Least square problem for polynomial regression

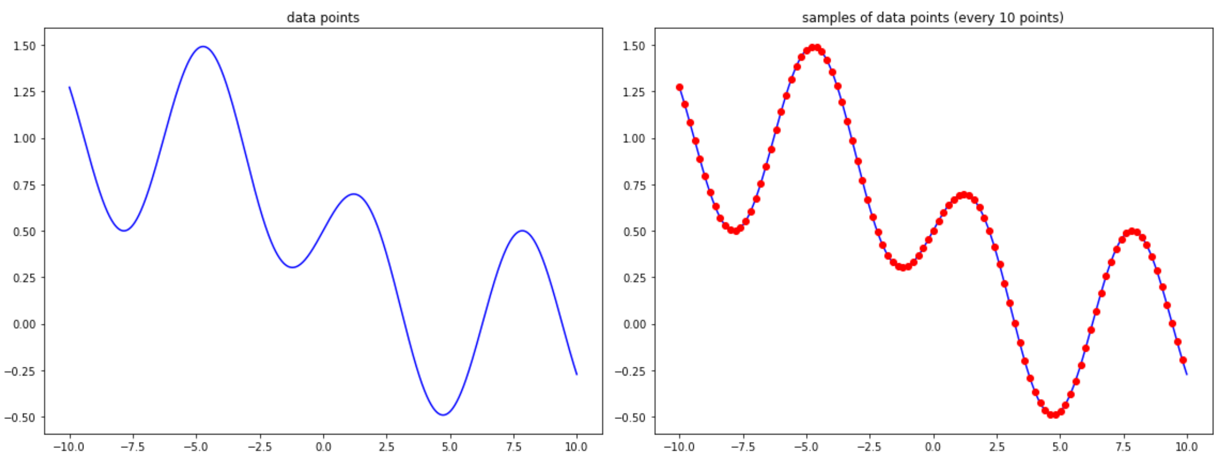
import library

In [ ]:
```
import numpy as np
import matplotlib.image as img
import matplotlib.pyplot as plt
import matplotlib.colors as colors
```

load data points

- $\{(x_i, y_i)\}_{i=1}^n$

In [ ]:
```
filename     = 'assignment_05_data.csv'
data         = np.loadtxt(filename, delimiter = ',')

x            = data[0, :]    # independent variable
y            = data[1, :]    # dependent variable

x_sample     = x[::10]
y_sample     = y[::10]

plt.figure(figsize=(16,6))

plt.subplot(121)
plt.plot(x, y, '-', color = 'blue')
plt.title('data points')

plt.subplot(122)
plt.plot(x, y, '-', color = 'blue')
plt.plot(x_sample, y_sample, 'o', color = 'red')
plt.title('samples of data points (every 10 points)')

plt.tight_layout()
plt.show()
```



In [ ]:
```
x_sample.shape
```

Out[ ]: (100,)

In [ ]:
```
x[::20].shape
```

Out[ ]: (50,)

solve a linear system of equation $Az = b$

$$A = \begin{bmatrix} x_1^0 & x_1^1 & \cdots & x_1^{p-1} \\ x_2^0 & x_2^1 & \cdots & x_2^{p-1} \\ \vdots & \vdots & \vdots & \vdots \\ x_n^0 & x_n^1 & \cdots & x_n^{p-1} \end{bmatrix}, \quad z = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_{p-1} \end{bmatrix}, \quad b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

In [ ]:

construct matrix A for the polynomial regression with power $p - 1$

- useful functions : `np.power`

In [ ]:

```
def construct_matrix_A(x, p):

    n = len(x)
    A = np.zeros([n, p])

    # ++++++++++++++++++++++++++++++++++++++++++++++++++
    # complete the blanks
    #

    for i in range(n):
        A[i,] = np.power(x[i],range(p))

    #
    # ++++++++++++++++++++++++++++++++++++++++++++++++++

    return A
```

In [ ]:

```
A = np.zeros([3, 5])
for i in range(3):
    A[i,] = np.power(x_sample[i],range(5))
print(A)
```

```
[[ 1.00000000e+00 -1.00000000e+01  1.00000000e+02 -1.00000000e+03
   1.00000000e+04]
 [ 1.00000000e+00 -9.79980000e+00  9.60360800e+01 -9.41134377e+02
   9.22292867e+03]
 [ 1.00000000e+00 -9.59960000e+00  9.21523202e+01 -8.84625413e+02
   8.49205011e+03]]
```

In [ ]:

```
np.power(x_sample[0],range(3))
```

Out[ ]: array([1., -10., 100.])

