Uncertainty About Probability: A Reconciliation with the Subjectivist Viewpoint

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Abstract— **The use of probability distributions to represent uncertainty about probabilities (rather than events) has long been a subject of controversy among theorists. Many well-known theorists, such as de Finetti, have concluded that it is inherently meaningless to be uncertain about a probability, because this appears to violate the subjectivists' assumption that individuals can develop unique and precise probability judgments. Others have found the concept of uncertainty about probability to be both intuitively appealing and potentially useful. This paper presents a resolution of this question, indicating that at least one type of uncertainty about probabilities (that arising from uncertainty about the underlying events on which those probabilities are conditioned) is consistent with the subjective theory of probability. Another type of uncertainty (namely, that arising from cognitive imprecision) appears not to be consistent with the axioms of subjective probability as they are currently formulated. Distinguishing between these two sources of uncertainty is an important step in resolving the persistent theoretical controversy over uncertainty about probability. We also show that uncertainty about probability will be potentially relevant to decision making whenever the expected value of the relevant performance measure is nonlinear in the probability in question.**

I. Introduction

According to the subjective theory of probability, each individual is assumed to be capable of summarizing his or her uncertainties in the form of unique and precise probability judgments. While this is generally accepted for uncertainties about events, the use of probability distributions to represent uncertainty about probabilities (rather than events) has long been a subject of controversy among theorists. Many well-known theorists, such as de Finetti [6], have concluded that it is inherently meaningless to be uncertain about a probability, apparently because this would violate the assumption that individuals can develop unique and precise probability judgments. Others (e.g., Good [13], Suppes [33]) have found the concept of uncertainty about probability to be both intuitively appealing and potentially useful.

However, even many of those who feel that uncertainty about probabilities is an important question still believe that such uncertainty is inconsistent with the subjective theory of probability. This paper is an attempt to resolve this apparent inconsistency. In particular, we identify two different types or sources of uncertainty about probability, one arising from uncertainty about underlying conditions and the other from inherent impreciso in our cognitive processes. We argue that the first type of uncertainty does not violate the axioms of subjective probability theory, and that this type of "conditional uncertainty" will often be more important in practice. Finally, we believe that the approach proposed for addressing conditional uncertainty can also offer pragmatic (although nonaxiomatic) guidance for dealing with the other type of uncertainty (cognitive imprecision) as well.

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II. The Debate Over Uncertainty About Probabilities

At least since Koopman [21], probability theorists have been discussing the issue of uncertainty about probabilities. Many well-known theorists have concluded that the concept of uncertainty about probabilities is inherently meaningless. Probably the most outspoken of these is de Finetti [6], who flatly states that "Any assertion concerning probabilities of events is merely the expression of somebody's opinion and not itself an event. There is no meaning, therefore, in asking whether such an assertion is true or false or more or less probable ... [S]peaking of unknown probabilities must be forbidden as meaningless."

De Finetti [6] acknowledges (p. 190) that "There might of course exist unknown objective facts or circumstances, the knowledge of which would influence our judgment of probability," and concedes (p. 211) that "the concept of 'unknown probability' is almost acceptable" in such cases. For example, if E is the selection of a white ball from an urn, H_i is the hypothesis that exactly i of the N balls in the urn are white, and P_i is the probability that hypothesis H_i is true, then de Finetti (p. 216) acknowledges that "everything is as if [this] distribution could be interpreted in the usual terminology as the distribution of the unknown probability," but concludes that it is preferable to avoid this terminology even in such apparently clear-cut cases.

This view has been widely adopted by other probability theorists and analysts. For example, Marschak [26] summarizes the consensus of a group of probabilists that "to assign probability to the truth of a probabilistic theory is to use a figure of speech, an 'as if,' that is justified *in the limit*." Similarly, in a discussion of uncertainties in risk analysis, Apostolakis [1] essentially paraphrases de Finetti, stating that "Events must be either true or false, and a statement involving the words 'probable' or 'likely' can never be proved to be such."

