Exercises

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Exercise 1: Bloch simulator Write a program to solve Bloch's equations (simple Euler integration is fine). It should take as inputs

$$\begin{aligned}
\mathbf{M}_{\text{init}} &- \text{initial magnetization} \\
T_1, T_2 &- \text{relaxation rates} \\
\omega_1(t) = \gamma B_1(t) &- \text{RF amplitude as function of time} \\
\delta\omega(t) = \omega_0 - \omega_{\text{rf}} &- \text{off-resonance frequency} \\
\phi &- \text{phase of } \mathbf{B}_1(t) = B_1(t)\Big(\cos(\phi)\hat{x}' + \sin(\phi)\hat{y}'\Big) \\
T &- \text{the total simulation time}
\end{aligned}$$

and output the magnetisation $\mathbf{M}(t)$ as function of time t. It is helpful first to rewrite Bloch's equations in matrix form. You can use a time step of $dt = 0.0001$ms, and $T_1 = T_2 = 1000$ms unless otherwise stated.
Check your code by simulation of a π-pulse and $\pi/2$-pulse if $T = 1$ ms duration, $\delta\omega = 0$ and various choices of ϕ in $\{0, \pm\pi/2, \pi\}$.
In the following you can use the values below (unless otherwise stated):

$$t_{\text{rf}} = 1 \text{ ms} \tag{1}$$
$$T_E = 16 \text{ ms} \tag{2}$$
$$N = 128 \tag{3}$$
$$L = 10 \text{ cm} \tag{4}$$
$$T_R = 35 \text{ ms} \tag{5}$$

Exercise 2: Slice selection This exercise illustrates "slice selection" in 1D for a sample $M_0(x) = 1$ with $x \in [-L, L]$. Take e.g. $L = 10$ cm. Implement the following RF pulse in the rotating coordinate system:

$$\mathbf{B}_1(t) = \begin{cases} b_1 \frac{\sin(2\pi t/t_w)}{2\pi t/t_w}, & \text{for} - \frac{1}{2}t_{\text{rf}} \le t \le \frac{1}{2}t_{\text{rf}} \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

Here $t_{\text{rf}} = n_z t_w$ is the total duration of the rf-pulse, where n_z is the number of zero-crossings. This is illustrated in Fig. 1, with $n_z = 10$.

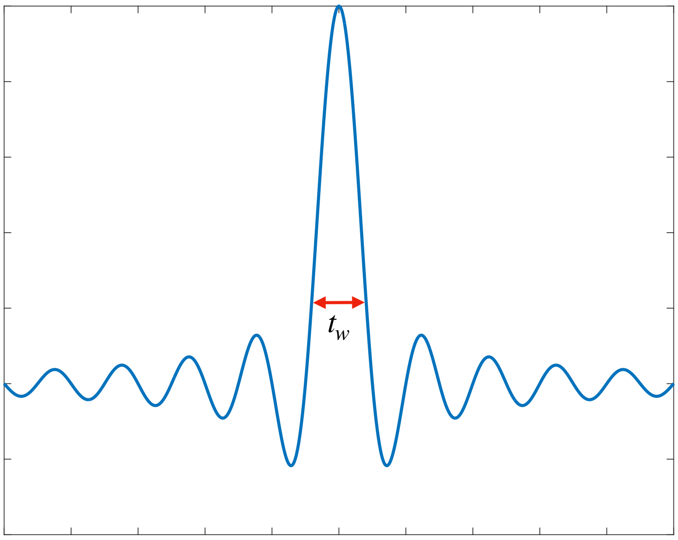

Figure 1: Illustration of $\frac{\sin(2\pi t/t_w)}{2\pi t/t_w}$. t_{rf} is the entire duration of the waveform.

1. Find the value of b_1 that produces a $\pi/2$-pulse with $n_z = 1$ and $t_w = 1$ ms.

Now add a gradient G so

$$B_z(x) = B_0 + Gx \tag{7}$$

with

$$G = \frac{2\pi}{\gamma t_w \Delta x}, \quad \Delta x = 2\,\text{mm}, \tag{8}$$

This produes a frequency offset depending on x as

$$\delta\omega(x) = \gamma G x. \tag{9}$$

Leave the gradient on for $-\frac{1}{2}t_{\text{rf}} \leq t \leq \frac{1}{2}t_{\text{rf}}$.

2. Plot the magnetization profile $|M_+(x,t)|$ vs. x at $t = t_{\text{rf}}/2$.
3. How is the profile affected when you vary you double or halve the gradient strength?
4. How would you spatially shift (translate) the slice?
5. How is the profile affected when you vary t_w?
6. How is the profile affected when you vary n_z?