construct vector b

In [ ]:

```
def construct_vector_b(y):

    n = len(y)
    b = np.zeros([n, 1])
```

```
    # ++++++++++++++++++++++++++++++++++++++++++++++++++
    # complete the blanks
    #

    b= y

    #
    # ++++++++++++++++++++++++++++++++++++++++++++++++++

    return b
```

solve the linear system of equation $Az = b$

- without regularization : $\min \frac{1}{2n} \|Az - b\|^2, \quad z = \left(A^T A\right)^{-1} A^T b$
- useful functions : `np.matmul, np.linalg.inv, np.sum`

In [ ]:
```
def solve_regression(x, y, p):

    z       = np.zeros([p, 1])
    loss    = 0

    # ++++++++++++++++++++++++++++++++++++++++++++++++++
    # complete the blanks
    #
    n = len(x)
    A= construct_matrix_A(x, p)
    b= construct_vector_b(y)
    z = np.matmul(np.linalg.inv(np.matmul(A.T,A)),np.matmul(A.T,b))

    loss = (np.sum(np.power(np.matmul(A,z) - b , 2)))/2*n

    #
    # ++++++++++++++++++++++++++++++++++++++++++++++++++

    return z, loss
```

- with regularization : $\min \frac{1}{2n} \|Az - b\|^2 + \frac{\alpha}{2} \|z\|^2, \quad z = \left(A^T A + n\alpha I\right)^{-1} A^T b$ where I denotes identity matrix
- useful functions : `np.matmul, np.linalg.inv, np.sum`

In [ ]:
```
def solve_regression_with_regularization(x, y, p, alpha):

    z       = np.zeros([p, 1])
    loss    = 0

    # ++++++++++++++++++++++++++++++++++++++++++++++++++
    # complete the blanks
    #
    n = len(x)
    A = construct_matrix_A(x, p)
    b = construct_vector_b(y)


    z = np.matmul(np.linalg.inv(np.matmul(A.T,A) + n*alpha*np.identity(p)) ,np.matmu

    loss = np.divide(np.sum(np.power(np.matmul(A,z) - b , 2)),2*n)  + (np.sum(np.pow
```

```
    #
    # ++++++++++++++++++++++++++++++++++++++++++++++++++

    return z, loss
```

In [ ]:
```
A = construct_matrix_A(x, 2)
```

In [ ]:
```
np.matmul(A.T,A).shape
```

Out[ ]:
```
(2, 2)
```

In [ ]:
```
b = np.array([1,2,3])
np.sum(np.power(b,2))
```

Out[ ]:
```
14
```

In [ ]:
```
a = np.array([[1,2,3],[4,5,6],[7,8,9]])
a + 3
```

Out[ ]:
```
array([[ 4,  5,  6],
       [ 7,  8,  9],
       [10, 11, 12]])
```

In [ ]:
```
3*np.identity(5)
```

Out[ ]:
```
array([[3., 0., 0., 0., 0.],
       [0., 3., 0., 0., 0.],
       [0., 0., 3., 0., 0.],
       [0., 0., 0., 3., 0.],
       [0., 0., 0., 0., 3.]])
```

approximate by polynomial regression

- $\hat{y} = Az^*$
- useful functions : `np.matmul`

In [ ]:
```
def approximate(x, y, p):

    n       = len(y)
    y_hat   = np.zeros([n, 1])
    loss    = 0

    # ++++++++++++++++++++++++++++++++++++++++++++++++++
    # complete the blanks
    #
    A = construct_matrix_A(x, p)
    z , loss = solve_regression(x, y, p)
    y_hat = np.matmul(A,z)


    #
    # ++++++++++++++++++++++++++++++++++++++++++++++++++

    return y_hat, loss
```

In [ ]:
```
def approximate_with_regularization(x, y, p, alpha):

    n       = len(y)
    y_hat   = np.zeros([n, 1])
    loss    = 0
```

```
    # ++++++++++++++++++++++++++++++++++++++++++++++++++
    # complete the blanks
    #
    A = construct_matrix_A(x, p)
    z , loss = solve_regression_with_regularization(x, y, p, alpha)
    y_hat = np.matmul(A,z)

    #
    # ++++++++++++++++++++++++++++++++++++++++++++++++++

    return y_hat, loss
```

---

functions for presenting the results

In [ ]:
```
def function_result_01():

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.title('data points')
    plt.show()
```

In [ ]:
```
def function_result_02():

    p           = 2
    (y_hat, _)  = approximate(x, y, p)

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:
```
def function_result_03():

    p           = 4
    (y_hat, _)  = approximate(x, y, p)

