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BICATEGORIES OF SPANS AND RELATIONS

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A new kind of bicategorical limit is used to characterize bicategories of the form Span($\mathscr{E}$) and Rel($\mathscr{E}$) where in the former case $\mathscr{E}$ is a category with pullbacks and in the latter $\mathscr{E}$ is a regular category. The characterization of Rel($\mathscr{E}$) differs from those in the literature which require involutions on the bicategories.

0. Introduction

Recent trends in enriched category theory [2] suggest the need to characterize bicategories of spans as defined by Bénabou [1]. Walters has observed that categories locally internal to $\mathscr{E}$ are categories enriched in Span($\mathscr{E}$); this example provided motivation for [6] and will be further developed in a forthcoming paper of Betti–Walters. Our characterizations of Span($\mathscr{E}$) and Rel($\mathscr{E}$) do not involve extra data such as involutions (compare [3], [7]) or tensor products on the bicategories, and in the case of Rel($\mathscr{E}$), we dispense with Freyd's modularity condition [3]. We exploit a new kind of lax limit for an arrow in a bicategory; we use Freyd's term 'tabulation' although his use involved the involution and local finite products [3].

1. Tabulation

An arrow $f: A \to B$ in a bicategory $\mathscr{B}$ will be called a *map* (after [6]) when it has a right adjoint $f^*: B \to A$; the unit and counit for $f \dashv f^*$ are denoted by $\varepsilon: ff^* \Rightarrow 1$, $\eta: 1 \Rightarrow f^*f$. Let $\mathscr{B}^*$ denote the sub-bicategory of $\mathscr{B}$ with the same objects, with maps as arrows, and with all 2-cells between these. We suppress the associativity 2-cells for composition in $\mathscr{B}$; so, for example if $\sigma: f \Rightarrow rs$, $\tau: st \Rightarrow g$ are 2-cells, we write $(r\tau)(\sigma t)$ for the composite

$$ft \overset{\sigma t}{\Longrightarrow} (rs)t \cong r(st) \overset{r\tau}{\Longrightarrow} rg.$$

A *tabulation* for an arrow $r: A \to B$ in $\mathcal{B}$ is a diagram (f, ϱ, g):

$$\begin{tikzcd} & R \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, phantom, "\overset{\varrho}{\Leftarrow}"] & \\ A \arrow[rr, "r"'] & {} & B \end{tikzcd}$$

satisfying the following conditions:

T0. f is a map.

T1. For all other such diagrams (u, ω, v) with u a map, there exist $w, \theta: fw \Rightarrow u$, and invertible $\nu: v \Rightarrow gw$ such that $\omega = (r\vartheta)(\varrho w)\nu$.

$$\begin{tikzcd} & X \arrow[dl, "u"'] \arrow[dr, "v"] \arrow[d, phantom, "\overset{\omega}{\Leftarrow}"] & \\ A \arrow[rr, "r"'] & {} & B \end{tikzcd} \quad = \quad \begin{tikzcd} & X \arrow[ddl, "u"'] \arrow[ddr, "v"] \arrow[d, "w"] & \\ {} \arrow[r, phantom, "\overset{\theta}{\Leftarrow}"] & R \arrow[dl, "f"] \arrow[dr, "g"'] \arrow[r, phantom, "\overset{\nu}{\cong}"] \arrow[d, phantom, "\overset{\varrho}{\Leftarrow}"] & {} \\ A \arrow[rr, "r"'] & {} & B \end{tikzcd}$$

T2. For all maps $u: X \to A$, arrows $w, w': X \to R$, and 2-cells $\theta: fw \Rightarrow u$, $\theta': fw' \Rightarrow u$, $\beta: gw \Rightarrow gw'$ such that $(r\theta)(\varrho w) = (r\theta')(\varrho w')\beta$, there exists a unique $\gamma: w \Rightarrow w'$ such that $\beta = g\gamma$, $\theta = \theta'(f\gamma)$.

