Approximate Probabilistic Linear Control on Product Features

Philipp Hennig

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We define a linear policy $\boldsymbol{\pi}$ mapping from D inputs $\boldsymbol{x} \in \mathbb{R}^D$ to U outputs $\boldsymbol{\pi} \in \mathbb{R}^U$. In contrast to a naï ve linear policy, we augment the variables by an intermediate variable $\boldsymbol{y} \in \mathbb{R}^{\frac{1}{2}D(D+1)}$ which contains all products of elements of $\boldsymbol{x}$ up to second order:

$$\boldsymbol{\pi}(\boldsymbol{y}) = \boldsymbol{w}\boldsymbol{x} + \boldsymbol{\omega}\boldsymbol{y} + \boldsymbol{b} \qquad y_k = x_i x_j \text{ for some map } (i,j) \to k \tag{1}$$

We further assume we have access to only a Gaussian belief (instead of exact values) on the value of $\boldsymbol{x}$:

$$p(\boldsymbol{x}) = \mathcal{N}(\boldsymbol{x}; \mathbf{m}, \mathbf{s}) \tag{2}$$

This document uses the summation convention[1]: subscripts on objects denote dimensions of that multi-dimensional object. Subscripts present twice or more times in a product (!) term are summed over.

Approximate Gaussian Belief on $\boldsymbol{y}$

The belief on $\boldsymbol{y}$ is not Gaussian (to see this, consider the simple example of the square value x_i^2, which is positive semidefinite). But we can can calculate the first two moments of the joint belief on $(\boldsymbol{x}, \boldsymbol{y})$, defining an approximate Gaussian belief. For this, we use ISSERLI's theorem[2] (a special case of WICK's theorem[3]), which states that, for Gaussian distributed variables, such as our $\boldsymbol{x}$, the higher moments are

$$\begin{aligned} \langle (x_1 - m_1)(x_2 - m_2) \cdots (x_{2n-1} - m_{2n-1}) \rangle &= 0 \qquad \text{and} \\ \langle (x_1 - m_1)(x_2 - m_2) \cdots (x_{2n} - m_{2n}) \rangle &= \sum_{\text{pairs}} \prod_{(i,j)} \langle (x_i - m_i)(x_j - m_j) \rangle \end{aligned} \tag{3}$$

where the notation on the right hand side denotes a sum over products of all possible combinations of the index set into pairs. The theorem also holds if indices are repeated (i.e. if terms are raised to a power). With this, we can easily find the moments of $\boldsymbol{y}$, and thus an approximate Gaussian belief

$$q(\boldsymbol{x}, \boldsymbol{y}) = \mathcal{N}\left[\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix}; \begin{pmatrix} \mathbf{m} \\ \boldsymbol{\mu} \end{pmatrix}, \begin{pmatrix} \mathbf{s} & \boldsymbol{\Xi}^\intercal \\ \boldsymbol{\Xi} & \boldsymbol{\Sigma} \end{pmatrix}\right] \tag{4}$$

For the parameters, we find, after some lengthy algebra,

$$\mu_{(ij)} = \langle x_i x_j \rangle = s_{ij} + m_i m_j \tag{5}$$

$$\Xi_{(ij)\ell} = \langle (x_i x_j) x_\ell \rangle - \langle x_i x_j \rangle \langle x_\ell \rangle = s_{i\ell} m_j + s_{j\ell} m_i \tag{6}$$

[1] A. Einstein, *"Die Grundlagen der allgemeinen Relativitätstheorie"*, Annalen der Physik **354** 7 (1916) pp. 769–822

[2] L. Isserlis, *"On a formula for the product-moment coefficient of a normal frequency distribution in any number of variables"*; Biometrika **12** 1/2 (Nov 1918), pp. 134–139

[3] G.C. Wick, *"The evaluation of the collision matrix"*; Phys. Rev. **80** 2 (Oct 1950), pp. 268–272

and

$$\begin{aligned}\Sigma_{(ij)(rt)} &= \langle (x_i x_j)(x_r x_t)\rangle - \langle x_i x_j\rangle\langle x_r x_t\rangle \\ &= s_{ir}s_{jt} + s_{it}s_{jr} + s_{ir}m_j m_t + s_{it}m_j m_r + s_{jr}m_i m_t + s_{jt}m_i m_r \\ &= s_{ir}\mu_{jt} + s_{it}\mu_{jr} + s_{jr}m_i m_t + s_{jt}m_i m_r.\end{aligned} \tag{7}$$

Belief on $\boldsymbol{\pi}$

Using the approximate Gaussian belief on $\boldsymbol{y}$, it is straightforward to obtain a belief on $\boldsymbol{\pi}$ by marginalizing.

