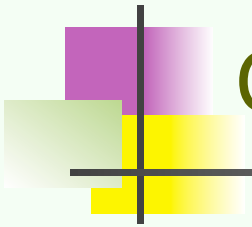
Constraint Satisfaction Problems

Chapter 5

Section 1 – 3



Constraint satisfaction problems (CSPs)

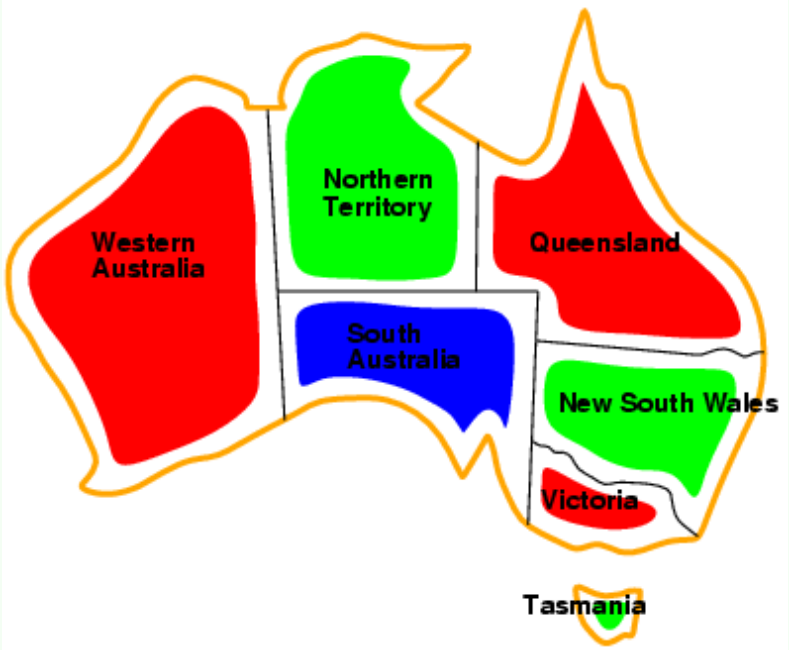
- CSP:
  - state is defined by variables X_i with values from domain D_i
  - goal test is a set of constraints specifying allowable combinations of values for subsets of variables

- Allows useful general-purpose algorithms with more power than standard search algorithms

Example: Map-Coloring

- Variables *WA, NT, Q, NSW, V, SA, T*
- Domains D_i = {red,green,blue}
- Constraints: adjacent regions must have different colors
- e.g., WA ≠ NT

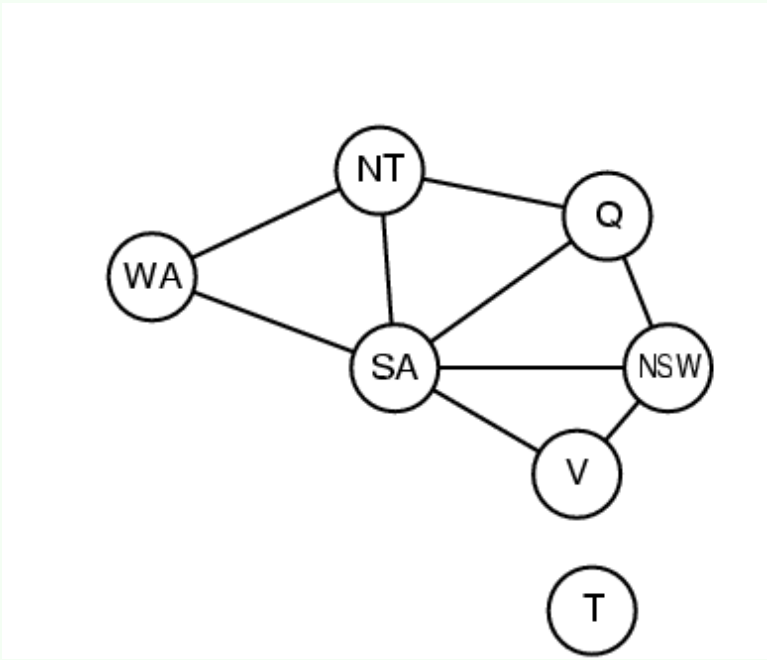
Example: Map-Coloring

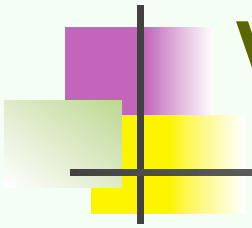


- Solutions are complete and consistent assignments, e.g., WA = red, NT = green,Q = red,NSW = green,V = red,SA = blue,T = green

Constraint graph

- Binary CSP: each constraint relates two variables
- Constraint graph: nodes are variables, arcs are constraints

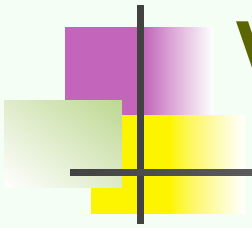




Varieties of CSPs

- Discrete variables
  - finite domains:
    - *n* variables, domain size *d* → $O(d^n)$ complete assignments
    - e.g., 3-SAT (NP-complete)
  - infinite domains:
    - integers, strings, etc.
    - e.g., job scheduling, variables are start/end days for each job:
      $StartJob_1 + 5 \leq StartJob_3$

- Continuous variables
  - linear objective & constraints solvable in polynomial time by linear programming
  - There are very good, off-the-shelves, methods for convex optimization problems.

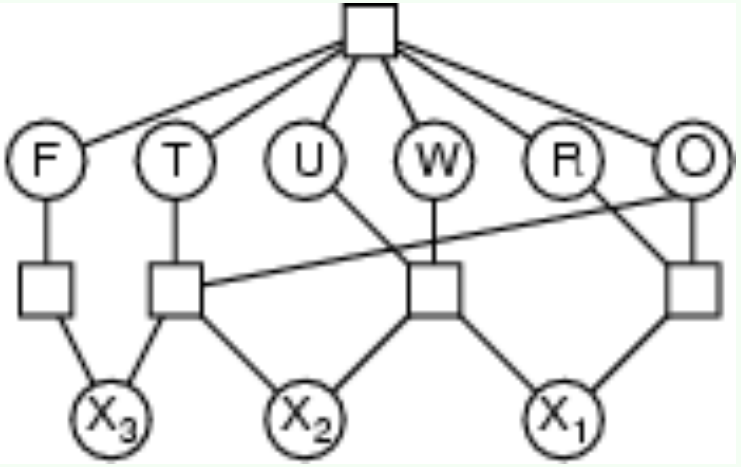


Varieties of constraints

- Unary constraints involve a single variable,
  - e.g., SA ≠ green
- Binary constraints involve pairs of variables,
  - e.g., SA ≠ WA
- Higher-order constraints involve 3 or more variables,
  - e.g., SA ≠ WA ≠ NT

Example: Cryptarithmetic

```
  T W O
+ T W O
-------
F O U R
```



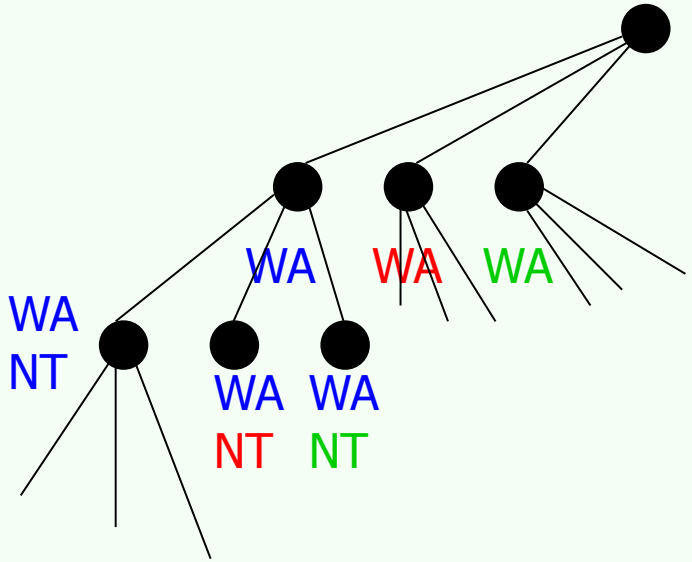
- Variables: *F T U W R O* $X_1\ X_2\ X_3$
- Domains: {*0,1,2,3,4,5,6,7,8,9*} {0,1}
- Constraints: *Alldiff (F,T,U,W,R,O)*
  - $O + O = R + 10 \cdot X_1$
  - $X_1 + W + W = U + 10 \cdot X_2$
  - $X_2 + T + T = O + 10 \cdot X_3$
  - $X_3 = F,\ T \neq 0,\ F \neq 0$

Backtracking (Depth-First) search

- Special property of CSPs: They are commutative:
  This means: the order in which we assign variables does not matter.

