

중성자별의 구조에 대한 수치적 풀이

- Solution of TOV equation

김진호 (한국천문연구원)

별

주계열성

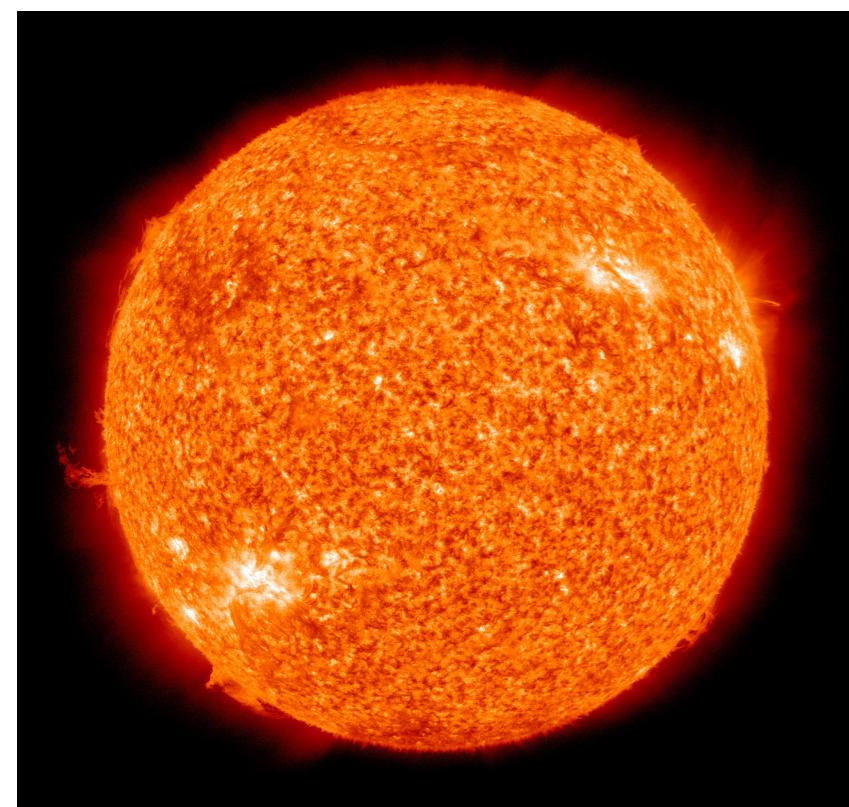


Image from NASA's
Solar Dynamics
Observatory

백색왜성

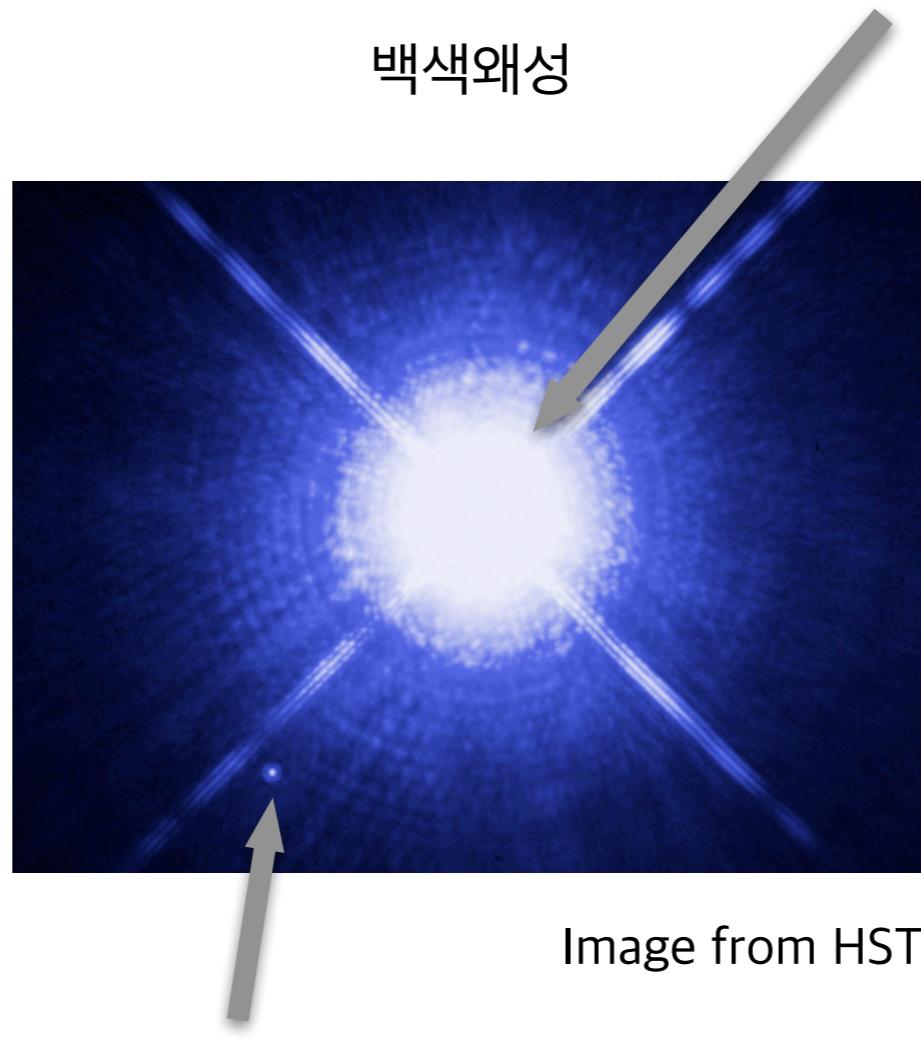
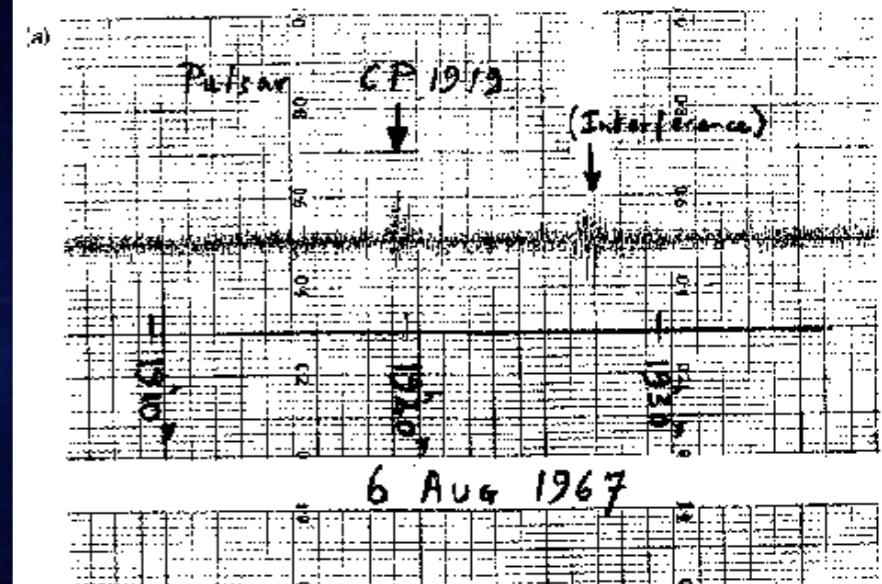


Image from HST

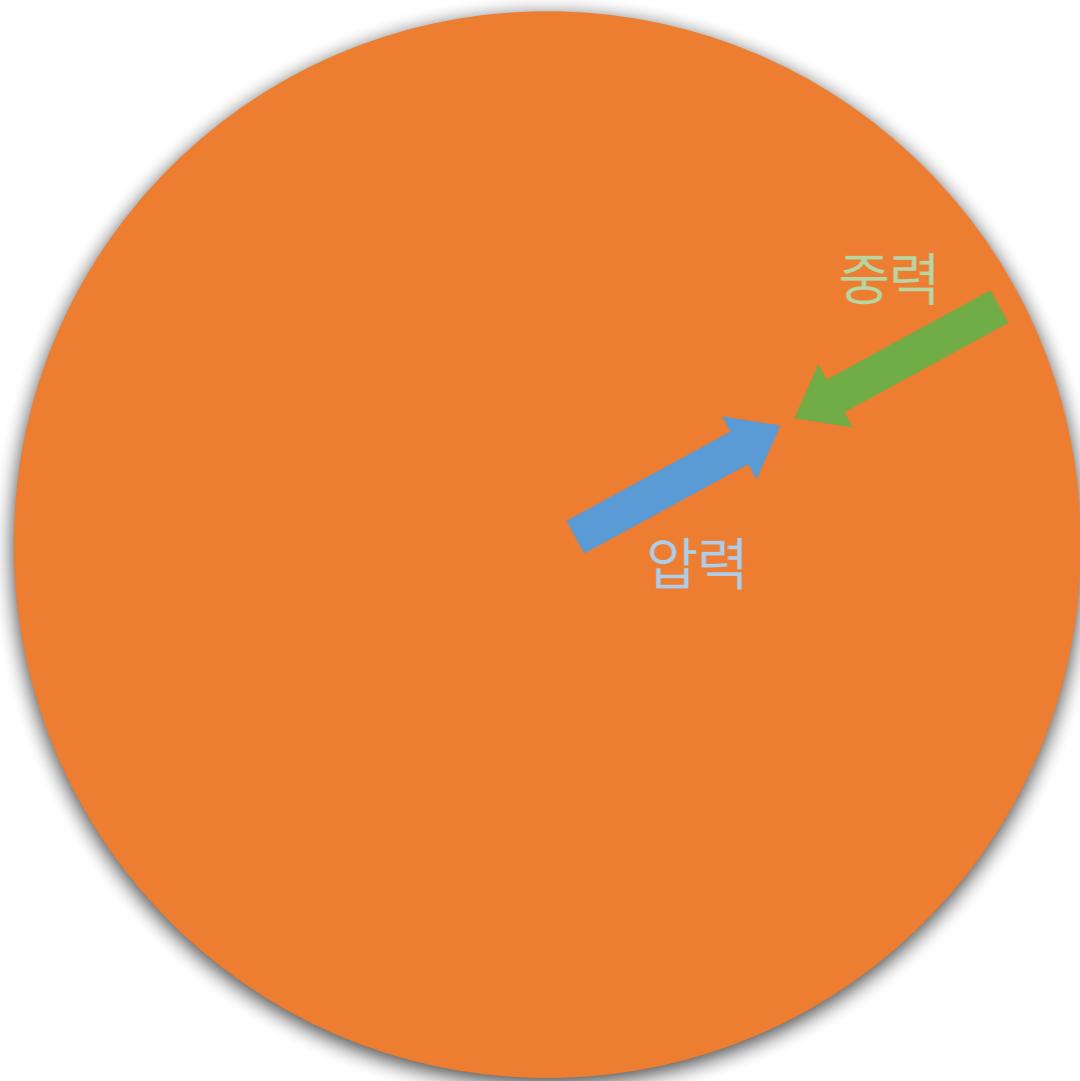
Sirius A

중성자별



Observation by Jocelyn Bell

별(항성)의 구조



항성

정역학적 평형 상태

압력

열압 (thermal): $P \sim T$
복사압 (radiation): $P \sim T^4$
축퇴압 (degenerate): $P \sim \rho^\alpha$

중력

뉴턴 중력
일반상대론

열압 vs 축퇴압



Thermal Pressure

characterized by kinetic motion



Degenerate Pressure

characterized by quantum states

Equation of State of White Dwarf

$$\begin{aligned} P &= \frac{\pi m_e^4 c^5}{h^3} \left[x_F (1 + x_F^2)^{\frac{1}{2}} \left(\frac{2}{3} x_F^2 - 1 \right) + \ln \left[x_F + (1 + x_F^2)^{\frac{1}{2}} \right] \right] \\ &= \frac{8\pi m_e^4 c^5}{15h^3} \left[x_F^5 - \frac{5}{14} x_F^7 + \frac{5}{24} x_F^9 + \dots \right] \quad \text{for } x_F \ll 1 && : \text{non-relativistic limit} \\ &= \frac{2\pi m_e^4 c^5}{3h^3} \left[x_F^4 - x_F^2 + \frac{3}{2} \ln(2x_F) + \dots \right] \quad \text{for } x_F \gg 1 && : \text{ultra-relativistic limit} \end{aligned}$$

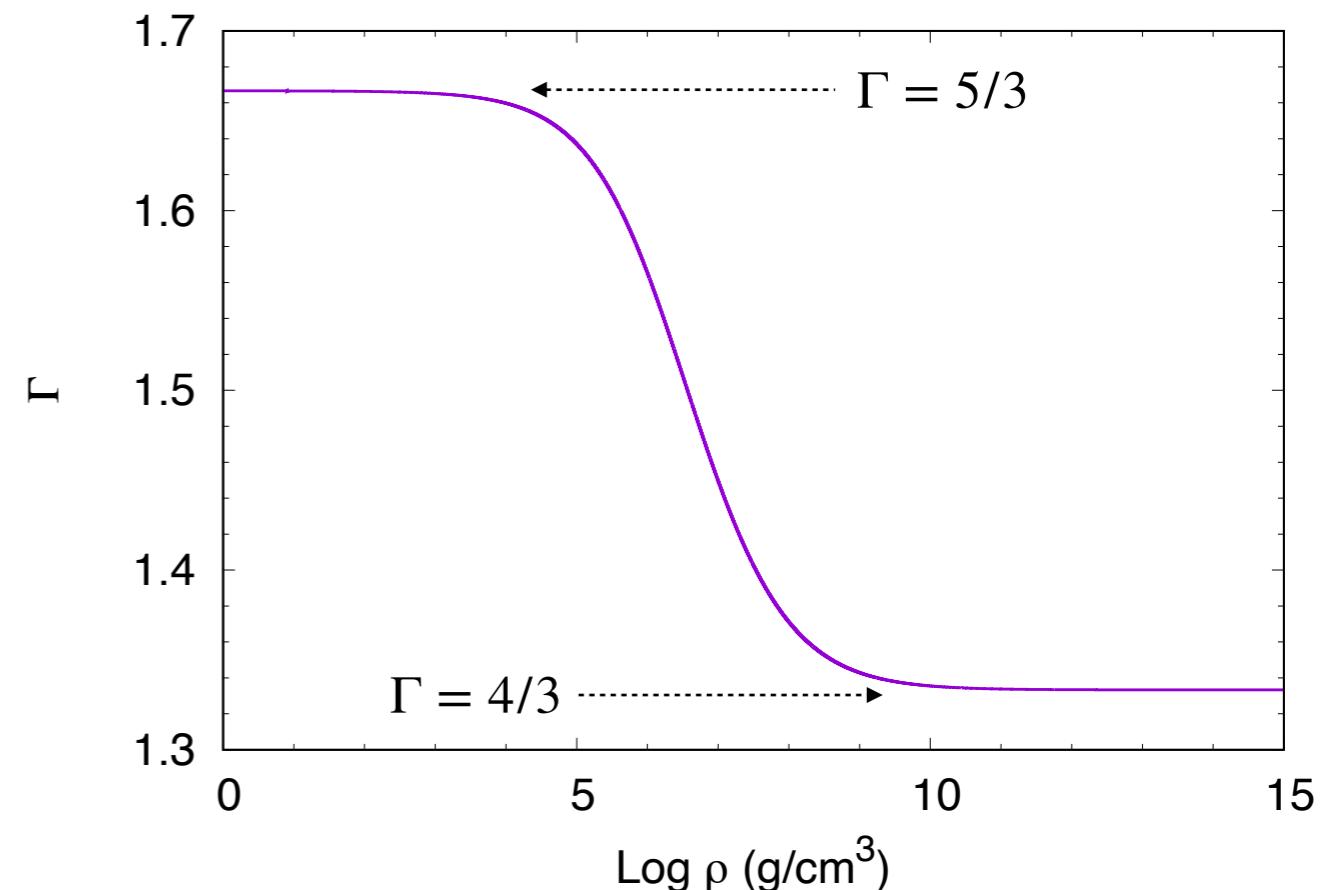
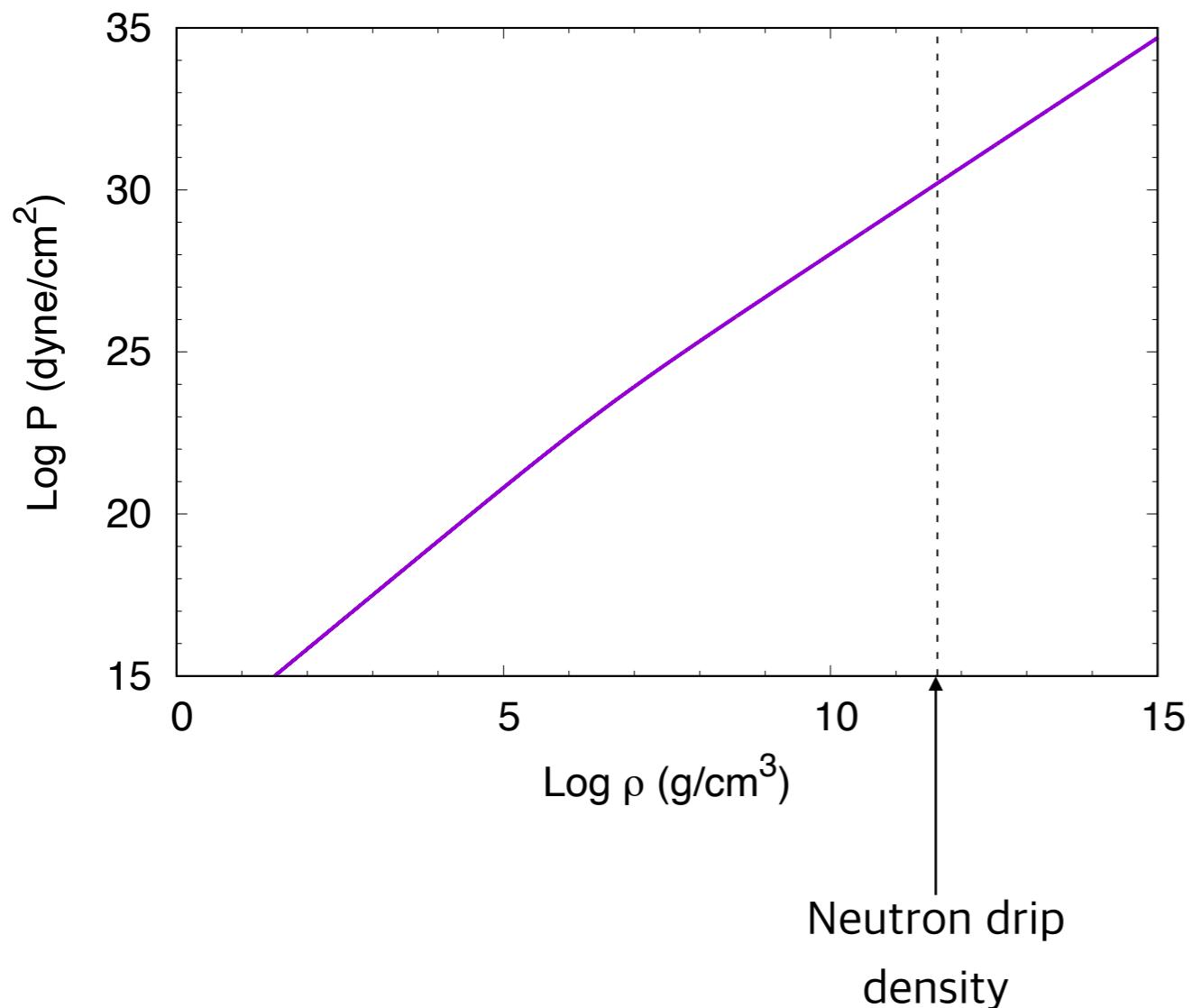
Recall that $p_F = m_e c x \sim n_e^{1/3} \sim \rho^{1/3}$.

