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LIMITS INDEXED BY CATEGORY-VALUED 2-FUNCTORS

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There is common agreement now on the correct general notion of limit for categories whose homs are enriched in a suitable category $\mathcal{V}$. The definition involves a $\mathcal{V}$-functor J: $\mathcal{A} \to \mathcal{V}$ which should be thought of as a $\mathcal{V}$-diagram scheme; for simplicity, we shall suppose $\mathcal{V}$ is complete enough so that the category of $\mathcal{V}$-valued $\mathcal{V}$-functors on $\mathcal{A}$ admits its natural enrichment to a $\mathcal{V}$-category $[\mathcal{A},\mathcal{V}]$. A *J-indexed limit* for a $\mathcal{V}$-functor S: $\mathcal{A} \to \mathcal{X}$ is then an object $\lim(J, S)$ of $\mathcal{X}$ together with a $\mathcal{V}$-natural isomorphism

$$\mathcal{X}(X, \lim(J,S)) = [\mathcal{A},\mathcal{V}](J,\mathcal{X}(X,S)).$$

This concept appears independently in Auderset [1], Borceux–Kelly [2], and Street–Walters (unpublished, 1971). The particular case where $\mathcal{V}$ is the category *Cat* of categories appears in Street [14, § 6], while the case where $\mathcal{V}$ is the category of abelian groups dates back to Freyd [7, Chapter 5, Exercise I]. Completeness for categories was discussed in Day–Kelly [4] where the notions of *end* and *cotensor product* were taken as fundamental; these are both instances of J-indexed limits for suitable J. On the other hand, for a general J, we have the formula

$$\lim(J,S) = \int_A JA \pitchfork SA,$$

expressing a J-indexed limit in terms of ends and cotensor products, provided each of the cotensor products $JA \pitchfork SA$ exists and the end exists. It is however possible for the left-hand side to exist without the existence of the integrand (see Proposition 1).

In this paper we are concerned with the case $\mathcal{V} = Cat$; then $\mathcal{V}$-categories are 2-categories. Very many of the familiar constructions of category theory are in fact examples of J-indexed limits for 2-functors into *Cat*: products of categories, equalizers of functors, functor categories, comma categories, subequalizers, Eilenberg–Moore constructions for monads, identifiers, inverters (dual to localizations), etc. Thus we are led to ask how far these constructions exist in 2-categories $\mathcal{X}$ other than *Cat*. In a mere category, we know that all limits can be constructed from products and equalizers. In a 2-category, ends can be constructed from products and equalizers (we always understand these to be preserved by the representable 2-functors into *Cat*), and cotensor products with small categories; these last can be constructed from products, equalizers and cotensor products with the arrow category **2**; so the above end-formula shows that we can construct $lim(J, S)$ from products, equalizers, and cotensor products with **2**. This naive procedure in fact leads to our Theorem 10 which states that: if $J: \mathcal{A} \to Cat$ is such that the set of 2-cells of $\mathcal{A}$ and the set of arrows of each JA have cardinality less than some regular cardinal α, and if $\mathcal{K}$ admits equalizers, cotensor products with **2**, and products over all indexing sets of cardinality $< \alpha$, then $\mathcal{K}$ admits all J-indexed limits.

When α is uncountable we can say no more than this; but consider the case where α is countable, so that $\mathcal{K}$ admits finite products, equalizers, and cotensor products with **2** (the term "representable 2-category" has been used in Gray [9] and Street [13] to mean essentially this). Such a $\mathcal{K}$ we call *finitely complete*. It is then fairly easy to see how to construct in $\mathcal{K}$ cotensor products with "*finitely-presented*" categories; this shows that certain J-indexed limits may exist in $\mathcal{K}$ even though the categories JA are *not* finite. Moreover, when $\mathcal{A}$ is a "*finitely-generated*" 2-category and J is constant at the terminal category, it is again possible to construct J-indexed limits in $\mathcal{K}$; here each JA is finite but $\mathcal{A}$ is not. So Theorem 10, which requires finiteness of $\mathcal{A}$ and of each JA, is much too weak in the case of *countable* α.

This leads us to ask which 2-functors $J: \mathcal{A} \to Cat$ are such that every 2-functor $S: \mathcal{A} \to \mathcal{K}$ into a finitely complete $\mathcal{K}$ admits a J-indexed limit. Our main result (Theorem 9) gives a sufficient condition on J for this. The condition is that a certain 2-category, called the *cone on J*, should admit a presentation by *computads* satisfying a certain finiteness condition. This provides many examples of J-indexed limit which exist although the integrand in the end-formula does not. One application of our results is the now-well-known fact that any finitely-complete 2-category $\mathcal{K}$ admits the construction of Eilenberg–Moore algebras in the sense of Street [12]. The indexing 2-functor for the Eilenberg–Moore construction is here denoted by $L_{Simp}: Simp \to Cat$; the 2-category *Simp* is certainly not finite, yet is "finitely presented" by computads in a very natural way.

When first attacking this problem, we used *Gph*-graphs rather than computads; a *Gph*-graph is simply a set of objects together with a graph for each ordered pair of objects. Roughly speaking, *Gph*-graphs are what one is led to when vertical and horizontal composition are taken as the basic operations in a 2-category; whereas computads result when "pasting" (see Kelly–Street [10]) is taken as the basic operation.

A different approach to the question of the appropriate notion of "limit" for 2-categories has been taken by Gray [9]. This approach depends on the existence of transformations more general than the obvious *Cat*-natural (= 2-natural) transformations between 2-functors. These more general transformations are here called *lax natural transformations* ("quasi-natural transformations" by Gray [9], and "catadeses" by Bourn [3]). Gray considers lax natural transformations between 2-functors out of $\mathcal{A}$ which are actually 2-natural (that is, the relevant 2-cells are identities) on some given subcategory $\mathcal{B}$ of $\mathcal{A}$, or pseudo-natural (that is, the relevant 2-cells are isomorphisms) on another given subcategory $\mathcal{B}'$ of $\mathcal{A}$. He then defines a *cartesian quasi-limit relative to $\mathcal{B}$ or to $\mathcal{B}'$*, for a 2-functor $S : \mathcal{A} \to \mathcal{K}$, to be an object X of $\mathcal{K}$ together with a 2-universal lax natural transformation, of the type specified in the last sentence, from the constant 2-functor at X to S.

We prove here that every cartesian quasi-limit is a J-indexed limit for a suitable J. Conversely, we show that each J-indexed limit is a cartesian quasi-limit (in fact, of the type involving a $\mathcal{B}$ rather than a $\mathcal{B}'$). This shows the equivalence of Gray's notion of limit with ours, in the context of 2-categories, = *Cat*-categories. Yet when *Cat* is replaced by a general $\mathcal{V}$, our notation continues to make sense, while "lax natural transformations" have no analogue.

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1. The definition

For any integer $n \geqslant 0$, we write $\mathbf{n}$ for the corresponding ordinal which is a category by virtue of its being a poset. A *terminal object* in a 2-category $\mathcal{K}$ is an object K such that $\mathcal{K}(X, K) \cong \mathbf{1}$ for all objects X.

For 2-categories $\mathcal{A}, \mathcal{K}$, we denote by $[\mathcal{A}, \mathcal{K}]$ the 2-category with the universal property that 2-functors $\mathcal{X} \to [\mathcal{A}, \mathcal{K}]$ are in natural bijection with 2-functors $\mathcal{X} \times \mathcal{A} \to \mathcal{K}$. The objects of $[\mathcal{A}, \mathcal{K}]$ are 2-functors $\mathcal{A} \to \mathcal{K}$, the arrows are 2-natural transformations and the 2-cells are modifications. Let $\Delta : \mathcal{K} \to [\mathcal{A}, \mathcal{K}]$ denote the 2-functor corresponding to the first projection $\mathcal{K} \times \mathcal{A} \to \mathcal{K}$. Let *Cat* denote a 2-category of categories large enough to receive the hom-2-functor $\mathcal{K}(\ ,\ ) : \mathcal{K}^{op} \times \mathcal{K} \to Cat$ of $\mathcal{K}$. Let *CAT* denote a 2-category of categories containing *Cat* and large enough to receive the hom-2-functor of $[\mathcal{A}, Cat]$.

Given a 2-functor $J : \mathcal{A} \to Cat$, let $\langle J, \mathcal{K} \rangle$ denote the 2-category described as follows. The objects are triples (K, κ, S) where K is an object of $\mathcal{K}$, where $S : \mathcal{A} \to \mathcal{K}$ is a 2-functor, and where $\kappa : J \to \mathcal{K}(K, S)$ is a natural transformation. A 2-cell

$$\begin{tikzcd}[column sep=huge] (K, \kappa, S) \arrow[r, bend left, "{(u, x)}"] \arrow[r, bend right, "{(v, y)}"'] \arrow[r, phantom, "\Downarrow (\sigma, \xi)"] & (K', \kappa', S') \end{tikzcd}$$

consists of a 2-cell $\begin{tikzcd} K \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\sigma"] & K' \end{tikzcd}$ in $\mathcal{K}$ and a modification $\begin{tikzcd} S \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow\xi"] & S' \end{tikzcd}$ such that the following composites are equal:

$$\begin{tikzcd} J \arrow[r, "\kappa"] & \mathcal{X}(K,S) \arrow[rr, bend left=40, "\mathcal{X}(1,x)"] \arrow[rr, bend right=40, "\mathcal{X}(1,y)"'] \arrow[rr, phantom, "\Downarrow \mathcal{X}(1,\xi)"] & & \mathcal{X}(K,S') \end{tikzcd} =$$

$$\begin{tikzcd} J \arrow[r, "\kappa'"] & \mathcal{X}(K',S') \arrow[rr, bend left=40, "\mathcal{X}(u,1)"] \arrow[rr, bend right=40, "\mathcal{X}(1,y)"'] \arrow[rr, phantom, "\Downarrow \mathcal{X}(\sigma,1)"] & & \mathcal{X}(K,S') \end{tikzcd}.$$

There are obvious projection 2-functors

$$\mathcal{X} \xleftarrow{d_0} \langle J, \mathcal{X}\rangle \xrightarrow{d_1} [\mathcal{A},\mathcal{X}].$$

Given a 2-functor $S: \mathcal{A} \to \mathcal{X}$, we can form the pullback

$$\begin{tikzcd} \langle J,\mathcal{X}\rangle_S \arrow[r] \arrow[d] & \langle J,\mathcal{X}\rangle \arrow[d, "d_1"] \\ \mathbf{1} \arrow[r, "\ulcorner S \urcorner"'] & {[\mathcal{A},\mathcal{X}]}. \end{tikzcd}$$

An object (K, κ) of $\langle J,\mathcal{X}\rangle_S$ is called a *J-indexed cone over S.* A terminal object of the 2-category $\langle J,\mathcal{X}\rangle_S$ is called a *J-indexed limit for S.* In other words, a J-indexed limit for S consists of an object L of $\mathcal{X}$ together with a 2-natural transformation $\lambda: J \to \mathcal{X}(L, S)$ which induces an isomorphism of categories

$$\mathcal{X}(X,L) \cong [\mathcal{A},Cat](J,\mathcal{X}(X,S)).$$

A particular choice of such an object L is denoted by $lim(J,S)$; it is uniquely determined up to a unique isomorphism in $\mathcal{X}$. (This is a specialization to the case $\mathcal{V} = Cat$ of the limit-notion discussed in [1] and [2].)

If $J: \mathcal{B}^{op} \times \mathcal{A} \to Cat$ and $S: \mathcal{A} \to \mathcal{X}$ are 2-functors such that $lim(J(B,-),S)$ exists for all B in $\mathcal{B}$, then we can define a 2-functor $L: \mathcal{B} \to \mathcal{X}$ with the property that there is an isomorphism of categories

$$\mathcal{X}(X,LB) \cong [\mathcal{A},Cat](J(B,-),\mathcal{X}(X,S))$$

which is 2-natural in X and B. In particular, for a 2-functor $U: \mathcal{A} \to \mathcal{B}$, if each $LB = lim(\mathcal{B}(B,U),S)$ exists, we obtain a 2-functor $L: \mathcal{B} \to \mathcal{X}$; a familiar "Yoneda's lemma" argument proves that L is a right Kan extension of S along U in the sense that there is a natural bijection between 2-natural transformations $M \to L$ and 2-natural transformations $MU \to S$. When U is the identity 2-functor on $\mathcal{A}$, the right Kan extension of S along U is clearly S, so there is an isomorphism $SB \cong lim(\mathcal{B}(B,-), S)$ which is 2-natural in B; in fact, *a priori* knowledge of the existence of the right-hand side of this isomorphism is not required.

Proposition 1. *For a representable 2-functor $J = \mathcal{A}(B,-)$: $\mathcal{A} \to Cat$ and any 2-functor S: $\mathcal{A} \to \mathcal{K}$, the object SB and the 2-natural transformation with components $S_{BA}: \mathcal{A}(B,A) \to \mathcal{K}(SB, SA)$ form a J-indexed limit for S.* □

In general, however, the existence of indexed cartesian limits is difficult to prove, and the usual procedure is to construct them from certain special kinds. We shall now discuss some of these special kinds.

