Nathaniel Rogalskyj November 29, 2016

Math 6230 Problem 5

Free Will Versus Efficiency

We assume that there is a one-dimensional beach where some parts are naturally more attractive than other parts. We denote the natural beauty as $N(x) = e^{-(x-0.5)^2}$. Furthermore the beauty of the beach is affected by the density at the position x given by C(q)>0 known as the crowding penalty. C(q) satisfies the conditions C(0)=1 and C'(q)<0 for all q>0. For this experiment we will use $C(q) = \frac{1}{1+q^2}$. We will also use $Q(x) = \int_0^x q(s)ds$ to denote the number of people on the interval [0,x] and focus on how the solution changes as we vary the total number of beachgoers Q(1)=P. Note that Q(0)=0 and Q(1)=P

(a) Suppose each beachgoer selects his own spot individually. Explain why people will keep moving around unless

$$\frac{d}{dx}A(x) = N'(x)C\big(q(x)\big) + N(x)C'\big(q(x)\big)q'(x) = 0 \; holds \; on \; x \in [0,1]$$

If the attractiveness A(x) is not constant with respect to position, then people will keep moving to the more attractive spot. The equality is found by apply the chain rule to A(x).

(b) Convert this to a two point BVP for Q(x) and solve it in Matlab for P=1, 0.95, 0.9 as far as you can go. Use uniform distribution as the initial guess.

We have that

$$N'(x) = (1-2x)e^{-(x-0.5)^2}$$

and that

$$C'(q) = \frac{-2q}{(1+q^2)^2}$$

So we get the following:

$$(1-2x)e^{-(x-0.5)^2}\frac{1}{1+q(x)^2} + e^{-(x-0.5)^2}\left(\frac{-2q(x)}{(1+q(x)^2)^2}\right)q'(x) = 0$$

Implies that:

$$[(1-2x)(1+q(x)^2) - 2q(x)q'(x)] = 0$$

We substitute q(x)=Q'(x) and q'(x)=Q''(x).

$$[(1-2x)(1+Q'(x)^2) - 2Q'(x)Q''(x)] = 0$$

Solving for $Q''(x)$ we get:

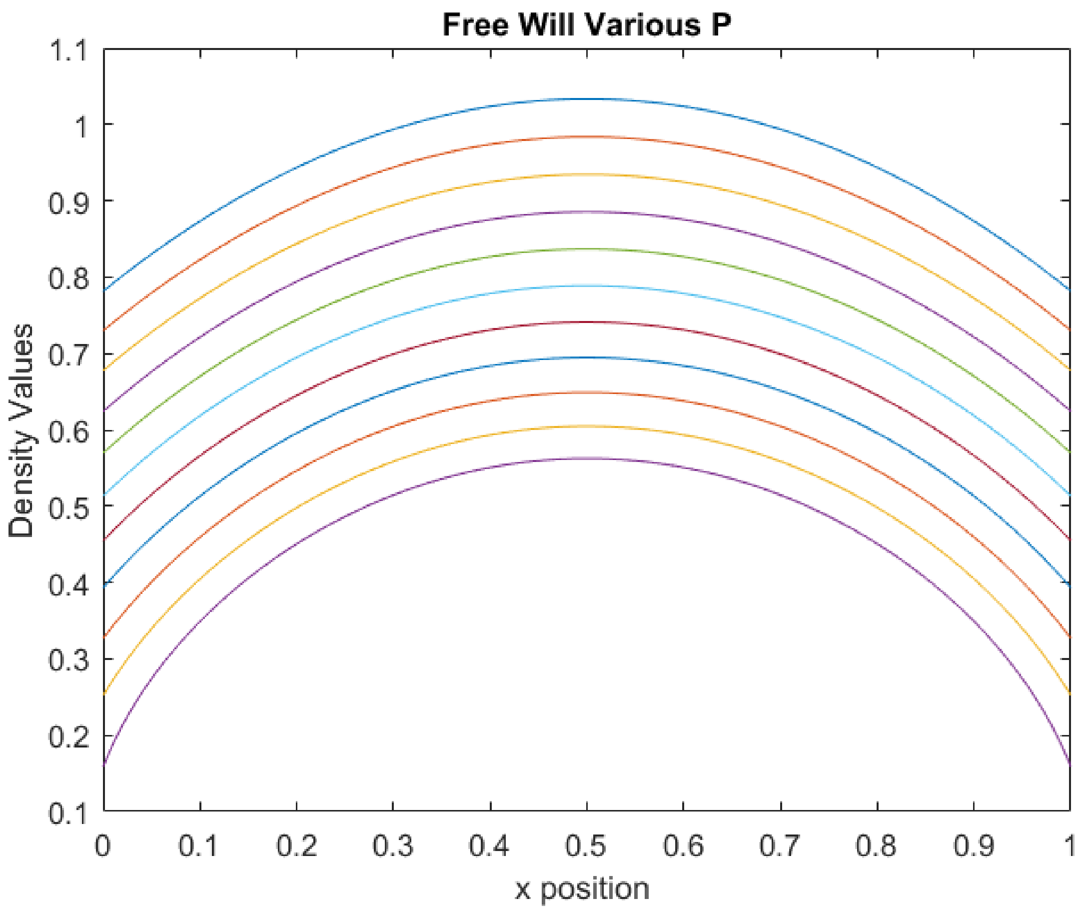
$$Q''(x) = \frac{(1-2x)(1+Q'(x)^2)}{2Q'(x)}$$