$$\begin{aligned}p(\boldsymbol{\pi}) &= \int p(\boldsymbol{\pi}\,|\,\boldsymbol{y})q(\boldsymbol{y}\,|\,\boldsymbol{\mu},\boldsymbol{\Sigma})\,\mathrm{d}\boldsymbol{y} \\ &= \mathcal{N}\left[\boldsymbol{\pi}; \begin{pmatrix}\boldsymbol{w} & \boldsymbol{\omega}\end{pmatrix}\begin{pmatrix}\mathbf{m}\\ \boldsymbol{\mu}\end{pmatrix}, \begin{pmatrix}\boldsymbol{w} & \boldsymbol{\omega}\end{pmatrix}\begin{pmatrix}\mathbf{s} & \boldsymbol{\Xi}^\intercal \\ \boldsymbol{\Xi} & \boldsymbol{\Sigma}\end{pmatrix}\begin{pmatrix}\boldsymbol{w}^\intercal \\ \boldsymbol{\omega}^\intercal\end{pmatrix}\right] \\ &= \mathcal{N}\left[\boldsymbol{\pi}; \boldsymbol{w}\mathbf{m} + \boldsymbol{\omega}\boldsymbol{\mu} + \boldsymbol{b}, \boldsymbol{w}\mathbf{s}\boldsymbol{w}^\intercal + \boldsymbol{\omega}\boldsymbol{\Xi}\boldsymbol{w}^\intercal + \boldsymbol{w}\boldsymbol{\Xi}^\intercal\boldsymbol{\omega}^\intercal + \boldsymbol{\omega}\boldsymbol{\Sigma}\boldsymbol{\omega}^\intercal\right] \\ &\equiv \mathcal{N}(\boldsymbol{\pi}, \boldsymbol{M}, \boldsymbol{S})\end{aligned} \tag{8}$$

Derivatives

To optimize the policy, we also need the derivatives of the output parameters $\boldsymbol{M}$, $\boldsymbol{S}$ with respect to $\mathbf{m}$ and $\mathbf{s}$, as well as those same derivatives for the variable $\boldsymbol{C}$, which is the product of $\mathbf{s}^{-1}$ and the input-output covariance

$$\begin{aligned}\boldsymbol{C} &= \mathbf{s}^{-1}(\langle \boldsymbol{x}\boldsymbol{\pi}^\intercal\rangle - \langle\boldsymbol{x}\rangle\langle\boldsymbol{\pi}^\intercal\rangle) \\ &= \mathbf{s}^{-1}(\langle \boldsymbol{x}(\boldsymbol{x}^\intercal\boldsymbol{w}^\intercal + \boldsymbol{y}^\intercal\boldsymbol{\omega} + \boldsymbol{b})\rangle - \langle\boldsymbol{x}\rangle\langle \boldsymbol{x}^\intercal\boldsymbol{w}^\intercal + \boldsymbol{y}^\intercal\boldsymbol{\omega}^\intercal + \boldsymbol{b}\rangle) \\ &= \mathbf{s}^{-1}(\mathbf{s}\boldsymbol{w}^\intercal + \boldsymbol{\Xi}^\intercal\boldsymbol{\omega}^\intercal) \\ &= \boldsymbol{w}^\intercal + \mathbf{s}^{-1}\boldsymbol{\Xi}^\intercal\boldsymbol{\omega}^\intercal \\ \boldsymbol{C}_{au} &= w_{ua} + s^{-1}_{a\ell}(s_{i\ell}m_j + s_{j\ell}m_i)\omega_{uij} \\ &= w_{ua} + (\delta_{ai}m_j + \delta_{aj}m_i)\omega_{uij}\end{aligned} \tag{9}$$

We can evaluate these derivatives using the chain rule, which will require the terms

$$\frac{\partial m_\ell}{\partial m_r} = \delta_{\ell r} \qquad \frac{\partial \mu_{(ij)}}{\partial m_r} = (\delta_{ir}m_j + \delta_{jr}m_i) \tag{10}$$

$$\frac{\partial m_i}{\partial s_{rt}} = 0 \qquad \frac{\partial \mu_{(ij)}}{\partial s_{rt}} = \delta_{(ij)(rt)} \tag{11}$$

$$\frac{\partial s_{k\ell}}{\partial m_r} = 0 \qquad \frac{\partial \Xi_{(ij)\ell}}{\partial m_r} = \delta_{ir}s_{j\ell} + \delta_{jr}s_{i\ell} \tag{12}$$