When $J = \Delta(1)$: $\mathcal{A} \to Cat$, a J-indexed limit is simply called a *limit*, and we write $limS$ instead of $lim(\Delta(1), S)$. The defining isomorphism becomes:

$$\mathcal{K}(X, lim\,S) \cong [\mathcal{A}, \mathcal{K}](\Delta(X), S).$$

If $\mathcal{A}$ is a set regarded as a 2-category with only identity arrows and 2-cells, then S amounts to a family of objects SA of $\mathcal{K}$ indexed by $\mathcal{A}$; a limit of S is called a *product*, and instead of $limS$ we write $\Pi_{A \in \mathcal{A}} SA$. If $\mathcal{A}$ is the empty set, a product is precisely a terminal object as defined before. If $\mathcal{A}$ is one of the 2-categories indicated by the following diagrams (where we have not drawn-in identity arrows or 2-cells)

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "f'"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] & B, \end{tikzcd}$$

a limit of a 2-functor S: $\mathcal{A} \to \mathcal{K}$ is called an *equalizer*, a *pullback*, an *identifier*, respectively, of the corresponding diagram in $\mathcal{K}$.

Given a category C and an object Z of the 2-category $\mathcal{K}$, a *cotensor product of Z with C* is an object $C \pitchfork Z$ of $\mathcal{K}$ together with an isomorphism of categories

$$\mathcal{K}(X, C \pitchfork Z) \cong [C, \mathcal{K}(X, Z)]$$

which is 2-natural in X. Clearly this precisely amounts to a $\Delta(C)$-indexed limit for $\Delta(Z)$: $1 \to \mathcal{K}$. Let $diag$: $Z \to C \pitchfork Z$ denote the arrow corresponding to the composite functor

$$C \longrightarrow 1 \xrightarrow{\ulcorner 1_X \urcorner} \mathcal{K}(X, X)$$

under the above isomorphism. Note that arrows $X \to 2 \pitchfork Z$ correspond to 2-cells $\begin{tikzcd} X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & Z \end{tikzcd}$; and, in particular, the identity arrow $2 \pitchfork Z \to 2 \pitchfork Z$ corresponds to a 2-cell $\begin{tikzcd} 2 \pitchfork Z \arrow[r, bend left, "d_0"] \arrow[r, bend right, "d_1"'] \arrow[r, phantom, "\Downarrow\lambda"] & Z. \end{tikzcd}$

Comma objects, the Eilenberg–Moore constructions, inverters (dual to localizations), subequalizers, etc. are all examples of indexed limits. These were treated in Street [14, pp. 166–168, 173], but will be discussed again at the end of this paper in the light of the present work.

2. Computads

A *graph* G is an ordered pair of functions $d_0, d_1 : G_1 \to G_0$ with the same source set G_1 and the same target set G_0. Elements of G_0 are called *objects* of G and elements of G_1 are called *arrows* in G; we write an arrow γ in G as $\gamma : g \to g'$ when $d_0(\gamma) = g$, $d_1(\gamma) = g'$; and we write $G(g, g')$ for the set of $\gamma : g \to g'$. A *map* $k : G \to H$ *of graphs* is just a map of diagrams of sets; so it assigns to each object g of G an object kf of H, and to each arrow $\gamma : g \to g'$ in G an arrow $k\gamma : kg \to kg'$ in H. This describes a category of graphs which is the target for an obvious forgetful functor from the category of categories. This forgetful functor (which we given no name) has a left adjoint which we denote by $\mathcal{F}$. Note that G and $\mathcal{F}F$ have the same objects, and the arrows of $\mathcal{F}G$ are strings $g_0 \to g_1 \to \ldots \to g_m$ (including the empty string when $g_0 = g_m$) of arrows in G. Call a graph G *trim* when $(d_0, d_1) : G_1 + G_1 \to G_0$ is surjective.

A *computad* $\mathcal{G}$ consists of a graph $|\mathcal{G}|$ together with, for each ordered pair of objects A, B of $|\mathcal{G}|$, a trim graph $\mathcal{G}(A, B)$ such that $\mathcal{G}(A, B)_0$ is a subset of $(\mathcal{F}|\mathcal{G}|)(A, B)$. Objects and arrows of $|\mathcal{G}|$ are called *objects* and *arrows* of $\mathcal{G}$; an arrow ρ in $\mathcal{G}(A, B)$ with

$$d_0(\rho) = (A \xrightarrow{f_0} \xrightarrow{f_1} \ldots \xrightarrow{f_m} B),$$

$$d_1(\rho) = (A \xrightarrow{f'_0} \xrightarrow{f'_1} \ldots \xrightarrow{f'_n} B),$$

is denoted by a diagram

$$\begin{tikzcd}
 & \cdot \arrow[r, "f_1"] & \cdots \arrow[dr, "f_m"] & \\
A \arrow[ur, "f_0"] \arrow[dr, "f'_0"'] & & \Downarrow \rho & B, \\
 & \cdot \arrow[r, "f'_1"'] & \cdots \arrow[ur, "f'_n"'] &
\end{tikzcd}$$

and called a *2-cell* in $\mathcal{G}$. A *map* $T : \mathcal{G} \to \mathcal{H}$ *of computads* consists of a map $T = |T| : |\mathcal{G}| \to |\mathcal{H}|$ of graphs together with a function which assigns, to each 2-cell

ρ in $\mathcal{G}$ as above, a 2-cell $T\rho$ in $\mathcal{H}$ as below:

$$
\begin{tikzcd}
 & \cdot \arrow[r, "Tf_1"] & \cdots & \cdot \arrow[dr, "Tf_m"] & \\
TA \arrow[ur, "Tf_0"] \arrow[dr, "Tf'_0"'] & & \Downarrow T\rho & & TB \\
 & \cdot \arrow[r, "Tf'_1"'] & \cdots & \cdot \arrow[ur, "Tf'_n"'] &
\end{tikzcd}
$$

This describes a category *Cmp* of computads.

Each 2-category $\mathcal{X}$ determines a computad $\mathcal{U}\mathcal{X}$ described as follows. The graph $|\mathcal{U}\mathcal{X}|$ is the graph underlying the category $|\mathcal{X}|$ obtained from $\mathcal{X}$ by ignoring the 2-cells; that is, the objects and arrows of $\mathcal{U}\mathcal{X}$ are those of $\mathcal{X}$. For objects A, B of $\mathcal{X}$, the set $(\mathcal{U}\mathcal{X})(A,B)_0$ is $(\mathcal{F}|\mathcal{U}\mathcal{X}|)(A,B)$. Strings of arrows in $\mathcal{X}$ can be composed, so we have a function $comp\colon (\mathcal{U}\mathcal{X})(A,B)_0 \to \mathcal{X}(A,B)$. Form the following inverse limit of sets:

$$
\begin{tikzcd}
 & & (\mathcal{U}\mathcal{X})(A,B)_1 \arrow[dll, dashed, "d_0"'] \arrow[d, dashed, "e"] \arrow[drr, dashed, "d_1"] & & \\
(\mathcal{U}\mathcal{X})(A,B)_0 \arrow[dr, "comp"'] & & \mathcal{X}(A,B)_1 \arrow[dl, "d_0"] \arrow[dr, "d_1"'] & & (\mathcal{U}\mathcal{X})(A,B)_0 \arrow[dl, "comp"] \\
 & \mathcal{X}(A,B)_0 & & \mathcal{X}(A,B)_0 &
\end{tikzcd}
$$

The graph $(\mathcal{U}\mathcal{X})(A,B)$ is the ordered pair of functions d_0, d_1 dotted in the above diagram. Each 2-functor $K\colon \mathcal{X} \to \mathcal{Y}$ induces a map $\mathcal{U}K\colon \mathcal{U}\mathcal{X} \to \mathcal{U}\mathcal{Y}$ of computads given by $|\mathcal{U}K| = |K|\colon |\mathcal{X}| \to |\mathcal{Y}|$, and, for $\xi \in (\mathcal{U}\mathcal{X})(A,B)_1$,

$$K\xi = ((\mathcal{F}|K|)d_0(\xi), Ke(\xi), (\mathcal{F}|K|)d_1(\xi)).$$

This describes a functor $\mathcal{U}\colon |2\text{-}Cat| \to Cmp$. (We emphasize for the last time that $|\mathcal{X}|$ denotes the category underlying the 2-category $\mathcal{X}$, or the graph underlying the computad $\mathcal{X}$, as appropriate.)

Given a computad $\mathcal{G}$, we shall now describe a 2-category $\mathcal{F}\mathcal{G}$ with the same objects as $\mathcal{G}$. For objects A, B of $\mathcal{G}$, two graphs $\mathcal{G}^1(A,B)$, $\mathcal{G}^2(A,B)$ are defined as follows:

$$\mathcal{G}^1(A,B)_0 = \mathcal{G}^2(A,B)_0 = (\mathcal{F}|\mathcal{G}|)(A,B),$$

$$\mathcal{G}^1(A,B)_1 = \{(s,\rho,r)|A \xrightarrow{r} A', B' \xrightarrow{s} B \text{ in } \mathcal{F}|\mathcal{G}|; \rho \in \mathcal{G}(A',B')_1\},$$

$$\mathcal{G}^2(A,B)_1 = \{(t,\sigma,s,\rho,r)|A \xrightarrow{r} A', A'' \xrightarrow{s} B'', B' \xrightarrow{t} B \text{ in } \mathcal{F}|\mathcal{G}|; \rho \in \mathcal{G}(A',A'')_1,$$
$$\sigma \in \mathcal{G}(B'',B')_1\},$$

$$d_i(s,\rho,r) = sd_i(\rho)r,\ d_i(t,\sigma,s,\rho,r) = td_i(\sigma)sd_i(\rho)r \text{ for } i = 0, 1.$$

Elements of $\mathcal{G}^1(A,B)_1$ can be written as diagrams:

$$\begin{tikzcd} A \arrow[r, "r"] & A' \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\rho"] & B' \arrow[r, "s"] & B, \end{tikzcd}$$

and elements of $\mathcal{G}^2(A,B)_1$ as diagrams:

$$\begin{tikzcd} A \arrow[r, "r"] & A' \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\rho"] & A'' \arrow[r, "s"] & B'' \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\sigma"] & B' \arrow[r, "t"] & B. \end{tikzcd}$$

Next form the coequalizer

$$\begin{tikzcd} \mathcal{F}(\mathcal{G}^2(A,B)) \arrow[r, shift left, "\theta"] \arrow[r, shift right, "\phi"'] & \mathcal{F}(\mathcal{G}^1(A,B)) \arrow[r, "{[\ ]}"] & (\mathcal{F}\mathcal{G})(A,B) \end{tikzcd}$$

in the category of categories, where θ, ϕ, [ ] are the identity on objects and

$$\theta(t,\sigma,s,\rho,r) = (t,\sigma,sd_1(\rho)r)(td_0(\sigma)s,\rho,r),$$

$$\phi(t,\sigma,s,\rho,r) = (td_1(\sigma)s,\rho,r)(t,\sigma,sd_0(\rho)r).$$

To complete the definition of the 2-category $\mathcal{F}\mathcal{G}$, it remains to describe the composition functor

$$(\mathcal{F}\mathcal{G})(B,C) \times (\mathcal{F}\mathcal{G})(A,B) \longrightarrow (\mathcal{F}\mathcal{G})(A,C).$$

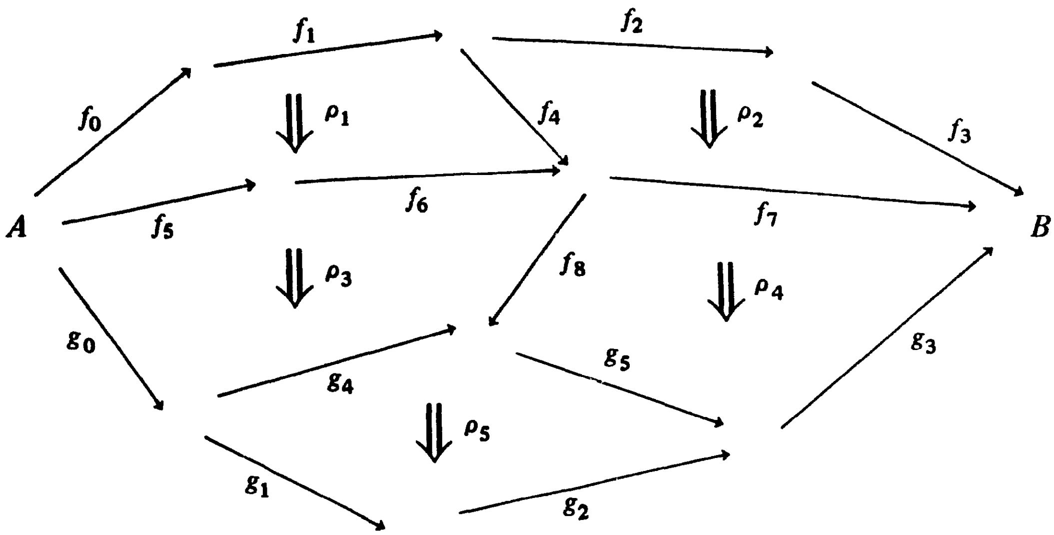
On objects, this functor is given by the composition function

$$(\mathcal{F}|\mathcal{G}|)(B,C) \times (\mathcal{F}|\mathcal{G}|)(A,B) \longrightarrow (\mathcal{F}|\mathcal{G}|)(A,C)$$

of the category $\mathcal{F}|\mathcal{G}|$. On arrows, it suffices to give the value on generators, and this is expressed by the assignment:

$$([s_1,\rho_1,r_1],[s,\rho,r]) \longmapsto [(s_1,\rho_1,r_1sd_1(\rho)r)(s_1d_0(\rho_1)r_1s,\rho,r)];$$

and note, this value is equal to $[(s_1d_1(\rho_1)r_1s,\rho,r)(s_1,\rho_1,r_1sd_0(\rho)r)]$. We leave it to the reader to supply the direct verifications that composition is well defined and associative, and that the identity 2-cell $A \underset{1_A}{\overset{1_A}{\Downarrow 1}} A$ in $\mathcal{F}\mathcal{G}$ is the equivalence class of the empty string in $\mathcal{F}(\mathcal{G}^1(A,A))$ from $1_A : A \to A$ to itself. Let $N : \mathcal{G} \to \mathcal{U}\mathcal{F}\mathcal{G}$ denote the map of computads which is the identity on objects, takes an arrow f in $\mathcal{G}$ to the string f in $\mathcal{F}|\mathcal{G}|$, and takes a 2-cell ρ in $\mathcal{G}$ to the 2-cell $(d_0(\rho), [1_B, \rho, 1_A], d_1(\rho))$ in $\mathcal{U}\mathcal{F}\mathcal{G}$. The 2-cells in $\mathcal{F}\mathcal{G}$ can be thought of as being built up from the 2-cells in $\mathcal{G}$ by the operation of "pasting" as loosely described in Kelly–Street [10, pp. 70–80]. For example, the diagram:



where all the arrows and 2-cells are in $\mathcal{G}$, represents the 2-cell

$$[(g_3, \rho_5, g_0)(g_3 g_5, \rho_3, 1_A)(1_B, \rho_4, f_6 f_5)(f_7, \rho_1, 1_A)(1_B, \rho_2, f_1 f_0)]$$

in $\mathcal{F}\mathcal{G}$.

