Improvement to Discounted Cash-Flow

Advances to Discounted Cash-Flow Relevant To High Interest Rate Environments

Nathaniel Rogalskyj, Spectral Technologies

January 2022

Contents

1 Executive Summary

The purpose of this report is to review the state of the art for the Discounted Cash-Flow which is a major tool for determining the value of a company or asset. It will also analyze improvements to its calculation using the novel application of powerful mathematical techniques, not known to the private equity industry, nor discussed in the financial literature.

1.1 Major Results

- Current Discounted Cash Flow has a problem that can be viewed in a series of different ways: (1) all cash-flow earned throughout a quarter or year (ie period) can be treated as being earned in aggregate at the last day of the quarter or year over which it was accumulated, (2) the discounting function that is used is discontinuous. A dollar earned at the end of quarter or year 1 is treated the same as a dollar earned at the beginning of quarter or year 1, and a dollar earned at the end of quarter or year 1 is treated entirely differently than a dollar earned at the beginning of quarter or year 2.

- Resolution of these problems are done using a standard, continuous, discounting function and by taking an analyst's projections of a cash-flow and using Calculus Of Variations to distribute the cash-flow across the periods in a continuous manner, and using this for further analysis. For a period discount rate of $r = 0.05$, the DCF error rate is decreased from 3.0 percent for traditional DCF to 0.6 percent for new, continuous DCF.

2 Introduction

The valuation of businesses via the Discounted Cash Flow (DCF) is essential to the Financial Industry. Billionaire Investor Bill Ackman has stated that it was one of the most important aspects of how to value an investment. In his "Theory of Investment Value" written over 50 years ago, John Burr Williams set forth the equation for value, which we condense here: "The value of any stock, bond or business today is determined by the cash inflows and outflows - discounted at an appropriate interest rate - that can be expected to occur during the remaining life of the asset". Billions of dollars can be said to flow through the discounted cash-flow. Any intellectual error in the formula ought to be addressed, less these errors improperly guide the flow of billions of dollars of assets, investment etc.

3 State of Discounted Cash Flow

A discounted cash-flow requires an analyst to make a projection about future cashflows, such as as a company will make 100 million in Q1, 200 million in

Q2, 300 million in Q3, and then -400 million in Q4. These estimates are highly uncertain in under certain regimes and not uncertain in others. Finally after making these projections, a discount rate will be used to state that positive cash-flows earned later are not as valuable as the same positive cash-flows earned earlier. This is because of the time-value of money. The link describing this concept is provided by: "https://en.wikipedia.org/wiki/Time_value_of_money". The risk-free discount rate is close to 1.02 per year, while higher discount rates are used for 'riskier' assets such as those in hedge-funds or private equity firms. It was believed by my managing director at Cerberus Capital Management that these regimes of the discount rate only exist in the 1-2 percent range, since this is the interest rate provided by banks, but hedge funds and venture capital firms often require much higher rates of return. Most good investment firms look for a yearly rate of return around 10 percent returned or 1.1 per year.

3.1 Standard Steps to a DCF

An outline of the concept of a DCF and the standard steps for a DCF (Discounted Cash Flow) are provided in the following youtube video, which was created by an investment banking analyst at JP Morgan. The video link is: `https://www.youtube.com/watch?v=OwbiEjINcpA`

3.2 DCF Outline

The outline provided for the rest of the video is provided below:

- Conceptual Understanding of a DCF
- Why are DCF(s) important and when to use them?
- How to do a DCF?
- Tesla DCF Example

3.2.1 Conceptual Understanding, Importance and Relevance of DCF

In the video, it is stated "The Discounted Cash Flow Analysis is a valuation methodology that measures the intrinsic value of a company based on the present value of its future free cash flows".

Further, it is stated in the video, "A company's true standalone value based on its ability to generate cash-flows rather than comparing to other assets.

This differs from relative valuation, where you compare one company's metric to another company's metric, such as P-E ratio, also known as "Price-To-Earnings" Ratio.

In the video, it is further stated that "A dollar today is worth more than a dollar tomorrow"

In the video, it is stated that "A company's value is based on the cash it's able to generate from today until the end of time".

Further, in the video, it is stated that "In the Theory of Investment Value, written over 50 years ago, John Burr Williams set forth the equation for value, which we condense here: The value of any stock, bond or business today is determined by the cash inflows and outflows - discounted at an appropriate interest rate - that can be expected to occur during the remaining life of the asset"

In the video, it is stated that "Many companies are cash-flow negative, potentially even for the next 5 to 10 years. He states that for something like that, you cannot really create a model where you are projecting out future cash flows."

In the video, it is stated that, "Discounted Cash Flow will most likely lead to a negative number, which means that someone will have to pay you in order for you to buy the company".