1D imaging In this part we simulate a $1D$ imaging experiment based on a gradient echo sequence, see Fig. 2.
We consider an ideal $\pi/2$ pulse at $t = 0$, infinitely short. Then we have the solution to the Bloch Equation at $t > 0$

$$\begin{aligned} M_+(\mathbf{r},t) &= M_+(\mathbf{r},0)\mathrm{e}^{-t/T_2}\mathrm{e}^{-i\theta(t)} \\ &= M_+(\mathbf{r},0)\mathrm{e}^{-t/T_2}\mathrm{e}^{-i\int_0^t \omega(t')\,dt'} \end{aligned} \tag{10}$$

For the sequence in Fig. 2, we find for $t_g + T_E/2 < t \le 3T_E/2$

$$\begin{aligned}\int_0^t dt'\,\omega(t') &= \omega_0 t - \gamma G T_E/2x + \gamma G x(t - t_g - T_E/2)\\ &= \omega_0 t + \gamma G x(t - (t_g + T_E))\\ &= \omega_0 t + kx, \quad \text{with } k \equiv \gamma G(t - (t_g + T_E)) = k(t). \end{aligned} \tag{11}$$

Thus

$$M_+(\mathbf{r}, t) = M_+(\mathbf{r}, 0)\mathrm{e}^{-t/T_2}\mathrm{e}^{-i\omega_0 t - ikx}, \tag{12}$$

With $t \ll T_2$, after demodulation (multiplication by $e^{i\omega_0 t}$) and in 1D

$$M_+(x, t) = M_+(x, 0)\mathrm{e}^{-ikx} \tag{13}$$

The signal $S(t)$ is just the "volume" integral of $M_+(x, t)$

$$S(t) = \int dx\, M_+(x, t)$$

or in terms of k

$$S(k) = \int dx\, M_+(x, 0)\mathrm{e}^{-ikx} \tag{14}$$

In many real sequences, often $t_g \ll T_E \ll T_2$, so

$$M_+(x, t) \approx M_+(x, 0)\mathrm{e}^{-T_E/T_2}\mathrm{e}^{-ikx} \tag{15}$$

$$\Rightarrow S(k) = \int dx\, M_+(x, 0)\mathrm{e}^{-T_E/T_2(x)}\mathrm{e}^{-ikx} \tag{16}$$

This means that if we take the Fourier Transform of our signal $S(k)$, we get our "image" $M_+(x, 0)\mathrm{e}^{-T_E/T_2(x)} \propto \rho(x)\mathrm{e}^{-T_E/T_2(x)}$. **See the "Introduction to MR for physicists" note for some things to keep in mind concerning discrete sampling.**

Exercise 3: 1D Imaging Simulation The idea of this exercise is to numerically verify the Fourier relationship in Eqs. 14-15. As "ground truth", we can take for example a profile of the Modified Shepp-Logan phantom. For instance in Matlab

```
M0 = abs(phantom('Modified Shepp-Logan',128));
M0 = M0(64,:)
```

See Fig. 3.

1. Show that the signal is the Fourier transform of M_0. E.g. by comparing $S(k)$ with `fft(`M_0`)`.
2. What happens if you decrease T_2?

Now take $t_g = 10$ ms, $M_0(x) = 1$ and take $T_2(x) = 15$ ms$+ax$, with $a = .5$ms/cm and take Fourier transform of your signal. What are we seeing? This is a sequence to generate T_2 contrast.

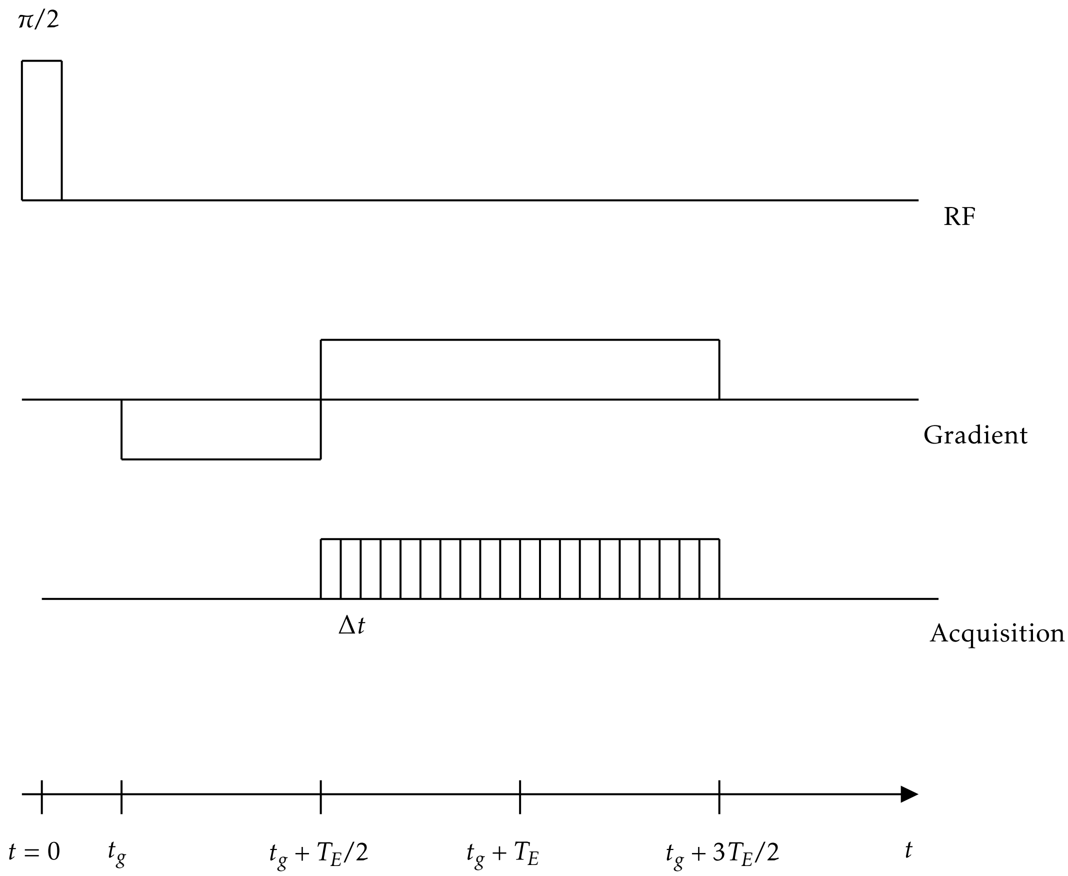

Figure 2: RF pulse, gradient and acquisition.

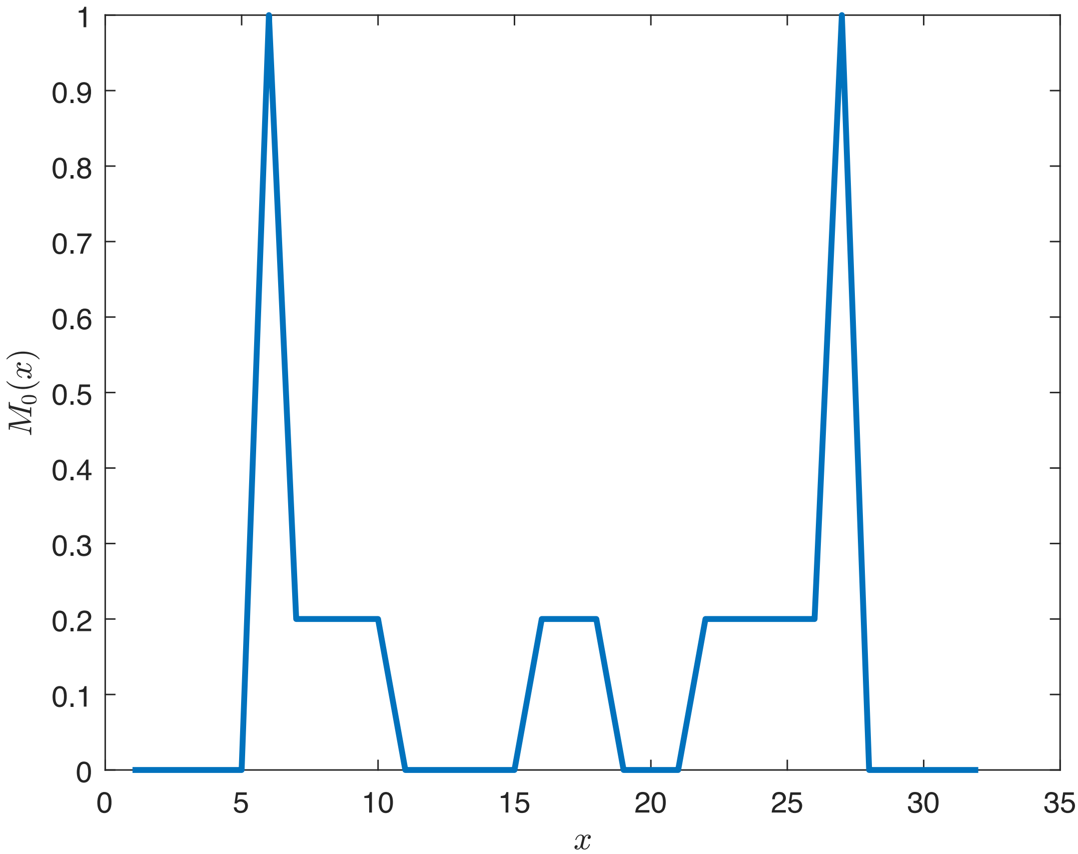

Figure 3: Profile