$$\frac{\partial s_{k\ell}}{\partial s_{rt}} = \delta_{(k\ell)(rt)} \qquad \frac{\partial \Xi_{(ij)\ell}}{\partial s_{rt}} = \delta_{ir}\delta_{\ell t}m_j + \delta_{jr}\delta_{\ell t}m_i \tag{13}$$

and

$$\begin{aligned}\frac{\partial \Sigma_{(ij)(k\ell)}}{\partial m_r} &= \delta_{ir}(s_{jk}m_\ell + s_{j\ell}m_k) + \delta_{jr}(s_{i\ell}m_k + s_{ik}m_\ell) + \delta_{\ell r}(s_{ik}m_j + s_{jk}m_i) + \delta_{kr}(s_{i\ell}m_j + s_{j\ell}m_i) \\ \frac{\partial \Sigma_{(ij)(k\ell)}}{\partial s_{rt}} &= \delta_{(ik)(rt)}(s_{j\ell} + m_j m_\ell) + \delta_{(j\ell)(rt)}(s_{ik} + m_i m_k) + \delta_{(i\ell)(rt)}(s_{jk} + m_j m_k) + \delta_{(jk)(rt)}(s_{i\ell} + m_i m_\ell) \\ &= \delta_{(ik)(rt)}\mu_{j\ell} + \delta_{(j\ell)(rt)}\mu_{ik} + \delta_{(i\ell)(rt)}\mu_{jk} + \delta_{(jk)(rt)}\mu_{i\ell}\end{aligned} \tag{14}$$

Using these intermediate results, we find

$$
\begin{aligned}
\frac{\partial M_u}{\partial m_k} &= w_{ui}\delta_{ik}m_i + \omega_{uij}(\delta_{ik}m_j + \delta_{jk}m_i) \\
&= w_{uk} + \omega_{ukj}m_j + \omega_{uik}m_i \\
\frac{\partial S_{ut}}{\partial m_r} &= \frac{\partial}{\partial m_r}\left(w_{ui}s_{ij}w_{tj} + \omega_{u(ij)}\Xi_{(ij)\ell}w_{t\ell} + w_{u\ell}\Xi_{(ij)\ell}\omega_{t(ij)} + \omega_{u(ij)}\Sigma_{(ij)(k\ell)}\omega_{t(k\ell)}\right) \\
&= \omega_{uij}(\delta_{ir}s_{j\ell} + \delta_{jr}s_{i\ell})w_{t\ell} + w_{u\ell}(\delta_{ir}s_{j\ell} + \delta_{jr}s_{i\ell})\omega_{tk\ell} + \\
&\quad \omega_{uij}\left[\delta_{ir}(s_{jk}m_\ell + s_{j\ell}m_k) + \delta_{jr}(s_{i\ell}m_k + s_{ik}m_\ell) + \delta_{\ell r}(s_{ik}m_j + s_{jk}m_i) + \delta_{kr}(s_{i\ell}m_j + s_{j\ell}m_i)\right]\omega_{tk\ell} \\
&= \omega_{urj}s_{j\ell}w_{t\ell} + \omega_{uir}s_{i\ell}w_{t\ell} + w_{u\ell}s_{j\ell}\omega_{trj} + w_{u\ell}s_{i\ell}\omega_{tir} + \\
&\quad \omega_{urj}(s_{jk}m_\ell + s_{j\ell}m_k)\omega_{tk\ell} + \omega_{uir}(s_{i\ell}m_k + s_{ik}m_\ell)\omega_{tk\ell} \\
&\quad \omega_{uij}(s_{ik}m_j + s_{jk}m_i)\omega_{tkr} + \omega_{uij}(s_{i\ell}m_j + s_{j\ell}m_i)\omega_{tr\ell} \\
&= \underbrace{(\omega_{urj} + \omega_{ujr})(\mathbf{s}\boldsymbol{w}^\intercal)_{jt} + (\omega_{trj} + \omega_{tjr})(\mathbf{s}\boldsymbol{w}^\intercal)_{ju}}_{\text{symmetric under } u \leftrightarrow t} + \\
&\quad (\underbrace{\omega_{urj}\omega_{tk\ell} + \omega_{uk\ell}\omega_{trj}}_{\text{symmetric under } t \leftrightarrow u} + \underbrace{\omega_{ujr}\omega_{tk\ell} + \omega_{uk\ell}\omega_{tjr}}_{\text{symmetric under } t \leftrightarrow u})(s_{jk}m_\ell + s_{j\ell}m_k)
\end{aligned}
\tag{15}
$$