In the video, it is stated that, "A good candidate for a discounted cash flow would be a company that does not change its business model too often such as General Electric versus a tiny little start-up."

In the video, it is stated, "That one gets to cater their analysis to what they believe, but that you need to make sure that your assumptions are correct."

In the video, it is stated, "At JP Morgan, he, as an analyst did indeed create DCF (Discounted Cash Flow) for biotech companies that had negative cash flows for the first few years, but would then have positive cash-flows over the next 20-30 years.

3.3 How to do a DCF?

In the video, it is further more stated that:

- Project out future free cash flows usually for the next 5-10 years.
- Calculate terminal value using either the exit multiple or perpetuity growth model
- Discount the free cash flow and terminal value back to present value using your WACC.
- Sum up the present value of the cash and terminal value to get enterprise value.

3.3.1 Setup

Increasingly, a business's cash-flow is accumulated instantaneously through time. Gone are the days where a businsess' cash-flow is restricted to standard 9am-5pm, which was the only time that money could be made. Increasingly, more than ever, the digitisation of commerce, also known as eccomerce, means that money can be accumulated at every moment of time. Hence a good model for a business cash-flow for the upcoming year is $c(t)$, where t is not years, quarters, or even days, but highly granular, down to the seconds. Indeed many financial firms ie market makers in financial securities trade and manage their trading books at ultra high frequencies in the fractions of seconds.

To setup notation for below, when it is stated that a cash-flow is accumulated throughout the period, I take it to be the following:

$$C(T) = C_T = \int_{T-1}^{T} c(t)dt$$

3.3.2 Calculation of WACC

WACC is very important to doing the calculation of the Perpetuity Growth Model and finding the Present Value of Cash Flows.

The calculation of WACC is given by:

$$WACC = \frac{\sum_{i=1}^{N} MV_i r_i}{\sum_{i=1}^{N} MV_i} \tag{1}$$

Above the different components can be expressed as:

- N is the number of sources of capital (securities, liabilities)
- r_i required rate return for a security
- i is the index of a particular security
- MV_i market value of all outstanding securities

WACC accounts for the different sources of capital.

- Cost Of Debt
- Cost Of Preferred Shares
- Cost of Equity

In the video, it is stated that "Suppose a company goes out and decides to issue some Bonds (Debt) for 30K and equity for 70K.

In the video, it is stated that the question becomes what percentage of the company is "debt" and what percentage of the company is "equity".

The view on taking on projects is that the financing of the projects comes from the debt and equity raised. This debt and equity may have a high required rate of return. If the company cannot meet the hurtle rate set by both the debt and equity (equity being determined by CAPM), then running a project that returns less than this destroys "wealth".

The video then states the value of 1000000 two years from now should be discounted by WACC, which is the weighted average cost of capital, which could be something like 1.079

$$DCF = \frac{1000000}{1.079^2} = 858928 \tag{2}$$

3.3.3 Present Value of Perpetuity Growth Model

The most used way of determining the exit value is the Perpetuity Growth Model as provided by the equation below:

$$PV_{PGM} = \frac{C_{N+1}(1 + TGR)}{(WACC - TGR)} \tag{3}$$

where C_{N+1} is the free cash flow at the end of the period, TGR is the terminal growth rate of the business, WACC is the weighted average cost of capital.

3.3.4 Present Value of Cash Flows

Calculating the present value of cash flows is another important aspect of finding out what the value of a stock is. The present value of cash flows is:

$$PV_{CF} = \sum_{i=1}^{N} \frac{C_i}{(1 + WACC^i)} \tag{4}$$

3.3.5 Net Present Value

The present value of all cash flows can then be found as:

$$PV = PV_{CF} + PV_{PGM} \tag{5}$$

PV noted as the present value of a company can be related to the stock price.

JP Morgan Analyst states that at a hedge fund it is not necessary to make everything look nice.

3.3.6 Tesla Net Present Value Calculations

Discounted Cash Flow (DCF) for Tesla provided below

https://site.financialmodelingprep.com/discounted-cash-flow-model-levered/TSLA

The reported WACC for Tesla was 0.1088. Alternatively, the WACC for FB is 0.0743.

4 Short Comings of Present Value Of Cash Flow Calculations

It seems that the formula presented in (3.3.4) is made by someone who may understand finance as it does indeed capture the time value of money, but does not understand computers or mathematics.

