The Pendulum Task

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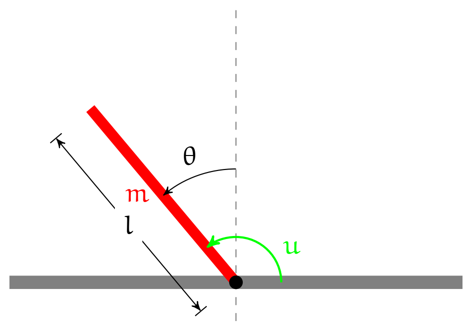
July 10th 2014



General

The pendulum system consists of pendulum (uniformly distributed mass m, length l) which is attached frictionless on one end. The pendulum angle, θ, is measured anti-clockwise from upright. A external moment u acting on the pendulum at the origin can be applied. Typical values are: $m = 1$kg, $l = 1$m. Two task can be considered

- For the pendulum starting in the upright position $\theta = 0$, the task is to stabilize it in this position. This can be achieved using a linear controller or a nonlinear controller.
- For the pendulum starting in the hanging down position $\theta = \pi$, the task is to swing-up in the upright position and stabilise there. This can only be achieved using a nonlinear controller.

Dynamics

The following equilibrium of moments acting on the pendulum must be fulfilled at all times:

$$\frac{d^2\theta}{dt^2} I = \frac{1}{2} mgl \sin(\theta) + u$$

where $I = \frac{1}{3}ml^2$ is the moment of inertia.
The state vector is defined as $z = \begin{bmatrix}\dot{\theta} & \theta\end{bmatrix}^T$, thus the differential equation is given as following

$$\frac{dz}{dt} = \begin{cases} \frac{3}{ml^2}\left(\frac{mgl}{2}\sin(z_2) + u\right) \\ z_1 \end{cases},$$

Linearized Dynamics

Linearising the dynamics around the goal state $z = \begin{bmatrix} 0 & 0 \end{bmatrix}^\top$, we can write the following approximation

$$\frac{dz}{dt} \simeq (Az + Bu), \qquad A = \begin{bmatrix} 0 & \frac{3g}{2l} \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} \frac{3}{ml^2} \\ 0 \end{bmatrix}$$

Loss Function

Currently exist two implementations for the loss functions

- The instantaneous loss is given by

  $$F = 1 - \exp(-\frac{d^2}{2a^2}),$$

  were a is the width parameter of the cost function and d is the Cartesian distance between the tip of the pendulum and the point at distance ℓ above the origin, The squared distance is

  $$d^2 = (\ell \sin\theta)^2 + (\ell - \ell\cos\theta)^2 = (\tilde{z} - \mu)^\top Q(\tilde{z} - \mu),$$

  where $\tilde{z}$ is z augmented by two coordinates $\sin\theta$ and $\cos\theta$ with

  $$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & l^2 & 0 \\ 0 & 0 & 0 & l^2 \end{bmatrix}, \qquad \mu = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

  Note that the instantaneous loss does not depend on the speed variable $\dot{\theta}$. This is currently implemented in loss.m

- The instantaneous loss is given by

  $$F = (\cos(\theta) + 1)/2$$

  i.e. the angular distance to the upright position. This is currently implemented in loss2.m.

The expected loss, averaging over the possibly uncertain states is therefore

$$\mathbb{E}[F(\tilde{z})] = 1 - \int F(\tilde{z})p(\tilde{z})d\tilde{z},$$

which can be evaluated in closed form for Gaussian $p(\tilde{z})$, see reward.pdf. Since the augmented state $\tilde{z}$ will not generally be Gaussian even if z is Gaussian, we project on to the closest Gaussian by matching first and second moments.