We let $v = Q'(x)$ and $u = Q(x)$. Solving we get the following system which we plug in to the Matlab BVP4C for solving:

$$\dot{u} = v$$
$$\dot{v} = \frac{(1-2x)(1+v^2)}{2v}$$

With U(0)=0 and U(1) = P.

We get the following:



The various curves correspond to various P values. The P values going from highest to lowest are:

```
P_list=[0.95, 0.90, 0.85, 0.80, 0.75, 0.70, 0.65, 0.60,
0.55, 0.50, 0.45]
```

For smaller values of P, the density becomes smaller both due to less People and a smaller penalty effect so the density at the ends becomes small at the ends of the interval, and we see that for small v, there is blowup on the right hand side when evaluating the BVP so the method becomes unstable.

(c) It is natural to define the average satisfaction of a beachgoer for each distribution:

$$S[q] = \frac{\int_0^1 N(x)C(q(x))dx}{\int_0^1 q(x)dx}$$

Compute $s[\tilde{q}]$ for each value of P you could handle.

The values of $s[\tilde{q}]$ are:

Values of P	Average Satisfaction Free Will	Average Satisfaction Benevolent
0.45	0.4835	0.4848
0.50	0.5081	0.5096
0.55	0.5338	0.5356
0.60	0.5605	0.5626
0.65	0.5881	0.5905
0.70	0.6165	0.6193
0.75	0.6454	0.6488
0.80	0.6746	0.6787
0.85	0.7037	0.7089
0.90	0.7323	0.7389
0.95	0.7597	0.7684

(d) Suppose now that the location of each individual is prescribed by a benevolent dictator, who strives to maximize the average satisfaction. Use the “calculus of variations” to show an optimal distribution would have to satisfy:

$$\frac{d}{dx}[N(x)C'(q(x))q(x) + C(q(x))]=0$$

We proceed in the following fashion:

First fix P. We wish to minimize the following function

$$S[q] = \frac{\int_0^1 N(x)C(q(x))q(x)dx}{\int_0^1 q(x)dx}$$

and q is function defined on [0,1] such that $\int_0^1 q(x)dx = P$. This is an integral constraint and not admissible to the general Calculus of Variations Framework as with the Brachistochrome Problem. We instead formulate the Problem as a Boundary value Problem where we have that we would like minimize over Q(x) where Q(0)=0 and Q(1)=P. $Q(x) := \int_0^x q(s)ds$

$$Y[Q] = \frac{\int_0^1 N(x)C\left(Q'^{(x)}\right)dx}{\int_0^1 Q'(x)dx} = \frac{\int_0^1 N(x)C\left(Q'^{(x)}\right)dx}{P}$$

We see that Because P is fixed the functional of actual concern is:

$$S[Q] = \int_0^1 N(x)C\left(Q'^{(x)}\right)dx$$

where Q has fixed endpoints: Q(0)=0 and Q(1)=P

Define

$$J\left(x, Q(x), Q'(x)\right) = N(x)C\left(Q'^{(x)}\right)$$

I walk through the derivation of the Euler-Lagrange Equations for my own enjoyment and for my own practice.

I consider the general problem find f such that f(a)=c and f(b)=d and I wish to maximize the integral constraint:

$$W(\varepsilon) = J[f + \varepsilon\eta] = \int_a^b F(f + \varepsilon\eta, f' + \varepsilon\eta', x)dx$$

I consider adding a small εη to f where η is admissible and I take a derivate with respect to ε and set it equal to zero, representing there is “zero” slope in function space in the functional for a at the maximum. (extremal).

