Cart and Double Pendulum Dynamics

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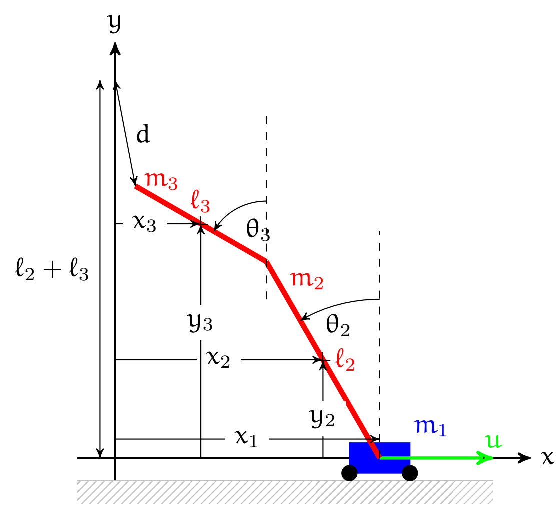
The cart and double pendulum dynamical system consists of a cart with mass m_1 and an attached double pendulum. The double pendulum consists of two pieces, the first of mass m_2 and length ℓ_2 and the second has mass m_3 and length ℓ_3. Both joints are frictionless and unactuated, the double pendulum can move freely in the vertical plane. The goal position is balancing the double pendulum vertically above $x = 0$, and the discrepancy is the distance d. The system is actuated by applying a force u horizontally to the cart. The state of the system is given by the horizontal position of the cart x_1, the two angles θ_2 and θ_3 (both measured anti-clockwise from vertically up) and the time derivatives of these three variables, $\dot{x}_1$, $\dot{\theta}_2$ and $\dot{\theta}_3$.



The cart can move horizontally, with an applied external force $-20\text{N} \leqslant u \leqslant 20\text{N}$, and coefficient of friction $b = 0.1\text{Ns/m}$. The other physical values are $m_1 = m_2 = m_3 = 0.5\text{kg}$ and $\ell_2 = \ell_3 = 0.6\text{m}$ (doesn't correspond to figure). Thus, the moment of inertia around the midpoint of each part of the pendulum is $I_2 = I_3 = 0.015\text{kg}\,\text{m}^2$.

Equations of Motion

The geometric relationships gives the following conditions for the coordinates of the midpoint of the two penduli:

$$x_2 = x_1 - \tfrac{1}{2}\ell_2 \sin\theta_2, \qquad y_2 = \tfrac{1}{2}\ell_2 \cos\theta_2, \tag{1}$$

$$x_3 = x_1 - \ell_2 \sin\theta_2 - \tfrac{1}{2}\ell_3 \sin\theta_3, \qquad y_3 = \ell_2 \cos\theta_2 + \tfrac{1}{2}\ell_3 \cos\theta_3. \tag{2}$$

The squared velocities of the centres of gravity are

$$v_1^2 = \dot{x}_1^2, \qquad v_2^2 = \dot{x}_1^2 - \ell_2 \dot{x}_1 \dot{\theta}_2 \cos\theta_2 + \tfrac{1}{4}\ell_2^2 \dot{\theta}_2^2, \tag{3}$$

$$v_3^2 = \dot{x}_1^2 - 2\ell_2 \dot{x}_1 \dot{\theta}_2 \cos\theta_2 - \ell_3 \dot{x}_1 \dot{\theta}_3 \cos\theta_3 + \ell_2^2 \dot{\theta}_2^2 + \tfrac{1}{4}\ell_3^2 \dot{\theta}_3^2 + \ell_2 \ell_3 \dot{\theta}_2 \dot{\theta}_3 \cos\theta_2 \cos\theta_3. \tag{4}$$