However, this view has not been unanimous, and there have been inconsistencies even among those who espouse it. For example, Marschak [25] states that "'Bayesian statisticians' use ... personal ('prior') probabilities defined, not on the set of events, but on the set of probability distributions over this set." Similarly, Press [29] specifically invokes exchangeability to justify the use of uncertain probabilities, stating that "de Finetti's theorem makes it possible to use the subjective (personalistic) interpretation of statements involving an unknown probability."

Despite these theoretical controversies, however, many people have found the concept of uncertainty about probability to be an appealing and useful one in practice. For example, Savage [31] acknowledges that "there seem to be some probability relations about which we feel relatively 'sure' as compared with others," and recognizes that there is "some temptation to introduce probabilities of a second order." Similarly, Good [13] finds that "type II probability is decidedly useful as an unofficial aid to the formulation of judgments of upper and lower probabilities of type I." He also acknowledges [14] that "it is not always possible to judge whether one subjective probability is greater than another." Along the same lines, Suppes [33] describes "some uneasiness with the problem of always asking for the next decimal of accuracy in the prior estimation of a probability." In response to the intuitive appeal of the idea, various approaches for dealing with the issue of uncertainty about probabilities have been proposed, and these are outlined in the following section.

A. Approaches for Addressing Uncertainty About Probabilities

Some analysts, torn between the theorists' prohibition of uncertain probabilities and a perceived need to address such uncertainty in practice, have adopted strategies that are essentially semantic in nature. For example, Koopman [21] replaces "the probability of a probability" by "the probabilities of different values of a physical parameter"; a similar approach is adopted by Apostolakis [2]. Along similar lines, Kaplan and Garrick [19] introduce a "probability of frequency" framework, in which the quantity about which we are uncertain is taken to be a long-run relative frequency, based on a (real or hypothetical) series of repeated trials, rather than a subjective probability. Such semantic approaches, while satisfying "the letter of the law" and possibly helping to avoid confusion in some instances, have in our opinion failed to give a clear explanation of exactly when a quantity ceases to be a probability and becomes a physical parameter or a relative frequency.

Others, such as Good [13], [14] and Howard [18], simply do not seem very troubled by the theoretical objections raised by de Finetti and others. They are content to allow the use of uncertain probabilities (at least on an ad hoc or informal basis) when practitioners find it useful, and do not offer detailed theoretical justifications for their approaches. Edwards [7] goes even farther, stating flatly that he does not see any fundamental difference between probabilities over probabilities and probabilities over events: "I see absolutely no formal difference between my opinions about the population parameter [i.e., the sampling probability] and about the identity of the next sample." The basis for this view seems to be that in both cases the probability reflects a personal judgment, and thus cannot be different in kind.

Finally, some theorists explicitly relax the axioms of probability and decision theory to accommodate uncertain probabilities. The earliest efforts along these lines involve the development of axioms for interval-valued probabilities. Axioms of this sort are presented by Koopman [21], as a method of deriving axioms for point-valued probabilities. Subsequent work by Good [13], Smith [32], and Suppes [33] explores the use of interval-valued probabilities as a way of "explaining the felt uneasiness of any attempts to seek exact measurements of belief" (Suppes [33]).

More recent work explores alternatives other than interval-valued probability. For example, Fishburn [10] develops axioms for comparative probabilities, and Nau [28] develops axioms for "indeterminate" probabilities, which he interprets as "type II probabilities" in the sense of Good [13]. In the area of decision analysis, Gärdenfors and Sahlin [12] propose a modified decision theory based on "epistemic reliability," which is based on the concept of uncertain probabilities. Similarly, Viscusi's "prospective reference theory" [34] extends conventional decision theory to handle risk perceptions that are characterized by probability distributions rather than single values, and Winkler [35] proposes incorporating preferences regarding ambiguous or uncertain probabilities into utility functions.

B. Theoretical Objections

In order to better understand the nature of the debate over uncertain probabilities, we must first explore the specific objections that have been raised by opponents of this concept. These fall into two major categories: arguments to the effect that allowing "unknown probabilities" would violate the subjectivist theory of probability; and concerns about possibly infinite hierarchies of probabilities.

