BICATEGORIES OF SPANS AND RELATIONS

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A new kind of bicategorical limit is used to characterize bicategories of the form Span(ℓ) and Rel(ℓ) where in the former case ℓ is a category with pullbacks and in the latter ℓ is a regular category. The characterization of Rel(ℓ) differs from those in the literature which require involutions on the bicategories.

0. Introduction

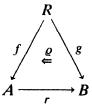
Recent trends in enriched category theory [2] suggest the need to characterize bicategories of spans as defined by Bénabou [1]. Walters has observed that categories locally internal to \mathcal{E} are categories enriched in Span(\mathcal{E}); this example provided motivation for [6] and will be further developed in a forthcoming paper of Betti-Walters. Our characterizations of Span(\mathcal{E}) and Rel(\mathcal{E}) do not involve extra data such as involutions (compare [3], [7]) or tensor products on the bicategories, and in the case of Rel(\mathcal{E}), we dispense with Freyd's modularity condition [3]. We exploit a new kind of lax limit for an arrow in a bicategory; we use Freyd's term 'tabulation' although his use involved the involution and local finite products [3].

1. Tabulation

An arrow $f: A \to B$ in a bicategory \mathcal{B} will be called a map (after [6]) when it has a right adjoint $f^*: B \to A$; the unit and counit for $f \to f^*$ are denoted by $\varepsilon: ff^* \Rightarrow 1$, $\eta: 1 \Rightarrow f^*f$. Let \mathcal{B}^* denote the sub-bicategory of \mathcal{B} with the same objects, with maps as arrows, and with all 2-cells between these. We suppress the associativity 2-cells for composition in \mathcal{B} ; so, for example if $\sigma: f \Rightarrow rs$, $\tau: st \Rightarrow g$ are 2-cells, we write $(r\tau)(\sigma t)$ for the composite

$$ft \stackrel{\sigma t}{\Longrightarrow} (rs)t \cong r(st) \stackrel{r\tau}{\Longrightarrow} rg.$$

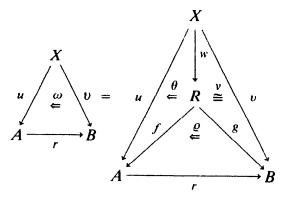
A tabulation for an arrow $r: A \rightarrow B$ in \mathcal{B} is a diagram (f, ϱ, g) :



satisfying the following conditions:

T0. f is a map.

T1. For all other such diagrams (u, ω, v) with u a map, there exist $w, \theta : fw \Rightarrow u$, and invertible $v : v \Rightarrow gw$ such that $\omega = (r\theta)(\varrho w)v$.



T2. For all maps $u: X \to A$, arrows $w, w': X \to R$, and 2-cells $\theta: fw \Rightarrow u$, $\theta': fw' \Rightarrow u$, $\beta: gw \Rightarrow gw'$ such that $(r\theta)(\varrho w) = (r\theta')(\varrho w')\beta$, there exists a unique $\gamma: w \Rightarrow w'$ such that $\beta = g\gamma$, $\theta = \theta'(f\gamma)$.

The diagram (f, ϱ, g) is called a *wide tabulation* for r when, in the definition above, T0 is deleted and T1, T2 are strengthened to allow u to be an arbitrary arrow (not just a map).

These definitions can be reformulated in terms of the bicategory $\mathscr{B}/\!\!/A$ whose objects are arrows $u: X \to A$, whose arrows $(h, \theta): u \to v$ consist of $h: X \to Y$, $\theta: vh \Rightarrow u$, and whose 2-cells $\sigma: (h, \theta) \Rightarrow (h', \theta')$ are $\sigma: h \Rightarrow h'$ with $\theta = \theta'(v\sigma)$. An arrow $r: A \to B$ induces a homomorphism of bicategories $r-: \mathscr{B}/\!\!/A \to \mathscr{B}/\!\!/B$ which takes u to ru and (h, θ) to $(h, r\theta)$. Let $\mathscr{B}/\!\!/*A$ denote the full sub-bicategory of $\mathscr{B}/\!\!/A$ consisting of the $u: X \to A$ which are maps.