Here, I analyze the shortcomings of equation (4), which is again presented below:

$$PV_{CF}(c(t)) = \sum_{i=1}^{N} \frac{C_i}{(1+WACC)^i} \tag{6}$$

where

$$C(T) = C_T = \int_{T-1}^{T} c(t)dt$$

First, cash actually earned only enters the formula via dependence on C_i. If $WACC > 0$, then we can now consider the PV_{CF} as a function of an ordered tuple of $C_1, C_1, ..., C_N$,

$$PV_{CF}((C_1, C_2, ..., C_N)) = \sum_{i=1}^{N} \frac{C_i}{(1+WACC)^i} \tag{7}$$

Hence $PV_{CF}(C_1, C_1, ..., C_N)$ no longer is a function of $c(t)$, but these aggregates of $c(t)$. One can see now that the information found in an infinite function $c(t)$ is now collapsed into only a handful of numbers via the aggregates C_i for $i = 1, ...N$.

Some significant consequences of this can be immediately found. Concretely, suppose that a business 1 has an earning pattern of $c_1(t) = \delta(T-\epsilon)$, where they earn 1 dollar at time $T - \epsilon$ but another business has an earning pattern of $c_2(t) = \delta(T+\epsilon)$, where they earn 1 dollar at $T+\epsilon$, where $\delta(t)$ is the dirac delta function. The value of business 1 is then $PV_{CF}(c_1(t)) = \frac{1}{(1+WACC)^T}$. The value of business 2 is then $PV_{CF}(c_1(t)) = \frac{1}{(1+WACC)^{T+1}}$. If $T = 1$ and WACC is 0.1, then value of business 1 is $\frac{1}{1.1} = 0.90$. If $T = 1$ and WACC is 0.1, then value of business 2 is $\frac{1}{1.1^2} = 0.82$. The fact that two business' $c(t)$'s are earning dollars arbitrarily close in time, but have different company values is disturbing.

Furthermore, not only is the Discounted Cash-Flow procedure unstable in mapping from cash-flows $c(t)$ to values of companies PV_{CF}, the estimates put out by analysts C_i specify nothing about how the cash-flows are distributed on the interval. No continuity of cash-flows $c(t)$ is enforced or expected, and this makes it difficult to understand the path that a business has taken throughout the period to generate the anticipated analyst outputs. For instance, an analyst may predict that a company such as Walmart will make 100 million dollars in Q1, and then 10 Billion in Q2, but not make any statement about what Walmart Cash-Flow generation potential during the transition between $Q1$ and $Q2$, which would point to the fact that most of the money is probably made at the end of Q1 and not at the beginning of Q1, and should have a discount rate closer to Q2's discount rate.

Furthermore, supposing that the true value of a cash-flow $c(t)$ is provided by

$$V = \int_{0}^{N} e^{-ln(1+r)t} c(t)dt$$

where

$$C_i = \int_{i-1}^{i} c(t)dt$$

then

$$V = \sum_{i=1}^{N} \int_{i-1}^{i} e^{-ln(1+r)t} c(t)dt$$

Supposing that $C(T) > 0$, I derive the following bounds:

$$\frac{C(i)}{(1+r)^i} = \int_{i-1}^{i} (\frac{1}{1+r})^i c(t)dt \leq \int_{i-1}^{i} e^{-ln(1+r)t} c(t)dt \leq \int_{i-1}^{i} (\frac{1}{1+r})^{i-1} c(t)dt = \frac{C(i)}{(1+r)^{i-1}}$$

I can then bound either side of the equation, and achieve the following bounds as seen below.

$$DCF_s[c(t)] = \sum_{i=1}^{N} C(i)\frac{1}{(1+r)^i} \leq V \leq \sum_{i=1}^{N} C(i)\frac{1}{(1+r)^{i-1}}$$

I achieve the following bound then:

$$DCF_s[c(t)] \leq V \leq DCF_s[c(t)](1+r)$$

When $r = 0$, the calculated $DCF_s[c(t)]$ and the true value V are exactly the same. Given that the calculated value can diverge from the true value by up to $1 + r$, there can be pretty significant difference for large r. In different areas of finance, there are different possible r's. The question is how much can r differ by for different types of assets. Having the standard formula that is provided is not bad, if indeed r is small. For some assets though r can be large and $c(t)$, can be continuous. As one can see below, there are potentially significant mis pricings in both both percentage and net amount.

