
- [Haa+17]: Distillation with optimal asymptotic input count
- [KT18]: Punctured polar codes from decreasing monomial codes
- [VB19]: Quantum Pin Codes
- ...

In this talk ...

1. Motivation and Related Work
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4. Stabilizer Codes Matched to QFD Gates

Overview

Pauli Group, Clifford Group and Symplectic Matrices

Heisenberg-Weyl Group $HW_N := \{\imath^{\kappa} E(a,b) \colon a, b \in \mathbb{F}_2^n, \kappa \in \mathbb{Z}_4\}$ $(\imath = \sqrt{-1})$

$E(a,b), a, b \in \mathbb{F}_2^n$: $\underbrace{X \otimes Z \otimes Y}_{n=3 \text{ qubits}} = E(\underbrace{1\,0\,1}_{a}, \underbrace{0\,1\,1}_{b})$

$a =$	1	0	1
$b =$	0	1	1
$E(a,b) =$	X_1	Z_2	Y_3

Symplectic Inner Product: $\langle [a,b],[c,d] \rangle_s := [a,b]\,\Omega\,[c,d]^T, \Omega := \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$

Clifford Group: All unitaries that map Paulis to Paulis under conjugation

Symplectic Matrices: If $g \in \text{Cliff}_N$ (Cliffords on $n = \log_2 N$ qubits) then

$$g\, E(a,b)\, g^{\dagger} = \pm E\left([a,b]F_g\right), \text{ where } F_g \Omega F_g^T = \Omega$$

$F_g \in \mathbb{F}_2^{2n \times 2n}$ is symplectic: preserves the symplectic inner product

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Stabilizer Codes ($N = 2^n$)

r-dimensional Stabilizer: Generated by r commuting Pauli operators:

$$S = \langle \epsilon_i E(a_i, b_i)\,;\ i = 1, \ldots, r \rangle,\ \ \epsilon_i \in \{\pm 1\},\ \ -I_N \notin S$$

$[\![n, k = n - r, d]\!]$ Stabilizer Code: The 2^k dimensional subspace, $V(S)$, jointly fixed by all elements of S

$$V(S) \coloneqq \left\{ |\psi\rangle \in \mathbb{C}^N : g\,|\psi\rangle = |\psi\rangle \ \text{ for all } g \in S \right\}$$

Example:

$[\![6, 4, 2]\!]$ CSS Code: $S \coloneqq \langle X^{\otimes 6} = E(a, 0), Z^{\otimes 6} = E(0, a) \rangle$, $a \coloneqq [\,1\,1\,1\,1\,1\,1\,]$

Generator Matrix: $G_S = \left[\begin{array}{cccccc|cccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

Logical Clifford Synthesis (LCS)

Synthesis of CZ_{12}^{L} for $[\![6,4,2]\!]$ Code

Logical Clifford
$g^L = \mathrm{CZ}_{12}^{L}$

Stabilizer

$$S = \langle X^{\otimes 6}, Z^{\otimes 6} \rangle$$
$$= \langle E(\underline{1}, \underline{0}), E(\underline{0}, \underline{1}) \rangle$$

LCS
Algorithm

All* circuits $\bar{g} \in \mathrm{Cliff}_{2^6}$
that realize g^L and fix S
(stabilizer freedom ignored)
* up to an equivalence class

Logical Pauli Operators

$$\bar{X}_j = X_1 X_{j+1} = E(\underline{e}_1 + \underline{e}_{j+1}, \underline{0})$$
$$\bar{Z}_j = Z_{j+1} Z_6 = E(\underline{0}, \underline{e}_{j+1} + \underline{e}_6)$$
$$(j = 1, 2, 3, 4)$$

Implementation: https://github.com/nrenga/symplectic-arxiv18a

Paper: https://arxiv.org/abs/1907.00310

Generalizing the LCS Algorithm

Main Ideas in LCS: Use $\bar{g}\, E(a,b)\, \bar{g}^\dagger = \pm E([a,b]F_{\bar{g}})$

- Implied logical action: $g^L X_j^L (g^L)^\dagger, g^L Z_j^L (g^L)^\dagger \Rightarrow \bar{g}\bar{X}_j\bar{g}^\dagger, \bar{g}\bar{Z}_j\bar{g}^\dagger$
- $\bar{g} \in \text{Cliff}_N$ must map stabilizers to stabilizers under conjugation
- Translate conjugation relations into symplectic constraints on $F_{\bar{g}}$