Proposition 1. (a) A tabulation for $r: A \rightarrow B$ is a birepresentation [5; (1.11)] for the homomorphism

$$(\mathcal{B}//^*A)^{\mathrm{op}} \xrightarrow{r-} (\mathcal{B}//B)^{\mathrm{op}} \xrightarrow{(\mathcal{B}//B)(-,1_B)} \mathrm{Cat}$$

and so is unique up to equivalence.

(b) A wide tabulation for $r: A \rightarrow B$ is a birepresentation for the homomorphism

$$(\mathcal{B}/\!\!/A)^{\mathrm{op}} \xrightarrow{r} (\mathcal{B}/\!\!/B)^{\mathrm{op}} \cdot \xrightarrow{(\mathcal{B}/\!\!/B)(-,1_B)} \mathrm{Cat}.$$

- (c) A wide tabulation for r satisfies T0 and so is a tabulation.
- (d) If (f, ϱ, g) is a tabulation for r, then $(r\varepsilon)(\varrho f^*): gf^* \Rightarrow r$ is invertible.
- (e) If f is a map, then $(f, \eta, 1)$ is a wide tabulation for f^* .

Proof. (a) A birepresentation for the homomorphism is an object $f: R \to A$ of $\mathcal{H} / ^*A$ and an equivalence

$$(\mathcal{B}//^*A)(u, f) \simeq (\mathcal{B}//B)(ru, 1_B)$$

which is a strong transformation in $u \in \mathcal{B} / A$. To give this equivalence is precisely to give $g: R \to B$ and $\varrho: g \Rightarrow rf$ satisfying T1, T2.

- (b) Delete '*' in the proof of (a).
- (c) Apply T1 with X = A, $u = 1_A$, v = r, $\omega = 1_r$ to obtain a candidate for f^* and a candidate for the counit. Apply the strong T2 with $w = 1_R$, $w'f^*f$ to obtain the unit and the adjunction conditions. (Note that $gf^* \cong r$ so (d) is clear here.)
- (d) Apply T1 with X = A, $u = 1_A$, v = r, $\omega = 1_R$, to obtain f', $\theta' : ff' \Rightarrow 1_A$, $v : r \cong gf'$ with $1_R = (r\theta')(\varrho f')v$. Apply T2 with $u = 1_A$, $w = f^*$, w' = f', $\theta = \varepsilon : ff^* \Rightarrow 1$, $\theta' : ff' \Rightarrow 1$, $\beta = v(r\varepsilon)(\varrho f^*)$ to obtain $\gamma : f^* \Rightarrow f'$ with $g\gamma = v(r\varepsilon)(\varrho f^*)\varepsilon = \theta'(f\gamma)$. The last equation implies $\gamma : f^* \Rightarrow f'$ is a split monic (coretraction), while the calculation:

$$(g\gamma)(gf^*\theta')(g\eta f') = \nu(r\varepsilon)(\varrho f^*)(gf^*\theta')(g\eta f')$$

$$= \nu(r\varepsilon)(rff^*\theta')(\varrho f^*ff')(g\eta f')$$

$$= \nu(r\theta')(r\varepsilon ff')(rf\eta f')(\varrho f')$$

$$= \nu(r\theta')(\varrho f') = 1_{gf'},$$

shows that $g\gamma$ is a split epic. So $g\gamma = v(r\varepsilon)(\varrho f^*)$: $gf^* \Rightarrow gf'$ is invertible. So $(r\varepsilon)(\varrho f^*) = v^{-1}(g\gamma)$ is invertible.

(e) Since 2-cells $\omega: v \Rightarrow f^*u$ are in bijection with 2-cells $\theta: fw \Rightarrow u$ with v = w, the stronger form of T1 follows; the strong form of T2 is clear since g = 1.