The diagram (f, ϱ, g) is called a *wide tabulation* for r when, in the definition above, T0 is deleted and T1, T2 are strengthened to allow u to be an arbitrary arrow (not just a map).

These definitions can be reformulated in terms of the bicategory $\mathcal{B}/\!\!/A$ whose objects are arrows $u: X \to A$, whose arrows $(h, \theta): u \to v$ consist of $h: X \to Y$, $\theta: vh \Rightarrow u$, and whose 2-cells $\sigma: (h, \theta) \Rightarrow (h', \theta')$ are $\sigma: h \Rightarrow h'$ with $\theta = \theta'(v\sigma)$. An arrow $r: A \to B$ induces a homomorphism of bicategories $r-: \mathcal{B}/\!\!/A \to \mathcal{B}/\!\!/B$ which takes u to ru and (h, θ) to $(h, r\theta)$. Let $\mathcal{B}/\!\!/{*A}$ denote the full sub-bicategory of $\mathcal{B}/\!\!/A$ consisting of the $u: X \to A$ which are maps.

Proposition 1. (a) *A tabulation for $r: A \to B$ is a birepresentation* [5; (1.11)] *for the homomorphism*

$$(\mathcal{B}/\!\!/{*A})^{\mathrm{op}} \xrightarrow[r-]{} (\mathcal{B}/\!\!/B)^{\mathrm{op}} \xrightarrow[(\mathcal{B}/\!\!/B)(-,1_B)]{} \mathrm{Cat}$$

and so is unique up to equivalence.

(b) *A wide tabulation for $r: A \to B$ is a birepresentation for the homomorphism*

$$(\mathcal{B}/\!\!/A)^{\mathrm{op}} \xrightarrow[r-]{} (\mathcal{B}/\!\!/B)^{\mathrm{op}} \xrightarrow[(\mathcal{B}/\!\!/B)(-,1_B)]{} \mathrm{Cat}.$$

(c) *A wide tabulation for r satisfies* T0 *and so is a tabulation.*
(d) *If* (f, ϱ, g) *is a tabulation for r, then* $(r\varepsilon)(\varrho f^*): gf^* \Rightarrow r$ *is invertible.*
(e) *If f is a map, then* $(f, \eta, 1)$ *is a wide tabulation for* f^*.

Proof. (a) A birepresentation for the homomorphism is an object $f: R \to A$ of $\mathscr{B} /\!/ {}^*A$ and an equivalence

$$(\mathscr{B} /\!/ {}^*A)(u, f) \simeq (\mathscr{B} /\!/ B)(ru, 1_B)$$

which is a strong transformation in $u \in \mathscr{B} /\!/ {}^*A$. To give this equivalence is precisely to give $g: R \to B$ and $\varrho: g \Rightarrow rf$ satisfying T1, T2.

(b) Delete '$*$' in the proof of (a).

(c) Apply T1 with $X = A$, $u = 1_A$, $v = r$, $\omega = 1_r$ to obtain a candidate for f^* and a candidate for the counit. Apply the strong T2 with $w = 1_R$, $w'f^*f$ to obtain the unit and the adjunction conditions. (Note that $gf^* \cong r$ so (d) is clear here.)

(d) Apply T1 with $X = A$, $u = 1_A$, $v = r$, $\omega = 1_R$, to obtain f', $\theta': ff' \Rightarrow 1_A$, $\nu: r \cong gf'$ with $1_R = (r\theta')(\varrho f')\nu$. Apply T2 with $u = 1_A$, $w = f^*$, $w' = f'$, $\theta = \varepsilon: ff^* \Rightarrow 1$, $\theta': ff' \Rightarrow 1$, $\beta = \nu(r\varepsilon)(\varrho f^*)$ to obtain $\gamma: f^* \Rightarrow f'$ with $g\gamma = \nu(r\varepsilon)(\varrho f^*)\varepsilon = \theta'(f\gamma)$. The last equation implies $\gamma: f^* \Rightarrow f'$ is a split monic (coretraction), while the calculation:

