WPI

Efficient Factor Graph Fusion for Multi-robot Mapping

Ramkumar Natarajan

Advisor: Prof. Michael A. Gennert

Outline

- Motivation
  - Factor Graphs
  - Mapping History
- Factor Graph SLAM
  - Variable Ordering
  - Bayes Trees
- Efficient Factor Graph Fusion
  - Fusion Ordering
  - Formal Verification
- Relative Pose Graphs
- Results

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  - Without GPS even more challenging.
- High and efficient processing requirements for multiple robots in large scale environment.

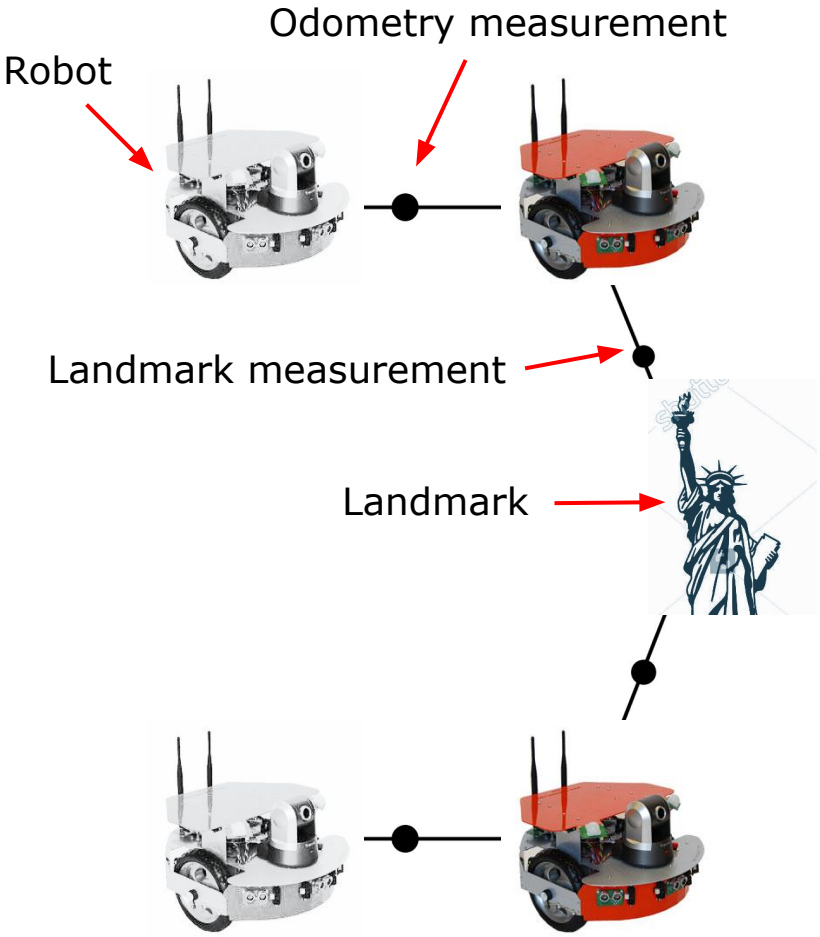
Multi-robot Mapping Problem (Simultaneous Localization and Mapping)

- Onboard sensors:
  - Wheel odometry
  - Inertial measurement unit
  - Laser range finder
  - Sonar
  - Camera
  - RGB-D sensor

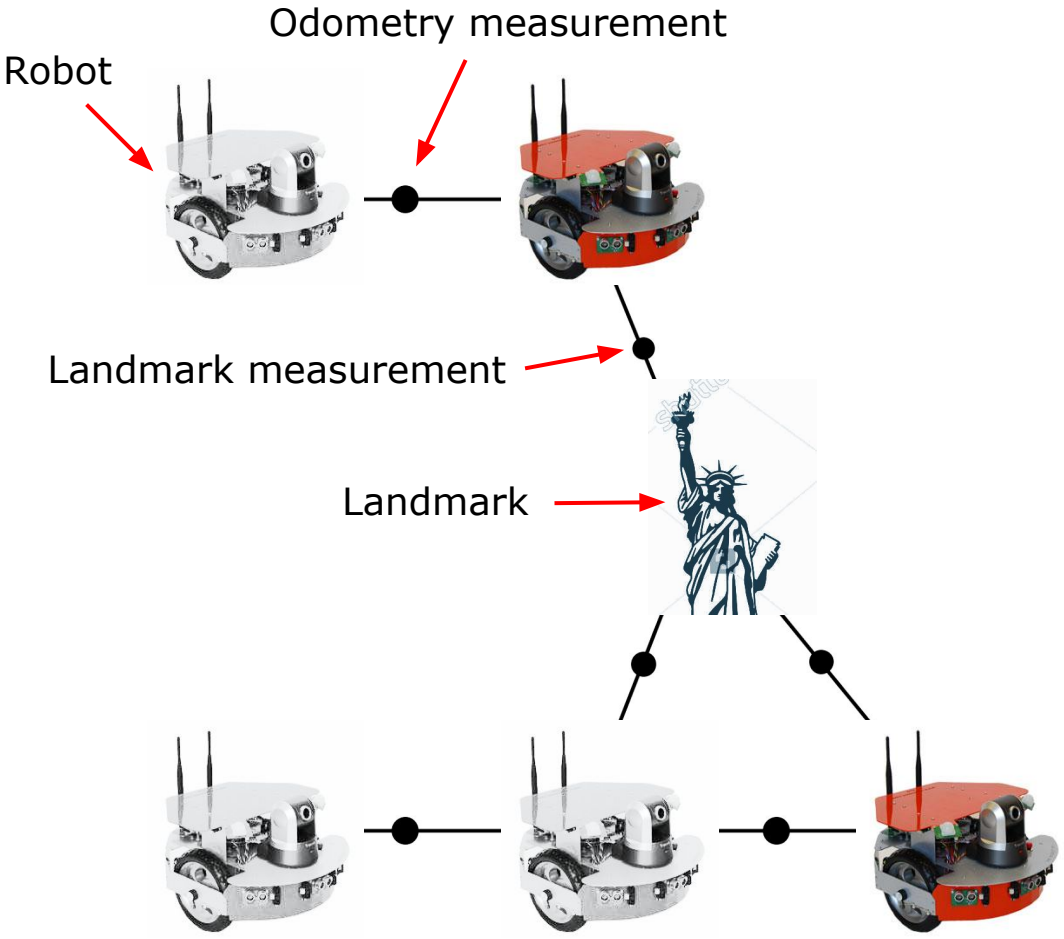
Multi-robot Mapping Problem (t=0)

Robot

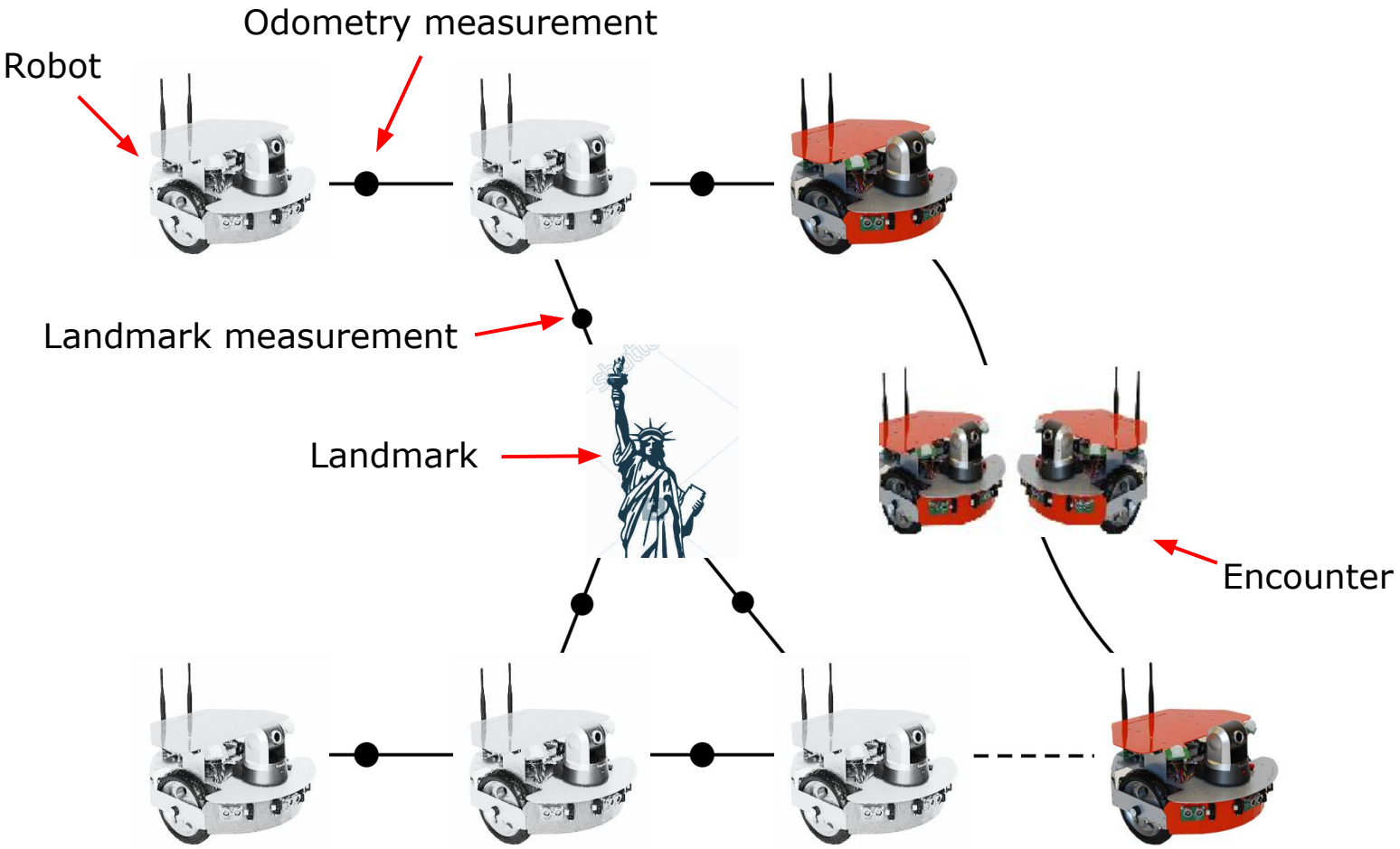
Multi-robot Mapping Problem (t=1)



Multi-robot Mapping Problem (t=2)



Multi-robot Mapping Problem (t=n-1)



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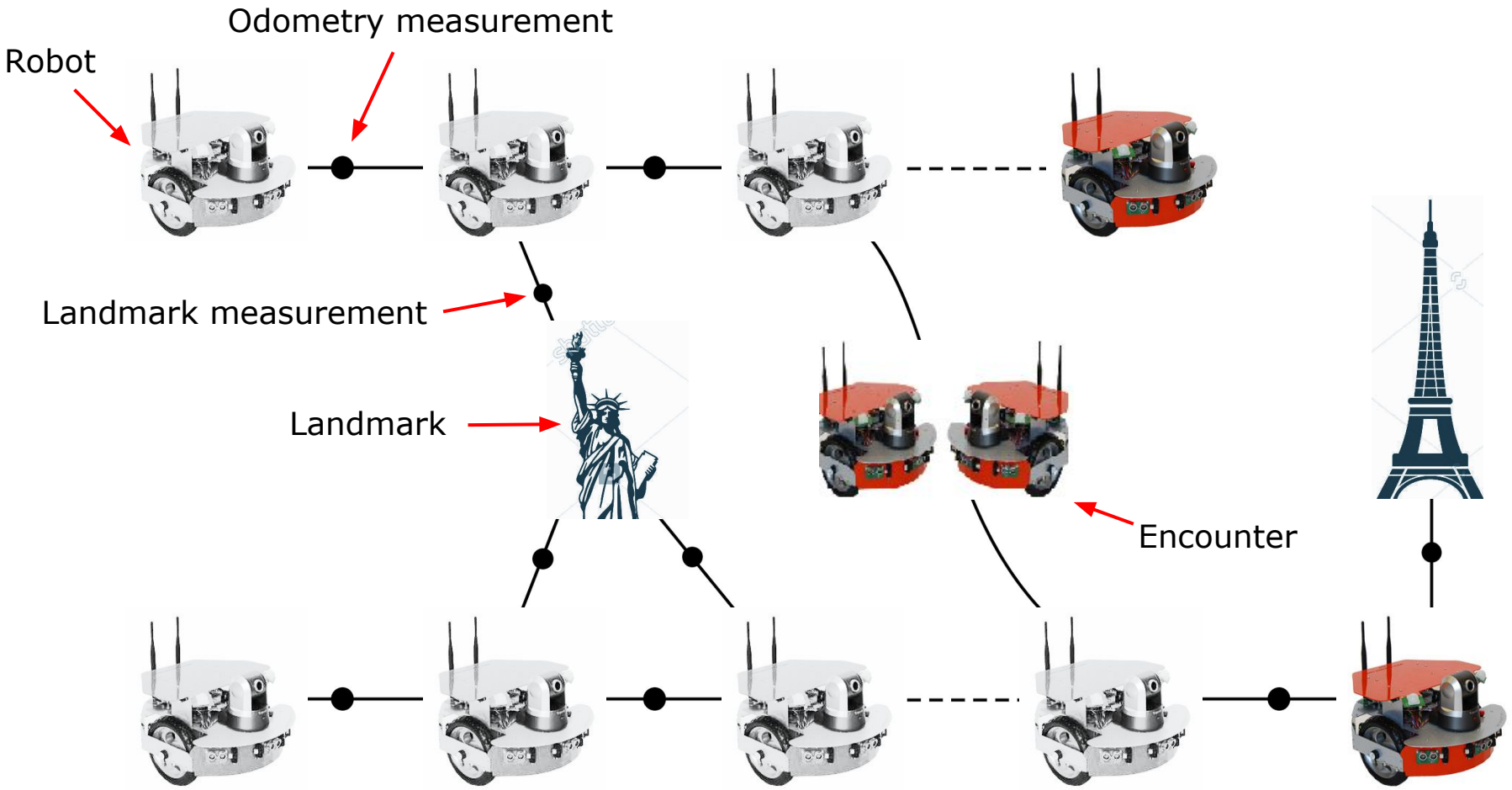
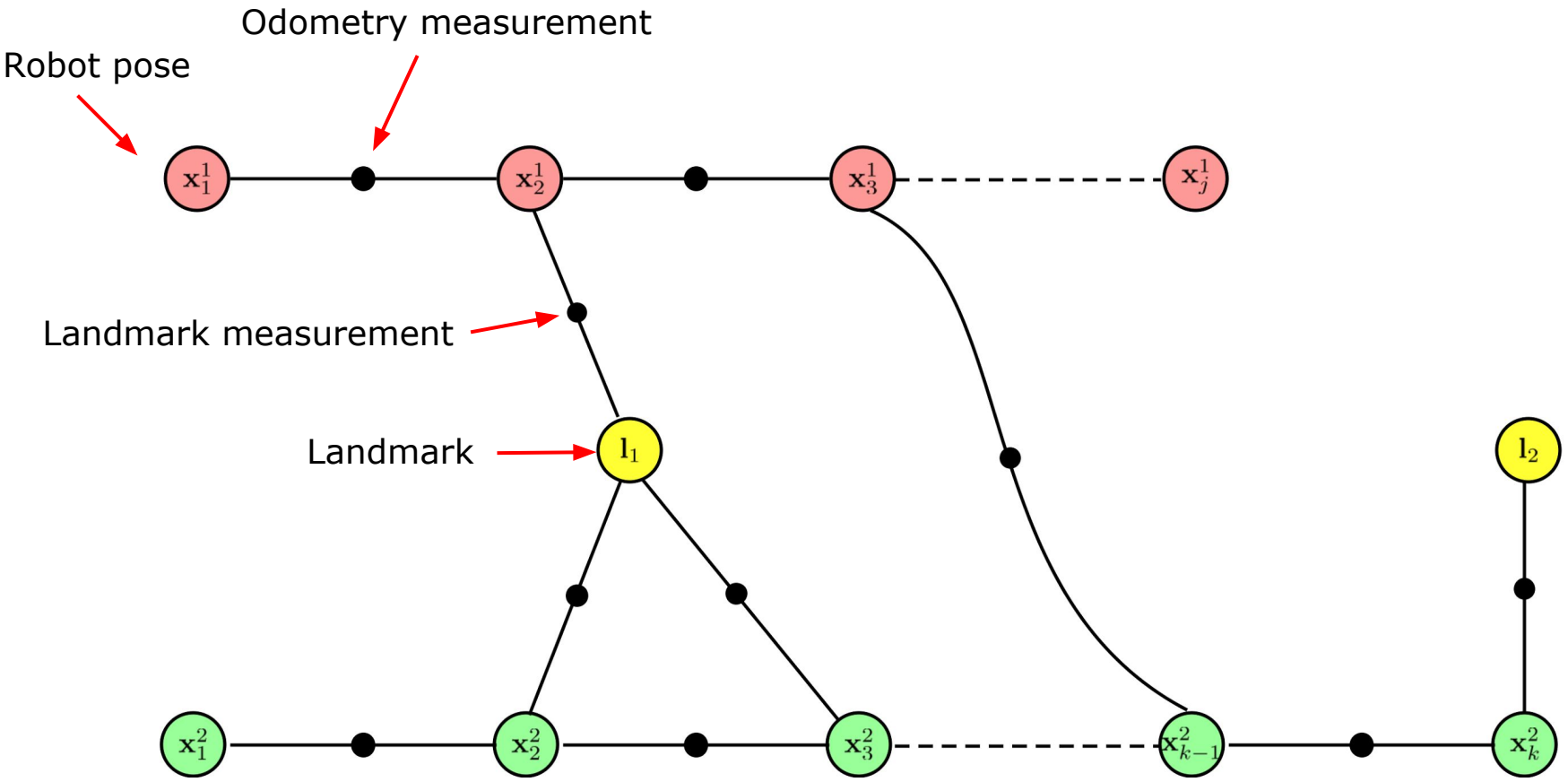


Factor Graph Representation [Lu & Milios 97, Folkesson 06]



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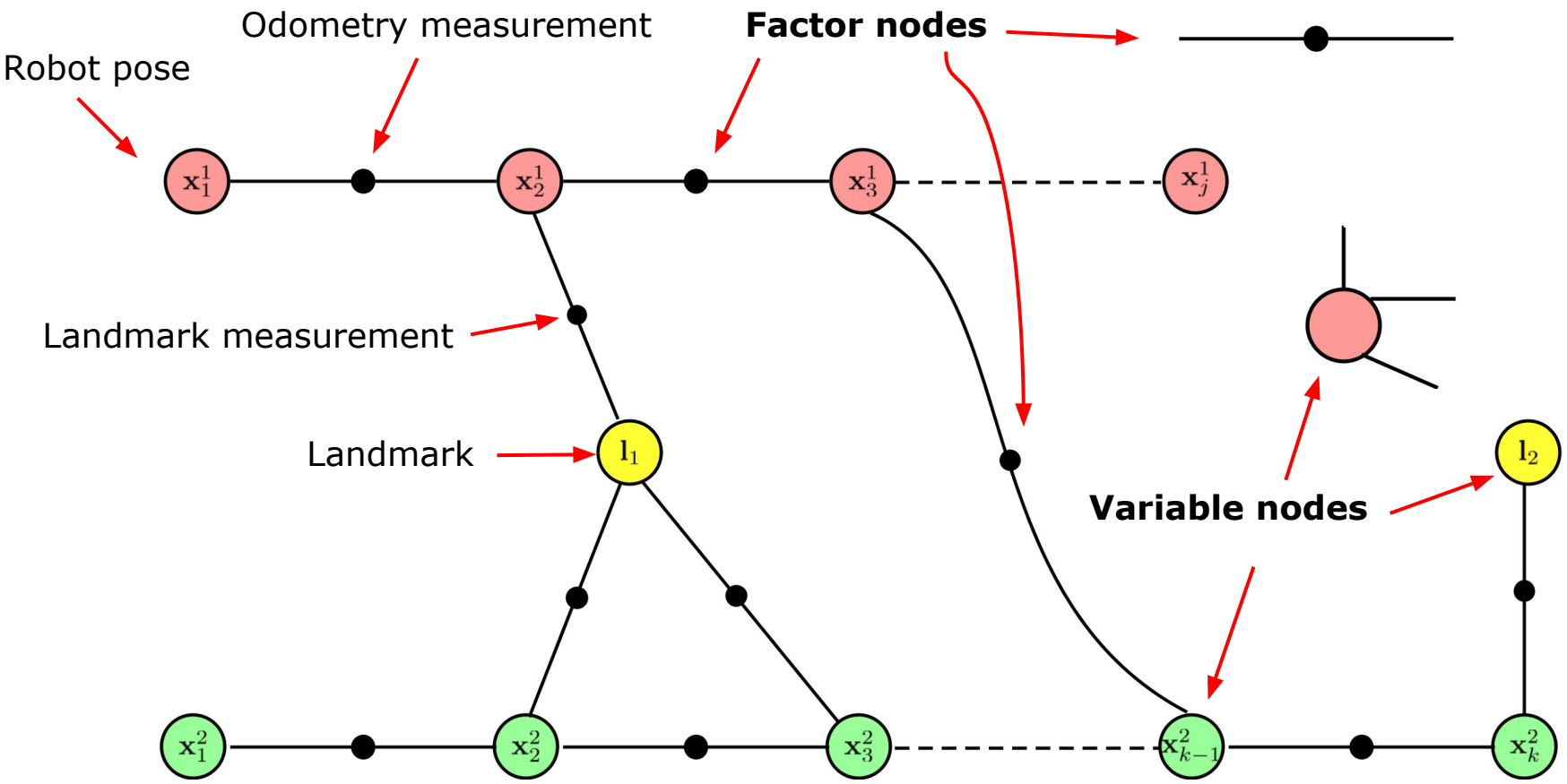

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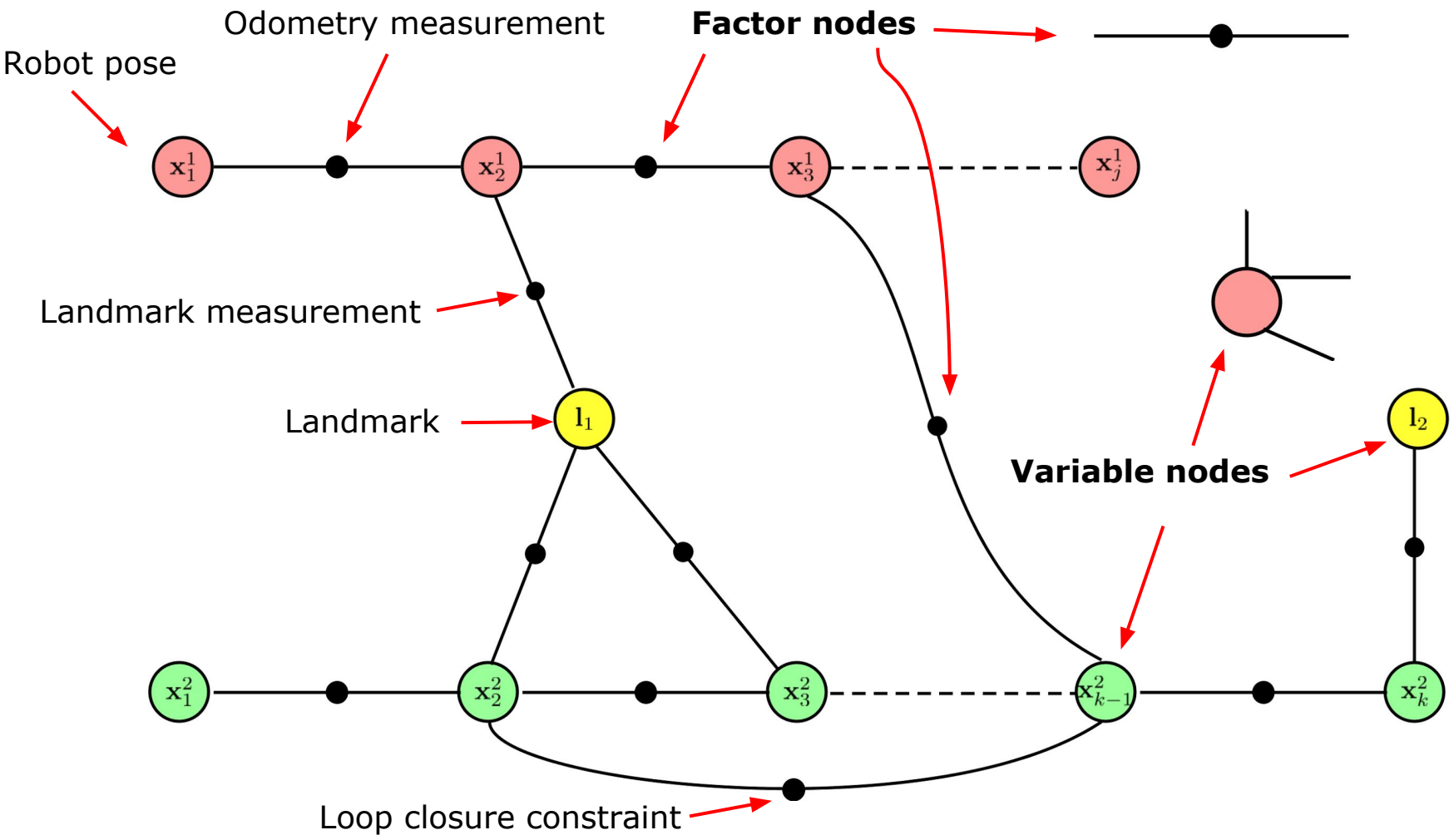
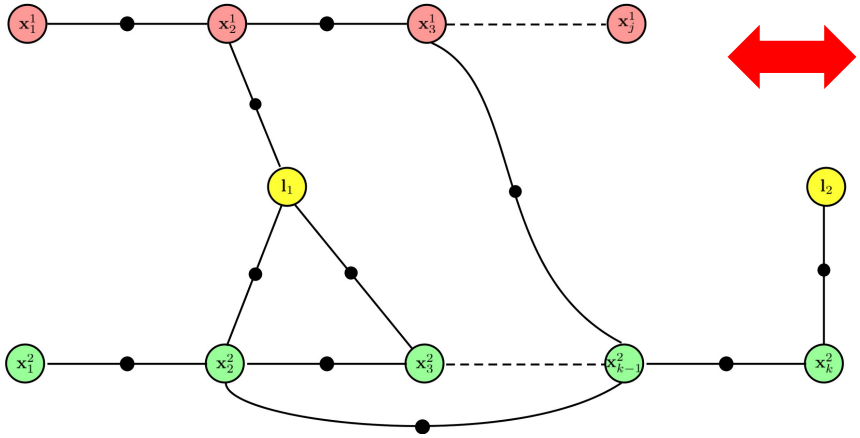


Nonlinear Least Squares



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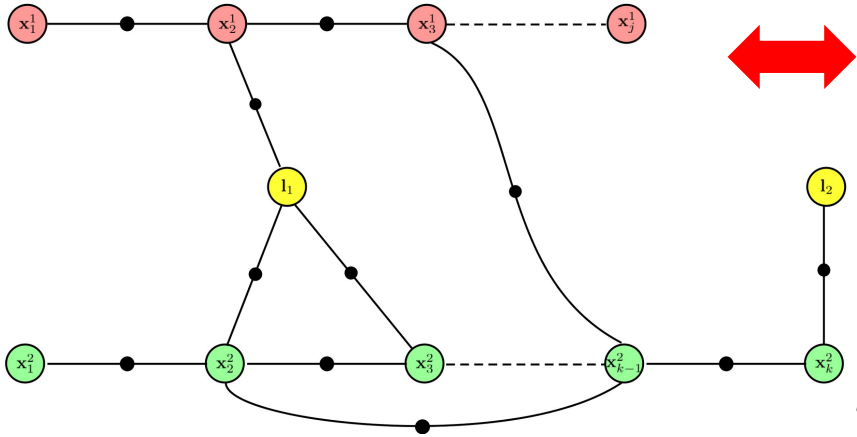

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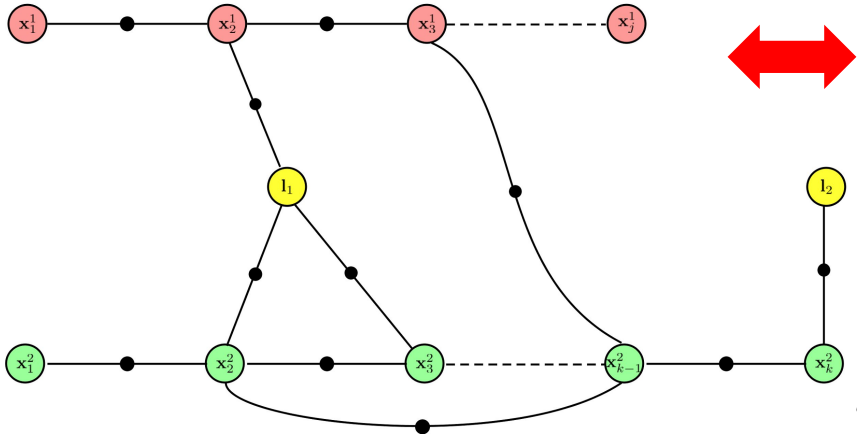