For the transformation to the last line, a few summation variables were re-named, and some matrix multiplications made explicit. Note that the second two terms are identical to the first two up to the "transposition" $u \leftrightarrow t$, and the same symmetry applies for the following four terms. It is important to point out here that the tensor ω_{uij} is *not* symmetric under $i \leftrightarrow j$, because we only have nonzero weights for $j \geq i$. With similar methods, we find further

$$
\begin{aligned}
\frac{\partial C_{ku}}{\partial m_r} &= \frac{\partial}{\partial m_r}\left(w_{uk} + s^{-1}_{k\ell}\Xi_{(ij)\ell}\omega_{u(ij)}\right) = s^{-1}_{k\ell}(\delta_{ir}s_{j\ell} + \delta_{jr}s_{i\ell})\omega_{uij} = s^{-1}_{k\ell}[s_{j\ell}(\omega_{urj} + \omega_{ujr})] \\
&= \delta_{kj}(\omega_{urj} + \omega_{ujr}) = \omega_{urk} + \omega_{ukr} \\
\frac{\partial M_u}{\partial s_{rt}} &= \omega_{uij}\frac{\partial \mu_{ij}}{\partial s_{rt}} = \omega_{uij}\delta_{(ij)(rt)} = \omega_{urt} \\
\frac{\partial S_{uv}}{\partial s_{rt}} &= w_{ui}\frac{\partial s_{ij}}{\partial s_{rt}}\omega_{vj} + (\omega_{uij}w_{v\ell} + w_{u\ell}\omega_{vij})\frac{\partial \Xi_{ij\ell}}{\partial s_{rt}} + \omega_{uij}\frac{\partial \Sigma_{ijk\ell}}{\partial s_{rt}}\omega_{vk\ell} \\
&= w_{ur}\delta_{(rt)(ij)}w_{vt} + (\omega_{uij}w_{v\ell} + w_{u\ell}\omega_{vij})(\delta_{ir}\delta_{lt}m_j + \delta_{jr}\delta_{\ell t}m_i) + \\
&\quad \omega_{uij}(\delta_{ir}\delta_{kt}\mu_{j\ell} + \delta_{jr}\delta_{\ell t}\mu_{ik} + \delta_{ir}\delta_{\ell t}\mu_{jk} + \delta_{jr}\delta_{kt}\mu_{il})\omega_{vkl} \\
&= w_{ur}w_{vt} + \omega_{urj}w_{vt}m_j + w_{ut}\omega_{vrj}m_j + \omega_{uir}w_{vt}m_i + w_{ut}\omega_{vir}m_i + \\
&\quad \omega_{urj}\omega_{vt\ell}\mu_{j\ell} + \omega_{uir}\omega_{vkt}\mu_{ik} + \omega_{urj}\omega_{vkt}\mu_{jk} + \omega_{uir}\omega_{vt\ell}\mu_{i\ell} \\
&= w_{ur}w_{vt} + [(\omega_{urj} + \omega_{ujr})w_{vt} + (\omega_{vrj} + \omega_{vjr})w_{ut}]m_j + [(\omega_{urj} + \omega_{ujr})(\omega_{vt\ell} + \omega_{v\ell t})]\mu_{j\ell}
\end{aligned}
\tag{16}
$$

The derivative of the input-output covariance with respect to the product weights holds a surprise. Using that, for invertible matrices[4] $\boldsymbol{A}$,

$$
\frac{\partial A^{-1}_{ij}}{\partial z} = -A^{-1}_{ik}\frac{\partial A_{k\ell}}{\partial z}A^{-1}_{\ell j}
\tag{17}
$$

[4] S. Roweis, *"Matrix Identities"*, Univ. of Toronto, 1999

we arrive at

$$\begin{aligned}
\frac{\partial C_{au}}{\partial s_{rt}} &= \frac{\partial s^{-1}_{a\ell}}{\partial s_{rt}} \Xi_{ij\ell}\omega_{uij} + s^{-1}_{a\ell}\frac{\partial \Xi_{ij\ell}}{\partial s_{rt}}\omega_{uij} \\
&= -s^{-1}_{ab}\delta_{br}\delta_{ct}s^{-1}_{c\ell}\Xi_{ij\ell}\omega_{uij} + s^{-1}_{a\ell}(\delta_{ir}\delta_{\ell t}m_j + \delta_{jr}\delta_{\ell t}m_i)\omega_{hij} \\
&= -s^{-1}_{ar}s^{-1}_{t\ell}\Xi_{ij\ell}\omega_{uij} + s^{-1}_{at}\omega_{urj}m_j + s^{-1}_{at}\omega_{uir}m_i \\
&= -s^{-1}_{ar}s^{-1}_{t\ell}\Xi_{ij\ell}\omega_{uij} + s^{-1}_{at}(\omega_{urj} + \omega_{ujr})m_j
\end{aligned} \tag{18}$$