NT = WA
WA = NT

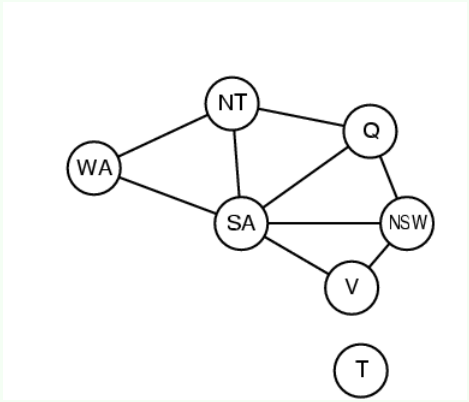
- Search tree: First order variables, then assign them values one-by-one.



D

D^2

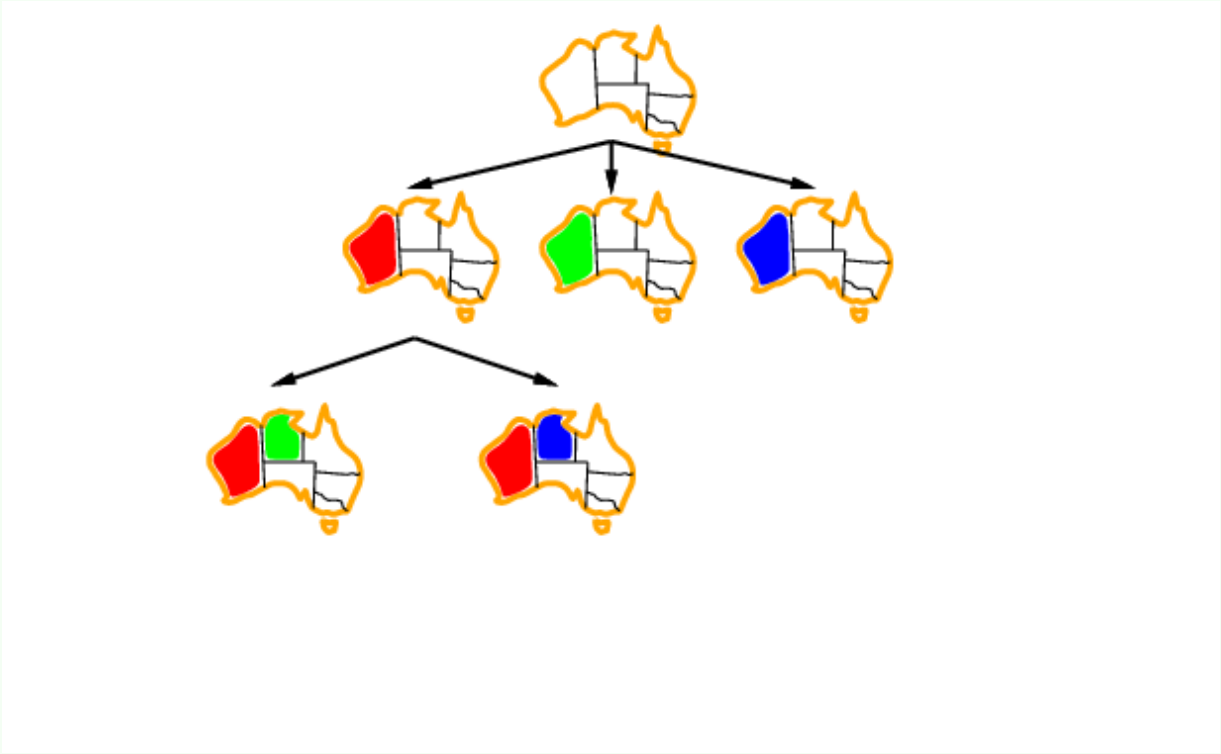
D^N



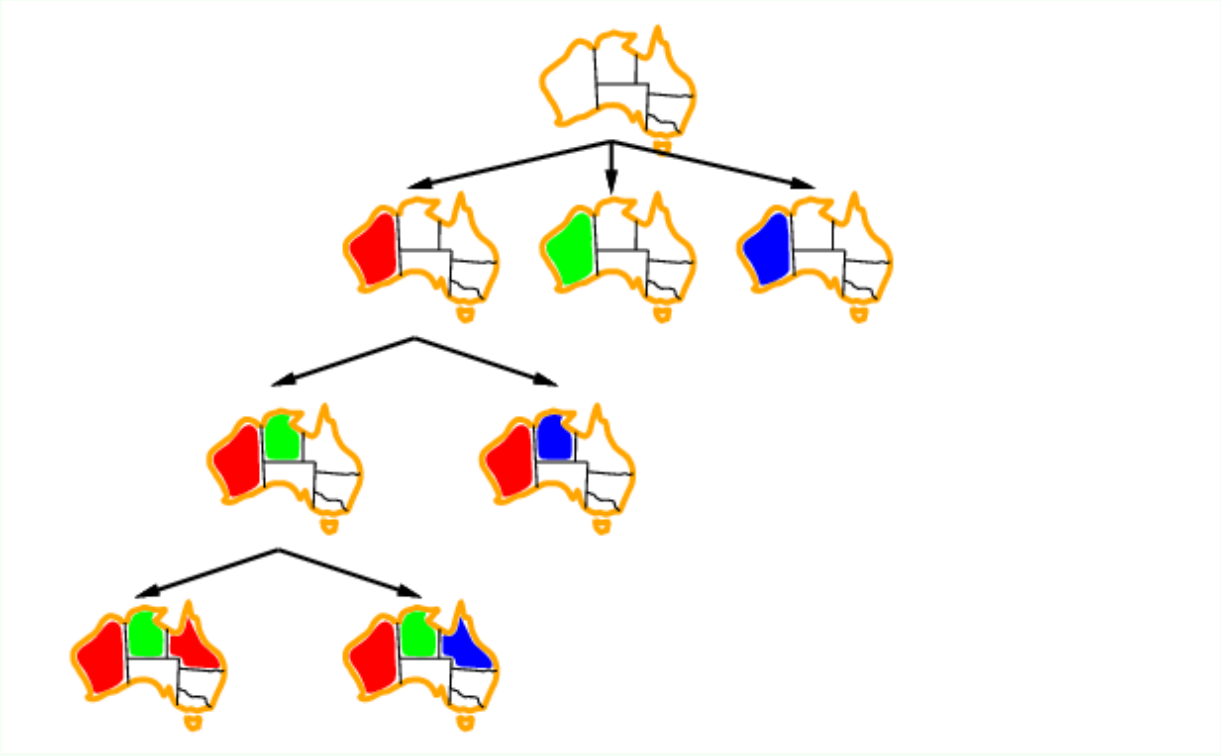
Backtracking example

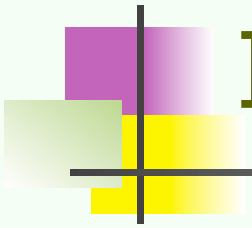
Backtracking example

Backtracking example



Backtracking example





Improving backtracking efficiency

- General-purpose methods can give huge gains in speed:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?

  - We'll discuss heuristics for all these questions in the following.