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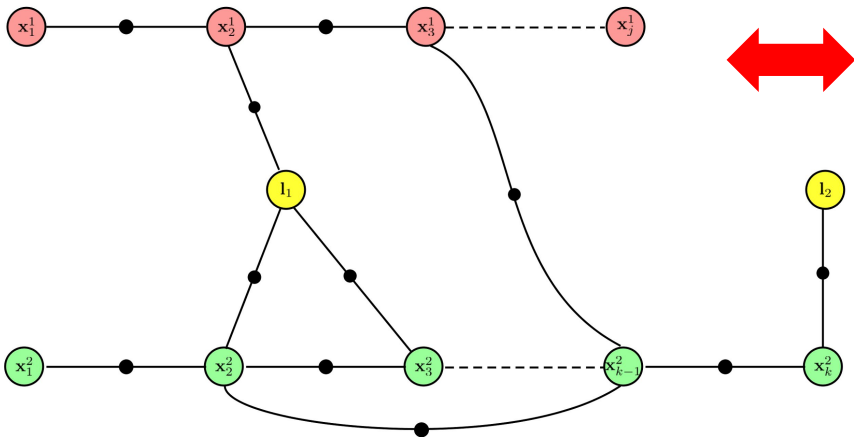

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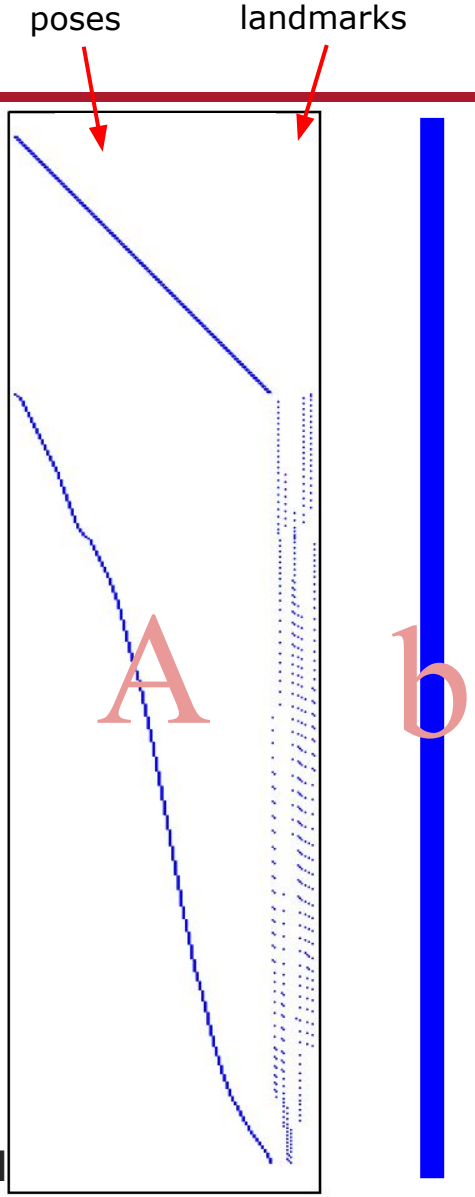

[Dellaert et al, 06]

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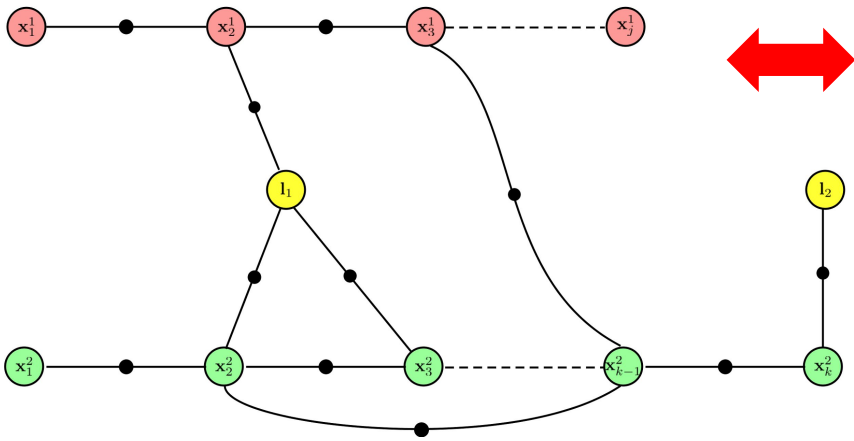

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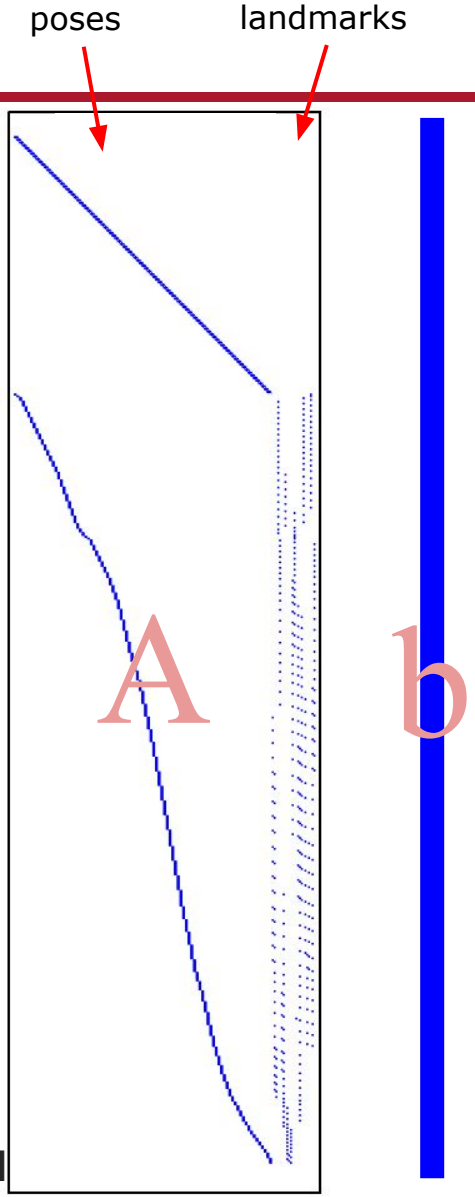

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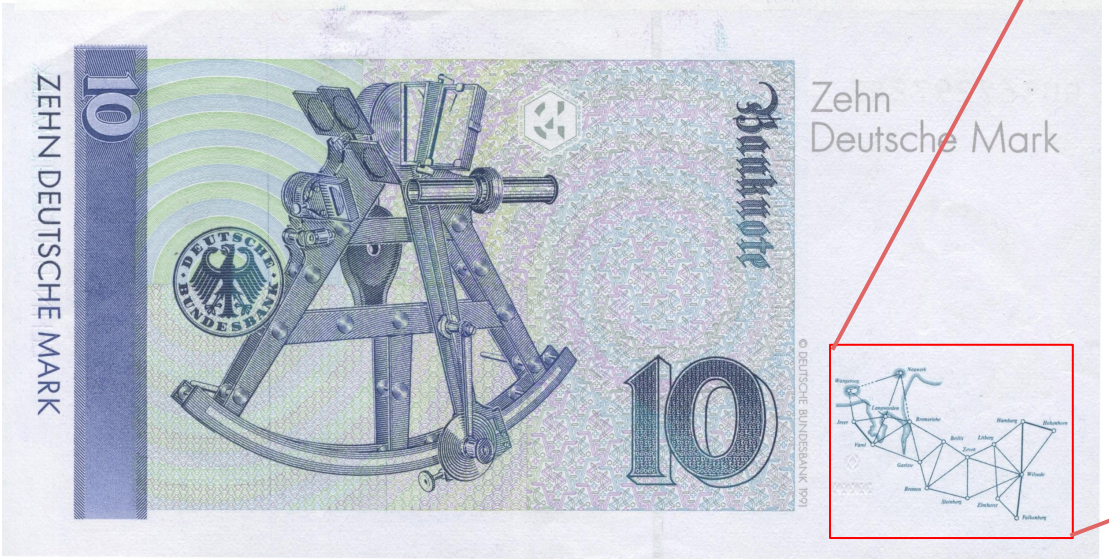
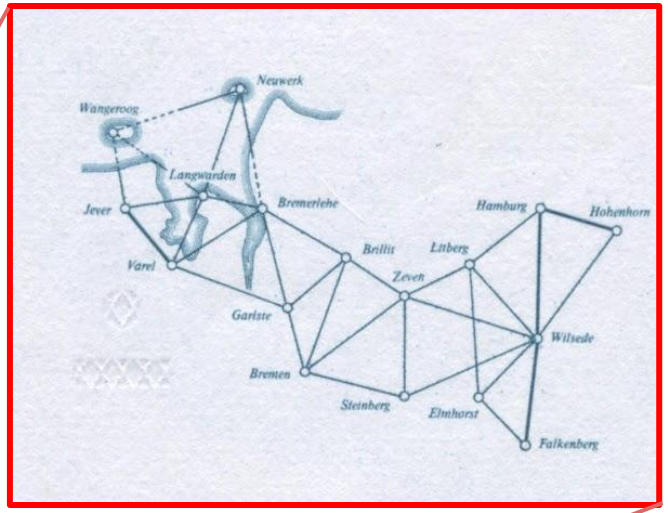


Carl Friedrich Gauss

1828: Triangulation of Hanover



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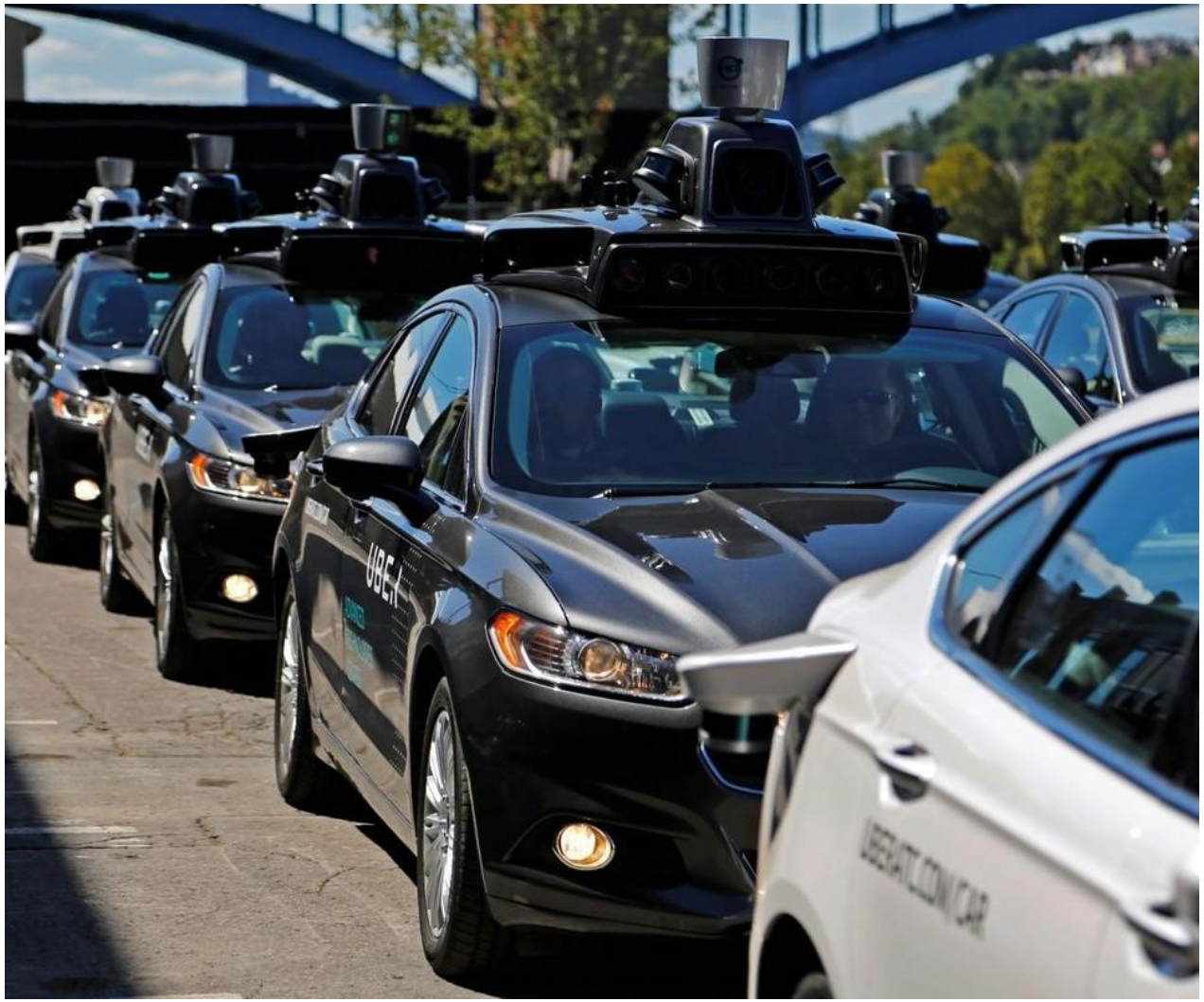

Efficient map sharing by UBER self-driving cars.

Congestion free traffic

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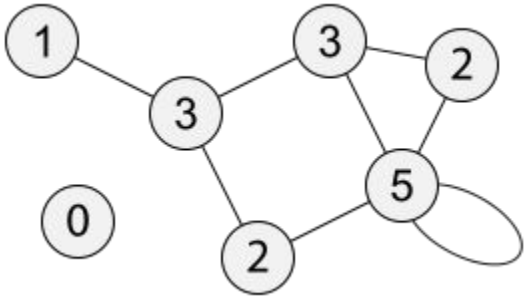
Graph Theory Glossary



Degree: Number of edges incident to a vertex.

[Image source: Wikipedia]

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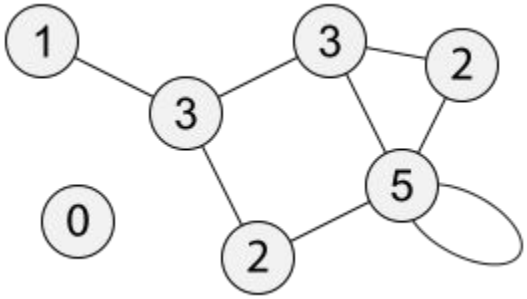

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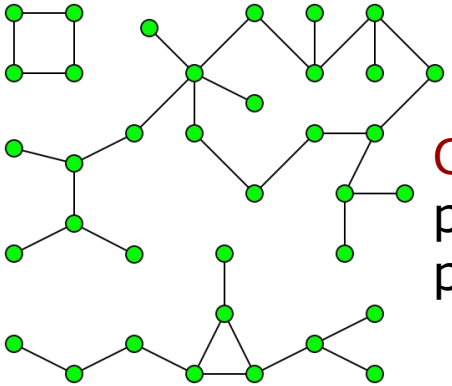

Connected component: A path exists between every pair of distinct vertices.

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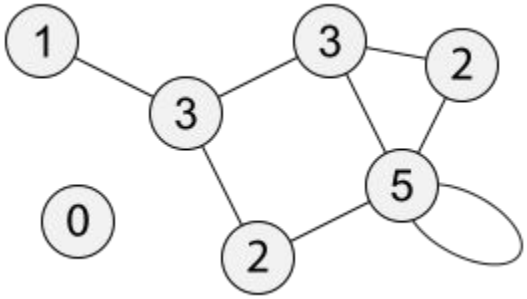

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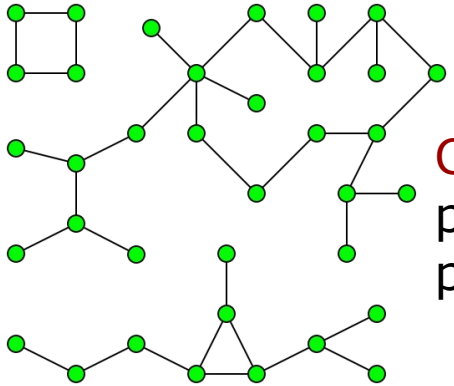

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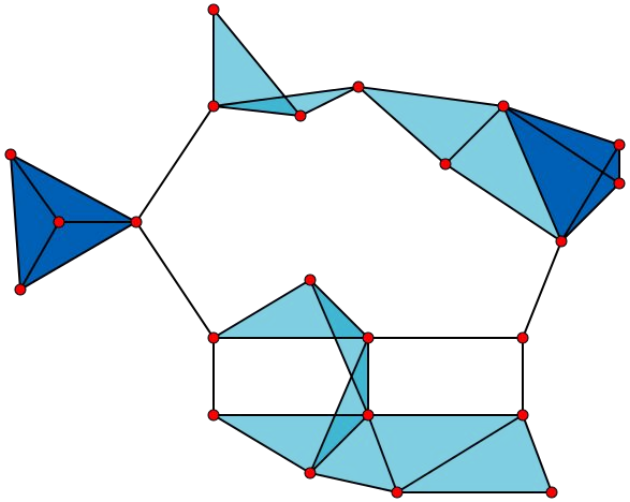

Clique: Fully connected (complete) subgraph.

[Image source: Wikipedia]

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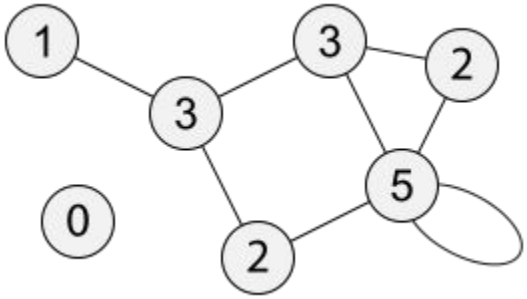

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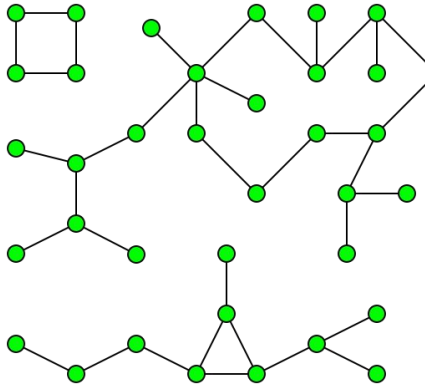

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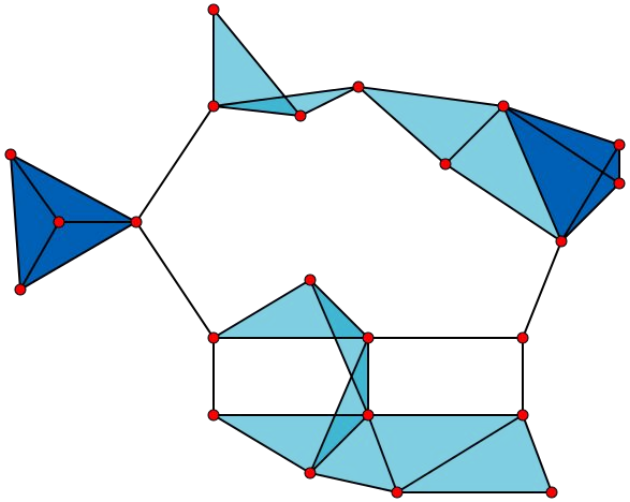

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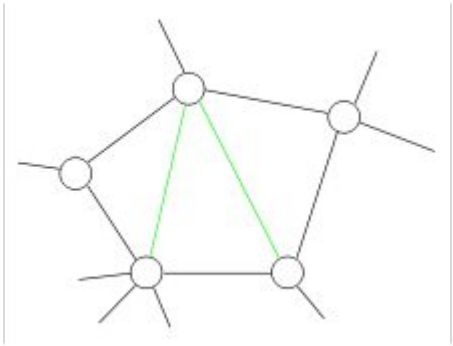

Chordal graph: Cycles of 4 or more vertices have a chord.

[Image source: Wikipedia]

Incremental Smoothing and Mapping [Kaess 08]

Repeatedly solve the linear system

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$$\left|\left|A_r\Theta_r - b_r\right|\right|^2 = \left|\left|Q_r^\top Q_r \begin{bmatrix} R_r \\ 0 \end{bmatrix} \Theta_r - Q_r^\top b_r\right|\right|^2$$

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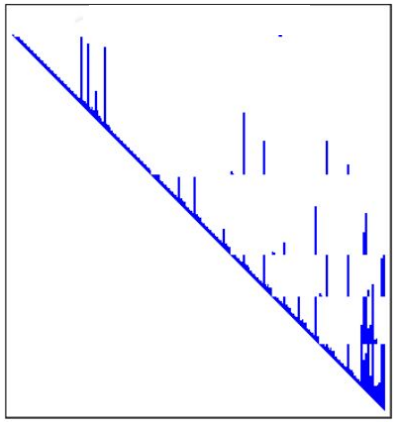

Square-root factor "R"

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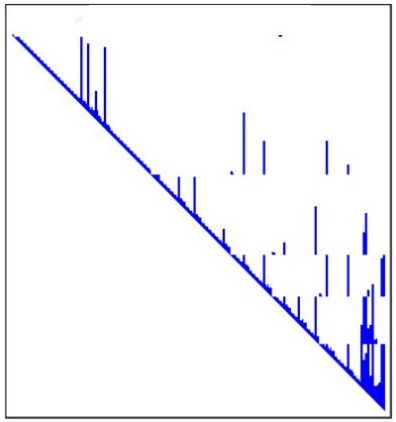

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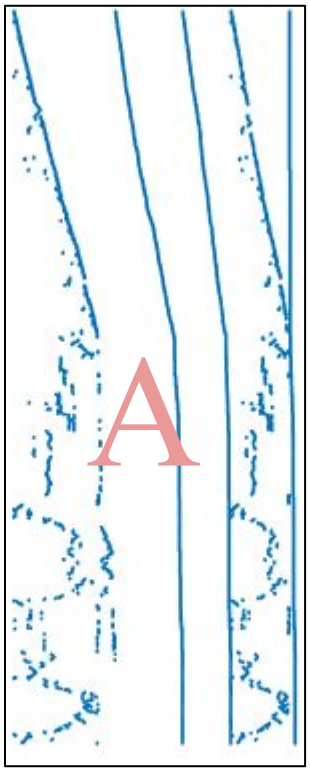
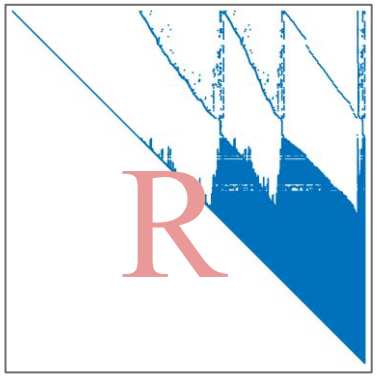
Why QR?

- Incremental update of R matrix using Givens rotation.
- Need not calculate the dense A^TA.
- Numerically stable.

Variable Ordering: Maintaining Sparsity

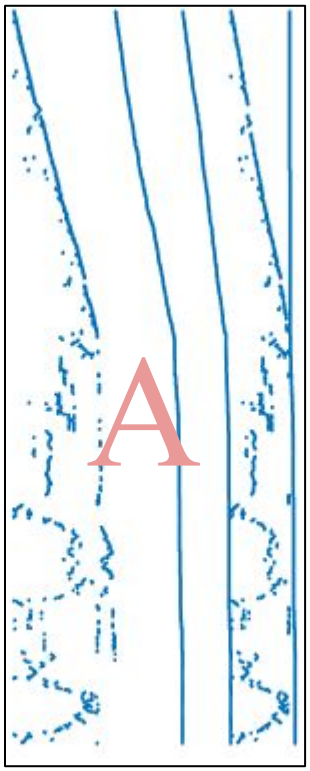


factorize



Default ordering as stored in memory (poses, landmarks)

Variable Ordering: Maintaining Sparsity



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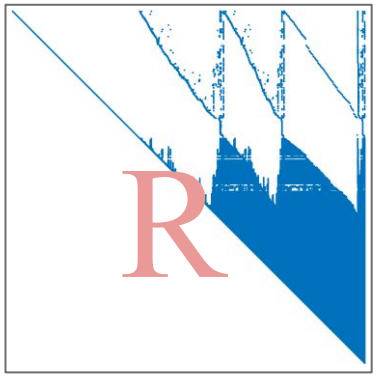

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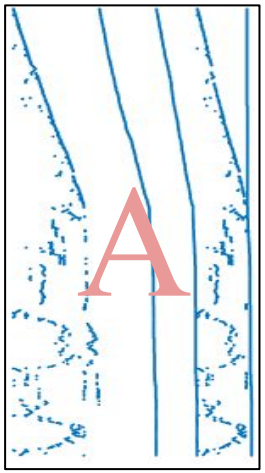
Variable ordering (permute)

factorize

COLAMD ordering
[Davis 04]

Optimal ordering is NP complete
[Arnborg et al, 87]

Matrix vs. Graph



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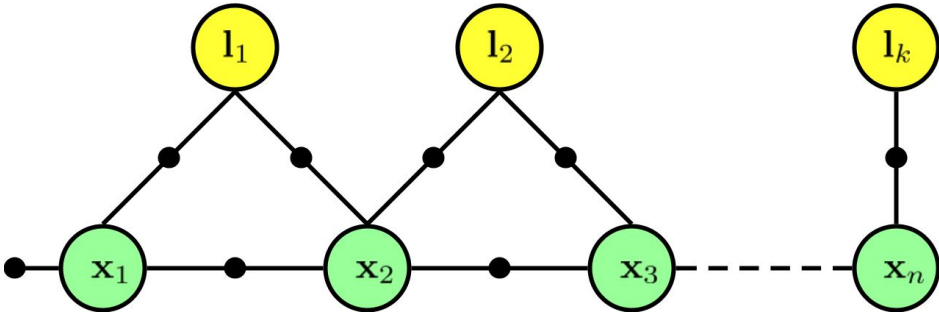

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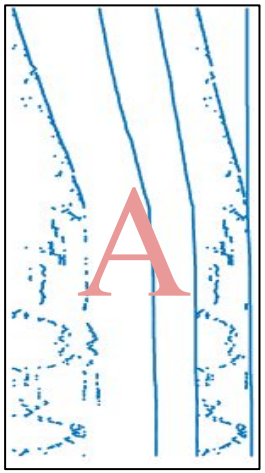

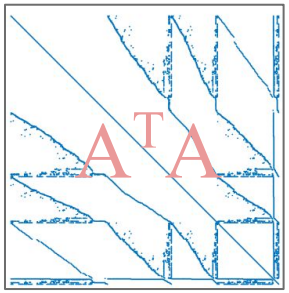

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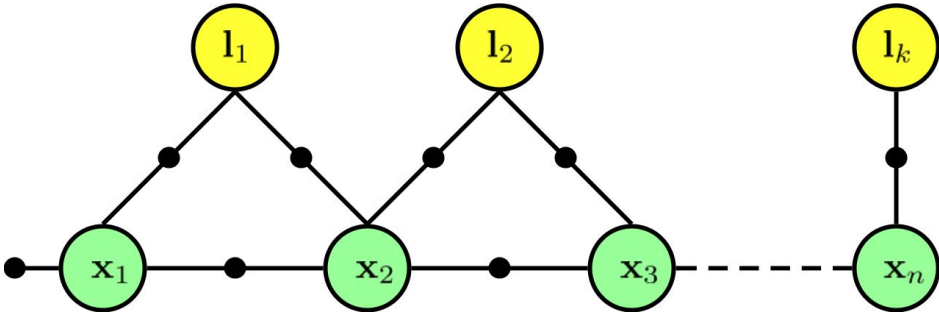
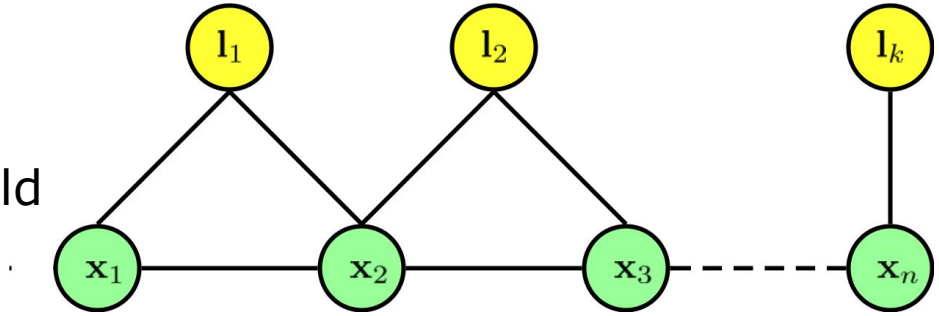
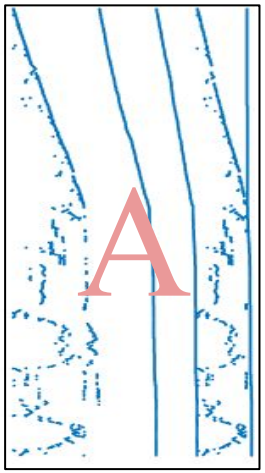