Asset	r (per year)	r (per quarter)	Size of Asset	Max Mis-Pricing
Tesla Stock	0.16	0.04	1×10^{12}	40B
FB Stock	0.0984	0.0246	1×10^{12}	20B
Series A Funding	0.3	0.075	15 million	1M
Series B Funding	0.2	0.05	33 million	1.5M
Series C Funding	0.2	0.05	118 million	5.9M
Growth Venture	0.2	0.05	50 million	2.5M

5 Lower Bounds For Discounted Cash Flow

I provide to $c(t)$ a lambda parameter λ, where we have that and $t_0 < t_1$

$$\frac{|c(t_1) - c(t_0)|}{|t_1 - t_0|} \leq \lambda$$

We look for a bound of the following form:

$$|DCF_s[c(t)] - DCF_c[c(t)]| > E(C_i, WACC, \lambda)$$

We do some algebra to make some progress:

$$|DCF_s[c(t)] - DCF_c[c(t)]|$$

$$|\sum_{i=1}^{N} \frac{C_i}{(1+WACC)^i} - \int_0^N e^{-ln(1+WACC)t} c(t)dt|$$

$$|\sum_{i=1}^{N} \int_{i-1}^{i} ((1+WACC)^{-i} - e^{-ln(1+WACC)t})c(t)dt|$$

I take a look specifically at the terms that can be found in the integral.

$$\int_{i-1}^{i} ((1+WACC)^{-i} - e^{-ln(1+WACC)t})c(t)|dt$$

Since $c(t) > 0$,

$$\int_{i-1}^{i} |((1+WACC)^{-i} - e^{-ln(1+WACC)t})c(t)|dt > 0$$

We can look at this term as taking values from a set of $c(t)$. There will be a $c(t)$ that minimizes the value of this term.

$$|\frac{C_i}{(1+WACC)^i} - \int_{i-1}^{i} e^{-ln(1+WACC)t})c(t)dt|$$

v

Furthermore, the value at which this attains the minimum is when the mass of c(t) is distributed to the right of the interval.

The minimizing c(t) is provided by:

$$c(t) = c(i) - \lambda(i - t)$$

$$c(i) = C_i + \frac{1}{2}\lambda$$

$$c(t) = C_i + \frac{1}{2}\lambda - \lambda(i - t)$$

for $i = 1, ...N$

6 Positives Of Discounted Cash-Flow

Reducing information about a business from $c(t)$ to just a handful of numbers, namely the C_T has some positives. First, since it allows for efficient computation on a single piece of paper. Second, it is also efficient for the analyst since instead of specifying a complicated function $c(t)$ into the future, he only needs to specify a handful of numbers. Furthermore, these analyst projections can be checked against the real numbers that companies report each quarter, which is excellent for checking the accuracy of said analyst. Hence, most of the modern finance world is reduced to a handful of numbers for praticality's sake.

7 Design Of New Procedure

An improvement to a discounted cash flow must only tinker with the algorithm and formula, not the workflow of that the analyst is using, otherwise adoption of it will require learning a new system. The idea here is not to change the system of the discounted cash-flow but instead change how the numbers are calculated.

The system will take in the analysts estimates $C_1, C_2 ... C_N$ and spit out a number that is not sensitive to the drawbacks discussed above. It will provide an inference for what $c(t)$ ought to be under certain assumptions, and display that if required. In this manner, the approach that will be taken is to take $C_1, C_2 ... C_N$, find an appropriate $c(t)$ and discount it correctly. Calcululs Of Variations allows one to find this $c(t)$, as the field is devoted to finding the optimal functions given some constraints and other things.

8 Calculus of Variations

There is large space of functions that satisfy the integral constraints that we see above, are continuous and have nice properties (ie not overly oscillatory). Finding a nice and appropriate $c(t)$ is one of minimizing, equivalently maximizing something over a space of functions.

One thing that can be minimized is E:

$$E[c(t)] = \int_0^N c'(t)^2 dt$$

subject to the constraints, provided by the analyst.

$$C(T) = \int_{T-1}^T c(t) dt$$

Calculus of Variations provides one with the means of solving this problem.

We can use the Euler Lagrange Differential Equation presented in the link below to solve for the appropriate $c(t)$. The reader should read the link to understnad what will occur in the rest of the derivation.

http://liberzon.csl.illinois.edu/teaching/cvoc/node38.html

For the reader, we pull from the notation provided in the link and denoting the following as for all $i > 0$

$$M_i(x, y, z) = (y - C_i)(1_{i-1<=x<=i})$$

$$\int_0^N M_i(t, c(t), c'(t))dt = 0$$

The augmented Lagrangian then becomes:

$$= \mathrm{L} + \textstyle\sum_{i=1}^{N} \lambda_i \mathrm{M}_i$$

In the above analysis, to reformulate the the function E, one sets the $L(t, c(t), c'(t)) = c'(t)^2$ and then one takes:

$$E[c(t)] = \int_0^N L(t, c(t), c'(t))dt$$

From here, one only needs to solve the Euler-Lagrange equation which is the following:

$$\hat{L}_y = \frac{d}{dt}(\hat{L})_{y'}$$

The problem then becomes one of solving the above differential equation. The above differential equation will allow for the calculation of a $c(t)$ that satisfies the above constraints and minimizes the appropriate functional.