Which variable should be assigned next? →minimum remaining values heuristic

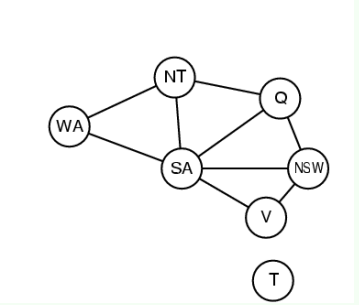
- Most constrained variable:

  choose the variable with the fewest legal values

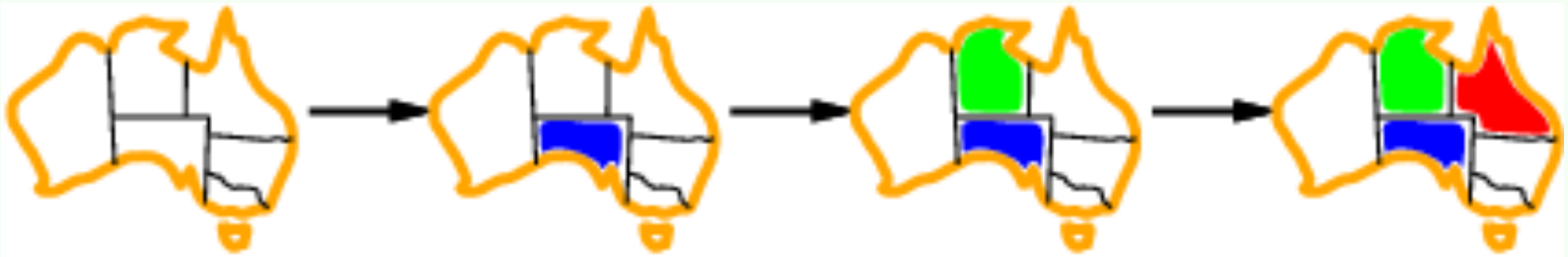
- a.k.a. minimum remaining values (MRV) heuristic

Which variable should be assigned next?
→ degree heuristic

- Tie-breaker among most constrained variables

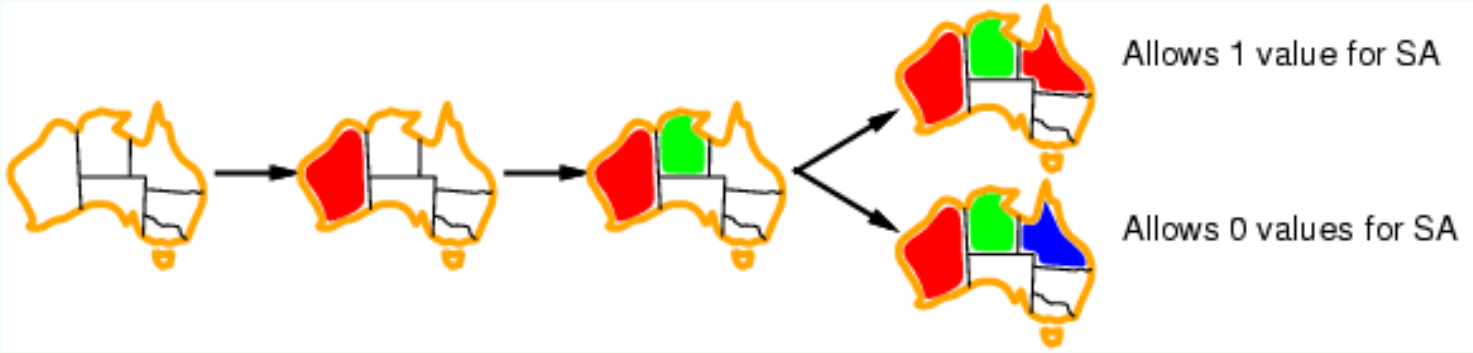


- Most constrain*ing* variable:
  - choose the variable with the most constraints on remaining variables (most edges in graph)



In what order should its values be tried?
→ least constraining value heuristic

- *Given a variable*, choose the least constraining value:
  - the one that rules out the fewest values in the remaining variables



- Leaves maximal flexibility for a solution.
- Combining these heuristics makes 1000 queens feasible

Rationale for MRV, DH, LCV

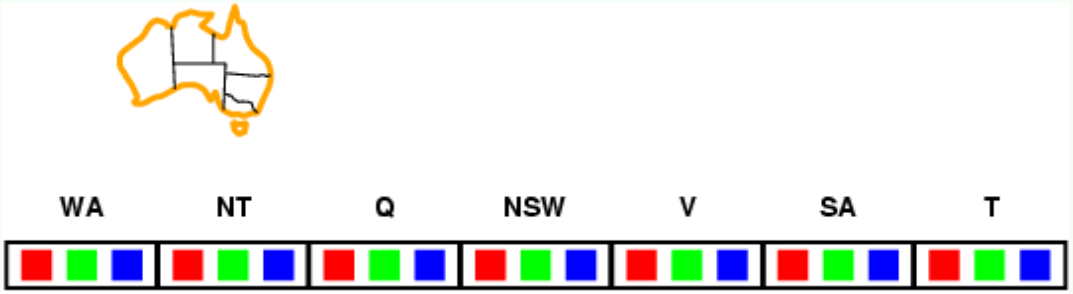
- In all cases we want to enter the most promising branch, but we also want to detect inevitable failure as soon as possible.

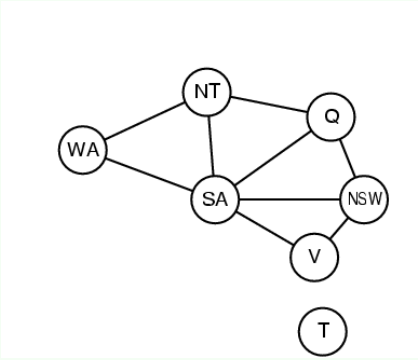
- MRV+DH: the variable that is most likely to cause failure in a branch is assigned first. The variable must be assigned at some point, so if it is doomed to fail, we'd better found out soon. E.g X1-X2-X3, values 0,1, neighbors cannot be the same.

- LCV: tries to avoid failure by assigning values that leave maximal flexibility for the remaining variables. We want our search to succeed as soon as possible, so given some ordering, we want to find the successful branch.

Can we detect inevitable failure early? → forward checking

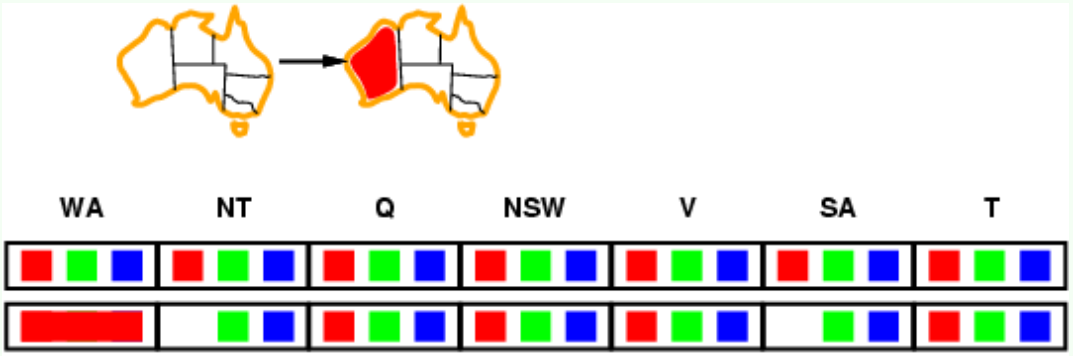
- Idea:
  - Keep track of remaining legal values for unassigned variables that are connected to current variable.
  - Terminate search when any variable has no legal values

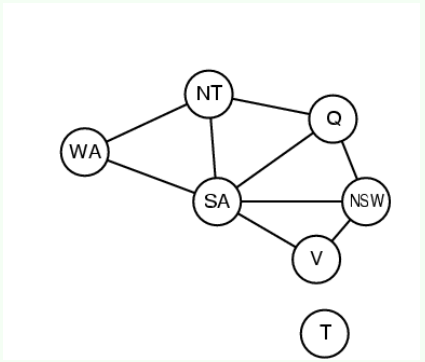




Forward checking

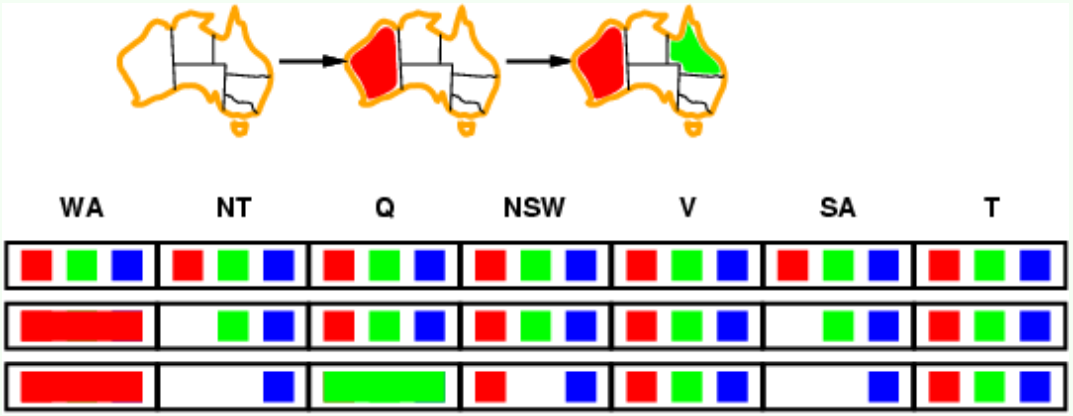
- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values