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:
```
def function_result_04():

    p           = 8
    (y_hat, _)  = approximate(x, y, p)

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:
```
def function_result_05():

    p           = 16
    (y_hat, _)  = approximate(x, y, p)
```

```
    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:
```
def function_result_06():

    p            = 32
    (y_hat, _)  = approximate(x, y, p)

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:
```
def function_result_07():

    p            = 2
    alpha        = 0.1
    (y_hat, _)  = approximate_with_regularization(x, y, p, alpha)

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:
```
def function_result_08():

    p            = 4
    alpha        = 0.1
    (y_hat, _)  = approximate_with_regularization(x, y, p, alpha)

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:
```
def function_result_09():

    p            = 8
    alpha        = 0.1
    (y_hat, _)  = approximate_with_regularization(x, y, p, alpha)

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:
```
def function_result_10():

    p            = 16
    alpha        = 0.1
    (y_hat, _)  = approximate_with_regularization(x, y, p, alpha)

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:
```
def function_result_11():
```

```
    p           = 32
    alpha       = 0.1
    (y_hat, _)  = approximate_with_regularization(x, y, p, alpha)

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:
```
def function_result_12():

    p           = 4
    (_, loss)   = approximate(x, y, p)

    print('loss = ', loss)
```

In [ ]:
```
def function_result_13():

    p           = 16
    (_, loss)   = approximate(x, y, p)

    print('loss = ', loss)
```

In [ ]:
```
def function_result_14():

    p           = 4
    alpha       = 0.1
    (_, loss)   = approximate_with_regularization(x, y, p, alpha)

    print('loss = ', loss)
```

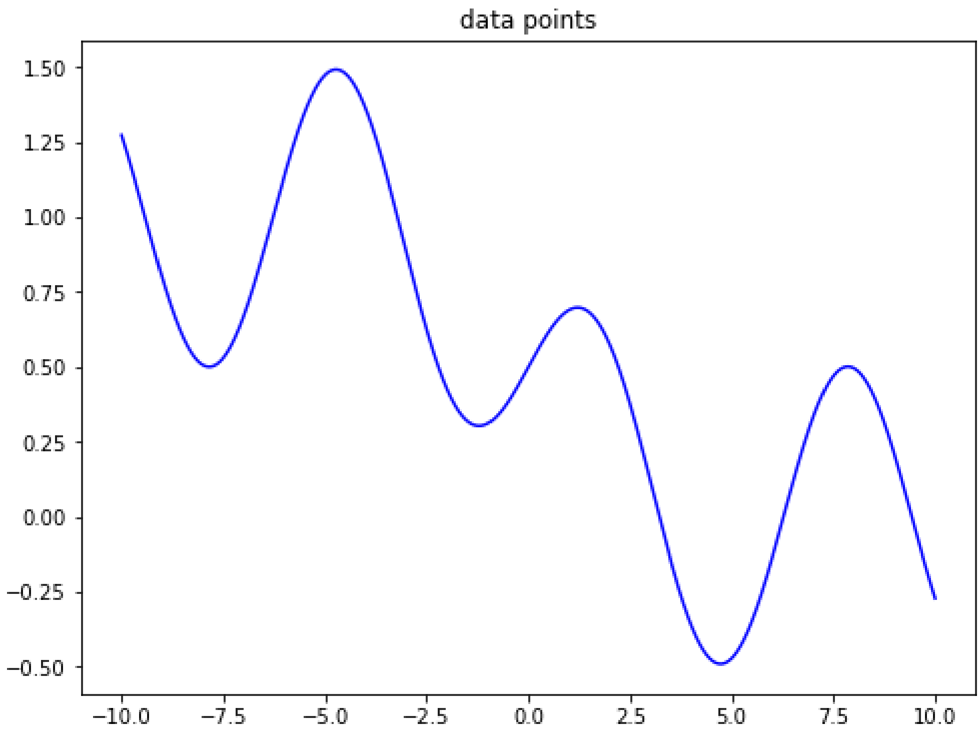
In [ ]:
```
def function_result_15():

    p           = 16
    alpha       = 0.1
    (_, loss)   = approximate_with_regularization(x, y, p, alpha)

    print('loss = ', loss)
```

