MATH 2LA3: Midterm 1 Cheat Sheet

Orthogonal Matrices

Q is an orthogonal matrix iff:

1. Columns of Q form an orthogonal set (all orthogonal to each other)
2. Magnitude of each column = 1

Properties:

1. $(Q\vec{x})^T(Q\vec{z}) = \vec{x}^T z$ (angle preserving)
2. $||Q\vec{x}|| = ||\vec{x}||$ (length preserving)
3. $Q^T = Q^{-1}$
4. $\det(Q) = \pm 1$

Markov Chains

Let A be a regular transition matrix: $A = \begin{bmatrix} 0.1 & 0.6 \\ 0.9 & 0.4 \end{bmatrix}$

To find the long term probability the system will be at a particular state, find the eigenvector corresponding to $\lambda = 1$ and scale it to a probability vector $\vec{p}$.

$[A|0] \sim \left[\begin{array}{cc|c} 1 & \frac{-2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \vec{x} = x_2 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$

$\Rightarrow \vec{p} = \frac{1}{\frac{2}{3}+1} \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \begin{matrix} \leftarrow \text{Long term } P(\text{in state 1}) \\ \leftarrow \text{Long term } P(\text{in state 2}) \end{matrix}$

Diagonalization

$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$

$P = \begin{bmatrix} \vec{\text{eigen}}_{\lambda_1} & \dots & \vec{\text{eigen}}_{\lambda_n} \end{bmatrix}$

A and D are similar and A is diagonalizable iff $A = PDP^{-1}$.

A is diagonalizable iff (equivalent statements):

- A has n linearly independent eigenvectors
- $\sum_i \text{GM}(\lambda_i) = n$. $\text{GM}(\lambda) :=$ number of eigenvectors for eigenvalue λ

A diagonalizable $\Leftrightarrow A^n = PD^nP^{-1}$

Orthogonal Compliment

Definition: if W is a subspace of $\mathbb{R}^n$, then $W^\perp$ contains all vectors $\in \mathbb{R}^n$ perpendicular to W

Properties:

1. $W^\perp = \text{Null}(A^T)$
2. $\dim(W) + \dim(W^\perp) = n$
3. $(W^\perp)^\perp = W$
4. $W \cap W^\perp = \{\vec{0}\}$
5. $(\mathbb{R}^n)^\perp = \{\vec{0}\}$; $\{\vec{0}\}^\perp = \mathbb{R}^n$

Subspaces of $\mathbb{R}^n$:

- $\text{Row}(A) = (\text{Nul}(A))^\perp$
- $\text{Nul}(A) = (\text{Row}(A))^\perp$
- $\dim(\text{Row}(A)) = \text{Rank}(A)$
- $\dim(\text{Nul}(A)) = n$ - $\text{Rank}(A)$

Subspaces of $\mathbb{R}^m$:

- $\text{Col}(A) = (\text{Nul}(A^T))^\perp$
- $\text{Nul}(A^T) = (\text{Col}(A))^\perp$
- $\dim(\text{Col}(A)) = \text{Rank}(A)$
- $\dim(\text{Nul}(A^T)) = m$ - $\text{Rank}(A)$

Example: find a basis for Row(A)

$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$, and $A\vec{v} = \vec{0}$

$A\vec{v} = \vec{0}$, so $\vec{v} \in \text{Nul}(A)$.

Let $W = \text{Nul}(A)$, so $W^\perp = \text{Row}(A)$

$\Rightarrow \text{Nul}(\vec{v}^T)$ is a basis for Row(A).

$$\text{Nul}(\begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

= basis for Row(A)

Vector Spaces and Subspaces

- $\text{Nul}(A) :=$ Solutions to $A\vec{x} = \vec{0}$. From RREF($[A|0]$)
- Basis of Col(A) cols of A w/ pivots
- Basis of Row(A) row vectors of RREF(A)

CR Factorization: $A = CR$

- $C = [\text{basis of } \vec{\text{col}}(\text{A})_1 \dots \text{basis of } \vec{\text{col}}(\text{A})_k]$
- $R = RREF(A)$

Descriptions of a subspace in $\mathbb{R}^n$:

- A span of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$
- $\vec{0}$
- $\mathbb{R}^n$ (any space is a subspace of itself)
- The solution set to $A\vec{x} = \vec{0}$ for any $n \times n$ matrix A
- Any line in $\mathbb{R}^n$ **Which must pass through the origin**

