MATH 2LA3: Midterm 2 Summary Sheet

Orthogonal Matrices

$Q_{n\times n}$ is an orthogonal matrix iff:
1. Columns of Q form an orthogonal set (all orthogonal to each other)
2. Magnitude of each column = 1

Properties:
1. $(Q\vec{x})^T(Q\vec{z}) = \vec{x}^T z$ (angle preserving)
2. $||Q\vec{x}|| = ||\vec{x}||$ (length preserving)
3. $Q^T = Q^{-1}$ (true iff Q is orthogonal)
4. Q is invertible
5. $\det(Q) = \pm 1$
6. $Q^T Q = I_n$

$\det(Q) = 1$ for rotation $\det(Q) = -1$ for reflection

Markov Chains

Let A be a regular transition matrix: $A = \begin{bmatrix} 0.1 & 0.6 \\ 0.9 & 0.4 \end{bmatrix}$

To find the long term probability the system will be at a particular state, find the eigenvector corresponding to $\lambda = 1$ and scale it to a probability vector $\vec{p}$.

$$[A|0] \sim \left[\begin{array}{cc:c} 1 & \frac{-2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \vec{x} = x_2 \begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$$

$$\Rightarrow \vec{p} = \frac{1}{\frac{2}{3}+1}\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \begin{matrix} \leftarrow \text{Long term } P(\text{in state 1}) \\ \leftarrow \text{Long term } P(\text{in state 2}) \end{matrix}$$

Diagonalization

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

$P = [\vec{\text{eigen}}_{\lambda_1} \quad \dots \quad \vec{\text{eigen}}_{\lambda_n}]$

A and D are similar and A is diagonalizable iff $A = PDP^{-1}$.

- A is similar to itself.

A is diagonalizable iff (equivalent statements):

- A has n linearly independent eigenvectors
- $\sum_i \text{GM}(\lambda_i) = n$. $\text{GM}(\lambda) :=$ number of eigenvectors for eigenvalue λ

A diagonalizable $\Leftrightarrow A^n = PD^nP^{-1}$

Orthogonal Compliment

Definition: if W is a subspace of $\mathbb{R}^n$, then $W^\perp$ contains all vectors $\in \mathbb{R}^n$ perpendicular to W

Properties:

1. $W^\perp = \text{Null}(A^T)$
2. $\dim(W) + \dim(W^\perp) = n$
3. $(W^\perp)^\perp = W$
4. $W \cap W^\perp = \{\vec{0}\}$
5. $(\mathbb{R}^n)^\perp = \{\vec{0}\}$; $\{\vec{0}\}^\perp = \mathbb{R}^n$

Subspaces of $\mathbb{R}^n$:

- $\text{Row}(A) = (\text{Nul}(A))^\perp$
- $\text{Nul}(A) = (\text{Row}(A))^\perp$
- $\dim(\text{Row}(A)) = \text{Rank}(A)$
- $\dim(\text{Nul}(A)) = n$ - $\text{Rank}(A)$

Subspaces of $\mathbb{R}^m$:

- $\text{Col}(A) = (\text{Nul}(A^T))^\perp$
- $\text{Nul}(A^T) = (\text{Col}(A))^\perp$
- $\dim(\text{Col}(A)) = \text{Rank}(A)$
- $\dim(\text{Nul}(A^T)) = m$ - $\text{Rank}(A)$

Example: find a basis for Row(A)

$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$, and $A\vec{v} = \vec{0}$

$A\vec{v} = \vec{0}$, so $\vec{v} \in \text{Nul}(A)$.
Let $W = \text{Nul}(A)$, so $W^\perp = \text{Row}(A)$
$\Rightarrow \text{Nul}(\vec{v}^T)$ is a basis for $\text{Row}(A)$.

$$\text{Nul}(\begin{bmatrix} 1 & 2 & 1 & 2 \end{bmatrix}) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$
$$= \text{basis for Row}(A)$$