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Issues in generalizing to non-Clifford gates:

- Translating logical non-Cliffords to physical non-Cliffords is hard: there is no clear symplectic connection
- Physical operation is not Clifford $\Rightarrow$ does not necessarily map stabilizers to stabilizers under conjugation

Overview

Gates for Universal Computation

$\text{Cliff}_N = \langle H, P, CZ \text{ or CNOT (on all qubits)}\rangle \longleftarrow$ Not universal!

Gate	Unitary Matrix	Action on Paulis	Symplectic Matrix
Hadamard	$H := \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$	$HXH^\dagger = Z$ $HZH^\dagger = X$	$F_H = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$
Phase	$P := \begin{bmatrix}1 & 0\\ 0 & \imath\end{bmatrix} = \sqrt{Z}$	$PXP^\dagger = Y$ $PZP^\dagger = Z$	$F_P = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$
Phase (P), Ctrl-Z (CZ)	$t_R := \sum_{v \in \mathbb{F}_2^n} \imath^{vRv^T} \lvert v\rangle\langle v\rvert$ (vRv^T computed over $\mathbb{Z}$)	CZ: $X_a \mapsto X_a Z_b$ $Z_a \mapsto Z_a$	$T_R = \begin{bmatrix}I_n & R\\ 0 & I_n\end{bmatrix}$ with R symmetric
T	$T := \begin{bmatrix}1 & 0\\ 0 & e^{\imath\pi/4}\end{bmatrix} = \sqrt{P}$	$TXT^\dagger = \frac{X+Y}{\sqrt{2}}$ $TZT^\dagger = Z$	?

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Quadratic Form Diagonal (QFD) Gates

S.X. Cui, D. Gottesman and A. Krishna, Phys. Rev. A, 2017

If $U \in \mathcal{C}^{(\ell)}$ is diagonal, then all entries are 2^{ℓ}-th roots of unity.

Examples:

$P \in \mathcal{C}^{(2)} \leftrightarrow R = [\,1\,]$ over $\mathbb{Z}_4$

$$\mathcal{C}^{(2)}:\ t_R = \sum_{v \in \mathbb{F}_2^n} \imath^{vRv^T} |v\rangle \langle v|$$

R is $n \times n$ symmetric with entries in $\mathbb{Z}_2$

$$\mathcal{C}^{(1)} = HW_N$$

$\mathrm{CZ} = \mathrm{diag}\,[1, 1, 1, -1] \in \mathcal{C}^{(2)}$

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$\mathcal{C}^{(\ell)}: \tau_R^{(\ell)} = \sum_{v \in \mathbb{F}_2^n} \xi^{vRv^T} |v\rangle \langle v|$

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$\leftrightarrow R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ over $\mathbb{Z}_4$

$\mathrm{CP} = \mathrm{diag}\,[1,1,1,\imath] \in \mathcal{C}^{(3)}$

$\leftrightarrow R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ over $\mathbb{Z}_8$

Diagonal Recursion for QFD Gates

Recollect: Clifford g acts as $g\,E(a,b)\,g^{\dagger} = \pm E([a,b]F_g)$, F_g symplectic.

How do QFD gates act on Pauli matrices under conjugation?

$$\tau_R^{(\ell)} E(a,b) \left(\tau_R^{(\ell)}\right)^{\dagger} = \phi(R,a,b,\ell) \cdot E\left([a,b]\begin{bmatrix} I_n & R \\ 0 & I_n \end{bmatrix}\right) \cdot \tau_{\tilde{R}(R,a,\ell)}^{(\ell-1)}$$

$\phi(R,a,b,\ell)$: Deterministic global phase

$\tilde{R}(R,a,\ell)$: New symmetric matrix with entries in $\mathbb{Z}_{2^{\ell-1}}$

All 1- and 2-local diagonal gates in $\mathcal{C}^{(\ell)}$ are QFD for any $\ell \geq 1$

Mølmer-Sørensen gates $\mathrm{MS}(\frac{\pi}{2^{\ell}})$ are QFD up to Hadamards

For details see: `https://arxiv.org/abs/1902.04022`

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Issues in generalizing to $\mathcal{C}^{(\ell)}, \ell > 2$:

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Issues in generalizing to $\mathcal{C}^{(\ell)}, \ell > 2$:

- Translating logical non-Cliffords to physical non-Cliffords is hard: there is no clear symplectic connection. QFD Gates!
- Physical operation is not Clifford $\Rightarrow$ does not necessarily map stabilizers to stabilizers. Preserve projector onto code subspace!