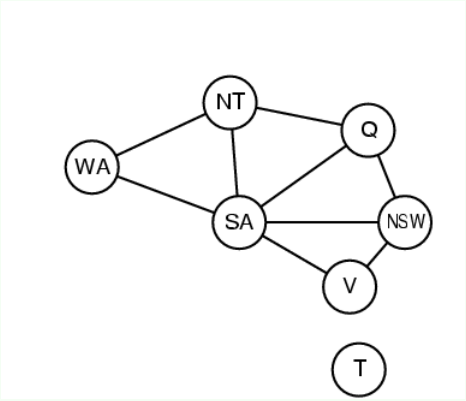




Forward checking

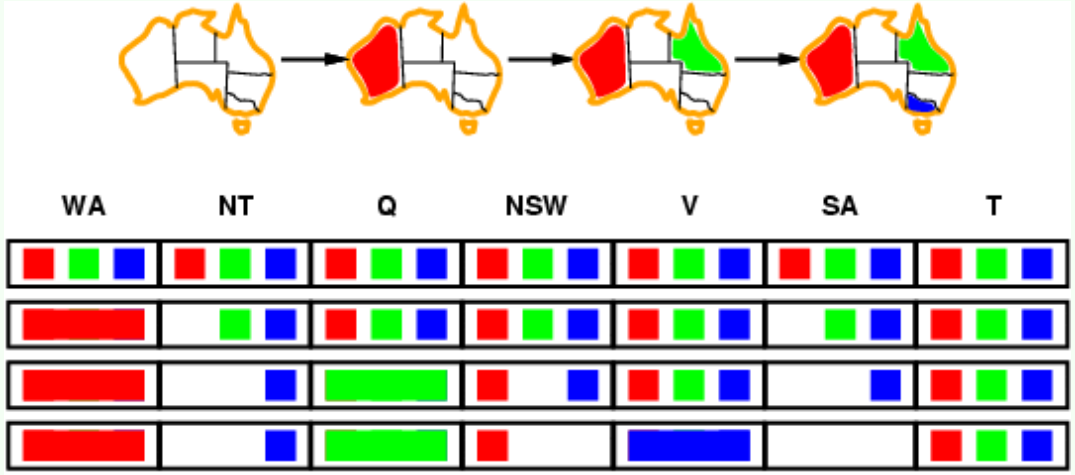
- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values

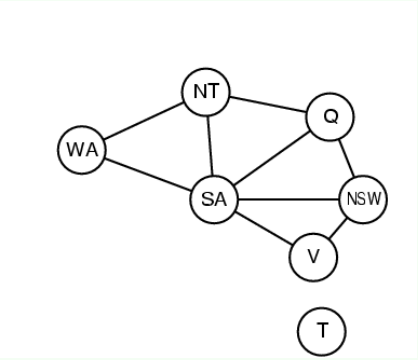




Forward checking

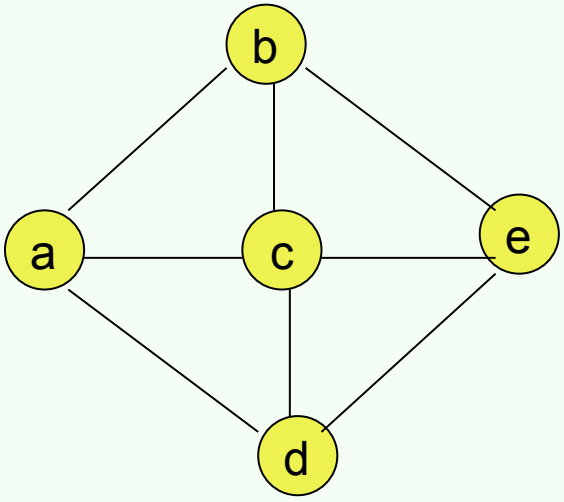
- Idea:
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values





2) Consider the constraint graph on the right.
The domain for every variable is [1,2,3,4].
There are 2 unary constraints:
- variable “a” cannot take values 3 and 4.
- variable “b” cannot take value 4.
There are 8 binary constraints stating that variables connected by an edge cannot have the same value.

Find a solution for this CSP by using the following heuristics: minimum value heuristic, degree heuristic, forward checking. Explain each step of your answer.

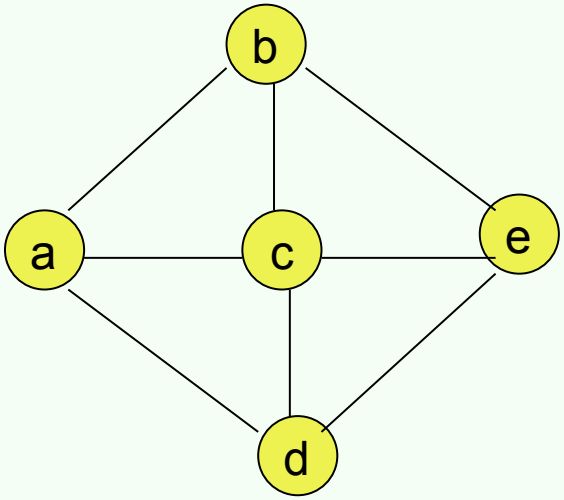


CONSTRAINT GRAPH

2) Consider the constraint graph on the right.
The domain for every variable is [1,2,3,4].
There are 2 unary constraints:
- variable “a” cannot take values 3 and 4.
- variable “b” cannot take value 4.
There are 8 binary constraints stating that variables connected by an edge cannot have the same value.

Find a solution for this CSP by using the following heuristics: minimum value heuristic, degree heuristic, forward checking. Explain each step of your answer.

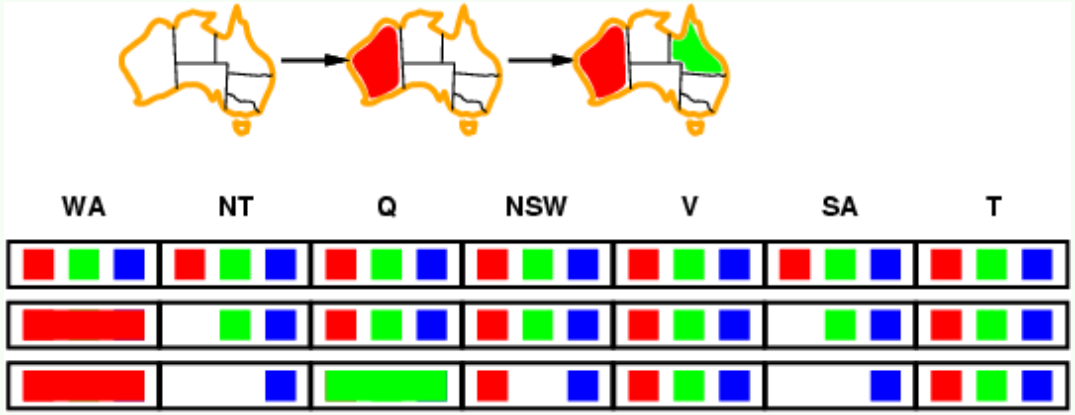
MVH→	*a=1 (for example)*
FC+MVH→	*b=2*
FC+MVH→	*c=3*
FC+MVH→	*d=4*
FC→	*e=1*

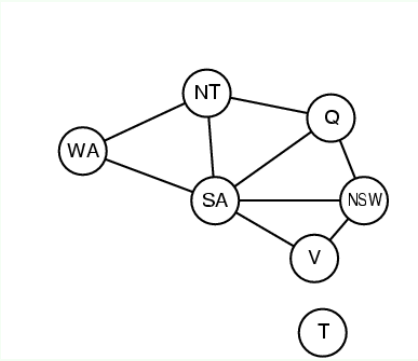


CONSTRAINT GRAPH

Constraint propagation

- Forward checking only checks consistency between assigned and non-assigned states. How about constraints between two unassigned states?