Matrix Properties

$A_{m\times n} \Rightarrow$ transformation $\vec{x} \rightarrow A\vec{x}$ is from $\mathbb{R}^n \rightarrow \mathbb{R}^m$
$\text{Rank}(A) + \text{nullity}(A) = n$

Determinants	$\det(AB) = \det(A)\det(B)$
	$\det(A^{-1}) = \frac{1}{\det(A)}$
	$\det(\text{adj}(A)) = \det(A)^{n-1}$
	$\det(A^n) = (\det(A))^n$
	$\det(cA) = c^n \det(A)$
	$\det(A^T) = \det(A)$
	$\det(A_{\text{tringl}}) = \prod_{i=1}^n A_{(i,i)}$
	$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$
Inverse	$(AB)^{-1} = B^{-1}A^{-1}$
	$(A^{-1})^{-1} = A$
	$(kA^{-1}) = \frac{1}{k}A$
	$(A^{-1})^{-1} = I_n$
	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
Transpose	$(A^T)^T = A$
	$(AB)^T = B^TA^T$
	$(kA^T) = kA^T$
	$(A+B)^T = A^T + B^T$

Eigenvalues and Eigenvectors

The Eigenvalues of a triangular or diagonal matrix are along the main diagonal.

- $\text{AM}(\lambda = 0) = \text{nullity}(A)$

To **Find Eigenvalues** of $A = \begin{bmatrix} -5 & 2 \\ -9 & 6 \end{bmatrix}$

Solve $|A - \lambda I_n| = 0 \Leftrightarrow \begin{bmatrix} -5-\lambda & 2 \\ -9 & 6-\lambda \end{bmatrix} = 0$

To **Find Eigenvectors for** λ_i, solve:
$$\left[\begin{array}{cc:c} -5-\lambda_i & 2 & 0 \\ -9 & 6-\lambda_i & 0 \end{array}\right]$$

Dot Product

- $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = (u_1)(v_1) + (u_2)(v_2)$
- $\vec{u} \cdot \vec{v} = \vec{u}^T\vec{v}$
- $\vec{u} \cdot \vec{u} = (||\vec{u}||)^2$
- $\vec{u} \cdot \vec{v} = 0 \Leftrightarrow \vec{u}, \vec{v}$ orthogonal $\Leftrightarrow ||\vec{u}|| + ||\vec{v}|| = ||\vec{u} + \vec{v}||$
- $(\vec{a} + \vec{b}) \cdot (\vec{c} + \vec{d}) = \vec{a} \cdot \vec{c} + \vec{a} \cdot \vec{d} + \vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{d}$ -
$|| \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} || = \sqrt{v_1^2 + v_2^2}$

Span

Def The span of a set of vectors is the set of linear combinations of the vectors.

Check if $\vec{w} \in \text{span}\{\vec{u},\vec{v}\}$:

$\left[\begin{array}{cc|c} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{array}\right]$ $\vec{w} \in \text{span}\{\vec{u},\vec{v}\}$ iff that system has a solution.

Check if $\text{span}\{\vec{u},\vec{v},\vec{w}\} = \mathbb{R}^3$:

(equivalent to $\text{col}[\vec{u},\vec{v},\vec{w}] = \mathbb{R}^3$)

$\begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$ True iff that matrix has a pivot in each row/column

Injectivity and Surjectivity

Injective: (one-to-one)

1. $f(x) = f(y) \leftarrow x = y$ (map to the same point)
2. $\text{rank}(A) = \text{n}$
3. pivot in every column

Surjective: (onto)

1. $\forall \vec{b} \in \mathbb{R}^m, \exists \vec{x} \in \mathbb{R}^n$ such that $f(\vec{x}) = \vec{b}$
2. $\text{rank}(A) = \text{m}$
3. pivot in every row

