Residential Location

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1 Residential location choice

We will calculate spacial equilibrium numerically for the simplest setup of the model—i.e., the one with homogeneous consumers and homogenous space.

Let's start by reviewing the main equations that we will use to compute the equilibrium. First, the consumers' utility maximization problem is

$$\max_{s,c}\{u(c,s)\}$$
$$pc + r(x)s = y - tx$$

Since we are interested in individual demand for land, we can use budget constrain to exclude consumption from the problem:

$$s^*(r(x),p,y,x) = \arg\max_s \left\{ u\left(\frac{y - tx - r(x)s}{p}, s\right)\right\}$$

The condition that the indirect utility must be indepenent of the location (otherwise, some locations will not be populated in equilibrium) gives us the equation that disciplines the difference in prices between two locations:

$$\frac{d}{dx}r(x) = -\frac{t}{s^*(r(x),p,y,x)}$$

or in the integral form

$$r(x) - r(0) = -\int_0^x \frac{tdz}{s^*(r(z),p,y,z)}$$

The endogenous city boundary x_B solves $r(x_B) = r_A$. The integral equation pins down prices at all locations except $x = 0$. In order to find the price in the center we solve the market clearing condition for land:

$$N = \int_0^{x_B} \frac{2\pi z dz}{s^*(r(z),p,y,z)}$$

1.0.1 Algorithm

We will use the following algorithm for computing the equilibrium: 1. For given prices we compute the individual demand for land. 2. Using the integral equation we compute the land prices.

3. Iterate 1. and 2. until the prices that we use as input coincide with the prices that solve the integral equation. 4. Solve the market clearing condition for $r(0)$.

We cannot work with the continuum of locations directly, so we will intoduce a deterministic location grid with a small increment.

We begin by defining consumer's preferences. In this example we will use Cobb-Duglas utility, but the script works for arbitrary utility functions (as long as the assumptions used in the model are satisfied).

```
In [1]: function utilityfn(c,s)
        #Utility function()
        alpha = 0.3
        rslt = (c.^alpha).*(s.^(1-alpha))
        end;
```

Now we can solve the utility maximization problem. Inputs for this problem are transportation costs t (variable `trc`), income y (variable `income`), the price of consumption good p (variable `p`), the discreet grid on the set of locations (variable `xx`) and the land prices (variable `rr`). Naturally, variables `rr` and `xx` should have the same dimensions. Function `dmnd` solves for demanded consumption and land for each location (so the outpus will be vectors with the same dimensions as `xx`) given inputs.

```
In [2]: # using Pkg
        # Pkg.add("Optim")
        using Optim

        function dmnd(trc, income, p, xx, rr)
        #demand
        nn = length(xx)
        s_temp = zeros(nn,1)
        c_temp = zeros(nn,1)
        for ii in 1:nn
            utilityfn_lambda(s) = -utilityfn((income-trc*xx[ii]-rr[ii]*s)/p,s)
            s_temp[ii,1] = Optim.minimizer(optimize(utilityfn_lambda,0,max(0,(income-trc*xx[ii])
            c_temp[ii,1] = (income-trc*xx[ii]-rr[ii]*s_temp[ii,1])/p
        end
        s_dmnd = s_temp
        c_dmnd = c_temp
        return (c_dmnd, s_dmnd)
        end;
```

Once we have individual demand functions, we can solve the integral equation for land prices. We will iterate 1. and 2. until we arrive at the solution: this iterations are implemented in the `while` loop. Note that the prices in the iterations are updated with some inertia (see `rr_old = (rr_old+rr_new)/2`). This is a heuristic to obtain the convergence of the iterations. The function `findrent` below finds prices $r(x)$ given the price in the center $r(0)$ (we will solve for it later).

```
In [3]: function findrent(trc, income, p, r_A, xx, r0)
        #search for the solution to dr/dx = -t/s[r]
        toler1 = 0.001 #this is a precision parameter for iterations on r(x)
        nn = length(xx)
        drr = 12345.0
        rr_old = fill(r0, nn, 1)#r0*ones(length(xx),1)
        rr_new = rr_old.+drr


        while drr>toler1
            (c_dmnd, s_dmnd) = dmnd(trc, income, p, xx, rr_old)
            rr_new[1] = r0
            for ii in 2:nn
                if s_dmnd[ii] > 0
                    rr_new[ii] = max(rr_new[ii-1]-trc*(xx[ii]-xx[ii-1])/s_dmnd[ii] , 0)
                else
                    rr_new[ii] = 0
                end
            end
            drr = maximum(abs.(rr_new-rr_old), dims=1)[1]
            #println(drr)
            rr_old = (rr_old+rr_new)/2
        end

        return rr_new

                end;
```