Above asymptotic limit of the pressure gives polytropic equation of state i.e., $P = K\rho^\Gamma$.

$$\Gamma = \frac{5}{3} : \text{non-relativistic}$$

$$\Gamma = \frac{4}{3} : \text{relativistic}$$

Equation of State for White Dwarf



중성자별의 상태방정식

내일 (화) 이창환 교수님 수업 참고

Equilibrium Structure of Star

Fluid equation (Euler equation).

1. Continuity Equation

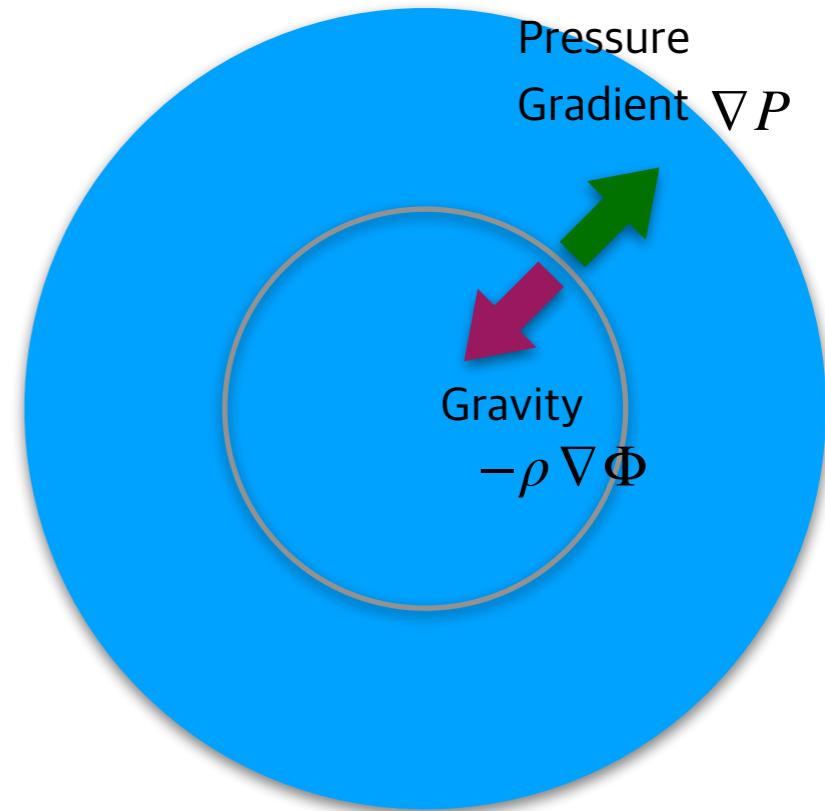
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

2. Momentum Equation

$$\rho \frac{d\vec{v}}{dt} + \rho \vec{v} \cdot \nabla \vec{v} + \nabla P = -\rho \nabla \Phi$$

3. Energy Equation

$$\rho \frac{de}{dt} + \vec{v} \cdot \nabla e + \frac{P}{\rho} \nabla \cdot \vec{v} = 0$$



For spherical star (non-rotating, non-magnetized), LHS of 1. Continuity and 3. Energy equation is 0. The remaining equation is 2. Momentum equation and can be rewritten in spherical coordinates as

$$\frac{dP}{dr} = -\rho \frac{GM_r}{r^2} \quad \text{where } M_r = 4\pi \int_0^r \rho(r) r^2 dr \quad \rightarrow \quad \frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G \rho$$

Lane-Emden Equation

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G \rho$$

Polytropic equation of state: $P = K\rho^\Gamma = K\rho^{1+1/N}$

This equation can further be reduced to dimensionless form by writing

$$\rho = \rho_c \theta^N, \quad r = a\xi, \quad a = \left[\frac{(N+1)K\rho_c^{1/N}-1}{4\pi G} \right]^{1/2}, \text{ where } \rho_c \text{ is the central density.}$$

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^N$$



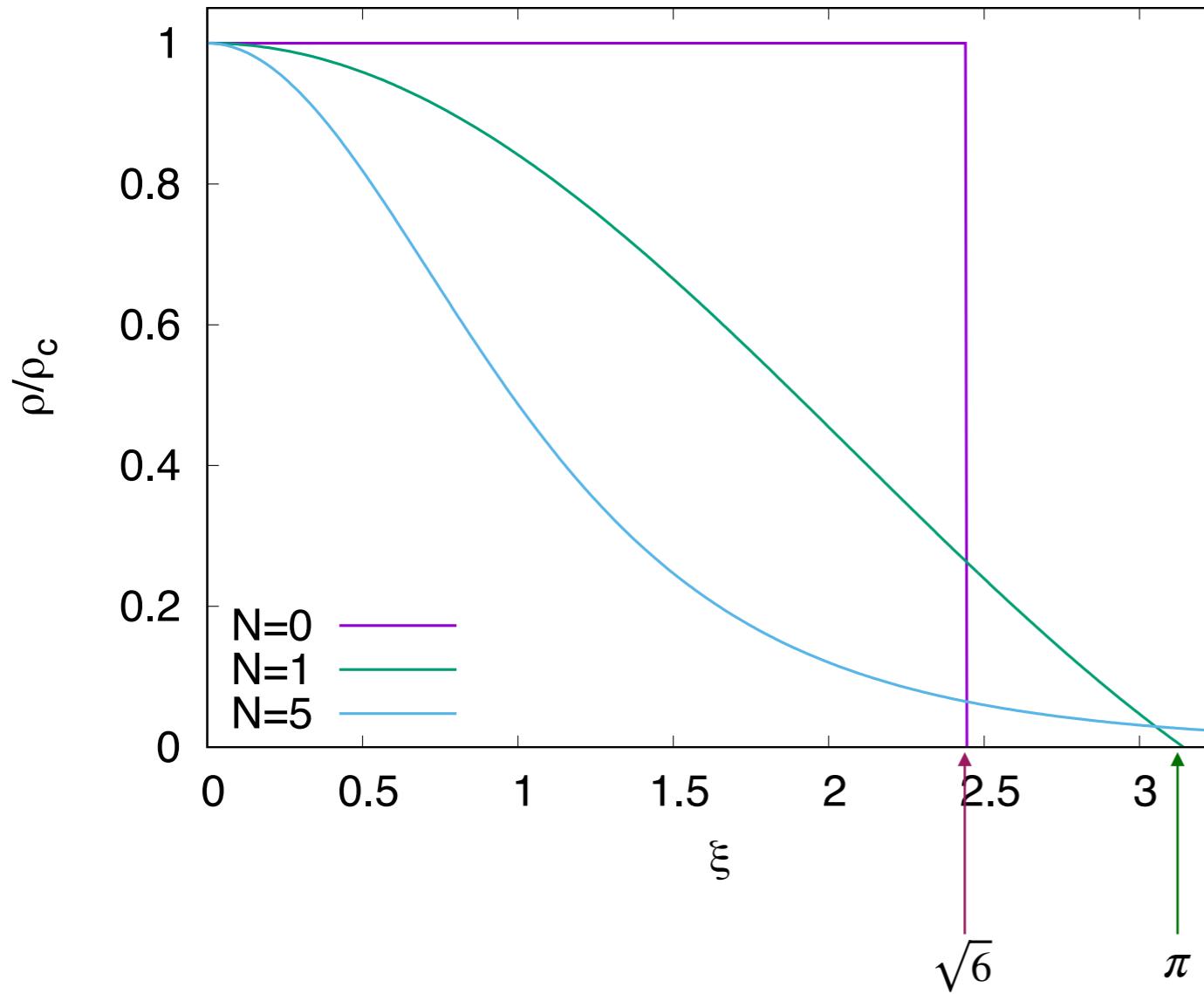
Lane-Emden Equation

$$\text{Boundary condition: } \theta(0) = 1, \frac{d\theta(0)}{dr} = 0.$$

Analytical solutions are available for particular N values (N=0, 1 and 5)

$$\text{For } N=0, \theta(\xi) = 1 - \frac{1}{6}\xi^2, \quad \text{For } N=1, \theta(\xi) = \frac{\sin \xi}{\xi}, \quad \text{For } N=5, \theta(\xi) = \frac{1}{\sqrt{1 + \xi^2/3}}$$

Solution of Lane-Emden Equation



1. Conventionally, the equation of state is
 - hard (stiff) when Γ is large or N is small.
 - soft when Γ is small or N is large.
2. Density gradient with respect to ξ is
 - small for hard EoS.
 - large for soft EoS.
3. Recall that the EoS of relativistic degenerate gas ($N=3$) is softer than that of non-relativistic gas ($N=1.5$).
 - Relativistic degenerate gas can form more compact star than non-relativistic counterpart.
4. $N=5$ solution can extend to infinity while the total mass of the solution is finite.

Mass - Radius Relation

$$R = a\xi_s = \sqrt{\frac{(N+1)K}{4\pi G}} \rho_c^{\frac{1-N}{2N}} \xi_s$$

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi a^3 \rho_c \int_0^{\xi_s} \xi^2 \theta^N d\xi$$

$$= -4\pi a^3 \rho_c \int_0^{\xi_s} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) d\xi$$

$$= -4\pi a^3 \rho_c \xi_s^2 \left| \theta'(\xi_s) \right|$$

$$= 4\pi \left[\frac{(N+1)K}{4\pi G} \right]^{3/2} \rho_c^{\frac{3-N}{2N}} \xi_s^2 \left| \theta'(\xi_s) \right|$$

A star dominated by degenerate pressure becomes smaller as the mass of the star increases. This is the opposite of common sense that we generally know. ($1 < N < 3$).



$$M(R) = 4\pi R^{\frac{3-N}{1-N}} \left[\frac{(N+1)K}{4\pi G} \right]^{\frac{N}{N-1}} \xi_s^{\frac{5-3N}{1-N}} \left| \theta'(\xi_s) \right|$$

Chandrasekhar Mass

As we see in the previous slide, total mass and radius of a star with polytropic EoS are independent on then central density when N=3 that corresponds to the relativistic Fermi electron gas.

$$R = 3.347 \times 10^4 \text{ km} \left(\frac{\rho_c}{10^6 \text{ g/cm}^{-3}} \right)^{-1/3} \left(\frac{Y_e}{0.5} \right)^{2/3}$$
$$M = M_{\text{ch}} = 1.457 M_{\odot} \left(\frac{Y_e}{0.5} \right)^2$$

The Chandrasekhar limit occurs when it asymptotically approach to $\rho_c \rightarrow \infty$ which correspond to the relativistic environment. Resulting radius of the star is zero.

White dwarf with larger than Chandrasekhar mass cannot exist since a star with N>3 is unstable under radial pulsation. This fact will appear soon.

Relativistic Approach

To quantify how relativistic the object is, we can consider two dimensionless quantities as follows:

$$\xi = \frac{GM}{Rc^2}, \quad \beta = \frac{v}{c}.$$

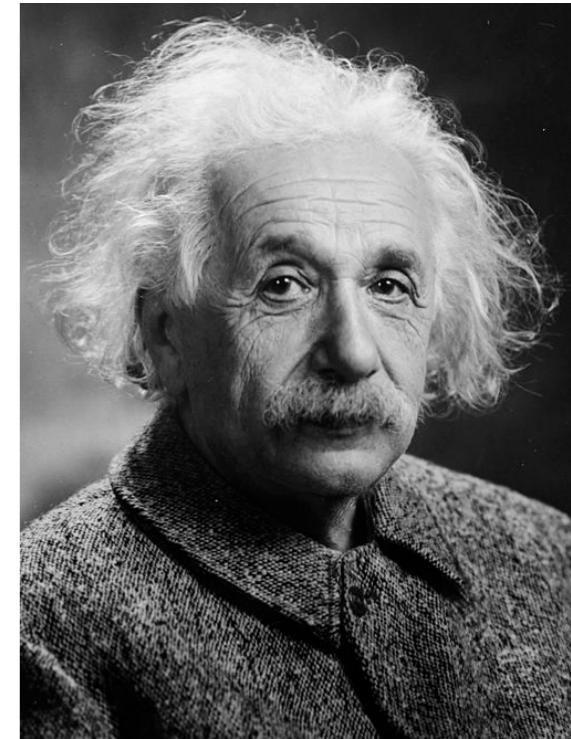
Relativistic Objects

- Black hole: $R = 2GM/c^2$ (Schwarzschild BH), $R = GM/c^2$ (Extreme Kerr BH) $\rightarrow \xi = 0.5 \sim 1$.
- Neutron Star: $M \sim 1.4M_\odot$, $R \sim 10\text{km}$ $\rightarrow \xi \sim 0.2$.
 $P_{\text{rot}} \sim 1\text{ms}$ $\rightarrow \beta \sim 0.2$.
- Jet: $\beta > 0.99$.

Newtonian Objects

- White Dwarf: $M \sim M_\oplus$, $R \sim R_\oplus$ $\rightarrow \xi \sim 0.0003$, $\beta \sim 0.0003$.
- Sun: $M \sim 1M_\odot$, $R \sim 1.4 \times 10^6\text{km}$ $\rightarrow \xi \sim 10^{-6} \ll 1$.

General Relativity



- Albert Einstein proposed a new concept for studying the gravity.
- His idea was summarized in one sentence by John Wheeler.
- “Space-time tells matter how to move, matter tells space-time how to curve.”
- In general relativity the stress energy tensor of perfect fluid and matter current are defined as

$$T^{ab} = \rho_0 h u^a u^b + P g^{ab}, J^a = \rho_0 u^a,$$

where ρ_0 , h and u^a are rest mass density, specific enthalpy and fluid 4-vector, respectively.

- Then the Einstein equation and fluid (Euler) equation in curved space-time can be written as

$$(1) G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab},$$

$$(2) \nabla_a T^{ab} = 0, \nabla_a J^a = 0.$$

- Equation (1) shows that “matter (fluid) tells space-time how to curve”.
- Equation (2) represents that “space-time tells matter (fluid) how to move”.