Note that graphs can be regarded as computads with no 2-cells, and in this way our use of $\mathcal{F}$ on graphs and computads is not ambiguous.

Theorem 2. *The functor $\mathcal{U} : |2\text{-}Cat| \to Cmp$ has a left adjoint whose value at the computad $\mathcal{G}$ is $\mathcal{F}\mathcal{G}$. The maps $N : \mathcal{G} \to \mathcal{U}\mathcal{F}\mathcal{G}$ of computads are the components of the unit of this adjunction.*

Proof. Let $\mathcal{X}$ be a 2-category and $T : \mathcal{G} \to \mathcal{U}\mathcal{X}$ a map of computads. We must define a 2-functor $K : \mathcal{F}\mathcal{G} \to \mathcal{X}$ unique with the property that $\mathcal{U}K \cdot N = T$. We are forced then to take $K : \mathcal{F}|\mathcal{G}| \to |\mathcal{X}|$ to be the unique extension of $|T|$ to $\mathcal{F}|\mathcal{G}|$, and to take $K[1_B, \rho, 1_A] = e(T\rho)$ (this e appears in the definition of $\mathcal{U}$). Since $[s, \rho, r] = s \cdot [1_B, \rho, 1_A] \cdot r$ in $\mathcal{F}\mathcal{G}$, and K is to be a 2-functor, we are forced to put $K[s, \rho, r] = |K| \cdot e(T\rho) \cdot |K| r$. That K is then well defined and a 2-functor is readily checked. □

The following result is a direct consequence of Duskin [5, Theorem 3.2, p. 89].

Suppose $\mathcal{U} : \mathcal{M} \to \mathcal{N}$ is a functor which admits a left adjoint, between finitely complete categories. If $\mathcal{U}$ reflects isomorphisms, if $\mathcal{M}$ admits coequalizers of equivalence pairs, and $\mathcal{U}$ preserves these coequalizers, then $\mathcal{U}$ is monadic.

We use *monadic* (where Duskin uses "tripleable") to mean the functor is equivalent to an underlying algebra functor. We remind the reader that a pair of arrows $h, k: A \to B$ in a category $\mathcal{M}$ with finite limits is called an *equivalence pair* when

(a) $\binom{h}{k}: A \to B \times B$ is a monomorphism,

(b) there exists $r: B \to A$ such that $hr = kr = 1_B$, and

(c) there exists $t : P \to A$ such that $ht = hp$, $kt = hq$ where the following square is a pullback:

$$\begin{tikzcd} P \arrow[r, "q"] \arrow[d, "p"'] & A \arrow[d, "k"] \\ A \arrow[r, "k"'] & B \end{tikzcd}$$

The above result applies directly to yield the next theorem.

Theorem 3. *The functor $\mathcal{U} : |2\text{-}Cat| \to Cmp$ is monadic.* □

Suppose α is a regular cardinal. A computad $\mathcal{G}$ is said to have *cardinality* $< \alpha$ when the sets $|\mathcal{G}|_0, |\mathcal{G}|_1, \mathcal{G}(A,B)_1$ all have cardinality $< \alpha$. In the case where α is countable, we say instead that $\mathcal{G}$ is *finite*.

A *presentation* of a 2-category $\mathcal{A}$ consists of computads $\mathcal{G}, \mathcal{H}$ together with a coequalizer

$$\mathcal{F}\mathcal{H} \underset{V}{\overset{U}{\rightrightarrows}} \mathcal{F}\mathcal{G} \xrightarrow{W} \mathcal{A}$$

in the category $|2\text{-}Cat|$, where U, V, W are the identity on objects. We say that $\mathcal{A}$ is *α-generated* when such a presentation exists in which $\mathcal{G}$ has cardinality $< \alpha$. We say that $\mathcal{A}$ is *α-presented* when such a presentation exists in which both $\mathcal{G}$ and $\mathcal{H}$ have cardinality $< \alpha$. A 2-category $\mathcal{A}$ is said to be *α-presented to the right of an object C* when a presentation exists in which $\mathcal{G}$ and each of the sets $|\mathcal{H}|(C,A), \mathcal{H}(C,A)_1$ has cardinality $< \alpha$. In the case where α is countable, we use the terms *finitely generated* and *finitely presented*. The next result shows that for uncountable α these distinctions disappear.

Proposition 4. *For a regular cardinal α, consider the following conditions on a 2-category $\mathcal{A}$ and an object C of $\mathcal{A}$:*

(i) *the set of 2-cells in* $\mathscr{A}$ *has cardinality* $<\alpha$;
(ii) $\mathscr{A}$ *is* α*-presented*;
(iii) $\mathscr{A}$ *is* α*-presented to the right of* C;
(iv) $\mathscr{A}$ *is* α*-generated.*
It is always the case that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). *If* α *is uncountable* (iv) $\Rightarrow$ (i).

Proof. Let $\mathscr{G}$ be the computad with the same objects, arrows and 2-cells as $\mathscr{A}$ (this is not $\mathscr{U}\mathscr{A}$ in general). Let us call an arrow in $\mathscr{F}\mathscr{G}$ *short* when it is a string involving 0, 1 or 2 arrows in $\mathscr{G}$; and call a 2-cell in $\mathscr{F}\mathscr{G}$ *short* when it is a horizontal string involving 0, 1 or 2 vertical strings of 0, 1 or 2 2-cells in $\mathscr{G}$. Let $\mathscr{H}$ be the computad whose objects are the same as $\mathscr{A}$ and whose arrows and 2-cells are the short ones in $\mathscr{F}\mathscr{G}$. Let $W: \mathscr{F}\mathscr{G} \to \mathscr{A}$ be the 2-functor "compose in $\mathscr{A}$" (where, of course, we understand the composite in $\mathscr{A}$ of an empty string of arrows or 2-cells to be the appropriate identity arrow or 2-cell in $\mathscr{A}$). Let $U: \mathscr{F}\mathscr{H} \to \mathscr{F}\mathscr{G}$ be the 2-functor whose restriction to $\mathscr{H}$ is just the inclusion. Let $V: \mathscr{F}\mathscr{H} \to \mathscr{F}\mathscr{G}$ be the 2-functor whose restriction to $\mathscr{H}$ is "compose in $\mathscr{A}$". Then W is a coequalizer of U, V. Assuming (i), both $\mathscr{G}$ and $\mathscr{H}$ have cardinality $<\alpha$. So (i) $\Rightarrow$ (ii). Trivially we have (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Looking at the construction of coequalizers and of $\mathscr{F}$ on computads, we see that only countable unions of given sets are taken; so if α is uncountable, (iv) $\Rightarrow$ (i). □

By way of example, consider the 2-category *Simp* with one object A, with $Simp(A, A)$ the category of finite ordinals and order-preserving maps, and with horizontal composition given by ordinal sum. We shall describe a presentation of *Simp* involving finite computads $\mathscr{G}$, $\mathscr{H}$ each with one object A and one arrow $t: A \to A$. The 2-cells of $\mathscr{G}$ are:

$$\begin{tikzcd} A \arrow[r, bend left, "1_A"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\eta"] & A \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dr, "t"] \arrow[d, phantom, "\Downarrow\mu"] & \\ A \arrow[ur, "t"] \arrow[rr, bend right, "t"'] & {} & A \end{tikzcd} .$$

In order to obtain *Simp*, we must impose the equalities:

$$\begin{tikzcd} A \arrow[r, bend left, "1"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\eta"] \arrow[dr, bend right=50, "t"'] & A \arrow[d, "t"] \arrow[dl, phantom, "\Downarrow\mu"] \\ & A \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "t"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow 1"] & A \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "1"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\eta"] \arrow[dr, phantom, "\mu\Downarrow"] & A \\ A \arrow[u, "t"] \arrow[ur, bend right=50, "t"'] & \end{tikzcd}$$

$$
\begin{tikzcd}
A \arrow[r, "t"] \arrow[dr, "t"'] \arrow[ddr, bend right=40, "t"'] \arrow[dr, phantom, bend left=40, "\Downarrow\mu"] & A \arrow[d, "t"] \\
{} \arrow[r, phantom, "\Downarrow\mu"] & A \arrow[d, "t"] \\
 & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "t"] \arrow[ddr, bend right=40, "t"'] \arrow[ddr, phantom, "\Downarrow\mu"] & A \arrow[d, "t"] \arrow[dd, bend right=20, "t"'] \\
 & A \arrow[d, "t"] \arrow[l, phantom, "\Downarrow\mu"] \\
 & A
\end{tikzcd}
$$

on $\mathcal{FG}$. This is done in the obvious way by taking $\mathcal{H}$ to have three 2-cells

$$
\begin{tikzcd}
A \arrow[r, bend left=50, "t"] \arrow[r, bend right=50, "t"'] \arrow[r, phantom, "\Downarrow\rho"] & A
\end{tikzcd},
\quad
\begin{tikzcd}
A \arrow[r, bend left=50, "t"] \arrow[r, bend right=50, "t"'] \arrow[r, phantom, "\Downarrow\lambda"] & A
\end{tikzcd},
\quad
\begin{tikzcd}
 & A \arrow[r, "t"] \arrow[d, phantom, "\Downarrow\alpha"] & A \arrow[dr, "t"] & \\
A \arrow[ur, "t"] \arrow[rrr, "t"'] & {} & & A
\end{tikzcd},
$$

one for each equality, taking U to be the 2-functor which assigns to each of these 2-cells one side of its equality, and taking V to be the 2-functor which assigns the other.

Remark. While this work was being exposed to the Sydney Category Seminar, certain questions arose at this point. It may be helpful if we explain here how computads relate to other possibilities and why we have introduced the notion of "finitely presented" in the above way.

Let *Gph* denote the category of graphs. There are underlying functors

$$|2\text{-}Cat| \to |Cat\text{-}Gph| \to Gph\text{-}Gph,$$

the first of which has a left adjoint by Wolff [16], and the second of which has a left adjoint since the forgetful functor $|Cat| \to Gph$ has a left adjoint. Indeed, a *Gph*-graph is just a computad $\mathcal{G}$ in which the elements of each $\mathcal{G}(A,B)_0$ are strings of arrows in $\mathcal{G}$ of length precisely 1, and the value of the composite of the above left adjoints at such a $\mathcal{G}$ is just $\mathcal{FG}$. Furthermore, using the same consequence of Duskin's theorem stated above, one can see that the underlying functor $|2\text{-}Cat| \to Gph\text{-}Gph$ is monadic.

It was suggested by John W. Gray that, for a general computad $\mathcal{G}$, the 2-category $\mathcal{FG}$ is not really "free" and can be obtained via a presentation

$$\mathcal{FM} \rightrightarrows \mathcal{FL} \to \mathcal{FG}$$

where $\mathcal{L}$, $\mathcal{M}$ are *Gph*-graphs with the same objects as $\mathcal{G}$, where $\mathcal{M}$ has no 2-cells, and where $\mathcal{L}, \mathcal{M}$ have cardinality $< \alpha$ if $\mathcal{G}$ does. Since a composite of regular epimorphisms in $|2\text{-}Cat|$ is not necessarily regular, it does not follow from this data that a finitely generated 2-category can be "generated" by a finite *Gph*-graph. The author conjectures that the 2-category *Simp* in the example above admits no presentation using finite *Gph*-graphs (although of course it is finitely *generated* by a *Gph*-graph).

It was pointed out by Robert F.C. Walters that, from a presentation $\mathcal{F}\mathcal{H} \rightrightarrows \mathcal{F}\mathcal{G} \to \mathcal{A}$ of a 2-category $\mathcal{A}$, one can construct a coequalizer $\mathcal{F}\mathcal{M} \rightrightarrows \mathcal{F}\mathcal{L} \to \mathcal{A}$ in $|2\text{-}Cat|$ where $\mathcal{L}, \mathcal{M}$ are *Gph*-graphs and, if $\mathcal{G}, \mathcal{H}$ have cardinality $< \alpha$ then so do $\mathcal{L}, \mathcal{M}$; however, $\mathcal{L}, \mathcal{M}$ need not have the same objects as $\mathcal{A}$. If $\mathcal{G}$ is finite and $\mathcal{H}$ is not, then $\mathcal{L}$ is not finite in this construction; so it does not follow that we can use *Gph*-graphs rather than computads if we allow the 2-functors in presentations to be non-identities on objects. The need for the 2-functors in a presentation to be the identity on objects should be clear from the later theorems in the present work. Also, finite presentations in the one-object case provide a standard way of generating a strict monoidal category from logical data using first-order predicate calculus.