$$\begin{aligned}(g\gamma)(gf^*\theta')(g\eta f') &= \nu(r\varepsilon)(\varrho f^*)(gf^*\theta')(g\eta f') \\ &= \nu(r\varepsilon)(rff^*\theta')(\varrho f^*ff')(g\eta f') \\ &= \nu(r\theta')(r\varepsilon ff')(rf\eta f')(\varrho f') \\ &= \nu(r\theta')(\varrho f') = 1_{gf'},\end{aligned}$$

shows that $g\gamma$ is a split epic. So $g\gamma = \nu(r\varepsilon)(\varrho f^*): gf^* \Rightarrow gf'$ is invertible. So $(r\varepsilon)(\varrho f^*) = \nu^{-1}(g\gamma)$ is invertible.

(e) Since 2-cells $\omega: v \Rightarrow f^*u$ are in bijection with 2-cells $\theta: fw \Rightarrow u$ with $v = w$, the stronger form of T1 follows; the strong form of T2 is clear since $g = 1$. $\square$

2. Spans

Let $\mathscr{E}$ denote a category with pullbacks. The bicategory Span($\mathscr{E}$) is defined as follows. The objects are those of $\mathscr{E}$. An arrow $r: A \to B$ is a *span* $r = (r_0, R, r_1)$:

$$A \xleftarrow{r_0} R \xrightarrow{r_1} B$$

in $\mathscr{E}$. Composition of $r: A \to B$, $s: B \to C$ is obtained by forming the pullback of r_1, s_0. A 2-cell $\sigma: r \Rightarrow r'$ is an arrow $\sigma: R \to R'$ in $\mathscr{E}$ such that $r_0'\sigma = r_0$, $r_1'\sigma = r_1$.

The next result was stated in [2] without proof.

Proposition 2. *An arrow* $r = (r_0, R, r_1): A \to B$ *in* Span($\mathscr{E}$) *is a map if and only if* $r_0: \mathrm{R} \to A$ *is invertible in* $\mathscr{E}$.

Proof. Any arrow isomorphic to a map is a map, so, in order to prove r is a map when r_0 is invertible, it suffices to assume $r=(1,A,f)$. Let $s=(f,A,1):B\to A$. Then $rs=(f,A,f)$ and $sr=(k_0,K,k_1)$ where k_0,k_1 form the kernel pair of f. Let $d:A\to K$ be the arrow in $\mathscr{E}$ with $k_0d=k_1d=1_A$. Then $f:rs\Rightarrow 1_B$, $d:1_A\Rightarrow sr$ are counit, unit for $r\dashv s$.

Conversely, suppose $r\dashv s$ with counit $\varepsilon:rs\Rightarrow 1$, unit $\eta:1\Rightarrow sr$. Form the pullbacks:

$$
\begin{tikzcd}
P \arrow[r, "p_0"] \arrow[d, "p_1"'] & S \arrow[d, "s_1"] \\
R \arrow[r, "r_0"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \arrow[r, "q_0"] \arrow[d, "q_1"'] & R \arrow[d, "r_1"] \\
S \arrow[r, "s_0"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[r, "t_0"] \arrow[d, "t_1"'] & Q \arrow[d, "q_1"] \\
P \arrow[r, "p_0"'] & S
\end{tikzcd}
$$