---

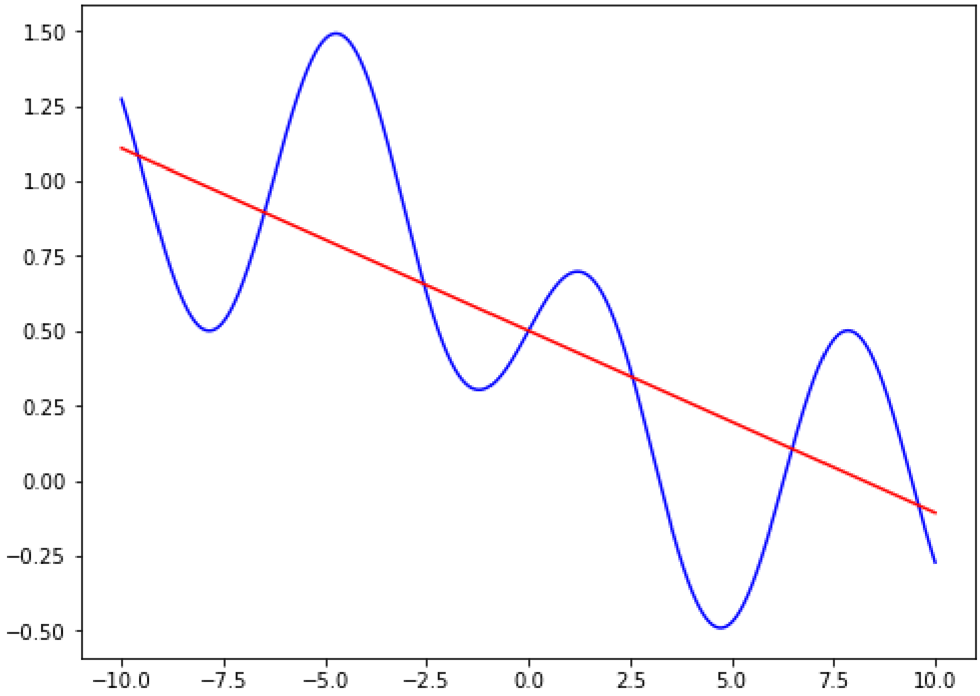
results

In [ ]:
```
number_result = 15

for i in range(number_result):
    title = '## [RESULT {:02d}]'.format(i+1)
    name_function = 'function_result_{:02d}()'.format(i+1)

    print('**************************************************')
    print(title)
    print('**************************************************')
    eval(name_function)
```

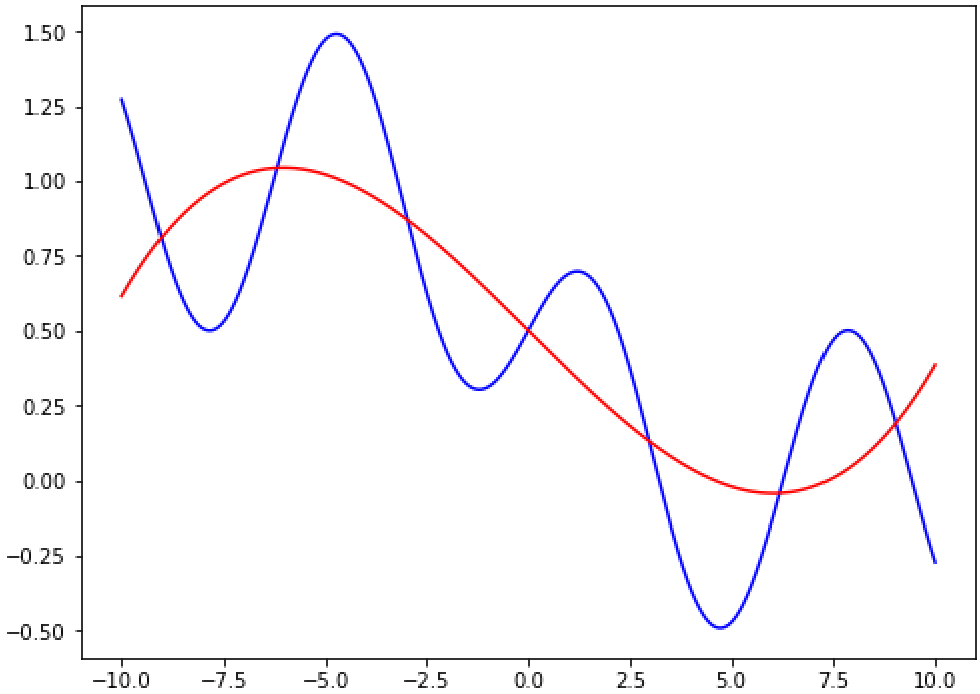
```
**************************************************
## [RESULT 01]
**************************************************
```