Invertible Matrix Theorem

1. A is row-equivalent to the $n \times n$ I_n.
2. A has n pivot positions.
3. The equation $Ax = 0$ has only the trivial solution $x = 0$.
4. The columns of A form a linearly independent set.
5. The linear transformation $x \mapsto Ax$ is one-to-one.
6. For each column vector b in $\mathbb{R}^n$, the equation $Ax = b$ has a unique solution.
7. The columns of A span $\mathbb{R}^n$.
8. The linear transformation $x \mapsto Ax$ is a surjection.
9. There is an $n \times n$ matrix C such that $CA = I_n$.
10. There is an $n \times n$ matrix D such that $AD = I_n$.
11. The transpose matrix A^T is invertible.
12. The columns of A form a basis for $\mathbb{R}^n$.
13. The column space of A is equal to $\mathbb{R}^n$.
14. The dimension of the column space of A is n.
15. The rank of A is n.
16. The null space of A is $\{0\}$.
17. The dimension of the null space of A is 0.
18. 0 fails to be an eigenvalue of A.
19. The determinant of A is not zero.
20. The orthogonal complement of the column space of A is $\{0\}$.
21. The orthogonal complement of the null space of A is $\mathbb{R}^n$.
22. The row space of A is $\mathbb{R}^n$.

Least Squares

$\hat{x} = (A^TA)^{-1}A^T\vec{b}$

- **Normal equations** for $A\vec{x} = \vec{b}$: $A^TA\vec{x} = A^T\vec{b}$
- $A^TA\vec{x} = A^T\vec{b}$ has same solutions as $\hat{x}$.
- $A\hat{x} = proj_w\vec{b}$
- $\hat{x} = R^{-1}Q^T\vec{b}$, where $A = QR$ factorization
- $\hat{x} = \vec{x}$ s.t. $A\hat{x}$ is as close as possible to $\vec{b}$

Projections

- Projection matrices are symmetric.
- $proj_{\vec{u}}\vec{v} = \frac{\vec{v}\cdot\vec{u}}{||\vec{u}||^2}\vec{u}$
- $proj_w\vec{b} = P\vec{b}$, $w = \text{col}(A)$
- $P_{\text{Row}(A)} = I - P_{\text{Col}(A)}$
- $\vec{v} \in \text{Nul}(A) \Rightarrow \forall\vec{u}$ s.t. $\vec{u}\cdot\vec{v} = 0, \vec{u} \in \text{Row}(A)$
- $\vec{v} \in w \Leftrightarrow proj_w\vec{v} = \vec{v}$, so $P\vec{v} = \vec{v} \Rightarrow v \in \text{Nul}(A)$
- $\vec{v} \in w^\perp \Leftrightarrow proj_w\vec{v} = 0$, so $P\vec{v} = 0 \Rightarrow v \in \text{Col}(A)$
- $\text{Rank}(P_{\text{subspace}}) = \dim(\text{subspace})$
- $||P_{\text{Row}(A)}\vec{x}||$ is shortest distance from $\vec{x}$ to $\text{Nul}(A)$
- $P = A(A^TA)^{-1}A^T$ In general (subspace)
- $P = \frac{(\vec{a})(\vec{a}^T)}{(\vec{a}^T)(\vec{a})}$ (for line only)

Shortcut for Orthonormal vectors:

$P\vec{b} = (\vec{a_1}\cdot\vec{b})\vec{a_1} + (\vec{a_2}\cdot\vec{b})\vec{a_2} + ... + (\vec{a_n}\cdot\vec{b})\vec{a_n}$

Shortcut for Orthogonal vectors:

$P\vec{b} = proj_{\vec{a_1}}b + proj_{\vec{a_2}}b + ... + proj_{\vec{a_n}}b$

Example: $P_W\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\-2\\-2\end{bmatrix}$, find W

$\vec{b} = \begin{bmatrix}1\\1\\1\end{bmatrix} \notin W, \vec{b} \notin W^\perp$

so $\vec{z} = \vec{b} - P\vec{b}$, and $\vec{z} \in W^\perp$

$\vec{z} = \begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}0\\-2\\-2\end{bmatrix} = \begin{bmatrix}1\\3\\3\end{bmatrix} \Rightarrow W$ given by $x+3y+3z = 0$