Then $\eta:A\to Q$ with $r_0q_0\eta=s_1q_1\eta=1$ and $\varepsilon:P\to B$ with $\varepsilon=s_0p_0=r_1p_2$. Moreover, $s\eta:R\to T$ is defined by $t_0(s\eta)=\eta r_0$, $p_1t_1(s\eta)=1$; and $\varepsilon s:T\to R$ is just q_0t_0. So the adjunction condition gives

$$1=(\varepsilon s)(s\eta)=(q_0t_0)(s\eta)=q_0\eta r_0.$$

Thus r_0 has inverse $q_0\eta$. $\square$

Recall [1], [5] that the *classifying category* $\mathrm{C}\,\mathscr{W}$ of a bicategory $\mathscr{W}$ has the same objects as $\mathscr{W}$ and has as arrows the isomorphism classes of arrows in $\mathscr{W}$. Proposition 2 gives an equivalence of categories:

$$\mathscr{E}\simeq \mathrm{C}\ \mathrm{Span}(\mathscr{E})^*.$$

Proposition 3. *Each arrow r in* Span($\mathscr{E}$) *has a wide tabulation (f,ϱ,g) where g is a map.*

Proof. Suppose $r=(r_0,R,r_1):A\to B$ and put $f=(1,R,r_0)$, $g=(1,R,r_1)$. Let k_0,k_1 form a kernel pair for r_0 and define ϱ by $k_0\varrho=k_1\varrho=1_R$. We must show that (f,ϱ,g) is a wide tabulation of r. Take $u=(u_0,U,u_1):X\to A$, $v=(v_0,V,v_1):X\to B$, $\omega:v\Rightarrow ru$ as in T1. Let P be the pullback of u_1,r_0.

$$
\begin{tikzcd}
 & & V \arrow[dddll, "v_0"'] \arrow[dddrr, "v_1"] \arrow[d, "\omega"] & & \\
 & & P \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & & \\
 & U \arrow[dl, "u_0"] \arrow[dr, "u_1"'] & & R \arrow[dl, "r_0"] \arrow[dr, "r_1"'] & \\
X & & A & & B
\end{tikzcd}
$$

Let $w=(v_0, V, p_1\omega): X\to R$, $\theta=p_0\omega: fw\Rightarrow u$, $\nu=1: v\Rightarrow gw$; so $\omega=(r\theta)(\varrho w)\nu$ as required.

Take $u, w, w', \theta, \theta', \beta$ as in T2 and note that $fw=(w_0, W, r_0 w_1)$, $gw=(w_0, W, r_1 w_1)$, etc. So $\beta: W\to W'$ in $\mathcal{C}$ satisfies $w_0=w_0'\beta$, $r_1 w_1 = r_1 w_1'\beta$. But the equation $(r\theta)(\varrho w) = (r\theta')(\varrho w')\beta$ gives $w_1=w_1'$. So $\gamma=\beta: w\Rightarrow w'$ is unique with $\beta = g\gamma$, $\theta=\theta'(f\gamma)$. □

Theorem 4. *A bicategory $\mathcal{B}$ is biequivalent to* Span($\mathcal{C}$) *for some category $\mathcal{C}$ with pullbacks if and only if $\mathcal{B}$ satisfies the following three conditions:*

(i) *Each arrow r is isomorphic to gf^* for some maps f, g.*

(ii) *For all maps f, g with the same source, there exist an arrow r and 2-cell $\varrho: g\Rightarrow rf$ such that (f, ϱ, g) is a tabulation of r.*

(iii) *Any two 2-cells $f\Rightarrow f'$ between maps f, f' are equal and invertible.*

Proof. Span($\mathcal{C}$) satisfies the conditions by Propositions 2 and 3. The conditions are invariant under biequivalence, so we have proved 'only if'.

Suppose $\mathcal{B}$ satisfies the conditions. It is useful to observe that, if g and gw are maps, then so is w (for, by (i) there are maps m, n with $w\cong nm^*$, so, by (ii), we have two tabulations $(1, \varrho, gw)$, (m, σ, gn) of gw; since tabulations are unique up to equivalence, m is invertible and $w\cong nm^*$ is a map).

From the remark preceding Proposition 3 we see that we must take $\mathcal{C} = \mathrm{C}\,\mathcal{B}^*$. Condition (iii) implies that $\mathcal{C}$ is biequivalent to $\mathcal{B}^*$.