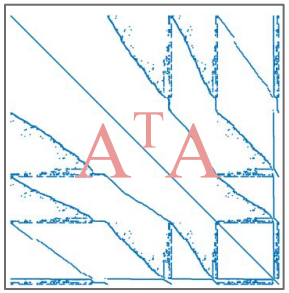
Markov Random Field



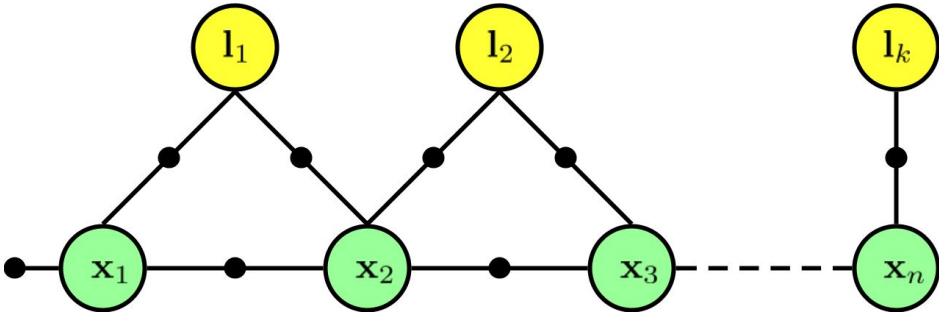
[Dellaert et al, 06]

Matrix vs. Graph

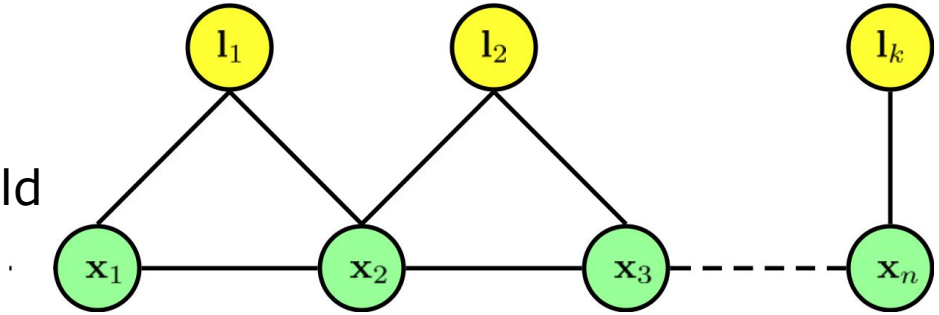




Factor Graph

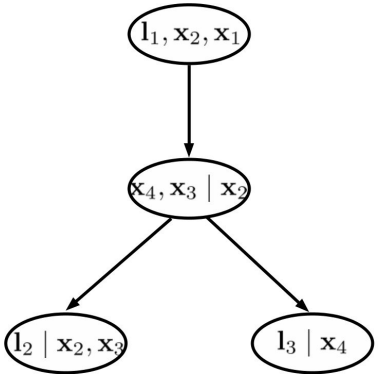


Markov Random Field



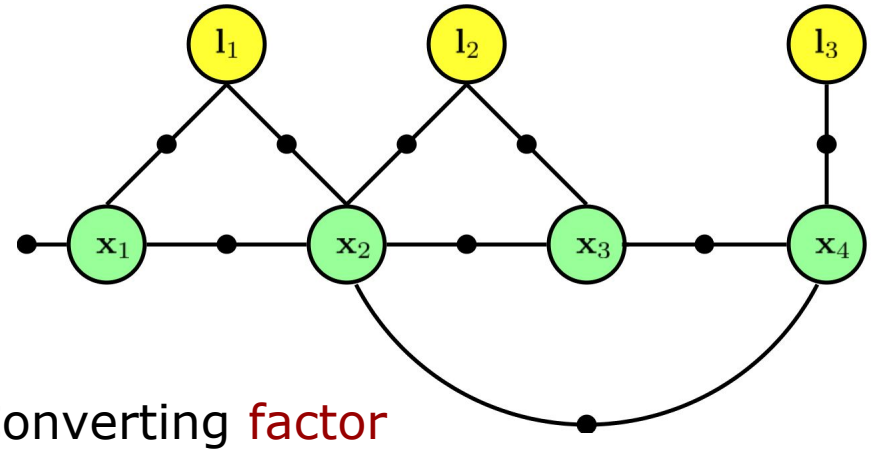
[Dellaert et al, 06]

Bayes Tree



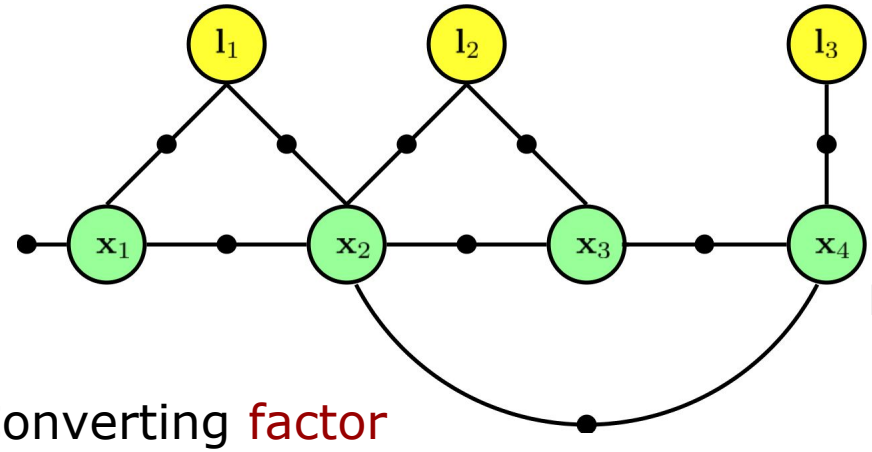
[Kaess et al, 10]

Factor Graph -> Bayes Network -> Bayes Tree



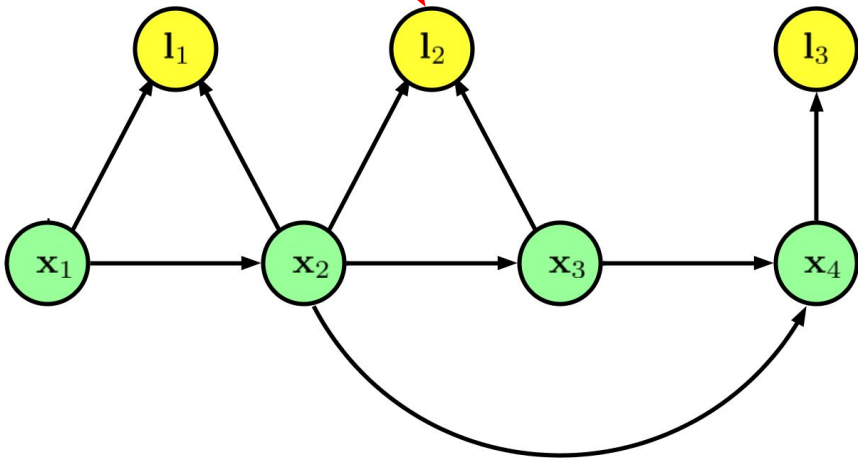
Converting factor graph to Bayes tree is same as QR factorizing A to get R

Factor Graph -> Bayes Network -> Bayes Tree



$$O = [l_3, l_2, l_1, x_4, x_3, x_2, x_1]$$

[Heggernes et al, 96]



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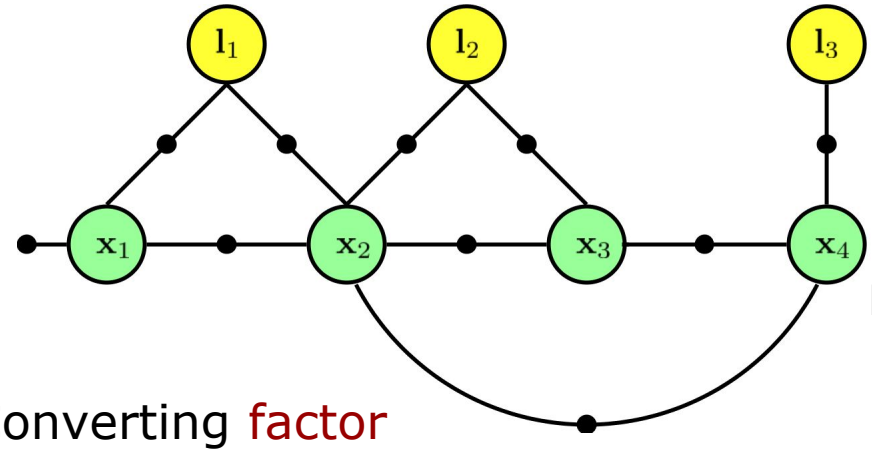

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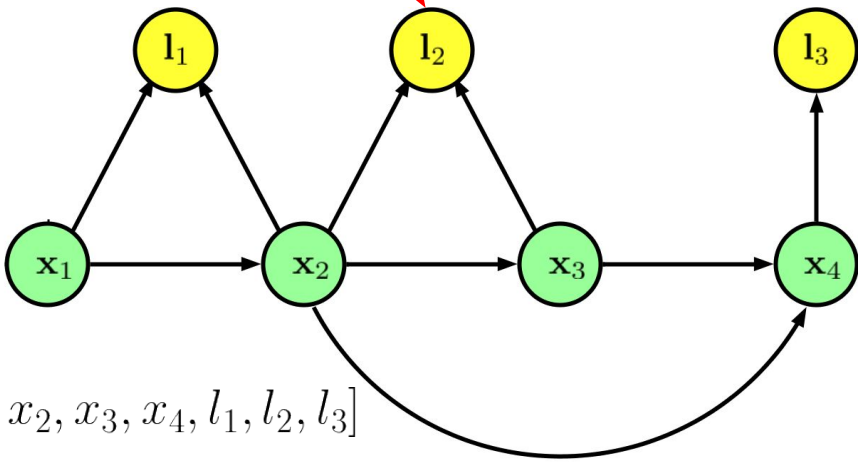

$$O^{reverse} = [x_1, x_2, x_3, x_4, l_1, l_2, l_3]$$

[Tarjan&Yannakakis, 84]
[Kaess, 10]

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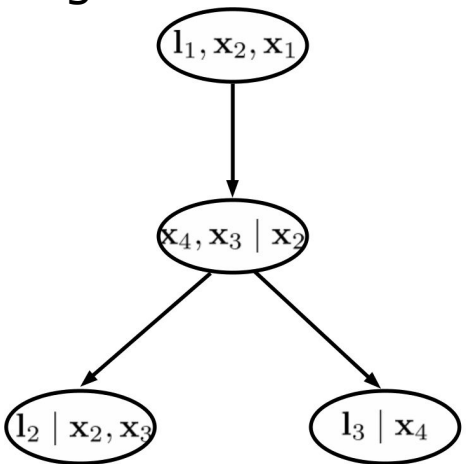
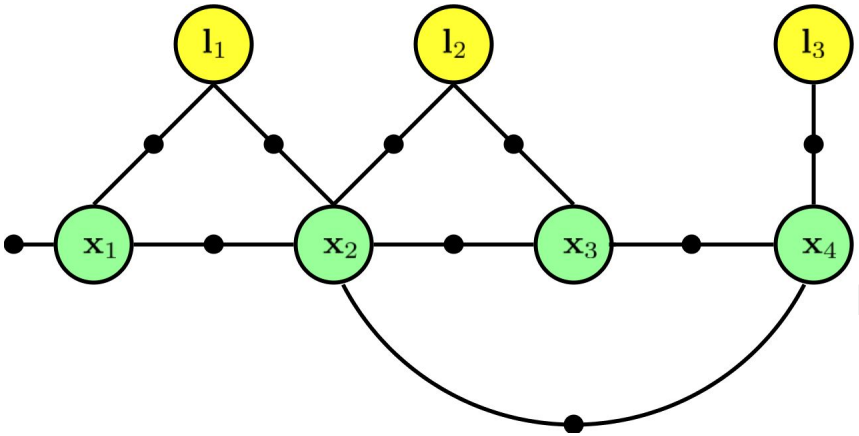


Bayes Tree: Incremental Variable Addition [Kaess 12]



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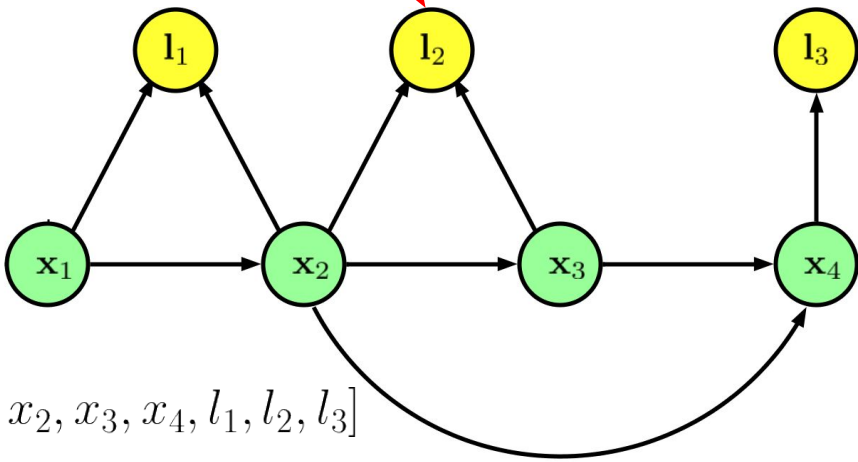
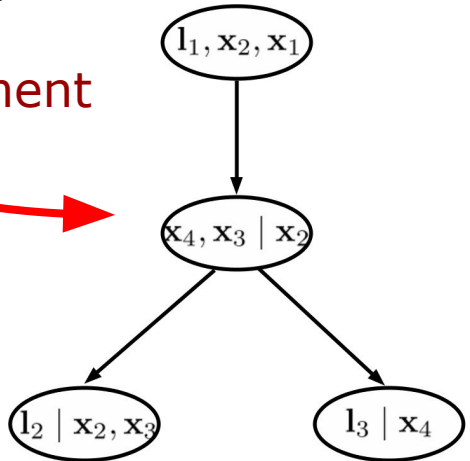
[Heggernes et al, 96]



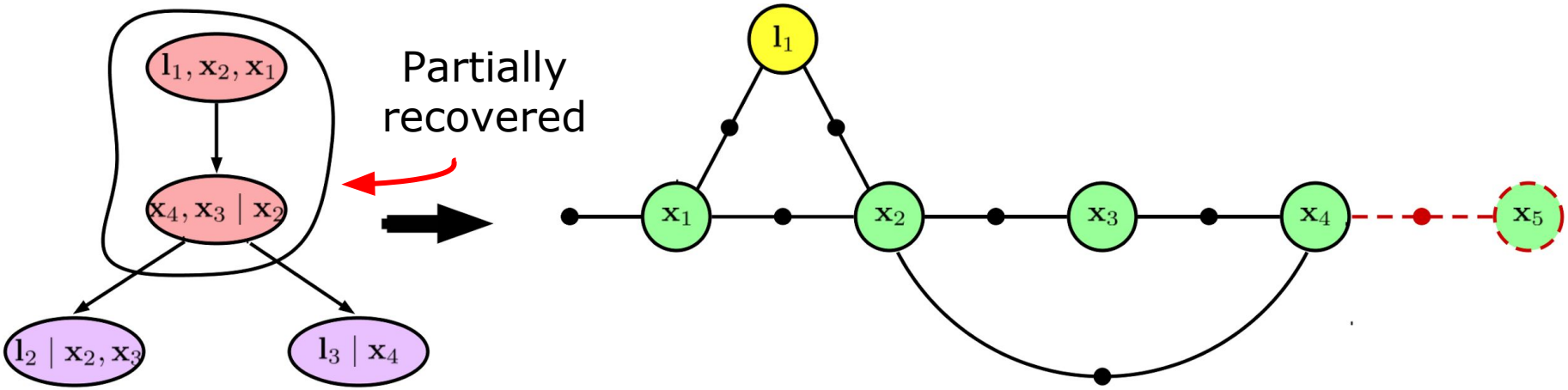
$$O^{reverse} = [x_1, x_2, x_3, x_4, l_1, l_2, l_3]$$

[Tarjan&Yannakakis, 84]
[Kaess, 10]

Add a new variable x_5 connecting the node x_4 by an odometry measurement



Bayes Tree: Incremental Update [Kaess 12]



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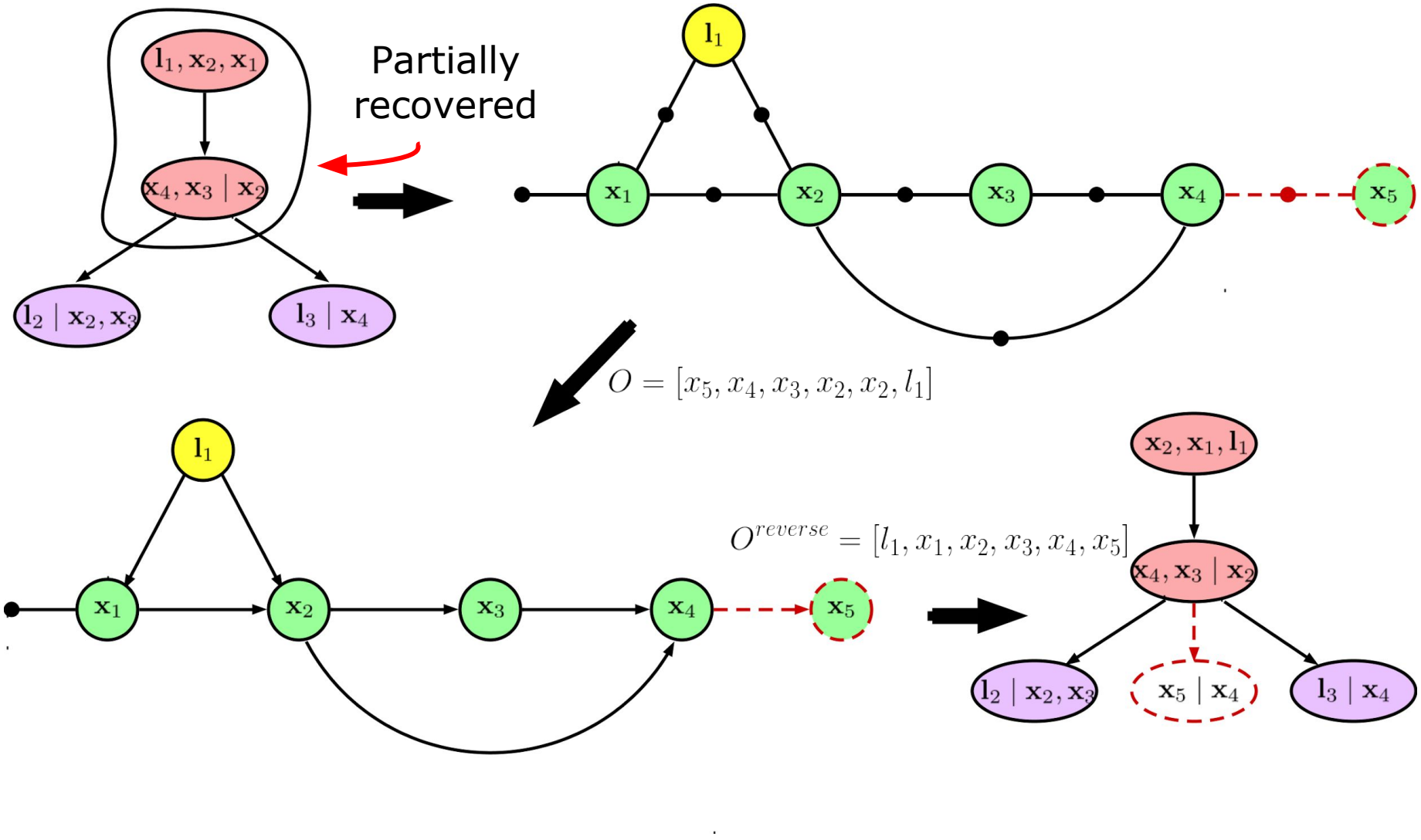

Outline

- Motivation
  - Factor Graphs
  - Mapping History
- Factor Graph SLAM
  - Variable Ordering
  - Bayes Trees
- Efficient Factor Graph Fusion
  - Fusion Ordering
  - Formal Verification
- Relative Pose Graphs
- Results

Related Work

Smoothing formulation
with Least-Squares
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Factor Graph Fusion

- Consider
  - Two bipartite graphs: $\mathcal{G}_i = (\mathcal{C}_i, \Theta_i, \mathcal{E}_i); \quad i = 1, 2$
    - $\mathcal{C}$ represents variable nodes
    - $\mathcal{E}$ represents the edges
    - Θ represents the factor nodes.

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  - Corresponding Bayes tree function: $\mathcal{B}_i(\mathcal{C}_i); \quad i = 1, 2$
    - Returns the set of ancestor variables in the tree for a given variable $\mathcal{C}_i$.

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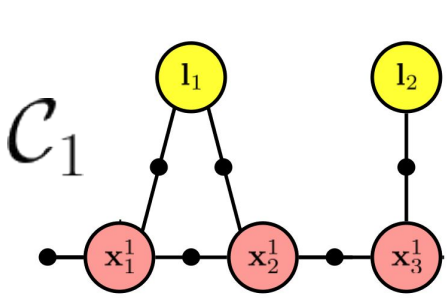
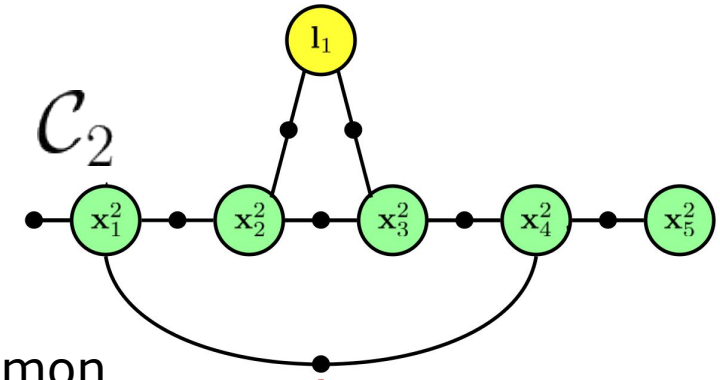
    - Returns the set of ancestor variables in the tree for a given variable $\mathcal{C}_i$.
- Graph merge condition: $\mathcal{C}_1 \cap \mathcal{C}_2 \neq \emptyset$

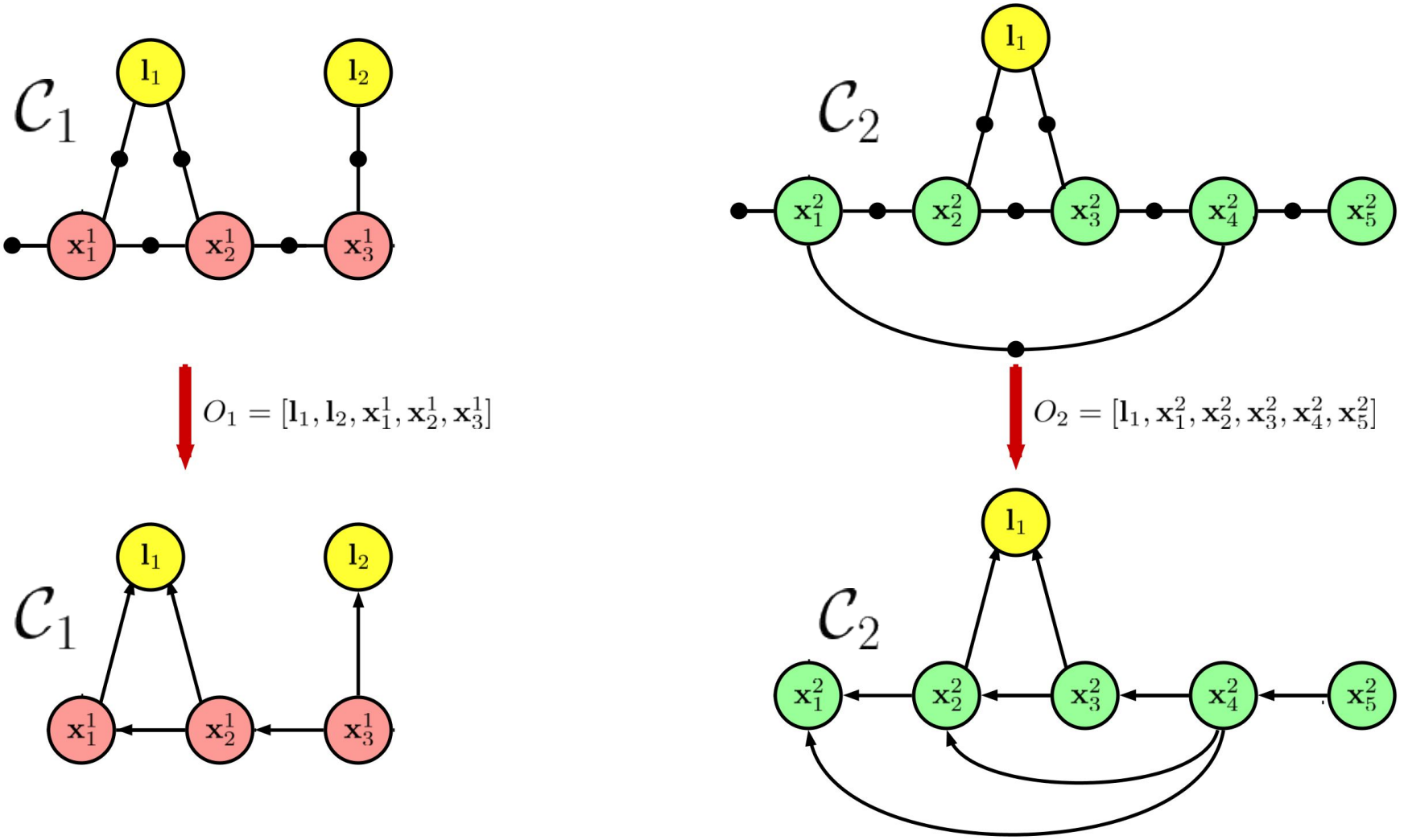
Example:





l_1 is the common variable

Factor Graph Fusion Example



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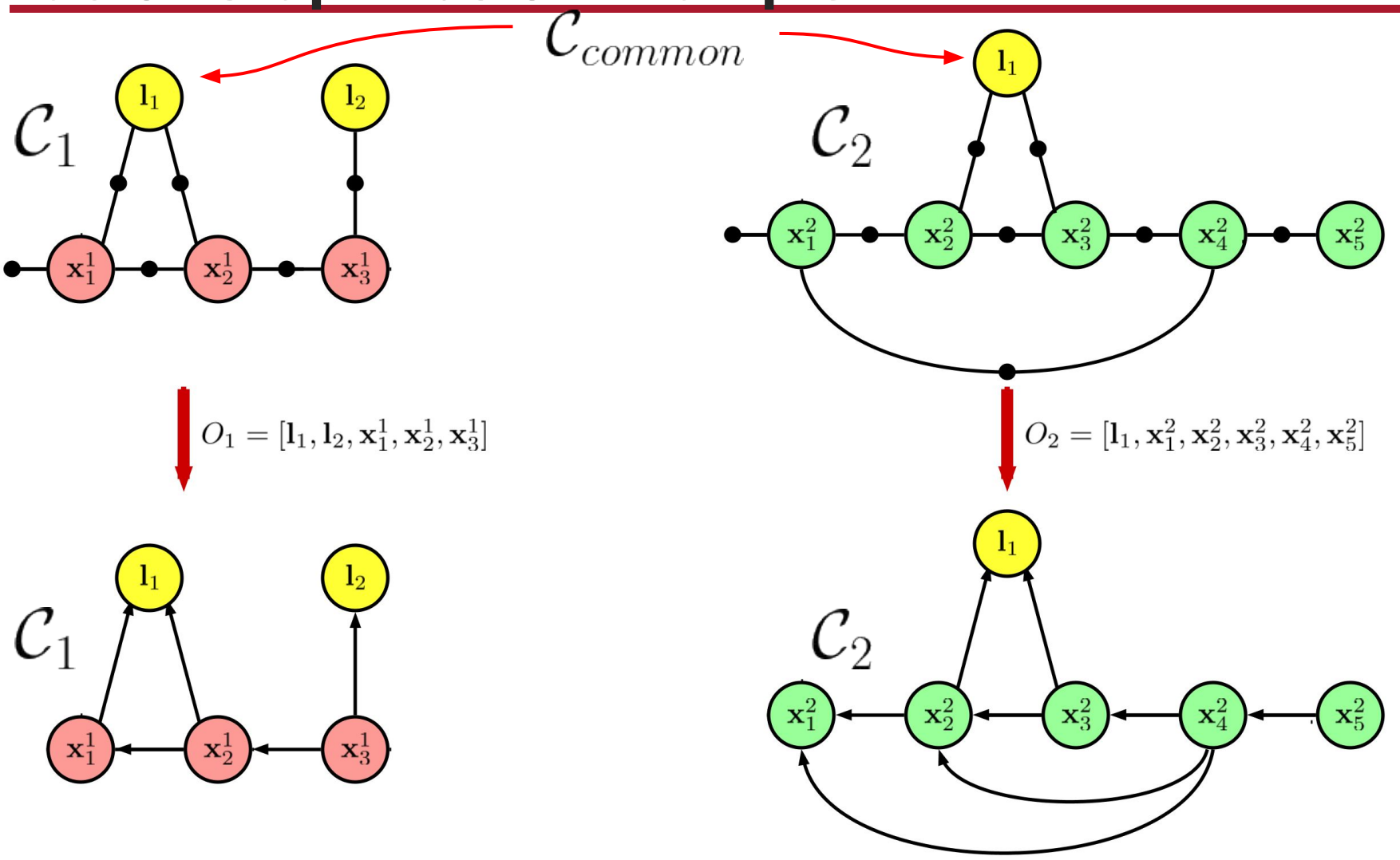