The system Lagrangian is

$$\begin{aligned} L &= T - V = \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2 + \tfrac{1}{2}m_3 v_3^2 + \tfrac{1}{2}I_2 \dot{\theta}_2^2 + \tfrac{1}{2}I_3 \dot{\theta}_3^2 - m_2 g y_2 - m_3 g y_3 \Rightarrow \\ L &= \tfrac{1}{2}(m_1 + m_2 + m_3)\dot{x}_1^2 - (\tfrac{1}{2}m_2 + m_3)\ell_2 \dot{x}_1 \dot{\theta}_2 \cos\theta_2 + (\tfrac{1}{6}m_2 + \tfrac{1}{2}m_3)\ell_2^2 \dot{\theta}_2^2 - \tfrac{1}{2}m_3 \ell_3 \dot{x}_1 \dot{\theta}_3 \cos\theta_3 + \\ &\quad \tfrac{1}{6}m_3 \ell_3^2 \dot{\theta}_3^2 + \tfrac{1}{2}m_3 \ell_2 \ell_3 \dot{\theta}_2 \dot{\theta}_3 \cos(\theta_2 - \theta_3) - (\tfrac{1}{2}m_2 + m_3) g \ell_2 \cos\theta_2 - \tfrac{1}{2}m_3 g \ell_3 \cos\theta_3, \end{aligned}$$

where $g = 9.82\text{m/s}^2$ is the accelleration of gravity.
The equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i,$$

where Q_i are the non-conservative forces. In our case

$$\frac{\partial L}{\partial \dot{x}_1} = (m_1 + m_2 + m_3)\dot{x}_1 - (\tfrac{1}{2}m_2 + m_3)\ell_2 \dot{\theta}_2 \cos\theta_2 - \tfrac{1}{2}m_3 \ell_3 \dot{\theta}_3 \cos\theta_3, \tag{5}$$

$$\frac{\partial L}{\partial x} = 0, \tag{6}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = (\tfrac{1}{3}m_2 + m_3)\ell_2^2 \dot{\theta}_2 - (\tfrac{1}{2}m_2 + m_3)\ell_2 \dot{x}_1 \cos\theta_2 + \tfrac{1}{2}m_3 \ell_2 \ell_3 \dot{\theta}_3 \cos(\theta_2 - \theta_3), \tag{7}$$

$$\frac{\partial L}{\partial \theta_2} = (\tfrac{1}{2}m_2 + m_3)\ell_2 (g + \dot{x}_1 \dot{\theta}_2)\sin\theta_2 - \tfrac{1}{2}m_3 \ell_2 \ell_3 \dot{\theta}_2 \dot{\theta}_3 \sin(\theta_2 - \theta_3), \tag{8}$$

$$\frac{\partial L}{\partial \dot{\theta}_3} = \tfrac{1}{3}m_3 \ell_3^2 \dot{\theta}_3 - \tfrac{1}{2}m_3 \ell_3 \dot{x}_1 \cos\theta_3 + \tfrac{1}{2}m_3 \ell_2 \ell_3 \dot{\theta}_2 \cos(\theta_2 - \theta_3), \tag{9}$$

$$\frac{\partial L}{\partial \theta_3} = \tfrac{1}{2}m_3 \ell_3 (g + \dot{x}_1 \dot{\theta}_3)\sin\theta_3 + \tfrac{1}{2}m_3 \ell_2 \ell_3 \dot{\theta}_2 \dot{\theta}_3 \sin(\theta_2 - \theta_3), \tag{10}$$