1) Violation of the Subjectivist Theory: The argument that uncertain probabilities violate the subjective theory of probability is the primary concern motivating de Finetti and most other opponents of this approach. In particular, according to the subjectivist theory, probabilities are essentially functions of our (subjective) state of knowledge, not objective properties of the real world. Thus, for example, the probability of getting a "head" on the next toss of a coin is considered a property of the person whose opinion is being assessed, not a property of the particular coin in question, or even of the coin-tossing process.

Since probabilities are taken to be functions of our state of knowledge, rather than properties of the real world, different people can justifiably assign different probabilities to the same event, as long as their probability assessments are "coherent" (in other words, consistent with their states of knowledge). Some probability judgments may be based on more knowledge than others; for example, a political forecaster may have more knowledge than a layperson about the outcome of the next election. However, this does not make the layperson's assessment "wrong," at least for use in decisions by that individual.

Similarly, the axiomatic bases of subjective theory do not permit one to abstain from making probability judgments (e.g., on the grounds of not having enough information). For example, Savage [31] states that "The postulates of personal probability imply that I can determine, to any degree of accuracy whatsoever, the probability (for me) that the next president will be a Democrat." In this framework, not knowing what probability to assign to some event is similar to not knowing one's favorite color. One may be indifferent between two or more colors, or one's preferences may change over time, but at any given point in time one's preferences are assumed to be perfectly knowable. Since subjective probabilities can be derived from preference judgments (e.g., Savage [31], Luce and Krantz [23]), probabilities must likewise be perfectly knowable.

From this point of view, de Finetti's comments can be seen as an attempt to ensure compliance with the axioms. In particular, he is concerned that speaking of "unknown probabilities" will foster the viewpoint that "a subjective, or personal, probability in realistic circumstances is ... an 'estimate' of the 'true' (objective) value of an unknown probability." In fact, de Finetti believed that the subjectivist theory could not accommodate uncertainty about probabilities in any form. In this context, it is perhaps surprising that many thinkers favorably disposed toward the concept of uncertainty about probability (e.g., Good [13], [14], Howard [18], Edwards [7]) have not explicitly addressed de Finetti's concerns. In Section III below, we provide a partial resolution of those concerns by showing that they do not apply to one important type of uncertainty about probabilities; namely, that arising from uncertainty about underlying conditions.

2) Infinite Hierarchies of Probabilities: Some theorists who have argued against the use of uncertain probabilities have been disturbed by the possibility that there might be an infinite hierarchy of such probabilities. For example, Savage [31] postulates that "once second order probabilities are introduced, the introduction of an endless hierarchy seems inescapable. Such a hierarchy seems very difficult to interpret, and it seems at best to make the theory less realistic, not more."

The idea of an infinite hierarchy of probability distributions also troubles Levi [22], who resorts to a distinction between "chance" (i.e., objective) and "credal" (i.e., subjective) probabilities, and permits uncertainty only about chance probabilities. In our opinion, however, this distinction is subject to some of the same problems as the semantic approaches discussed earlier; in particular, it may be difficult to distinguish between chance and credal probabilities in practice.

Good [13] recognizes the possibility of an infinite hierarchy of probabilities, but does not find this prospect to be particularly troublesome. In fact, he observes that "the notion of an infinite sequence of types of probability does have the philosophical use of providing a rationale for the lack of precision of upper and lower probabilities." However, Good's approach to the issue is primarily intuitive and pragmatic, and he does not advocate the use of distributions over probabilities on a formal basis. In Section V, we present criteria for deciding when uncertain probabilities are potentially relevant, and we believe that according to these criteria, infinite hierarchies of probabilities will never be necessary in practice.

C. Sources of Uncertainty about Probability

The various theorists discussed above have generally failed to distinguish between different sources of uncertainty about probabilities. In particular, they have failed to distinguish between two very different reasons that judgments of subjective probability may be uncertain: uncertainty about the underlying events, propositions, or hypotheses on which those probabilities are conditioned; and imprecision resulting from inherent cognitive limitations.