Gram-Schmidt Process

Gram-Schmidt Process: (orthogonalize basis)

$\vec{v_1} = \vec{x_1}$ (remember to normalize at end)

$\vec{v_2} = \vec{x_2} - \frac{\vec{x_2}\cdot\vec{v_1}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1}$

$\vec{v_p} = \vec{x_p} - \frac{\vec{x_p}\cdot\vec{v_1}}{\vec{v_1}\cdot\vec{v_1}}\vec{v_1} - \frac{\vec{x_p}\cdot\vec{v_2}}{\vec{v_2}\cdot\vec{v_2}}\vec{v_2} - ...\frac{\vec{x_p}\cdot\vec{v_{p-1}}}{\vec{v_{p-1}}\cdot\vec{v_{p-1}}}\vec{v_{p-1}}$

QR Factorization ($A = QR$):

$A_{n\times q}$ has independent columns. Apply Gram-Schmidt on columns to get $\{\vec{v_1},..,\vec{v_q}\}$, then normalize to $\{\vec{u_1},..,\vec{u_q}\}$

- $Q = [\vec{u_1},..,\vec{u_q}]$
- $R = Q^TA$
- $R_{(i,j)} = u_i \cdot a_j$ ($R_{q\times q}$ will be upper triangular)

Orthogonal Diagonalization & Spec. Decomp.

- $A = QDQ^T$ for orthogonal Q
- A is orthogonally diagonalizable iff $A = A^T$
- Symmetric A has n real eigenvalues, w/ multiplicity
- $\vec{Eigen}$ for different λ orthogonal
- $A^T = A \Rightarrow A\vec{x} = (A\vec{x})^T$

Spectral Decomposition:

$A = \sum_{i=1}^n \lambda_i\vec{u_i}\vec{u_i}^T \Leftrightarrow (Q = [\vec{u_1},\vec{u_2},...,\vec{u_n}])$

Quadratic Forms of Symmetric Matrices

$A_{n\times n}$ has quadratic form $Q(\vec{x}) = \vec{x}^TA\vec{x}$ for $\vec{x} \in \mathbb{R}^n$

$Q(x) = a_1x_1^2 + a_2x_1x_2 + a_3x_2^2 \Leftrightarrow A = \begin{bmatrix} a_1 & \frac{a_2}{2} \\ \frac{a_2}{2} & a_2 \end{bmatrix}$

- Definite: $Q(x) < 0\forall x \neq 0 \Leftrightarrow \lambda < 0\forall\lambda$

- Semidefinite: $Q(x) \leq 0\forall x \neq 0 \Leftrightarrow \lambda \leq 0\forall\lambda$

+ Definite: $Q(x) > 0\forall x \neq 0 \Leftrightarrow \lambda > 0\forall\lambda$

+ Semidefinite: $Q(x) \geq 0\forall x \neq 0 \Leftrightarrow \lambda \geq 0\forall\lambda$

- Definite matrices are symmetric
- $Q(x)$ of $A_{n\times n}$ has n x_i^2 terms
- $||\vec{x}|| = 1 \Rightarrow \min Q(\vec{x}) = \min(\lambda)$, $\max Q(\vec{x}) = \max(\lambda)$
- Max occurs at $\vec{x} = \frac{1}{||\vec{\text{eigen}}||}\vec{\text{eigen}}$ for $\max(\lambda)$

Principal Axes Theorem:

Symmetric A has orthogonal change of vars $\vec{x} = P\vec{y}$, transforms $\vec{x}^TA\vec{x}$ into $\vec{y}^TQ\vec{y}$ w/ no cross-prod term.

- After change of vars, coeff's on y_i^2 are eigenvalues

Orthogonal Decomposition

W a subspace of $\mathbb{R}^n \Leftrightarrow y = \hat{y} + z$, s.t. $\hat{y} \in W, z \in W^\perp$

$\hat{y} = proj_{u_1}y + ... + proj_{u_p}y = proj_Wy = \sum_{i=1}^p \frac{y\cdot u_i}{||u_i||^2}$