CostSuper Function Derivations

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1 CostSat

Define the saturating cost of a state s as:

$$\mathsf{costSat}(s) \quad \doteq \quad 1 - \exp(-(s-z)^\top W(s-z)/2) \tag{1}$$

2 CostSat Covariance

The goal is to compute covariance of two costSat function outputs, each inputted with a different state:

$$C \quad \doteq \quad \mathbb{C}[\mathsf{costSat}(s_1), \mathsf{costSat}(s_2)], \tag{2}$$

where s_1 is one state, and s_2 is another state. Note the covariance C is non-zero if and only if states s_1 and s_2 covary. Our aim is to re-use the functionality of `costSat` already provided as much as possible. Let us denote

$$\begin{aligned} q_i &\doteq -(s_i - z)^\top W(s_i - z)/2,\ i \in \{1,2\}, && (3)\\ \mu_i &\doteq \mathbb{E}\left[\mathsf{costSat}(s_i)\right],\ i \in \{1,2\}, && (4)\\ &= \mathbb{E}\left[1 - \exp(q_i)\right]. && (5) \end{aligned}$$

Note the existing `costSat` function is able to compute the expectation μ_i. We begin with the covariance definition:

$$\begin{aligned} C &= \mathbb{E}\left[\mathsf{costSat}(s_1)\cdot\mathsf{costSat}(s_2)\right] - \mu_1\mu_2 && (6)\\ &= \mathbb{E}\left[\big(1-\exp(q_1)\big)\big(1-\exp(q_2)\big)\right] - \mu_1\mu_2 && (7)\\ &= \mathbb{E}\left[\exp(q_1)\exp(q_2)\right] - (1-\mu_1)(1-\mu_2) && (8)\\ &= (1-\mu) - (1-\mu_1)(1-\mu_2) && (9) \end{aligned}$$

where μ is the expectation output of the `costSat` function with augmented parameters $\hat{z} = \begin{bmatrix} z \\ z \end{bmatrix}$, $\hat{W} = \begin{bmatrix} W & 0 \\ 0 & W \end{bmatrix}$ and concatenated input $\hat{s} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$.

3 CostSat Moments

Using again the definition:

$$\mathsf{costSat}(s; z, W) \quad \doteq \quad 1 - \exp(-\tfrac{1}{2}(s-z)^\top W(s-z)) \tag{10}$$

Let $s \sim \mathcal{N}(m, \Sigma)$, then:

$$\begin{aligned} M = \mathbb{E}_s[\mathsf{costSat}(s;z,W)] &= 1 - \det(I + \Sigma W)^{-1/2} \exp\Big(-\tfrac{1}{2}(m-z)^\top W(I+\Sigma W)^{-1}(m-z)\Big) && (11)\\ S = \mathbb{V}_s[\mathsf{costSat}(s;z,W)] &= \det(I + 2\Sigma W)^{-1/2} \exp\Big(-(m-z)^\top W(I+2\Sigma W)^{-1}(m-z)\Big) - (M-1)^2 && (12)\\ C = \Sigma^{-1}\mathbb{C}_s[s,\ \mathsf{costSat}(s;z,W)] &= (M-1)\Big(Wz - W(I+\Sigma W)^{-1}(\Sigma W z + m)\Big) && (13) \end{aligned}$$

4 CostSat Hierarchical-Moments

Let $s \sim \mathcal{N}(\mu, V)$ and $\mu \sim \mathcal{N}(m, \Sigma)$, and $W' = W(I + VW)^{-1}$, and $W'' = 2W(I + 2VW)^{-1}$:

$$
\begin{aligned}
M' = \mathbb{E}_\mu[M] &= \mathbb{E}_\mu\Big[1 - \det(I + VW)^{-1/2} \exp\Big(-\tfrac{1}{2}(\mu - z)^\top W'(\mu - z)\Big)\Big] && (14)\\
&= 1 - \det(I + VW)^{-1/2}\Big(1 - \mathbb{E}_\mu\big[\mathsf{costSat}(\mu; z, W')\big]\Big) && (15)\\
&= 1 - \det\big((I + VW)(I + \Sigma W')\big)^{-1/2} \exp\Big(-\tfrac{1}{2}(m - z)^\top W'(I + \Sigma W')^{-1}(m - z)\Big) && (16)\\
&= 1 - \det\big(I + (\Sigma + V)W)\big)^{-1/2} \exp\Big(-\tfrac{1}{2}(m - z)^\top W(I + (\Sigma + V)W)^{-1}(m - z)\Big) && (17)\\
&= \mathbb{E}[\mathsf{costSat}(\mathcal{N}(m, \Sigma + V)\ ;\ z, W)] && (18)\\
S' = \mathbb{V}_\mu[M] &= \mathbb{V}_\mu\Big[\det(I + VW)^{-1/2} \exp\Big(-\tfrac{1}{2}(\mu - z)^\top W'(\mu - z)\Big)\Big] && (19)\\
&= \det(I + VW)^{-1}\mathbb{V}_\mu\big[\mathsf{costSat}(\mu; z, W')\big] && (20)\\
&= \det(I + VW)^{-1}\left(\mathbb{E}_\mu\Big[\exp\Big(-(\mu - z)^\top W'(\mu - z)\Big)\Big] - \mathbb{E}_\mu\Big[\exp\Big(-\tfrac{1}{2}(\mu - z)^\top W'(\mu - z)\Big)\Big]^2\right) && (21)\\
&= \det(I + VW)^{-1}\det(I + 2\Sigma W')^{-1/2}\exp\Big(-(m - z)^\top W'(I + 2\Sigma W')^{-1}(m - z)\Big) - (M' - 1)^2 && (22)\\
&= \det(I + VW)^{-1/2}\det(I + (V + 2\Sigma)W)^{-1/2}\exp\Big(-(m - z)^\top W(I + (V + 2\Sigma)W)^{-1}(m - z)\Big) - (M' - 1)^2 && (23)\\
&= \det(I + VW)^{-1/2} \cdot \big(\mathbb{V}[\mathsf{costSat}(\mathcal{N}(m, (V + 2\Sigma)/2); z, W)] + (\mathbb{E}[\cdot] - 1)^2\big) - (M' - 1)^2 && (24)\\
V' = \mathbb{E}_\mu[S] &= \mathbb{V}[\mathsf{costSat}(\mathcal{N}(m, \Sigma + V); z, W)] - \mathbb{V}[\mathsf{costSat}(\mathcal{N}(m, \Sigma); z, W)] && (25)
\end{aligned}
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