8.1 Calculating $\hat{L}_y$

The question here is how do we calculate $\hat{L}_y$.

$$\frac{d}{dc}\hat{L} = \frac{d}{dc}(L + \sum_{i=1}^{N} \lambda_i M_i) = \frac{d}{dc}L + \sum_{i=1}^{N} \lambda_i \frac{d}{dc} M_i = \sum_{i=1}^{N} \lambda_i 1_{i-1<=t<=i}$$

8.2 Calculating $\hat{L}_{y'}$

$$\frac{d}{dc'}\hat{L}(t, c, c') = \frac{d}{dc'}(L + \sum_{i=1}^{N} \lambda_i M_i) = 2c'(t) \tag{8}$$

8.3 Calculating $\frac{d}{dt}\hat{L}_{y'}$

$$\frac{d}{dt}(2c'(t)) = 2c''(t) \tag{9}$$

8.4 Full Differential Equation

The differential equations is provided below.

$$2c''(t) = \sum_{i=1}^{N} \lambda_i 1_{i-1<=t<=i}$$

$$c''(t) = 0.5 \sum_{i=1}^{N} \lambda_i 1_{i-1<=t<=i}$$

8.5 Derivation of $c(t)$

$$c'(s) = c'(0) + \int_0^s c''(t)dt = c'(0) + \int_0^s 0.5 \sum_{i=1}^{N} \lambda_i 1_{i-1<=t<=i}$$

$$c'(s) = c'(0) + \sum_{i=1}^{\lfloor s \rfloor} \lambda_i + (s - \lfloor s \rfloor)\lambda_{\lfloor s \rfloor}$$

$$c(t) = c(0) + c'(0)t + \int_0^t c'(s)ds = c(0) + c'(0)t + \int_0^t \sum_{i=1}^{\lfloor s \rfloor} \lambda_i + (s - \lfloor s \rfloor)\lambda_{\lfloor s \rfloor} ds$$

$$c(t) = c(0) + c'(0)t + \int_0^t \sum_{i=1}^{\lfloor s \rfloor} \lambda_i + (s - \lfloor s \rfloor)\lambda_{\lfloor s \rfloor} ds$$

What is the value of the following term?

$$\int_0^t \sum_{i=1}^{\lfloor s \rfloor} \lambda_i + (s - \lfloor s \rfloor)\lambda_{\lfloor s \rfloor} ds$$

Its the following:

$$= \sum_{i=1}^{\lfloor t \rfloor - 1} (\sum_{j=1}^{i} \lambda_j + \lambda_{i+1}/2) + min(t^2, 1)\frac{\lambda_1}{2} + 1_{t>1}(0.5(t - \lfloor t \rfloor)\lambda_{\lceil t \rceil} + \sum_{i=1}^{\lfloor t \rfloor} \lambda_i)(t - \lfloor t \rfloor)$$

8.6 Solving for λ_i

For $T = 1, 2, 3...N$,

$$\int_{T-1}^{T} c(t)dt = C_T$$

$$\int_{T-1}^{T} [c(0)+c'(0)t+\sum_{i=1}^{\lfloor t \rfloor -1}(\sum_{j=1}^{i}\lambda_j+\lambda_{i+1}/2)+min(t^2,1)\frac{\lambda_1}{2}+1_{t>1}(0.5(t-\lfloor t \rfloor)\lambda_{\lceil t \rceil}+\sum_{i=1}^{\lfloor t \rfloor}\lambda_i)(t-\lfloor t \rfloor)]dt = C_T$$

Further simplifying the integral

$$c(0)t|_{T-1}^{T}+\frac{1}{2}c'(0)t^2|_{T-1}^{T}+\sum_{i=1}^{\lfloor T-1 \rfloor -1}(\sum_{j=1}^{i}\lambda_j+\lambda_{i+1}/2)|_{T-1}^{T}+\int_{T-1}^{T} min(t^2,1)\frac{\lambda_1}{2}dt+\int_{T-1}^{T} 1_{t>1}(0.5(t-\lfloor t \rfloor)\lceil \lambda \rceil+\sum_{i=1}^{\lfloor t \rfloor}\lambda_i)(t-\lfloor t \rfloor)dt = C_T$$

We now explore the value of the following terms:
Term 1

$$\int_{T-1}^{T} min(t^2,1)\frac{\lambda_1}{2}dt = f(T)$$

$$f(T) = \begin{cases} \frac{\lambda_1}{6} & T \leq 1 \\ \frac{\lambda_1}{2} & T > 1 \end{cases} /]$$