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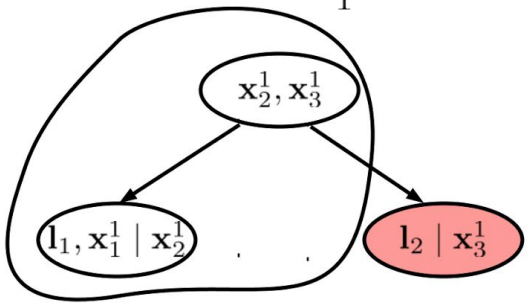
$$\mathcal{C}_1^{affected} = \mathcal{B}_1(\mathcal{C}_{common})$$



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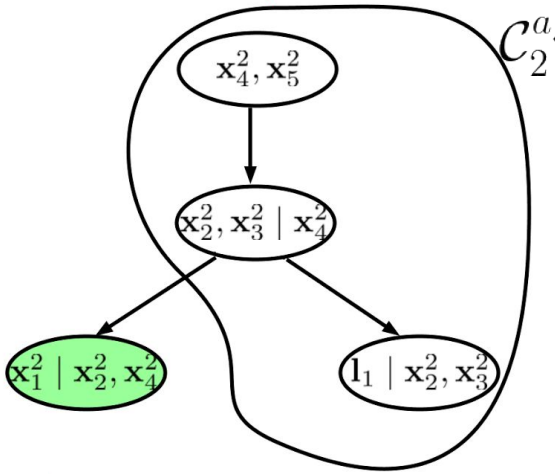

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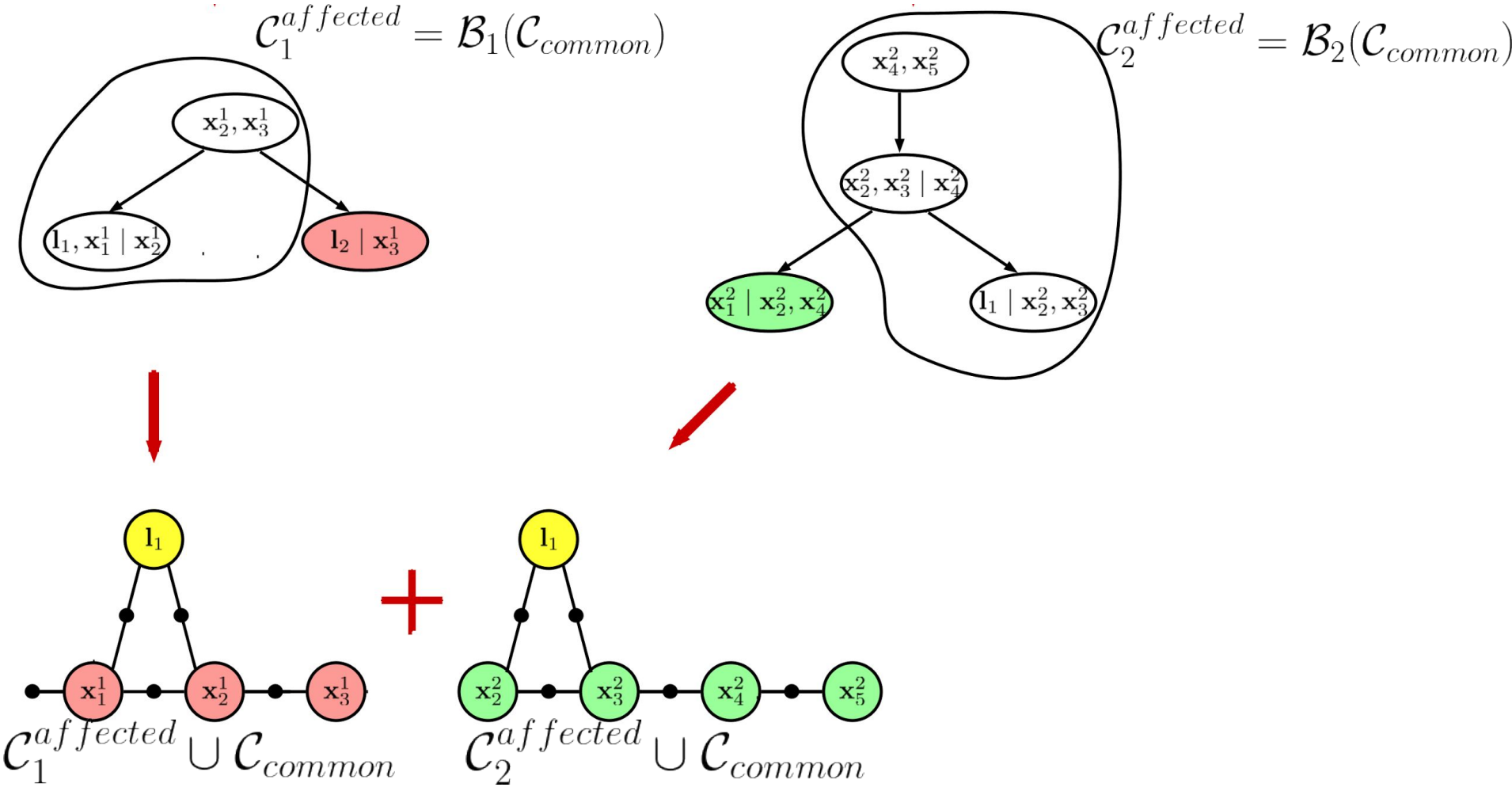

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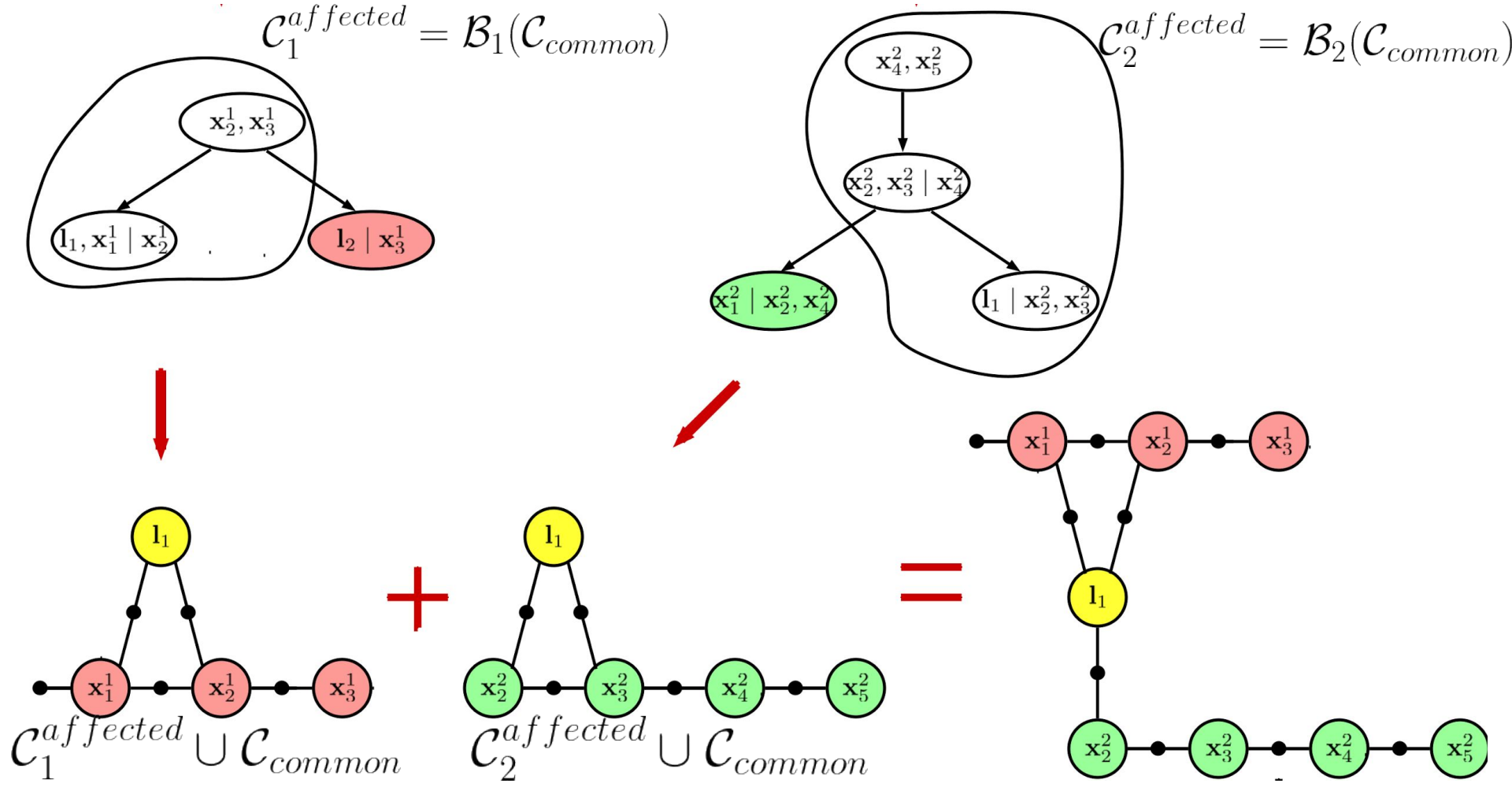

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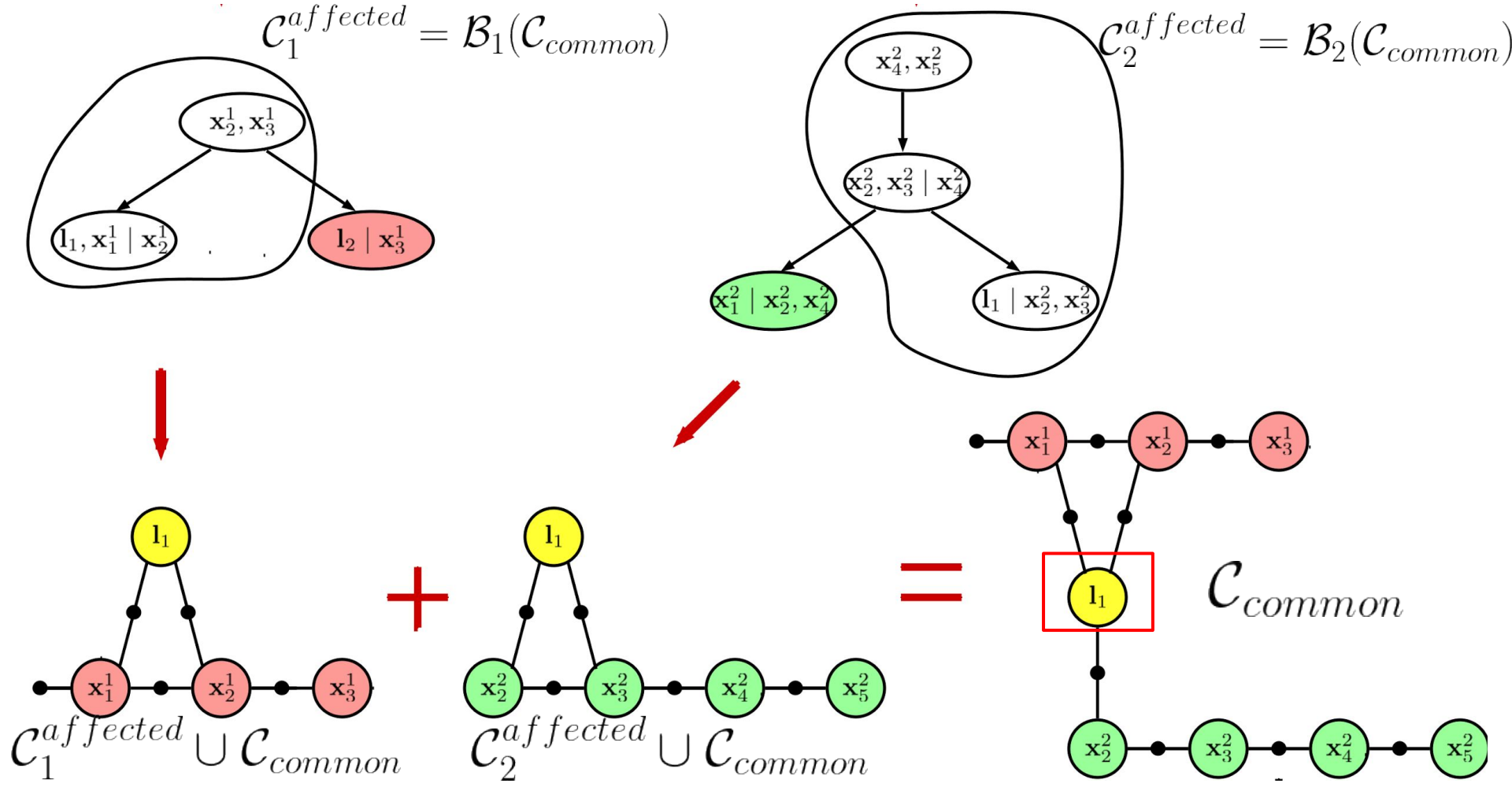

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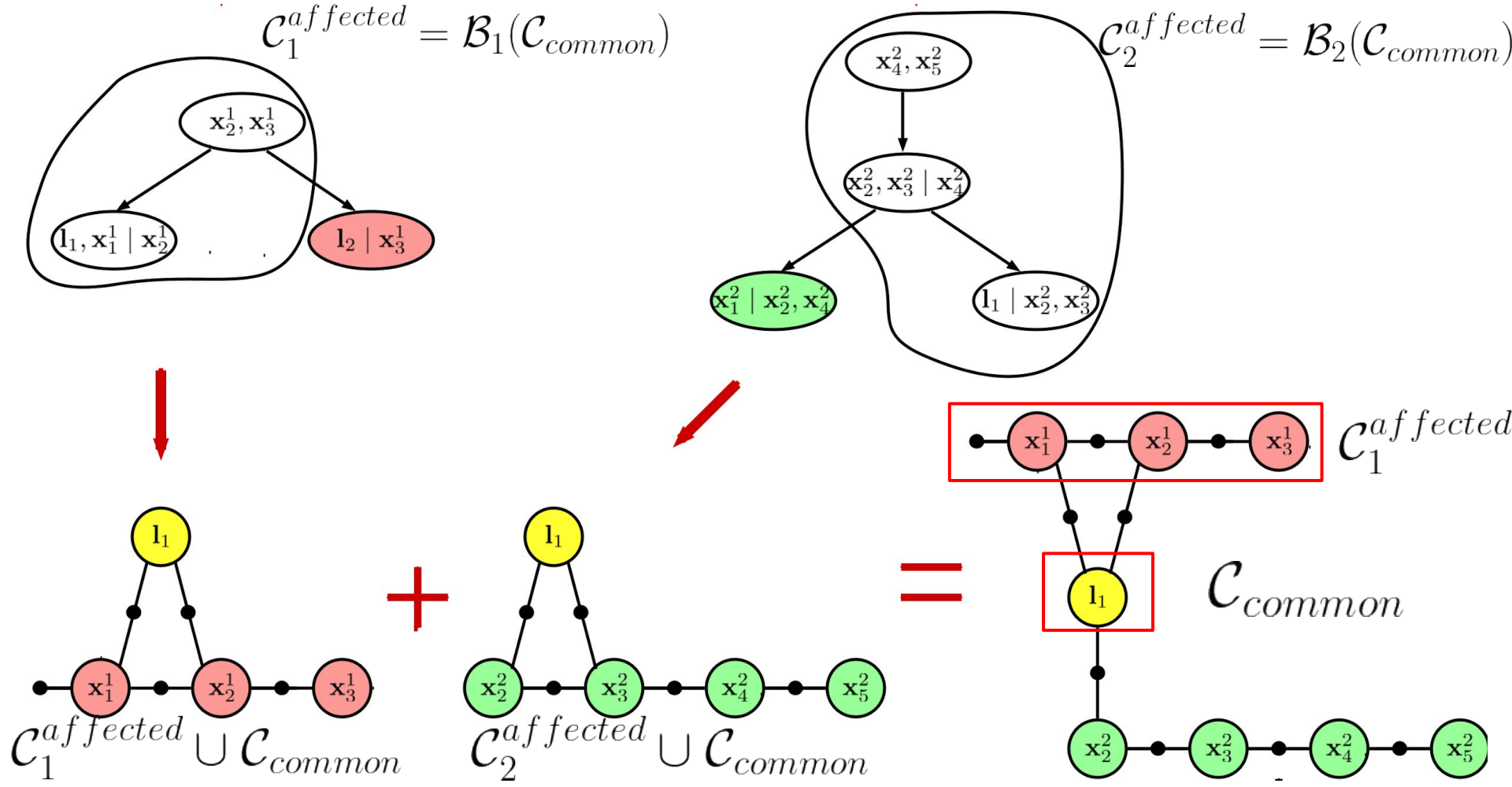

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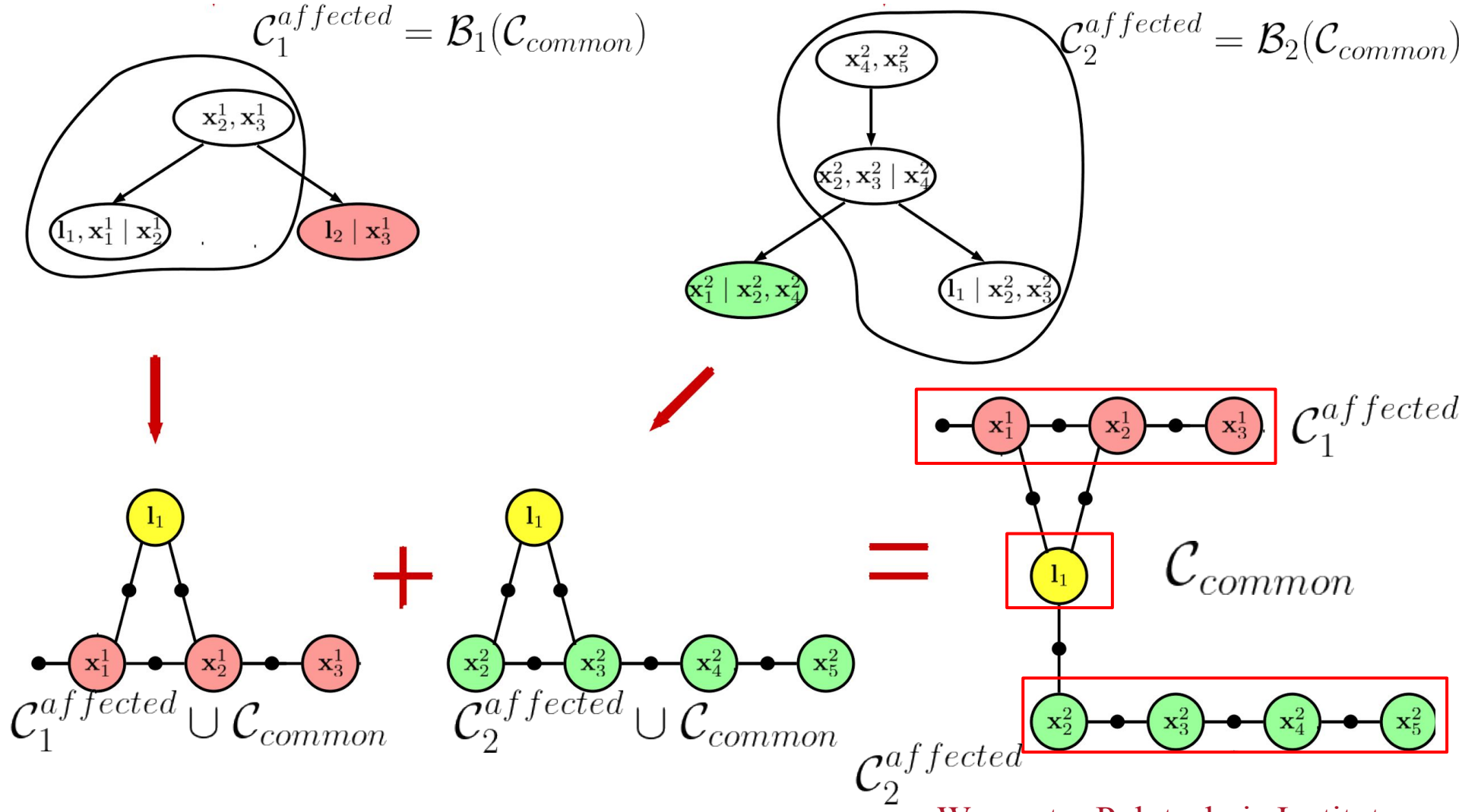

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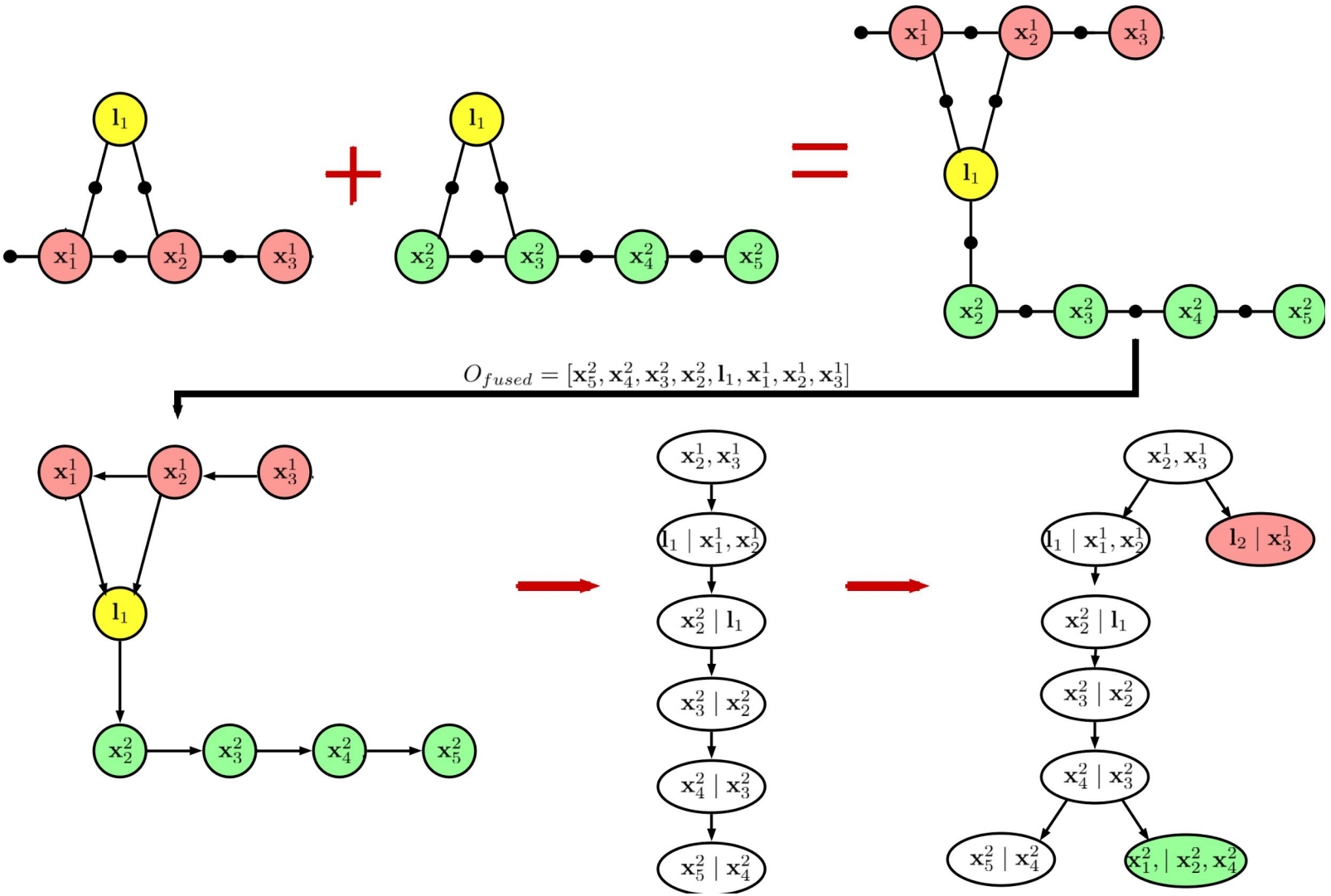

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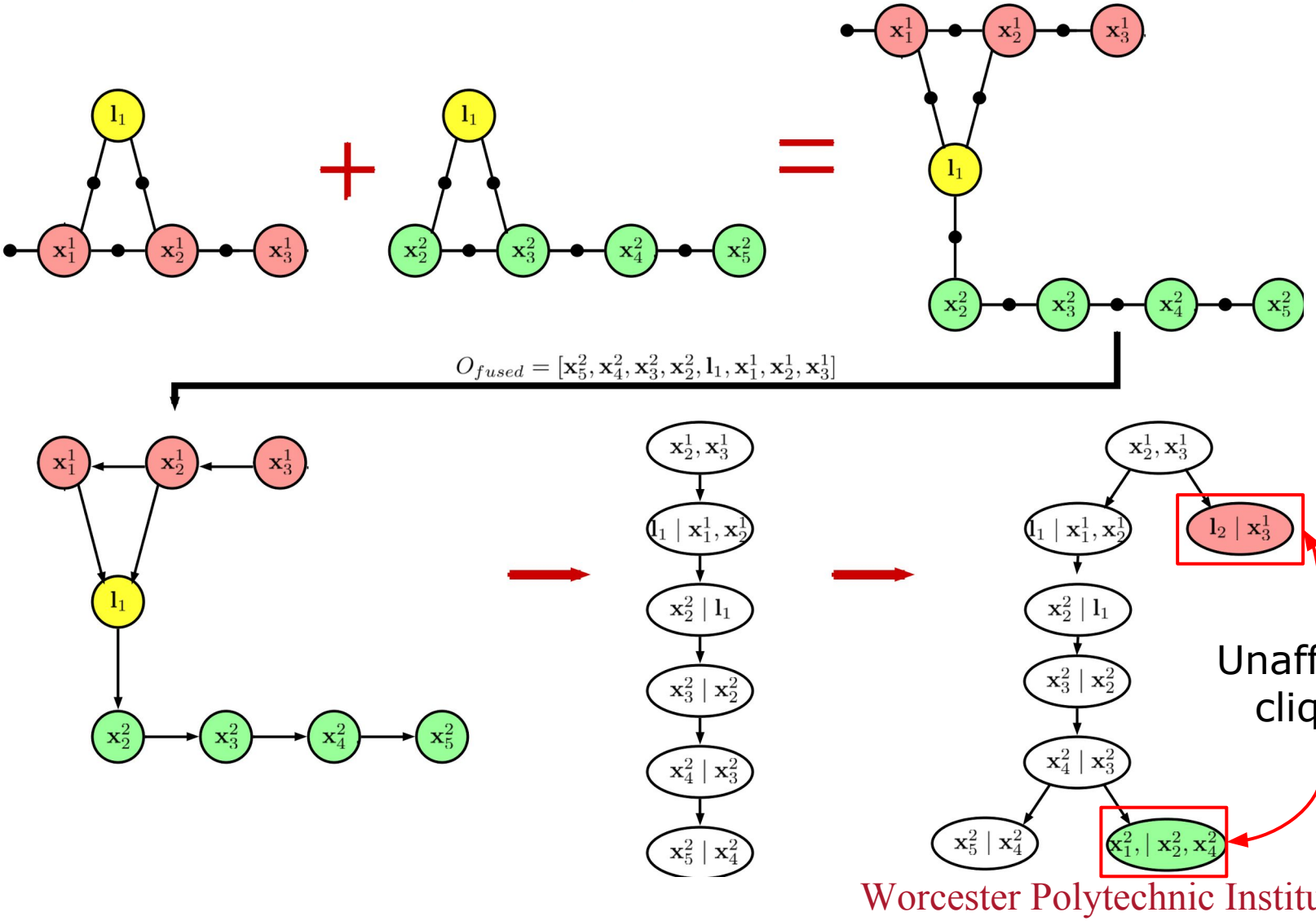