Equilibrium Structure

- General relativistic counterpart of the Lane-Emden equation Tolmann-Oppenheimer-Volkoff (TOV) equation which is written as follows (Tolman 1939, Oppenheimer & Volkoff 1939)

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho \left(1 + \frac{P}{\rho c^2}\right) \left(1 + \frac{4\pi r^3 P}{mc^2}\right) \left(1 - \frac{2Gm}{rc^2}\right)^{-1}, \quad \frac{dM}{dr} = 4\pi r^2 \rho.$$

where $\rho = \rho_0 + \rho_0 \epsilon = \rho_0 h - P$. Here ρ contains all the energy sources (rest mass as well as internal energy).

- The space-time metric in this equation is

$$ds^2 = -e^\nu c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

where ν is determined by the constraint,

$$\frac{d\nu}{dr} = -\left(\frac{2}{P + \rho c^2}\right) \frac{dP}{dr}.$$

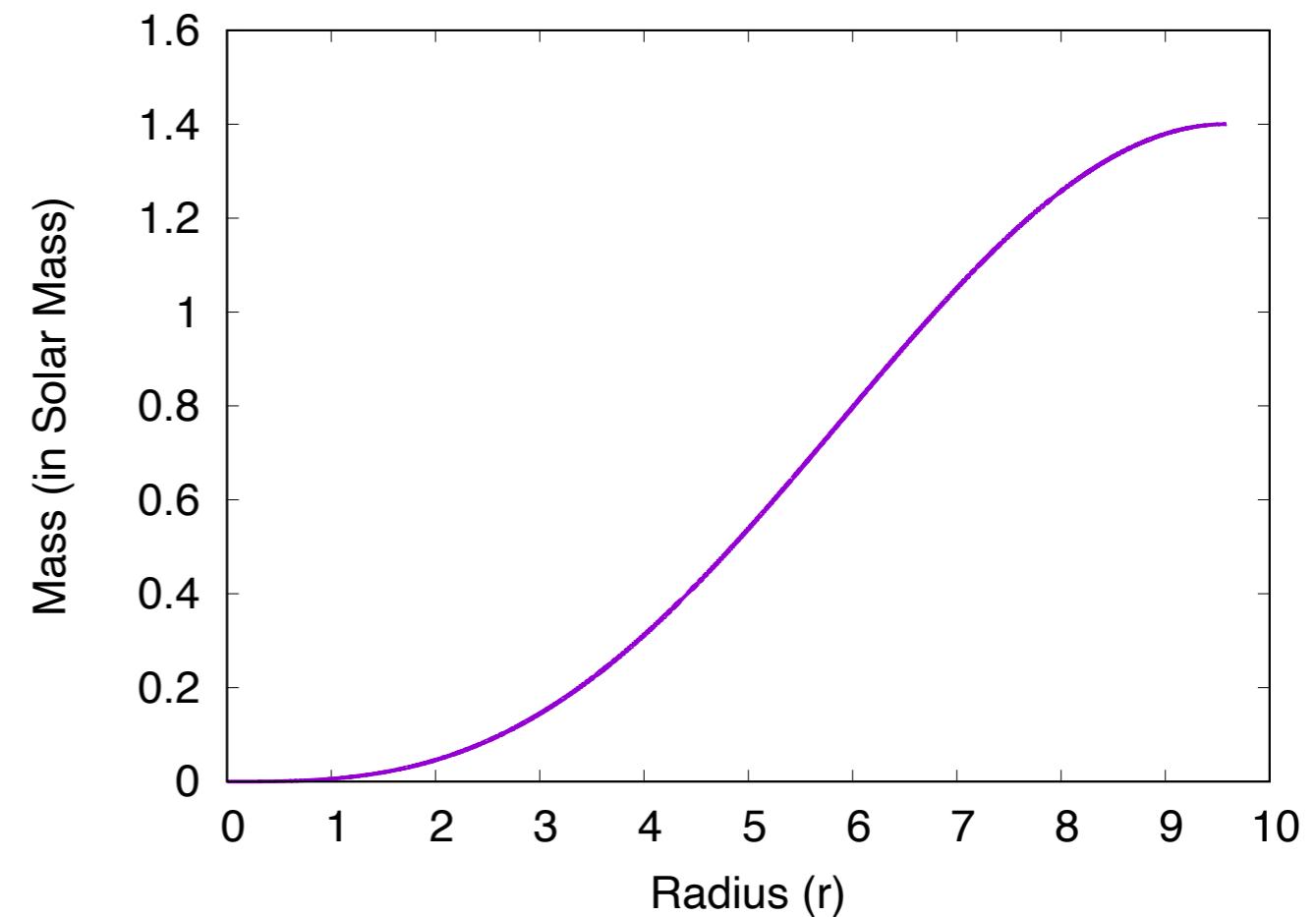
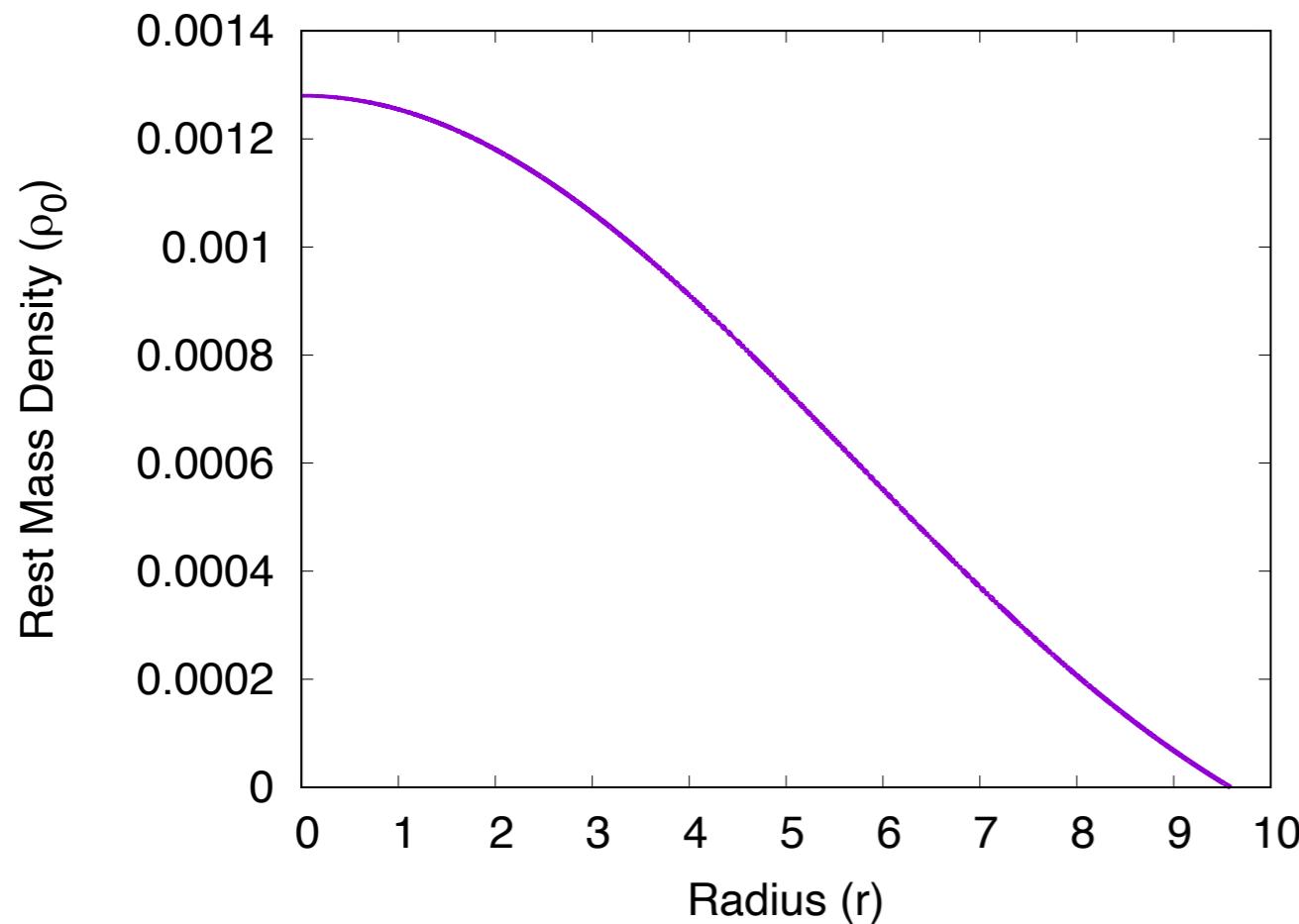
- Schwarzschild metric can be imposed for the boundary condition at the surface of the star.

$$e^\nu = 1 - \frac{2GM}{rc^2}.$$

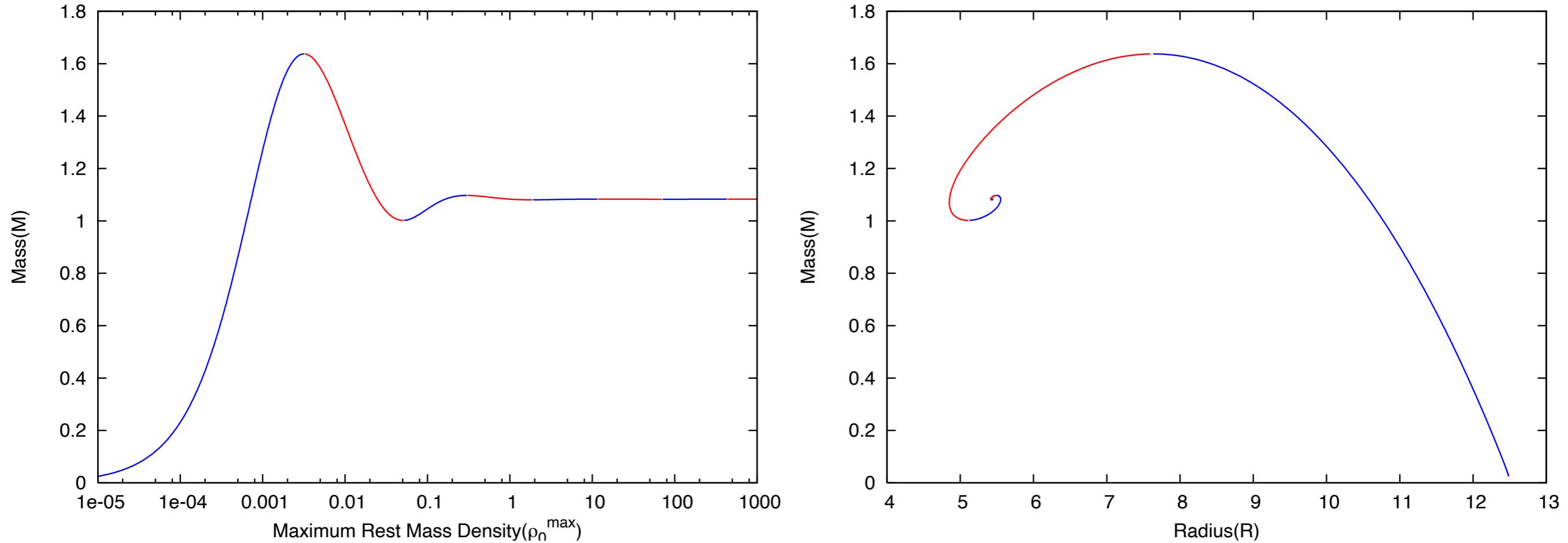
- Detailed derivation will not be covered in this lecture. Please read reference books if you want.

Solution of TOV Equation

Solution of TOV equation using polytropic equation of state with N=1, K=100.



Sequence of Given EoS



- Same turning point method (even in relativistic calculation) can be applied to the neutron star.

$$\frac{dM}{d\rho_{\text{max}}} > 0 \text{ or } \frac{dM}{dR} < 0: \text{Stable},$$

$$\frac{dM}{d\rho_{\text{max}}} < 0 \text{ or } \frac{dM}{dR} > 0: \text{Unstable}.$$

Realistic EoS

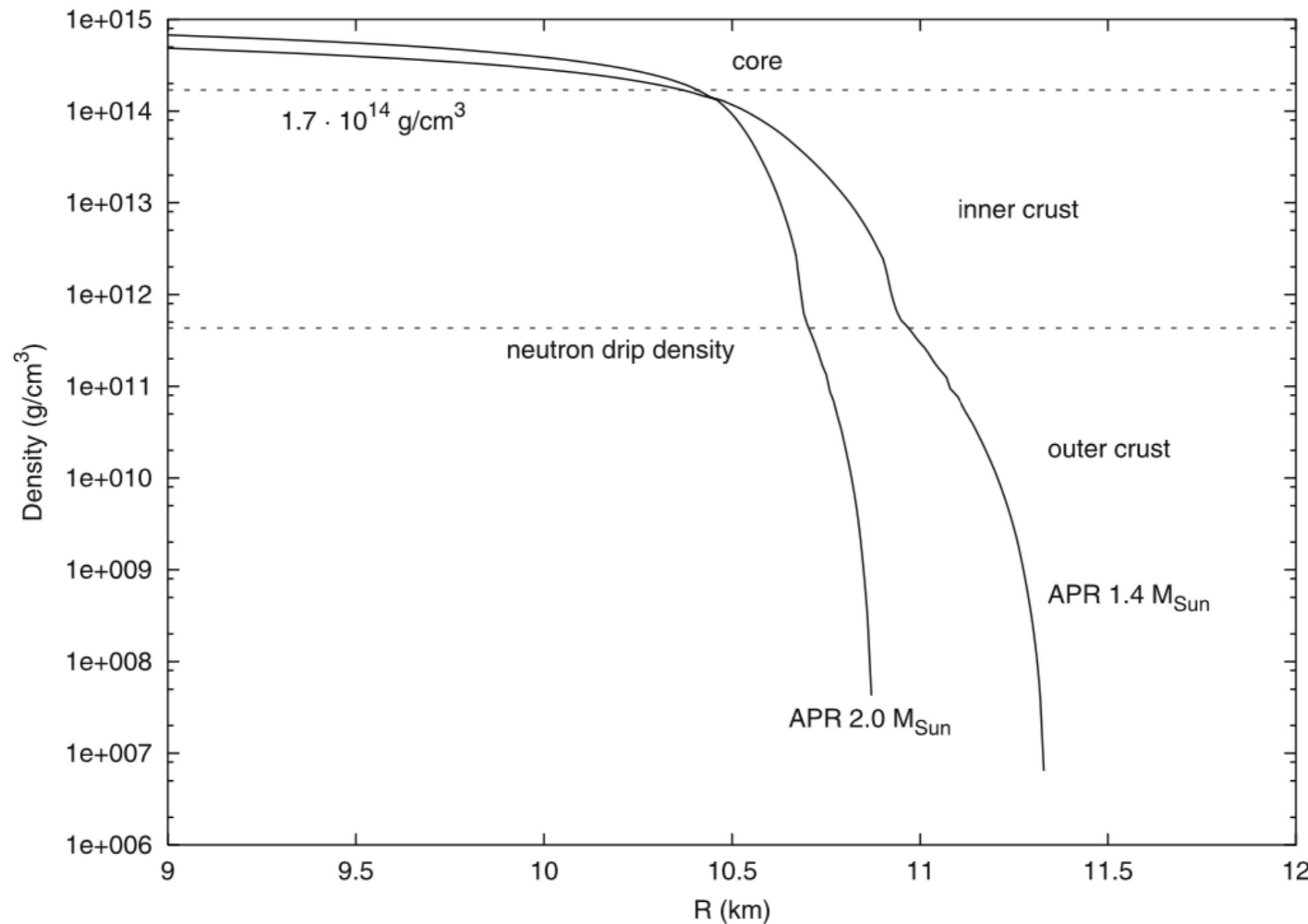
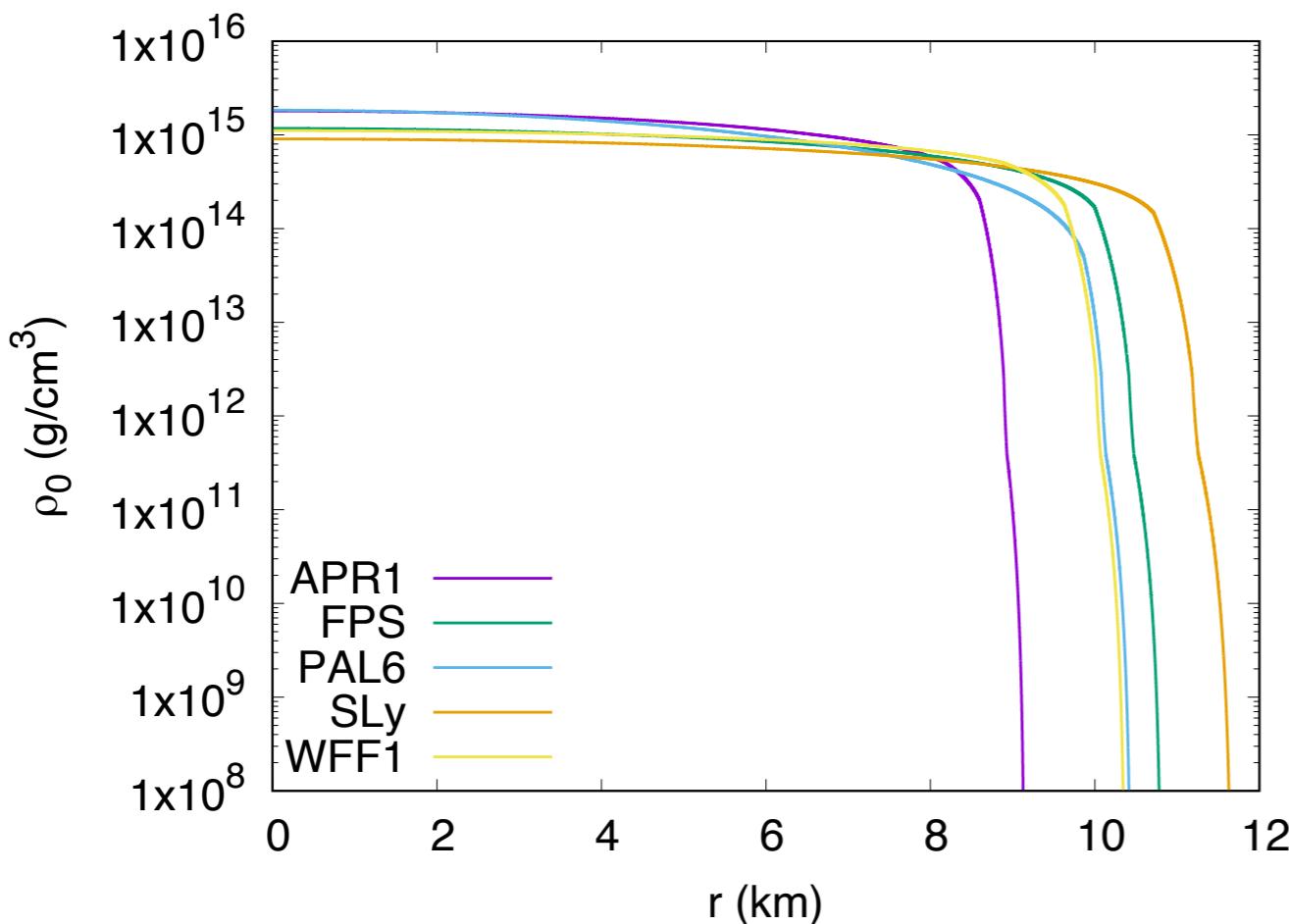