Reverse LCS Strategy for Physical T Gates

QECC: Quantum Error Correcting Code

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Transversal T as a Logical Operator

Question: When is transversal T a logical operator for a stabilizer code?
What is the induced logical operation?

Stabilizer: $S = \langle \epsilon_i E(a_i, b_i)\,;\, i = 1, 2, \ldots, r \rangle,\ \epsilon_i \in \{\pm 1\}$

Code Projector: $\Pi_s = \prod_{i=1}^{r} \dfrac{I_N + \epsilon_i E(a_i, b_i)}{2} = \dfrac{1}{2^r} \sum_{a,b \in S} \epsilon_{a,b} E(a, b)$

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Calculation: Using QFD recursion we get, [hard for general QFD!]

$$T^{\otimes n} E(a, b) \left(T^{\otimes n}\right)^{\dagger} = \frac{1}{2^{\mathrm{wt}_H(a)/2}} \sum_{y \preceq a} (-1)^{by^T} E(a, b \oplus y)$$

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$T^{\otimes n}$ is a logical operator iff $T^{\otimes n} \Pi_S (T^{\otimes n})^{\dagger} = \Pi_S$: [also hard in general!]

$$\frac{1}{2^r} \sum_{a,b \in S} \frac{\epsilon_{a,b}}{2^{\mathrm{wt}_H(a)/2}} \sum_{y \preceq a} (-1)^{b y^T} E(a, b \oplus y) = \frac{1}{2^r} \sum_{a,b \in S} \epsilon_{a,b} E(a, b)$$

Transversal T Example: $[\![6,2,2]\!]$ CSS Code

Theorem: $T^{\otimes n}$ commutes with the code projector if and only if:

1. For each $\epsilon_{a,b} E(a,b) \in S$ with $a \neq 0$, we have $w_H(a) \equiv 0 \pmod 2$.
2. $Z_S := \{$ binary vectors z that produce Z-type stabilizers $\epsilon_z E(0,z)$ $\}$. For any $\epsilon_{a,b} E(a,b) \in S$ with $a \neq 0$, Z_S contains a dimension $w_H(a)/2$ self-dual code Z_a that is supported on $a \in \mathbb{F}_2^n$.
3. For all $\epsilon_{a,b} E(a,b) \in S$, for all $z \in Z_a$, we have $\imath^{w_H(z)} E(0,z) \in S$.

$$G_S = \left[\begin{array}{cccccc|cccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} & G_1^{\perp} \\ \hline G_2 & \end{array}\right]$$

$$S = \langle X^{\otimes 6}, -Z_1 Z_2, -Z_3 Z_4, -Z_5 Z_6 \rangle$$

CSS-T Codes and Two Corollaries

Transversal T preserves the code subspace of a $\mathrm{CSS}(X, C_2; Z, C_1^{\perp})$ code iff:

1. For all $x \in C_2$, $w_H(x)$ is even.
2. For each $x \in C_2$, $C_1^{\perp}$ consists of a dimension $w_H(x)/2$ self-dual code Z_x supported on x. New classical coding problem!
3. For all $x \in C_2$ and for each $z \in Z_x$, $\imath^{w_H(z)} E(0, z) \in S$.