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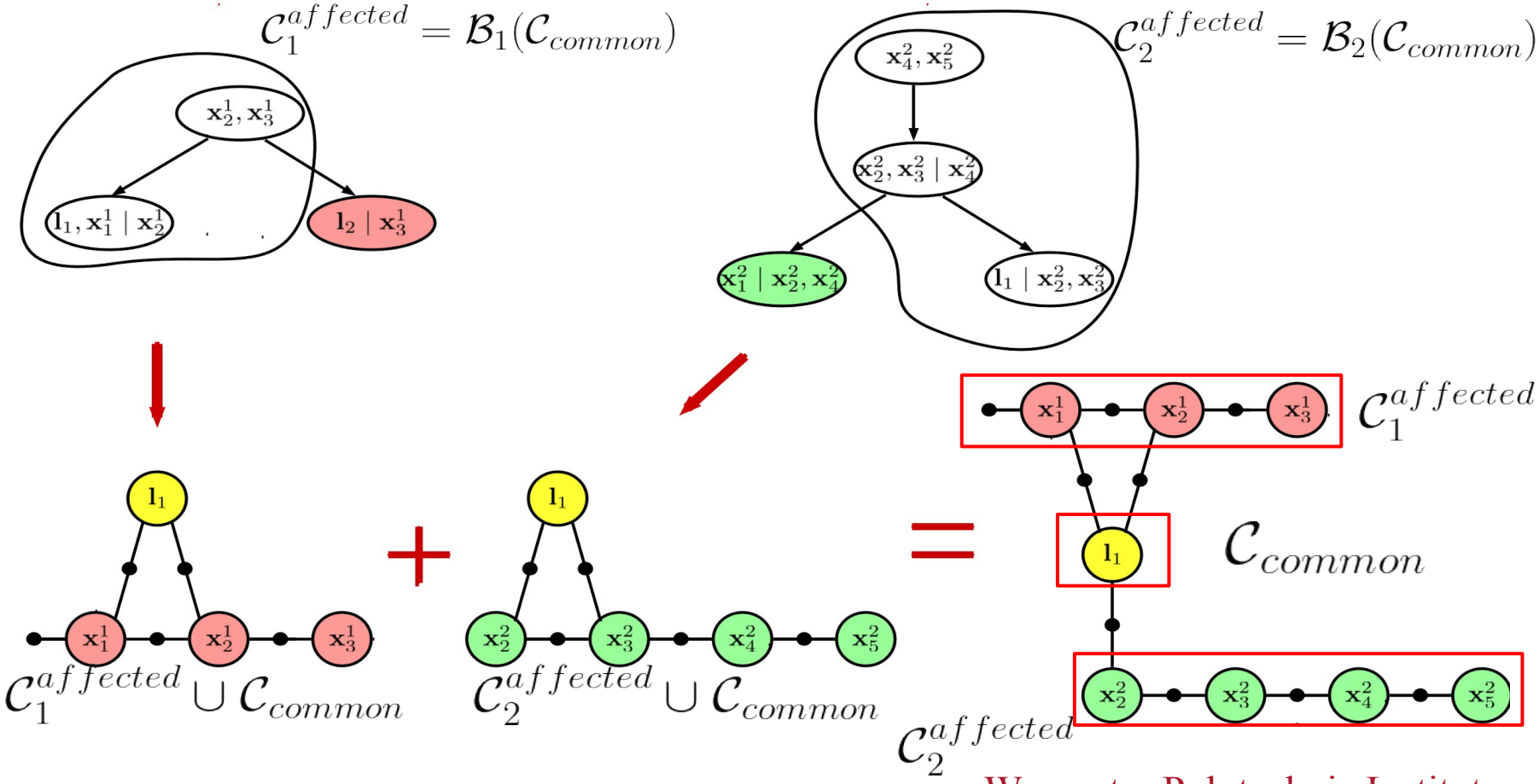

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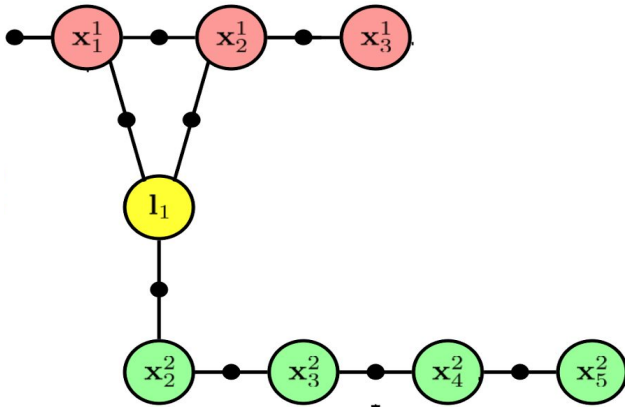

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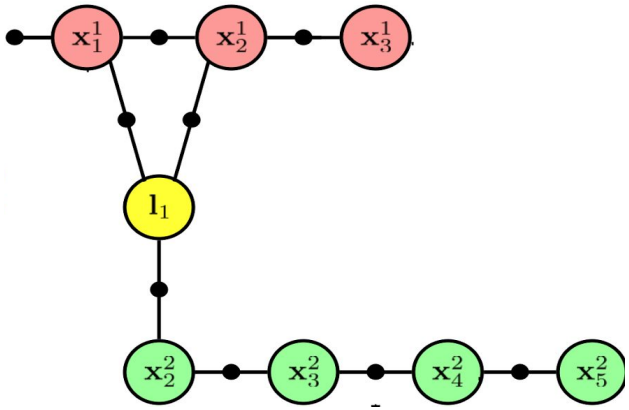

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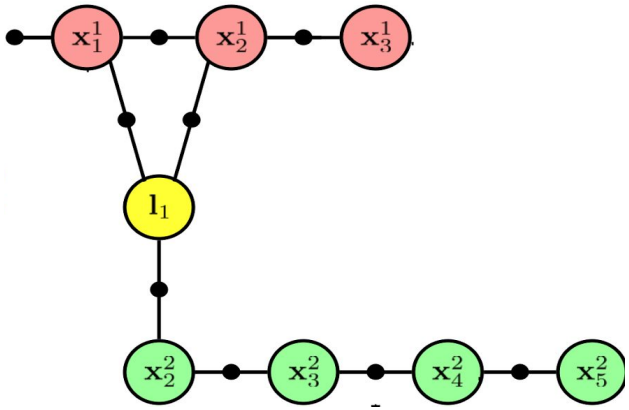
There is always a factor of form p(b|a)

b - last but one variable
a - last variable



Fusion Ordering

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$$O_i = [o_i^{c_i^1}, o_i^{c_i^2}, \ldots, o_i^{c_i^{n_i}}]; \quad i = 1, 2$$

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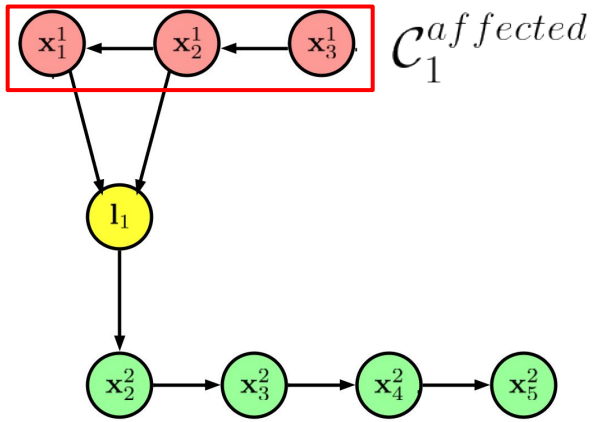

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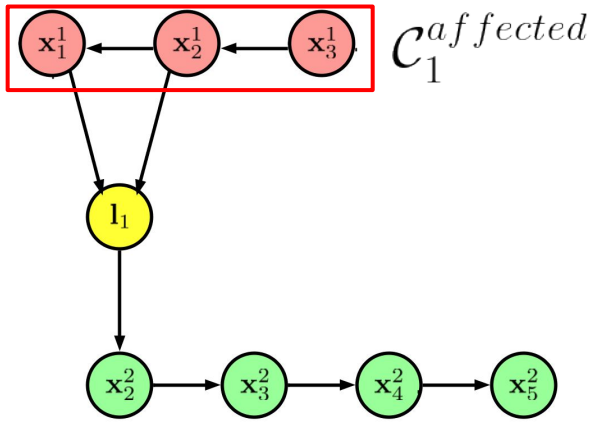

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Eg: [41,23,12,8,22] →[4,2,5,2,1]

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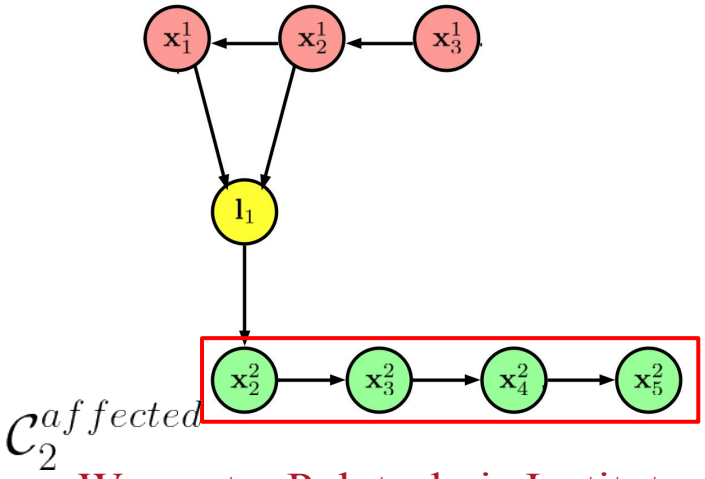

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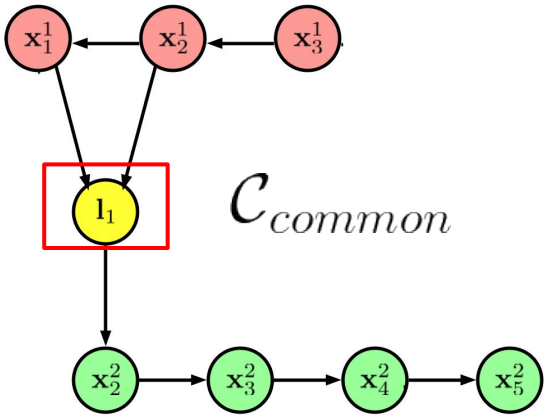

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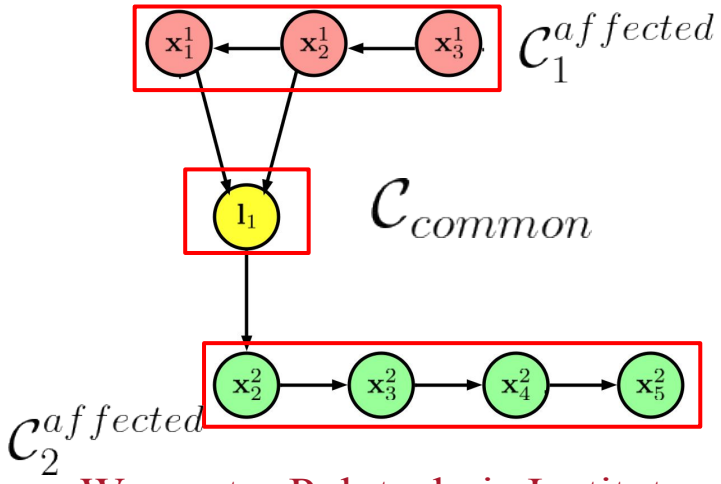

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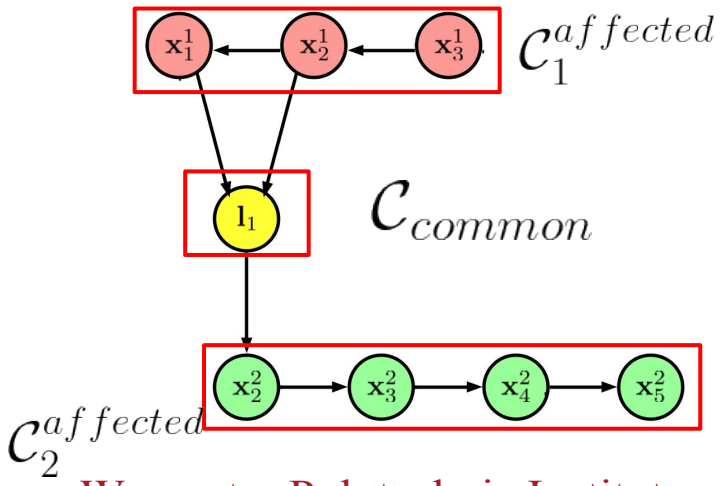
$$P^1_{j,k} := \begin{cases} 1 & \text{if } j = O_k^{fused} \\ 0 & \text{otherwise} \end{cases}$$

where $N = |O^{fused}|$



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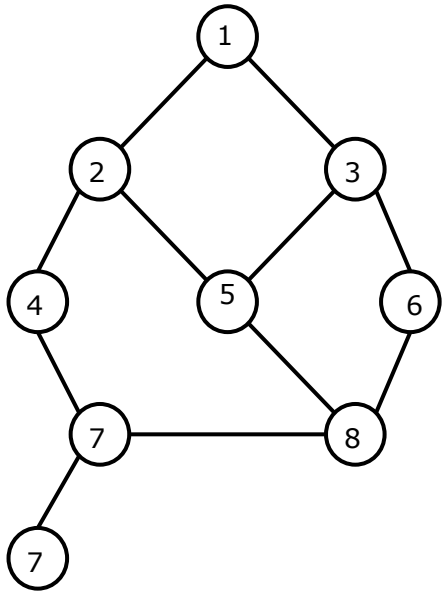

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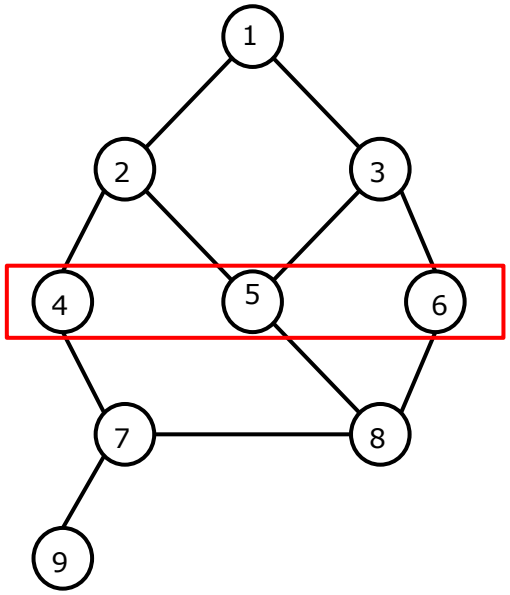

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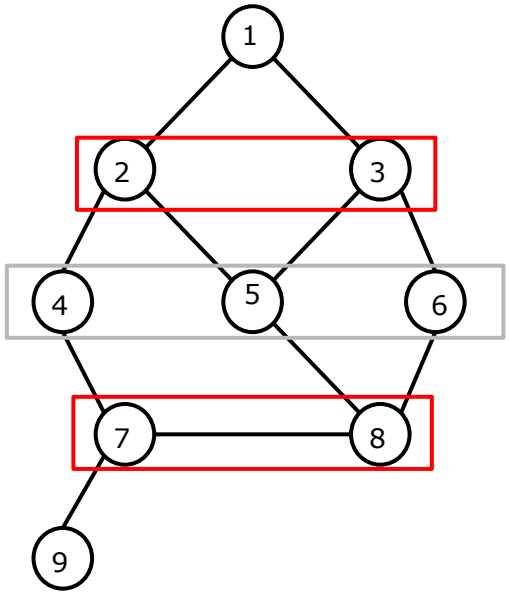

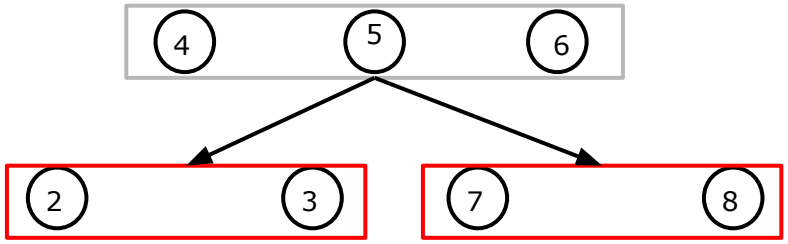

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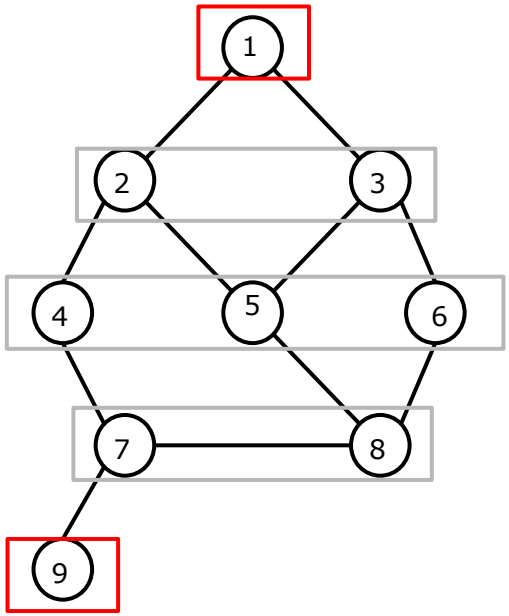

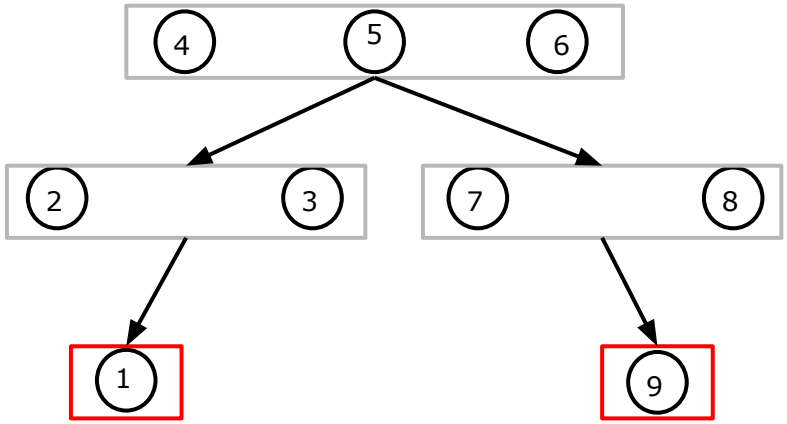

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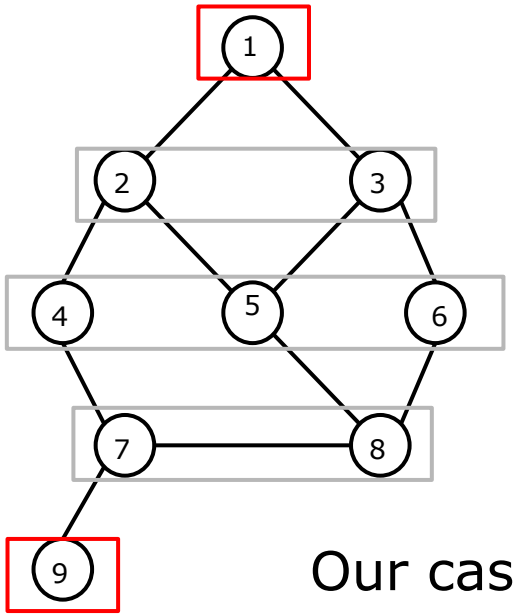

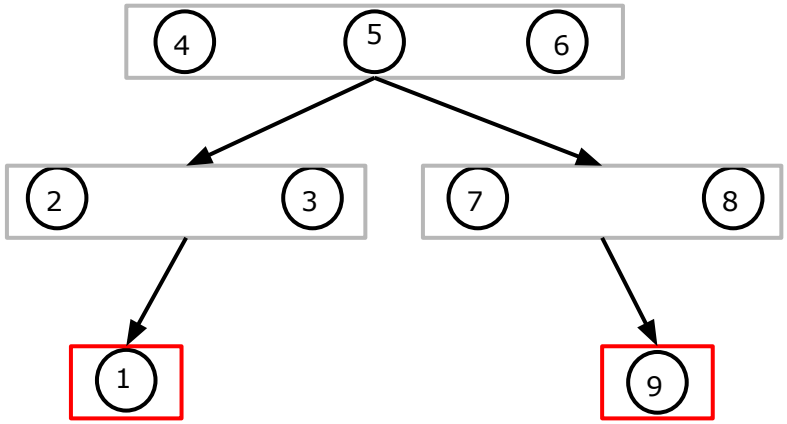

Our case is a depth 1 nested dissection.

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Pivoting: Common technique to add numerical stability.

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Finding an element to null the remaining elements in its row or column

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x	y	z	rhs
*3	2	-4	3
2	3	3	15
5	-3	1	14

$3R_2-2R_1 \rightarrow R_2$ & $3R_3-5R_1 \rightarrow R_3$

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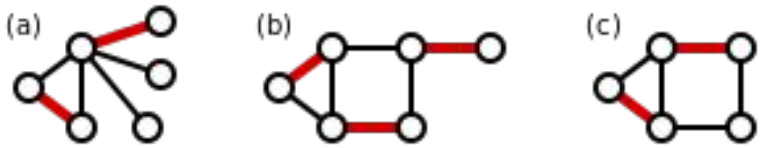
Matching problem: Finding pairwise non-adjacent edges

Numerical Stabilization

Matching: Finding an edge set such that no two edges are incident to the same node.



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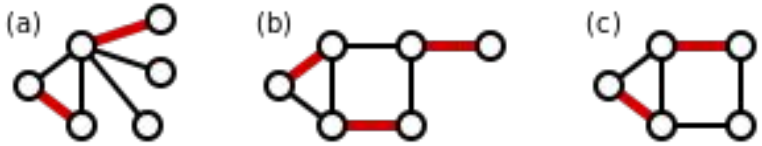

HSL routine MC64: Bipartite matching routine for MATLAB

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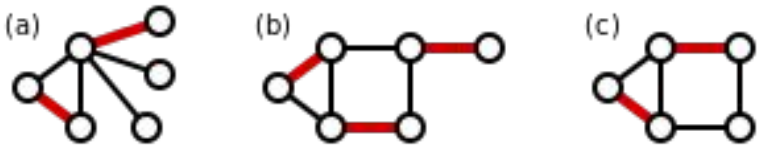
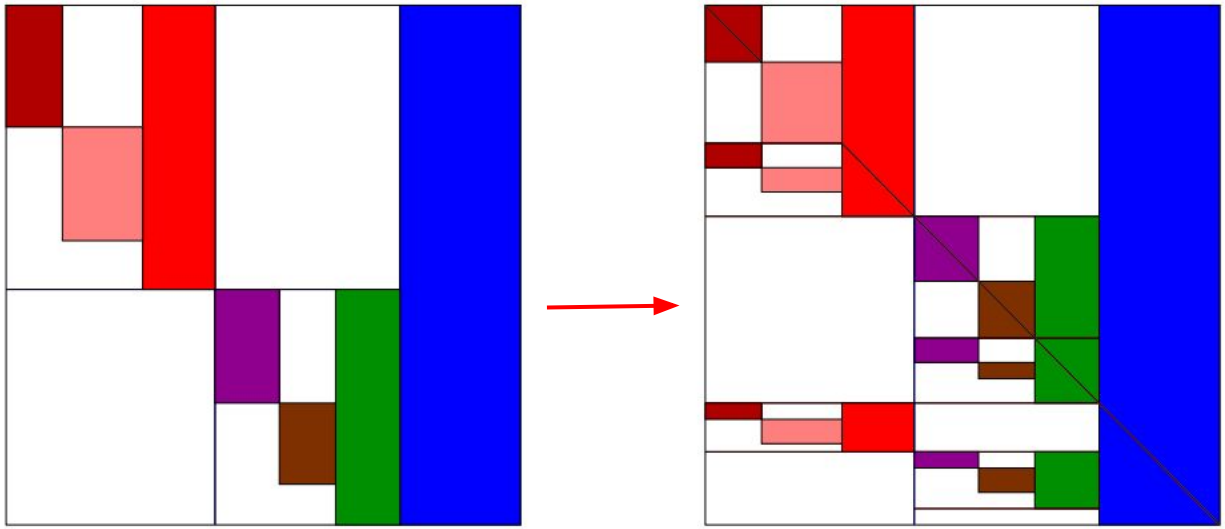
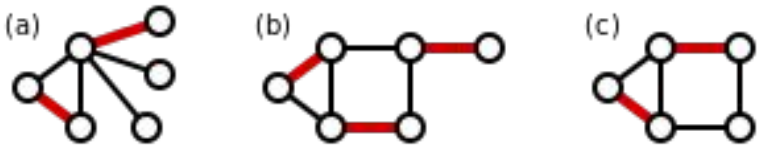


HSL routine MC64: Bipartite matching routine for MATLAB



Numerical Stabilization

Matching: Finding an edge set such that no two edges are incident to the same node.



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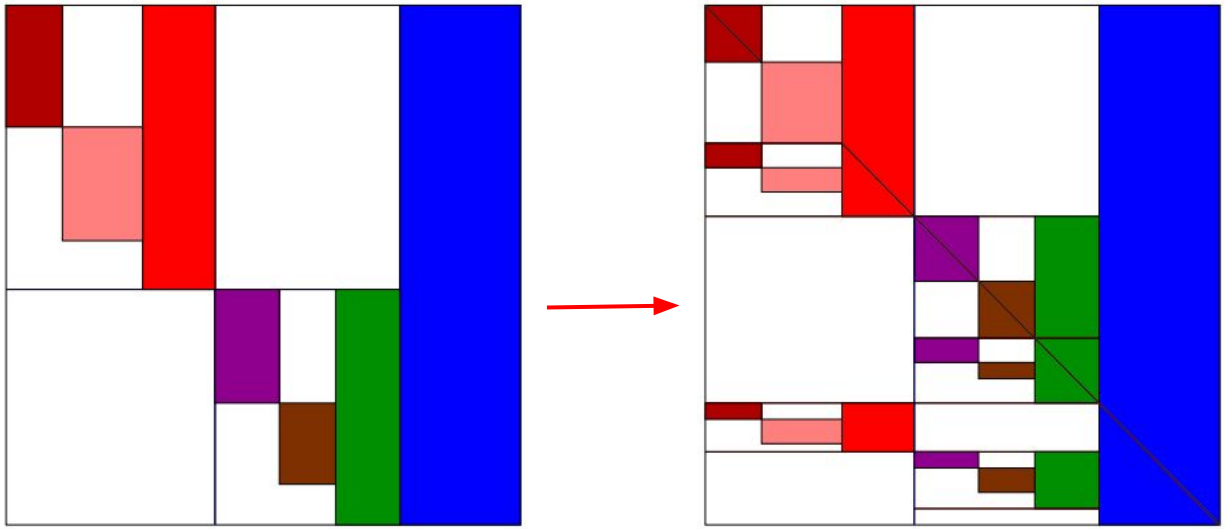
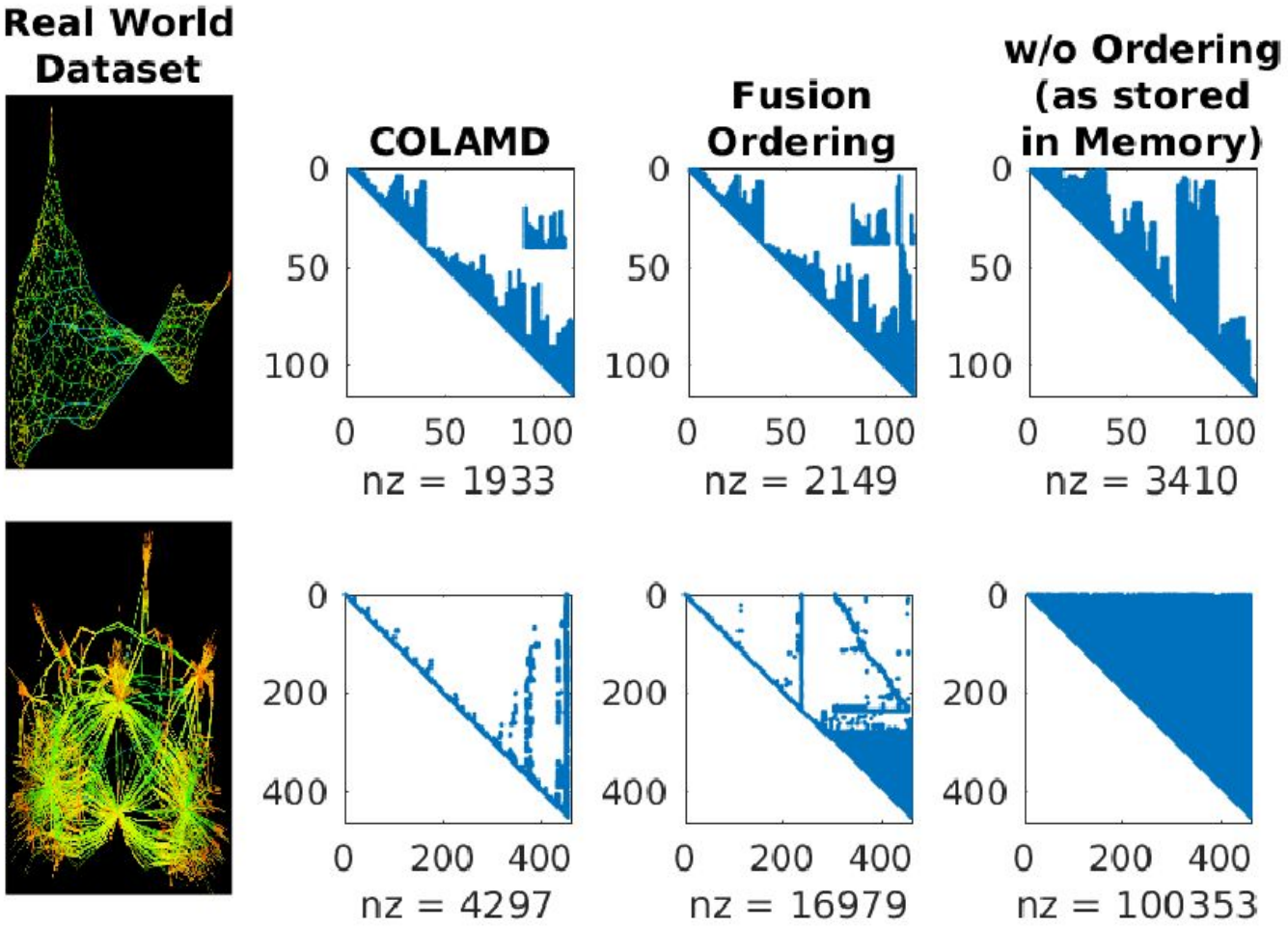


$$A_r^{ordered} = P^{matching} A_r P^1$$

SuiteSparse Results



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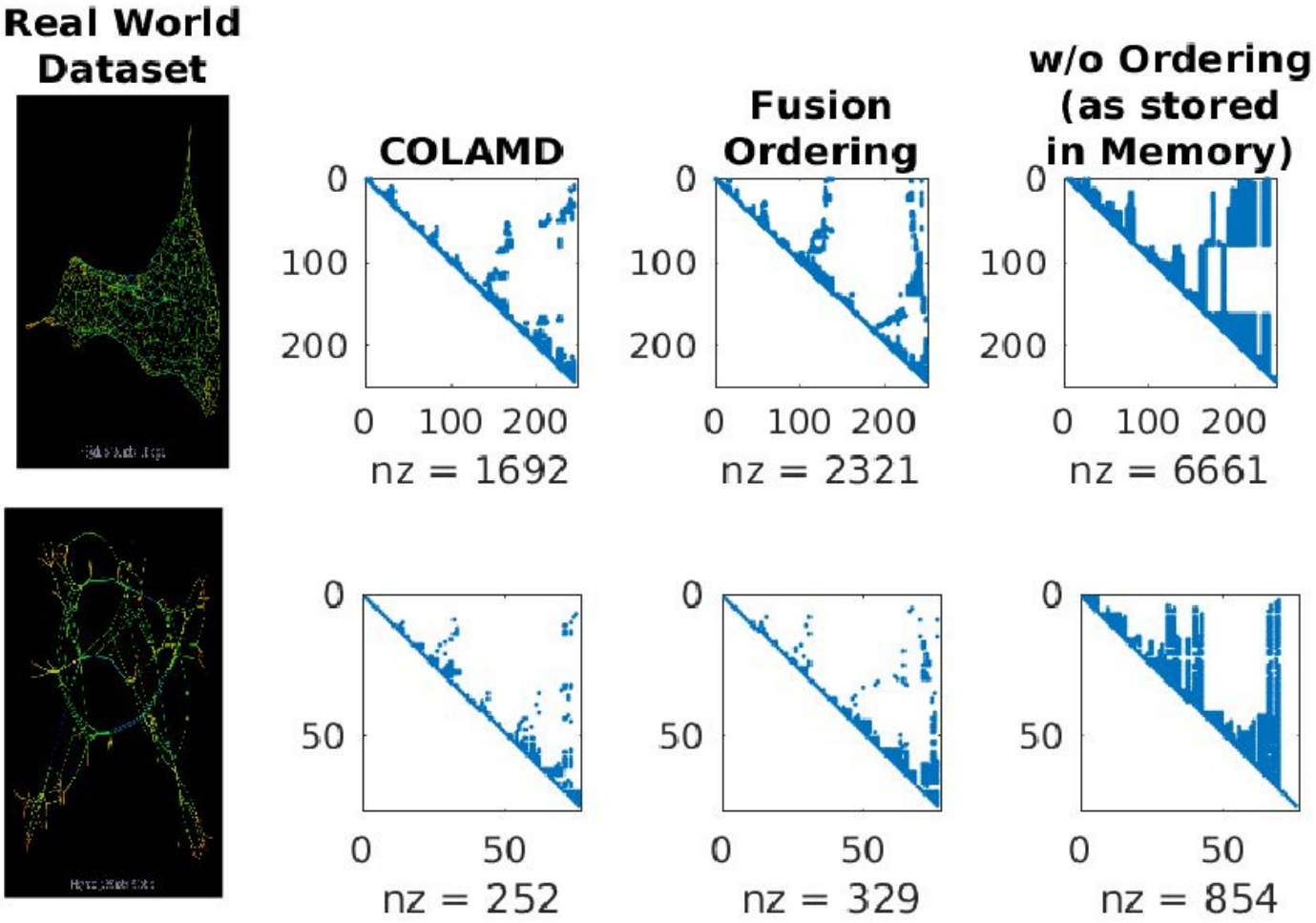

