Linear Model with Uncertainties on Parameters

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1 Model

With both parameters A and b unknown the stochastic linear model is defined as,

$$\boldsymbol{x}^{(t+1)} \;=\; A\,\boldsymbol{x}^{(t)} \;+\; \boldsymbol{b} \tag{1}$$

where $\boldsymbol{x}^{(t)}, \boldsymbol{x}_{t-1} \in \mathbb{R}^n$ are consecutive states, $\boldsymbol{b} \in \mathbb{R}^n$ is the offset $A \in \mathbb{R}^{n\times n}$ is the slope.

$$\begin{bmatrix} \boldsymbol{x}^{(t)} \\ \boldsymbol{b} \\ A \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mathbb{E}[\boldsymbol{x}^{(t)}] \\ \mathbb{E}[\boldsymbol{b}] \\ \mathbb{E}[A] \end{bmatrix}, \begin{bmatrix} \mathbb{V}[\boldsymbol{x}^{(t)}] & \mathbb{C}[\boldsymbol{x}^{(t)}, \boldsymbol{b}] & \mathbb{C}[\boldsymbol{x}^{(t)}, A] \\ \mathbb{C}[\boldsymbol{b}, \boldsymbol{x}^{(t)}] & \mathbb{V}[\boldsymbol{b}] & \mathbb{C}[A, \boldsymbol{b}] \\ \mathbb{C}[A, \boldsymbol{x}^{(t)}] & \mathbb{C}[\boldsymbol{b}, A] & \mathbb{V}[A] \end{bmatrix} \right) \tag{2}$$

2 Useful Relationships

$$\mathbb{C}[A\boldsymbol{x}, \boldsymbol{b}] = \mathbb{E}[A]\mathbb{C}[\boldsymbol{x}, \boldsymbol{b}] + \sum_i \mathbb{E}[x_i]\mathbb{C}[\boldsymbol{a}_i, \boldsymbol{b}] \tag{3}$$

$$\begin{aligned} \mathbb{V}[A\boldsymbol{x}] = \sum_{l,k} & \mathbb{C}[\boldsymbol{a}_l, \boldsymbol{a}_k]\mathbb{C}[x_l, x_k] + \mathbb{C}[\boldsymbol{a}_l, x_k]\mathbb{C}[x_l, \boldsymbol{a}_k] + \mathbb{E}[\boldsymbol{a}_l]\mathbb{E}[\boldsymbol{a}_k]\mathbb{C}[x_l, x_k] + \mathbb{E}[x_l|\mathbb{E}[x_k]\mathbb{C}[\boldsymbol{a}_l, \boldsymbol{a}_k] && (4) \\ & + \mathbb{E}[x_l]\mathbb{E}[\boldsymbol{a}_k]\mathbb{C}[x_k, \boldsymbol{a}_l] + \mathbb{E}[x_k]\mathbb{C}[\boldsymbol{a}_k, x_l]\mathbb{E}[\boldsymbol{a}_l]^T && (5) \end{aligned}$$

$$\mathbb{E}[A\boldsymbol{x}] = \mathbb{E}[A]\mathbb{E}[\boldsymbol{x}] + \sum_i \mathbb{C}[\boldsymbol{a}_i, x_i] \tag{6}$$

where $\boldsymbol{a}_i$ denotes the i-th column of matrix A .

3 Expected Value

$$\begin{aligned} \mathbb{E}[\boldsymbol{x}^{(t+1)}] &= \mathbb{E}[A\boldsymbol{x}^{(t)} + \boldsymbol{b}] = \mathbb{E}[A\boldsymbol{x}^{(t)}] + \mathbb{E}[\boldsymbol{b}] \\ &= \mathbb{E}[A]\ \mathbb{E}[\boldsymbol{x}^{(t)}] + \sum_i \mathbb{C}[\boldsymbol{a}_i, x_i^{(t)}] + \mathbb{E}[\boldsymbol{b}] \end{aligned} \tag{7}$$

The expected values $\mathbb{E}[A]$ and $\mathbb{E}[\boldsymbol{b}]$ remain constant.

4 Variance

$$
\begin{aligned}
\mathbb{V}[\boldsymbol{x}^{(t+1)}] =& \mathbb{V}[A\boldsymbol{x}^{(t)} + \boldsymbol{b}] = \mathbb{V}[\boldsymbol{b}] + \mathbb{C}[A\boldsymbol{x}^{(t)}, \boldsymbol{b}] + \mathbb{C}[\boldsymbol{b}, A\boldsymbol{x}^{(t)}] + \mathbb{V}[A\boldsymbol{x}^{(t)}] && (8)\\
=& \mathbb{V}[\boldsymbol{b}]\\
&+ \mathbb{E}[A]\mathbb{C}[\boldsymbol{x}^{(t)}, \boldsymbol{b}] + \sum_i \mathbb{E}[x_i^{(t)}]\mathbb{C}[\boldsymbol{a}_i, \boldsymbol{b}]\\
&+ \mathbb{C}[\boldsymbol{b}, \boldsymbol{x}^{(t)}]\mathbb{E}[A]^T + \sum_i \mathbb{E}[x_i^{(t)}]\mathbb{C}[\boldsymbol{b}, \boldsymbol{a}_i]\\
&+ \sum_{l,k} \Big(\mathbb{C}[\boldsymbol{a}_l, \boldsymbol{a}_k]\mathbb{C}[x_l^{(t)}, x_k^{(t)}] + \mathbb{C}[\boldsymbol{a}_l, x_k^{(t)}]\mathbb{C}[x_l^{(t)}, \boldsymbol{a}_k] + \mathbb{E}[\boldsymbol{a}_l]\mathbb{E}[\boldsymbol{a}_k]\mathbb{C}[x_l^{(t)}, x_k^{(t)}] + \mathbb{E}[x_l^{(t)}]\mathbb{E}[x_k^{(t)}]\mathbb{C}[\boldsymbol{a}_l, \boldsymbol{a}_k] && (9)\\
&+ \mathbb{E}[x_l^{(t)}]\mathbb{E}[\boldsymbol{a}_k]\mathbb{C}[x_k^{(t)}, \boldsymbol{a}_l] + \mathbb{E}[x_k^{(t)}]\mathbb{C}[\boldsymbol{a}_k, x_l^{(t)}]\mathbb{E}[\boldsymbol{a}_l]^T \Big) && (10)
\end{aligned}
$$