Distinguishing between these two sources of uncertainty is an important step in resolving the persistent theoretical controversy over uncertainty about probability. In particular, due to the failure of past theorists to distinguish between conditional uncertainty and imprecision (discussed in Sections III and IV, respectively), none of the viewpoints put forth to date provide a complete explanation of the phenomenon of uncertain probabilities. We believe that the ideas presented in this paper provide a needed conceptual underpinning for such an explanation.

III. Uncertainty About Underlying Conditions

In our view, uncertainty about probabilities frequently arises from uncertainty about the underlying events, propositions, or hypotheses on which those probabilities are conditioned. In other words, if $P(E|X)$ is the probability of an event E given a set of conditions X, then the quantity $P(E|X)$ can be viewed as a function of X, which may itself be uncertain (and would therefore be a random variable). Since every function of a random variable is itself a random variable, it follows that $P(E|X)$ is a random variable and is therefore described by a probability distribution. In particular, one can construct the distribution for the quantity $P(E|X)$ from the following pieces of information: 1) the form of the function $P(E|X)$ and 2) the underlying probability distribution over X.

In this formulation, we agree with de Finetti that it is inconsistent with the subjective theory of probability to be uncertain about the value of $P(E|X)$ when the conditions X are fully known. Similarly, it is also inconsistent to be uncertain about the expected or unconditional probability of the event in question, $P(E)$, since the unconditional probability can be found by simply taking the expectation of $P(E|X)$ over all possible values of X. However, as will be shown in Section V, there are situations where use of the unconditional (or expected) probability is incorrect, and the conditional probability is needed. Since the conditions X may not always be known, it must clearly be admissible to be uncertain about the conditional probability $P(E|X)$ in such cases.

A. Example

As an illustration, let us consider an example in which the assessor is asked to give the probability of observing a head in the first toss of a fair coin. Coherence in this case requires the assessor's response to be 0.5. Let us now replace the coin and inform our subject that the coin has either two heads or two tails (each with probability 0.5). Again, we would expect the probability of observing a head in the next trial to be 0.5.

However, this value is the *expected* probability that the coin has two heads. The *conditional* probability of heads would be 1.0 if the coin had two heads, and 0.0 if the coin had two tails.

To formalize this, let X be the unknown condition of the coin; i.e.,

$$X = \begin{matrix} HH & \text{if two heads} \\ TT & \text{if two tails} \end{matrix}$$

Then the conditional probability of observing a head given X is

$$P(\text{head}|\, X = HH) = 1 \,(\text{with probability } 0.5)$$
$$P(\text{head}|\, X = TT) = 0 \,(\text{with probability } 0.5)$$

In other words, $P(\text{head}|X)$ is now a random variable taking on values 0 and 1, each with probability 0.5. This constitutes a probability distribution over the conditional probability of a head on the next toss.

Naturally, the unconditional probability of observing a head in the next trial is simply equal to 0.5, since this is the expected value of the conditional probability, $P(\text{head}|X)$, taken over all possible values of X. This is exactly the same as the unconditional probability of heads for the fair coin. However, there are clearly a great many decision problems (e.g., cases involving multiple tosses of the same coin) for which the two coins are far from equivalent. Some similar cases are discussed in Section V below.

B. Some Observations

The notion of uncertainty about underlying conditions is a simple but powerful explanation for the phenomenon of uncertainty about probability. In particular, we believe that uncertainty about probabilities in practice often arises from uncertainty about such underlying conditions. Since this explanation depends only on the concept of conditional probability, such conditional uncertainties are clearly consistent with the axioms of subjective probability theory. Therefore, uncertainties of this type do not necessitate the development of new axioms, as is done in some of the approaches discussed in Section II.

It is worth noting that this type of conditional uncertainty is neither a "higher" nor a "lower" order probability, but simply the probability of another event or verifiable proposition. Since the uncertainty in such cases stems not from some "failure of introspection" but rather from uncertainty about the conditioning event, the validity of a given conditional probability is therefore verifiable, based on whether the conditioning event actually occurs. Viewed from this perspective, conditional probabilities are verifiable propositions, about which uncertainty is permitted. (The case of imprecision, in which uncertainty arises from inherent cognitive limitations rather than lack of knowledge about underlying conditions, is discussed in Section IV.)