2. Spans

Let ℓ denote a category with pullbacks. The bicategory Span (ℓ) is defined as follows. The objects are those of ℓ . An arrow $r: A \rightarrow B$ is a span $r = (r_0, R, r_1)$:

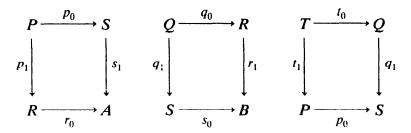
$$A \stackrel{r_0}{\longleftarrow} R \stackrel{r_1}{\longrightarrow} B$$

in $\& composition of <math>r: A \to B$, $s: B \to C$ is obtained by forming the pullback of r_1, s_0 . A 2-cell $\sigma: r \Rightarrow r'$ is an arrow $\sigma: R \to R'$ in & condots such that $r_0'\sigma = r_0$, $r_1'\sigma = r_1$. The next result was stated in [2] without proof.

Proposition 2. An arrow $r = (r_0, R, r_1) : A \rightarrow B$ in Span(ϵ) is a map if and only if $r_0 : R \rightarrow A$ is invertible in ϵ .

Proof. Any arrow isomorphic to a map is a map, so, in order to prove r is a map when r_0 is invertible, it suffices to assume r = (1, A, f). Let $s = (f, A, 1) : B \to A$. Then rs = (f, A, f) and $sr = (k_0, K, k_1)$ where k_0, k_1 form the kernel pair of f. Let $d: A \to K$ be the arrow in f with $k_0 d = k_1 d = 1_A$. Then $f: rs \Rightarrow 1_B$, $d: 1_A \Rightarrow sr$ are counit, unit for $r \to s$.

Conversely, suppose $r \dashv s$ with counit $\varepsilon : rs \Rightarrow 1$, unit $\eta : 1 \Rightarrow sr$. Form the pullbacks:



Then $\eta: A \to Q$ with $r_0 q_0 \eta = s_1 q_1 \eta = 1$ and $\varepsilon: P \to B$ with $\varepsilon = s_0 p_0 = r_1 p_2$. Moreover, $s\eta: R \to T$ is defined by $t_0(s\eta) = \eta r_0$, $p_1 t_1(s\eta) = 1$; and $\varepsilon s: T \to R$ is just $q_0 t_0$. So the adjunction condition gives

$$1 = (\varepsilon s)(s\eta) = (q_0 t_0)(s\eta) = q_0 \eta r_0.$$

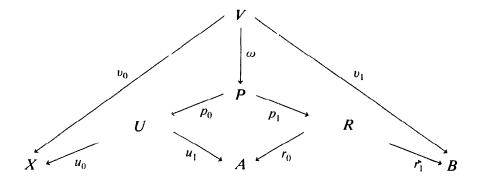
Thus r_0 has inverse $q_0\eta$. \square

Recall [1], [5] that the *classifying category* C # of a bicategory # has the same objects as # and has as arrows the isomorphism classes of arrows in #. Proposition 2 gives an equivalence of categories:

$$\delta \simeq C \operatorname{Span}(\delta)^*$$
.

Proposition 3. Each arrow r in Span(δ) has a wide tabulation (f, ϱ, g) where g is a map.

Proof. Suppose $r = (r_0, R, r_1) : A \to B$ and put $f = (1, R, r_0)$, $g = (1, R, r_1)$. Let k_0, k_1 form a kernel pair for r_0 and define ϱ by $k_0 \varrho = k_1 \varrho = 1_R$. We must show that (f, ϱ, g) is a wide tabulation of r. Take $u = (u_0, U, u_1) : X \to A$, $v = (v_0, V, v_1) : X \to B$, $\omega : v \Rightarrow ru$ as in T1. Let P be the pullback of u_1, r_0 .



Let $w = (v_0, V, p_1\omega) : X \to R$, $\theta = p_0\omega : fw \Rightarrow u$, $v = 1 : v \Rightarrow gw$; so $\omega = (r\theta)(\varrho w)v$ as required.

