Complex Variables

Week 1:
Introduction to Complex Numbers

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1 What is a Complex Number?

The encounter with complex numbers is usually presented as a solution to the equation: $x^2 = -1 \implies x = \pm\sqrt{-1}$

Though this is an intuitive approach for a high school algebra class, it can be misleading. Consider the following:

$$i = \sqrt{-1} = \frac{\sqrt{-1}}{1} = \frac{\sqrt{-1}}{\sqrt{1}} = \sqrt{\frac{-1}{1}} = \sqrt{\frac{1}{-1}} = \frac{1}{i}$$
$$\therefore i = \frac{1}{i} \text{ ?}$$

As you can see from this example, the imaginary number i has properties that are different from real numbers.

1.1 A Better Definition

The complex field $\mathbb{C}$ is the set of ordered pairs of real numbers (a, b) with the following properties:

Addition: $(a, b) + (c, d) = (a + c, b + d)$
Multiplication: $(a, b)(c, d) = (ac - bd, ad + bc)$

With this definition, we can now consider the ordered pair $(0, 1)$ which we will call i. If we multiply i by itself:

$$(0, 1)(0, 1) = (-1, 0) = -1$$

... we can conclude that $i = \sqrt{-1}$.

1.2 Notation

The most common way to represent complex numbers is: $z = (x, y) = x + iy$
ALL complex numbers can (and should) be written in this form.

There are three graphical ways to represent complex numbers. First, they can be represented as points on a 2D plane:

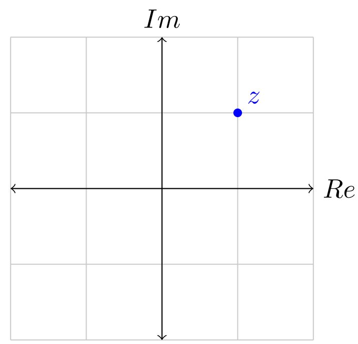

Figure 1: The point $z = (1, 1) = 1 + i$

Secondly, complex numbers can also be represented as vectors beginning at the origin.

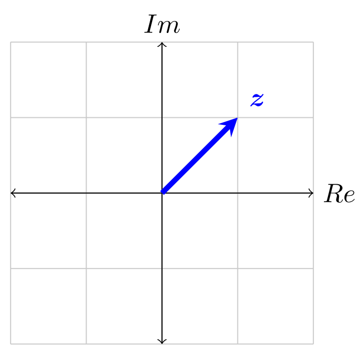

Figure 2: The vector $\vec{z} = (1, 1) = 1 + i$

Thirdly, complex numbers can be represented in polar coordinates:
$z = re^{i\theta} = r\text{cis}(\theta) = r(\cos\theta + i\sin\theta)$

θ is known as the argument of z (sometimes written as $\arg z$) and can be computed from Cartesian coordinates (x, y) by $\theta = \arctan \frac{y}{x}$. The radius r can be computed by $r = |z| = \sqrt{x^2 + y^2}$

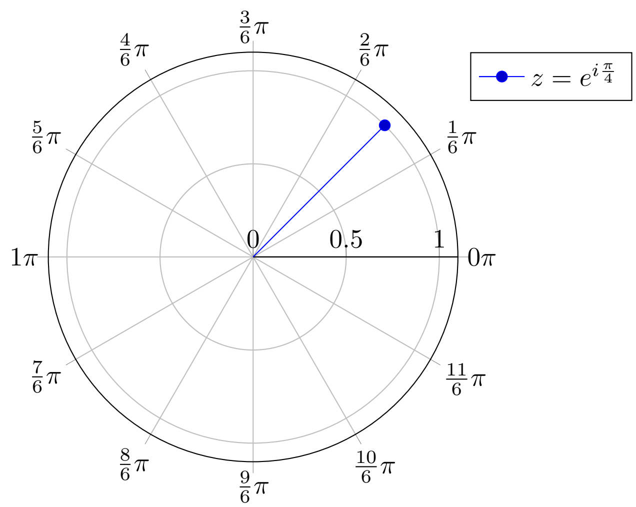

Figure 3: Polar representation

1.3 Definitions of Terms

arg(z): Argument of z, also known as θ, $\arg z = \arctan \frac{y}{x}$

cis(θ): Short-hand for $\cos\theta + i\sin\theta$

Complex Conjugate: Written as $\bar{z}$ it is the reflection about the *Re* axis (i.e. $z = x + iy \implies \bar{z} = x - iy$)

Im(z): The imaginary part of z (i.e. the y term)
Magnitude/Modulous: Distance from the origin, represented as the radius r (in polar coordinates) and $|z| = \sqrt{x^2 + y^2}$ (in Cartesian coordinates)

Re(z): The real part of z (i.e. the x term)

Triangle Inequality: $|z_1 + z_2| \leq |z_1| + |z_2|$ which follows from graphical representation of vector addition

2 Algebra with Complex Numbers

2.1 Addition

Addition of complex numbers is done by adding the real parts and imaginary parts: $z_1 + z_2 = (x_1 + y_2 i) + (x_2 + y_2 i) = (x_1 + x_2) + i(y_1 + y_2)$ This can be visualized in the same way as vector addition (”tip-to-tail” method).