This approach is also fully consistent with the view adopted in the subjective theory of probability that probability assessments are conditional on the assessor's state of knowledge. In the proposed framework, in assessing a conditional probability $P(E|X)$, the overall uncertainty about whether E will occur is simply partitioned into two segments: 1) uncertainty about the underlying conditions, X; and 2) any remaining uncertainty about whether E will occur after the conditions X have been completely specified.

When assessors seem unable or unwilling to state a unique probability for an event, we believe that this is often because they intuitively feel that the uncertainty about the underlying conditions X should be taken into account, and hence are unwilling to integrate over this source of uncertainty to derive an expected or unconditional probability. Therefore, the assessor may prefer to give a range of probabilities for the event E, each of which could be correct depending on which set of conditions X is found to be applicable. An unconditional probability could be derived by the analyst in such cases if one was needed, based on the probability assigned to each set of conditions. (Of course, the probabilities of the conditions may also be uncertain, and so on, creating a possibly infinite hierarchy of probabilities, as discussed by Savage [30]. However, this is not a problem in practical decision making, for the reasons discussed in Section V.)

To summarize, if the only source of uncertainty is of the conditional type discussed here, uncertainty about probability does not violate the axioms of subjective probability theory (as given, for example, by Savage [31] or Luce and Krantz [23]). According to the proposed framework, any individual should be able to specify a unique unconditional probability for any well-defined event, and also a unique conditional probability, provided that all relevant conditions have been specified. This is sufficient to ensure compliance with the theory of subjective probability, since no set of axioms to date compels us to derive such probabilities in the face of unknown underlying conditions.

Of course, the axioms of the subjective theory of probability are violated by "ambiguity aversion," as discussed by Ellsberg [8]; e.g., if the subject prefers to bet on a single toss of a fair coin than a single toss of a coin that is equally likely to have two heads or two tails. However, cases of Ellsberg's paradox involve not only uncertainty about probabilities, but also an aversion to that uncertainty. Such aversion is not implied by the present framework, as discussed in Section V.

IV. Imprecision

In the above section, we have discussed uncertainty about probability arising from uncertainty about some underlying condition or event. However, there is another source of uncertainty or ambiguity, arising from cognitive limitations in our ability to assign unique numerical values to states of knowledge. As a result of these limitations, most probability judgments are subject to a certain degree of inherent imprecision, described by Suppes [33] as "inexact measurement" of beliefs. In other words, people are often unable to distinguish between different numbers intended to represent a given state of knowledge; the different probabilities simply don't "feel" sufficiently different. This can be viewed as analogous to an inability to distinguish between similar colors. In fact, a certain degree of imprecision is probably inherent in the capabilities of the human brain, just as any other physical system (e.g., a computer or a measuring device) is also subject to certain limitations on its numerical accuracy.

In particular, these cognitive limitations reflect the nature of our computational and perceptual abilities, and our methods of storing and retrieving information. Thus, for example, Suppes [33] states that:

> it is a mistake to think of beliefs as being stored in some fixed and inert form ... [T]he constructive processes in the case of belief ... are easily disturbed by slight variations in the situation in which the constructive processes are operating. This kind of view backs up the layman's view that it is ridiculous to seek exact measurements of belief ... [or] to ask for the next decimal in a measurement of subjective probability.

Similar problems also arise in the assessment of preferences. For example, Luce and Raiffa [24] point out that someone who is indifferent between a trip to Rome and a trip to Paris may also claim to be indifferent between a trip to Rome and a trip to Paris plus $20. However, that person would almost certainly prefer a trip to Paris plus $20 over a trip to Paris by itself. In this case, our representations of the attractiveness of trips to Paris and Rome are sufficiently imprecise that we cannot distinguish between differences on the order of $20, and different methods of elicitation may therefore give rise to different expressed preferences.