$$\frac{d}{d\varepsilon}W(\varepsilon)_{\varepsilon=0} = 0$$

$$\frac{d}{d\varepsilon}W(\varepsilon)_{\varepsilon=0} = \frac{d}{d\varepsilon}\left(W(0) + W'(\varepsilon)\varepsilon + O(\varepsilon^2)\right)_{\varepsilon=0} = W'(\varepsilon)$$

$$W'(\varepsilon) = \frac{d}{d\varepsilon}J[f + \varepsilon\eta]_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_a^b F(f + \varepsilon\eta, f' + \varepsilon\eta', x)dx_{\varepsilon=0} = 0$$

$$\int_a^b \frac{d}{d\varepsilon}F(f + \varepsilon\eta, f' + \varepsilon\eta', x)_{\varepsilon=0}dx$$

$$\int_a^b \frac{dF}{da}\eta + \eta'\frac{dF}{db}dx = 0$$

$$\frac{dF}{da}\ \textit{represents derivate with respect to first argument}$$

$$\frac{dF}{d\mathrm{b}}\ represents\ derivative\ with\ respect\ to\ second\ argument$$

Integrate by parts on the second term to get:

$$\int_a^b \frac{dF}{d\mathrm{a}}\eta + \frac{\mathrm{d}}{\mathrm{dx}}\frac{dF}{d\mathrm{b}}\eta\, dx = 0$$

We see that if this were to hold for all possible η then:

$$\frac{dF}{d\mathrm{a}} + \frac{\mathrm{d}}{\mathrm{dx}}\frac{dF}{d\mathrm{b}} = 0$$

We apply the Euler-Lagrange Equation to J and we get that:

$$\frac{\mathrm{d}}{\mathrm{dx}}\frac{dF}{d\mathrm{b}} = 0$$

$$\frac{dF}{d\mathrm{b}} = \frac{d}{d\mathrm{b}}N(x)C(b)b = N(x)[C'(b)b + C(b)]$$

and

substituting b=Q'(x)

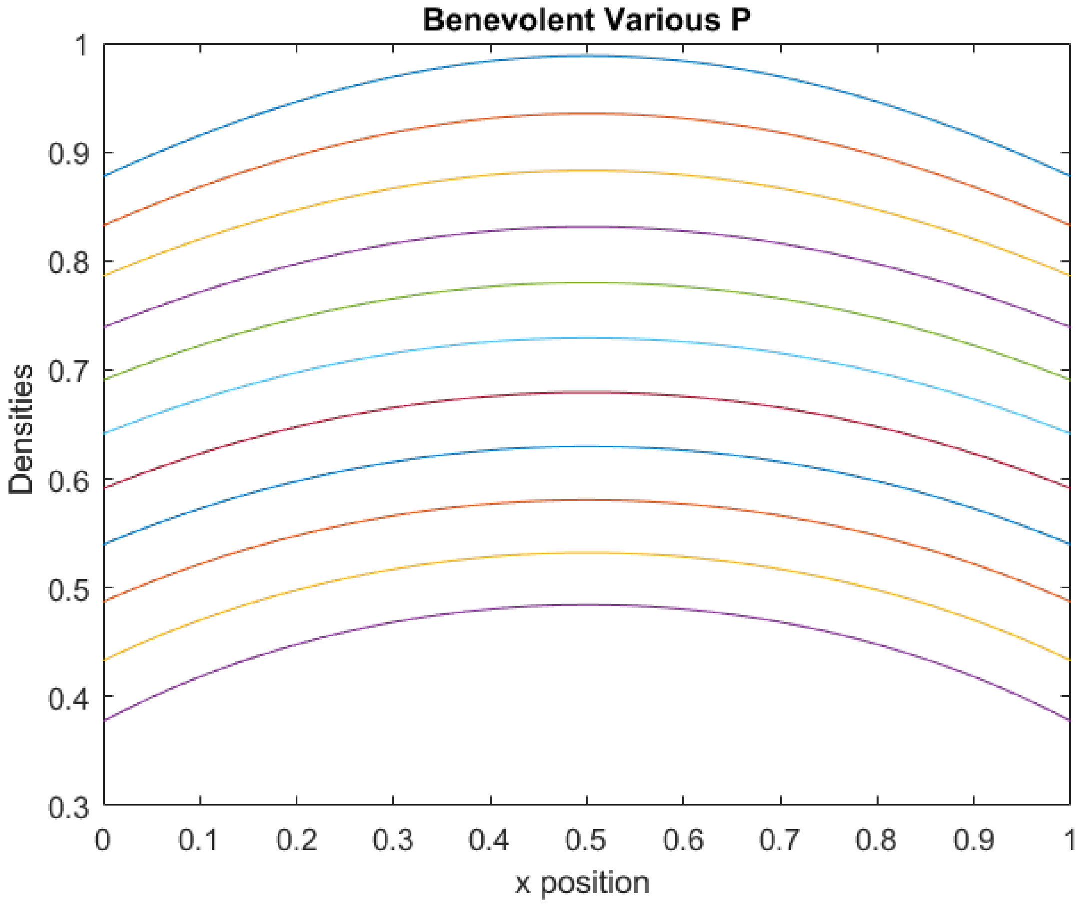
$$\frac{\mathrm{d}}{\mathrm{dx}}[N(x)[C'\big(Q'(x)\big)Q'(x) + C(Q'(x))]$$

$$\frac{\mathrm{d}}{\mathrm{dx}}[N(x)\big[C'\big(q(x)\big)q(x) + C\big(q(x)\big)\big] = 0$$