Term 2

$$\int_{T-1}^{T} 1_{t>1}(0.5(t-\lfloor t \rfloor)\lceil \lambda \rceil + \sum_{i=1}^{\lfloor t \rfloor}\lambda_i)(t-\lfloor t \rfloor)dt = g(T)$$

If we take $s = t - \lfloor t \rfloor$, then we get

$$\int_0^1 1_{t>1}(0.5s^2)\lceil \lambda \rceil + \sum_{i=1}^{T-1}\lambda_i s ds$$

$$g(T) = \begin{cases} 0 & T \leq 1 \\ \frac{1}{6}\lceil \lambda \rceil + \frac{1}{2}\sum_{i=1}^{T-1}\lambda_i & T > 1 \end{cases} /]$$

For $T = 1, 2, 3...N$, one can create linear constraints that map the λ_i into the C_i. Linear constraints for $c'(0)$ and $c''(0)$ have yet to been solved for nor have the boundary conditions been matched.

8.7 Full Statement Of C(t)

For $T = 1...N$, we have the following linear equations that map λ_i into the C_T

$$c(0)t|_{T-1}^{T} + \frac{1}{2}c'(0)t^2|_{T-1}^{T} + \sum_{i=1}^{\lfloor T-1 \rfloor -1}(\sum_{j=1}^{i}\lambda_j + \lambda_{i+1}/2)|_{T-1}^{T} + f(t) + g(t) = C_T$$

These will provide the λ_i for $i = 1...N$. These will be provided to us by a linear solve. There are still two missing unknowns in order to get the $c(t)$ that is provided below. These are $c'(0)$ and $c(0)$. From above, we have that:

$$c(t) = c(0) + c'(0)t + \int_0^t \sum_{i=1}^{\lfloor s \rfloor}\lambda_i + (s - \lfloor s \rfloor)\lambda_{\lfloor s \rfloor} ds$$

These will be matched by applying appropriate boundary conditions. One set of boundary conditions that could be applied is setting $c(0) = C_0$ and $c(N) = C_N$.

8.8 Discount Continuous C(t)

After deriving the continuous $c(t)$, it is then necessary to discount it in a continuous fashion.

Finally, calculate the value V with :

$$V = \int_0^N c(t)e^{-rt}$$

where

$$r = ln(1 + WACC)$$

8.8.1 Integral Calculation Provided Here

As before we had that:

$$c(t) = c(0)+c'(0)t+\sum_{i=1}^{\lfloor t\rfloor-1}(\sum_{j=1}^{i}\lambda_j+\lambda_{i+1}/2)+min(t^2,1)\frac{\lambda_1}{2}+1_{t>1}(0.5(t-\lfloor t\rfloor)\lambda_{\lceil t\rceil}+\sum_{i=1}^{\lfloor t\rfloor}\lambda_i)(t-\lfloor t\rfloor)$$

The key to looking at the integral:

$$V = \int_0^N c(t)e^{-rt}$$

In solving the above equation, we note that the function is piecewise in the time periods of $t = 1...N$. Hence, a good maneuver to calculate the integral on the different periods and sum the solution together. Hence make the following dedeuctions:

$$V = \sum_{i=1}^{N}\int_{i-1}^{i} c(t)e^{-rt}$$

In order the calculate V, I need to break up the terms that are in $c(t)$. The terms are defined as follows:

- $F_1(t) = c(0)$
- $F_2(t) = c'(0)t$
- $F_3(t) = \sum_{i=1}^{\lfloor t\rfloor-1}(\sum_{j=1}^{i}\lambda_j) + \lambda_{i+1}/2)$
- $F_4(t) = \frac{\lambda_1}{2}min(t^2, 1)$
- $F_5(t) = 1_{t>1}(0.5(t-\lfloor t\rfloor)\lambda_{\lceil t\rceil} + \sum_{i=1}^{\lfloor t\rfloor}\lambda_i(t-\lfloor t\rfloor))$

8.8.2 Integrate $\mathbf{F}_5(t)$

The following is important to the calculation:

$$1_{t>1}(0.5(t-\lfloor t\rfloor)\lambda_{\lceil t\rceil}+\sum_{i=1}^{\lfloor t\rfloor}\lambda_i(t-\lfloor t\rfloor))$$

I would like to calculate the following value:

$$\int_i^{i+1} e^{-rt}(1_{t>1}(0.5(t-\lfloor t\rfloor)\lambda_{\lceil t\rceil}+\sum_{i=1}^{\lfloor t\rfloor}\lambda_i(t-\lfloor t\rfloor)))dt$$

I take

$$s=t-\lfloor t\rfloor=t-i$$

I can take a derivative of $\frac{ds}{dt}=1$.