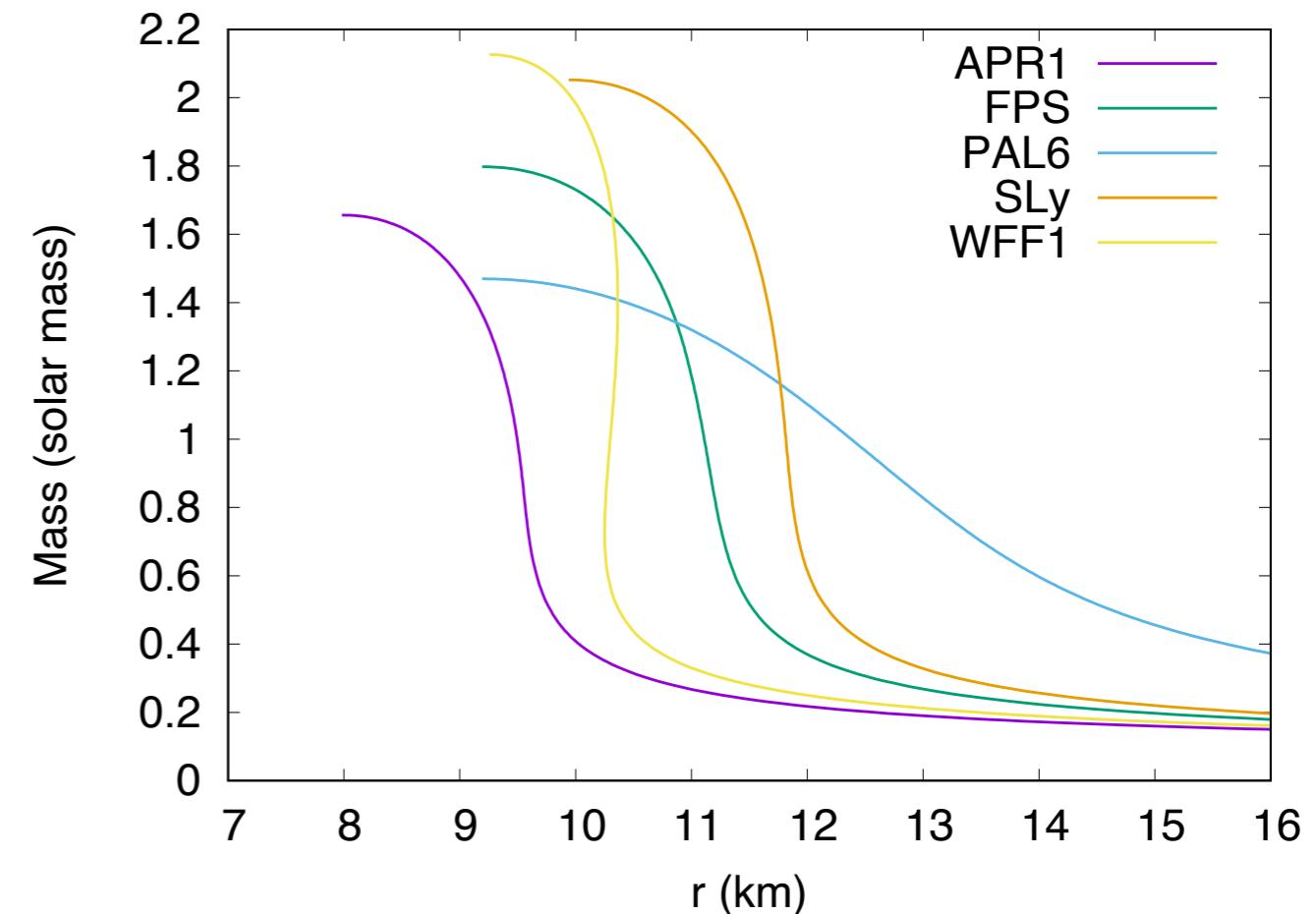
Image from Bauswein (2006)

Different EoS

1.4 solar mass neutron star with various EoSs



M-R relation



등방좌표계

Einstein field equation

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab}$$

$$dx^2 + dy^2 + dz^2$$

Metric assumption

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2)$$

Non-zero Christoffel symbols

$$\Gamma_{tr}^t = \Gamma_{rt}^t = \nu',$$

$$\Gamma_{tt}^r = e^{2(\nu - \lambda)}\nu', \quad \Gamma_{rr}^r = \lambda', \quad \Gamma_{\theta\theta}^r = -r^2\lambda' - r, \quad \Gamma_{\phi\phi}^r = -\sin^2\theta(r^2\lambda' + r),$$

$$\Gamma_{r\theta}^\theta = \Gamma_{\theta r}^\theta = \lambda' + \frac{1}{r}, \quad \Gamma_{\phi\phi}^\theta = -\sin\theta\cos\theta,$$

$$\Gamma_{r\phi}^\phi = \Gamma_{\phi r}^\phi = \lambda' + \frac{1}{r}, \quad \Gamma_{\theta\phi}^\phi = \Gamma_{\phi\theta}^\phi = \frac{\cos\theta}{\sin\theta}$$

등방좌표계

Einstein field equation

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab}$$

Metric assumption

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2)$$

Einstein tensor

$$\begin{aligned} G_t^t &= e^{-2\lambda} \left[(\lambda')^2 + 2\lambda'' + \frac{4}{r}\lambda' \right], \\ G_r^r &= e^{-2\lambda} \left[2\nu'\lambda' + (\lambda')^2 + \frac{2}{r}(\nu' + \lambda') \right], \\ G_\theta^\theta = G_\phi^\phi &= e^{-2\lambda} \left[(\nu')^2 + \nu'' + \lambda'' + \frac{1}{r}(\nu' + \lambda') \right]. \end{aligned}$$

We will choose

$$\begin{aligned} G_t^t &= e^{-2\lambda} \left[(\lambda')^2 + 2\lambda'' + \frac{4}{r}\lambda' \right] \\ &= 8\pi T_t^t = -8\pi(\rho h - P), \\ G_i^i - G_t^t &= 2e^{-2\lambda} \left[\nu'' + (\nu')^2 + \nu'\lambda' + \frac{2}{r}\nu' \right] \\ &= 8\pi(T_i^i - T_t^t) = 8\pi(\rho h + 2P) \end{aligned}$$

유체역학

Ideal fluid

$$T^{ab} = \rho h u^a u^b + P g^{ab}$$

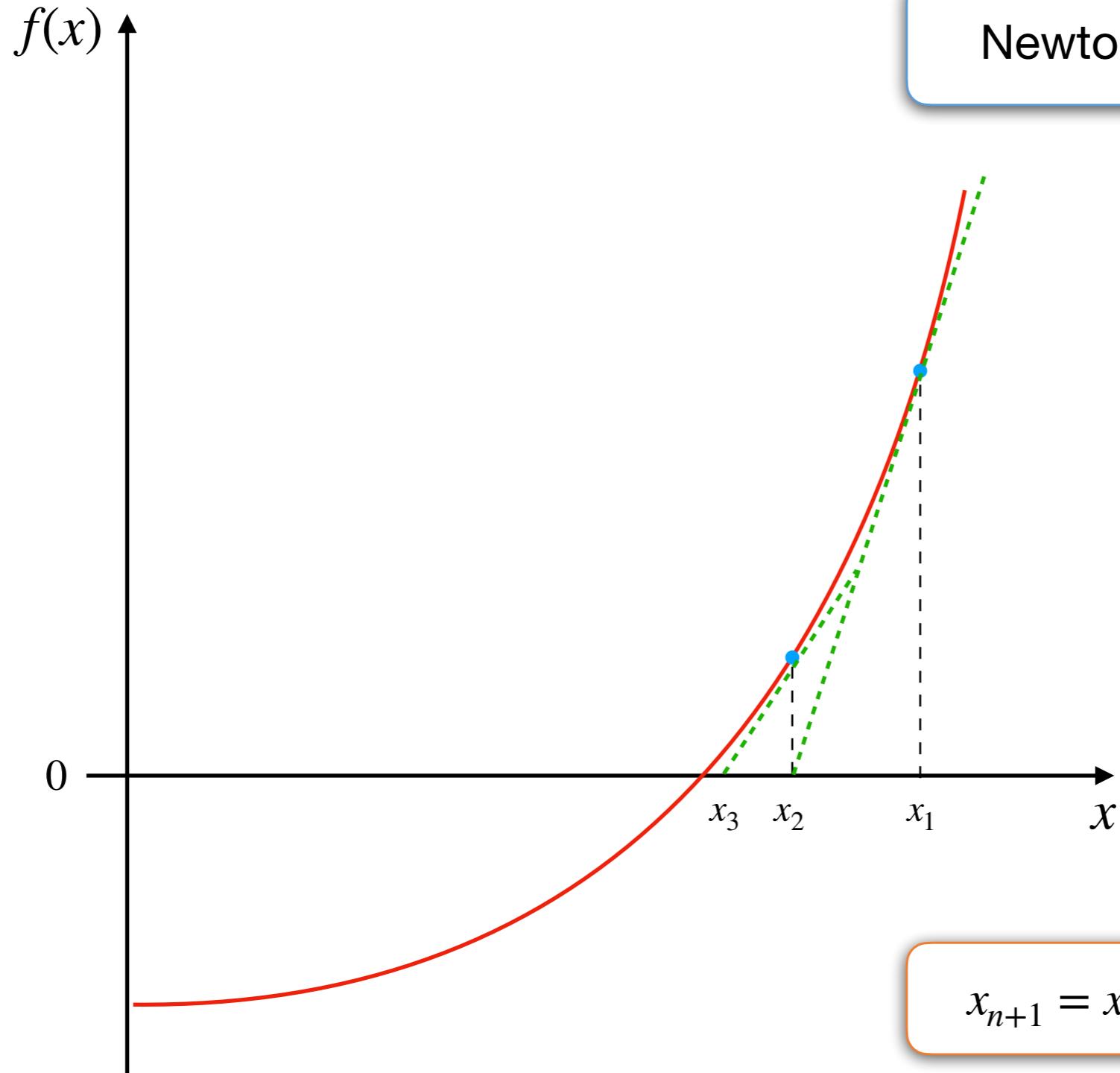
$$\nabla_a T^{ab} = 0$$

$$\begin{aligned}\gamma_r^b \nabla_a T_b^a \\&= \partial_a T_r^a + \Gamma_{ab}^a T_r^b - \Gamma_{ra}^b T_b^a \\&= \frac{1}{\sqrt{-g}} \partial_a \left(\sqrt{-g} T_r^a \right) - \Gamma_{ra}^b T_b^a \\&= P' + \rho h \nu' = 0\end{aligned}$$

$$\ln h + \nu = \text{const.}$$

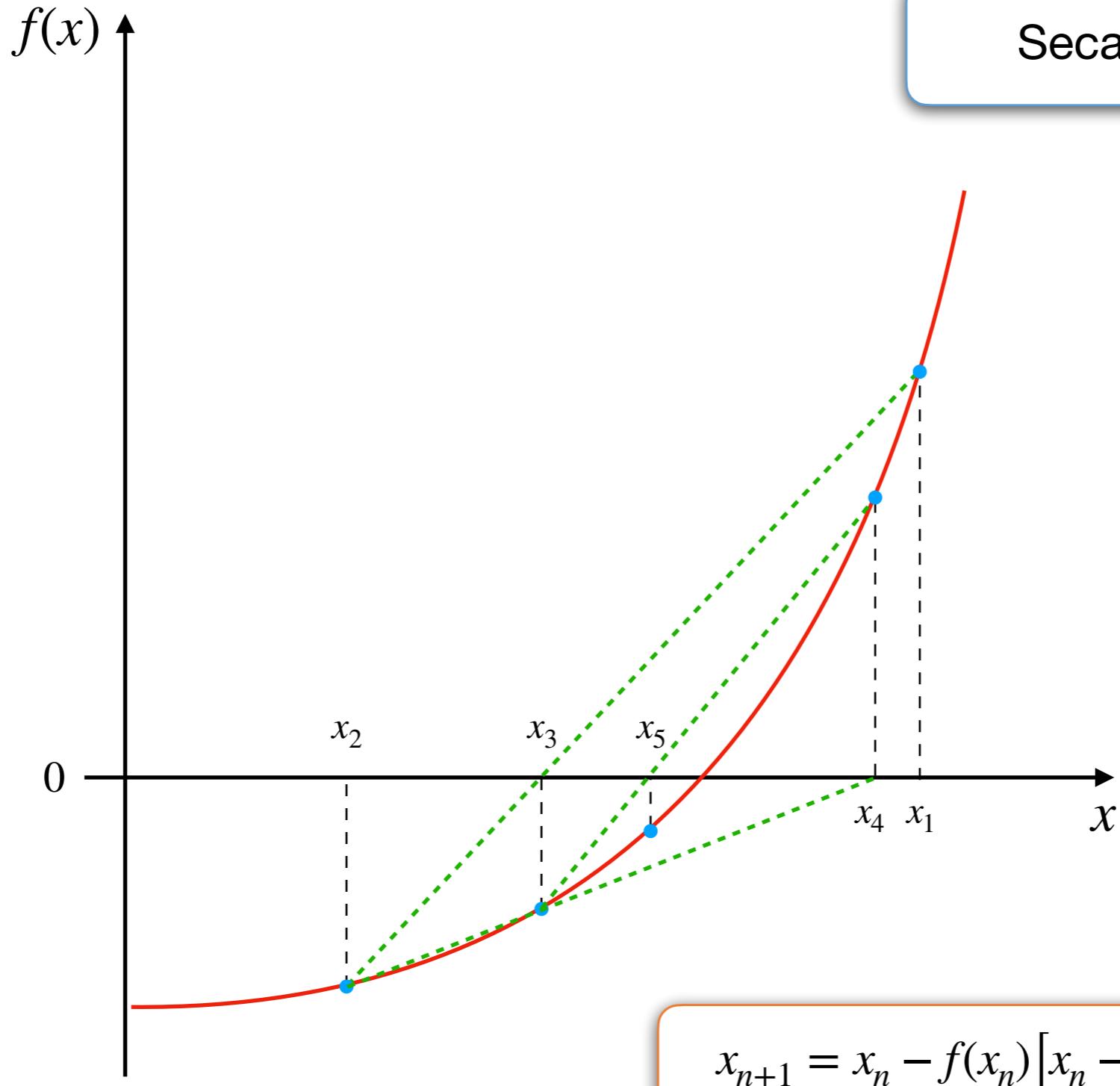
수치적 방법

수치방법을 통한 방정식 풀이



$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$

수치방법을 통한 방정식 풀이



Secant Method

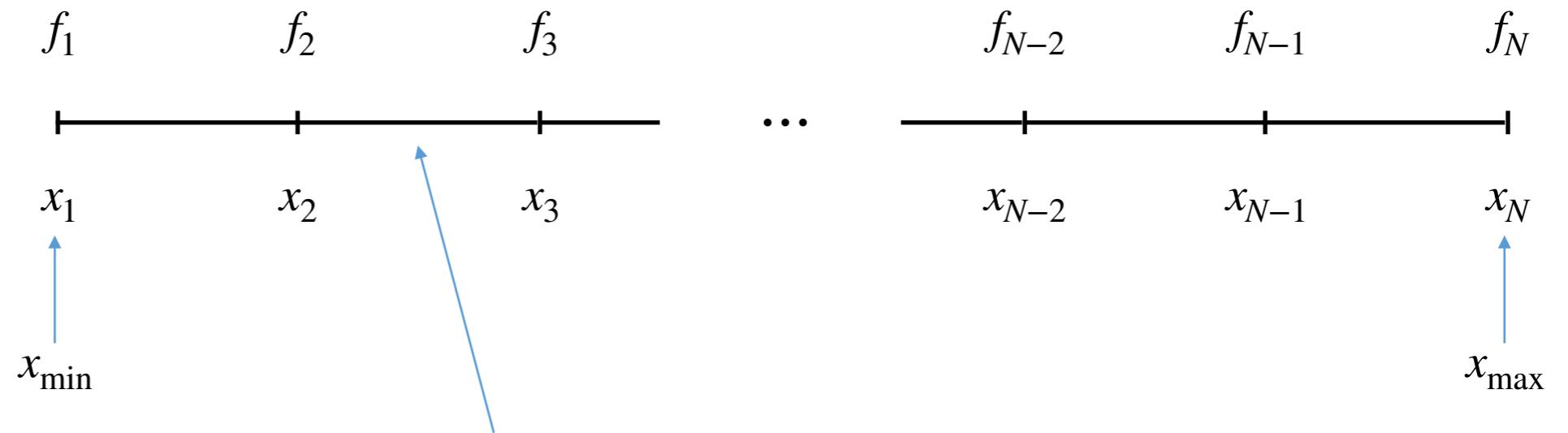
$$x_{n+1} = x_n - f(x_n)[x_n - x_{n-1}] / [f(x_n) - f(x_{n-1})]$$

할선법의 특징

- 할선법은 뉴턴 방법과 근접한 해의 수렴성을 갖는다 (bisection 방법에 비해 매우 빠름).
- Bisection 방법에 비해 잘못된 해를 줄 가능성이 높다.
- 할선법은 방정식의 미분 값을 알 수 없을때 유용하다.