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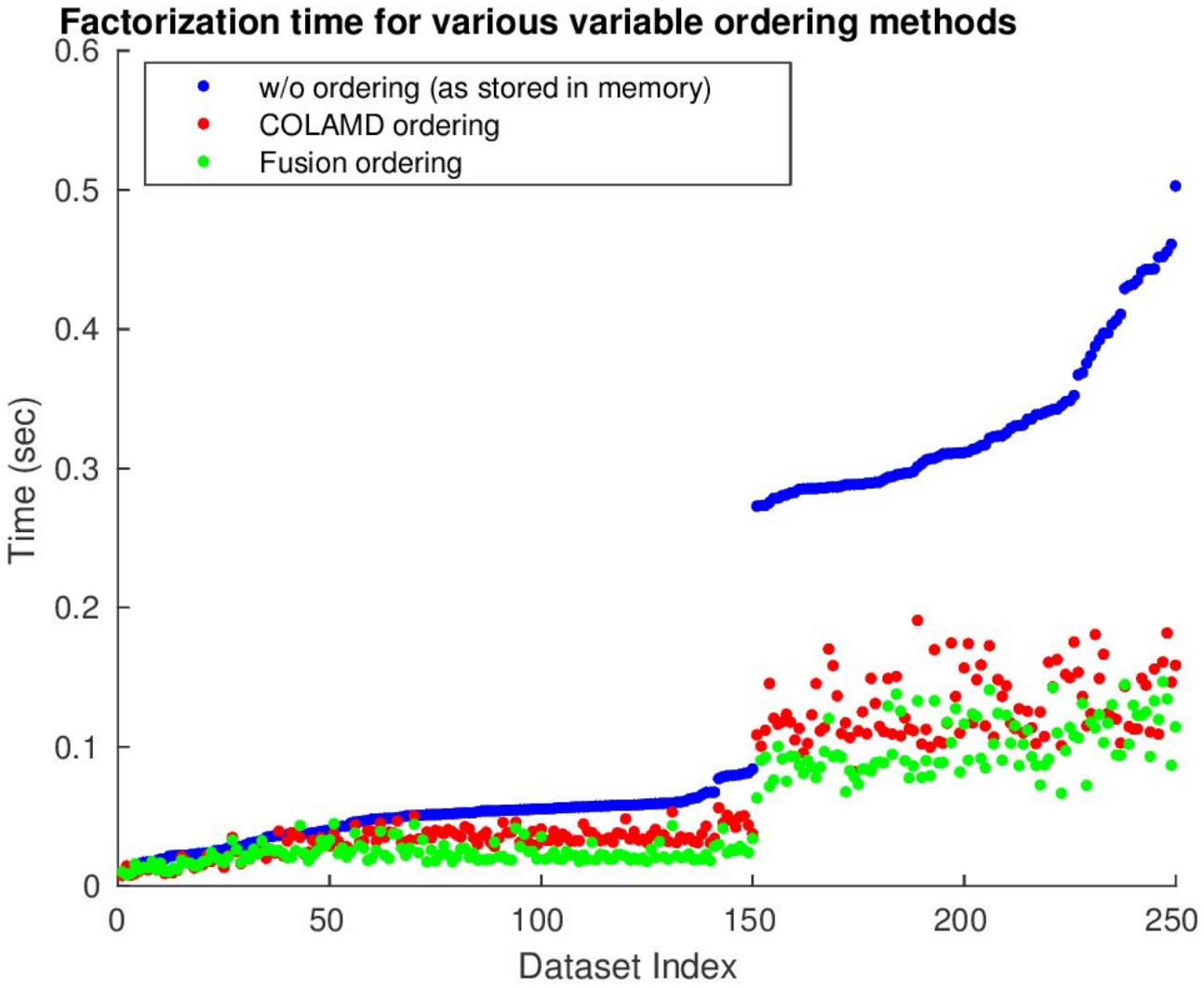

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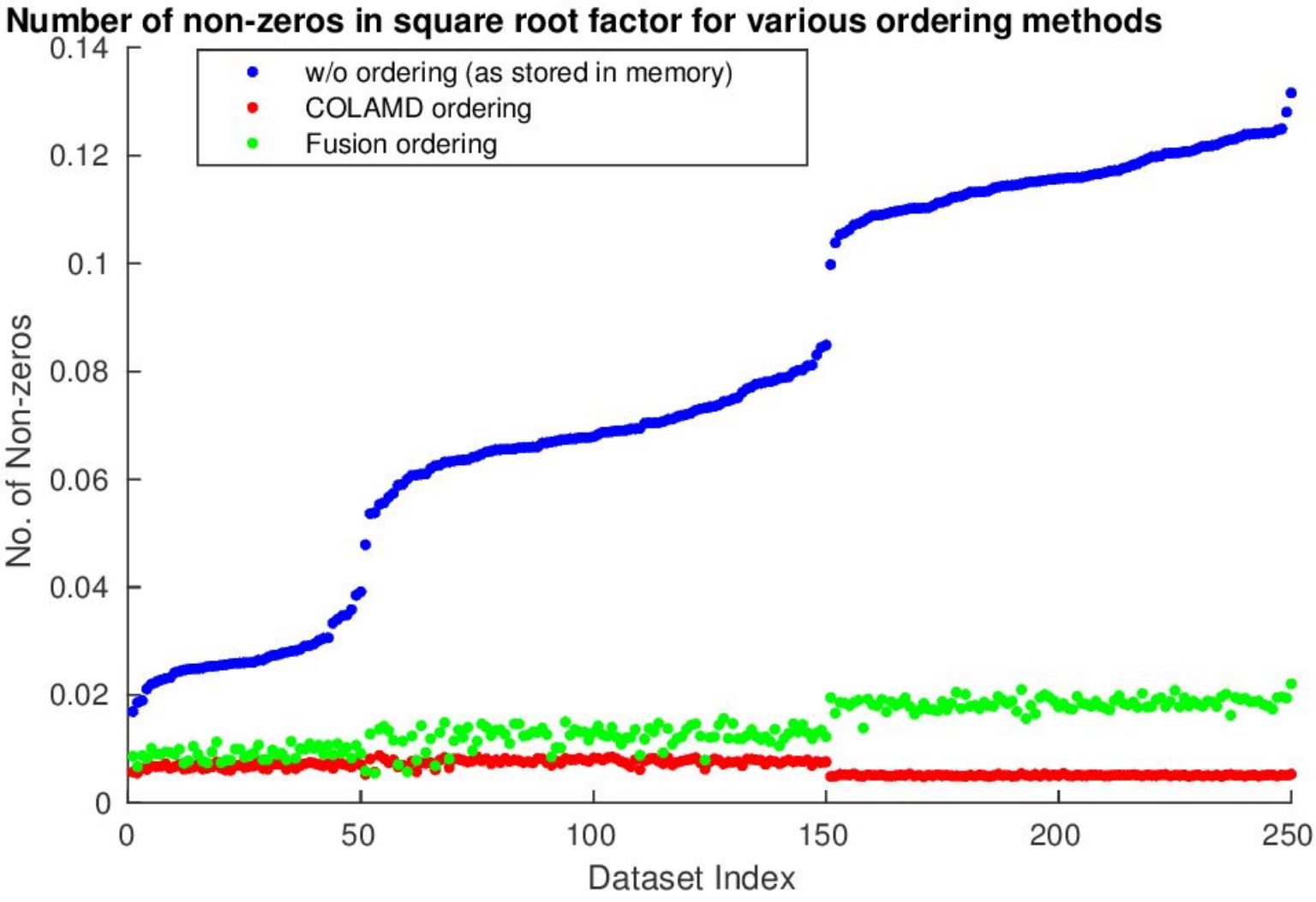
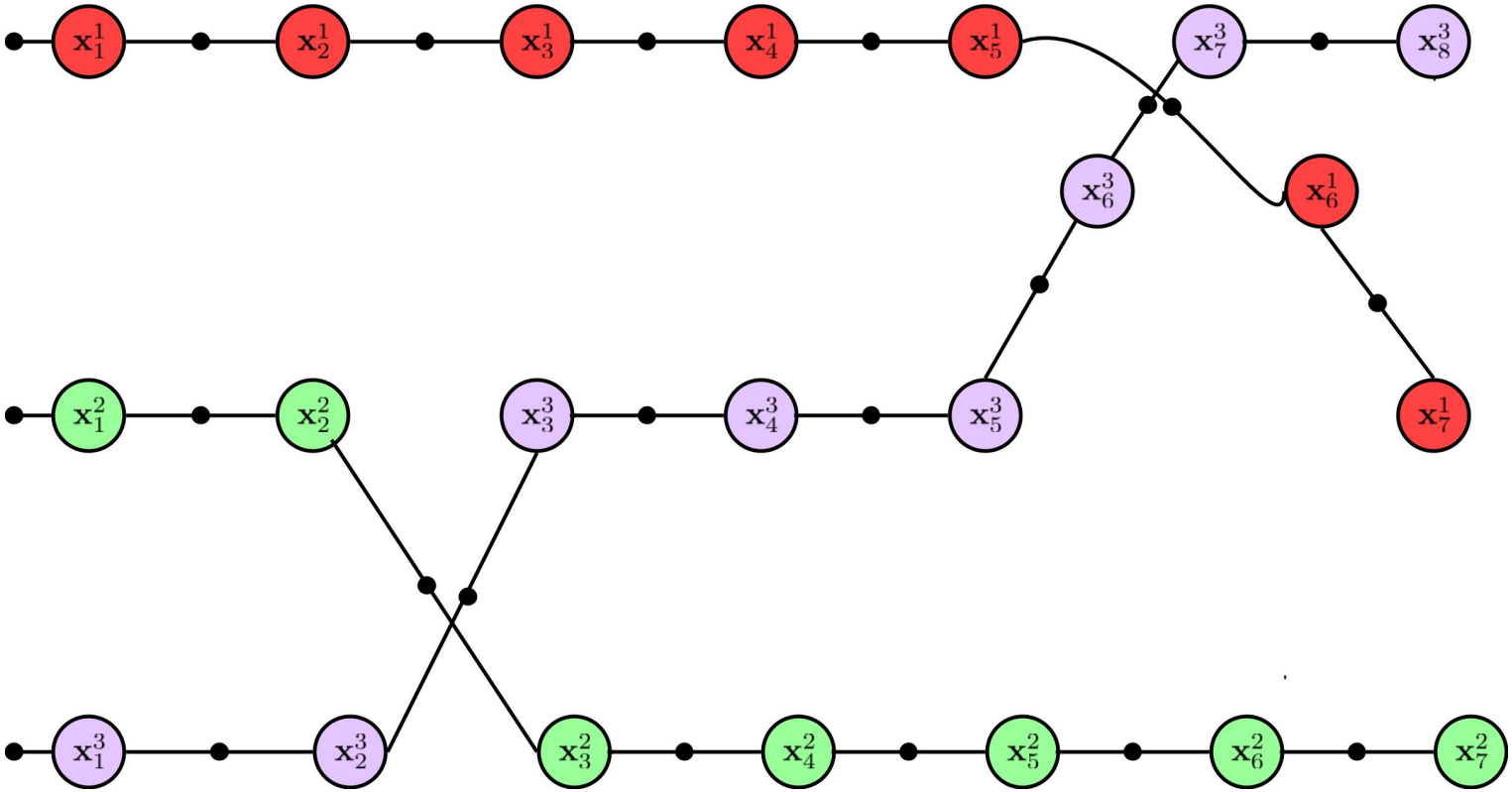


Outline

- Motivation
  - Factor Graphs
  - Mapping History
- Factor Graph SLAM
  - Variable Ordering
  - Bayes Trees
- Efficient Factor Graph Fusion
  - Fusion Ordering
  - Formal Verification
- Relative Pose Graphs
- Results

Relative Pose Graph Initialization



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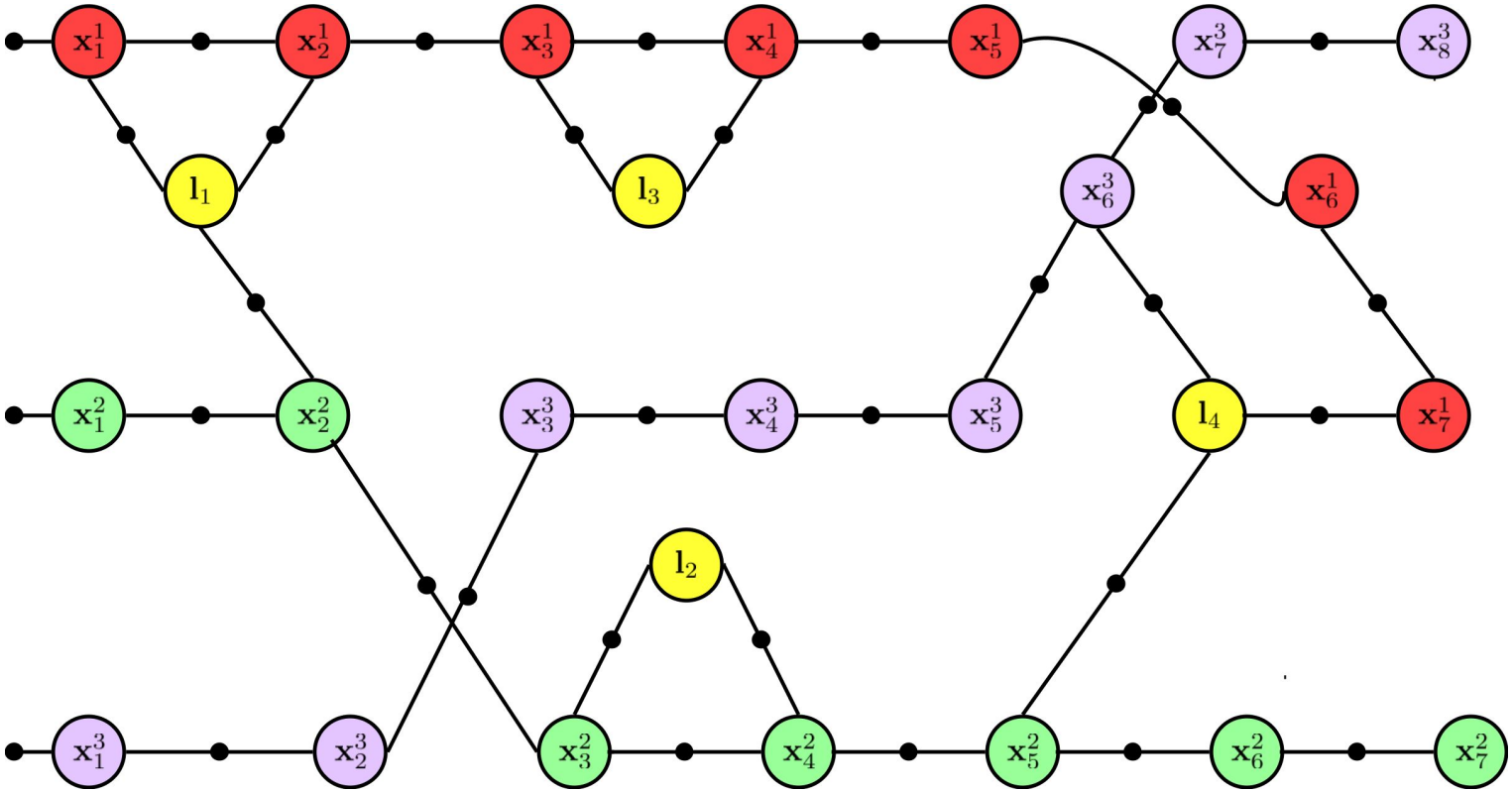

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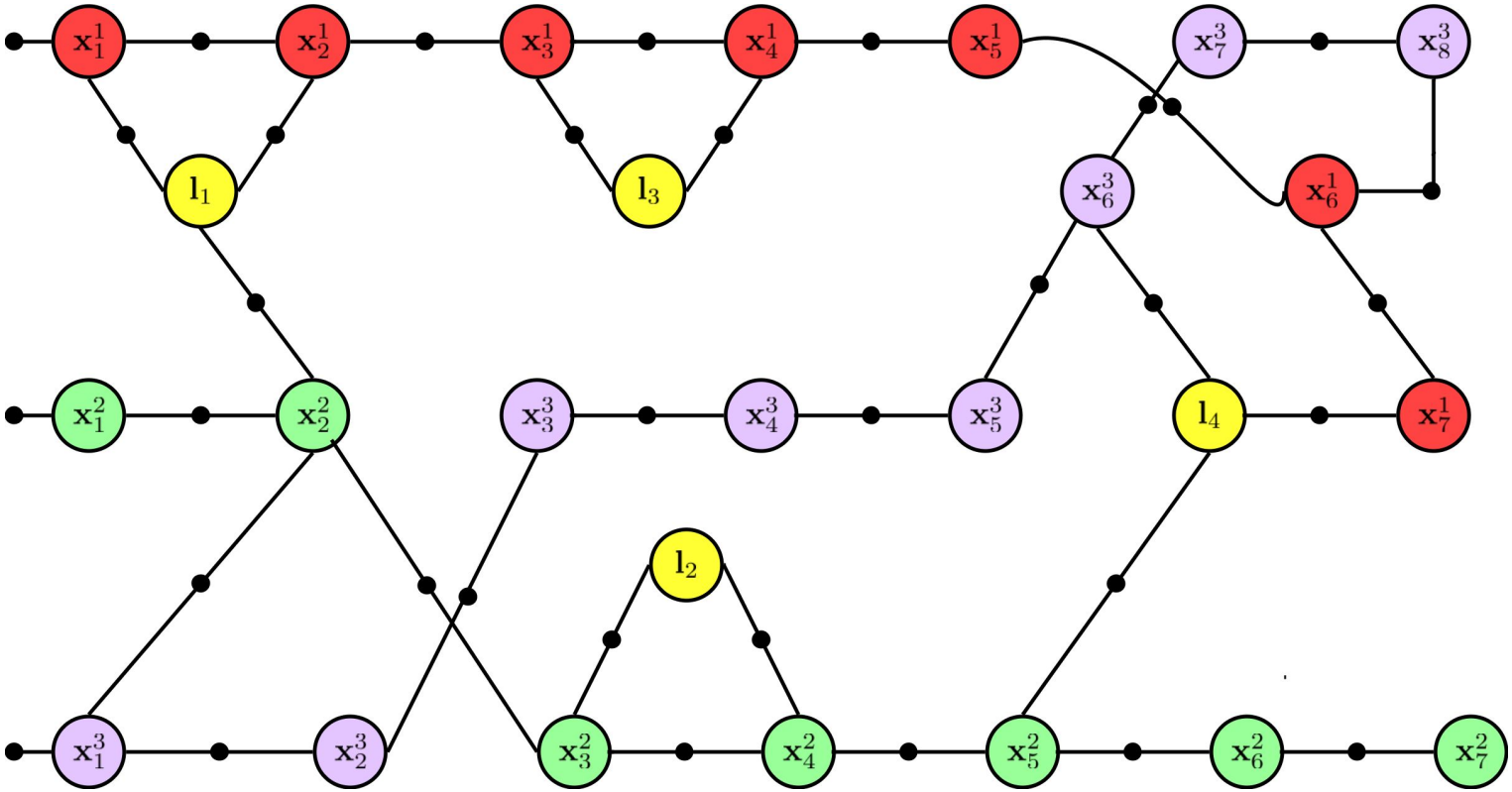

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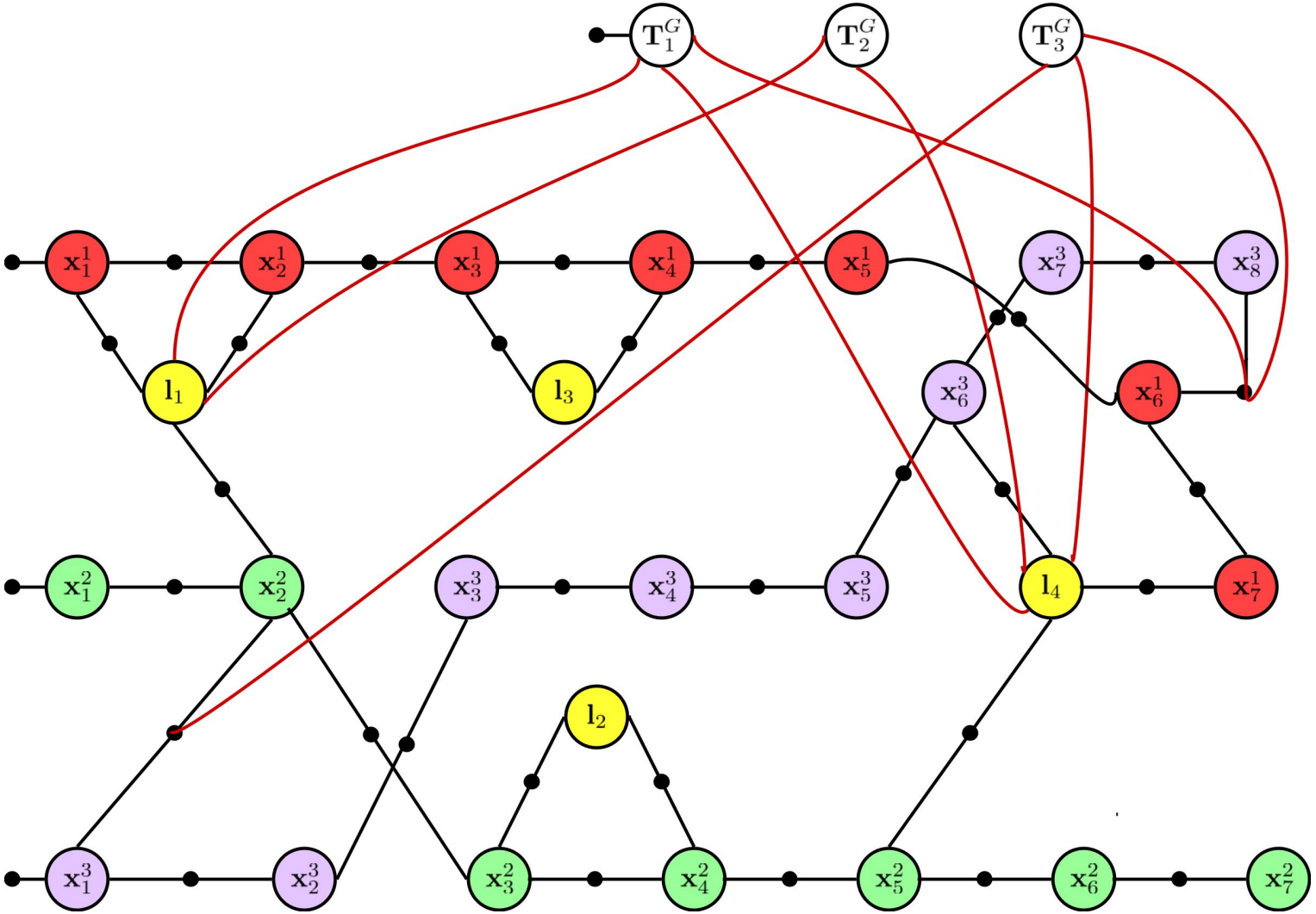

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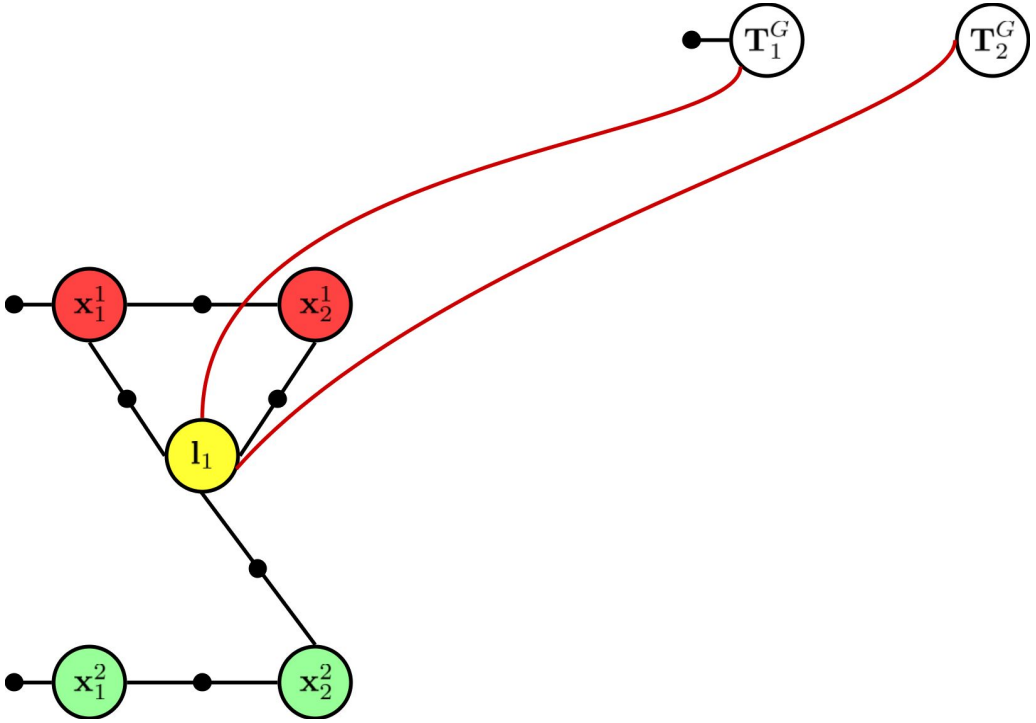