3. Existence theorems

Our first existence theorem generalizes the result from the folklore of category theory which states that limits can be constructed from equalizers and products.

Theorem 5. *Suppose α is a regular cardinal and $\mathcal{K}$ is a 2-category which admits*:
(i) *a product for any family of objects indexed over a set of cardinality* $< \alpha$;
(ii) *a limit for any diagram in $\mathcal{K}$ of the form*

$$\begin{tikzcd} X \arrow[r, bend left=50, "u"] \arrow[r, "v"] \arrow[r, bend right=50, "w"'] \arrow[r, phantom, bend left=25, "\Downarrow\sigma"] & Y \end{tikzcd} .$$

If $\mathcal{A}$ is an α-generated 2-category, then any 2-functor $S : \mathcal{A} \to \mathcal{K}$ admits a limit.

Proof. Suppose $\mathcal{F}\mathcal{H} \underset{V}{\overset{U}{\rightrightarrows}} \mathcal{F}\mathcal{G} \xrightarrow{W} \mathcal{A}$ is a presentation of $\mathcal{A}$ in which $\mathcal{G}$ has cardinality $< \alpha$. For any object K of $\mathcal{K}$, 2-natural transformations α: $\Delta(K) = \Delta(K)W \Rightarrow SW$ automatically equalize U and V (since U, V are the identity on objects), and so are in bijection with 2-natural transformations β: $\Delta(K) \Rightarrow S$ via $\alpha = \beta W$ (since W is a coequalizer of U, V in the 2-category 2-*Cat*). So we may suppose that $\mathcal{A} = \mathcal{F}\mathcal{G}$ and that W is the identity. Form the products

$$P = \Pi_{A \in |\mathcal{G}|_0} SA, \qquad Q = \Pi_{f \in |\mathcal{G}|(A,B)} SB,$$

with projections $p_A : P \to SA$, $q_f : Q \to SB$. Define arrows $h, k : P \to Q$ by the equations

$q_f h = Sf \cdot p_A$, $q_f k = p_B$. Applying (ii) to the diagram

$$\begin{tikzcd} P \arrow[r, bend left=60, "h"] \arrow[r, "h"'] \arrow[r, bend right=60, "k"'] \arrow[r, phantom, bend left=30, "\Downarrow 1"] & Q, \end{tikzcd}$$

we obtain the equalizer of h, k, which is also the limit X of the functor $|S|:|\mathscr{A}| \to |\mathscr{X}|$. Let $x_A: X \to SA$ denote the projection. Form the product

$$Y = \Pi_{\rho \in \mathcal{G}(A,B)_1} SB,$$

and let $y_\rho: Y \to SB$ be the projection. Define a diagram as in (ii) by the following equalities:

$$\begin{tikzcd} X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\sigma"] & Y \arrow[r, "y_\rho"] & SB \end{tikzcd} = \begin{tikzcd} X \arrow[r, "x_A"] & SA \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow S\rho"] & SB, \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] & Y \arrow[r, "y_\rho"] & SB \end{tikzcd} = \begin{tikzcd} X \arrow[r, "x_B"] & SB. \end{tikzcd}$$

It is readily seen that the equality

$$\begin{tikzcd} K \arrow[r, "kA"] & SA \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow S\rho"] & SB \end{tikzcd} = \begin{tikzcd} K \arrow[r, bend left, "kB"] \arrow[r, bend right, "kB"'] \arrow[r, phantom, "\Downarrow 1"] & SB \end{tikzcd}$$

for all 2-cells ρ in $\mathcal{G}$ implies the equality for all 2-cells ρ in $\mathscr{A}$; so (ii) gives a limit for S. □

Theorem 6. *If $\mathscr{X}$ admits equalizers and identifiers then any diagram in $\mathscr{X}$ of the form*

$$\begin{tikzcd} X \arrow[r, bend left=60, "u"] \arrow[r, "v"'] \arrow[r, bend right=60, "w"'] \arrow[r, phantom, bend left=30, "\sigma\Downarrow"] & Y \end{tikzcd}$$

admits a limit.

Proof. Take the equalizer $k: E \to X$ of v and w, and then the identifier of the 2-cell σk. □

Theorem 7. *If $\mathscr{X}$ admits pullbacks and cotensor products with* **2** *then $\mathscr{X}$ admits identifiers.*

Proof. A 2-cell $X \mathrel{\substack{\longrightarrow \\ \Downarrow\sigma \\ \longrightarrow}} Y$ corresponds to an arrow $s: X \to \mathbf{2} \pitchfork Y$, and we can calculate an identifier of σ as a pullback of *diag*: $Y \to \mathbf{2} \pitchfork Y$ along s. □

A 2-category will be called *α-complete* for a regular cardinal α when it admits equalizers, cotensor products with $\mathbf{2}$, and products over indexing sets of cardinality $<\alpha$. If α is countable we say *finitely complete*. The last three theorems have the following

Corollary 8. *Any 2-functor from an α-generated 2-category to an α-complete 2-category admits a limit.* □

It is possible to construct many more indexed limits in an α-complete 2-category than those asserted in the above corollary. We shall now proceed to prove this. We shall distinguish the "finite" case since the case of uncountable $\mathcal{A}$ is really much simpler.

Given a 2-functor $J: \mathcal{A} \to \mathit{Cat}$ the 2-category $\mathcal{C}J$, called *the cone on J*, is described as follows. The objects are all the objects of $\mathcal{A}$ together with a further object Ω, called the *apex*. For A, B in $\mathcal{A}$, we have

$$(\mathcal{C}J)(A,B) = \mathcal{A}(A,B), \qquad (\mathcal{C}J)(\Omega, A) = JA,$$

$$(\mathcal{C}J)(\Omega,\Omega) = \mathbf{1}, \qquad (\mathcal{C}J)(A,\Omega) = \mathbf{0}.$$

The composition functor $(\mathcal{C}J)(A,B) \times (\mathcal{C}J)(\Omega, A) \to (\mathcal{C}J)(\Omega, B)$ corresponds, under cartesian closedness in $|\mathit{Cat}|$, to the functor $J_{AB}: \mathcal{A}(A,B) \to \mathit{Cat}(JA, JB)$, and the other composition functors are such that $\mathcal{A}$ is a 2-full sub-2-category of $\mathcal{C}J$; let $\partial_1: \mathcal{A} \to \mathcal{C}J$ denote the inclusion. There is an isomorphism of spans

$$\begin{tikzcd}
 & {[\mathcal{C}J, \mathcal{K}]} \arrow[dl, "eval_\Omega"'] \arrow[dr, "{[\partial_1, 1]}"] \arrow[dd, "\cong"'] & \\
\mathcal{K} & & {[\mathcal{A}, \mathcal{K}]} \\
 & \langle J, \mathcal{K} \rangle \arrow[ul, "d_0"] \arrow[ur, "d_1"'] &
\end{tikzcd}$$

for any 2-category $\mathcal{K}$. To give a J-indexed limit for $S: \mathcal{A} \to \mathcal{K}$ is precisely to give a right Kan extension of S along $\partial_1: \mathcal{A} \to \mathcal{C}J$; the value of a right extension at Ω is a J-indexed limit for S.

A 2-functor $J: \mathcal{A} \to \mathit{Cat}$ is said to be *finitary* when the cone $\mathcal{C}J$ on J is finitely presented to the right of its apex.

Theorem 9. *Any finitely complete 2-category $\mathcal{K}$ admits all J-indexed limits for any finitary 2-functor $J: \mathcal{A} \to \mathit{Cat}$.*

Proof. For any object Z of the finitely complete 2-category $\mathcal{X}$ and any finite set Λ, a cotensor product $\Lambda \pitchfork Z$ is given by the product of Λ copies of Z. For any finite graph G, a limit for the following diagram in $\mathcal{X}$ is clearly a cotensor product $\mathcal{F}G \pitchfork Z$; such a limit exists by Corollary 8,

$$\begin{tikzcd}
 & & G_1 \pitchfork (2 \pitchfork Z) \arrow[dl, "1 \pitchfork d_0"'] \arrow[dd, "1 \pitchfork d_1"] \\
 & G_1 \pitchfork Z & \\
G_0 \pitchfork Z \arrow[ur, "d_0 \pitchfork 1"] \arrow[rr, "d_1 \pitchfork 1"'] & & G_1 \pitchfork Z .
\end{tikzcd}$$

Suppose $J : \mathcal{A} \to Cat$ is a finitary 2-functor; so we have a presentation

$$\begin{tikzcd}
\mathcal{F}\mathcal{M} \arrow[r, shift left, "U"] \arrow[r, shift right, "V"'] & \mathcal{F}\mathcal{L} \arrow[r, "W"] & \mathcal{C}J
\end{tikzcd}$$

which is finite to the right of Ω. We may as well suppose there are no arrows and no 2-cells into Ω in $\mathcal{L}$ or $\mathcal{M}$. For any 2-functor $S : \mathcal{A} \to \mathcal{X}$, we have objects:

$$P = \Pi_{A \in |\mathcal{A}|_0} \mathcal{F}(\mathcal{L}(\Omega, A)) \pitchfork SA, \qquad Q = \Pi_{A \in |\mathcal{A}|_0} |\mathcal{L}|(\Omega, A) \pitchfork SA,$$

and projections $p_t : P \to SA$, $q_s : Q \to SA$ for each t in $\mathcal{L}(\Omega, A)_0$ and each $s : \Omega \to A$ in $|\mathcal{L}|$. Each element ρ in $\mathcal{L}(\Omega, A)_1$ can be written uniquely as a 2-cell

$$\begin{tikzcd}
 & B_0 \arrow[dr, "r"] \arrow[dd, phantom, "\Downarrow \rho"] & \\
\Omega \arrow[ur, "s_0"] \arrow[dr, "s_1"'] & & A \\
 & B_1 \arrow[ur, "r'"'] &
\end{tikzcd}$$

in $\mathcal{L}$, where s_0, s_1 are arrows in $|\mathcal{L}|$ and r, r' are arrows in $\mathcal{F}|\mathcal{L}|$ (note that r, r' are strings which do not involve the object Ω and that B_0, B_1, A are objects of $\mathcal{A}$). Let

R denote a limit for the finite diagram

$$\begin{tikzcd}
P \arrow[dr, "p_{rs_0}"'] \arrow[drrr, "p_{r's_1}"] & & & \\
& SA & & SA, \\
& SB_0 \arrow[u, "SWr"] & & SB_1 \arrow[u, "SWr'"'] \\
Q \arrow[ur, "q_{s_0}"] \arrow[urrr, "q_{s_1}"'] & & &
\end{tikzcd}$$

$$\rho \in \mathcal{L}(\Omega, A)_1.$$

For any object X of $\mathcal{X}$, let MX denote the category described as follows. An object θ consists of a family of arrows $\theta s\colon X \to SA$ in $\mathcal{X}$ indexed by the arrows $s\colon \Omega \to A$ in $|\mathcal{L}|$ with source Ω, and a family of 2-cells

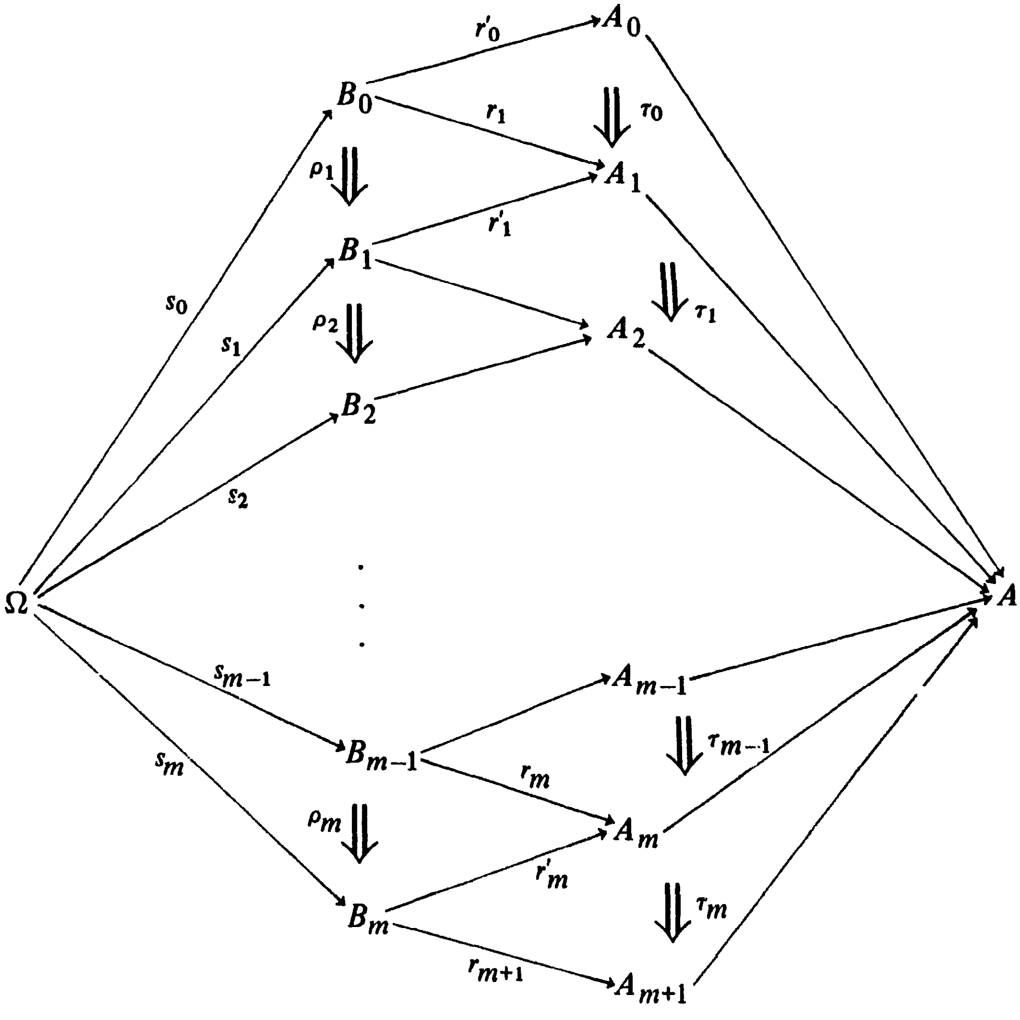
$$\begin{tikzcd}
& SB_0 \arrow[dr, "SWr"] \arrow[dd, phantom, "\theta\rho \Downarrow"] & \\
X \arrow[ur, "\theta s_0"] \arrow[dr, "\theta s_1"'] & & SA \\
& SB_1 \arrow[ur, "SWr'"'] &
\end{tikzcd}$$