This type of imprecision is inherently context-dependent, since our abilities to represent and process information will vary from one context to another. Thus, in attempting to estimate the probability of getting two heads in a row from a coin where $P(\text{head})$ is equal to 0.537, our range of imprecision may be relatively narrow, subject only to limitations in our ability to multiply three-digit numbers in our heads. By contrast, in estimating the probability of an event that has never happened, the demands on our information-processing capabilities are much more complex, and the range of imprecision may therefore be much greater. Imprecision may also be motivation-dependent. In other words, we may be able to do a better job if we choose to; e.g., by performing a few simple calculations, or by introspecting more deeply.

Finally, calibration training may be able to reduce imprecision, at least in some contexts. However, the benefits of calibration training in one context may not generalize to other estimation tasks. Thus, weather forecasters may be well-calibrated with regard to the probability of rain tomorrow, but less well-calibrated with regard to nonmeteorological events. Similarly, risk analysts may be reasonably well-calibrated in estimating component failure probabilities on the order of 10^{-2}–10^{-4}, but may nonetheless be poorly calibrated for more commonplace events such as tomorrow's weather.

In contrast to the case of conditional uncertainty, as discussed in Section III, the axioms of subjective probability are not adequate to explain imprecision. Further research on possible axiomatic bases of such imprecision may therefore be worthwhile. For consistency with the treatment of conditional uncertainties presented in Section III, we would particularly favor axiomatic systems that yield probability distributions over imprecise probabilities (e.g., Nau [28]) rather than upper and lower probabilities.

In practice, we believe that most cases in which uncertainty about probabilities is large enough to be important are likely to reflect primarily the effects of conditional uncertainty rather than imprecision, and uncertainty due to imprecision will often be sufficiently small that it can simply be neglected. Therefore, even if one does not accept the use of probability distributions to represent imprecision, distributions over probabilities may still be useful in a wide range of cases.

In some contexts, however, the uncertainty due to imprecision may actually be as large as that due to lack of knowledge about underlying conditions, and imprecision might then need to be taken into account in decision making along with conditional uncertainty. We believe that probability distributions can be used in practice to represent imprecision in such cases. In order to understand our reasons for adopting this view, it is important to distinguish between two different approaches to axiomatizing the theory of probability. Early formulations of probability theory such as Kolmogorov's work (Kolmogorov [20], Fine [9]) started from certain mathematical axioms that probabilities were assumed to obey (e.g., nonnegativity, finite additivity, and so on). Other theorists (e.g., Savage [31], Ramsey [30], de Finetti [5]) developed axioms of choice from which the mathematical laws of probability could be derived. This work was essentially an attempt to axiomatize what is meant by "rational behavior."

While imprecision may violate the axioms of "rational behavior," we see no reason that it should not obey the mathematical axioms of probability, and thus no reason that probability distributions should not be used to represent imprecision. Therefore, at present we would recommend the use of probability distributions over alternative formalisms (e.g., interval-valued probabilities) to represent imprecision in practice, again for consistency with the conditional uncertainty discussed in Section III.

V. Relevance to Decision Making

At first glance, it may seem that uncertainty about probabilities, while theoretically valid, has no relevance to practical decision making. However, this is not true. In fact, uncertainty about a probability p will in general be relevant to decision making when the expected value of the relevant performance measure is nonlinear in p. Uncertainty about p is, of course, irrelevant when the expected values of all relevant performance measures are strictly linear in p. Henrion [16], [17] has noted that uncertainty is irrelevant when the decision to be made is selection of an estimator, and the decision criterion is minimization of a quadratic loss function; more general conditions under which uncertainty will be irrelevent have been identified by Bier [3].

However, performance measures for which uncertainty is relevant can arise in many ways. One is when the outcomes of interest are compound events whose probabilities must be computed from some underlying or elemental probability. Such cases arise frequently in risk and reliability analysis. For example, let the probability of a component failure on any particular trial be equal to p, and let p be uncertain. In this case, Howard [18] has shown that the expected probability of observing no failures in n trials will depend in general not only on the expected failure probability, $E(p)$, but on the first n moments of p. Therefore, two components that have the same expected failure probability, $E(p)$, but differing degrees of uncertainty about p (as reflected, for example, by the variance of p) will in general have different probabilities of performing successfully on n consecutive trials.