유한 차분법

그리드 구조



$$\Delta x = \frac{x_{\max} - x_{\min}}{N - 1}$$

유한 차분법

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x + \frac{1}{2}f''(x_0)\Delta x^2 + \frac{1}{6}f'''(x_0)\Delta x^3$$

$$f(x_0 - \Delta x) = f(x_0) - f'(x_0)\Delta x + \frac{1}{2}f''(x_0)\Delta x^2 - \frac{1}{6}f'''(x_0)\Delta x^3$$



$$f'(x_0) = \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x}$$

$$f''(x_0) = \frac{f(x_0 + \Delta x) - 2f(x_0) + f(x_0 - \Delta x)}{\Delta x^2}$$

Centered differencing

유한 차분법

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x + \frac{1}{2}f''(x_0)\Delta x^2 + \frac{1}{6}f'''(x_0)\Delta x^3$$

$$f(x_0 + 2\Delta x) = f(x_0) + 2f'(x_0)\Delta x + 2f''(x_0)\Delta x^2 + \frac{4}{3}f'''(x_0)\Delta x^3$$

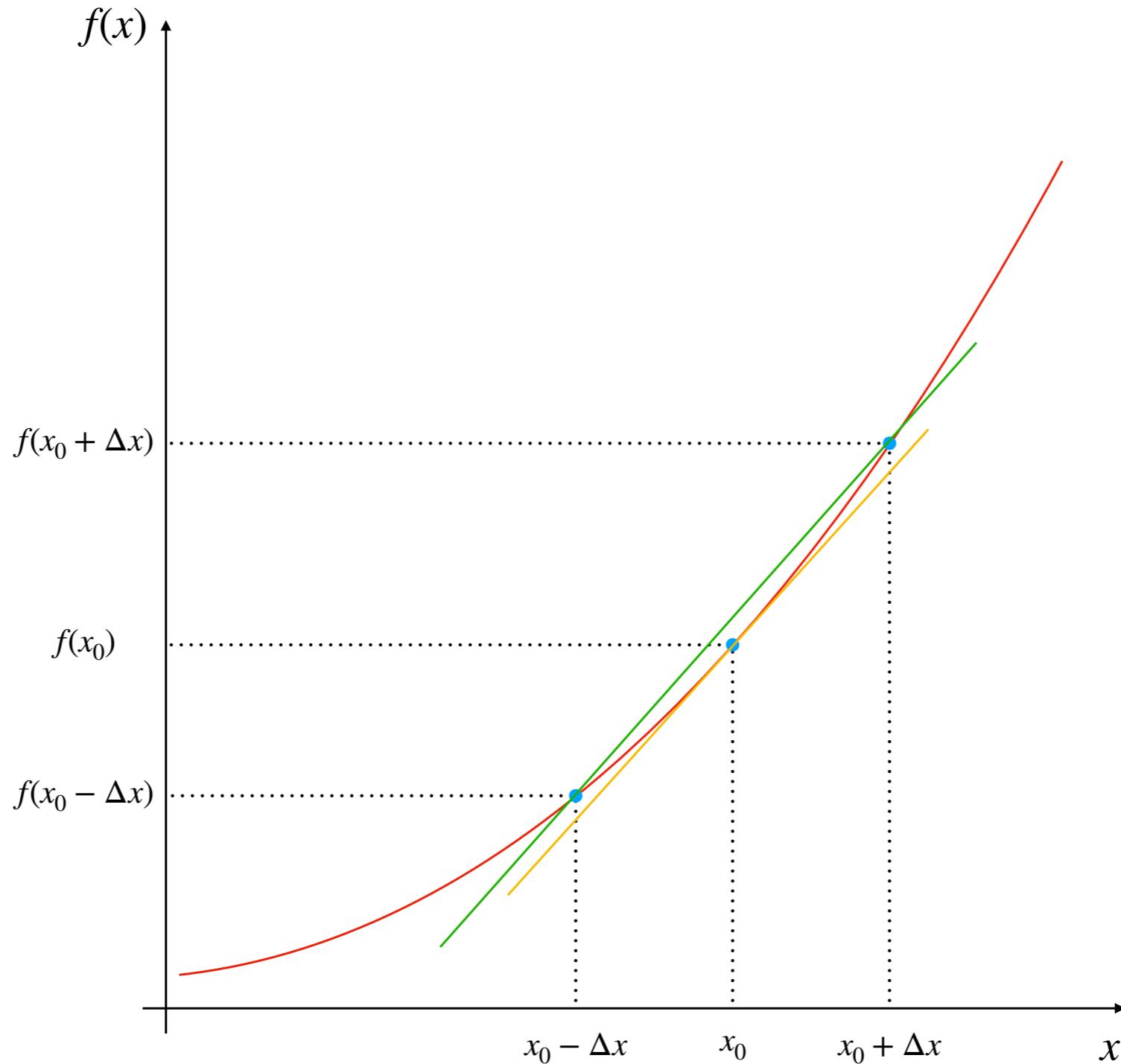


$$f'(x_0) = \frac{-f(x_0 + 2\Delta x) + 4f(x_0 + \Delta x) - 3f(x_0)}{2\Delta x}$$

$$f''(x_0) = \frac{f(x_0 + 2\Delta x) - 2f(x_0 + \Delta x) + f(x_0)}{\Delta x^2}$$

Forward differencing

유한 차분법

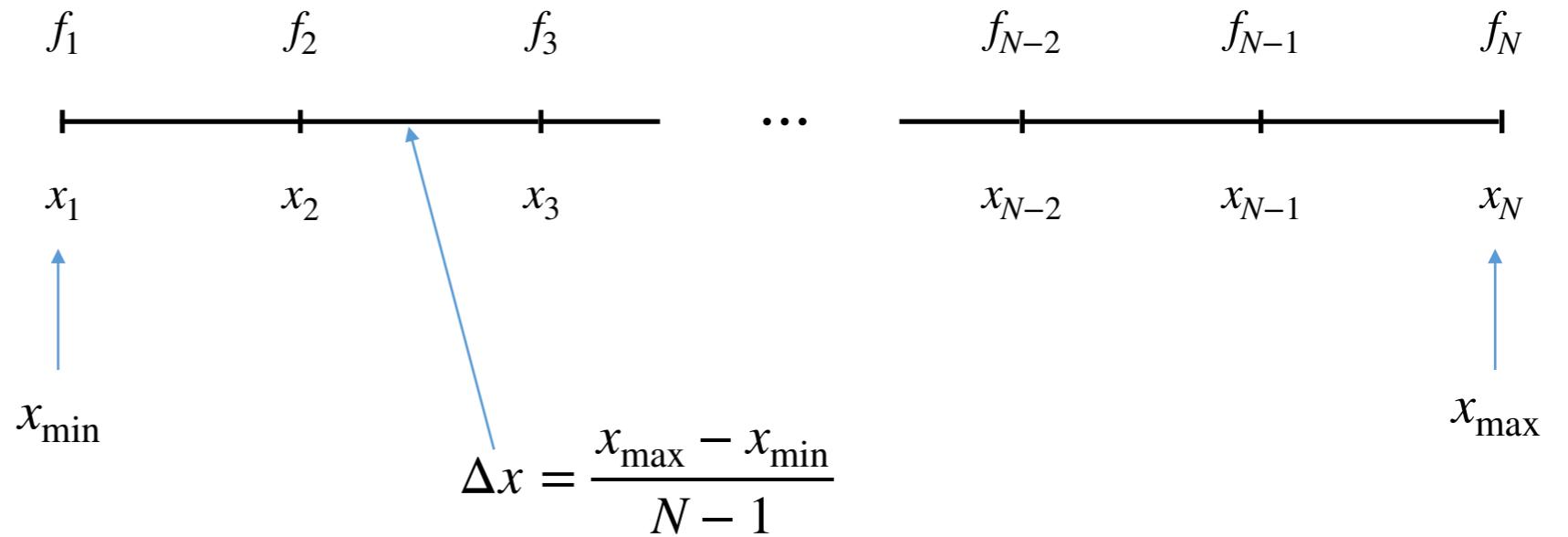


완화법

$$\frac{d^2f}{dx^2} = g(x)$$



$$r_i = f_{i+1} - 2f_i + f_{i-1} - g_i \Delta x^2$$



$$f_i^{n+1} = f_i^n - r_i / \frac{\partial r_i}{\partial f_i}$$

완화법의 특징

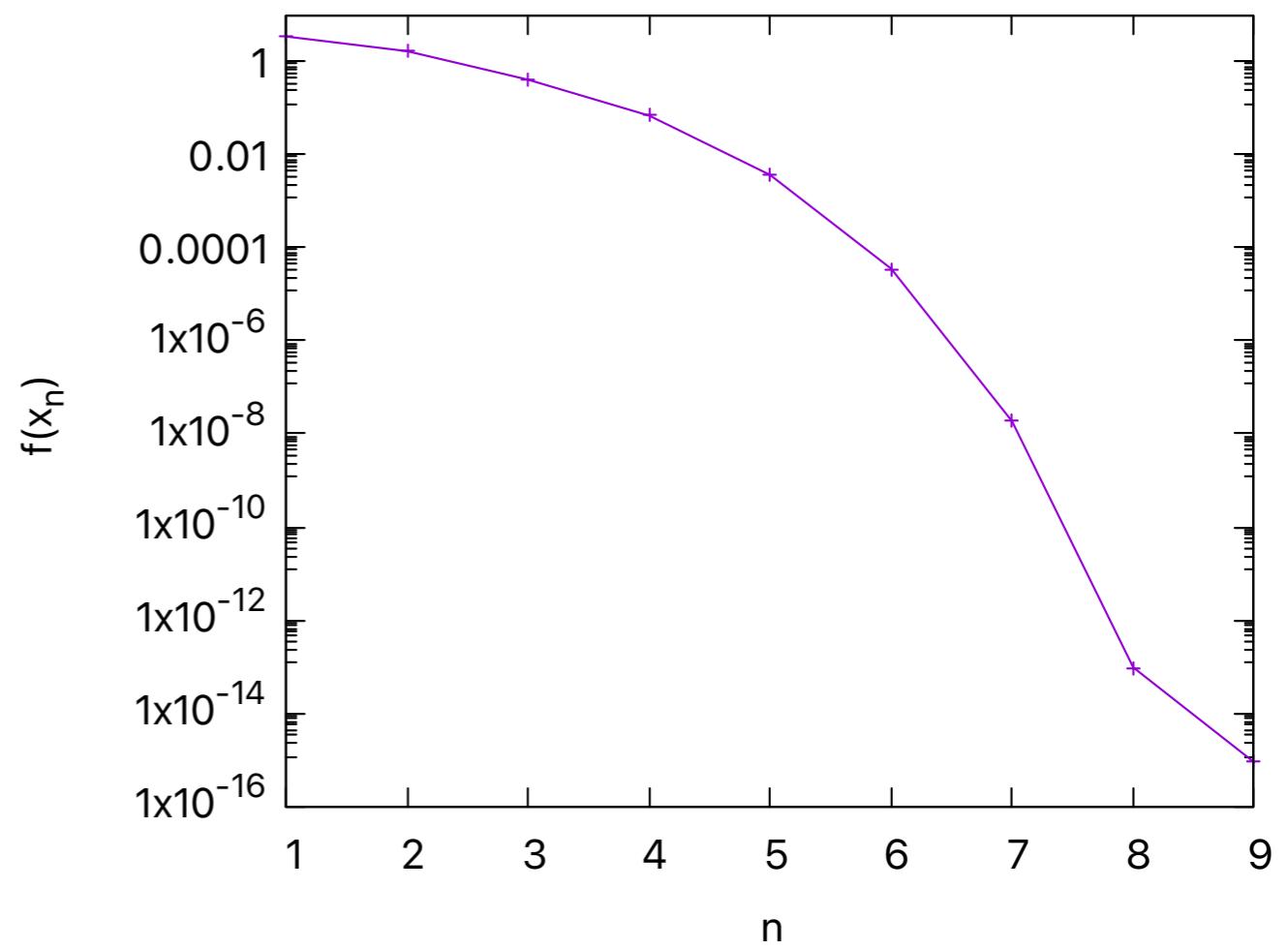
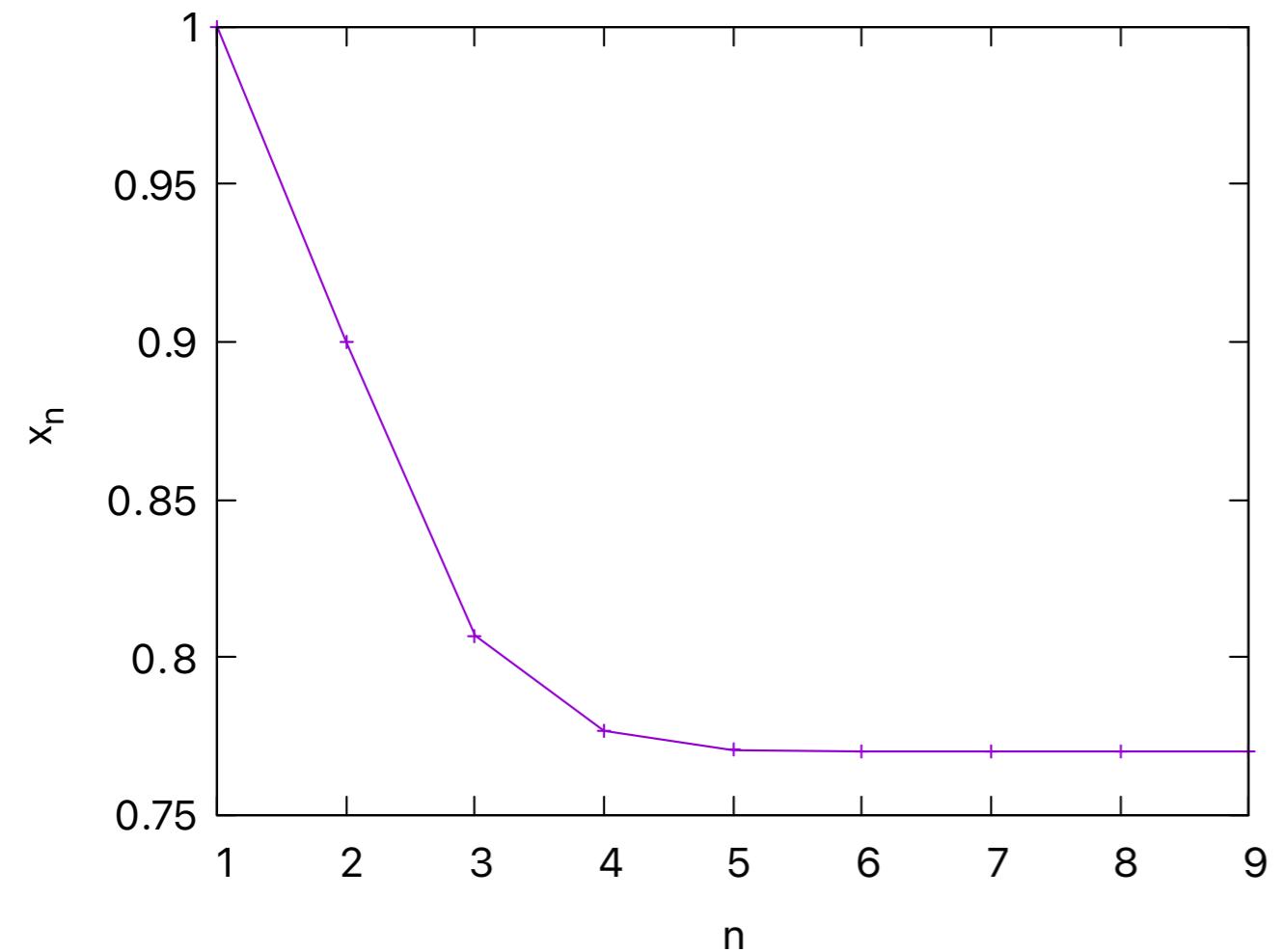
- 완화법을 통한 풀이는 매우 안정적이다.
- 하지만 수렴성이 매우 느리다 (계산 속도가 매우 느리다).
- 그리드의 크기에 비례하는 에러는 빠르게 걸러주는 반면 전체적인 큰 변화는 매우 천천히 해에 수렴한다.
- 이 특성을 이용한 계층적인 그리드 구조를 가지는 계산을 통해 수렴성을 크게 개선할 수 있다 (Multigrid method).