indexed by the arrows ρ in $\mathcal{L}(\Omega, A)$ as above. An arrow $\omega\colon \theta \to \phi$ in MX consists of a family of 2-cells
$\begin{tikzcd} X \arrow[r, bend left, "\theta s"] \arrow[r, bend right, "\phi s"'] \arrow[r, phantom, "\Downarrow \omega s"] & SA \end{tikzcd}$
in $\mathcal{X}$ indexed by the arrows $s\colon \Omega \to A$ in $|\mathcal{L}|$,
such that the following equality holds for all ρ as above:

$$\begin{tikzcd}
& SB_0 \arrow[dr, "SWr"] \arrow[dd, phantom, "\Downarrow \theta\rho"] & \\
X \arrow[ur, "\theta s_0"] \arrow[dr, bend left, "\theta s_1"] \arrow[dr, bend right, "\phi s"'] \arrow[dr, phantom, "\Downarrow \omega s"] & & SA \\
& SB_1 \arrow[ur, "SWr'"'] &
\end{tikzcd}
=
\begin{tikzcd}
& SB_0 \arrow[dr, "SWr"] \arrow[dd, phantom, "\Downarrow \phi\rho"] & \\
X \arrow[ur, bend left, "\theta s_0"] \arrow[ur, bend right, "\phi s_0"'] \arrow[ur, phantom, "\Downarrow \omega s_0"] \arrow[dr, "\phi s_1"'] & & SA \\
& SB_1 \arrow[ur, "SWr'"'] &
\end{tikzcd}$$

In an obvious way, the assignment $X \longmapsto MX$ extends to a 2-functor $M : \mathcal{X}^{op} \to CAT$. It is clear that there is a 2-natural isomorphism $\mathcal{X}(\ \ , R) \cong M$. Let μ denote the object of MR obtained by evaluating this isomorphism at the identity of R.

Any 2-cell $\Omega \overset{t}{\underset{t'}{\Downarrow\tau}} A$ in $\mathcal{FL}$ can be written uniquely in the following form:



where each s_i is an arrow in $|\mathcal{L}|$, each ρ_i is an arrow in $\mathcal{L}(\Omega, A_i)$, and each τ_i is a 2-cell in $\mathcal{FL}$ (not involving the object Ω). For any object θ of MX, let

$X \overset{\theta t}{\underset{\theta t'}{\Downarrow\theta\tau}} SA$ denote the following composite 2-cell in $\mathcal{X}$:

$$
\begin{tikzcd}
 & & SA_0 \arrow[dd, phantom, "\Downarrow SW\tau_0"] \arrow[ddddddr] & \\
 & SB_0 \arrow[ur, "SWr'_0"] \arrow[dr] \arrow[dd, phantom, "\Downarrow \theta\rho_1"] & & \\
 & & SA_1 \arrow[dd, phantom, "\Downarrow SW\tau_1"] \arrow[ddddr] & \\
 & SB_1 \arrow[ur] \arrow[dr] \arrow[dd, phantom, "\Downarrow \theta\rho_2"] & & \\
 & & SA_2 \arrow[ddr] & \\
 & SB_2 \arrow[ur] & & \\
X \arrow[uuuuur, "\theta s_0"] \arrow[uuur] \arrow[ur] \arrow[ddr] \arrow[ddddr, "\theta s_m"'] & \vdots & & SA\,. \\
 & & SA_{m-1} \arrow[ur] \arrow[dd, phantom, "\Downarrow SW\tau_{m-1}"] & \\
 & SB_{m-1} \arrow[ur] \arrow[dr] \arrow[dd, phantom, "\Downarrow \theta\rho_m"] & & \\
 & & SA_m \arrow[uuur] \arrow[dd, phantom, "\Downarrow SW\tau_m"] & \\
 & SB_m \arrow[ur] \arrow[dr, "SWr_{m+1}"'] & & \\
 & & SA_{m+1} \arrow[uuuuur] &
\end{tikzcd}
$$

Note then that, for 2-cells,

$$
\begin{tikzcd}
\Omega \arrow[r, bend left=80, "t"] \arrow[r, "t'" description] \arrow[r, bend right=80, "t''"'] \arrow[r, phantom, shift left=4, "\Downarrow\tau"] \arrow[r, phantom, shift right=4, "\Downarrow\tau'"] & A \arrow[r, bend left, "r"] \arrow[r, bend right, "r'"'] \arrow[r, phantom, "\Downarrow\zeta"] & A
\end{tikzcd}
$$

in $\mathcal{FL}$, we have the equations:

$$
\theta 1_t = 1_{\theta t}, \qquad \theta(\tau'\tau) = (\theta\tau')(\theta\tau), \qquad \theta(\zeta,\tau) = SW\zeta \cdot \theta\tau.
$$

For any arrow $\omega : \theta \to \phi$ in MX, we put

$$
\begin{tikzcd}
X \arrow[r, bend left, "\theta t"] \arrow[r, bend right, "\phi t"'] \arrow[r, phantom, "\Downarrow \omega t"] & SA
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, bend left, "\theta s"] \arrow[r, bend right, "\phi s"'] \arrow[r, phantom, "\Downarrow \omega s"] & SB \arrow[r, "SWr"] & SA
\end{tikzcd}
$$

where $t = rs: \Omega \to A$ in $\mathcal{FL}$ and s is an arrow in $|\mathcal{L}|$. We then have the equations:

$$\omega(\bar{r}t) = SW\bar{r} \cdot \omega t, \qquad (\omega t')(\theta\tau) = (\theta\tau)(\omega t).$$

Let $\mathcal{G}, \mathcal{H}$ be the 2-full subcomputads of $\mathcal{L}, \mathcal{M}$ (respectively) consisting of the objects of $\mathcal{A}$. Then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{FH} \arrow[r, shift left, "\hat{U}"] \arrow[r, shift right, "\hat{V}"'] \arrow[d, "\partial_1"'] & \mathcal{FG} \arrow[r, "\hat{W}"] \arrow[d, "\partial_1"] & \mathcal{A} \arrow[d, "\partial_1"] \\
\mathcal{FM} \arrow[r, shift left, "U"] \arrow[r, shift right, "V"'] & \mathcal{FL} \arrow[r] & \mathcal{C}J
\end{tikzcd}$$

in which the vertical 2-functors are inclusions and the rows are presentations. The equations of the last paragraph precisely say that $\theta: (\mathcal{FL})(\Omega, \partial_1) \to \mathcal{K}(X, S\hat{W})$ is an arrow and $\omega: \theta \to \phi$ is a 2-cell in the 2-category $[\mathcal{FG}, Cat]$. Clearly then we have described an isomorphism of categories

$$MX \cong [\mathcal{FG}, Cat]((\mathcal{FL})(\Omega, \partial_1), \mathcal{K}(X, S\hat{W}))$$

which is 2-natural in X. The right-hand side of this isomorphism is a limit of the following diagram of 2-categories, 2-naturally in X:

$$\begin{tikzcd}
\mathbf{1} \arrow[dr, "\ulcorner S\hat{W} \urcorner"'] & & {[\mathcal{FL}, \mathcal{K}]} \arrow[dl, "{[\partial_1, 1]}"'] \arrow[dr, "eval_\Omega"] & & \mathbf{1} \arrow[dl, "\ulcorner X \urcorner"] \\
& {[\mathcal{FG}, \mathcal{K}]} & & \mathcal{K} &
\end{tikzcd}$$

For each arrow $b: \Omega \to B$ in $|\mathcal{M}|$, we have two arrows $\mu(Ub), \mu(Vb): R \to SB$ in $\mathcal{K}$. For each 2-cell $\Omega \overset{\Downarrow \nu}{\rightrightarrows} A$ in $\mathcal{M}$, we have two 2-cells $\mu(U\nu), \mu(V\nu): R \overset{\Downarrow}{\rightrightarrows} SA$ in $\mathcal{K}$ which correspond to two arrows $u_\nu, v_\nu: R \to \mathbf{2} \pitchfork SA$ in $\mathcal{K}$. Take the simultaneous equalizer L of all the pairs $\mu(Ub), \mu(Vb)$ and all the pairs u_ν, v_ν; this exists by Corollary 8 since there are only finitely many such b and ν. There is an arrow $k: L \to R$ in $\mathcal{K}$ such that the composite functor

$$\mathcal{K}(X, L) \xrightarrow{\mathcal{K}(1,k)} \mathcal{K}(X, R) \cong MX$$

induces an isomorphism between $\mathcal{K}(X, L)$ and the subcategory NX of MX consisting of those objects θ such that $\theta(Ub) = \theta(Vb)$ and $\theta(U\nu) = \theta(V\nu)$, and those arrows $\omega: \theta \to \phi$ in MX such that $\omega(Ub) = \omega(Vb)$.

From the presentation coequalizers for $\mathcal{A}$ and $\mathcal{C}J$, it is clear that the following arrow of diagrams

$$\begin{tikzcd}
1 \arrow[dr, "\ulcorner S \urcorner"] \arrow[dddr, "\ulcorner S\hat{W} \urcorner"'] & & {[\mathcal{C}J, \mathcal{X}]} \arrow[dl, "{[\partial_1, 1]}"'] \arrow[dd, "{[W, 1]}"] \arrow[dr, "eval_\Omega"] & & 1 \arrow[dl, "\ulcorner X \urcorner"'] \arrow[dddl, "\ulcorner X \urcorner"] \\
& {[\mathcal{A}, \mathcal{X}]} \arrow[dd, "{[\hat{W}, 1]}"] & & \mathcal{X} \arrow[dd, "1"] & \\
& & {[\mathcal{F}\mathcal{L}, \mathcal{X}]} \arrow[dl, "{[\partial_1, 1]}"] \arrow[dr, "eval_\Omega"'] & & \\
& {[\mathcal{F}\mathcal{G}, \mathcal{X}]} & & \mathcal{X} &
\end{tikzcd}$$

induces an isomorphism of categories between the limit $[\mathcal{A}, Cat](J, \mathcal{X}(X, S))$ of the top diagram and the subcategory NX of the limit MX of the bottom diagram, and that this isomorphism is 2-natural in X. So $\mathcal{X}(X, L)$ is a limit of the top diagram 2-naturally in X, as required for L to be a J-indexed limit for S. □

A 2-functor J: $\mathcal{A} \to Cat$ is said *to have cardinality* $< \alpha$, where α is a regular cardinal, when the set of 2-cells of $\mathcal{A}$ and the set of arrows of JA for each object A of $\mathcal{A}$ all have cardinality $< \alpha$.