Similarly, one may have to consider uncertainty about probabilities in expected utility decision making (at least for nonexpected-value decision makers), even when the attribute over which a utility function is assessed is simply the number of successes in n trials. Hazen [15] has exploited this property to demonstrate that "ambiguity aversion" can be consistent with maximizing expected utility for risk-averse decision makers when the decision may have multiple implementations or trials.

It is interesting to note here that the component with the smaller uncertainty will not necessarily always be preferred. For example, the expected probability of observing two consecutive successes will in fact be greater for the component with the greater variance. This is analogous to the case of a risk-preferring utility function. In such cases, we do not generally prefer the uncertainty itself, but rather the chance for a better outcome. Thus, if there were a way to resolve the uncertainty before making a decision (e.g., by testing the more uncertain component to obtain a better estimate of its failure probability), we would in general be willing to pay for such information.

Another case where uncertainty about probability will be relevant is the value of additional information about the uncertain probability p. The value of additional information is a well-established concept in decision analysis (e.g., Morgan and Henrion [27]). However, decision analysts have typically assumed that the information to be obtained relates to some quantity other than a probability (e.g., market share, or the cost of electric power). When the new information instead relates directly to a probability (e.g., the probability of success in a business venture, or the probability of an explosion at a chemical factory), then the prior uncertainty about that probability will clearly be relevant in determining the value of the information. For example, let us assume that the decision maker will proceed with the proposed venture only if p is greater than some threshold value, p_0. In this case, the value of additional information about p will depend on: 1) the probability that p will exceed p_0; and 2) the conditional expectation of p, given that it exceeds p_0.

Uncertainty about probabilities is obviously also relevant in incorporating new information via Bayes' theorem, as noted by observers such as Howard [18] and Press [29]. Thus, the effect of observing a particular event on our estimate of its probability will vary, depending on how uncertain we were about the probability in the first place. For example, if the event is one that we know for certain to be quite unlikely (e.g., a meteorite strike), our mean estimate of its probability might not change very much even if several such events actually occurred. We would simply interpret the evidence as an unlucky coincidence. By contrast, if we had initially been very uncertain about the probability of the event (as might be the case, for example, about events such as human error or sabotage), then our mean estimate of its probability might increase quite substantially after only a single observed occurrence. One example of this is the analysis of accident precursors presented in Bier and Mosleh [3].

Finally, uncertainty about probabilities can be relevant even if the performance measure of interest is linear in each individual probability, but involves the product of two or more uncertain probabilities that depend on the same unknown condition (and hence is nonlinear in that respect). To see this, consider the joint probability that a new business venture will be successful, and a Republican will be elected president in 1996. Let us assume that the probabilities of both events depend on the gross national product (GNP) during 1996, but that the events are conditionally independent for any given level of GNP. If the GNP is known, then one can simply multiply the probability of business success by the probability of a Republican president (both conditional on the level of GNP) to obtain the desired joint probability.

However, if the level of GNP is unknown, then the two required conditional probabilities will be uncertain. In general, these uncertain probabilities will also be correlated with each other, since the level of GNP is likely to induce some dependence between the two events. The usual way of dealing with this is, of course, to replace the unconditional probability of a Republican president by the corresponding conditional probability, conditioned on the success of the new business. However, an alternative approach would be to specify the joint distribution of the two uncertain probabilities, and compute the expected value of their product. Fryback *et al.* (in press) discuss this phenomenon in the context of medical decision making, and show that "The resulting error [e.g., if the prior probability of disease is correlated with the probability of detection] may affect computed expected utilities sufficiently to change decision recommendations."

To summarize, with few exceptions, uncertainty about probabilities will be relevant to decision making when the relevant performance measure is nonlinear in some uncertain probability, p, and also when the performance measure involves the product of two or more uncertain probabilities that are correlated with each other. Note that this explicitly excludes "ambiguity aversion" as discussed by Ellsberg [8]; i.e., aversion to uncertain probabilities in bets on one-time events. Thus, one can accept uncertainty about probability without necessarily accepting ambiguity aversion.