$$\int_i^{i+1} e^{-rt}(1_{t>1}(0.5(t-\lfloor t\rfloor)\lambda_{\lceil t\rceil}+\sum_{i=1}^{\lfloor t\rfloor}\lambda_i(t-\lfloor t\rfloor)))dt$$

If $i\geq 1$

$$\int_0^1 e^{-r(s+i)}(0.5(s)\lambda_{\lceil t\rceil}+\sum_{i=1}^{\lfloor t\rfloor}\lambda_i s))ds$$

$$\int_0^1 (0.5\lambda_{i+1}+\sum_{x=1}^{i}\lambda_x))(e^{-r(s+i)}s)ds$$

$$(0.5\lambda_{i+1}+\sum_{x=1}^{i}\lambda_x))\int_0^1 (e^{-r(s+i)}s)ds$$

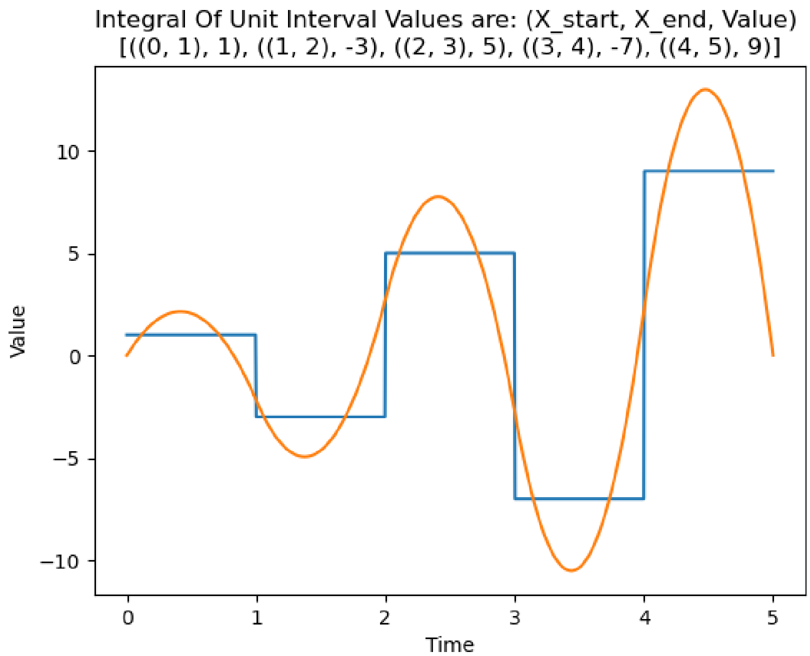
$$(0.5\lambda_{i+1}+\sum_{x=1}^{i}\lambda_x))\frac{(e^r-1)e^{-r(i+1)}}{r}$$

9 Example of Inference of C(t)

A concrete example is provided here of how I go from predicted cash-flows of 1 for period 1, -3 for period 2, 5 for period 3, -7 for period 4, 9 for period 4 to a $c(t)$ that matches the analysts projections but can then be integrated against and discounted against in a continuous fashion.

The blue curve shows a traditional financial analysts view of the world, while the orange curve shows the invention and/or continuous version of the world that has been derived above.

The continuous curve shows a traditional financial analysts view of the world, while the orange curve shows the invention and/or continuous version of the world that has been derived above.



Above, I showed the primitive version of the discounted-cash flow, where I look to interpolate the blue curve with the orange curve. In the below, I show that an interesting numeric property where if the first cash flows in the sequence are differing, then over time the c(t) re-converge to be close to being equal.

10 Simulation Results

The algorithm's ability to infer the true discounted cash-flow value can be tested by simulating cash-flows in the background and calculating their true discounted cash-flow values. One can then compare the true values against the values calculated via the standard discounted cash-flow procedure and the procedure that has been laid out above that uses the Calculus Of Variations. Results of the simulation are shown below. The r or $WACC$ value is 0.05 per period. Uniform Cash Flow treats every day as being drawn from the same distribution between $[-1000, 1000]$. And the binary cash-flow takes a random walk, either going up one or down one from the previous day and starting at 0. From the chart below, Traditional and Calculus of Variations perform same on Uniform. However, Calculus of Variations outperforms Traditional by a significant factor on binary tree cash-flows. The error reduction is about 5 fold.