Example:

- $w_1 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + w_2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is a subspace (all vectors have weights)
- $w_1 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ isn't a subspace

Matrix Properties

$A_{m \times n} \Rightarrow$ transformation $\vec{x} \to A\vec{x}$ is from $\mathbb{R}^n \to \mathbb{R}^m$

$\text{Rank}(A) + \text{nullity}(A) = n$

Determinants	$\det(AB) = \det(A)\det(B)$ $\det(A^{-1}) = \frac{1}{\det(A)}$ $\det(\text{adj}(A)) = \det(A)^{n-1}$ $\det(A^n) = (\det(A))^n$ $\det(cA) = c^n \det(A)$ $\det(A^T) = \det(A)$ $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$
Inverse	$(AB)^{-1} = B^{-1}A^{-1}$ $(A^{-1})^{-1} = A$ $(kA^{-1}) = \frac{1}{k}A$ $(A^{-1})^{-1} = I_n$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
Transpose	$(A^T)^T = A$ $(AB)^T = B^TA^T$ $(kA^T) = kA^T$ $(A+B)^T = A^T + B^T$

Eigenvalues and Eigenvectors

The Eigenvalues of a triangular or diagonal matrix are along the main diagonal.

To **Find Eigenvalues** of $A = \begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$

Solve $|A - \lambda I_n| = \vec{0} \Leftrightarrow \begin{bmatrix} -5-\lambda & 2 \\ -9 & 6-\lambda \end{bmatrix} = \vec{0}$

To **Find Eigenvectors for** λ_i, solve:

$\left[\begin{array}{cc:c} -5-\lambda_i & 2 & 0 \\ -9 & 6-\lambda_i & 0 \end{array}\right]$

- AM($\lambda = 0$) = nullity(A)

Dot Product

- $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (u_1)(v_1) + (u_2)(v_2)$
- $\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$
- $\vec{u} \cdot \vec{u} = (||\vec{u}||)^2$
- if $\vec{u} \cdot \vec{v} = 0$, then $\vec{u}$ and $\vec{v}$ are orthogonal
- $(\vec{a} + \vec{b}) \cdot (\vec{c} + \vec{d}) = \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{d} + \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{d}$

Span

Def The span of a set of vectors is the set of linear combinations of the vectors.

Check if $\vec{w} \in$span$\{\vec{u}, \vec{v}\}$:

$\left[\begin{array}{cc:c} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{array}\right]$ $\vec{w} \in$span$\{\vec{u}, \vec{v}\}$ iff that system has a solution.

Check if span$\{\vec{u}, \vec{v}, \vec{w}\} = \mathbb{R}^3$:

(equivalent to col$[\vec{u}, \vec{v}, \vec{w}] = \mathbb{R}^3$)

$\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$ True iff that matrix has a pivot in each row/column

Vector Spaces

$v_1, v_2 \in V$

1. $v_1 + v_2 \in V$
2. $k \in \mathbb{F}, kv_1 \in V$
3. $v_1 + v_2 = v_2 + v_1$
4. $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$
5. $\forall v \in V, 0 \in V \mid 0 + v_1 = v_1 + 0 = v_1$
6. $\forall v \in V, \exists - v \in V \mid v + (-v) = (-v) + v = 0$
7. $\forall v \in V, 1 \in \mathbb{F} \mid 1 * v = v$
8. $\forall v \in V, k, l \in \mathbb{F}, (kl)v = k(lv)$
9. $\forall k \in \mathbb{F}, k(v_1 + v_2) = kv_1 + kv_2$
10. $\forall v \in V, k, l \in \mathbb{F}, (k + l)v = kv + lv$

Sample Questions

PART E: Which the following is true about c?

(a) c is orthogonal to every vector in col(A)
(b) c is orthogonal to every vector in row(A)
(c) c is orthogonal to every vector in nul(A)
(d) c is orthogonal to every vector in nul(A^T)

Solution. Since $Ax = c$ has a solution, that means $c \in$col(A). From the Four Fundamental Subspaces of A, we know that the orthogonal complement of col(A) is nul(A^T). This means every vector in col(A) is orthogonal to every vector in nul(A^T). So the answer is (d).