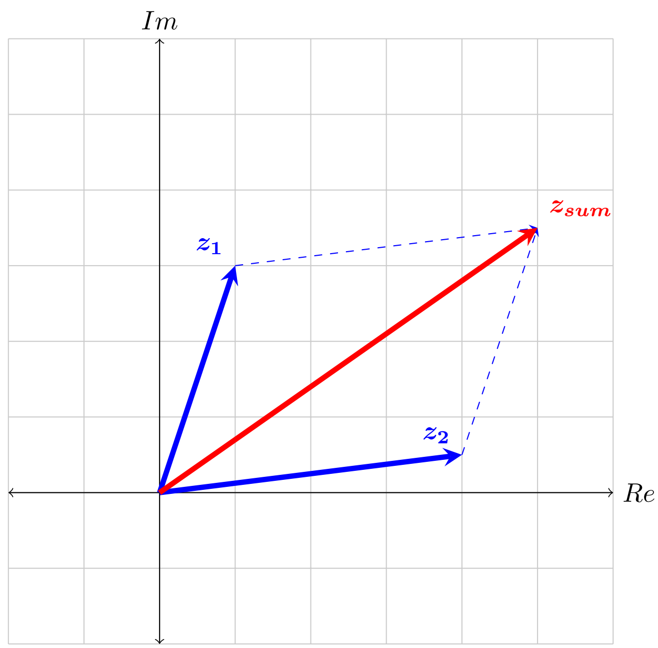

Figure 4: The sum $\vec{z_1} + \vec{z_2}$ is the diagonal through the parallelogram formed by z_1 and z_2

2.2 Subtraction

Subtraction of complex numbers is done by adding the real parts and imaginary parts of z_1 and $-z_2$: $z_1 - z_2 = (x_1 + y_2 i) - (x_2 + y_2 i) = (x_1 + y_2 i) + (-x_2 - y_2 i) = (x_1 - x_2) + i(y_1 - y_2)$ This can be visualized in the same way as vector subtraction ("triangle" method).

2.3 Multiplication

Multiplication of complex numbers can be done in the same way as real numbers (e.g. expansion), except that i has the special property: $i^2 = -1$.

For polynomials of high degree, expansion can be cumbersome. The complex exponential form ($z = re^{i\theta}$) may be more efficient in these situations because it simplifies to addition of angles (i.e. $\text{cis}(\theta_1)\text{cis}(\theta_2) = \text{cis}(\theta_1 + \theta_2)$).

Example: Simplify $(-\frac{1}{2} + i\frac{\sqrt{3}}{2})^4$

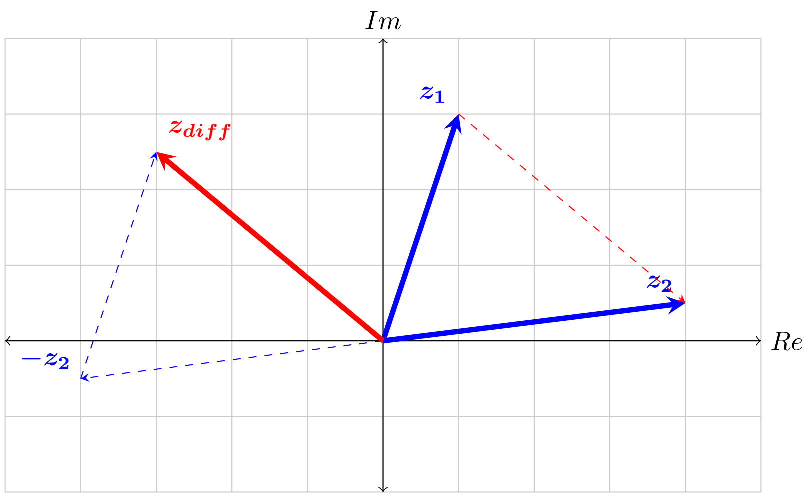

Figure 5: The difference $\vec{z_1} - \vec{z_2}$ is the third side of the triangle formed by z_1 and $-z_2$

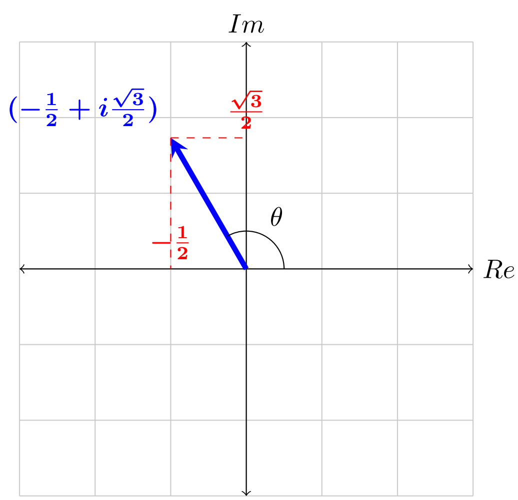

Figure 6: From the example, $(-\frac{1}{2} + i\frac{\sqrt{3}}{2})$ broken into (x, y) components

$$r = \sqrt{(-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = 1 \quad (1)$$

$$\theta = \pi - \arctan \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \frac{2\pi}{3} \quad (2)$$

$$
\begin{aligned}
(-\frac{1}{2} + i\frac{\sqrt{3}}{2})^4 &= (re^{i\theta})^4 && (3)\\
&= (1e^{i\frac{2\pi}{3}})^4 && (4)\\
&= (e^{i\frac{2\pi}{3}})^4 && (5)\\
&= e^{i\frac{8\pi}{3}} && (6)\\
&= e^{i\frac{2\pi}{3}} \quad (because \frac{8\pi}{3} - 2\pi = \frac{2\pi}{3}) && (7)
\end{aligned}
$$

2.4 Division

When it comes to division of complex numbers in Cartesian coordinates, the first step is usually to multiply by the denominator's complex conjugate (i.e. $\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{\bar{z_2}}{\bar{z_2}} = \frac{z_1\bar{z_2}}{\mathrm{Re}\{z_2\}^2+\mathrm{Im}\{z_2\}^2}$)

3 Topological Properties of the Complex Plane

3.1 Definitions

3.2 Weierstrass M-Test

3.3 Stereographic Projection