To prove $\mathcal{C}$ has pullbacks, take $h: A\to C$, $k: B\to C$ to be maps in $\mathcal{B}$. By (i), (ii), the arrow k^*h has a tabulation (f, ϱ, g) with g a map.

$$\begin{tikzcd} R \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\varrho", near start] & B \arrow[d, "k"] \\ A \arrow[ur, "k^*h"] \arrow[r, "h"'] \arrow[ur, phantom, "\Downarrow\varepsilon h", near end, shift right=3] & C \end{tikzcd}$$

By (iii) we have $kg\cong hf$. Taking isomorphism classes of maps, we obtain a commutative square in $\mathcal{C}$. To see that this is a pullback, take maps $u: X\to A$, $v: X\to B$ with $hu\cong kv$. By T1, there is $w: X\to R$ with $v\cong gw$ and $fw\Rightarrow u$. Since g, gw are maps, w is too. Then $fw\Rightarrow u$ is invertible by (iii). To prove uniqueness of w in $\mathcal{C}$, suppose $fw'\cong u$, $gw'\cong v$ with w' a map. Let β be the composite $gw\cong v\cong gw'$. In order to apply T2, we must verify the compatibility condition which involves the equality of two 2-cells $gw\Rightarrow k^*hu$. Such 2-cells correspond to 2-cells $kgw\Rightarrow hu$ and there is at most one such by (iii). So T2 applies to yield $\gamma: w\Rightarrow w'$ which is invertible by (iii). So w, w' become equal in $\mathcal{C}$.

It remains to define a biequivalence $F: \mathcal{B}\to$Span($\mathcal{C}$). On objects it is the identity. An arrow $r: A\to B$ in $\mathcal{B}$ is taken to the span $Fr: A\to B$ made up of the isomorphism classes of maps f, g with $r\cong gf^*$ (this uses (i) and makes a choice). Suppose $r\cong gf^*$, $s\cong kh^*$ in $\mathcal{B}(A, B)$ are obtained by applying (i) to r, s. Then 2-cells $\sigma: r\Rightarrow s$ are in bijection with 2-cells $gf^* \Rightarrow kh^*$ which are in bijection with 2-cells $g \Rightarrow kh^*f$ (using $f \dashv f^*$). By (ii) and T1, such 2-cells lead to arrows w with $g \cong kw$, $hw \Rightarrow f$. Since k, kw are maps, w is a map; and, by (iii), $hw \cong f$. Put $F\sigma : Fr \Rightarrow Fs$ equal to the isomorphism class of w. Using T2 and (iii), we see that the functor

$$F : \mathscr{B}(A, B) \to \mathrm{Span}(\mathscr{E})(A, B)$$

is fully faithful, and so is clearly an equivalence. From the description above of pullbacks in $\mathscr{E}$ it is also clear that $F : B \to \mathrm{Span}(\mathscr{E})$ really is a homomorphism. A homomorphism which is bijective on objects and a local equivalence is certainly a biequivalence. □

Remarks. (1) For categories $\mathscr{E}, \mathscr{E}'$ with pullbacks, it follows that the category of pullback preserving functors $\mathscr{E} \to \mathscr{E}'$ is biequivalent to the bicategory of tabulation preserving homomorphisms $\mathrm{Span}(\mathscr{E}) \to \mathrm{Span}(\mathscr{E}')$. Furthermore, tabulation preserving implies wide tabulation preserving in this case.

(2) It is easy to see that $\mathscr{B}$ is biequivalent to $\mathrm{Span}(\mathscr{E})$ for some $\mathscr{E}$ with finite limits if and only if $\mathscr{B}$ satisfies (i), (ii), (iii) of the Theorem and:

(iv) *There exists an object* 1 *of* $\mathscr{B}$ *such that each hom-category* $\mathscr{B}(A, 1)$ *has a terminal object which is a map.*

It follows that each hom-category $\mathscr{B}(A, B)$ is finitely complete.