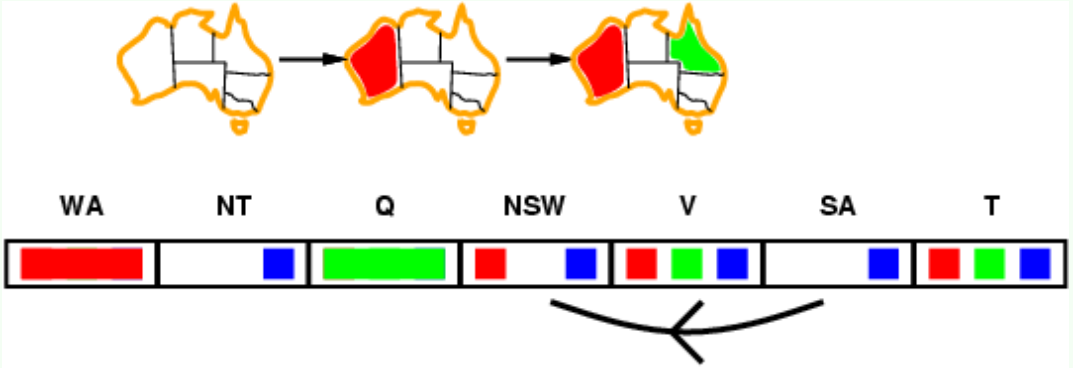


- NT and SA cannot both be blue!
- Constraint propagation repeatedly enforces constraints locally

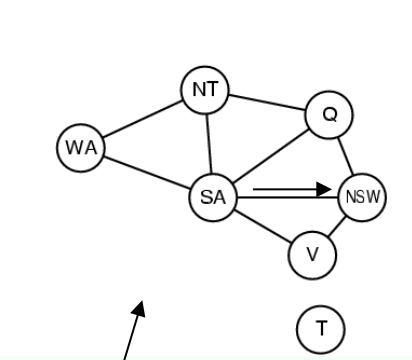
Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff

  for every value x of X there is some allowed y of Y



consistent arc.

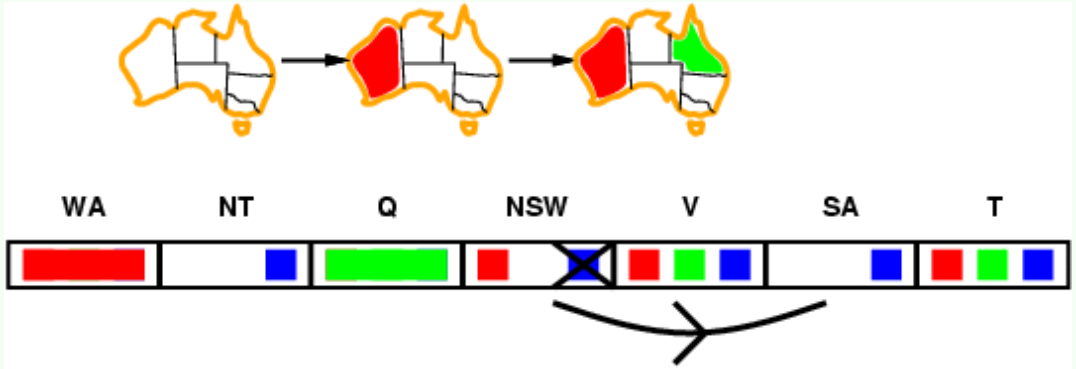


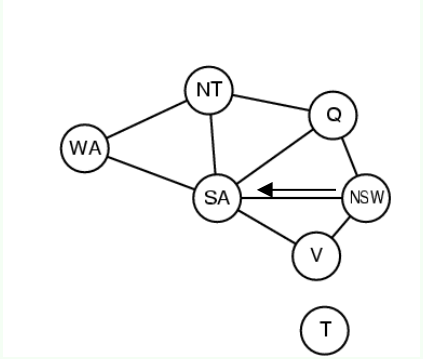
constraint propagation propagates arc consistency on the graph.

Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff

  for every value x of X there is some allowed y



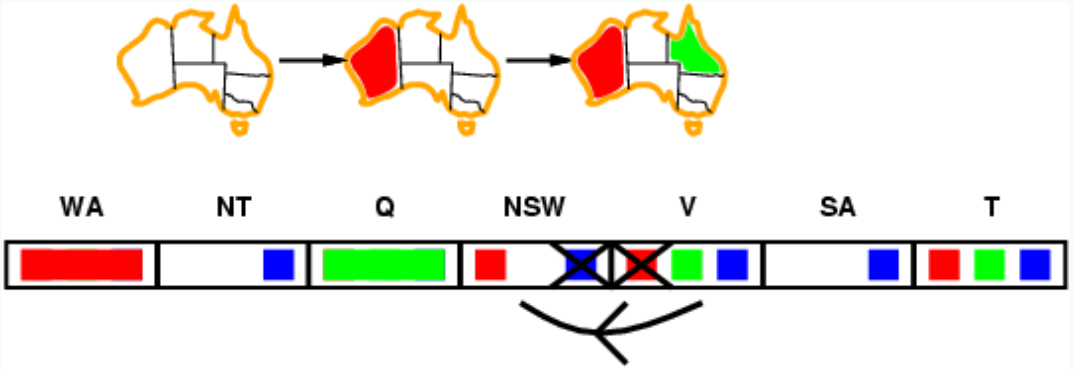


inconsistent arc.
remove blue from source→ consistent arc.

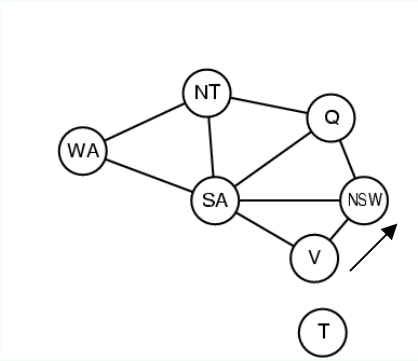
Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff

  for every value x of X there is some allowed y



this arc just became inconsistent



- If X loses a value, neighbors of X need to be rechecked:

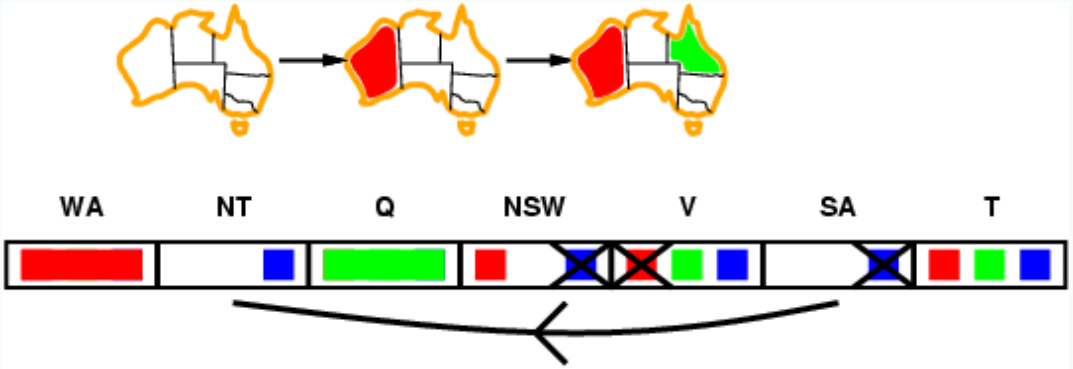
  i.e. incoming arcs can become inconsistent again

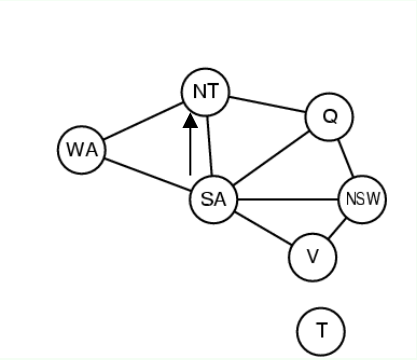
  (outgoing arcs will stay consistent).

Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff

  for every value x of X there is some allowed y

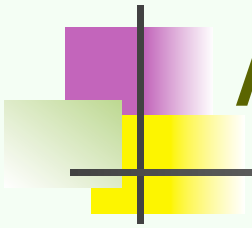




- If X loses a value, neighbors of X need to be rechecked
- Arc consistency detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment
- Time complexity: $O(n^2 d^3)$

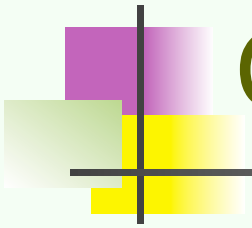
  # arcs

  d^2 for checking, each node can be checked d times at most



Arc Consistency

- This is a propagation algorithm. It's like sending messages to neighbors on the graph! How do we schedule these messages?
- Every time a domain changes, all incoming messages need to be re-send. Repeat until convergence → no message will change any domains.
- Since we only remove values from domains when they can never be part of a solution, an empty domain means no solution possible at all → back out of that branch.
- Forward checking is simply sending messages into a variable that just got its value assigned. First step of arc-consistency.