Finally, we can use market clearing condition to solve for $r(0)$. We use the binary search since we know that excess demand is monotone in $r(0)$. The first `while` look finds $r(0)$ large enough for the excess demand to be negative (recall, that this value is required to start the binary search). the second `while` loop solves the market clearing condition.

```
In [22]: function findeq(trc, income, p, r_A, xx, N)
         toler2 = 0.0001
         rr = similar(xx)
         dxx = similar(xx)
         dxx[1] = 0
         dxx[2:end] = xx[2:end]-xx[1:end-1]
         xxdxx = xx.*dxx



         r0h = 0.0 # this value must be low enough otherwise the solution will not be found
         r0l = 0.0
         rhs = 0.0 # arbitrary value less than N

         #Find the rough bound on r(0) that result in excess supply and excess demand for land
         while rhs[1]<N
             r0l = r0h
             r0h = 2*r0h+0.1
```

```
            rr = findrent(trc, income, p, r_A, xx, r0h)
            ii_B = argmin(abs.(rr.-r_A), dims=1)[1][1]
            (c_dmnd, s_dmnd) = dmnd(trc, income, p, xx, rr)
            rhs_mat = 2*pi*xxdxx./s_dmnd
            rhs = sum(rhs_mat[1:ii_B], dims=1)
            println("Right hand side is $rhs")
        end

        #r0l = 0
        r0 = r0l
        rhs = 0.0

        while abs(r0h-r0l)>toler2
            if rhs[1]>N
                r0h = r0
            else()
                r0l = r0
            end
            r0 = (r0h+r0l)/2
            rr = findrent(trc, income, p, r_A, xx, r0)
            ii_B = argmin(abs.(rr.-r_A), dims=1)[1][1]
            (c_dmnd, s_dmnd) = dmnd(trc, income, p, xx, rr)
            rhs_mat = 2*pi*xxdxx./s_dmnd
            rhs = sum(rhs_mat[1:ii_B], dims=1)
            println("$r0l $r0h")
        end

        return rr

        end;
```

Once all this functions are defined we can run them on some parameters. For instance, we can set

$$\begin{aligned} t &= 1 \\ y &= 5 \\ p &= 1 \\ r_A &= 0.1 \\ N &= 20 \end{aligned}$$

and run the code to compute the prices.

```
In [23]: #include("dmnd.jl")
         #include("findrent.jl")
         #include("utilityfn.jl")
         #include("findeq.jl")
         # using Pkg
         # Pkg.add("Plots")
```

```julia
using Plots
pyplot()
# using Pkg
# Pkg.add("LaTeXStrings")
using LaTeXStrings
#gr()

trc = 1.0
income = 5.0
p = 1.0
r_A = 0.1
N = 10.0
grid_size = 300
maxX = income/trc

xx = [(maxX/grid_size)*(ii-1) for ii in (1:grid_size)]

rr_bid = findeq(trc, income, p, r_A, xx, N)
r_A_mat = r_A*ones(length(xx),1)
rr = maximum([rr_bid r_A_mat], dims=2)
ii_B = argmin(abs.(rr_bid.-r_A), dims=1)[1][1]
border_x = xx[ii_B]

(c_dmnd, s_dmnd) = dmnd(trc, income, p, xx, rr)

V = utilityfn(c_dmnd[1],s_dmnd[1])

dxx = copy(xx)
dxx[1] = 0
dxx[2:end] = xx[2:end]-xx[1:end-1]
xxdxx = xx.*dxx

WW = 2*pi*xxdxx.*(rr.-r_A)

W = sum(WW[1:ii_B])

println("====================================")

println("utility:       $V")

println("value of land: $W")
```