문제 1: 할선법을 이용한 비선형 방정식의 해 구하기

$$f(x) = 3 + x^2 - xe^{2x} \quad x_1 = 1, \quad x_2 = 0.9$$

n	x_n	$f(x_n)$
1	1.000000000000000E+00	-3.3890560989306504E+00
2	9.000000000000002E-01	-1.6346827179716521E+00
3	8.0682241672647359E-01	-4.0015661075950071E-01
4	7.7662003728413431E-01	-6.7758198284421312E-02
	:	

$|f(x_n)| < \text{tol} = 1 \times 10^{-14}$
이 될 때까지 계산을 지속



문제 2: 유한 차분법과 완화법을 이용한 뿌아송 방정식 풀이

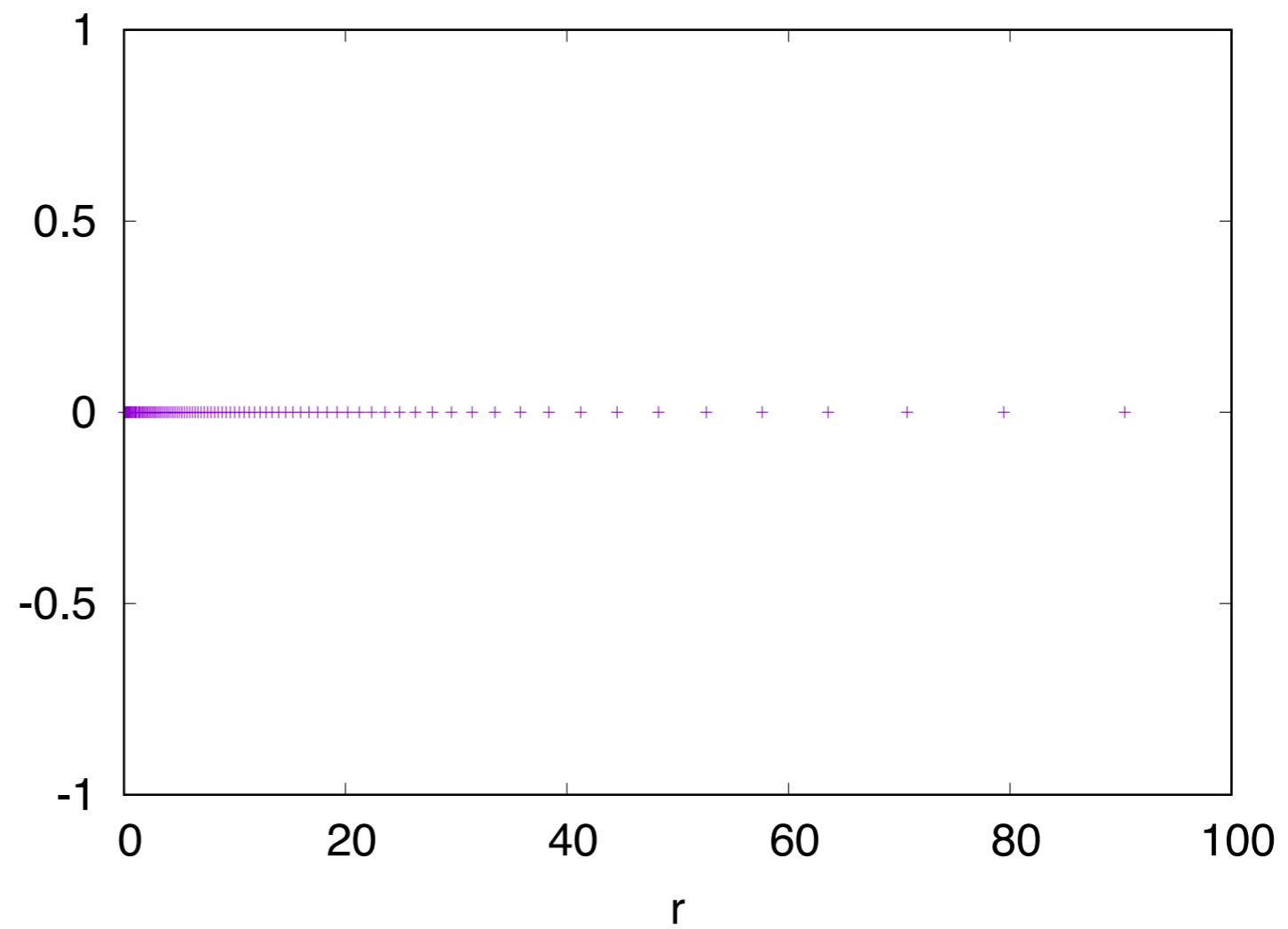
$$\frac{d^2\Phi}{dr^2} + \frac{2}{r} \frac{d\Phi}{dr} = 4\pi\rho_0$$

좌표변환

$$\frac{d^2\Phi}{ds^2} + \frac{2}{s} \frac{d\Phi}{ds} = 4\pi\rho_0 r_s^2 (1 - s)^{-4}$$

$$r = r_s \left(\frac{s}{1 - s} \right)$$

$$\begin{aligned}s &= 0 \rightarrow r = 0 \\s &= 1/2 \rightarrow r = r_s \\s &= 1 \rightarrow r = \infty\end{aligned}$$



$$\frac{d^2\Phi}{ds^2} + \frac{2}{s} \frac{d\Phi}{ds} = 4\pi\rho_0 r_s^2 (1-s)^{-4}$$



$$\frac{\Phi_{i+1} - 2\Phi_i + \Phi_{i-1}}{\Delta s^2} + \frac{2}{s} \frac{\Phi_{i+1} - \Phi_{i-1}}{2\Delta s} = 4\pi\rho_0 r_s^2 (1-s)^{-4}$$



$$r_i = \Phi_{i+1} - 2\Phi_i + \Phi_{i-1} + \frac{1}{s} (\Phi_{i+1} - \Phi_{i-1}) \Delta s - 4\pi\rho_0 r_s^2 (1-s)^{-4} \Delta s^2$$

경계조건

$$s = 0 \ (r = 0)$$



$$\frac{d\Phi}{ds} = 0$$



$$\frac{-f_{i+2} + 4f_{i+1} - 3f_i}{2\Delta s} = 0$$

$$s = 1 \ (r = \infty)$$

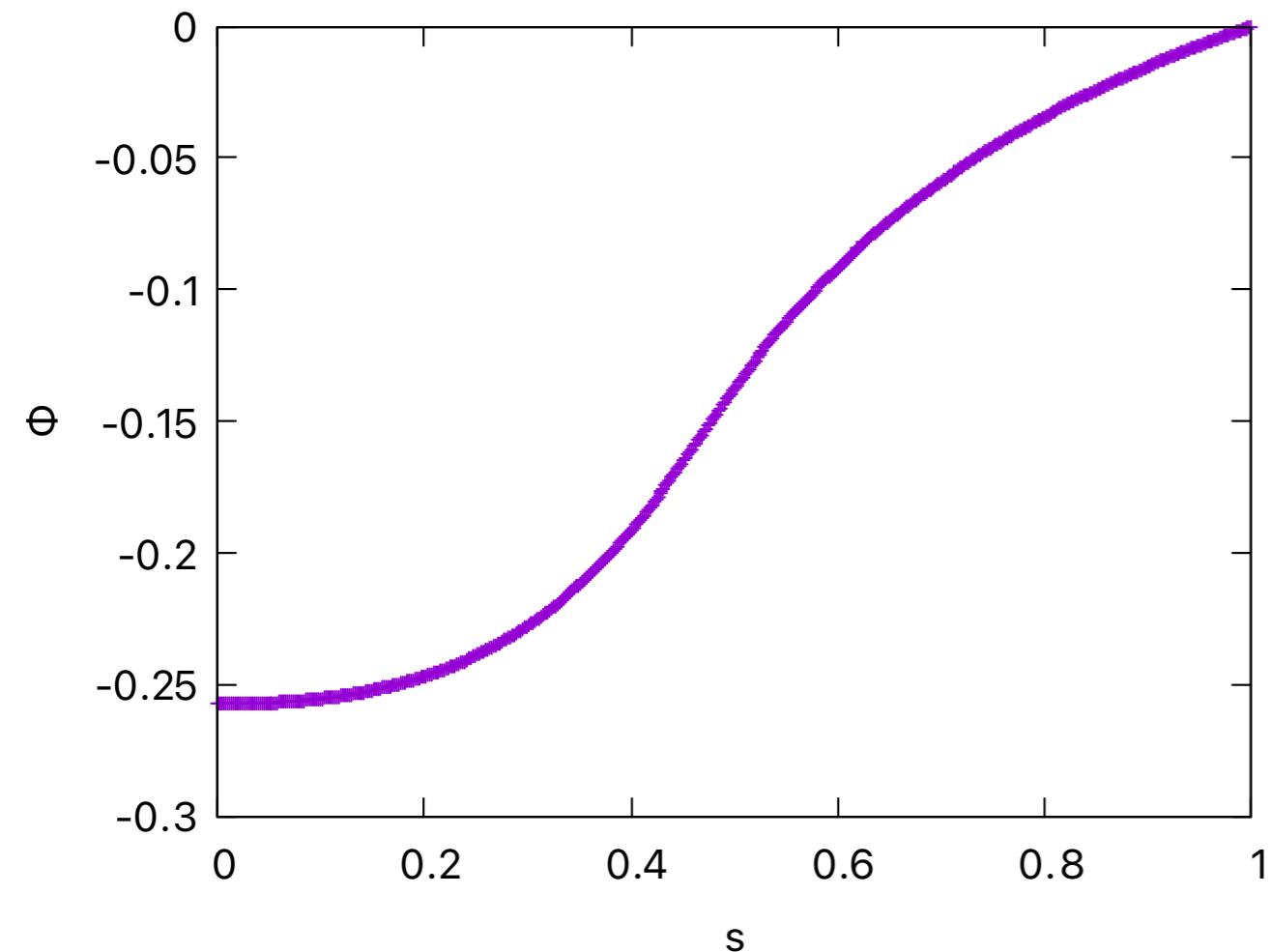


$$\Phi = 0$$

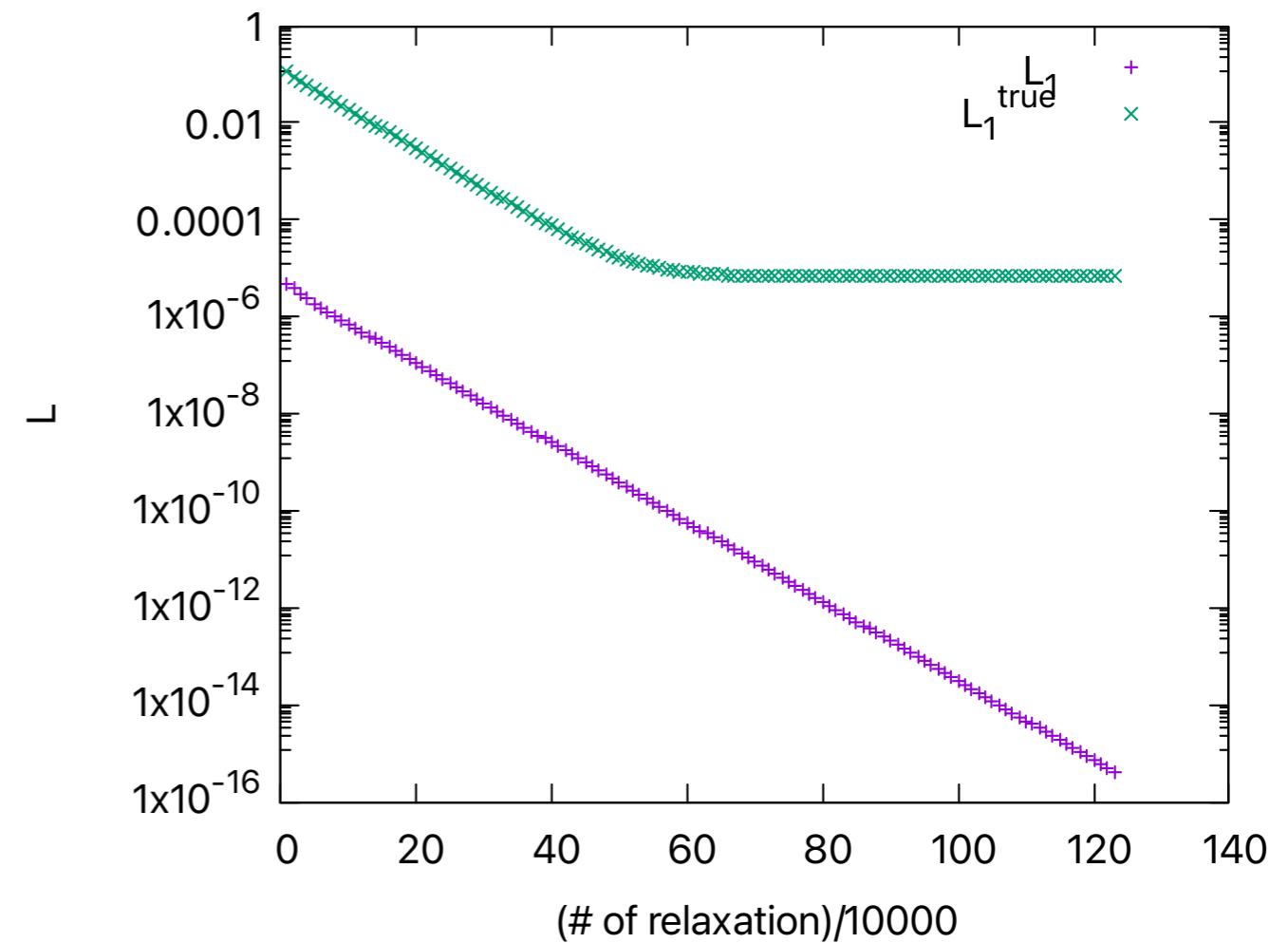
	$n/10000$	L_1 norm of r_i	L_∞ norm of r_i	L_1 norm of difference (true - numerical)	L_∞ norm of difference (true - numerical)
1	5.0109932547092184E-06	1.6133744119961602E-05	1.1259869242093096E-01	2.1184500361626862E-01	
2	3.8591442333689623E-06	1.213673208235122E-05	9.0539141656252264E-02	1.6433087322862749E-01	
3	2.9931845283822996E-06	8.7691916915710877E-06	7.3568761366881286E-02	1.2989197442695077E-01	
4	2.3765451864331352E-06	6.5927485466410474E-06	6.0251564330984088E-02	1.0455402059800048E-01	
5	1.9191721100935895E-06	5.1359617012636249E-06	4.9583982910048688E-02	8.5161077466815338E-02	
6	1.5663280301966552E-06	4.0998301837902318E-06	4.0920979065003055E-02	6.9862119717043952E-02	
7	1.2864450203351027E-06	3.3231354575891103E-06	3.3826902160134162E-02	5.7551665757009929E-02	
8	1.0604616573810848E-06	2.7183906710082084E-06	2.7989006839431500E-02	4.7525131531390319E-02	
9	8.7602641062685398E-07	2.2356701832038084E-06	2.3171178239209157E-02	3.9299937145575836E-02	
10	7.2454471800758594E-07	1.8443851974581094E-06	1.9188684916534245E-02	3.2524217445069364E-02	
11	5.9967123671666484E-07	1.5242951679761418E-06	1.5893621655597892E-02	2.6929100161045494E-02	