Theorem 10. *For any regular cardinal α, any α-complete 2-category admits all J-indexed limits for any 2-functor $J: \mathcal{A} \to Cat$ of cardinality $< \alpha$.*

Proof. We could follow through the proof of Theorem 9 since under these conditions $\mathcal{C}J$ is α-presented to the right of Ω (see Proposition 4). However, a simpler proof is possible in this case. For any object Z of the α-complete 2-category $\mathcal{X}$, we can construct $\mathcal{F}G \pitchfork Z$ for any graph G of cardinality $< \alpha$ (as at the beginning of the proof of Theorem 9). Here we use the fact that any category C of cardinality $< \alpha$ can be expressed as a coequalizer

$$\mathcal{F}H \rightrightarrows \mathcal{F}G \to C$$

in Cat, where G, H are graphs of cardinality $< \alpha$, to obtain $C \pitchfork Z$ as an equalizer

$$C \pitchfork Z \to \mathcal{F}G \pitchfork Z \rightrightarrows \mathcal{F}H \pitchfork Z.$$

Then, for any 2-functor S: $\mathcal{A} \to \mathcal{X}$, we can use an idea of Day–Kelly [4, p. 180] and take a limit L of the diagram

$$\begin{tikzcd}
& JA \pitchfork SA \arrow[d, "s_{AB}"] \\
JB \pitchfork SB \arrow[r, "t_{AB}"'] & \mathcal{A}(A, B) \pitchfork (JA \pitchfork SB)
\end{tikzcd}
\qquad A, B \in \mathcal{A},$$

where s_{AB} corresponds to $JA \pitchfork S-: \mathcal{A}(A,B) \to \mathcal{K}(JA \pitchfork SA, JA \pitchfork SB)$ and t_{AB} corresponds to $J- \pitchfork SB$: $\mathcal{A}(A,B) \to \mathcal{K}(JB \pitchfork SB, JA \pitchfork SB)$: the existence of this limit is assured by Corollary 8. Then $L = \int_A JA \pitchfork SA = lim(J,S)$. □

4. Lax limits

Given 2-functors S, T: $\mathcal{A} \to \mathcal{K}$, recall that a *lax natural transformation* θ: $S \to T$ consists of a family of arrows $\theta A: SA \to TA$ in $\mathcal{K}$ indexed by the objects A of $\mathcal{A}$, and a family of 2-cells

$$\begin{tikzcd}
SA \arrow[r, "\theta A"] \arrow[d, "Sf"'] \arrow[dr, phantom, "\theta f \Downarrow"] & TA \arrow[d, "Tf"] \\
SB \arrow[r, "\theta B"'] & TB
\end{tikzcd}$$

indexed by the arrows f: $A \to B$ in $\mathcal{A}$, satisfying the following equalities:

$$\begin{tikzcd}
SA \arrow[r, "\theta A"] \arrow[d, "S1"'] \arrow[dr, phantom, "\theta 1 \Downarrow"] & TA \arrow[d, "T1"] \\
SA \arrow[r, "\theta A"'] & TA
\end{tikzcd}
=
\begin{tikzcd}
SA \arrow[r, "\theta A"] \arrow[d, "1"'] \arrow[dr, phantom, "1 \Downarrow"] & TA \arrow[d, "1"] \\
SA \arrow[r, "\theta A"'] & TA
\end{tikzcd},$$

$$\begin{tikzcd}
SA \arrow[r, "\theta A"] \arrow[d, "Sf"'] \arrow[dr, phantom, "\theta f \Downarrow"] & TA \arrow[d, "Tf"] \\
SA' \arrow[r, "\theta A'"] \arrow[d, "Sf'"'] \arrow[dr, phantom, "\theta f' \Downarrow"] & TA' \arrow[d, "Tf'"] \\
SA'' \arrow[r, "\theta A''"'] & TA''
\end{tikzcd}
=
\begin{tikzcd}
SA \arrow[r, "\theta A"] \arrow[d, "S(f'f)"'] \arrow[dr, phantom, "\theta(f'f) \Downarrow"] & TA \arrow[d, "T(f'f)"] \\
SA'' \arrow[r, "\theta A''"'] & TA''
\end{tikzcd},$$

$$\begin{tikzcd}
SA \arrow[r, "\theta A"] \arrow[d, "Sf"] \arrow[d, bend right=60, "Sg"'] \arrow[d, phantom, shift right=4ex, "\overset{S\alpha}{\Leftarrow}"] \arrow[dr, phantom, "\Downarrow \theta f"] & TA \arrow[d, "Tf"] \\
SA' \arrow[r, "\theta A'"'] & TA'
\end{tikzcd}
=
\begin{tikzcd}
SA \arrow[r, "\theta A"] \arrow[d, "Sg"'] \arrow[dr, phantom, "\theta g \Downarrow"] & TA \arrow[d, "Tg"'] \arrow[d, bend left=60, "Tf"] \arrow[d, phantom, shift left=4ex, "\overset{T\alpha}{\Leftarrow}"] \\
SA' \arrow[r, "\theta A'"'] & TA'
\end{tikzcd}.$$

A *modification* $S \underset{\theta'}{\overset{\theta}{\Downarrow\omega}} T$ of lax natural transformations consists of a family of 2-cells $SA \underset{\theta A'}{\overset{\theta A}{\Downarrow\omega A}} TA$ in $\mathscr{X}$ indexed over the objects A of $\mathscr{A}$, such that

$$\begin{tikzcd} SA \arrow[r, bend left, "\theta A"] \arrow[r, bend right, "\theta A'"'] \arrow[r, phantom, "\Downarrow\omega A"] \arrow[d, "Sf"'] & TA \arrow[d, "Tf"] \\ SA' \arrow[r, "\theta' A'"'] \arrow[ur, phantom, "\Downarrow\theta' f"] & TA' \end{tikzcd} = \begin{tikzcd} SA \arrow[r, "\theta A"] \arrow[d, "Sf"'] \arrow[dr, phantom, "\Downarrow\theta f"] & TA \arrow[d, "Tf"] \\ SA' \arrow[r, bend left, "\theta A'"] \arrow[r, bend right, "\theta' A'"'] \arrow[r, phantom, "\Downarrow\omega A'"] & TA' \end{tikzcd}.$$

Let $[\![\mathscr{A},\mathscr{X}]\!]$ denote the 2-category whose objects are 2-functors from $\mathscr{A}$ to $\mathscr{X}$, whose arrows are lax natural transformations, and whose 2-cells are modifications (compositions are the obvious ones). Note that $[\mathscr{A},\mathscr{X}]$ is a sub-2-category of $[\![\mathscr{A},\mathscr{X}]\!]$; a 2-natural transformation is a lax natural transformation θ for which θf is an identity.

A right lifting of an object S of $[\![\mathscr{A},\mathscr{X}]\!]$ through the 2-functor $\Delta : \mathscr{X} \xrightarrow{\Delta} [\mathscr{A},\mathscr{X}] \to [\![\mathscr{A},\mathscr{X}]\!]$ is called a *lax limit* for S. In other words, a lax limit for S is an object *lax lim* S of $\mathscr{X}$ together with an isomorphism of categories

$$\mathscr{X}(X, \mathit{lax\,lim}\, S) \cong [\![\mathscr{A},\mathscr{X}]\!](\Delta(X), S)$$

which is 2-natural in X.

We have often regarded sets as categories with only identity arrows; this inclusion *Set* $\to$ *Cat* has a left adjoint π which assigns to each category C the set πC of path components of C. Since π is finite-product preserving, it is a closed functor and so, in the notation of Eilenberg–Kelly [6, p. 449], we have a 2-functor $\pi_* : 2\text{-}\mathit{Cat} \to \mathit{Cat}$. For a 2-category $\mathscr{A}$, the category $\pi_*\mathscr{A}$ has the same objects as $\mathscr{A}$, and $(\pi_*\mathscr{A})(A,B) \cong \pi(\mathscr{A}(A,B))$.

Let $\mathbf{1} \underset{\partial_1}{\overset{\partial_0}{\Downarrow}} \mathbf{2}$ denote the only such non-identity natural transformation. For any 2-category $\mathscr{A}$, consider the 2-functor $d_1 = [\![\partial_1, 1]\!] : [\![\mathbf{2}, \mathscr{A}]\!] \to [\![\mathbf{1}, \mathscr{A}]\!] = \mathscr{A}$. Taking fibres with respect to this 2-functor determines a 2-functor $\mathscr{A} \to 2\text{-}\mathit{Cat}$. Let $L_{\mathscr{A}} : \mathscr{A} \to \mathit{Cat}$ denote the composite $\mathscr{A} \to 2\text{-}\mathit{Cat} \xrightarrow{\pi_*} \mathit{Cat}$. A more explicit description of $L_{\mathscr{A}}$ will be helpful. The category $L_{\mathscr{A}}A$, where A is an object of $\mathscr{A}$, has pairs (B, u) as objects, where $u : B \to A$ is an arrow in $\mathscr{A}$, and has equivalence classes $[h, \gamma]$ of pairs $(h, \gamma) : (B, u) \to (C, v)$ as arrows, where $h : B \to C$ is an arrow and $\gamma : u \to vh$ is a 2-cell in $\mathscr{A}$, and two such pairs $(h, \gamma), (k, \delta) : (B, u) \to (C, v)$ are equivalent if there is a 2-cell $B \overset{h}{\underset{k}{\Downarrow}} C$ which when pasted onto γ at h yields δ. For a 2-cell $A \overset{f}{\underset{g}{\Downarrow\alpha}} A'$ in $\mathscr{A}$, we have:

$$(L_{\mathscr{A}}f)(B,u) = (B, fu), \qquad (L_{\mathscr{A}}f)[h,\gamma] = [h, f\gamma], \qquad (L_{\mathscr{A}}\alpha)(B,u) = [1_B, \alpha u].$$

Theorem 11. *For any 2-category $\mathscr{A}$, an $L_{\mathscr{A}}$-indexed limit for a 2-functor $S: \mathscr{A} \to \mathscr{K}$ is precisely a lax limit for S. Symbolically,*

$$\lim(L_{\mathscr{A}}, S) \cong \text{lax} \lim S.$$

Proof. From the representability definitions of lax limits and $L_{\mathscr{A}}$-indexed limits, the problem reduces to that of finding an isomorphism of categories

$$\Gamma: [\mathscr{A}, Cat](L_{\mathscr{A}}, S) \cong [\![\mathscr{A}, Cat]\!](\Delta(\mathbf{1}), S)$$

which is 2-natural in 2-functors $S: \mathscr{A} \to Cat$. Write L for $L_{\mathscr{A}}$.

Given a 2-natural transformation $\nu: L \to S$, put

$$\theta A = (\nu A)(A, 1_A): \mathbf{1} \to SA$$

for each object A of $\mathscr{A}$. From the naturality of ν, we have $(\nu A')(A,f) = Sf \cdot \theta A$ for any $f: A \to A'$ in $\mathscr{A}$; put $\theta f = (\nu A')[f, 1_f]: Sf \cdot \theta A \to \theta A'$ in SA'. We claim that the data θA, θf determine a lax natural transformation $\Gamma(\nu) = \theta: \Delta(\mathbf{1}) \to S$. The first two conditions follow from the functoriality of the νA. For any 2-cell $A \overset{f}{\underset{g}{\Downarrow\alpha}} A'$ in $\mathscr{A}$, the 2-naturality of ν implies that $(\nu A')[1_A, \alpha] = S\alpha \cdot \theta A: Sf \cdot \theta A \to Sg \cdot \theta A$; also $[f, 1_f] = [g, \alpha]: (A, f) \to (A', 1_{A'})$, so the third condition follows from the calculation

$$\begin{aligned}\theta g \cdot (S\alpha \cdot \theta A) &= (\nu A')[f, 1_f] \cdot (\nu A')[1_A, \alpha] \\ &= (\nu A')([f, 1_f] \cdot [1_A, \alpha]) \\ &= (\nu A')[g, \alpha] = (\nu A')[f, 1_f] = \theta f.\end{aligned}$$

Each modification ϵ between 2-natural transformations from L to S gives a modification ω between lax natural transformations from $\Delta(\mathbf{1})$ to S via the formula $\omega A = (\epsilon A)(A, 1_A)$. From the naturality of $\epsilon A'$ and the modification property of ϵ

we have:

$$\begin{aligned}\omega A' \cdot \theta f &= (\epsilon A')(A', 1_{A'}) \cdot (\nu A')[f, 1_f] \\ &= (\nu' A')[f, 1_f] \cdot (\epsilon A')(A, f) \\ &= \theta' f \cdot (Sf)(\omega A);\end{aligned}$$

so ω is a modification $\Gamma(\epsilon)$ as required. That Γ is a functor and 2-natural in S is easily checked.

The inverse for Γ is given as follows. Given a lax natural transformation $\theta\colon \Delta(1) \to S$, we can define $\nu\colon L \to S$ by:

$$\begin{tikzcd}
(\nu A)(B,u) \arrow[rr, "(\nu A)[h,\gamma]"] \arrow[d, equal] & & (\nu A)(C,v) \arrow[d, equal] \\
Su \cdot \theta B \arrow[r, "S\gamma \cdot \theta B"'] & Sv \cdot Sh \cdot \theta B \arrow[r, "Sv \cdot \theta h"'] & Sv \cdot \theta C .
\end{tikzcd}$$

Given a modification $\omega\colon \theta \to \theta'$, we can define $\epsilon\colon \nu \to \nu'$ by $(\epsilon A)(B,u) = Su \cdot \omega B$. We leave it to the reader to verify that ν is 2-natural and ϵ is a modification, and to supply the "Yoneda-like" argument required to show that these assignments $\theta \to \nu$, $\omega \to \epsilon$ describe Γ^{-1}. $\square$

Theorem 12. *If $\mathcal{A}$ is a 2-category which is either finitely generated or finitely presented then so is $\mathcal{C}L_{\mathcal{A}}$.*

Proof. Given a presentation $\mathcal{F}\mathcal{H} \underset{V}{\overset{U}{\rightrightarrows}} \mathcal{F}\mathcal{G} \xrightarrow{W} \mathcal{A}$ of $\mathcal{A}$, we shall describe a presentation $\mathcal{F}\mathcal{M} \underset{\tilde V}{\overset{\tilde U}{\rightrightarrows}} \mathcal{F}\mathcal{L} \xrightarrow[\tilde W]{} \mathcal{C}L_{\mathcal{A}}$ of $\mathcal{C}L_{\mathcal{A}}$ in which $\mathcal{L}, \mathcal{M}$ contain $\mathcal{G}, \mathcal{H}$ as 2-full subcomputads, and $\tilde U, \tilde V, \tilde W$ restrict to U, V, W respectively.