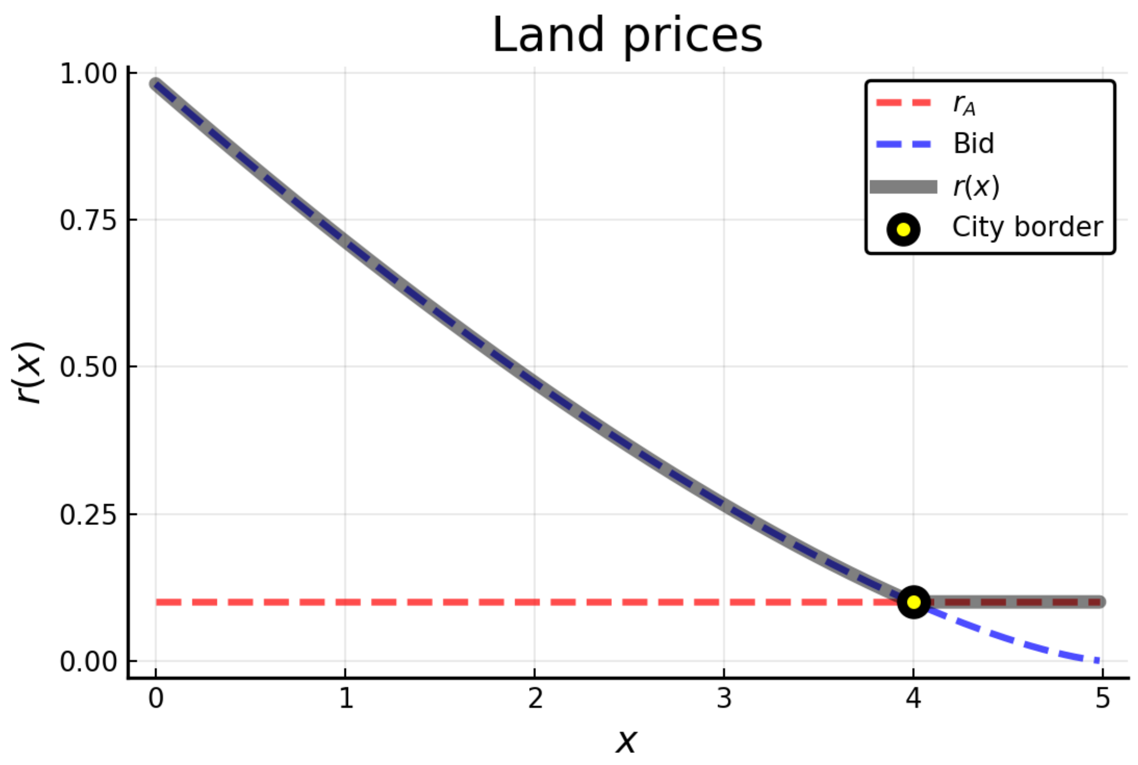
```
Right hand side is [0.0]
Right hand side is [1.60608]
Right hand side is [6.40852]
Right hand side is [16.615]
0.7000000000000001 1.5000000000000002
0.7000000000000001 1.1
```

```
0.9000000000000001 1.1
0.9000000000000001 1.0
0.9500000000000001 1.0
0.9750000000000001 1.0
0.9750000000000001 0.9875
0.9812500000000001 0.9875
0.9812500000000001 0.984375
0.9812500000000001 0.9828125000000001
0.9812500000000001 0.9820312500000001
0.9812500000000001 0.981640625
0.9812500000000001 0.9814453125
0.9812500000000001 0.9813476562500001
====================================
utility:       2.7505156009621485
value of land: 12.409670238478299
```

Now we can use the output to plot a graph of $r(x)$.

```
In [24]: plot(xx, r_A_mat,
                 seriestype = :line,
                 linestyle = :dash,
                 linealpha = 0.7,
                 linewidth = 2,
                 linecolor = :red,
                 label = L"r_A")
         xlabel!(L"x")
         ylabel!(L"r(x)")
         title!("Land prices")
         plot!(xx, rr_bid,
                 seriestype = :line,
                 linestyle = :dash,
                 linealpha = 0.7,
                 linewidth = 2,
                 linecolor = :blue,
                 label = "Bid")
         plot!(xx, rr,
                 seriestype = :line,
                 linestyle = :solid,
                 linealpha = 0.5,
                 linewidth = 4,
                 linecolor = :black,
                 label = L"r(x)")
         scatter!([border_x], [r_A],
                 markershape = :circle,
                 markersize = 8,
                 markeralpha =1,
                 markercolor = :yellow,
                 markerstrokewidth = 3,
                 markerstrokealpha = 1,
                 markerstrokecolor = :black,
                 markerstrokestyle = :solid,
                 dpi = 180,
             #        size = (1200,800),
                 thickness_scaling = 1.3,
                 label = "City border")
```

Out[24]:



```
'c' argument looks like a single numeric RGB or RGBA sequence, which should be avoided as value-
```