(e) I take as given without proof from the problem statement.

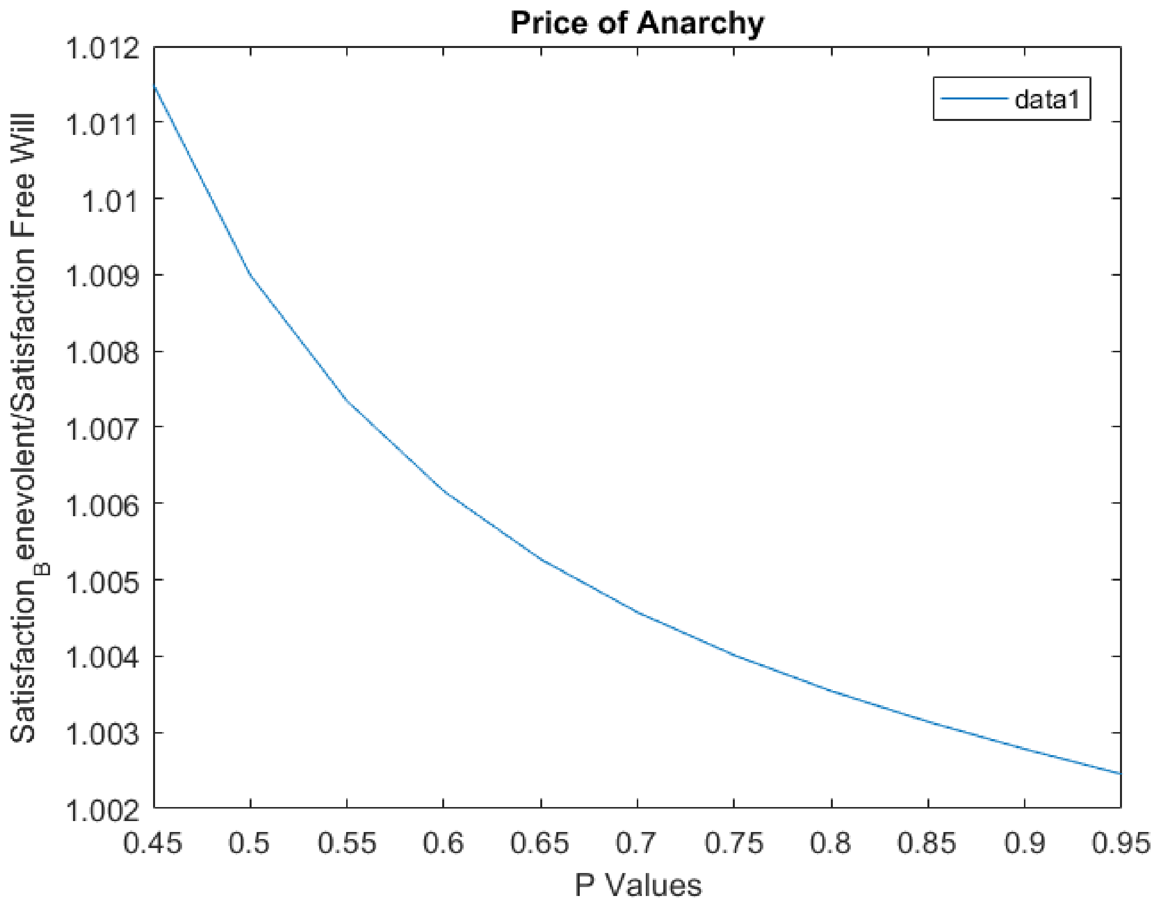
$$Q''(x) = \frac{N'(x)\big(1 - Q'(x)\big)^4)}{N(x)(6q(x) - 2\big(q(x)\big)^3}$$

(f) I plot the Benevolent Distributions below:



We can see that the benevolent dictator forces people out to the sides so as to not cause bunching.

(g) We now see that the Benevolent distributions are better by graphing the benevolent/free_will for the values of P and see that they are above 1.



The price of Anarchy for the problem seems low.