The rest of the description of $\mathcal{L}$ is as follows:

$$|\mathcal{L}|(\Omega,\Omega) = |\mathcal{L}|(A,\Omega) = 0, \qquad |\mathcal{L}|(\Omega, A) = \{e_A\},$$

$$\mathcal{L}(\Omega, A)_1 = \sum_B |\mathcal{G}|(B,A);$$

and the functions $d_0, d_1\colon \mathcal{L}(\Omega, A)_1 \to (\mathcal{F}|\mathcal{L}|)(\Omega, A)$ are given by $d_0(s) = se_B$, $d_1(s) = e_A$ for $s \in |\mathcal{G}|(B, A)$. It will be convenient to introduce the following notation in $\mathcal{F}\mathcal{L}$: for an arrow $t = (B \xrightarrow{s_0} B_1 \xrightarrow{s_1} \dots \xrightarrow{s_m} C)$ in $\mathcal{F}\mathcal{G}$, where each s_i is in $\mathcal{G}$, let $\begin{tikzcd} \Omega \arrow[r, bend left, "te_B"] \arrow[r, bend right, "e_C"'] \arrow[r, phantom, "\Downarrow t"] & C \end{tikzcd}$ denote the following 2-cell in $\mathcal{FL}$:

$$\begin{tikzcd}
 & & & \Omega \arrow[dlll, "e_B"'] \arrow[dll, "e_{B_1}"] \arrow[dl, "e_{B_2}"] \arrow[dr, "e_{B_m}"] \arrow[drr, "e_C"] & & \\
B \arrow[r, "s_0"'] \arrow[r, phantom, shift left=4ex, "\overset{\Rightarrow}{s_0}"] & B_1 \arrow[r, "s_1"'] \arrow[r, phantom, shift left=4ex, "\overset{\Rightarrow}{s_1}"] & B_2 \arrow[r, "s_2"'] & \cdots \arrow[r] & B_m \arrow[r, "s_m"'] \arrow[r, phantom, shift left=4ex, "\overset{\Rightarrow}{s_m}"] & C;
\end{tikzcd}$$

when t is the empty string this is taken to mean the identity 2-cell $\begin{tikzcd} \Omega \arrow[r, bend left, "e_B"] \arrow[r, bend right, "e_B"'] \arrow[r, phantom, "\Downarrow 1"] & B \end{tikzcd}$, of course.

The rest of the description of $\mathcal{M}$ is as follows:

$$|\mathcal{M}|(\Omega,\Omega) = |\mathcal{M}|(A,\Omega) = 0, \qquad |\mathcal{M}|(\Omega,A) = \{e_A\} + \sum_B \mathcal{G}(B,A)_0,$$

$$\mathcal{M}(\Omega,A)_1 = \sum_B (\mathcal{G}(B,A)_1 + |\mathcal{H}|(B,A));$$

and the functions $d_0, d_1 : \mathcal{M}(\Omega,A)_1 \to (\mathcal{F}|\mathcal{M}|)(\Omega,A)$ are given by $d_0(\rho) = te_B$, $d_1(\rho) = e_A$ for $\begin{tikzcd} B \arrow[r, bend left, "t"] \arrow[r, bend right, "t'"'] \arrow[r, phantom, "\Downarrow\rho"] & A \end{tikzcd}$ in $\mathcal{G}$, and $d_0(b) = be_B$, $d_1(b) = e_A$ for $b \in |\mathcal{H}|(B,A)$.

To complete the description of our presentation of $\mathcal{C}L_{\mathcal{A}}$, put:

$\tilde{W}e_A = (A, 1_A) : \Omega \to A$, $\quad \tilde{W}s = [s,1] : \begin{tikzcd} \Omega \arrow[r, bend left, "{(B,s)}"] \arrow[r, bend right, "{(A,1)}"'] \arrow[r, phantom, "\Downarrow"] & A \end{tikzcd}$ for $s \in |\mathcal{G}|(B,A)$;

$\tilde{U}e_A = \tilde{V}e_A = e_A : \Omega \to A$, $\quad \tilde{U}t = \tilde{V}t = te_B : \Omega \to A$ for $t \in \mathcal{G}(B,A)_0$;

$$\tilde{U}(\rho) = \begin{tikzcd} \Omega \arrow[r, "e_B"] \arrow[rr, bend right=60, "e_A"'] \arrow[rr, phantom, bend right=30, "t'\Downarrow"] & B \arrow[r, bend left, "t"] \arrow[r, bend right, "t'"'] \arrow[r, phantom, "\Downarrow\rho"] & A \end{tikzcd}, \qquad \tilde{V}(\rho) = \begin{tikzcd} \Omega \arrow[r, "e_B"] \arrow[rr, bend right=50, "e_A"'] \arrow[rr, phantom, bend right=25, "t\Downarrow"] & B \arrow[r, "t"] & A \end{tikzcd} \text{ for } \begin{tikzcd} B \arrow[r, bend left, "t"] \arrow[r, bend right, "t'"'] \arrow[r, phantom, "\Downarrow\rho"] & A \end{tikzcd} \text{ in } \mathcal{G};$$

$$\tilde{U}(b) = \begin{tikzcd} & B \arrow[dd, "Ub"] \\ \Omega \arrow[ur, "e_B"] \arrow[dr, "e_A"'] \arrow[r, phantom, "\Downarrow Ub"] & \\ & A \end{tikzcd}, \qquad \tilde{V}(b) = \begin{tikzcd} & B \arrow[dd, "Vb"] \\ \Omega \arrow[ur, "e_B"] \arrow[dr, "e_A"'] \arrow[r, phantom, "\Downarrow Vb"] & \\ & A \end{tikzcd} \text{ for } b \in |\mathcal{H}|(B, A).$$

Then $\tilde{W}\tilde{U}(\rho) = [Wt', W\rho] = [Wt, 1] = \tilde{W}\tilde{V}(\rho)$ and $WU = WV$, so $\tilde{W}\tilde{V} = \tilde{W}\tilde{U}$. Given any 2-functor K: $\mathcal{FL} \to \mathcal{X}$ satisfying $K\tilde{U} = K\tilde{V}$, we must show that there is a unique 2-functor H: $\mathcal{C}L_{\mathcal{A}} \to \mathcal{X}$ such that $K = HW$. Since W is the identity on objects, H must agree with K on objects. The restriction of K to $\mathcal{FG}$ equalizes U, V and so H is uniquely determined on the full sub-2-category $\mathcal{A}$ of $\mathcal{C}L_{\mathcal{A}}$. Consider a diagram

$$\begin{tikzcd} B \arrow[rr, "h"] \arrow[dr, "u"'] & \arrow[d, phantom, "\overset{\gamma}{\Rightarrow}"] & C \arrow[dl, "v"] \\ & A & \end{tikzcd}$$

in $\mathcal{A}$. There is an arrow $t: B \to C$ in $\mathcal{FG}$ with $Wt = h$. In $\mathcal{C}L_{\mathcal{A}}$ we have the equality

$$\begin{tikzcd} \Omega \arrow[r, bend left, "{(B,u)}"] \arrow[r, bend right, "{(C,v)}"'] \arrow[r, phantom, "\Downarrow {[h,\gamma]}"] & A \end{tikzcd} = \begin{tikzcd} & B \arrow[dd, "h"] \arrow[dr, "u"] & \\ \Omega \arrow[ur, "{(B,1)}"] \arrow[dr, "{(C,1)}"'] \arrow[r, phantom, "\Downarrow {[h,1]}"] & \arrow[r, phantom, "\Downarrow \gamma"] & A \\ & C \arrow[ur, "v"'] & \end{tikzcd};$$

and so, since H is to be a 2-functor, we must put

$$H\Big(\begin{tikzcd} \Omega \arrow[r, bend left, "{(B,u)}"] \arrow[r, bend right, "{(C,v)}"'] \arrow[r, phantom, "\Downarrow {[h,\gamma]}"] & A \end{tikzcd}\Big) = \begin{tikzcd} & KB \arrow[dd, "Hh"] \arrow[dr, "Hu"] & \\ K\Omega \arrow[ur, "Ke_B"] \arrow[dr, "Ke_C"'] \arrow[r, phantom, "Kt \Downarrow"] & \arrow[r, phantom, "\Downarrow H\gamma"] & KA \\ & KC \arrow[ur, "Hv"'] & \end{tikzcd}.$$

That this definition of H is independent of the choice of t satisfying $Wt = h$ follows from the equalities $K\tilde{U}(b) = K\tilde{V}(b)$ for all $b \in |\mathcal{G}|(B', A') \subset \mathcal{M}\ (\Omega, A')_1$; and that it is independent of the choice of (h, γ) in $[h, \gamma]$ follows from the equalities $K\tilde{U}(\rho) = K\tilde{V}(\rho)$ for all $\rho \in \mathcal{G}(B', A')_1 \subset \mathcal{M}\ (\Omega, A')$. So H is well-defined and unique, it is clearly a 2-functor. □

Combining Theorems 9, 11 and 12, we obtain:

Corollary 13. *If $\mathcal{A}$ is a finitely presented 2-category and $\mathcal{X}$ is a finitely complete 2-category then any 2-functor S: $\mathcal{A} \to \mathcal{X}$ has a lax limit.* □

Remark. We should point out that $L_{\mathcal{A}}$ is the lax colimit of the Yoneda representation 2-functor Y: $\mathcal{A}^{op} \to [\mathcal{A}, Cat]$. That this is what $L_{\mathcal{A}}$ must be follows from the fact that the isomorphism Γ at the beginning of the proof of Theorem 11 can be obtained as the composite:

$$\begin{aligned}[\mathcal{A}, Cat]\,(lax\ col\, Y, S) &\cong lax\ lim\ [\mathcal{A}, Cat](Y-, S)\\ &\cong lax\ lim\ S \cong Cat(\mathbf{1}, lax\ lim\ S)\\ &\cong [\![\mathcal{A}, Cat]\!](\Delta(\mathbf{1}), S).\end{aligned}$$

Now lax colimits in Cat are constructed in Gray [9, p. 201], so we can obtain lax colimits in $[\mathcal{A}, Cat]$ by the usual pointwise procedure. It will be seen that our explicit description of $L_{\mathcal{A}}$ agrees with this construction.

5. Further examples

If a category $\mathcal{X}$ is α-complete then the 2-category $\mathcal{X}^{co}$ obtained from $\mathcal{X}$ by reversing 2-cells is also α-complete. The 2-categories Cat, Cat^{op} admit equalizers, cotensoring with $\mathbf{2}$, and products over small indexing sets; so Cat, Cat^{op}, Cat^{co}, Cat^{coop} are α-complete for all small regular cardinals α. If $\mathcal{X}$ is α-complete and $\mathcal{D}$ is a small 2-category then clearly $[\mathcal{D}, \mathcal{X}\ ]$ is also α-complete. If U: $\mathcal{X} \to \mathcal{X}'$ is a 2-functor with a left 2-adjoint and $lim(J, S)$ exists in $\mathcal{X}$ then $lim(J, US)$ exists in $\mathcal{X}'$ and is canonically isomorphic to $U\, lim(J, S)$.

Let $\mathcal{A}$ denote the 2-category indicated by $A \xrightarrow{f} C \xleftarrow{g} B$, and let J: $\mathcal{A} \to Cat$ denote the 2-functor given by

$$J(A \xrightarrow{\quad f \quad} C \xleftarrow{\quad g \quad} B) = (\mathbf{1} \xrightarrow{\quad \partial_0 \quad} \mathbf{2} \xleftarrow{\quad \partial_1 \quad} \mathbf{1}).$$

If, for a 2-functor S: $\mathcal{A} \to \mathcal{X}$, we put

$$S(A \xrightarrow{\quad f \quad} C \xleftarrow{\quad g \quad} B) = (X \xrightarrow{\quad u \quad} Z \xleftarrow{\quad v \quad} Y),$$

then $lim(J, S)$ is precisely a *comma object* u/v; that is, there is a 2-cell

$$\begin{tikzcd} u/v \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & Y \arrow[d, "v"] \\ X \arrow[r, "u"'] & Z \end{tikzcd}$$

with the obvious universal property. Since $\mathscr{A}$ is finite and JA, JB, JC are finite, it follows from Theorem 10 that any finitely-complete 2-category admits comma objects. This is easily seen directly; in fact, cotensoring with **2** and pullbacks are sufficient.

Let $\mathscr{A}$ denote the subcategory of $|Cat|$ generated by the diagram

$$\begin{tikzcd} \mathbf{1} \arrow[r, shift left=2, "\partial_0"] \arrow[r, shift right=0, "\partial_1"'] & \mathbf{2} \arrow[l, bend left=40] \arrow[r, shift left=3, "\partial_0"] \arrow[r, "\partial_1"] \arrow[r, shift right=3, "\partial_2"'] & \mathbf{3} \end{tikzcd}$$

and let $J: \mathscr{A} \to Cat$ denote the inclusion. Suppose $\mathcal{V}$ is a closed 2-category and D is a $\mathcal{V}$-monad on a $\mathcal{V}$-category $\mathcal{X}$ with unit $j: 1 \to D$ and multiplication $m: DD \to D$. For D-algebras $(A, DA \xrightarrow{a} A)$, $(B, DB \xrightarrow{b} B)$, we have the following diagram in $\mathcal{V}$:

$$\begin{tikzcd}[column sep=huge] \mathcal{X}(A,B) \arrow[r, shift left=2, "\mathcal{X}(a,1)"] \arrow[r, shift right=1, "\mathcal{X}(1,b)\cdot D"'] & \mathcal{X}(DA,B) \arrow[l, bend left=40, "\mathcal{X}(jA,1)"] \arrow[r, shift left=3, "\mathcal{X}(Da,1)"] \arrow[r, "\mathcal{X}(mA.1)"] \arrow[r, shift right=3, "\mathcal{X}(1,b)\cdot D"'] & \mathcal{X}(DDA,B) \end{tikzcd}$$

and this can be regarded as a 2-functor S_{AB}: $\mathscr{A} \to \mathcal{V}$. Since J is finite, it follows from Theorem 10 that, if $\mathcal{V}$ is finitely complete, $lim(J, S_{AB})$ exists. One readily checks that the category of D-algebras and lax D-homomorphisms (see Street [13]) becomes a $\mathcal{V}$-category with $lim(J, S_{AB})$ as its $\mathcal{V}$-valued *hom*.