5 Covariances

$$
\begin{aligned}
\mathbb{C}[\boldsymbol{x}^{(t+1)}, A] &= \mathbb{C}[Ax + b, A] = \mathbb{C}[Ax, A] + \mathbb{C}[b, A] \\
&= \mathbb{E}[A]\mathbb{C}[\boldsymbol{x}^{(t)}, A] + \sum_i \mathbb{E}[x_i^{(t)}]\mathbb{C}[\boldsymbol{a}_i, A] + \mathbb{C}[b, A]
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\mathbb{C}[\boldsymbol{x}^{(t+1)}, \boldsymbol{b}] &= \mathbb{C}[A\boldsymbol{x}^{(t)} + \boldsymbol{b}, \boldsymbol{b}] = \mathbb{C}[A\boldsymbol{x}^{(t)}, \boldsymbol{b}] + \mathbb{V}[\boldsymbol{b}] \\
&= \mathbb{E}[A]\mathbb{C}[\boldsymbol{x}^{(t)}, \boldsymbol{b}] + \sum_i \mathbb{E}[x_i^{(t)}]\mathbb{C}[\boldsymbol{a}_i, \boldsymbol{b}] + \mathbb{V}[\boldsymbol{b}]
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\mathbb{C}[\boldsymbol{x}^{(t+1)}, \boldsymbol{x}^{(t)}] &= \mathbb{C}[A\boldsymbol{x}^{(t)} + \boldsymbol{b}, \boldsymbol{x}^{(t)}] = \mathbb{C}[A\boldsymbol{x}^{(t)}, \boldsymbol{x}^{(t)}] + \mathbb{C}[\boldsymbol{b}, \boldsymbol{x}^{(t)}] \\
&= \mathbb{E}[A]\mathbb{V}[\boldsymbol{x}^{(t)}] + \sum_i \mathbb{E}[x_i^{(t)}]\mathbb{C}[\boldsymbol{a}_i, \boldsymbol{x}^{(t)}] + \mathbb{C}[\boldsymbol{b}, \boldsymbol{x}^{(t)}]
\end{aligned} \tag{13}
$$

The covariances $\mathbb{C}[\boldsymbol{b}, A]$ and $\mathbb{C}[A, \boldsymbol{b}]$ remain constant.

6 Implementation

6.1 Inputs

$$
m = \begin{bmatrix} \mathbb{E}[\boldsymbol{x}^{(t)}] \\ \mathbb{E}[\boldsymbol{b}] \\ \mathbb{E}[A] \end{bmatrix}, \qquad s = \begin{bmatrix} \mathbb{V}[\boldsymbol{x}^{(t)}] & \mathbb{C}[\boldsymbol{x}^{(t)}, \boldsymbol{b}] & \mathbb{C}[\boldsymbol{x}^{(t)}, A] \\ \mathbb{C}[\boldsymbol{b}, \boldsymbol{x}^{(t)}] & \mathbb{V}[\boldsymbol{b}] & \mathbb{C}[A, \boldsymbol{b}] \\ \mathbb{C}[A, \boldsymbol{x}^{(t)}] & \mathbb{C}[\boldsymbol{b}, A] & \mathbb{V}[A] \end{bmatrix}
$$

6.2 Outputs

$$
M = \begin{bmatrix} \mathbb{E}[\boldsymbol{x}^{(t+1)}] \\ \mathbb{E}[\boldsymbol{b}] \\ \mathbb{E}[A] \end{bmatrix}, \quad S = \begin{bmatrix} \mathbb{V}[\boldsymbol{x}^{(t+1)}] & \mathbb{C}[\boldsymbol{x}^{(t+1)}, \boldsymbol{b}] & \mathbb{C}[\boldsymbol{x}^{(t+1)}, A] \\ \mathbb{C}[\boldsymbol{b}, \boldsymbol{x}^{(t+1)}] & \mathbb{V}[\boldsymbol{b}] & \mathbb{C}[A, \boldsymbol{b}] \\ \mathbb{C}[A, \boldsymbol{x}^{(t+1)}] & \mathbb{C}[\boldsymbol{b}, A] & \mathbb{V}[A] \end{bmatrix}, \quad V = \begin{bmatrix} \mathbb{C}[\boldsymbol{x}^{(t)}, \boldsymbol{x}^{(t+1)}] & \mathbb{C}[\boldsymbol{x}^{(t)}, \boldsymbol{b}] & \mathbb{C}[\boldsymbol{x}^{(t)}, A] \\ \mathbb{C}[\boldsymbol{b}, \boldsymbol{x}^{(t+1)}] & \mathbb{V}[\boldsymbol{b}] & \mathbb{C}[A, \boldsymbol{b}] \\ \mathbb{C}[A, \boldsymbol{x}^{(t+1)}] & \mathbb{C}[\boldsymbol{b}, A] & \mathbb{V}[A] \end{bmatrix}
$$