(3) Recall that a category $\mathscr{E}$ is called *internally complete* ('locally cartesian closed' or 'closed span') when each $\mathscr{E}/A$ is cartesian closed. It follows now from [4] that $\mathscr{B}$ is biequivalent to $\mathrm{Span}(\mathscr{E})$ for some internally complete $\mathscr{E}$ if and only if $\mathscr{B}$ satisfies (i), (ii), (iii), (iv) and:

(v) *All right extensions exist.*

3. Relations

A *relation* $r : A \to B$ in a category $\mathscr{E}$ is a span $r : A \to B$ such that any two 2-cells $s \Rightarrow r$ in $\mathrm{Span}(\mathscr{E})$ are equal. If $a : X \to A$, $b : X \to B$ are arrows in $\mathscr{E}$ we write $a(r)b$ when there exists a 2-cell $(a, X, b) \Rightarrow r$ in $\mathrm{Span}(\mathscr{E})$; we say that *$a$ is r-related to b.*

An arrow $e : Y \to X$ in $\mathscr{E}$ is called *strong epic* when, for all relations $r : A \to B$ and arrows $a : X \to A$, $b : X \to B$, if $ae(r)be$, then $a(r)b$. In the presence of pullbacks, strong epic implies epic. A strong epic which is monic is invertible.

A category $\mathscr{E}$ is called *regular* when:

R1. Pullbacks exist.

R2. For each span $s = (s_0, S, s_1) : A \to B$, there exists a relation $r = (r_0, R, r_1) : A \to B$ and a strong epic $e : S \to R$ such that $r_0 e = s_0$, $r_1 e = s_1$.

R3. Each pullback of a strong epic is strong epic.

For a regular category $\mathscr{E}$, there is a bicategory $\mathrm{Rel}(\mathscr{E})$ defined as follows. The objects are those of $\mathscr{E}$. An arrow $r : A \to B$ is a relation. Composition of relations $r : A \to B$, $s : B \to C$ is obtained by composing as spans and then applying R2 to obtain a relation $sr: A \to C$; it is easily seen using R3 that $a(sr)c$ if there are b and strong epic e with $ae(r)b$ and $b(s)ce$. The 2-cells are those of spans; however, note that $\mathrm{Rel}(\mathscr{E})(A,B)$ is an ordered set.

Proposition 5. *An arrow $r=(r_0,R,r_1): A \to B$ in* $\mathrm{Rel}(\mathscr{E})$ *is a map if and only if $r_0: R \to A$ is invertible in $\mathscr{E}$.*

Proof. If r_0 is invertible, then the reverse relation (r_1,R,r_0) provides the right adjoint for r [3], [7].

Conversely, suppose $r \dashv s$. The unit condition is: for all $a: X \to A$, there exists b and a strong e with $ae(r)b$ and $b(s)ae$.

The counit condition amounts to:

$$b(s)a, a(r)b' \quad \text{imply} \quad b=b'.$$

From the former with $a=1_A$ we get $e(r)b$ with e strong epic. So r_0 is strong epic. It remains to prove r_0 monic. Take $x, s': X \to R$ with $r_0x=r_0x'$. Apply the unit condition with $a=r_0x$ to obtain b and strong epic e with $r_0xe(r)b$, $b(s)r_0xe$. Apply the counit condition to $b(s)r_0xe$, $r_0xe(r)r_1xe$ to obtain $b=r_1xe$; and similarly $b=r_1x'e$. Since r_0,r_1 are jointly monic, $x'e=xe$. Since e is epic, $x=x'$. □

Proposition 6. *Each arrow r in* $\mathrm{Rel}(\mathscr{E})$ *has a tabulation (f,ϱ,g) where g is a map.*

Proof. By Proposition 5 (the easy direction!), we have maps $f=(1,R,r_0)$, $g=(1,R,r_1)$. Assume $c(g)b$. Then $r_1c=b$; so we have $c(f)r_0c$, $r_0c(r)b$ which implies $c(rf)b$. Thus $g \le rf$.