Let $\mathscr{A}$ denote the 2-category indicated by $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$. Let $J: \mathscr{A} \to Cat$ denote the 2-functor given by

$$J\Big(\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}\Big) = \begin{tikzcd} \mathbf{1} \arrow[r, bend left, "\partial_0"] \arrow[r, bend right, "\partial_1"'] \arrow[r, phantom, "\Downarrow\iota"] & Iso \end{tikzcd}$$

where Iso is the category with two objects $0, 1$ and arrows $0 \to 1$, $1 \to 0$ which are mutually inverse isomorphisms and ι corresponds to the arrow $0 \to 1$ in Iso. A 2-functor S: $\mathscr{A} \to \mathcal{X}$ amounts to a 2-cell $\begin{tikzcd} X \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\sigma"] & Y \end{tikzcd}$ in $\mathcal{X}$. Then $lim(J, S) = K$ is an *inverter* for σ; that is, there is an arrow $k: K \to X$ which is universal with the property that σk is an isomorphism. If $\mathcal{X}$ is finitely complete, inverters exist by Theorem 10 again. Dually, if $\mathcal{X}^{op}$ is finitely complete then coinverters (= *localizations*) exist in $\mathcal{X}$ (see Gabriel–Zisman [8] for the case $\mathcal{X} = Cat$; see Wolff [15] for the case $\mathcal{X} = \mathcal{V} - Cat$).

Let *Simp* be the 2-category described at the end of Section 2. A 2-functor $S: Simp \to \mathcal{X}$ precisely amounts to a monad (X, s) in $\mathcal{X}$ (in the sense of Street [12]). It is immediate that a lax limit for S is precisely an Eilenberg–Moore object X^s for the monad (X, s); so X^s is also $lim(L_{Simp}, S)$ (compare Lawvere [11]) by Theorem 11. Since *Simp* is finitely presented, Theorem 12, Proposition 4, and Theorem 9 imply that the Eilenberg–Moore construction exists in any finitely complete 2-category (this fact appears in Gray [9]).

We shall now show how to obtain all Gray's "cartesian quasi-limits" as limits indexed by category-valued 2-functors (see Gray [9, pp. 188–189]).

Suppose $\mathcal{A}$ is a 2-category and $\mathcal{B}$ is a subcategory of $|\mathcal{A}|$. Let $M_{\mathcal{A}}^{\mathcal{B}}: \mathcal{A} \to Cat$ denote the 2-functor described as follows. For each A of $\mathcal{A}$, the category $M_{\mathcal{A}}^{\mathcal{B}}A$ has objects pairs (b, u) where $b: B' \to B$ is an arrow in $\mathcal{B}$ and $u: B \to A$ is an arrow in $\mathcal{A}$, and has arrows $(h', h, \gamma): (b, u) \to (c, v)$ given by diagrams

$$\begin{tikzcd}
B' \arrow[rr, "h'"] \arrow[d, "b"'] & & C' \arrow[d, "c"] \\
B \arrow[rr, "h"] \arrow[dr, "u"'] & \overset{\gamma}{\Rightarrow} & C \arrow[dl, "v"] \\
& A &
\end{tikzcd}$$

in $\mathcal{A}$. Given a 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & A' \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$ in $\mathcal{A}$, put

$$(M_{\mathcal{A}}^{\mathcal{B}}f)(b, u) = (b, fu), \qquad (M_{\mathcal{A}}^{\mathcal{B}}\alpha)(b, u) = (h', h, \alpha\gamma).$$

Let $R_0A: M_{\mathcal{A}}^{\mathcal{B}}A \to L_{\mathcal{A}}A$ be the functor which takes $(h', h, \gamma): (b, u) \to (c, v)$ to $[h', \gamma]: (B', ub) \to (C', vc)$. Let $R_1A: M_{\mathcal{A}}^{\mathcal{B}}A \to L_{\mathcal{A}}A$ be the functor which takes $(h', h, \gamma): (b, u) \to (c, v)$ to $[h, \gamma]: (B, u) \to (C, v)$. Let $\rho A: R_0A \to R_1A$ be the natural transformation whose component at (b, u) is $(\rho A)(b, u) = [b, 1]: (B', ub) \to (B, u)$. One readily checks that $R_0A, R_1A, \rho A$ are the components of a 2-natural transformation R_0, a 2-natural transformation R_1, and a modification

$$\begin{tikzcd} M_{\mathcal{A}}^{\mathcal{B}} \arrow[r, bend left, "R_0"{name=U}] \arrow[r, bend right, "R_1"'{name=D}] & L_{\mathcal{A}} \arrow[from=U, to=D, phantom, "\Downarrow\rho"] \end{tikzcd}.$$

Now $[\mathscr{A}, Cat]$ is α-cocomplete for any small regular cardinal α, so the above modification ρ has both a coidentifier $L_{\mathscr{A}}^{idt\,\mathscr{B}}$ and a coinverter $L_{\mathscr{A}}^{inv\,\mathscr{B}}$.

Let $[\![\mathscr{B}, \mathscr{A}; \mathscr{K}, idt]\!]$ (respectively, $[\![\mathscr{B}, \mathscr{A}; \mathscr{K}, inv]\!]$) denote the sub-2-category of $[\![\mathscr{A}, \mathscr{K}]\!]$ with the same objects, with arrows those lax natural transformations θ such that θf is an identity (respectively, isomorphism) for all arrows f in $\mathscr{B}$, and with 2-cells all modifications between such transformations.

Theorem 14. *Let $S: \mathscr{A} \to \mathscr{K}$ be a 2-functor and let $\mathscr{B}$ be a subcategory of $|\mathscr{A}|$.*

(a) *An $L_{\mathscr{A}}^{idt\,\mathscr{B}}$-indexed limit for S is precisely a right lifting of S through $\Delta: \mathscr{K} \to [\![\mathscr{B}, \mathscr{A}; \mathscr{K}, idt]\!]$.*

(b) *An $L_{\mathscr{A}}^{inv\,\mathscr{B}}$-indexed limit for S is precisely a right lifting of S through $\Delta: \mathscr{K} \to [\![\mathscr{B}, \mathscr{A}; \mathscr{K}, inv]\!]$.* □

In Gray's notation, this theorem states that there are isomorphisms:

$$lim(L_{\mathscr{A}}^{idt\,\mathscr{B}}, S) \cong Cart\, q\text{-}\underset{\leftarrow\, \mathscr{A}\text{-}id\,\mathscr{B}}{lim}\; S,$$

$$lim(L_{\mathscr{A}}^{inv\,\mathscr{B}}, S) \cong Cart\, q\text{-}\underset{\leftarrow\, \mathscr{A}\text{-}iso\,\mathscr{B}}{lim}\; S.$$

We shall not give a general existence theorem for $L_{\mathscr{A}}^{idt\,\mathscr{B}}$-, or $L_{\mathscr{A}}^{inv\,\mathscr{B}}$-indexed limits. In the examples we know of we are able to identify the category-valued 2-functor and to apply Theorem 9 or 10 directly. Note however that:

(i) If $\mathscr{B}$ is empty, $[\![\mathscr{B}, \mathscr{A}; \mathscr{K}, idt]\!] = [\![\mathscr{A}, \mathscr{K}]\!]$, so that $L_{\mathscr{A}}^{idt\,\mathscr{B}}$-indexed limits are lax limits;

(ii) If $\mathscr{B} = |\mathscr{A}|$, then $[\![\mathscr{B}, \mathscr{A}; \mathscr{K}, idt]\!] = [\mathscr{A}, \mathscr{K}]$, so that $L_{\mathscr{A}}^{idt\,\mathscr{B}}$-indexed limits are limits.

This suggests that there should be some condition on the pair $\mathscr{B}$, $\mathscr{A}$ under which $lim(L_{\mathscr{A}}^{idt\,\mathscr{B}}, S)$ exists for all $S: \mathscr{A} \to \mathscr{K}$ with $\mathscr{K}$ finitely complete and that this condition should reduce to "$\mathscr{A}$ finitely presented" when $\mathscr{B}$ is empty and reduce to "finitely generated" when $\mathscr{B} = |\mathscr{A}|$.

The next result will complete the proof of the "equivalence" of J-indexed limits and cartesian quasi-limits.

Let $J: \mathscr{A} \to Cat$ be a 2-functor. Let $El\,J$ denote the 2-category whose objects are pairs (A, s) where A is an object of $\mathscr{A}$ and s is an object of JA, whose arrows $(f, \rho): (A, s) \to (A', s')$ consist of an arrow $f: A \to A'$ in $\mathscr{A}$ and an arrow $\rho: (Jf)s \to s'$ in JA', and whose 2-cells $\alpha: (f, \rho) \to (g, \sigma)$ are 2-cells $\alpha: f \to g$ in $\mathscr{A}$ such that $\sigma \cdot (J\alpha)s = \rho$. The assignment of the projection $P: El\,J \to \mathscr{A}$ to the 2-functor J is the well-known Grothendieck construction. There is a lax natural transformation

$$\begin{tikzcd}
El\,J \arrow[r, "P"] \arrow[d] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & \mathscr{A} \arrow[d, "J"] \\
1 \arrow[r, "\Delta(1)"'] & Cat
\end{tikzcd}$$

which is the universal such with $\Delta(\mathbf{1})$, J fixed; that is, $El J$ is the lax comma 2-category of $\Delta(\mathbf{1})$, J. It is well-known that $\Delta(\mathbf{1})$: $\mathbf{1} \to Cat$ is "lax dense" in the sense that, for all such J, composition with λ sets up an isomorphism of categories (compare Street [14, p. 148]):

$$[\![\mathcal{A}, Cat]\!](J,H) \cong [\![El J, Cat]\!](\Delta(\mathbf{1}), HP).$$

It follows that, for any 2-functor S: $\mathcal{A} \to \mathcal{X}$ and any object X of $\mathcal{X}$, there is an isomorphism of categories:

$$[\![\mathcal{A}, Cat]\!](J, \mathcal{X}(X,S)) \cong [\![El J, \mathcal{X}]\!](\Delta(X), SP).$$

Let $el J$ denote the subcategory of $|El J|$ with the same objects but only those arrows (f, ρ) for which ρ is an identity. One readily checks that the latter isomorphism restricts to an isomorphism:

$$[\mathcal{A},\ Cat](J, \mathcal{X}(X,S)) \cong [\![el J, El J; \mathcal{X}, idt]\!](\Delta(X), SP).$$

This proves:

Theorem 15. *For any 2-functor* $J: \mathcal{A} \to Cat$, *a* J*-indexed limit for a 2-functor* $S: \mathcal{A} \to \mathcal{X}$ *is precisely a* $L_{El J}^{idt\, el J}$*-indexed limit.* □

Using Gray's notation, we see that this amounts to an isomorphism:

$$lim(J,S) \cong Cart\ q\text{-}lim_{\leftarrow El J\text{-}idel J}\ S.$$

Any indexed limit is a cartesian quasi-limit of the special type which asks certain 2-cells to be identities (Theorem 15), and any cartesian quasi-limit can be obtained as an indexed limit (Theorem 14), so it follows that all cartesian quasi-limits can be obtained from that special type.

Suppose J: $\mathcal{A} \to Cat$ is a 2-functor. A J-indexed limit for a 2-functor S: $\mathcal{A} \to \mathcal{X}^{op}$ is also called a *J-indexed colimit* for S^{op}: $\mathcal{A}^{op} \to \mathcal{X}$ and denoted by $col(J, S^{op})$.

Theorem 16. *Suppose* $R: \mathcal{B} \to Cat$ *is a 2-functor and* $\mathcal{B}$ *is small. Then any 2-functor* $T: \mathcal{B}^{op} \to [\mathcal{A}, Cat]$ *admits an R-indexed colimit* $col(R, T) = J$: $\mathcal{A} \to Cat$. *Furthermore if, for each object B of* $\mathcal{B}$, *the 2-functor* $S: \mathcal{A} \to \mathcal{X}$ *admits a TB-indexed limit then S admits a J-indexed limit precisely when* $lim(T, S)$ *admits an R-indexed limit*; *in this case*:

$$lim(J, S) \cong lim(R, lim(T, S)).\ \square$$

For a finitely generated 2-category $\mathcal{A}$, consider the class of all 2-functors J: $\mathcal{A} \to Cat$ for which any 2-functor S: $\mathcal{A} \to \mathcal{X}$ into a finitely complete 2-category $\mathcal{X}$ admits a J-indexed limit. By Theorem 9, this class contains all the finitary J and by the above theorem is closed under finite coproducts, coequalizers and tensor products with **2**. The extent to which the class of finitary J is closed under these colimits will be examined in a future paper. It may be helpful to note that the construction of the cone on J provides a 2-functor

$$\mathcal{C}: [\mathcal{A}, Cat] \to Opspn(\mathbf{1}, \mathcal{A})$$

which preserves and reflects all indexed colimits (the target 2-category here is the obvious 2-category of opspans $\mathbf{1} \xrightarrow{X} \mathcal{X} \xleftarrow{S} \mathcal{A}$ such that $\mathcal{X}(X, S)$ has values in Cat). In fact, $\mathcal{C}$ reflects isomorphisms and has a right 2-adjoint whose value at (X, S) is $\mathcal{X}(X, S)$: $\mathcal{A} \to Cat$.

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