Take $u, w, w', \theta, \theta', \beta$ as in T2 and note that $fw = (w_0, W, r_0 w_1), gw = (w_0, W, r_1 w_1),$ etc. So $\beta : W \to W'$ in δ satisfies $w_0 = w'_0 \beta$, $r_1 w_1 = r_1 w'_1 \beta$. But the equation $(r\theta)(\varrho w) = (r\theta')(\varrho w')\beta$ gives $w_1 = w'_1$. So $\gamma = \beta : w \Rightarrow w'$ is unique with $\beta = g\gamma$, $\theta = \theta'(f\gamma)$.

Theorem 4. A bicategory \mathcal{B} is biequivalent to Span(\mathcal{E}) for some category \mathcal{E} with pullbacks if and only if \mathcal{B} satisfies the following three conditions:

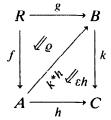
- (i) Each arrow r is isomorphic to gf^* for some maps f, g.
- (ii) For all maps f, g with the same source, there exist an arrow r and 2-cell $\varrho: g \Rightarrow rf$ such that (f, ϱ, g) is a tabulation of r.
 - (iii) Any two 2-cells $f \Rightarrow f'$ between maps f, f' are equal and invertible.

Proof. Span(\mathcal{E}) satisfies the conditions by Propositions 2 and 3. The conditions are invariant under biequivalence, so we have proved 'only if'.

Suppose \mathcal{B} satisfies the conditions. It is useful to observe that, if g and gw are maps, then so is w (for, by (i) there are maps m, n with $w \cong nm^*$, so, by (ii), we have two tabulations $(1, \varrho, gw)$, (m, σ, gn) of gw; since tabulations are unique up to equivalence, m is invertible and $w \cong nm^*$ is a map).

From the remark preceding Proposition 3 we see that we must take $\ell = C \Re^*$. Condition (iii) implies that ℓ is biequivalent to \Re^* .

To prove ℓ has pullbacks, take $h: A \rightarrow C$, $k: B \rightarrow C$ to be maps in \mathcal{A} . By (i), (ii), the arrow k*h has a tabulation (f, ϱ, g) with g a map.



By (iii) we have $kg \cong hf$. Taking isomorphism classes of maps, we obtain a commutative square in \mathcal{E} . To see that this is a pullback, take maps $u: X \to A$, $v: X \to B$ with $hu \cong kv$. By T1, there is $w: X \to R$ with $v \cong gw$ and $fw \Rightarrow u$. Since g, gw are maps, w is too. Then $fw \Rightarrow u$ is invertible by (iii). To prove uniqueness of w in \mathcal{E} , suppose $fw' \cong u$, $gw' \cong v$ with w' a map. Let β be the composite $gw \cong v \cong gw'$. In order to apply T2, we must verify the compatibility condition which involves the equality of two 2-cells $gw \Rightarrow k*hu$. Such 2-cells correspond to 2-cells $kgw \Rightarrow hu$ and there is at most one such by (iii). So T2 applies to yield $\gamma: w \Rightarrow w'$ which is invertible by (iii). So w, w' become equal in \mathcal{E} .

It remains to define a biequivalence $F: \mathcal{B} \to \operatorname{Span}(f)$. On objects it is the identity. An arrow $r: A \to B$ in \mathcal{B} is taken to the span $Fr: A \to B$ made up of the isomorphism classes of maps f, g with $r \cong gf^*$ (this uses (i) and makes a choice). Suppose $r \cong gf^*$, $s \cong kh^*$ in $\mathcal{B}(A, B)$ are obtained by applying (i) to r, s. Then 2-cells $\sigma: r \Rightarrow s$ are in

bijection with 2-cells $gf^* \Rightarrow kh^*$ which are in bijection with 2-cells $g \Rightarrow kh^*f$ (using $f \rightarrow f^*$). By (ii) and T1, such 2-cells lead to arrows w with $g \cong kw$, $hw \Rightarrow f$. Since k, kw are maps, w is a map; and, by (iii), $hw \cong f$. Put $F\sigma: Fr \Rightarrow Fs$ equal to the isomorphism class of w. Using T2 and (iii), we see that the functor

$$F: \mathcal{B}(A, B) \rightarrow \operatorname{Span}(\mathcal{E})(A, B)$$

is fully faithful, and so is clearly an equivalence. From the description above of pullbacks in $\mathscr E$ it is also clear that $F: B \to \operatorname{Span}(\mathscr E)$ really is a homomorphism. A homomorphism which is bijective on objects and a local equivalence is certainly a biequivalence. \square