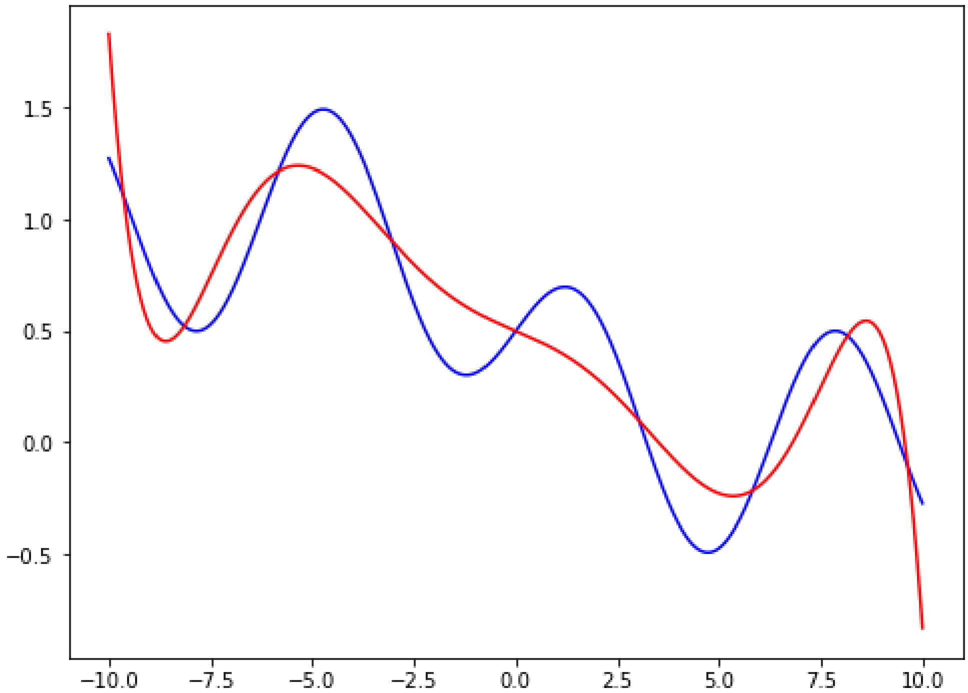
****
[RESULT 02]
****



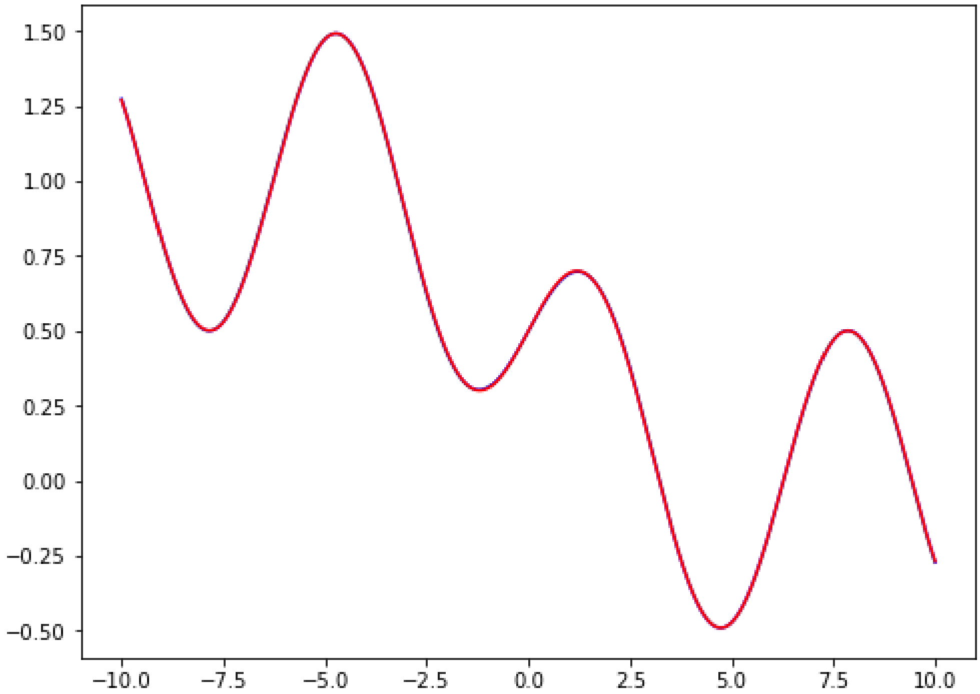
****
[RESULT 03]
****



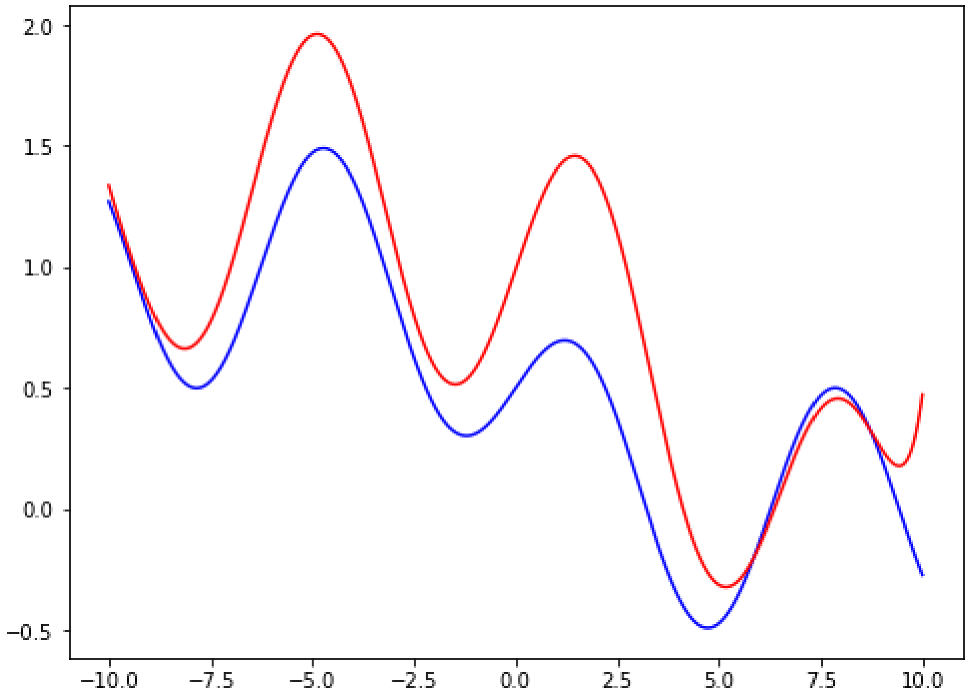

```
**************************************************
## [RESULT 