Suppose $u: X \to A$, $v: X \to B$ are relations with $v \le ru$. Then we can define a relation $w: X \to R$ by $x(w)c$ if and only if $x(u)r_0c$ and $x(v)r_1c$. Assume $x(v)b$. Since $v \le ru$, there exist a and strong epic e with $xe(u)a$, $a(r)be$. Let c be such that $r_0c=a$, $r_1c=be$. So $xe(w)c$, $c(g)be$. So $xe(gw)be$. So $x(gw)b$. This proves $v \le gw$. Reversing these steps we get $gw \le v$. So $v \cong gw$. If $x(fw)a$, then $xe(w)c$, $r_0c=ae$ for some c and strong epic e. So $xe(u)r_0c$. So $x(u)a$. So $fw \le u$. This proves T1 (in fact, in the stronger form!).

Suppose u, w, w', $fw \le u$, $fw' \le u$, $gw \le gw'$ as in T2. We must prove $w \le w'$. So take $x(w)c$. Then $fw \le u$, $x(w)c$, $c(f)r_0c$ imply $x(u)r_0c$. Also $gw \le gw'$, $x(w)x$, $c(g)r_1c$ imply $x(gw')r_1c$. So there are c' and strong epic e with $xe(w')c'$, $c'(g)r_1ce$. So $r_1c'=r_1ce$. But $fw' \le u$, $xe(w')c'$, $c'(g)r_0c'$ imply $xe(u)r_0c'$. So we have $xe(u)r_0c'$, $xe(u)r_0ce$. *Since u is a map* it follows that $r_0c'=r_0ce$. Since $r_0,, r_1$ are jointly monic, $c'=ce$. So we have $xe(w')ce$ which implies $x(w')c$ since e is strong epic. □

In a bicategory $\mathscr{B}$ for which each $\mathscr{B}(A,B)$ is an ordered set, equations between 2-cells such as those in T2 hold automatically. This means that T2 is a condition on the pair f,g independent of ϱ. Thus one cannot expect general pairs of maps f,g with the same source to form a tabulation as in Theorem 4(ii) except in very special cases (such as $\mathrm{Rel}(\mathscr{E})$ where $\mathscr{E}$ is an ordered set).

A pair of maps f, g in $\mathscr{B}$ is called *ripe* when f, g have the same source C and, for all maps $a, b : X \to C$ and 2-cells $\alpha : fa \Rightarrow fb$, $\beta : ga \Rightarrow gb$, there exists a unique $\gamma : a \Rightarrow b$ with $f\gamma = \alpha$, $g\gamma = \beta$. Clearly, if $\mathscr{B}$ is locally ordered then each tabulation (f, ϱ, g) has f, g ripe.

Theorem 7. *A bicategory $\mathscr{B}$ is biequivalent to* Rel($\mathscr{E}$) *with $\mathscr{E}$ a regular category if and only if $\mathscr{B}$ satisfies the following three conditions:*

(i) *Each arrow r is isomorphic to gf^* for some ripe pair of maps f, g.*

(ii) *For all ripe pairs of maps f, g there exist an arrow r and a 2-cell $\varrho : g \Rightarrow rf$ such that (f, ϱ, g) is a tabulation of r.*

(iii) *Any two 2-cells with the same source and target arrows are equal, and all 2-cells between maps are invertible.*

Proof. Clearly Rel($\mathscr{E}$) satisfies (iii). For a bicategory $\mathscr{B}$ satisfying (iii), ripeness of a pair of maps f, g amounts to: for maps a, b, if $fa \cong fb$, $ga \cong gb$ then $a \cong b$. So Rel($\mathscr{E}$) satisfies (i), (ii) by Propositions 5 and 6.