- **Remarks.** (1) For categories δ , δ' with pullbacks, it follows that the category of pullback preserving functors $\delta \to \delta'$ is biequivalent to the bicategory of tabulation preserving homomorphisms $\operatorname{Span}(\delta) \to \operatorname{Span}(\delta')$. Furthermore, tabulation preserving implies wide tabulation preserving in this case.
- (2) It is easy to see that \mathcal{B} is biequivalent to Span(δ) for some δ with finite limits if and only if \mathcal{B} satisfies (i), (ii), (iii) of the Theorem and:
- (iv) There exists an object 1 of \mathcal{B} such that each hom-category $\mathcal{B}(A, 1)$ has a terminal object which is a map.
- It follows that each hom-category $\mathcal{B}(A, B)$ is finitely complete.
- (3) Recall that a category \mathcal{E} is called *internally complete* ('locally cartesian closed' or 'closed span') when each \mathcal{E}/A is cartesian closed. It follows now from [4] that \mathcal{B} is biequivalent to Span(\mathcal{E}) for some internally complete \mathcal{E} if and only if \mathcal{B} satisfies (i), (ii), (iii), (iv) and:
 - (v) All right extensions exist.

3. Relations

A relation $r: A \to B$ in a category ℓ is a span $r: A \to B$ such that any two 2-cells $s \Rightarrow r$ in Span(ℓ) are equal. If $a: X \to A$, $b: X \to B$ are arrows in ℓ we write a(r)b when there exists a 2-cell $(a, X, b) \Rightarrow r$ in Span(ℓ); we say that a is r-related to b.

An arrow $e: Y \to X$ in \mathcal{E} is called *strong epic* when, for all relations $r: A \to B$ and arrows $a: X \to A$, $b: X \to B$, if ae(r)be, then a(r)b. In the presence of pullbacks, strong epic implies epic. A strong epic which is monic is invertible.

A category & is called regular when:

- R1. Pullbacks exist.
- R2. For each span $s = (s_0, S, s_1) : A \rightarrow B$, there exists a relation $r = (r_0, R, r_1) : A \rightarrow B$ and a strong epic $e : S \rightarrow R$ such that $r_0 e = s_0$, $r_1 e = s_1$.
 - R3. Each pullback of a strong epic is strong epic.

For a regular category δ , there is a bicategory Rel(δ) defined as follows. The objects are those of δ . An arrow $r: A \rightarrow B$ is a relation. Composition of relations $r: A \rightarrow B$, $s: B \rightarrow C$ is obtained by composing as spans and then applying R2 to ob-

tain 3 relation $sr: A \rightarrow C$; it is easily seen using R3 that a(sr)c if there are b and strong epic e with ae(r)b and b(s)ce. The 2-cells are those of spans; however, note that Rel(c)(A, B) is an ordered set.

Proposition 5. An arrow $r = (r_0, R, r_1) : A \rightarrow B$ in Rel(ϵ) is a map if and only if $r_0 : R \rightarrow A$ is invertible in ϵ .

Proof. If r_0 is invertible, then the reverse relation (r_1, R, r_0) provides the right adjoint for r [3], [7].

Conversely, suppose $r \dashv s$. The unit condition is: for all $a: X \rightarrow A$, there exists b and a strong e with ae(r)b and b(s)ae.

The counit condition amounts to:

$$b(s)a, a(r)b'$$
 imply $b = b'$.