Of course, the importance of uncertainty in practice will also depend on the extent of the uncertainty, the degree of nonlinearity in the relevant performance measure, and the importance of the decision to be made (i.e., whether the potential stakes are large or small). In many cases, the impact of the uncertainty may be sufficiently small to be neglected in practice.

These criteria also allow us to put to rest Savage's concern [31] that "once second order probabilities are introduced, the introduction of an endless hierarchy ... seems very difficult to interpret." Under the framework proposed here, an infinite hierarchy of probabilities is certainly possible. However, based on the criteria outlined above, it is difficult to imagine situations in which anything higher than second-order probabilities (or perhaps at most third-order probabilities) would actually be needed in decision making. As Good [14] remarked, "I stop [defining higher-order probability distributions] when the expected utility of going further becomes negative if the cost is taken into account," and this expected utility must clearly depend on the types of factors discussed. It is hard to imagine realistic performance measures that would be nonlinear in third- or higher-order probabilities (using Savage's terminology). Thus, while an endless hierarchy of probabilities is theoretically possible under the proposed framework, we disagree with Savage that such a hierarchy would pose difficulties in any practical decision problem.

VI. Conclusions

This paper distinguishes between two possible reasons that judgments of subjective probability may be uncertain: uncertainty about the conditions underlying judgments of probability; and imprecision resulting from inherent cognitive limitations. We have shown that the first type of uncertainty is consistent with the axioms of subjective probability theory, but these axioms are not adequate to explain cognitive imprecision. Therefore, from a theoretical perspective, there is a place for new axiomatic bases for subjective probability that would account for imprecision.

However, theorists commenting on the issue of uncertainty about probabilities have failed to distinguish between these two sources of uncertainty, giving rise to confusion. For example, strict subjectivists such as de Finetti objected primarily to those uncertainties arising from imprecision. de Finetti implicitly recognized that conditional uncertainty exists (e.g., in his ball and urn example), but apparently failed to recognize the widespread importance of such uncertainties in practice. Because conditional uncertainties are virtually ubiquitous, the terminology recommended by de Finetti and others (e.g., Kaplan and Garrick [19]) rapidly becomes cumbersome and unwieldy.

On the other hand, the "pragmatists," such as Good and Edwards, do not explicitly address the concerns raised by de Finetti and Savage. In particular, these thinkers have not addressed de Finetti's view that uncertainty about probability violates the axioms of subjective probability. Finally, new axiomatic approaches (e.g., interval-valued probabilities) could potentially explain the phenomenon of imprecision, but have nothing to say about conditional uncertainties. The focus of many researchers on the development of new axiomatic systems prevented the realization that uncertainty about probability was in fact already consistent with subjective probability theory in a great many practical cases. The question of imprecision is still of theoretical interest, but we believe has relatively little practical significance in most cases.

The failure to distinguish between conditional uncertainty and imprecision in the theoretical literature has also posed problems for those practitioners who were faced with large conditional uncertainties; e.g., in fields like risk analysis. Practitioners in such fields have often had correct intuitions about the need to address conditional uncertainties in their analyses. However, they frequently labored under the misimpression that this was somehow theoretically inappropriate. In addition, they were faced with nearly complete lack of guidance on how and when to address such uncertainties.

In this paper, we have provided guidance on the conditions under which uncertain probabilities are relevant to decision making. In general, uncertain probabilities will be relevant when the performance measure of interest is in some way nonlinear in the uncertain probabilities. Of course, in practice the importance of taking such uncertainties into account will also depend on the magnitude of the uncertainties, the extent of the nonlinearity in performance measures, and the importance of the decision to be made.

Based on an implicit recognition of these criteria, the importance of uncertain probabilities has already been recognized in some fields, such as risk analysis of nuclear power plants. However, current practice in other probabilistic disciplines should be reviewed according to these criteria, to assess the need for explicit treatment of uncertain probabilities in those fields.

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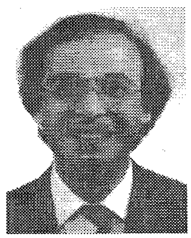

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