04]
**************************************************
```



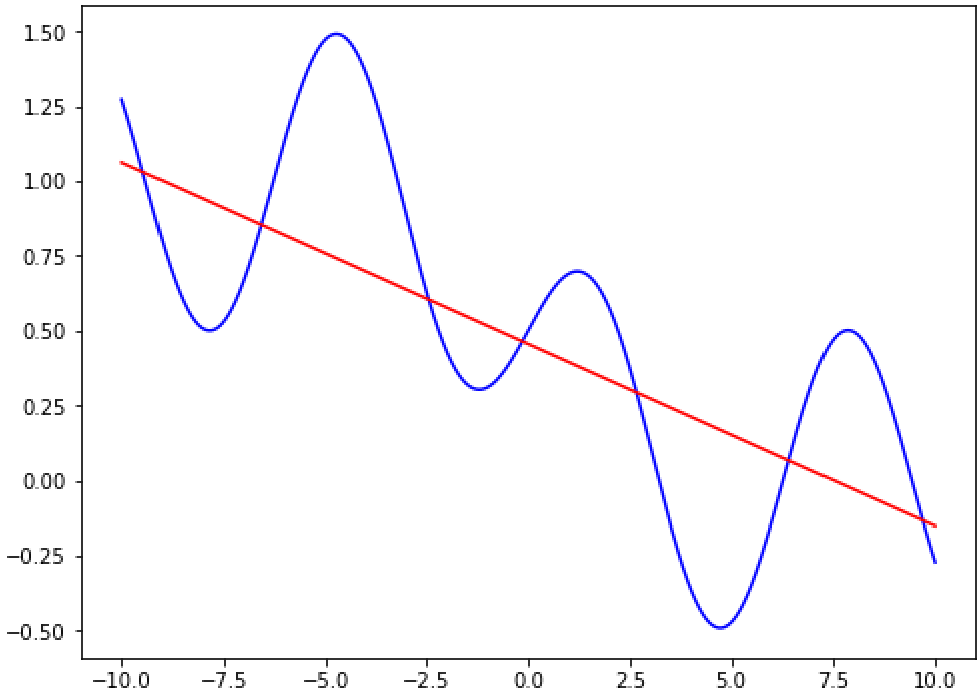

```
**************************************************
## [RESULT 05]
**************************************************
```