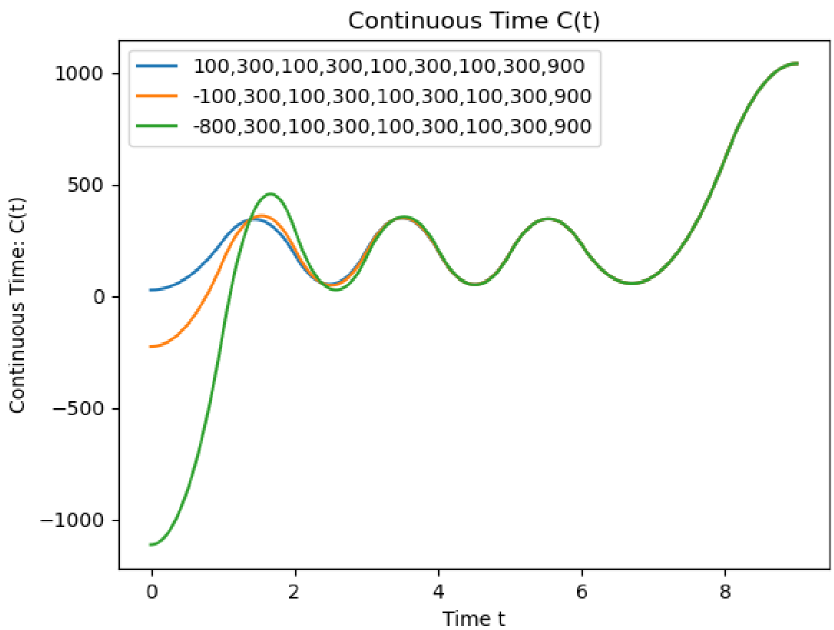


Figure 1: Convergence of Variational Methods

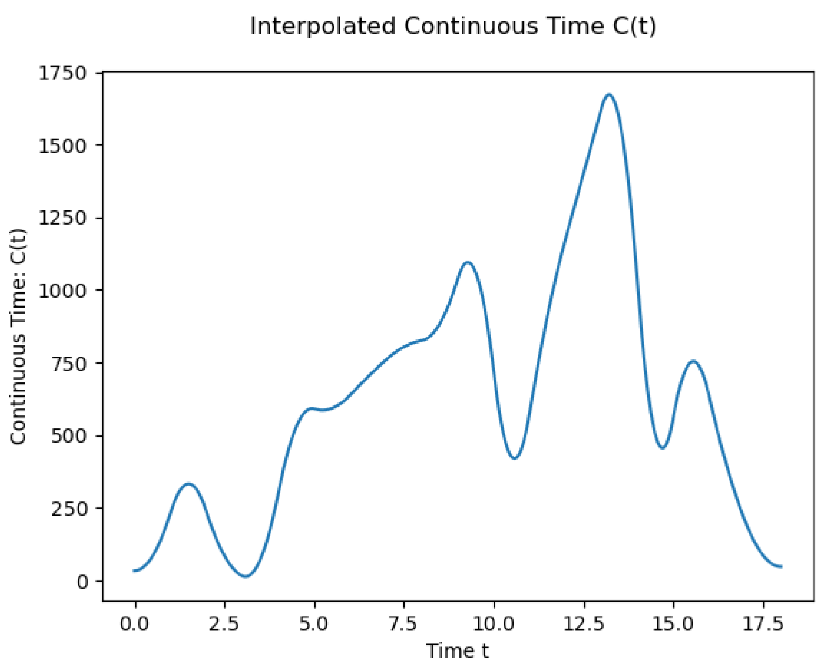


Figure 2: Bumpy Cash Flows

Stochastic Process	Traditional DCF Percent Error	Continuous DCF Percent Error
Uniform	7.0	7.0
Binary Tree	3.0	0.6

Table 1: Simulation Of Traditional and Continuous DCF

11 Results

My method provides a means of doing continuous discounting given period estimates already provided by an analyst, and hence can be used to provide a continuous internal rate of return and continuous discounted cash flow estimate.

There is an expectation that this method should provide a 1 percent difference against the other method that has been provided.

12 Notes

- Financial Modelling preparation link: https://site.financialmodelingprep.com/discounted-cash-flow-model/TSLA

13 Appendix

$$\int_i^{i+1} t^2 e^{-rt} dt$$

$$\int_i^{i+1} t^2 e^{-rt} dt = \frac{-1}{r} t^2 e^{-rt} - \int_i^{i+1} 2t(\frac{-1}{r}) e^{-rt} dt$$

$$u = -rt$$

$$\frac{du}{dt} = -r$$

$$dt = \frac{du}{-r}$$

$$\int_i^{i+1} t^2 e^{-rt} dt = \frac{-1}{r} t^2 e^{-rt} + \frac{2}{r} \int_i^{i+1} t e^{-rt} dt$$

$$\int_i^{i+1} t e^{-rt} dt = \frac{e^{-rt}(rt+1)}{r^2}$$

$$\int_i^{i+1} t^2 e^{-rt} dt = \frac{-1}{r} t^2 e^{-rt} + \frac{2}{r} \frac{e^{-rt}(rt+1)}{r^2} = F(t)$$