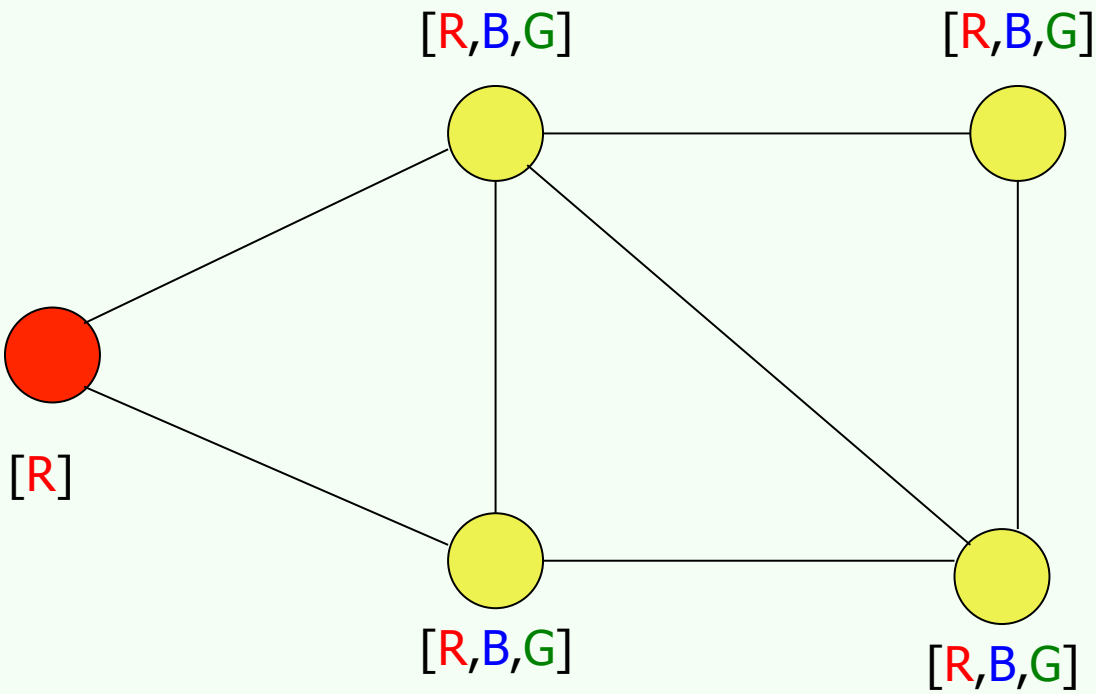


Constraint Propagation Algorithm

- Maintain all allowed values for each variable.
- At each iteration pick the variable with the fewest remaining values
- For variables with equal nr of remaining values, break ties by checking which variable has the largest nr of constraints with unassigned variables
- After we picked a variable, tentatively assign it to each of the remaining values in turn and run constraint propagation to convergence.

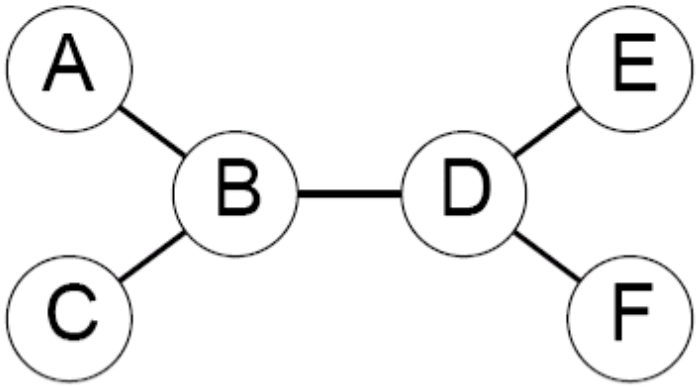
  (This involves iteratively making all arcs consistent that flow into domains that just have been changed, beginning with the neighbors of the variable you just assigned a value to and iterating until no more changes occur.)
- Among all checked values, pick the one that removed the least values from other domains using constraint propagation.
- Now run constraint propagation once more (or recall it from memory) for the assigned value and remove the the values from the domains of the other variables.
- When domains get empty, back out of that branch.
- Iterate until a solution has been found.
- *(as an alternative you only do constraint propagation after an assignment to prune domains of other variables but avoid doing it for all values. Use simply forward checking with the LCV heuristic to pick a value)*

Try it yourself



Use all heuristics including arc-propagation to solve this problem.

Tree-structured CSPs

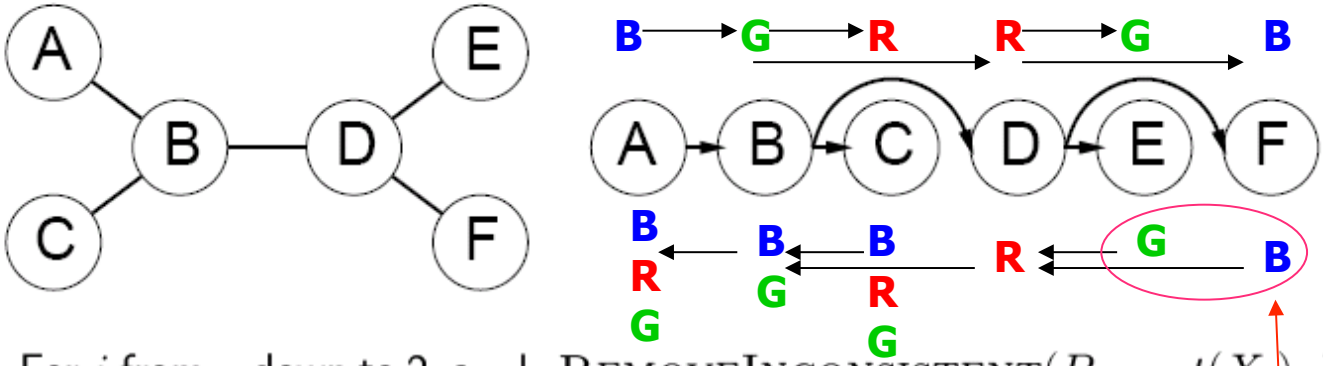


Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\,d^2)$ time

Compare to general CSPs, where worst-case time is $O(d^n)$

Algorithm for tree-structured CSPs

1. Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering



a priori constrained nodes

2. For j from n down to 2, apply REMOVEINCONSISTENT$(Parent(X_j), X_j)$

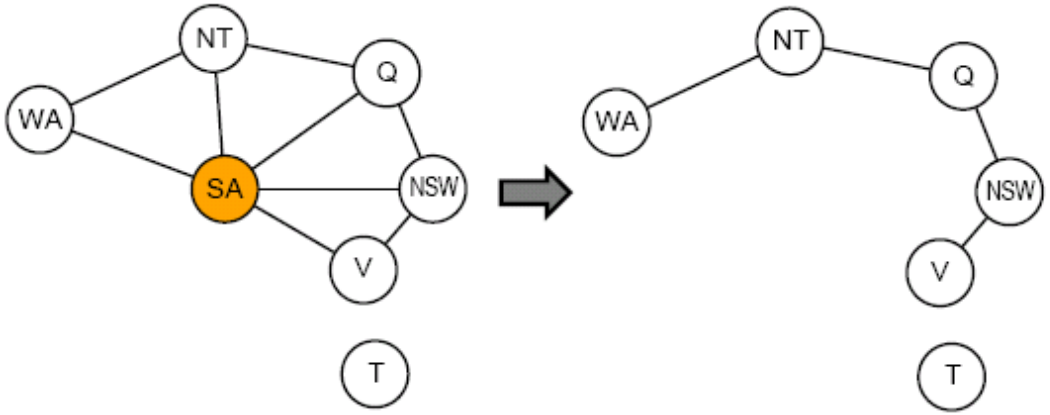
3. For j from 1 to n, assign X_j consistently with $Parent(X_j)$

Note: After the backward pass, there is guaranteed to be a legal choice for a child node for *any* of its leftover values.

This removes any inconsistent values from Parent(Xj), it applies arc-consistency moving backwards.

