Augment Gaussian with Trigonometric Functions

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October 20th 2008

In several contexts it is useful to be able to augment a joint Gaussian distribution with the trigonometric functions sine and cosine of one or more of its coordinates. The resulting distribution is not Gaussian, but we can compute exactly the first and second (central) moments of the augmented distribution. Additionally, the derivative of these moments wrt. the parameters of the joint distribution are also computed.
Let x be a D dimensional Gaussian

$$\mathbf{x} \sim \mathcal{N}(\mathbf{a}, A),$$

which we want to augment by sine and cosine of $x_i, \forall i \in I$, where d is the number of elements in I, resulting in the $D+2d$ dimensional joint Gaussian

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{z} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}\right).$$

Below we derive expressions for the elements $\mathbf{b}, B$ and C.
For the means, we have for $i = 1, \ldots, d$

$$\begin{aligned} b_{2i-1} &= \mathbb{E}[\sin(x_{I(i)})] = \exp(-A_{I(i),I(i)}/2)\sin(a_{I(i)}), \\ b_{2i} &= \mathbb{E}[\cos(x_{I(i)})] = \exp(-A_{I(i)I(i)}/2)\cos(a_{I(i)}). \end{aligned}$$

For the covariances we have for $i = 1, \ldots, d, j = 1, \ldots, D$

$$\begin{aligned} B_{j,2i-1} &= \exp(-\tfrac{1}{2}A_{I(i),I(i)})\cos(a_{I(i)})A_{j,I(i)}, \\ B_{j,2i} &= -\exp(-\tfrac{1}{2}A_{I(i),I(i)})\sin(a_{I(i)})A_{j,I(i)}, \end{aligned}$$

and for $i, j = 1, \ldots, d$, $i \neq j$

$$\begin{aligned} C_{2i-1,2i-1} &= q_i(1+\exp(-A_{I(i),I(i)})\cos(2a_{I(i)})) \\ C_{2i,2i} &= q_i(1-\exp(-A_{I(i),I(i)})\cos(2a_{I(i)})) \\ C_{2i,2i-1} = C_{2i-1,2i} &= -q_i\exp(-A_{I(i),I(i)})\sin(2a_{I(i)}) \\ C_{2i-1,2j-1} &= q_{ij}([\exp(A_{I(i),I(j)})-1]\cos(a_{I(i)}-a_{I(j)}) - [\exp(-A_{I(i),I(j)})-1]\cos(a_{I(i)}+a_{I(j)})) \\ C_{2i,2j} &= q_{ij}([\exp(A_{I(i),I(j)})-1]\cos(a_{I(i)}-a_{I(j)}) + [\exp(-A_{I(i),I(j)})-1]\cos(a_{I(i)}+a_{I(j)})) \\ C_{2i,2j-1} &= q_{ij}(-[\exp(A_{I(i),I(j)})-1]\sin(a_{I(i)}-a_{I(j)}) + [\exp(-A_{I(i),I(j)})-1]\sin(a_{I(i)}+a_{I(j)})) \\ C_{2i-1,2j} &= q_{ij}([\exp(A_{I(i),I(j)})-1]\sin(a_{I(i)}-a_{I(j)}) + [\exp(-A_{I(i),I(j)})-1]\sin(a_{I(i)}+a_{I(j)})), \end{aligned}$$

where $q_i = (1-\exp(-A_{I(i),I(i)}))/2$, and $q_{ij} = \exp(-\frac{1}{2}(A_{I(i),I(i)} + A_{I(j),I(j)}))/2$.