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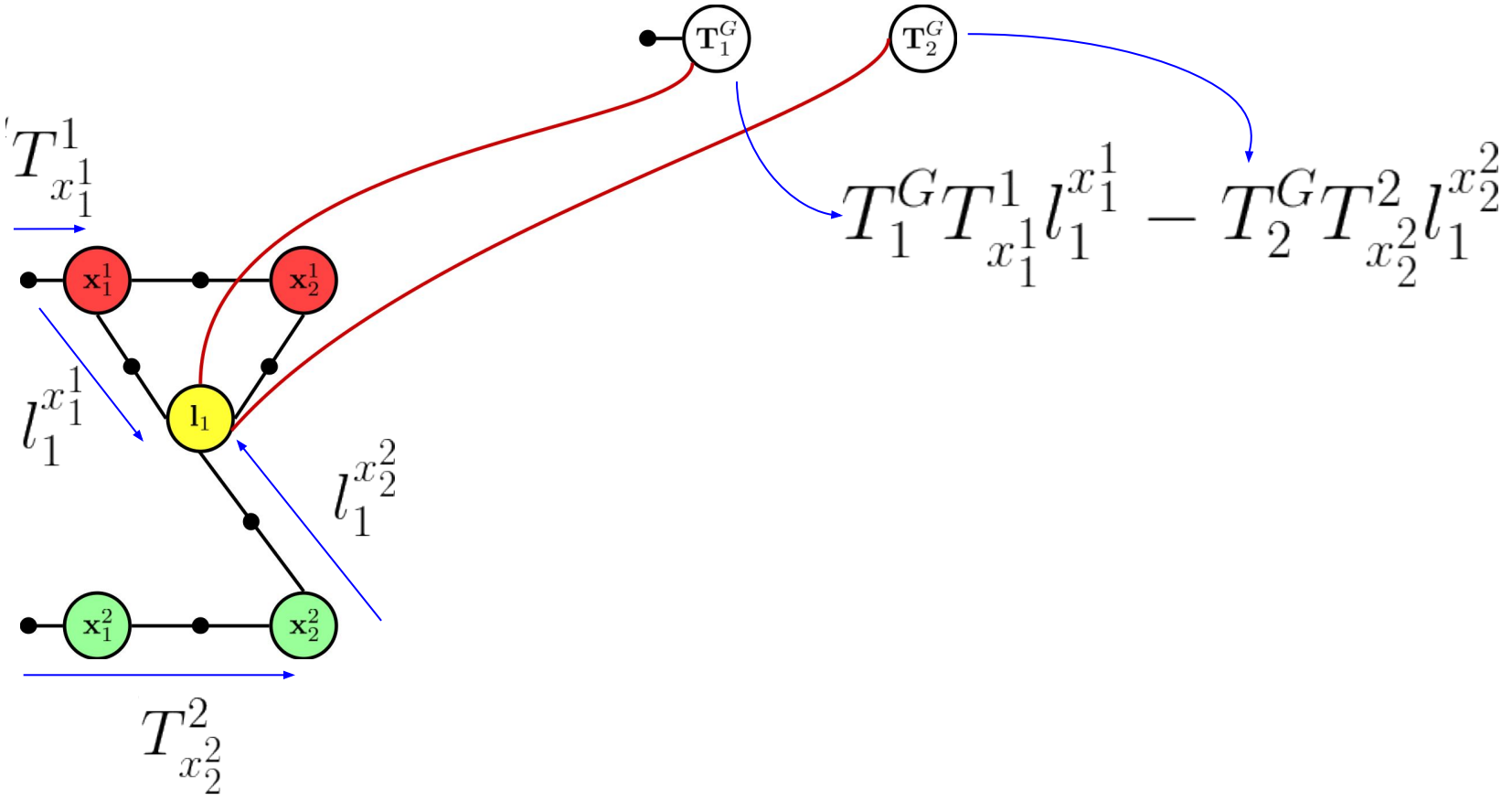

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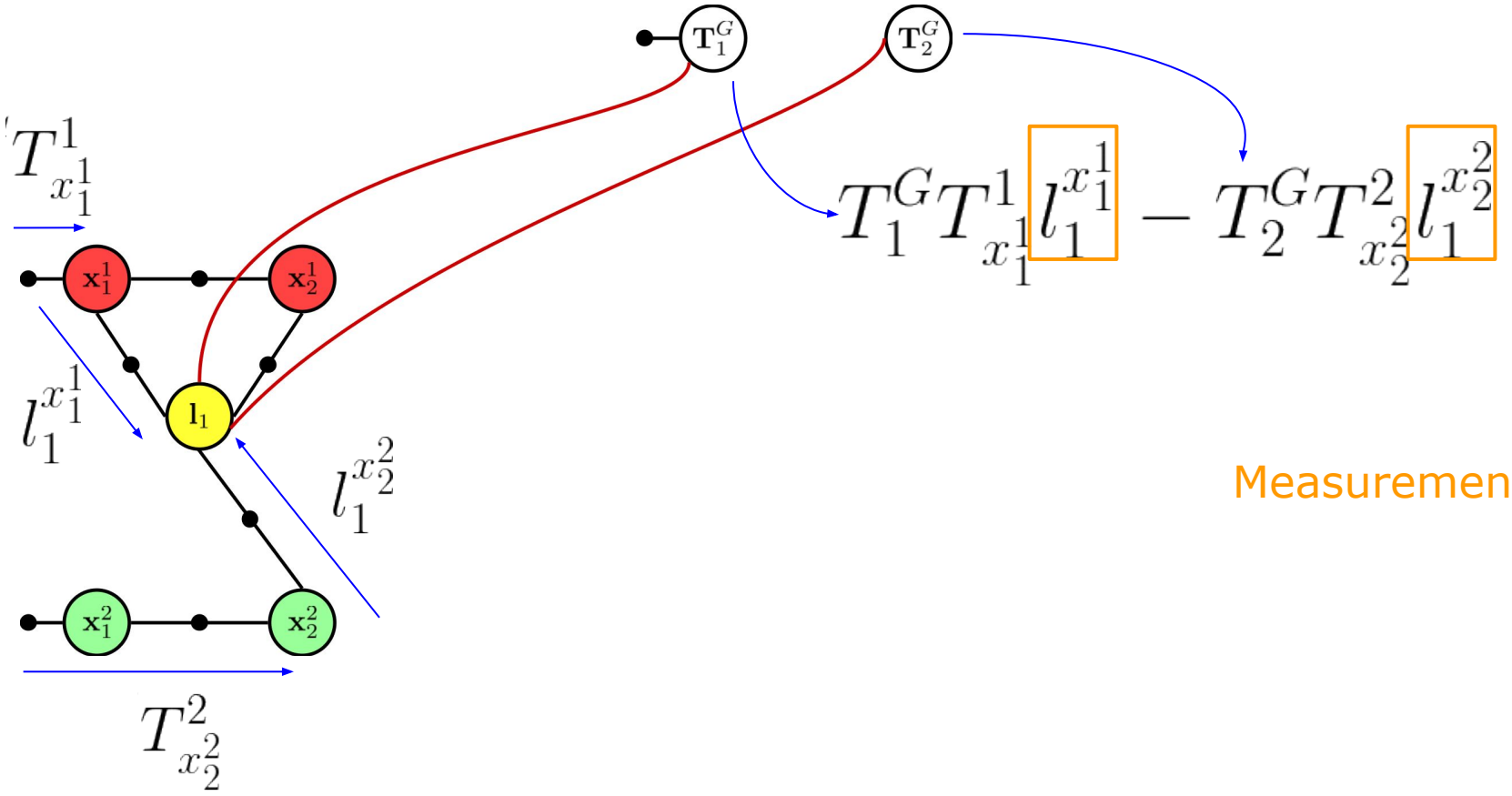

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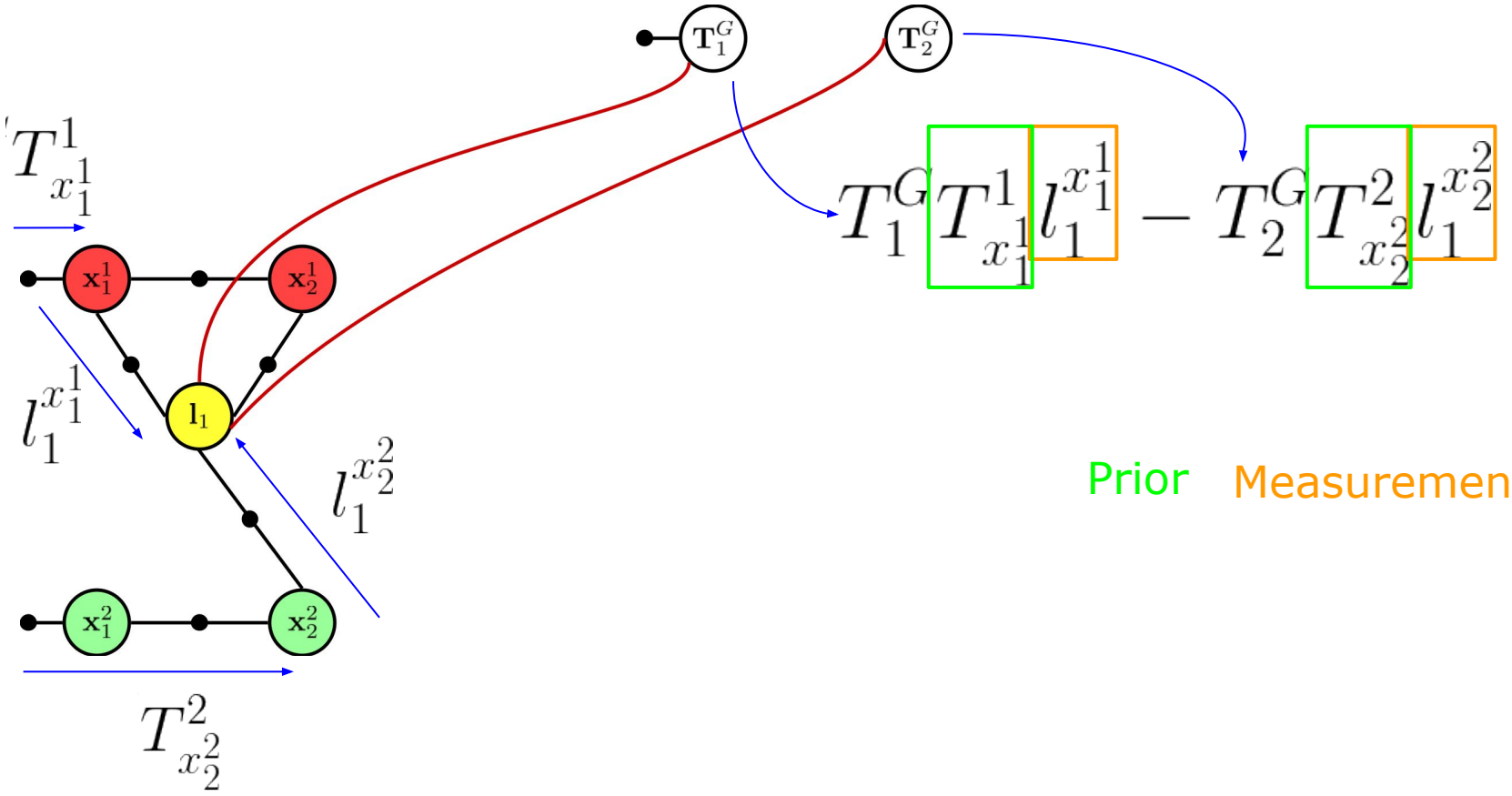

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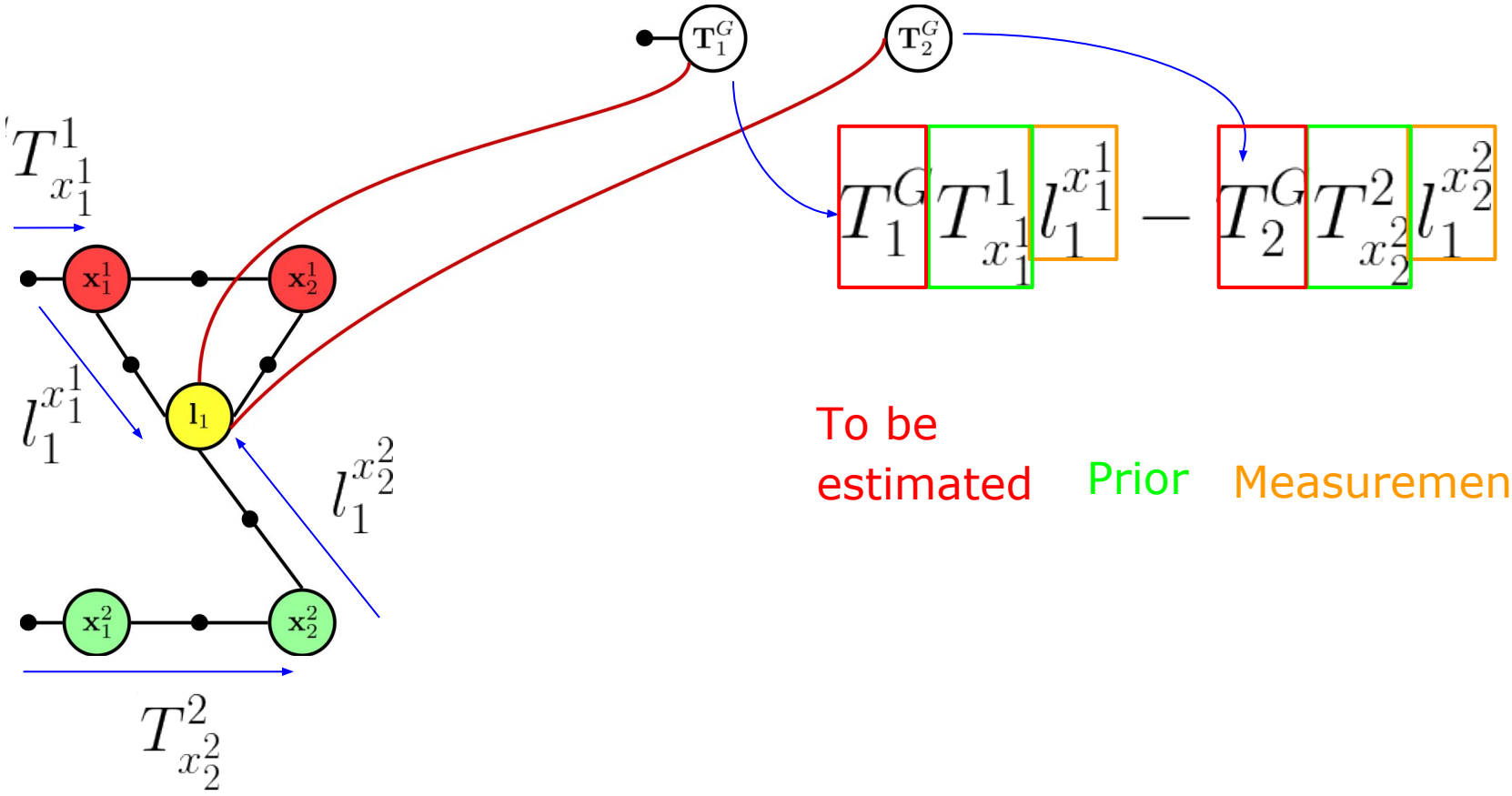

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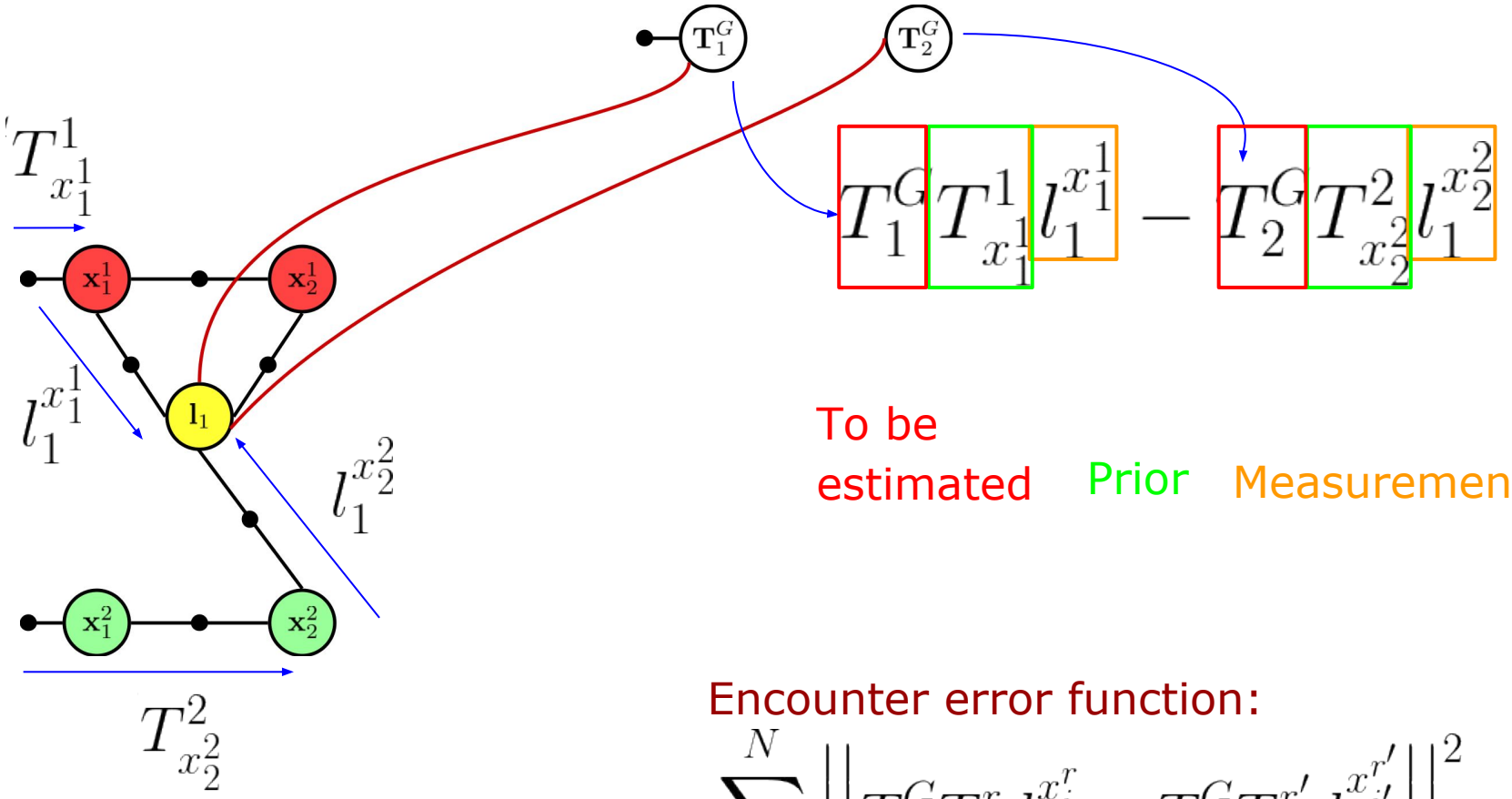

Encounter error function:

$$\sum_{n=1}^{N} \left|\left| T_r^G T_{x_i^r}^r l_n^{x_i^r} - T_{r'}^G T_{x_{i'}^{r'}}^{r'} l_1^{x_{i'}^{r'}} \right|\right|_{\Gamma_n}^2$$

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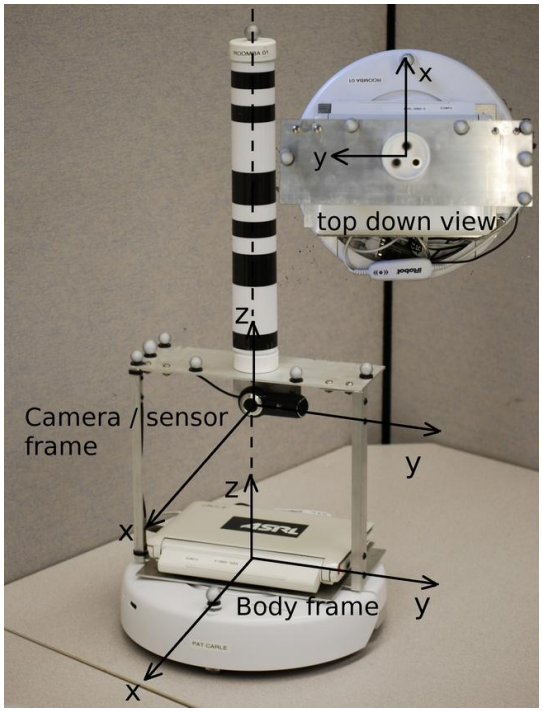
Experimental Results

- AP Hill multi-robot dataset
- 4 robots
  - Odometry (0.40 m/sec)
  - LIDAR (20m range with 1080 indices)
  - Camera
  - IMU
- Fiducial on top of each robot



Experimental Results

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- Variables to be tuned
  - Odometry model parameters - 3
  - Odometry velocity model parameters - 2
  - Scan matching parameters - ~20
  - Fiducial detection classifier parameters - ~5
  - Odometry covariance - 3
  - Odometry velocity model covariance - 3
  - Fiducial detection covariance - 3
  - Encounter covariance - 3
  - Scan matching keyframe size - 1
  - Keyframe type - 3
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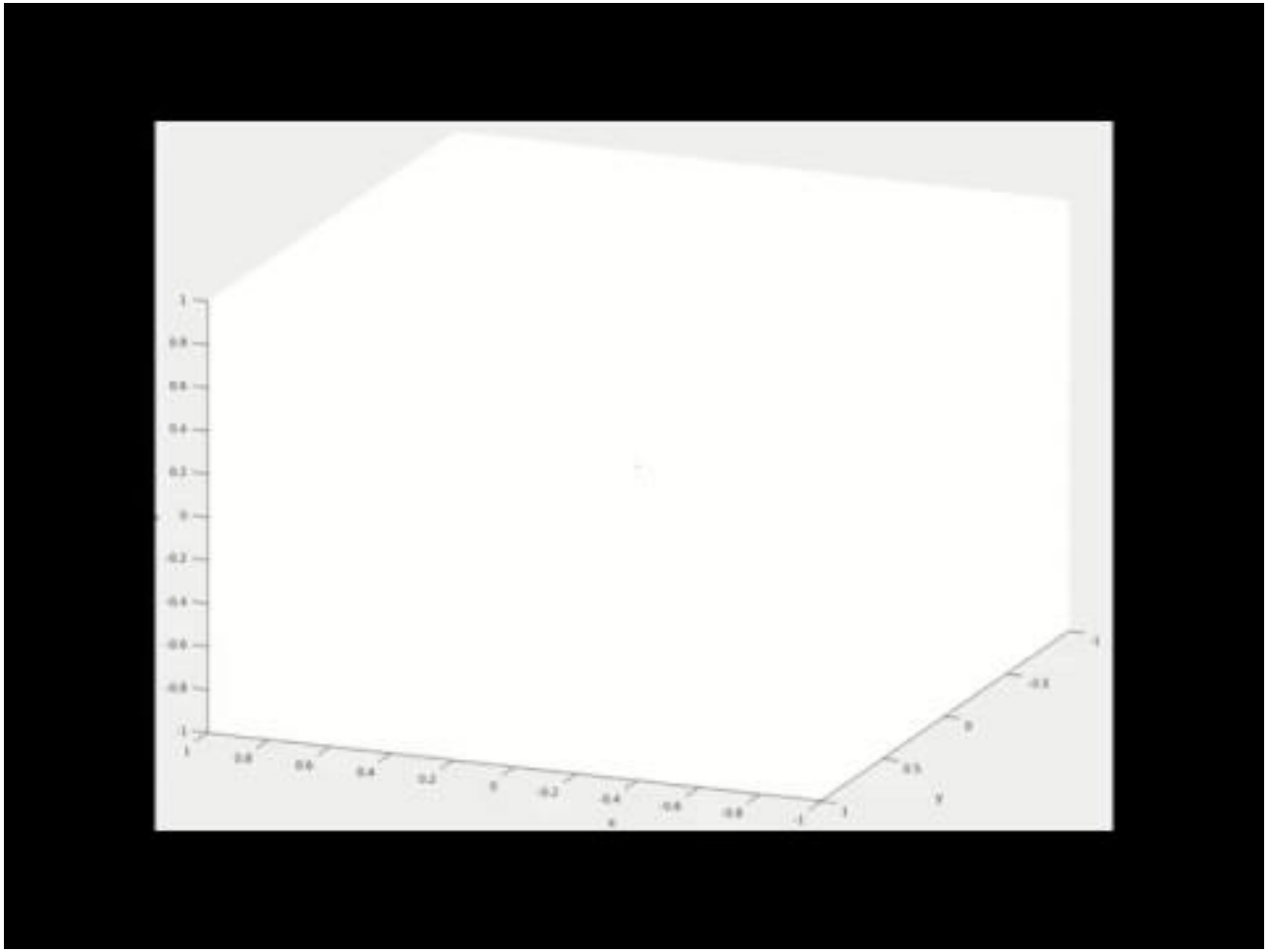
Variables/robot ~ 50

Total no. of variables

~200

Experimental Results



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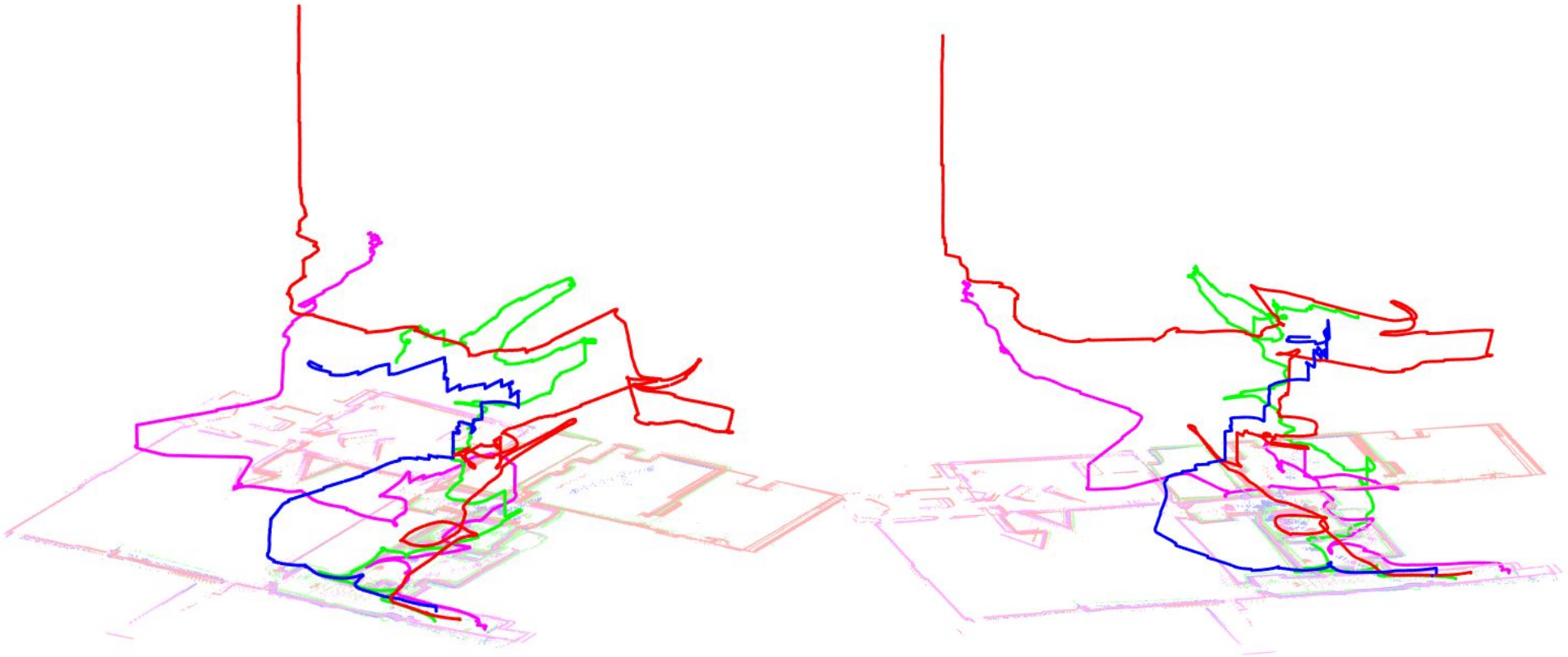