$L_\infty < \text{tol} = 1 \times 10^{-14}$
 이 될 때까지 계산을 지속

최종결과



Convergence



문제 3: 주어진 밀도 분포로 인한 시공간의 휘어짐 구하기

$$e^{-2\lambda} \left[(\lambda')^2 + 2\lambda'' + \frac{4}{r}\lambda' \right] = -8\pi(\rho_0 h - P)$$

$$2e^{-2\lambda} \left[\nu'' + (\nu')^2 + \nu'\lambda' + \frac{2}{r}\nu' \right] = 8\pi(\rho_0 h + 2P)$$

↑
좌표변환

$$r = r_s \left(\frac{s}{1-s} \right)$$

$$\frac{d^2\lambda}{ds^2} + \frac{2}{s} \frac{d\lambda}{ds} + \frac{1}{2} \left(\frac{d\lambda}{ds} \right)^2 = -4\pi e^{2\lambda} r_s^2 (1-s)^{-4} (\rho_0 h - P),$$

$$\frac{d^2\nu}{ds^2} + \frac{2}{s} \frac{d\nu}{ds} + \left(\frac{d\nu}{ds} \right)^2 + \left(\frac{d\nu}{ds} \right) \left(\frac{d\lambda}{ds} \right) = 4\pi e^{2\lambda} r_s^2 (1-s)^{-4} (\rho_0 h + 2P)$$

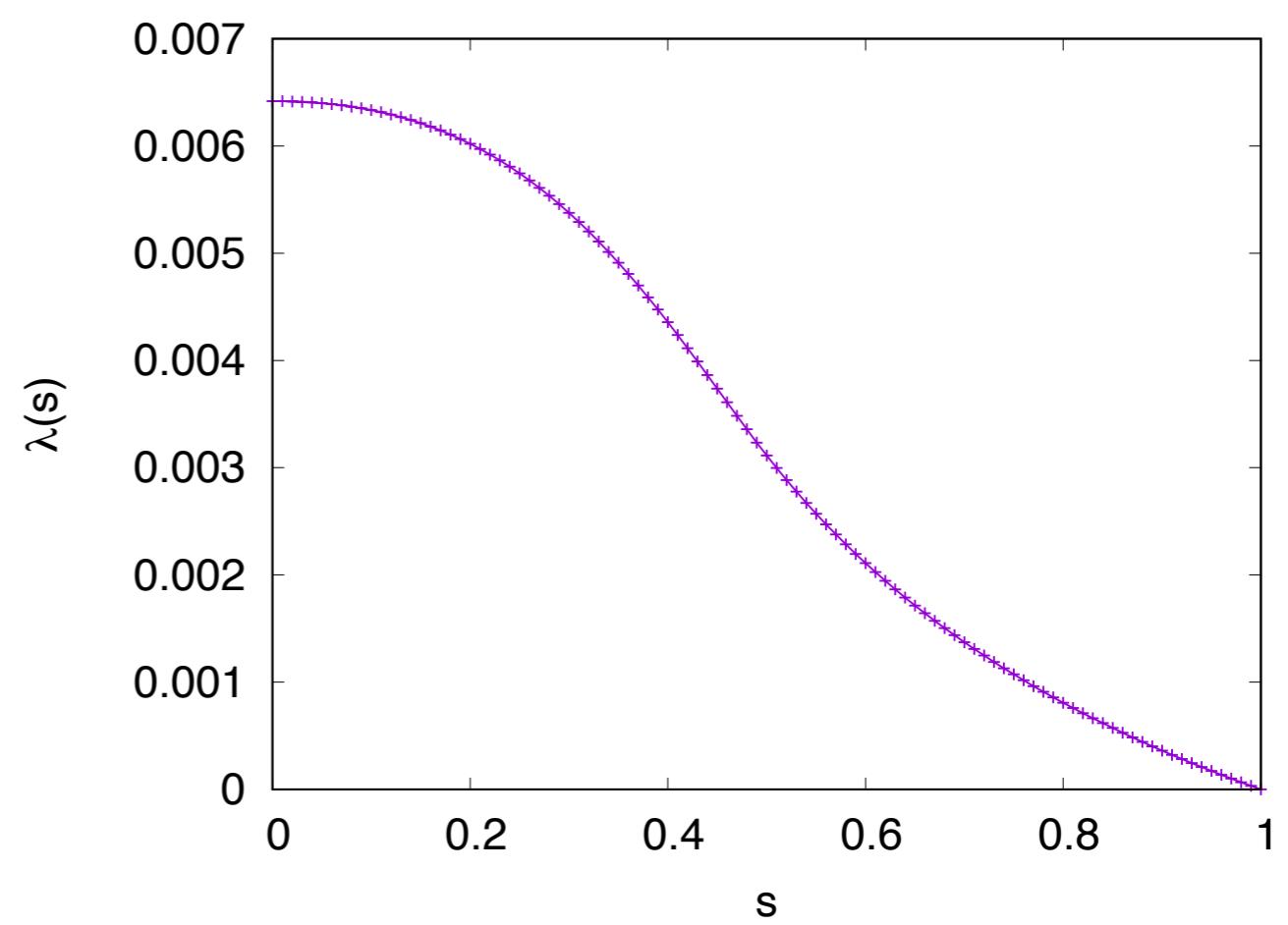
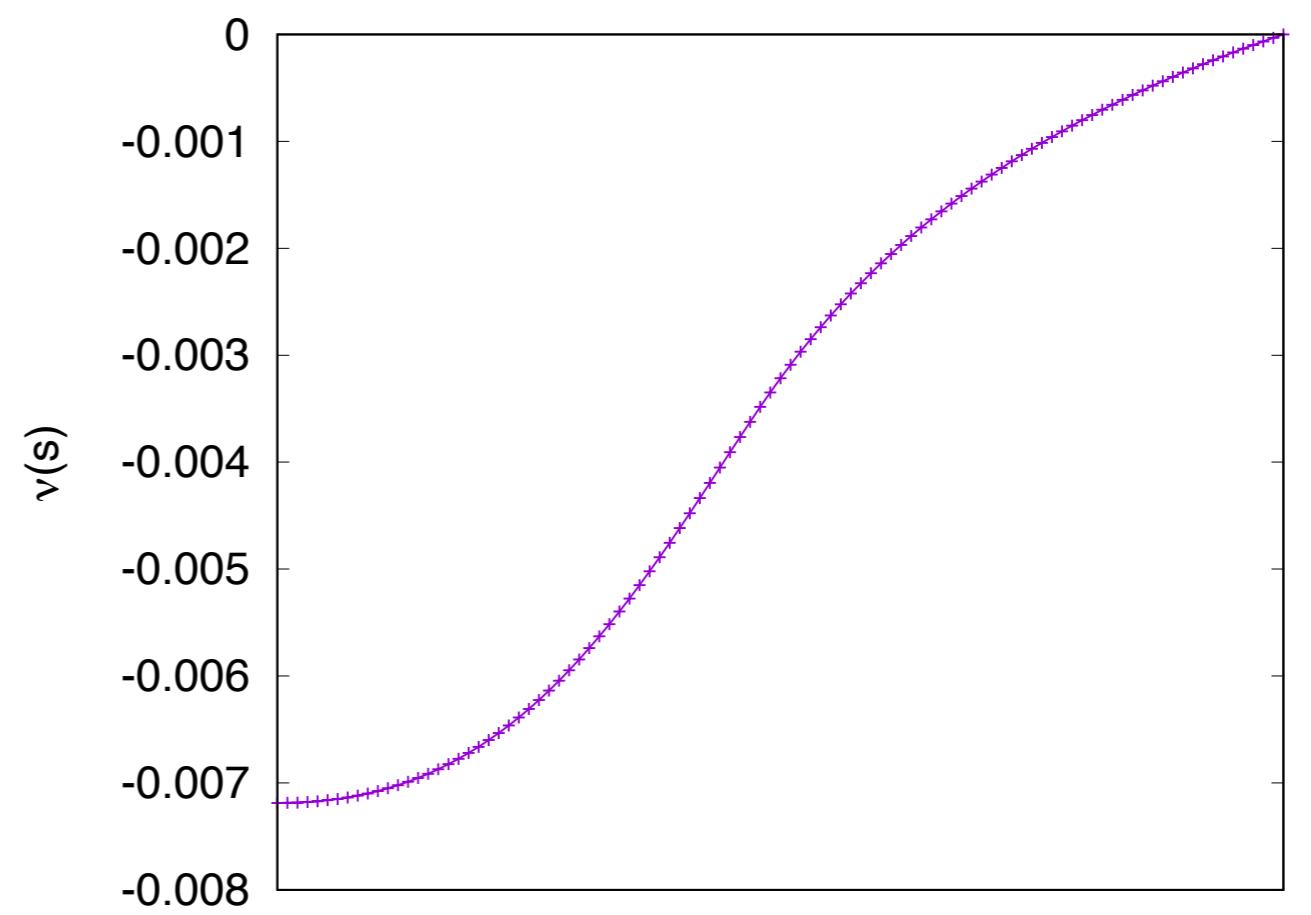
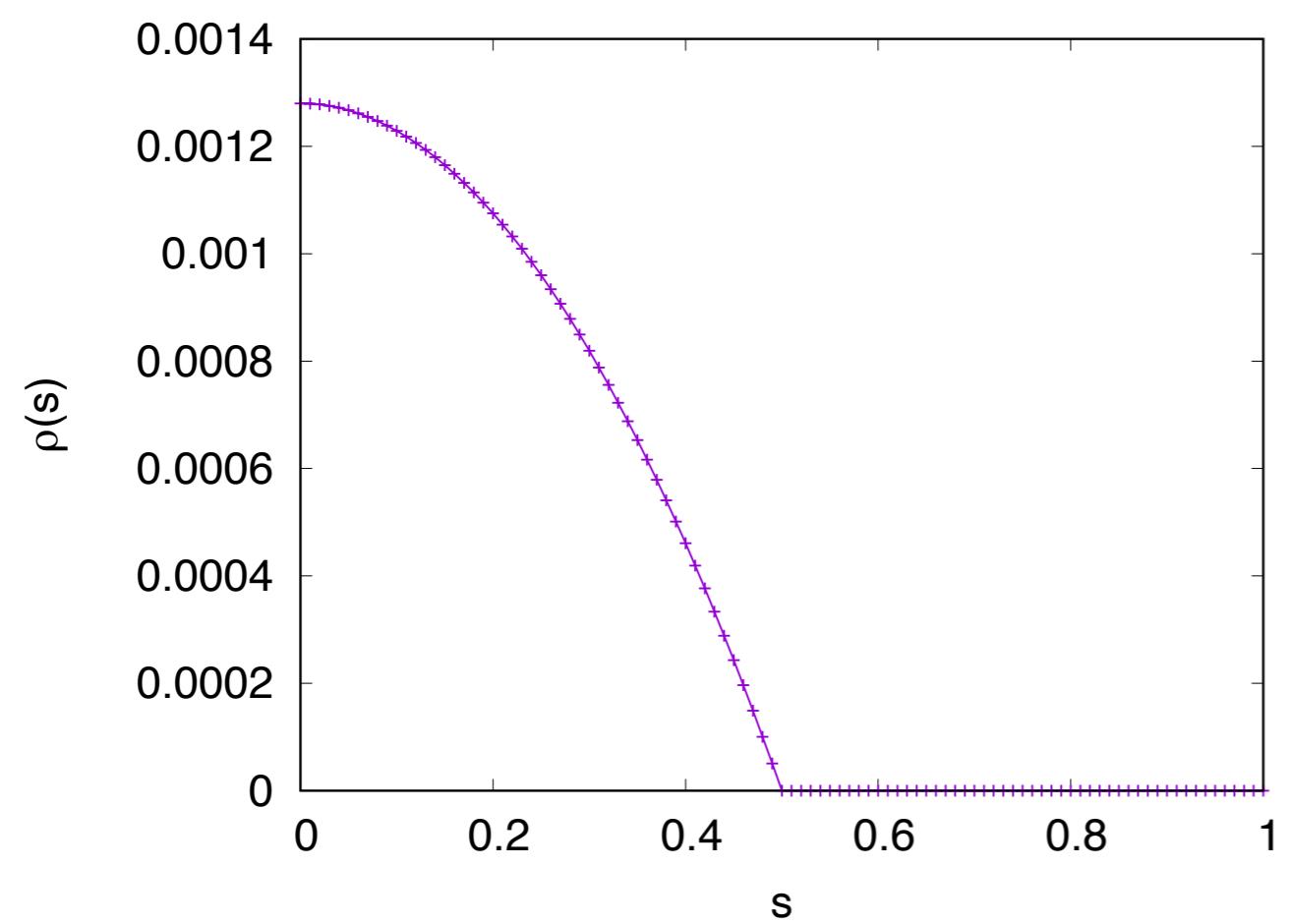
차분화 및 경계조건은 문제 2와 동일한 방법 적용
밀도 분포는 s좌표를 기준으로 포물선 프로파일 적용 (문제 2와 차이가 있음)
해석적 해가 존재하지 않음으로 r_i 기준으로 계산 종료 시점 판단
 λ 에 대한 방정식은 λ 만의 함수이므로 λ 를 먼저 구한 후 이를 이용하여 ν 를 구함

	$n/10000$	L_1 norm of r_i	L_∞ norm of r_i
1	5.0799548586573619E-06	9.8803576553951888E-06	
2	3.2202696213533162E-06	5.9347481291827098E-06	
3	2.2089399002408043E-06	4.0689724747910994E-06	
4	1.5713587571708805E-06	2.9061377953554235E-06	
5	1.1462630179799290E-06	2.1274400670012739E-06	
6	8.5209658794172813E-07	1.5858618482855746E-06	
7	6.4263290284324557E-07	1.1986002645403282E-06	
8	4.9008880395815977E-07	9.1561637211023950E-07	
9	3.7700054629666103E-07	7.0525861817216651E-07	
10	2.9196987825495831E-07	5.4675004212967337E-07	
	⋮		
1	7.7899537371440152E-06	1.4640843484403643E-05	
2	3.2301892172685863E-06	5.5800749306222919E-06	
3	1.3846650795534565E-06	2.3843111986443510E-06	
4	5.8858647176740414E-07	1.0161815761633441E-06	
5	2.4864667721528005E-07	4.2993841503635721E-07	
6	1.0473556145890568E-07	1.8122387507313320E-07	
7	4.4061685406107786E-08	7.6262235804147593E-08	
8	1.8526672371964029E-08	3.2070042843557189E-08	
9	7.7881907994528742E-09	1.3482216509874689E-08	
10	3.2736698440460150E-09	5.6672065373852831E-09	
	⋮		

Residual of
 λ equation

Residual of
 ν equation

$L_\infty < \text{tol} = 1 \times 10^{-14}$
 이 될 때까지 계산을 지속



문제 4: 할선법을 이용한 중성자별 크기 구하기

$$\begin{aligned}\nabla_a T_r^a &= \frac{1}{\sqrt{-g}} \partial_a \left(\sqrt{-g} T_r^a \right) - \Gamma_{ar}^c T_c^a \\ &= P' + \rho_0 h \nu' = 0\end{aligned}$$