leading to the equations of motion

$$\begin{aligned} (m_1 + m_2 + m_3)\ddot{x}_1 - (\tfrac{1}{2}m_2 + m_3)\ell_2(\ddot{\theta}_2 \cos\theta_2 - \dot{\theta}_2^2 \sin\theta_2) - \tfrac{1}{2}m_3 \ell_3(\ddot{\theta}_3 \cos\theta_3 - \dot{\theta}_3^2 \sin\theta_3) &= u - b\dot{x}_1, \\ (\tfrac{1}{3}m_2 + m_3)\ell_2 \ddot{\theta}_2 - (\tfrac{1}{2}m_2 + m_3)(\ddot{x}_1 \cos\theta_2 + g\sin\theta_2) + \tfrac{1}{2}m_3 \ell_3[\ddot{\theta}_3 \cos(\theta_2 - \theta_3) + \dot{\theta}_3^2 \sin(\theta_2 - \theta_3)] &= 0, \\ \tfrac{1}{3}\ell_3 \ddot{\theta}_3 - \tfrac{1}{2}(\ddot{x}_1 \cos\theta_3 + g\sin\theta_3) + \tfrac{1}{2}\ell_2[\ddot{\theta}_2 \cos(\theta_2 - \theta_3) - \dot{\theta}_2^2 \sin(\theta_2 - \theta_3)] &= 0. \end{aligned}$$

Collecting the six state variables $z = (\dot{x}_1, \dot{\theta}_2, \dot{\theta}_3, x_1, \theta_2, \theta_3)$ the equations of motion can be conveniently expressed as six coupled differential equations

$$\frac{dz}{dt} = \begin{matrix} A^{-1} & B \\ I & 0 \end{matrix}$$

Linearized Dynamics

Linearizing the dynamics aroud the goal state, we can write the following approximation

$$\frac{dz}{dt} \simeq q^{-1}(Az+bu), \qquad q = 2\ell_2\ell_3(m_1(4m_2+3m_3)+m_2(m_2+m_3)), \qquad b = \begin{bmatrix} 2\ell_2\ell_3(4m_2+3m_3) \\ 6\ell_3(2m_2+m_3) \\ -6\ell_2 m_2 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and

$$A = \begin{bmatrix} -2\ell_2\ell_3(4m_2+3m_3)b & 0 & 0 & 0 & 3\ell_2\ell_3(m_2+2m_3)(2m_2+m_3)g & -3\ell_2\ell_3 m_2 m_3 g \\ -6\ell_3(2m_2+m_3)b & 0 & 0 & 0 & 3\ell_3(m_2+2m_3)(4m_1+4m_2+m_3)g & -9\ell_3 m_3(2m_1+m_2)g \\ 6\ell_2 m_2 b & 0 & 0 & 0 & -9\ell_2(m_2+2m_3)(2m_1+m_2)g & 3\ell_2(4m_1(m_2+3m_3)+m_2(m_2+4m_3))g \\ q & 0 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 \end{bmatrix}.$$

Loss Function

The instantaneous loss is given by

$$F = 1 - \exp(-\frac{d^2}{2a^2}),$$

were d^2 is the squared distance between the tip of the pendulum and the point at distance $\ell_2 + \ell_3$ above the origin, and a is the width parameter. Note that the instantaneous loss does not depend on the speed variables, $\dot{x}_1$, $\dot{\theta}_2$ and $\dot{\theta}_3$. The squared distance is

$$d^2 = (x_1 - \ell_2 \sin\theta_2 - \ell_3 \sin\theta_3)^2 + (\ell - \ell_2\cos\theta_2 + \ell_3\cos\theta_3)^2 = (\tilde{z}-\mu)^\top Q(\tilde{z}-\mu),$$

where $\tilde{z}$ is z augmented by four coordinates $\sin\theta_2, \sin\theta_3$ and $\cos\theta_2, \cos\theta_3$ and

$$Q = C^\top C, \quad C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & -\ell_1 & 0 & -\ell_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ell_1 & 0 & \ell_2 \end{bmatrix}, \quad \mu = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}^\top.$$

The expected loss, averaging over the possibly uncertain states is therefore

$$\mathbb{E}[F(\tilde{z})] = 1 - \int F(\tilde{z})p(\tilde{z})d\tilde{z},$$

which can be evaluated in closed form for Gaussian $p(\tilde{z})$, see reward.pdf. Since the augmented state $\tilde{z}$ will not generally be Gaussian even if z is Gaussian, we project on to the closest Gaussian by matching first and second moments.