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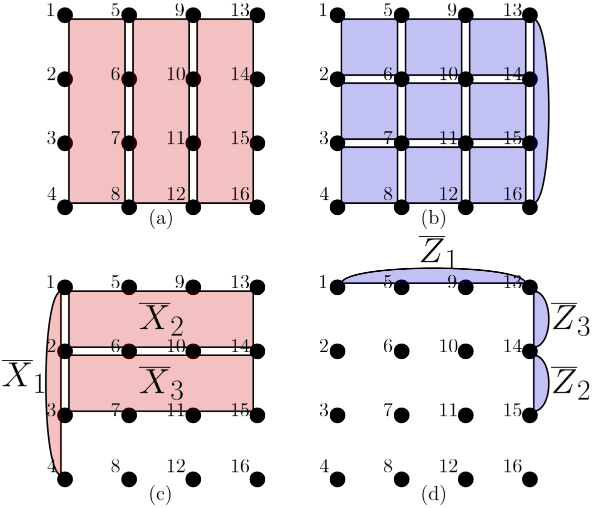
Two Corollaries: (Non-degenerate $\Rightarrow$ each stabilizer has weight $\geq d$)

1. Triorthogonal codes form the only CSS family with $T^{\otimes n} \equiv \bar{T}^{\otimes k}$.
2. For each $[\![n, k, d]\!]$ non-degenerate stabilizer code that supports transversal T, there is an $[\![n, k, d]\!]$ CSS-T code that does too.

$[\![16,3,2]\!]$ CSS-T Code: Logical CCZ

Can equivalently be constructed using decreasing monomial codes!



Procedure for General QFD Gates

Pauli Coeff.: $c_{\tilde{R},x}^{(\ell-1)} \coloneqq \mathrm{Tr}\left[\tau_{\tilde{R}}^{(\ell-1)} \cdot \frac{1}{\sqrt{2^n}} E(0,x)\right] = \frac{1}{\sqrt{2^n}} \sum_{v \in \mathbb{F}_2^n} (-1)^{vx^T} \xi^{v\tilde{R}v^T \bmod 2^{\ell-1}}$

$$\Rightarrow \tau_{\tilde{R}}^{(\ell-1)} = \frac{1}{\sqrt{2^n}} \sum_{x \in \mathbb{F}_2^n} c_{\tilde{R},x}^{(\ell-1)} \cdot E(0,x) \longleftarrow \text{Expand in Pauli basis}$$

Procedure for General QFD Gates

$$\text{Pauli Coeff.: } c_{\tilde{R},x}^{(\ell-1)} := \operatorname{Tr}\left[\tau_{\tilde{R}}^{(\ell-1)} \cdot \frac{1}{\sqrt{2^n}} E(0,x)\right] = \frac{1}{\sqrt{2^n}} \sum_{v \in \mathbb{F}_2^n} (-1)^{vx^T} \xi^{v\tilde{R}v^T \bmod 2^{\ell-1}}$$

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$$\begin{aligned} \Rightarrow \tau_R^{(\ell)} E(a,b) (\tau_R^{(\ell)})^{\dagger} &= \phi(R,a,b,\ell) \cdot E(a, b + aR) \cdot \tau_{\tilde{R}}^{(\ell-1)} \longleftarrow \text{ QFD formula} \\ &= \frac{1}{\sqrt{2^n}} \phi(R,a,b,\ell) \sum_{x \in \mathbb{F}_2^n} c_{\tilde{R},x}^{(\ell-1)} \imath^{-ax^T} E(a, b + aR + x). \end{aligned}$$

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Transversal Z-Rotations: $\tau_R^{(\ell)} = \exp\left(\frac{-\imath\pi}{2^\ell} Z\right)^{\otimes n} \in \mathcal{C}^{(\ell)}$ with $R = I_n$ satisfies

$$\tau_{I_n}^{(\ell)} E(a,b) (\tau_{I_n}^{(\ell)})^\dagger = \left(\cos\frac{2\pi}{2^\ell}\right)^{w_H(a)} \sum_{y \preceq a} \left(\tan\frac{2\pi}{2^\ell}\right)^{w_H(y)} (-1)^{by^T} E(a, b \oplus y).$$

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  - Paper: Extensions to finer angle Z-rotations and Reed-Muller codes
- Full extension to general Z-rotations and associated logical gates?
- Given logical diagonal gate, what code realizes it via transversal $Z^{2^{-\ell}}$?
- Universal QC: State distillation, gauge fixing, combine with flags?

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Thank you!

For details see: http://arxiv.org/abs/1910.09333

QFD Gates: http://arxiv.org/abs/1902.04022

LCS Algorithm: http://arxiv.org/abs/1907.00310
Code at https://github.com/nrenga/symplectic-arxiv18a

Any feedback is much appreciated.

A (near-term?) quantum communication advantage: visit poster #289 !