Nearly tree-structured CSPs

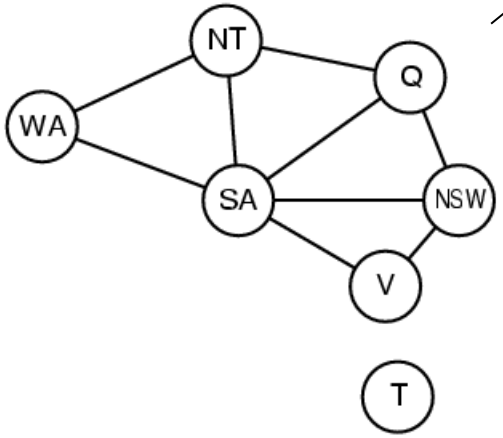
Conditioning: instantiate a variable, prune its neighbors' domains

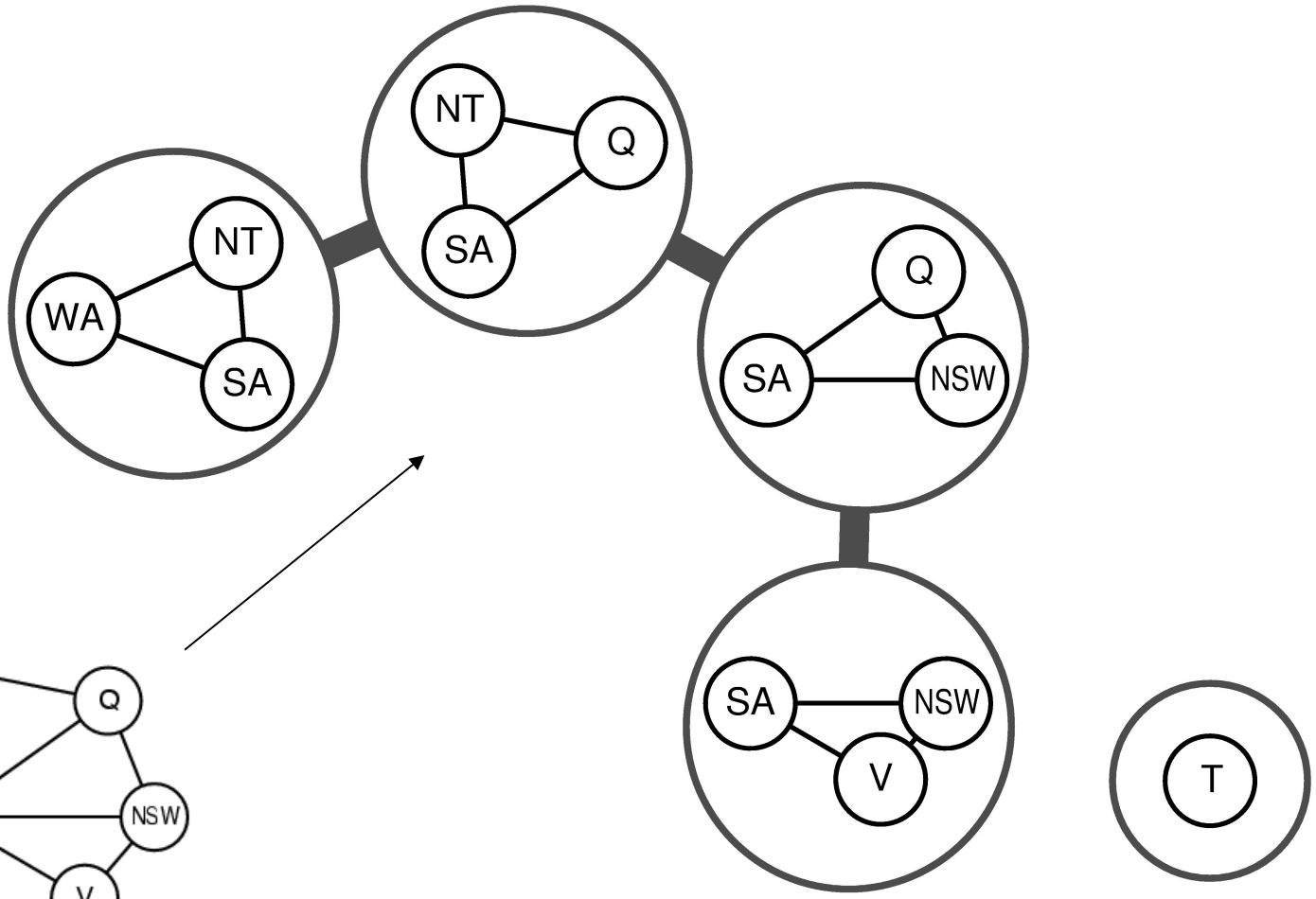


Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree

Cutset size c $\Rightarrow$ runtime $O(d^c \cdot (n-c)d^2)$, very fast for small c

Junction Tree Decompositions





Local search for CSPs

- Note: The path to the solution is unimportant, so we can apply local search!

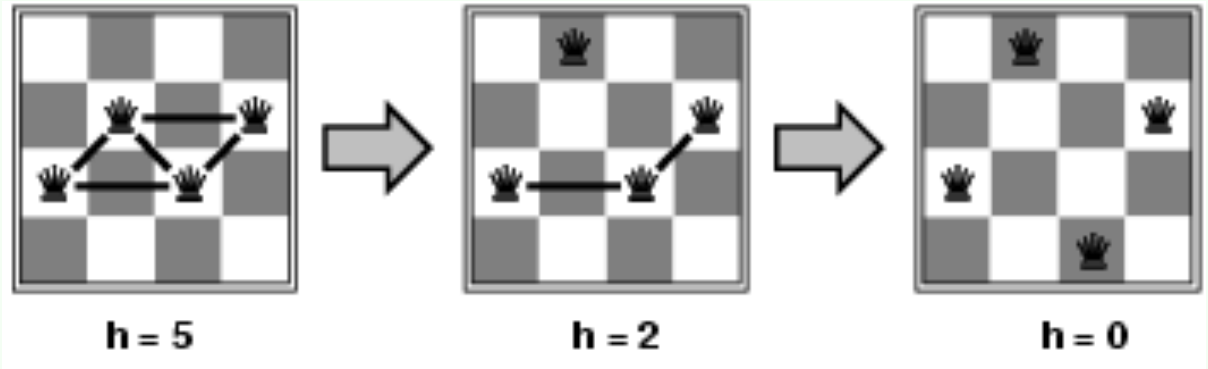
- To apply to CSPs:
  - allow states with unsatisfied constraints
  - operators reassign variable values

- Variable selection: randomly select any conflicted variable

- Value selection by min-conflicts heuristic:
  - choose value that violates the fewest constraints
  - i.e., hill-climb with *h(n)* = total number of violated constraints

Example: 4-Queens

- States: 4 queens in 4 columns (4^4 = 256 states)
- Actions: move queen in column
- Goal test: no attacks
- Evaluation: *h(n)* = number of attacks

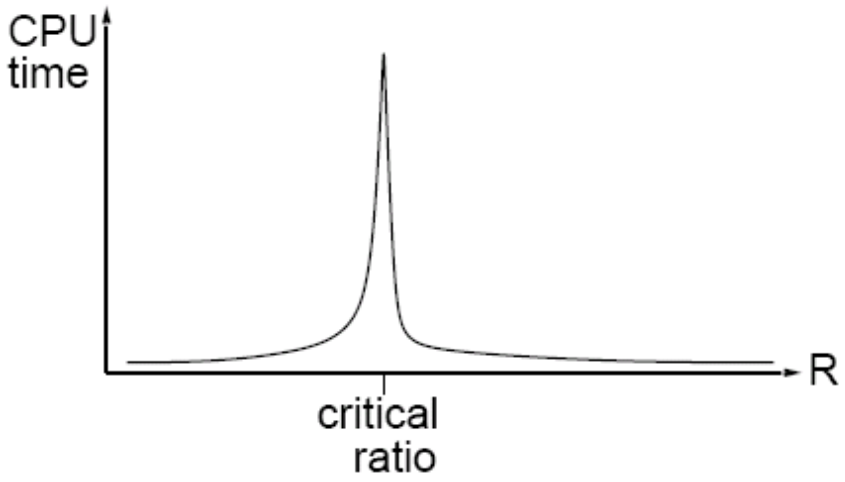


Performance of min-conflicts

Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., $n = 10{,}000{,}000$)

