Complex Variables
Week 1:
Introduction to Complex Numbers

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1 What is a Complex Number?

The encounter with complex numbers is usually presented as a solution to the equation: $x^2 = -1 \implies x = \pm\sqrt{-1}$

Though this is an intuitive approach for a high school algebra class, it can be misleading. Consider the following:

$$i = \sqrt{-1} = \frac{\sqrt{-1}}{1} = \frac{\sqrt{-1}}{\sqrt{1}} = \sqrt{\frac{-1}{1}} = \sqrt{\frac{1}{-1}} = \frac{1}{i}$$
$$\therefore i = \frac{1}{i} \ ?$$

As you can see from this example, the imaginary number i has properties that are different from real numbers.

1.1 A Better Definition

The complex field $\mathbb{C}$ is the set of ordered pairs of real numbers (a, b) with the following properties:

Addition: $(a, b) + (c, d) = (a + c, b + d)$
Multiplication: $(a, b)(c, d) = (ac - bd, ad + bc)$

With this definition, we can now consider the ordered pair $(0, 1)$ which we will call i. If we multiply i by itself:

$$(0, 1)(0, 1) = (-1, 0) = -1$$

... we can conclude that $i = \sqrt{-1}$.

1.2 Notation

The most common way to represent complex numbers is: $z = (x, y) = x + iy$
ALL complex numbers can (and should) be written in this form.

There are three graphical ways to represent complex numbers. First, they can be represented as points on a 2D plane:

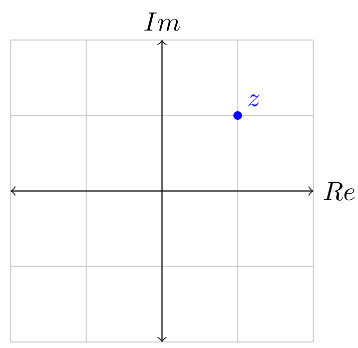

Figure 1: The point $z = (1, 1) = 1 + i$

Secondly, complex numbers can also be represented as vectors beginning at the origin.

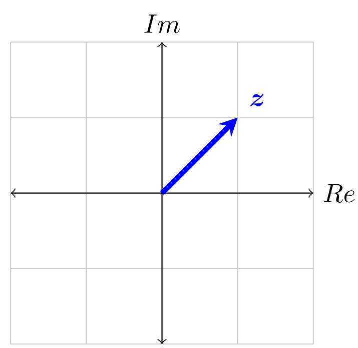

Figure 2: The vector $\vec{z} = (1, 1) = 1 + i$

Thirdly, complex numbers can be represented in polar coordinates:
$z = re^{i\theta} = r\text{cis}(\theta) = r(\cos\theta + i\sin\theta)$

θ is known as the argument of z (sometimes written as $\arg z$) and can be computed from Cartesian coordinates (x, y) by $\theta = \arctan\frac{y}{x}$. The radius r can be computed by $r = |z| = \sqrt{x^2 + y^2}$

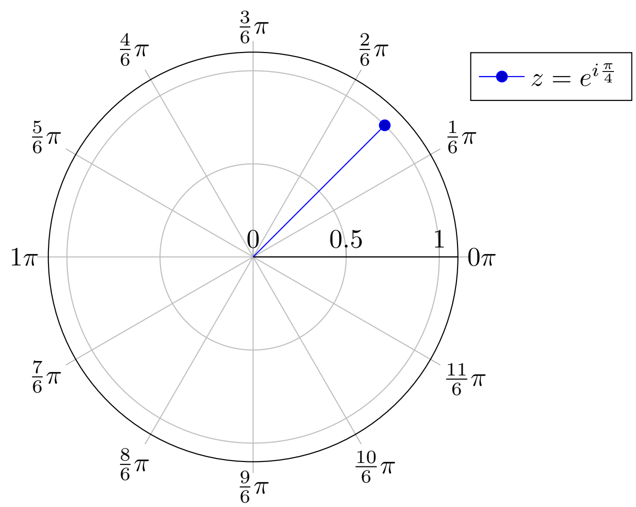

Figure 3: Polar representation

1.3 Definitions of Terms

arg(z): Argument of z, also known as θ, $\arg z = \arctan \frac{y}{x}$

cis(θ): Short-hand for $\cos\theta + i\sin\theta$

Complex Conjugate: Written as $\bar{z}$ it is the reflection about the *Re* axis (i.e. $z = x + iy \implies \bar{z} = x - iy$)

Im(z): The imaginary part of z (i.e. the y term)
Magnitude: Distance from the origin, represented as the radius r (in polar coordinates) and $|z| = \sqrt{x^2 + y^2}$ (in Cartesian coordinates)

Re(z): The real part of z (i.e. the x term)

2 Algebra with Complex Numbers

2.1 Addition

Addition of complex numbers is done by adding the real parts and imaginary parts: $z_1 + z_2 = (x_1 + y_2 i) + (x_2 + y_2 i) = (x_1 + x_2) + i(y_1 + y_2)$ This can be visualized in the same way as vector addition ("tip-to-tail" method).