With Equation (5), we can expand this expression further to find

$$\begin{aligned}
\frac{\partial C_{au}}{\partial s_{rt}} &= -s^{-1}_{ar}s^{-1}_{t\ell}(s_{i\ell}m_j s_{j\ell}m_i)\omega_{uij} + s^{-1}_{at}(\omega_{urj} + \omega_{ujr})m_j \\
&= -s^{-1}_{ar}\delta_{ti}(\omega_{uij} + \omega_{uji})m_j + s^{-1}_{at}(\omega_{urj} + \omega_{ujr})m_j \\
&= -s^{-1}_{ar}(\omega_{utj} + \omega_{ujt})m_j + s^{-1}_{at}(\omega_{urj} + \omega_{ujr})m_j
\end{aligned} \tag{19}$$

In particular, Equation (19) implies that $\partial \boldsymbol{C}/\partial s_{rt} = -\partial \boldsymbol{C}/\partial s_{tr}$. However, because $\mathbf{s}$ is positive definite, we also have $s_{rt} \equiv s_{tr}$ and hence $\partial \boldsymbol{C}/\partial s_{rt} = \partial \boldsymbol{C}/\partial s_{tr}$. It follows thence that this derivative vanishes:

$$\frac{\partial C_{au}}{\partial s_{rt}} \equiv 0 \tag{20}$$

A few of the required derivatives are straightforward:

$$\frac{\partial M_u}{\partial b_v} = \delta_{uv} \qquad \frac{\partial S_{uv}}{\partial b_h} = 0 \qquad \frac{\partial C_{au}}{\partial b_h} = 0 \tag{21}$$

$$\frac{\partial M_u}{\partial w_{vr}} = \delta_{vu}m_r \qquad \frac{\partial C_{au}}{\partial w_{vr}} = \delta_{uv}\delta_{ar} \qquad \frac{\partial M_u}{\partial \omega_{vrs}} = \delta_{uv}\delta_{ri}\delta_{sj}\mu_{ij} \tag{22}$$

The remaining necessary results are slightly more involved, but also not problematic:

$$\begin{aligned}
\frac{\partial S_{uv}}{\partial w_{hr}} &= \frac{\partial}{\partial w_{hr}}\left[w_{ui}s_{ij}w_{vj} + (\omega_{uij}w_{vk} + w_{uk}\omega_{vij})\Xi_{ijk}\right] \\
&= \delta_{uh}(s_{rj}w_{vj} + \omega_{vij}\Xi_{ijr}) + \delta_{vh}(s_{ir}w_{ui} + \omega_{uij}\Xi_{ijr}) \\
&= \delta_{uh}(s_{rj}w_{vj} + \omega_{vij}\Xi_{ijr}) + \delta_{vh}(s_{rj}w_{uj} + \omega_{uij}\Xi_{ijr}).
\end{aligned} \tag{23}$$

Also

$$\begin{aligned}
\frac{\partial S_{uv}}{\omega_{hrs}} &= \frac{\partial}{\partial \omega_{hrs}}\left[w_{ui}s_{ij}w_{vj} + (\omega_{uij}w_{vk} + w_{uk}\omega_{vij})\Xi_{ijk}\right] \\
&= \delta_{uh}(w_{vk}\Xi_{rsk} + \Sigma_{rsk\ell}\omega_{vk\ell}) + \delta_{vh}(w_{uk}\Xi_{rsk} + \Sigma_{ijrs}\omega_{uij}) \\
&= \delta_{uh}(w_{vk}\Xi_{rsk} + \Sigma_{rsk\ell}\omega_{vk\ell}) + \delta_{vh}(w_{uk}\Xi_{rsk} + \Sigma_{rsk\ell}\omega_{uk\ell})
\end{aligned} \tag{24}$$

And finally:

$$\begin{aligned}
\frac{\partial C_{au}}{\partial \omega_{vrs}} &= \left(\mathbf{s}^{-1}\boldsymbol{\Xi}^{\intercal}\right)_{aij}\delta_{uv}\delta_{(ij)(rs)} \\
&= \delta_{uv}(\delta_{ur}m_s + \delta_{us}m_r)
\end{aligned} \tag{25}$$

In particular, notice that none of the final results requires the explicit inverse of $\mathbf{s}$. This is important not only for computational efficiency, but also for numerical stability: If $\mathbf{s}$ is singular, the inverse is not defined; but there is nothing wrong with predicting the product of variables known with infinite precision.