The same appears to be true for any randomly-generated CSP
except in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$

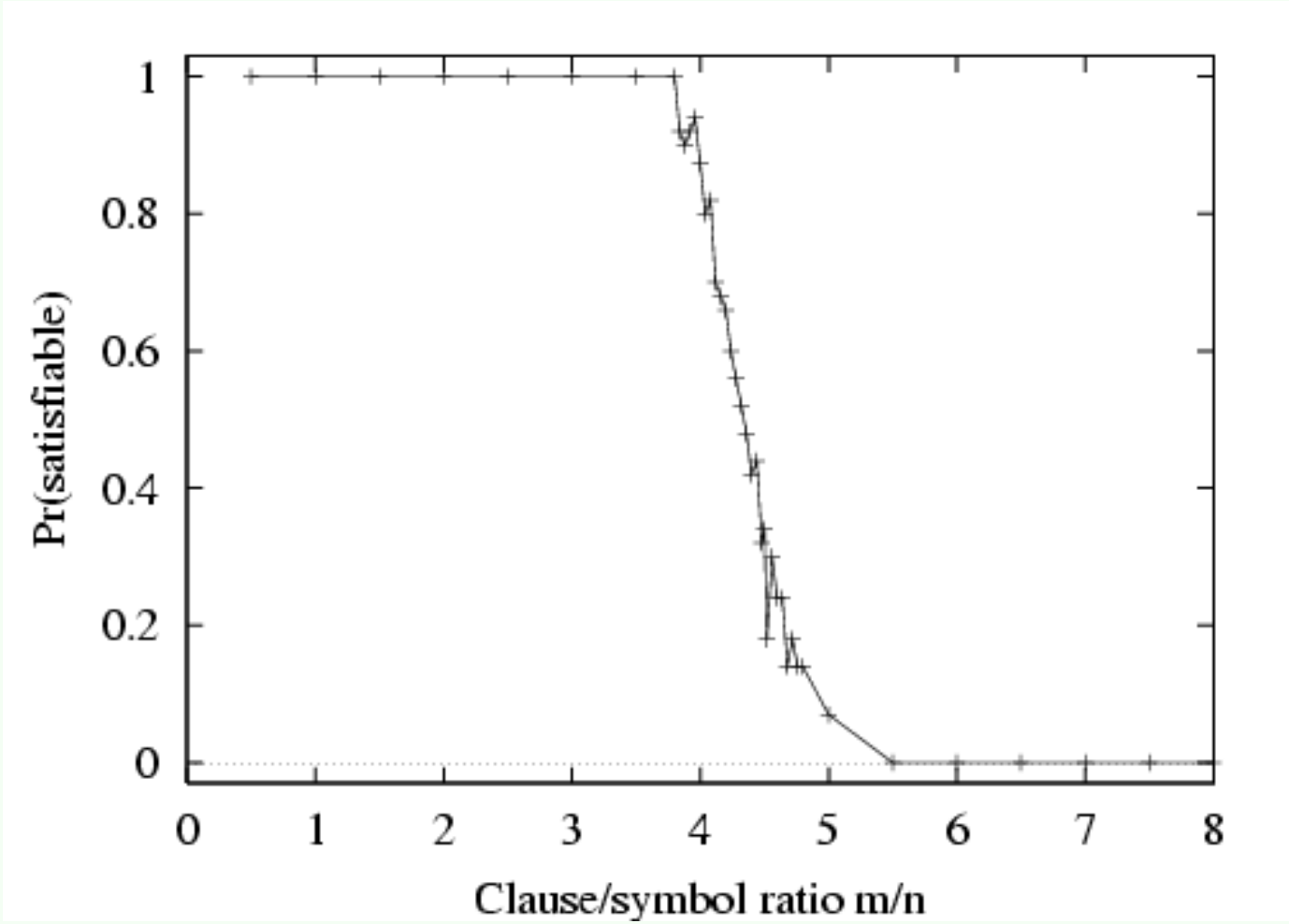


Hard satisfiability problems

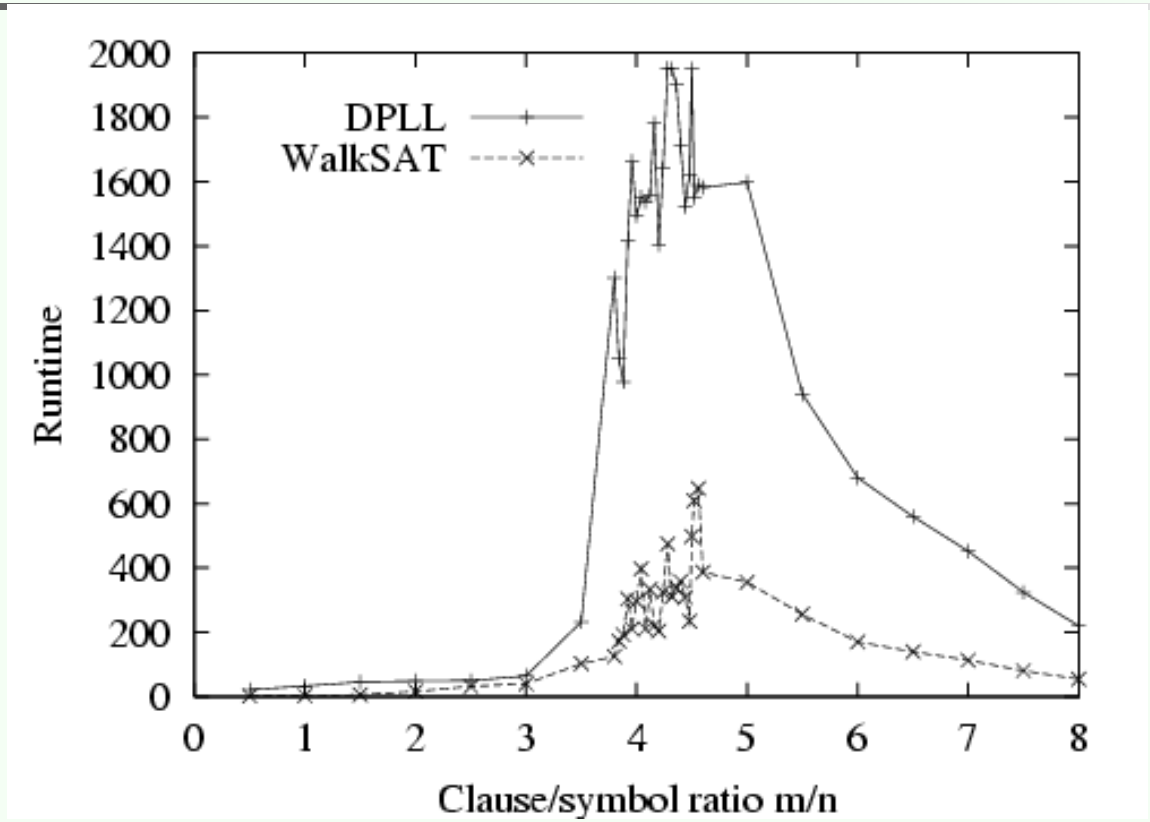
- A,B,C,D,E can take value (true, false).
- ¬A=true means that A must be false.
- (B ∨ ¬A ∨ ¬C) =true means that B=true or A=false or C=false
- Consider *random* conjunctions of constraints:
  (¬D ∨ ¬B ∨ C)=true ∧ (B ∨ ¬A ∨ ¬C)=true ∧ (¬C ∨ ¬B ∨ E)=true
  ∧ (E ∨ ¬D ∨ B)=true ∧ (B ∨ E ∨ ¬C)=true
- We want to find assignments that make all constraints true
  m = number of clauses (5)
  n = number of symbols (5)
- Hard problems seem to cluster near *m/n* = 4.3 (critical point)

Implementing algorithms for random 3-SAT problems will be your project

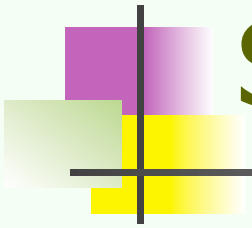
Hard satisfiability problems



Hard satisfiability problems



- Median runtime for 100 satisfiable random 3-CNF sentences, $n = 50$



Summary

- CSPs are a special kind of search problem:
  - states defined by values of a fixed set of variables
  - goal test defined by constraints on variable values
- Backtracking = depth-first search with one variable assigned per level.
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure
- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- Iterative min-conflicts is usually effective in practice