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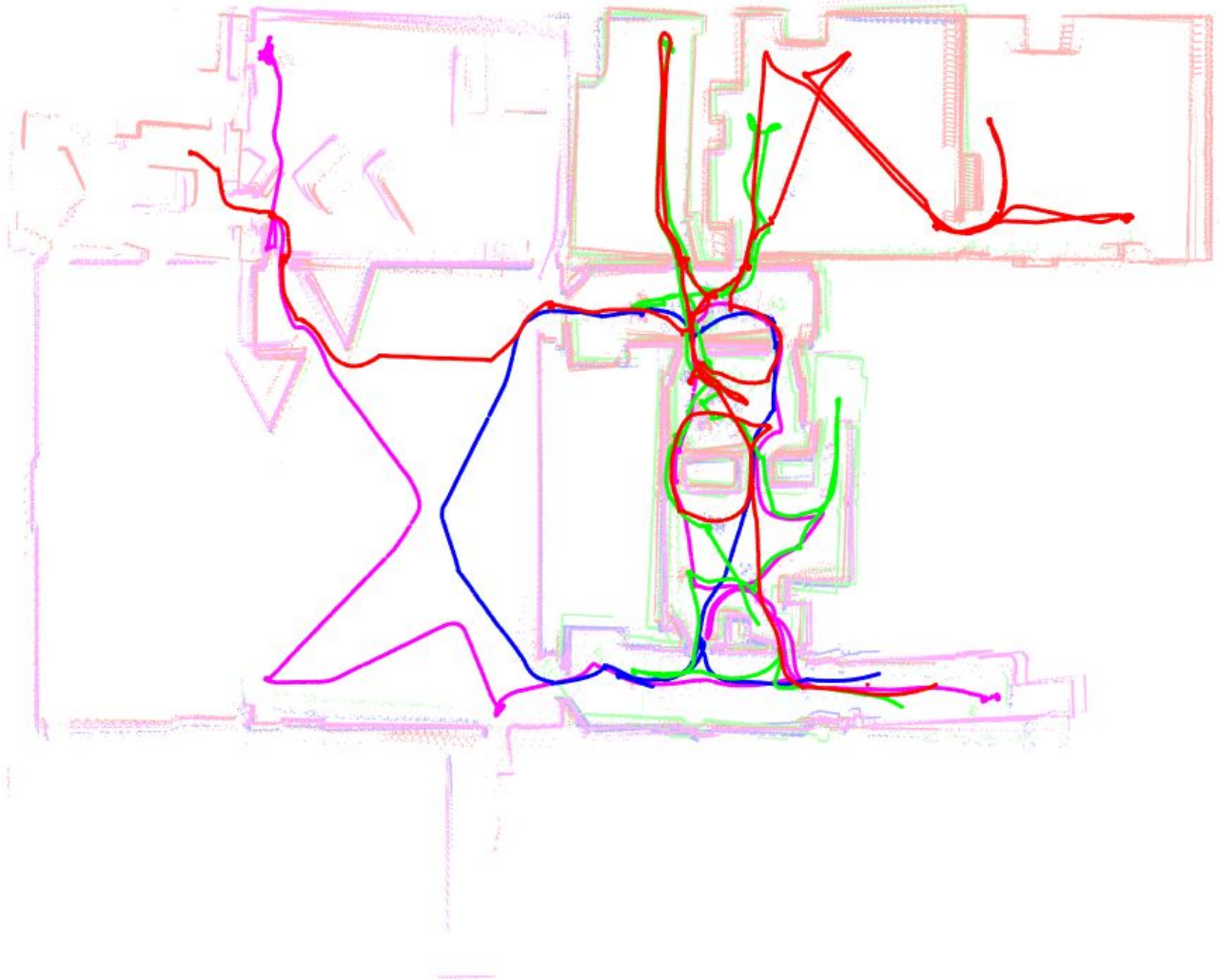
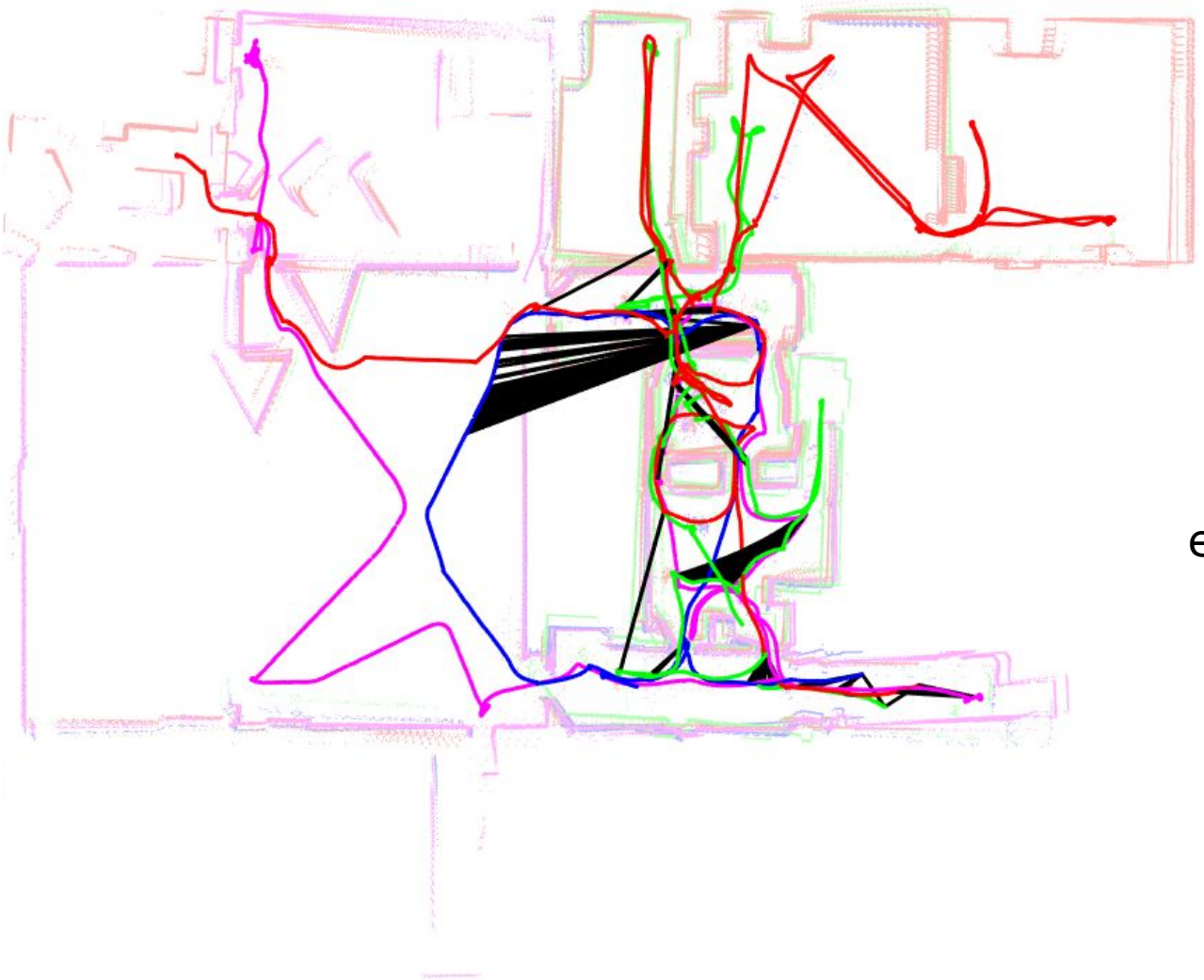


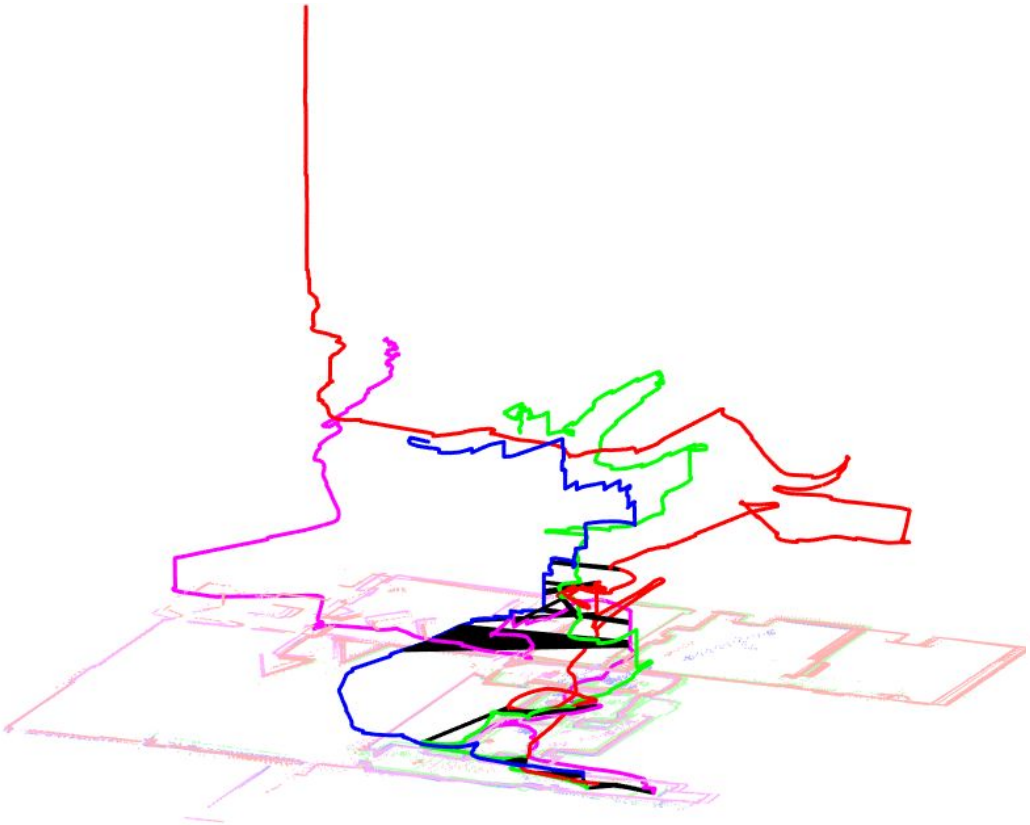
Top view

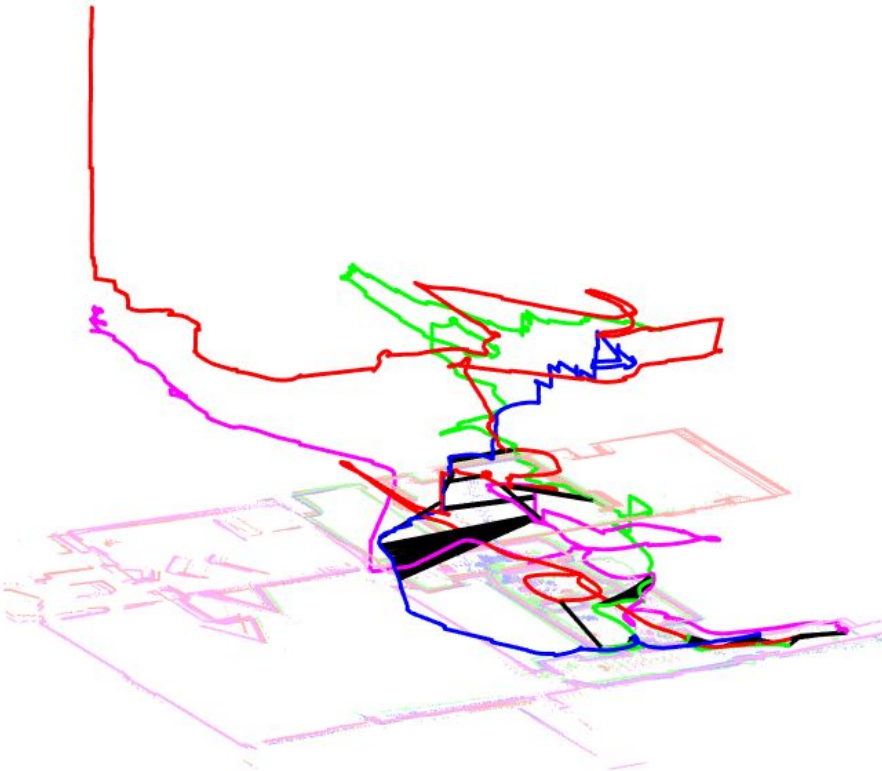
Experimental Results



Top view with encounters

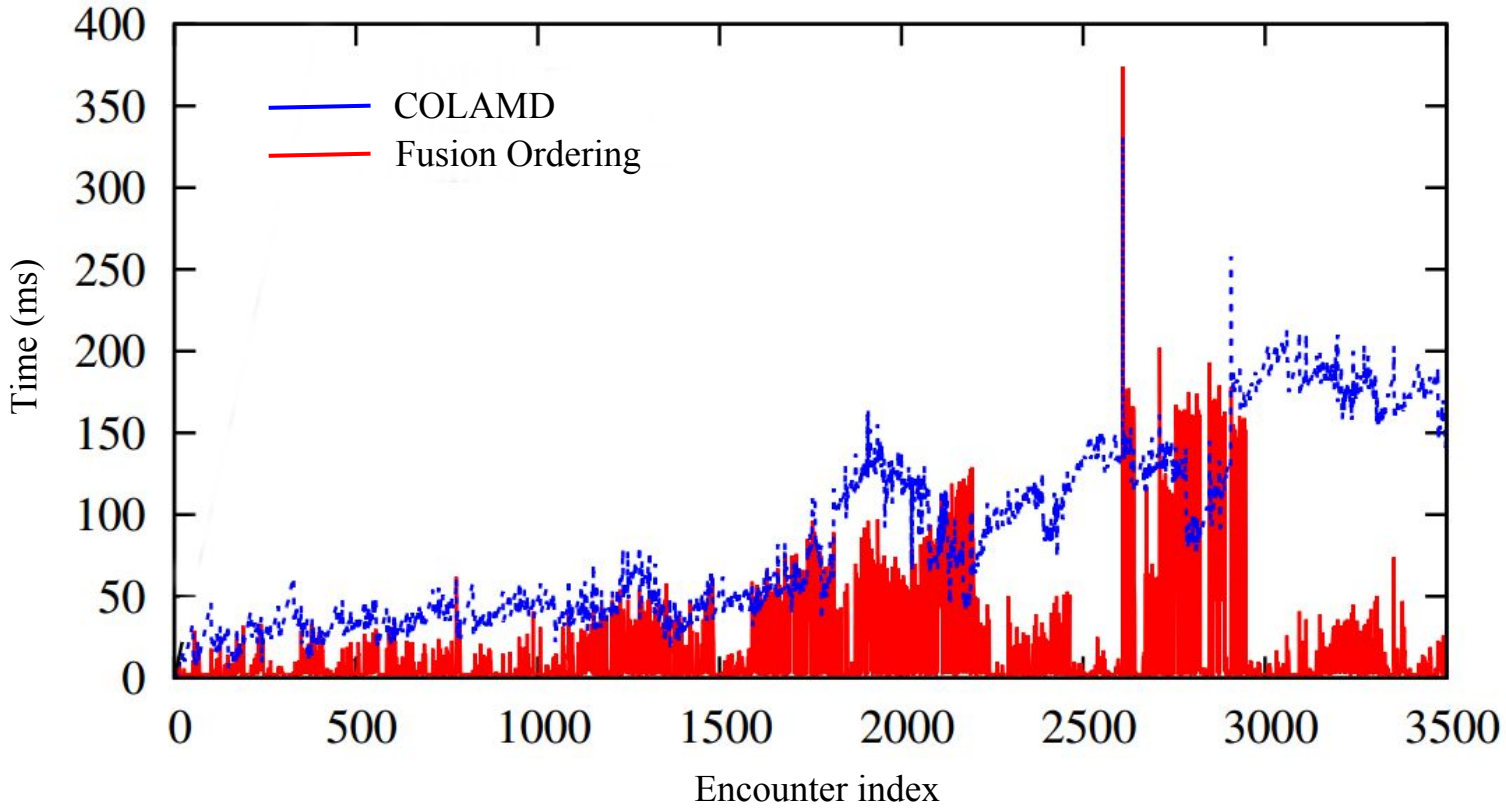
Experimental Results





Experimental Results

Time comparison between the ordering strategy



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- Relative pose graph initialization
- Results on standard real-world datasets
  - SuiteSparse
  - AP hill multi-robot dataset

Future Work

- Upper bound on additional fill-in in using Fusion Ordering.
- Use Fusion Ordering for Concurrent Filtering and Mapping (CFS) **[Williams, 2014]** to extend to multiple robots.
- Investigate communication bandwidth constraints.
- Incorporate partial encounter constraints like range-only or bearing-only.

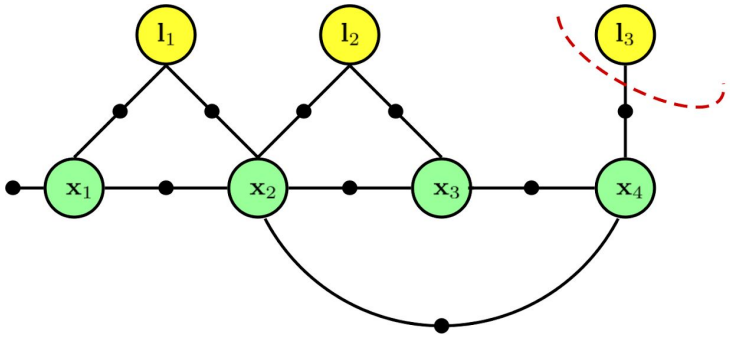
Thank you

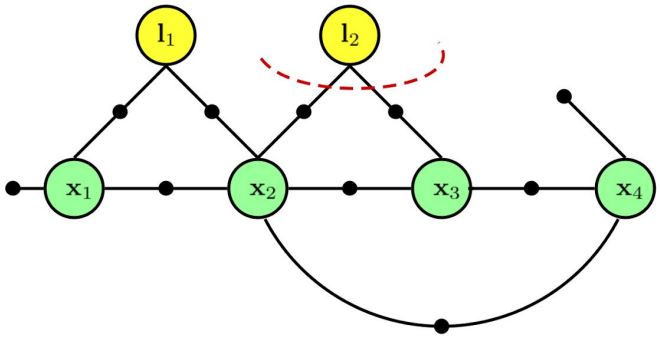
Questions?

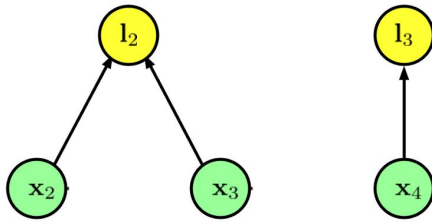
csm_params

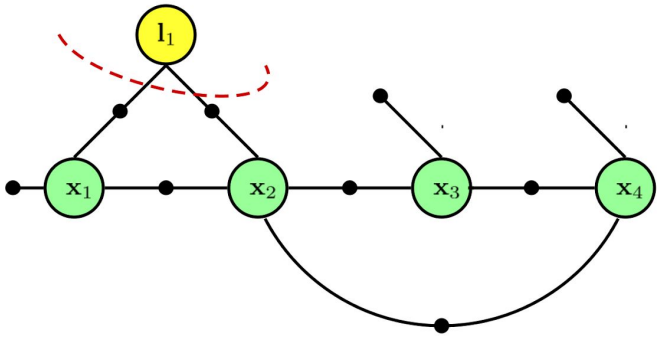
```
// Create a CSM parameters structure from the supplied configuration

sm_params csm_params;

csm_params.use_point_to_line_distance = true;

csm_params.use_corr_tricks = true;

csm_params.max_iterations = 1000;

csm_params.max_angular_correction_deg = 1.57 * 180.0 / M_PI;

csm_params.max_linear_correction = 2.0;

csm_params.max_correspondence_dist = 2.0;

csm_params.sigma = 0.05;

csm_params.epsilon_xy = 0.0001;

csm_params.epsilon_theta = 0.0001;

csm_params.outliers_maxPerc = 0.95;

csm_params.outliers_adaptive_order = 0.70;

csm_params.outliers_adaptive_mult = 2.0;

csm_params.outliers_remove_doubles = true;

csm_params.do_visibility_test = false;

csm_params.do_alpha_test = false;

csm_params.do_alpha_test_thresholdDeg = 0.35 * 180.0 / M_PI;

csm_params.clustering_threshold = 0.05;

csm_params.orientation_neighbourhood = 3;
```

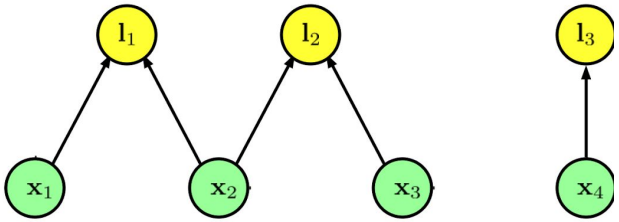
Factor Graph -> Bayes Network -> Bayes Tree











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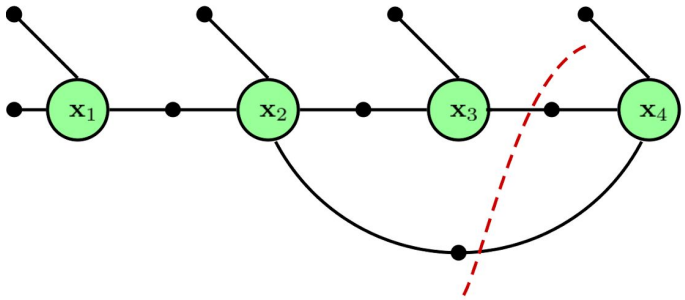

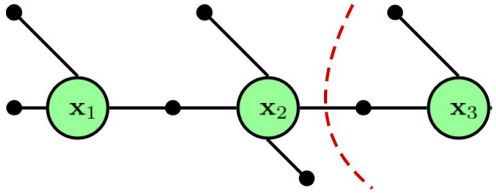

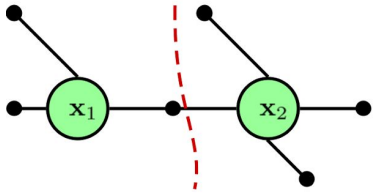

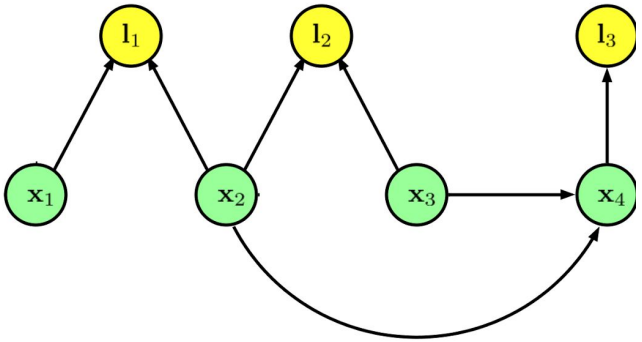

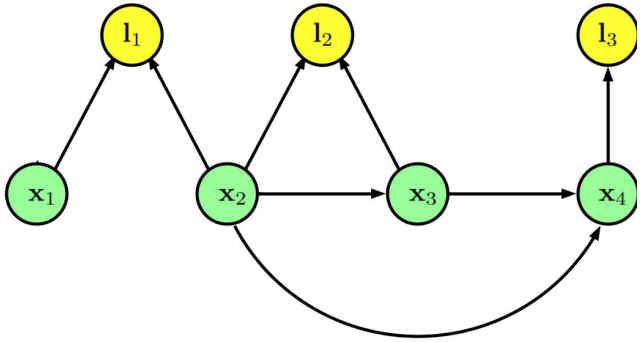

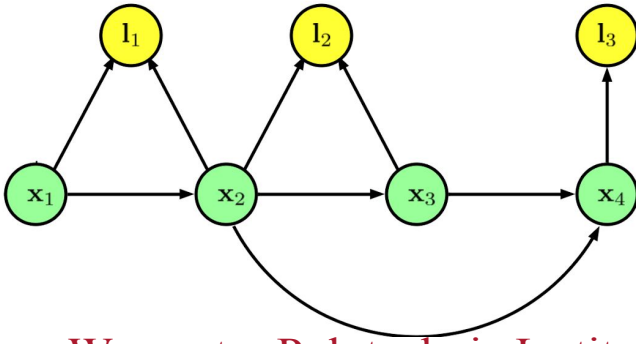

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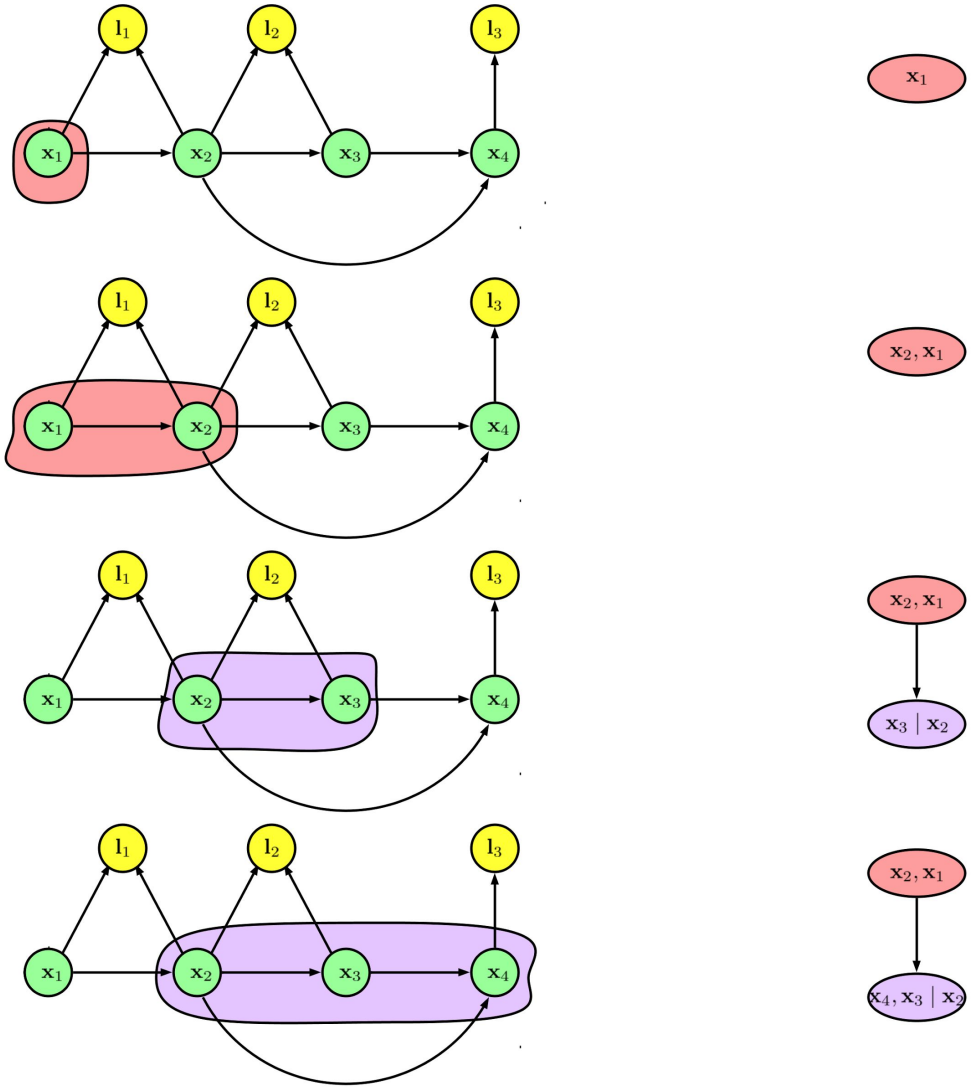

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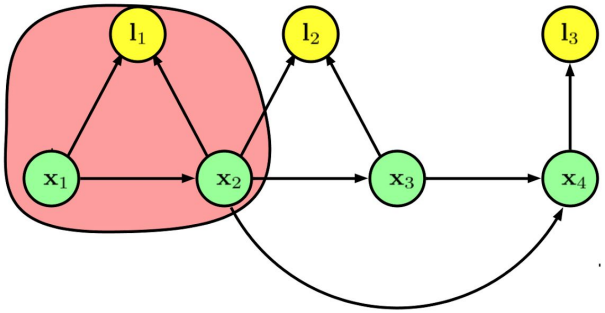

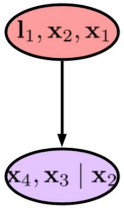

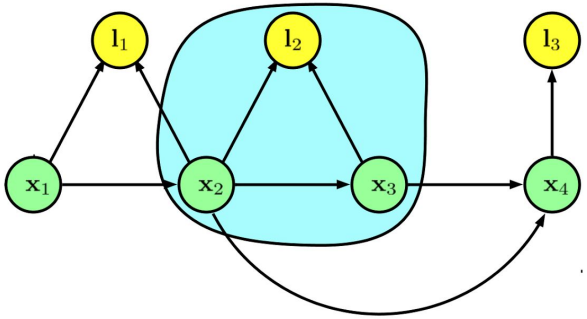

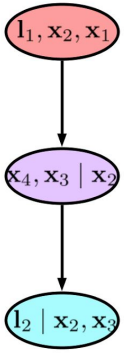

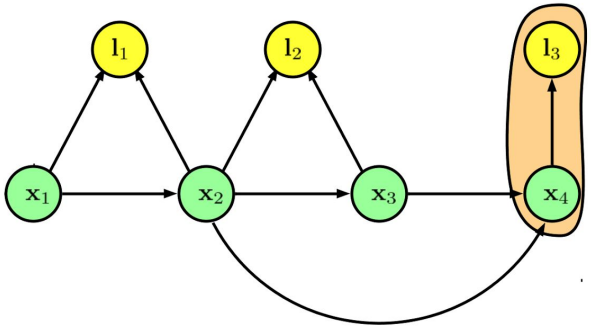

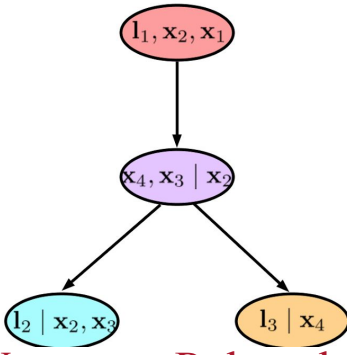

Ordering matters! - Intuition

Mapping: History & Motivation