Integral representation

$$\ln h + \nu = C$$

$$\ln h + \nu = C$$

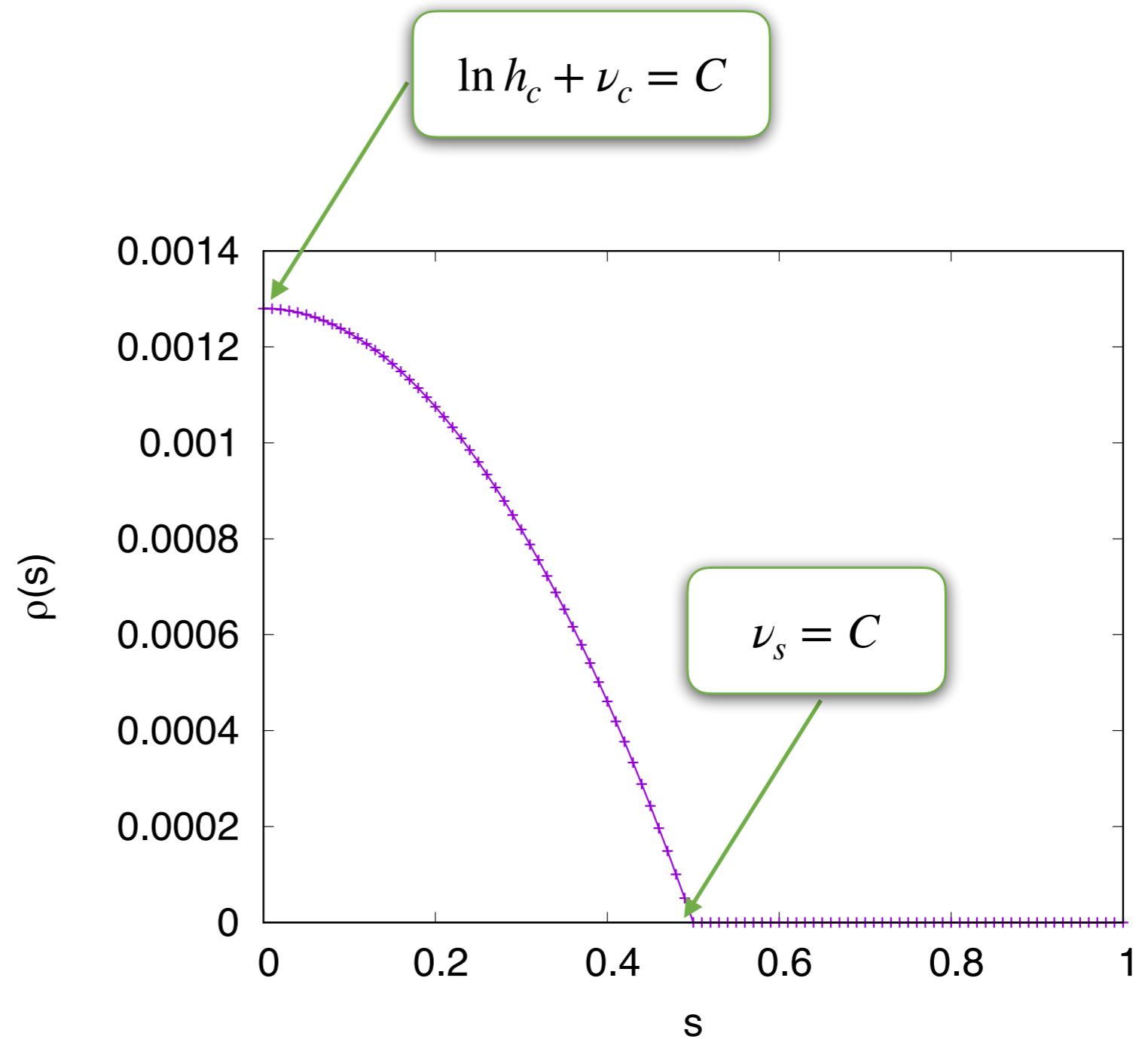
← Integral representation

경계조건

$$r_s, C$$

$$f = \ln h_c + \nu_c - \nu_s$$

$$r_s^2$$



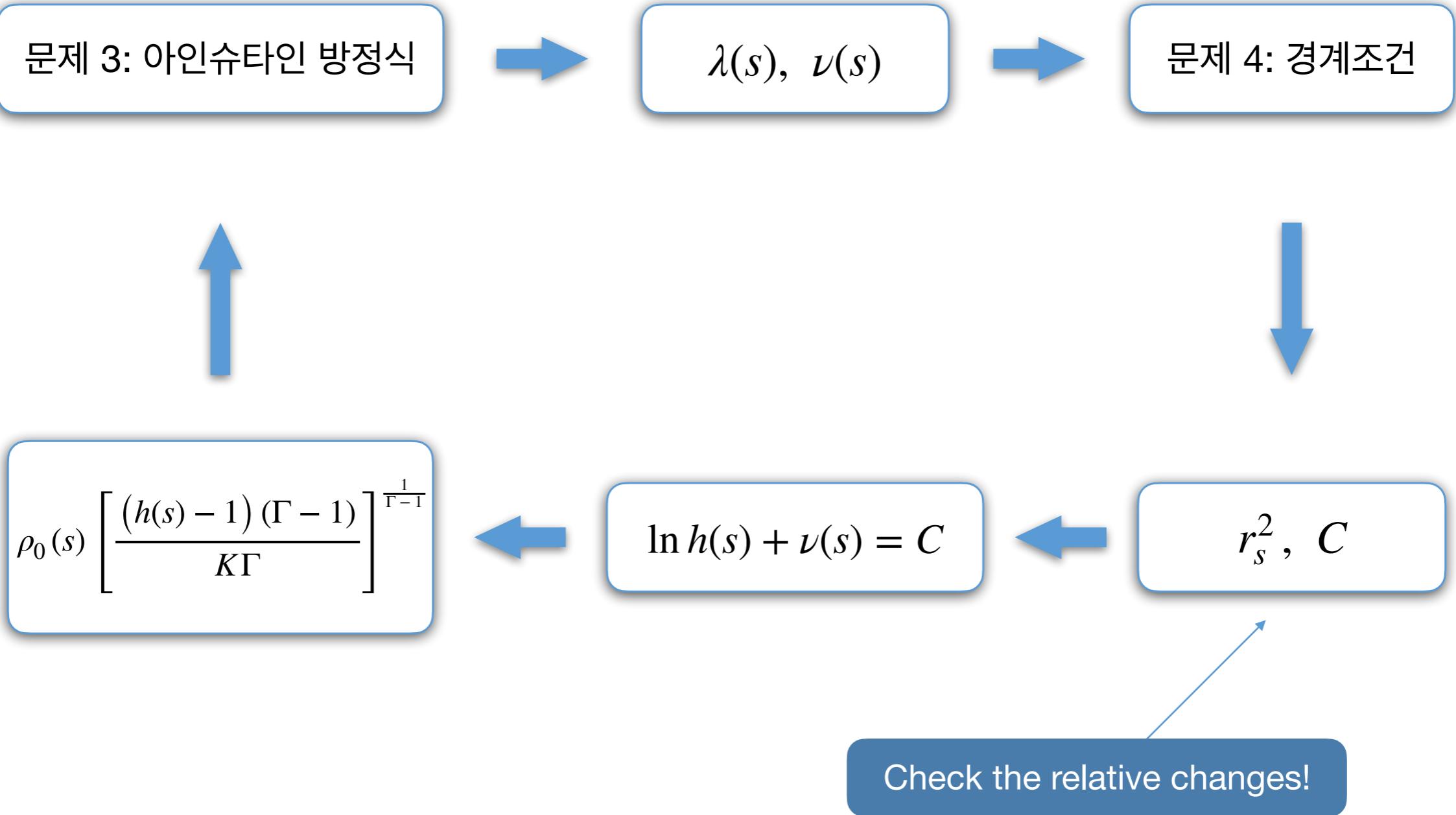
	n	r_s^2	f_n
1	6.40000000000000E+01	-2.6521504560154285E-02	
2	6.335999999999999E+01	-1.8229607860883512E-02	
3	6.1952969804846759E+01	-2.4113176571275707E-03	
4	6.1738484123399402E+01	-2.2496171238187146E-04	
5	6.1716414951041742E+01	-2.8505992679950243E-06	
		\vdots	

$$|f_n| < \text{tol} = 1 \times 10^{-10}$$

이 될 때까지 계산을 지속

문제 5: 반복을 통한 최종 중성자별 구조 구하기

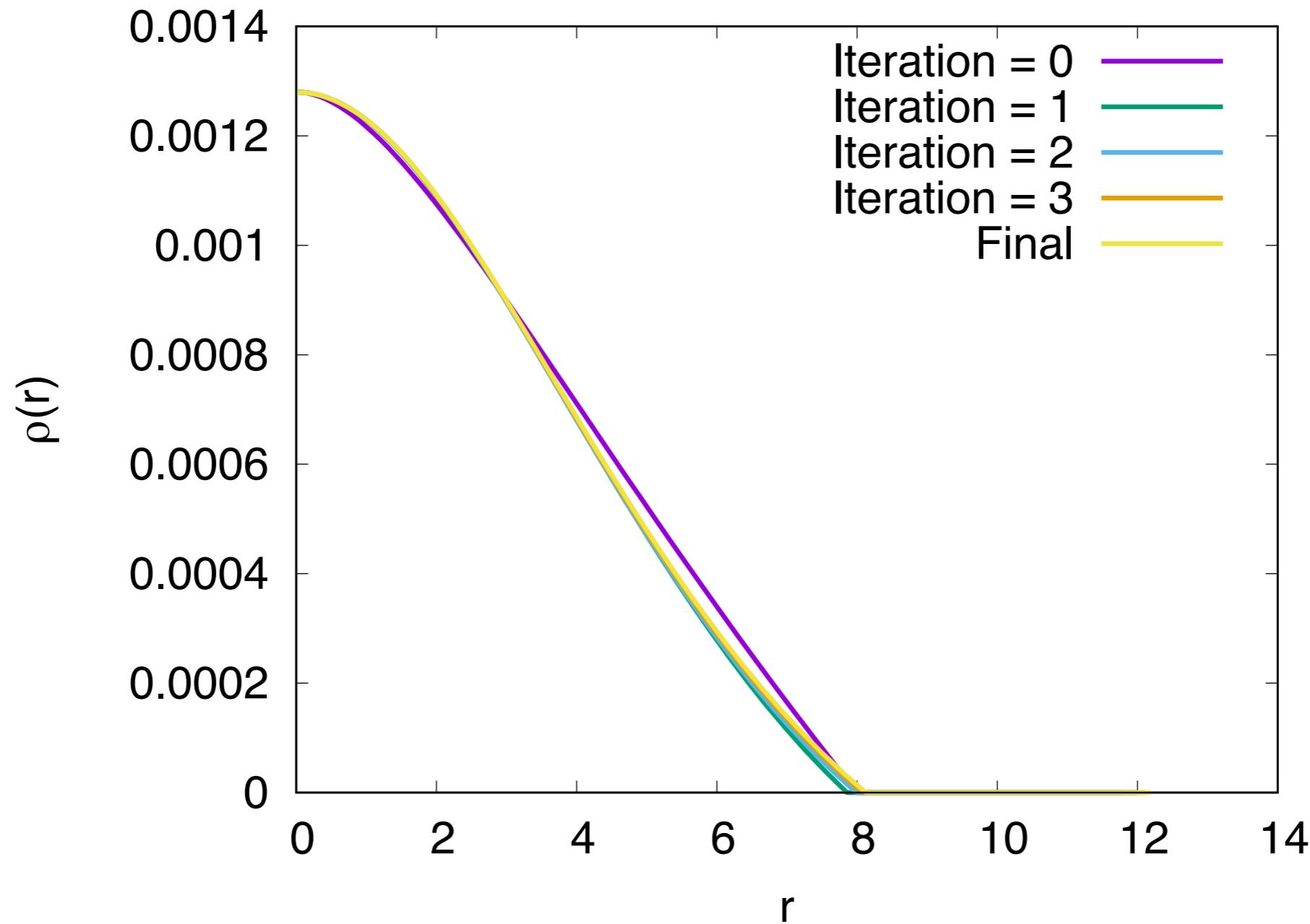
Solution procedure



	n	r_s^2	C	$\left \frac{[r_s^2]_n - [r_s^2]_{n-1}}{[r_s^2]_n} \right $	$\left \frac{C_n - C_{n-1}}{C_n} \right $
1	6.1716124772187584E+01	-1.8660983143549453E-01	3.7006134721563826E-02	1.000000000000000E+00	
2	6.3857598366076225E+01	-1.7632476946919279E-01	3.3535141450391277E-02	5.8330216436770881E-02	
3	6.5261852274811176E+01	-1.7372033138927334E-01	2.1517224224984872E-02	1.4992131658345790E-02	
4	6.5784162318657792E+01	-1.7301276928606002E-01	7.9397536646670635E-03	4.0896524928945308E-03	
5	6.5949732907096632E+01	-1.7281959351806733E-01	2.5105573766625996E-03	1.1177885797566793E-03	
⋮					

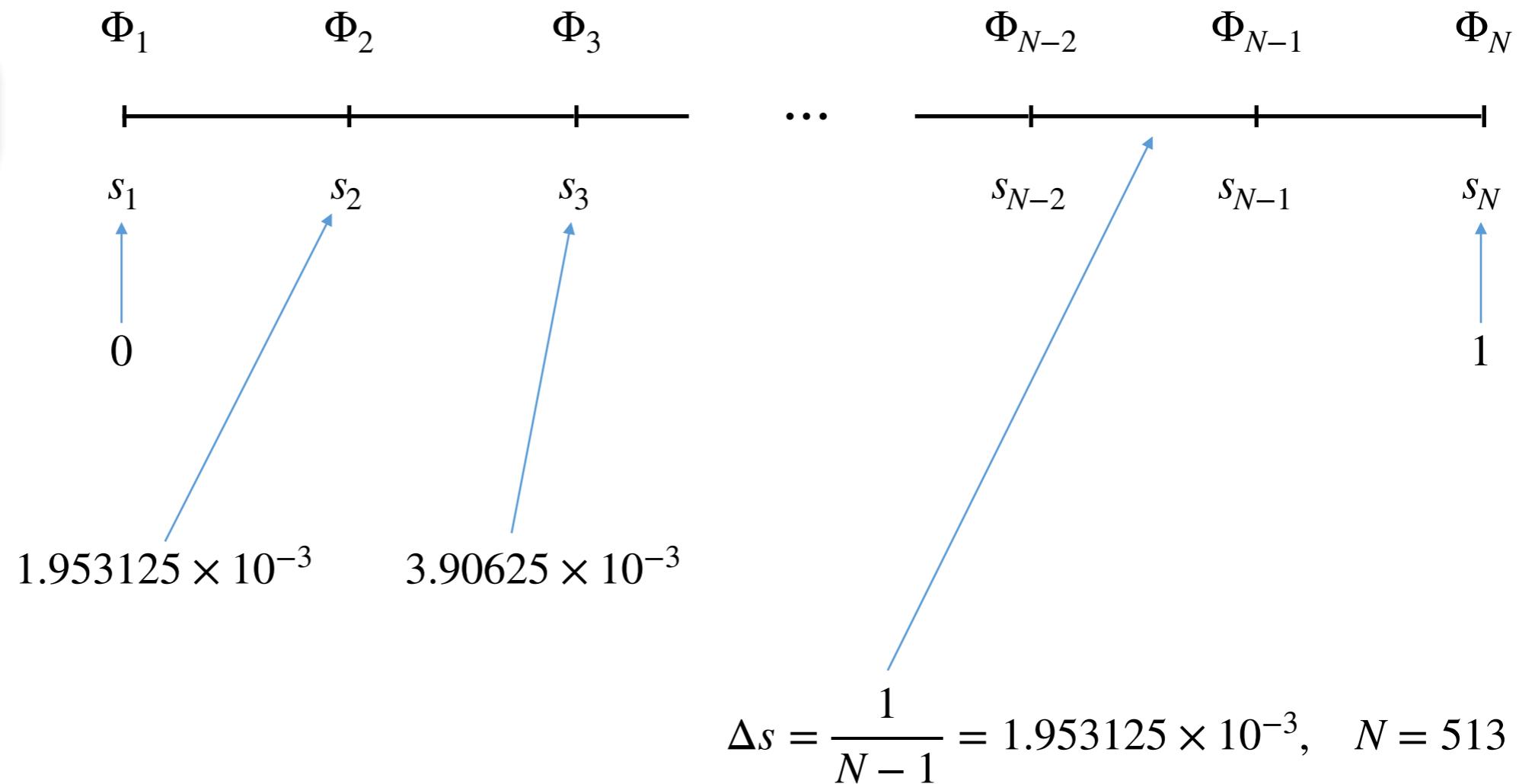
Relative change $< \text{tol} = 1 \times 10^{-6}$
이 될 때까지 계산을 지속

최종결과



완화법 힌트

그리드 구조



초기조건

$$\Phi_i^1 = 0 \quad (i = 1, \dots, N)$$

경계조건

$$r_1^1 = -\Phi_3^1 + 4\Phi_2^1 - 3\Phi_1^1, \quad \Phi_N = 0$$

Residual

$$r_i^1 = \Phi_{i+1}^1 - 2\Phi_i^1 + \Phi_{i-1}^1 + \frac{1}{s_i} (\Phi_{i+1}^1 - \Phi_{i-1}^1) \Delta s - 4\pi r_s^2 \rho_i (1 - s_i)^{-4} \Delta s^2 \quad (i = 2, \dots, N-1)$$

Update
 $\Phi_i^1 \rightarrow \Phi_i^2$

$$\Phi_i^2 = \Phi_i^1 - r_i^1 / \left(\frac{\partial r_i^1}{\partial \Phi_i^1} \right)$$

초기조건

$$\Phi_i^2 \quad (i = 1, \dots, N)$$

경계조건

$$r_1^2 = -\Phi_3^2 + 4\Phi_2^2 - 3\Phi_1^2, \quad \Phi_N = 0$$

Residual

$$r_i^2 = \Phi_{i+1}^2 - 2\Phi_i^2 + \Phi_{i-1}^2 + \frac{1}{s_i} (\Phi_{i+1}^2 - \Phi_{i-1}^2) \Delta s - 4\pi r_s^2 \rho_i (1 - s_i)^{-4} \Delta s^2 \quad (i = 2, \dots, N-1)$$

⋮

주의점

- Residual (r_i) 계산
- Φ_i 업데이트

위험한 예



안전한 예



또 다른 방법