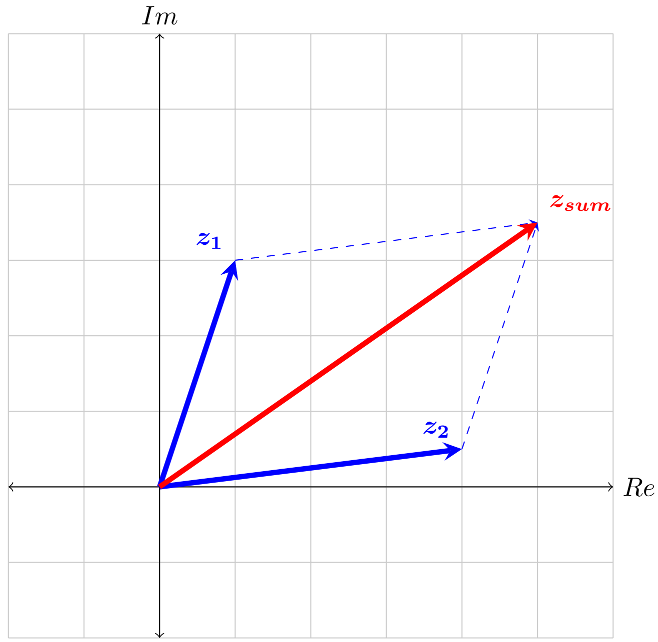

Figure 4: The sum $\vec{z_1} + \vec{z_2}$ is the diagonal through the parallelogram formed by z_1 and z_2

2.2 Subtraction

Subtraction of complex numbers is done by adding the real parts and imaginary parts of z_1 and $-z_2$: $z_1 - z_2 = (x_1 + y_2 i) - (x_2 + y_2 i) = (x_1 + y_2 i) + (-x_2 - y_2 i) = (x_1 - x_2) + i(y_1 - y_2)$ This can be visualized in the same way as vector subtraction ("triangle" method).

2.3 Multiplication

2.4 Division

3 Topological Properties of the Complex Plane

3.1 Definitions

3.2 Weierstrass M-Test

3.3 Stereographic Projection

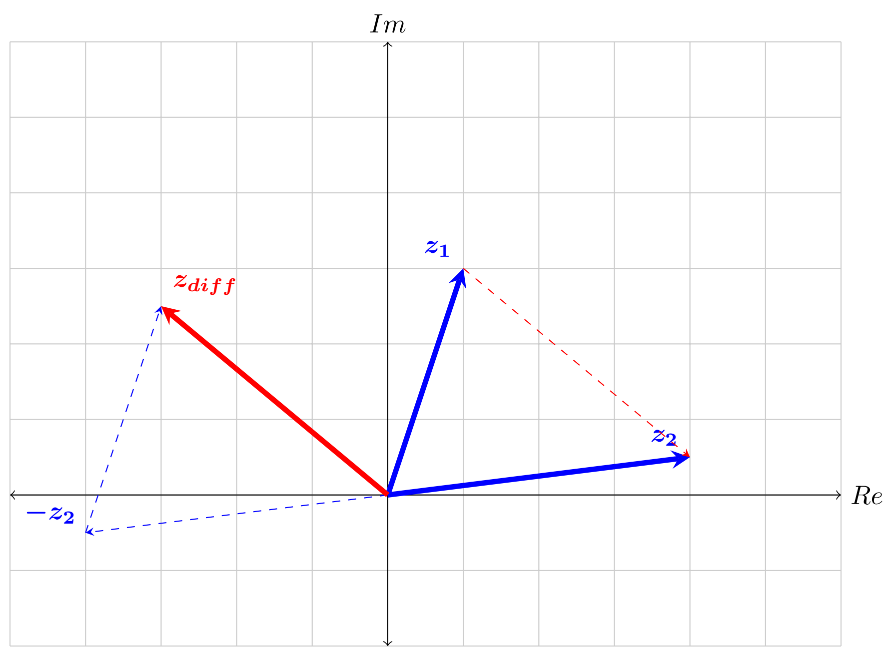

Figure 5: The difference $\vec{z_1} - \vec{z_2}$ is the third side of the triangle formed by z_1 and $-z_2$