Conversely, suppose $\mathscr{B}$ satisfies the conditions. Since (iii) implies $\mathscr{B}$ and C$\mathscr{B}$ are biequivalent we may assume all invertible 2-cells in $\mathscr{B}$ are identities. Each arrow in $\mathscr{B}$ does have a tabulation by (i) and (ii). It is important to observe that, if (f, ϱ, g) is a tabulation of r, then, in T2, the arrow w is a map when v is (and so $u = fw$ using (iii)). To see this, let (m, σ, n) be a tabulation of w. Since $m \dashv m^*$, $fnm^* \cong fw \leq u$ implies $fn \leq um$; so $fn = um$ by (iii). The pair of maps m, gn is ripe; for $ma = nb$, $gna = gnb$ imply $fna = uma = umb = fnb$, and so we have $na = nb$ (since f, g are ripe), so $a = b$ (since m, n are ripe). By (ii), m, gn tabulate $gnm^* = gw = v$. But $1, v$ tabulate v. So n is an isomorphism. So $w = nm^*$ is a map.

Let $\mathscr{E} = \mathscr{B}_*$. We shall show that $\mathscr{E}$ is a regular category. To prove R1 take maps $h : A \to C$, $k : B \to C$ and let f, g tabulate k^*h. So $g \leq k^*hf$ implies $kg \leq hf$ which means $kg = hf$ by(iii). That f, g provide a pullback for h, k now follows from the last paragraph.

To prove R2, take a span $(u, S, v) : A \to B$ in $\mathscr{E}$. Let f, g tabulate vu^*; ripeness means (f, R, g) is a relation in $\mathscr{E}$. By the second last paragraph there exists a map e with $ge = v$, $fe = u$. We claim $ee^* = 1$. To see this, let m, n tabulate ee^*. Then $nm^* = ee^* \leq 1$ gives $n \leq m$ which, using (iii), gives $n = m$. Since m, n form a relation in $\mathscr{E}$ (ripeness), this means m is monic. So fm, m form a ripe pair and so tabulate $m(fm)^* = mm^*f^* = ee^*f^* = e(fe)^* = eu^*$. But T2 applies to give $eu^* = f^*$ since $g(eu^*) = vu^* = gf^*$, $f(eu^*) = uu^* \leq 1$, and $ff^* \leq 1$. So fm, m tabulate f^*. But $f, 1$ tabulate f^*. So m is an isomorphism. So $ee^* = mm^* = 1$. This means R2 will be proved once we prove that any map with identity counit is strong epic in $\mathscr{E}$.

Let $e : Y \to X$ be a map in $\mathscr{B}$ with $ee^* = 1$. Take a relation $(f, R, g) : A \to B$ in $\mathscr{E}$ and a, b in $\mathscr{E}$ with $ae = fc$, $be = gc$. Then $a = aee^* = fce^*$, $b = bee^* = gce^*$. By the third last paragraph, ce^* is a map. So e is a strong epic in $\mathscr{E}$.

Suppose $e : Y \to X$ is a strong epic in $\mathscr{E}$. The reflection of the span $(e, Y, e) : X \to X$ into the subcategory of relations from X to X is the identity relation $(1, X, 1)$. By the last two paragraphs this reflection is also given by the tabulation of ee^*. So $1_X, 1_X$ tabulate ee^*. So $ee^*=1$. Thus strong epics are precisely maps with identity counits.

Now we prove R3. Recall the construction of pullbacks in the proof of R1 above. Suppose further that $h \dashv h^*$ has identity counit. Then $gg^*k^* = gf^*h^* = k^*hh^* = k^*$. This means that the reflection of the span (kg, R, g) into relations from C to B is $(k, B, 1)$. Thus the underlying map g of the 2-cell $(kg, R, g) \Rightarrow (k, B, 1)$ is strong epic in $\mathscr{E}$.

Thus $\mathscr{E}$ is a regular category. The homomorphism $\mathscr{B} \to \mathrm{Rel}(\mathscr{E})$, which is the identity on objects and takes each arrow to a tabulating relation, is clearly a bi-equivalence. □

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