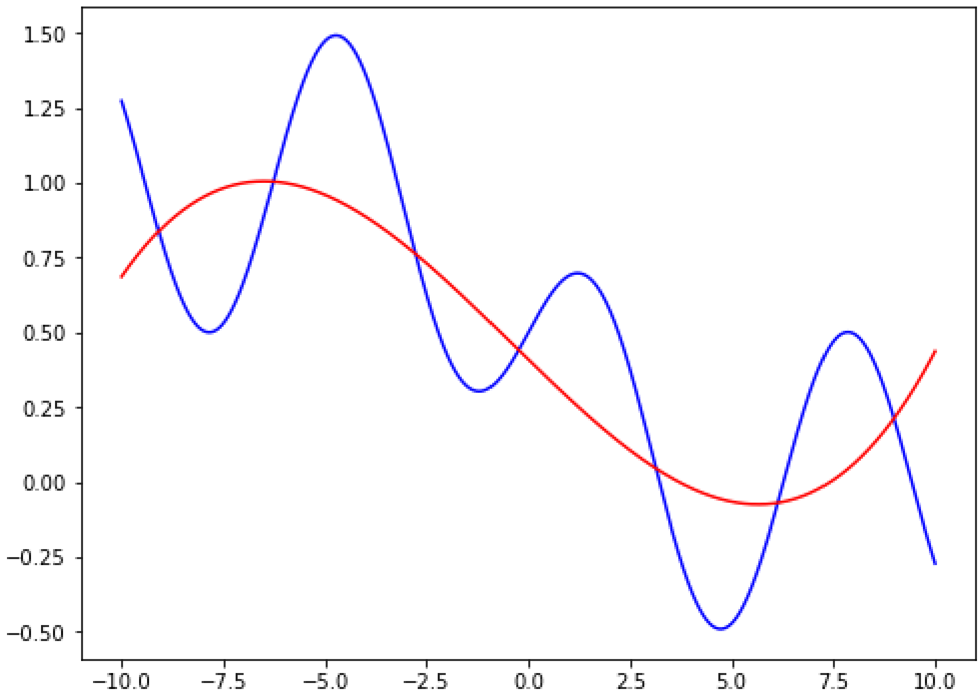
****
[RESULT 06]
****



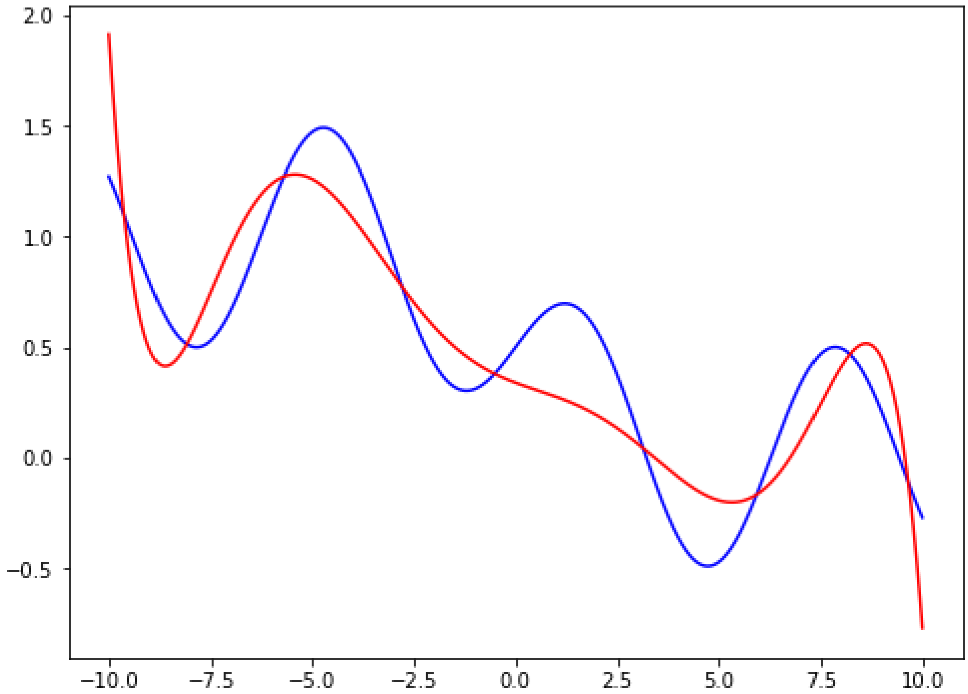
****
[RESULT 07]
****



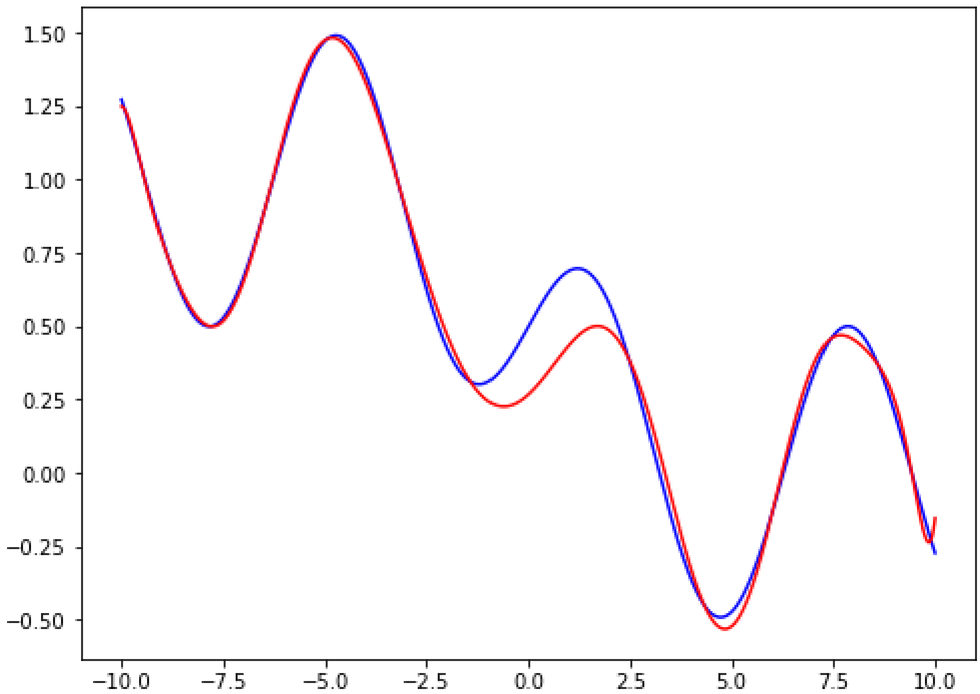
****
[RESULT 08]
****



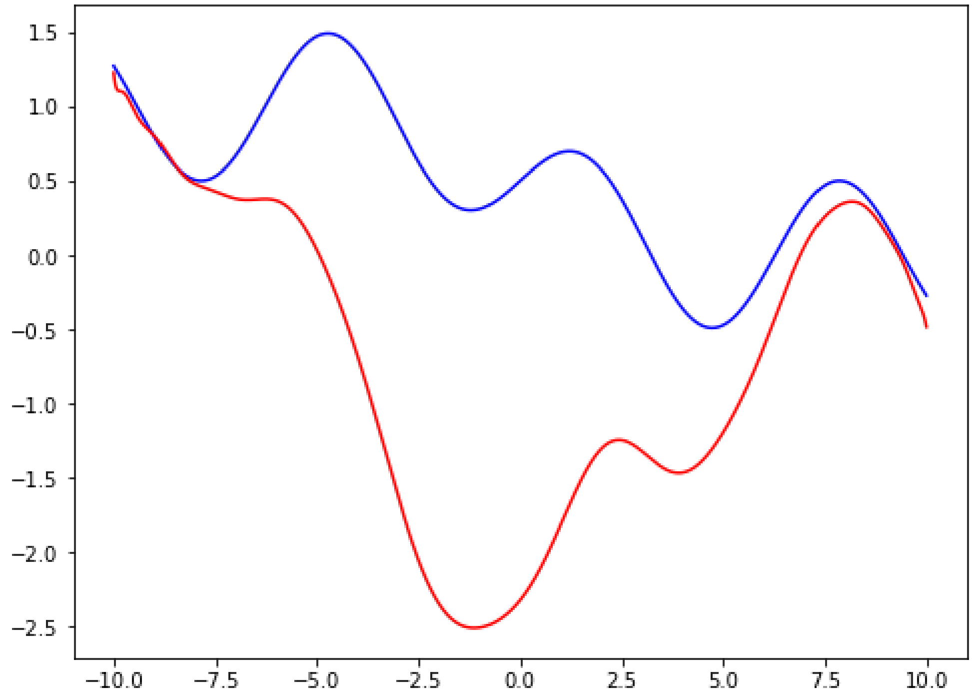
****
[RESULT 09]
****



****
[RESULT 10]
****



****
[RESULT 11]
****




```
***************************************************
## [RESULT 12]
***************************************************
loss =  105.39561372651886
***************************************************
## [RESULT 13]
***************************************************
loss =  0.0024164118115065835
***************************************************
## [RESULT 14]
***************************************************
loss =  0.06379867384617093
***************************************************
## [RESULT 15]
***************************************************
loss =  0.008444929356995277
```

In [ ]: