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Final CS 6210

1. Argue that the iteration:

$$M_{k+1} = A^T M_k A + I$$

converges to the unique solution to $M - A^T M_k A = I$ when $\|A\|_2 < 1$. Also argue that if M_0 is symmetric and positive definite then each successive iterate M_k is symmetric and positive definite.

Let M denote the solution to:

$$M - A^T M_k A = I$$

We take the iteration which is equal to

$$M_{k+1} - A^T M_k A = I$$

And we subtract the first equation from the second to get the solution which is then:

$$(M_{k+1} - M) - (A^T M_k A - A^T M_k A) = 0$$

Define:

$$E_{k+1} = M_{k+1} - M$$

And

$$E_k = M_k - M$$

And transforming the equation above we get:

$$E_{k+1} = A^T E_k A$$

Which implies when we are taking the norm of both sides that

$$E_{k+1} = A^T E_k A$$

where we consider the fact that to the x can be bounded by a chain where x the maximizer of Ax becomes $\hat{b}$ which when normalized is the maximizer of E_k and finally $\hat{c}$ equal to the product of E_k when normalized is the maximizer of A^T.

$$\max_{||x||=1} \|E_{k+1}x\| = \max_{||x||=1} \|A^T E_k A x\| \le \left|\left|A\right|\right| \left\|A^T E_k \hat{b}\right\| \le \left|\left|A\right|\right| \|E_k\| \|A^T \hat{c}\| \le \|A^T\| \|E_k\| \|A\|$$

Hence we have that

$$\|E_{k+1}\| \le \|A^T\| \|E_k\| \|A\|$$

We are given that $\|A\| < 1$ which implies the $\sigma_{A,max} = \|A\| = \sigma_{A^T,max} = \|A^T\| < 1$ So we conclude that the norm of

$$\|E_{k+1}\| \le \|A^T\| \|E_k\| \|A\| < \|E_k\|$$

Goes to zero and hence we have that M_{k+1} convergences to the solution.

We get uniqueness of the solution by using the Banach Fixed Point Theorem, where we define g to be the operator $g(M_k) = A^T M_k A + I$. We then quote the uniqueness of the fixed point iteration to give us that the convergence of the iteration and the solution are unique.

We show that if M_0 is symmetric and positive definite then each successive iterate M_k is symmetric and positive definite.

First notice that if M_k is symmetric and positive definite then it admits a Cholesky factorization of the form: $M_k = LL^T$ and so we have that:

$$M_{k+1} = A^T LL^T A + I$$

We see that if $W = L^T A$ then

$$M_{k+1} = W^T W + I$$

We know that $W^T W$ is symmetric and that addition of symmetric matrices are symmetric so M_{k+1} is symmetric, now we consider whether it is spd.

We consider the matrix

$$\begin{bmatrix} M_k^{-1} & A \\ A^T & I \end{bmatrix}$$

We immediately see that the above matrix is SPD since if we subtract λI off the diagonal where λ is an eigenvalue of M_k^{-1} which is SPD since inverse of SPD is SPD or 1 the eigenvalue of the identity, then the matrix is singular, hence all the eigenvalues of the matrix are postivie, and the matrix is symmetric so it is spd. Any schur complement of an spd matrix is also spd and so we find the Schur complement which is

$$\begin{bmatrix} M_k^{-1} & A \\ 0 & I + A^T M_k A \end{bmatrix}$$

And so we have that

$$I + A^T M_k A = M_{k+1}$$

is symmetric and positive definite.

2. Write an $O(n \log n)$ iterative solve for $Ax = b$ for any shifted Hankel Matrix $A \in R^{nxn}$ of the form $a_{ij} = d_{ij} + u_{i+j-1}$ where $\|u\|_1 = \frac{1}{2}$. Your method should converge in the ∞ norm at a rate independent of n. Your code should take the form ….

My analysis for the problem shows that A can be written in the form:

$$A = I + U_{ij}$$

where U_{ij} can be written as:

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_n \\ & u_3 & & \\ & & \ddots & \\ u_n & \cdots & \cdots & u_{2n-1} \end{bmatrix}$$

and so the A matrix takes the form:

$$\begin{bmatrix} u_1 + 1 & u_2 & \cdots & u_n \\ & u_3 + 1 & & \\ & & \ddots & \\ u_n & \cdots & \cdots & u_{2n-1} + 1 \end{bmatrix}$$

We propose using Jacobi iteration on the matrix where we take

$$A = M - N$$

Where $M = D$ the diagonal matrix of A and $N = L + U$ where L is the strictly lower triangular parts of A and U is the strictly upper triangular parts of A and so we get the iteration:

$$x_{k+1} = x_k + M^{-1}(b - Ax_k)$$

$$x_{k+1} = x_k + D^{-1}(b - Ax_k)$$

We assume in the above that multiplication by A takes $O(n \log n)$ for the moment. I will later show how to do this. Continuing then we have that

The error matrix is given by

$$R = M^{-1}N$$

$$R = D^{-1}(L + U)$$

We are given that $\|e_{k+1}\| \leq \|R\|\|e_k\|$. We now attempt to bound the the infinity norm $\|R\|_\infty$ given that $\|u\|_1 < 1/2$.

This requires us to bound the absolute value sum of each row j by 1.

Suppose the max row is j, then we are given that:

$$\sum_{\substack{i=0 \\ i \neq j-1}}^{n-1} \left|\frac{u_{j+i}}{1 + u_{2j-1}}\right|$$

then we have that:

$$\sum_{\substack{i=0\\i\neq j-1}}^{n-1} \left|\frac{u_{j+i}}{1+u_{2j-1}}\right| = \frac{1}{1+u_{2j-1}}\sum_{\substack{i=0\\i\neq j-1}}^{n-1} |u_{j+i}| < 1$$

The value $1 + u_{2j-1}$ is strictly greater than ½ since $|u_{2j-1}| < 1/2$ and so we can pull it out. We see then that the numerator is bounded by ½ and So then we get that and so it converges. However what we really want is that is bounded by something less than 1 and so we can see that the convergence is upper bounded by some fixed number of iterations given a fixed initial guess error.

We see that the maximum value that R can take on is:

$$\frac{1}{1+u_{2j-1}}\sum_{\substack{i=0\\i\neq j-1}}^{n-1} |u_{j+i}|$$

Without loss of generality $u_{2j-1} > 0$, $\sum_{\substack{i=0\\i\neq j-1}}^{n-1} u_{j+i}$ where the summands are greater than 0. We can set $x = u_{2j-1}$ and $y = \sum_{\substack{i=0\\i\neq j-1}}^{n-1} u_{j+i}$ where x+y=1/2, since we are trying to create the greatest value in this row and where we are maximizing now:

$$\frac{y}{1+x}$$

And for the bounds we put up we see that the maximum of this function where x>0 and y>0 is 1/2 . Hence we have achieved a bound on the norm of R, namely ½ and so we are guaranteed convergence in a fixed number of steps independent of n.

The fast matrix multiplication by A is given by doing the following. First the identity matrix takes no time to multiply and the Hankel matrix vector product can be found by getting a Toeplitz matrix flipud(U) and then multiplying by the flipud(x) multiplying using Toeplitz fast matrix multiply in Homework 1 and then flipping the result.

3. Write a routine to solve

$$\min_{x\neq 0}\frac{x^TAx}{x^TMx}\ st.\sum_j x_j = 0$$

We solve by taking the Cholesky decomposition of $M = LL^T$ which simplifying yields:

$$\min_{x\neq 0}\frac{x^TAx}{x^TLL^Tx}\ st.\, C^Tx = 0, and\ C^T = [\,1\ 1 \ldots 1\,]$$

and we denote = $y = L^Tx$ and so we get:

$$\min_{x\neq 0}\frac{y^TL^{-1}AL^{-T}y}{y^Ty}\ st.\, C^TL^{-T}y = 0$$

To get rid of the constraint I use the null space method.

$$D = C^T L^{-T} = [Q_1 \; Q_2] \begin{bmatrix} R_1 \\ 0 \end{bmatrix}, y_p = Q_1 R_1^{-T} 0 = 0.$$

We take Q_2 as a reduced basis for the space. We write $z = Q_2 y$, and our goal is to minimize:

$$minimize \; \frac{z^T Q_2^T L^{-1} A L^{-T} Q_2 z}{z^T z}$$

Which is equivalent to an eigenvalue problem where the stationary points are given by the eigenvectors of $Q_2^T L^{-1} A L^{-T}$. So we attain the eigenvectors, check for minimum and then back solve for y and then x by solving linear systems.

4. Interesting involutions. Let the matrix $A \in R^{nxn}$ be an involution $A = A^{-1}$ be orthonormal bases with associated invariant subspaces of A with eigenvalues of 1 and -1, respectively. Given the bases, write a routing to compute a Schur decomposition of

$$A[U \; W] = [U \; W] \begin{bmatrix} I & X \\ 0 & -I \end{bmatrix}$$

We first need to orthongalize V against U. The best way to do this is to subtract off the projection of W onto the row space of A which is achieved by subtracting from V U*the projection of V onto U attained using the pseudo inverse of U. One then orthongonalizes to an othonormal basis by computing the QR decomposition of the resulting matrix. We are now provided that Q is orthonormal basis and is orthogonal to U. Hence we set W = Q. We can then solve for X. Ultimately we get:

$$T = (I - U(U^T U)^{-1} U^T V) R^{-1}$$

$$T = QR$$

$$W = Q = TR^{-1}$$

From the matrix equation of Schur decomposition:

$$AU = U$$

$$AW = UX - W$$

$$AW + W = UX$$

$$X = U^{-1}((A + I)W)$$

$$X = U^{-1}((A + I)[VR^{-1} - U(U^T U)^{-1} U^T V R^{-1})$$

Since VR^{-1} is in the span of V, and $A + I$ sends these vectors to zero by def of eigenspace of eigenvalue -1 we see that $(A + I)VR^{-1} = 0$ and so X becomes noting AU=U

$$X = -2U^{-1}((U(U^T U)^{-1} U^T V R^{-1})$$

which we can solve for directly.

5. Suppose $Ax = b$ and x satisfy the constraint that $\sum_j \ x_j = 1$ where x is in the Kyrlov subspace $K_k(A, b)$

We run the Arnoldi code to calculate a basis for Q_k for $K_k(A, b)$ and we form the Kuhn tucker minimization equations and solve with appropriate constraint.

We attain Q_k from Arnoldi and then

$$argmin \left|\left|AQ_k y - b\right|\right|$$

Subject to $C^T Q_k y = 1$ where $C^T = [1 \ldots..1]$.

Denoting $D = AQ_k$ and $Y^T = C^T Q_k$. Solve the marix

$$\begin{bmatrix} D^T D & Y \\ Y & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} D^T b \\ 1 \end{bmatrix}$$

We then solve for $x = Q_k y$

Normally, if you wanted to make this faster you would exploit the structure of the Hessenberg factorization to bring the complexity down from $O(k^3)$ to $O(k^2)$ when solving.