From the former with $a = 1_A$ we get e(r)b with e strong epic. So r_0 is strong epic. It remains to prove r_0 monic. Take $x, s' : X \to R$ with $r_0x = r_0x'$. Apply the unit condition with $a = r_0x$ to obtain b and strong epic e with $r_0xe(r)b$, $b(s)r_0xe$. Apply the counit condition to $b(s)r_0xe$, $r_0xe(r)r_1xe$ to obtain $b = r_1xe$; and similarly $b = r_1x'e$. Since r_0, r_1 are jointly monic, x'e = xe. Since e is epic, x = x'. \square

Proposition 6. Each arrow r in Rel(δ) has a tabulation (f, ϱ, g) where g is a map.

Proof. By Proposition 5 (the easy direction!), we have maps $f = (1, R, r_0)$, $g = (1, R, r_1)$. Assume c(g)b. Then $r_1c = b$; so we have $c(f)r_0c$, $r_0c(r)b$ which implies c(rf)b. Thus $g \le rf$.

Suppose $u: X \to A$, $v: X \to B$ are relations with $v \le ru$. Then we can define a relation $w: X \to R$ by x(w)c if and only if $x(u)r_0c$ and $x(v)r_1c$. Assume x(v)b. Since $v \le ru$, there exist a and strong epic e with xe(u)a, a(r)be. Let c be such that $r_0c = a$, $r_1c = be$. So xe(w)c, c(g)be. So xe(gw)be. So x(gw)b. This proves $v \le gw$. Reversing these steps we get $gw \le v$. So v = gw. If x(fw)a, then xe(w)c, $r_0c = ae$ for some c and strong epic e. So $xe(u)r_0c$. So x(u)a. So $fw \le u$. This proves T1 (in fact, in the stronger form!).

Suppose $u, w, w', fw \le u, fw' \le u, gw \le gw'$ as in T2. We must prove $w \le w'$. So take x(w)c. Then $fw \le u, x(w)c, c(f)r_0c$ imply $x(u)r_0c$. Also $gw \le gw', x(w)x, c(g)r_1c$ imply $x(gw')r_1c$. So there are c' and strong epic e with $xe(w')c', c'(g)r_1ce$. So $r_1c' = r_1ce$. But $fw' \le u, xe(w')c', c'(g)r_0c'$ imply $xe(u)r_0c'$. So we have $xe(u)r_0c', xe(u)r_0ce$. Since u is a map it follows that $r_0c' = r_0ce$. Since r_0, r_1 are jointly monic, c' = ce. So we have xe(w')ce which implies x(w')c since e is strong epic.

In a bicategory \mathcal{B} for which each $\mathcal{B}(A, B)$ is an ordered set, equations between 2-cells such as those in T2 hold automatically. This means that T2 is a condition on the pair f, g independent of ϱ . Thus one cannot expect general pairs of maps f, g with the same source to form a tabulation as in Theorem 4(ii) except in very special cases (such as $\text{Rel}(\mathcal{E})$ where \mathcal{E} is an ordered set).

A pair of maps f, g in \mathcal{B} is called *ripe* when f, g have the same source C and, for all maps $a, b: X \to C$ and 2-cells $\alpha: fa \Rightarrow fb$, $\beta: ga \Rightarrow gb$, there exists a unique $\gamma: a \Rightarrow b$ with $f\gamma = \alpha$, $g\gamma = \beta$. Clearly, if \mathcal{B} is locally ordered then each tabulation (f, ϱ, g) has f, g ripe.

Theorem 7. A bicategory \mathcal{B} is biequivalent to Rel(\mathcal{E}) with \mathcal{E} a regular category if and only if \mathcal{B} satisfies the following three conditions:

- (i) Each arrow r is isomorphic to gf* for some ripe pair of maps f, g.
- (ii) For all ripe pairs of maps f, g there exist an arrow r and a 2-cell ϱ : $g \Rightarrow rf$ such that (f, ϱ, g) is a tabulation of r.
- (iii) Any two 2-cells with the same source and target arrows are equal, and all 2-cells between maps are invertible.

Proof. Clearly Rel(\mathcal{E}) satisfies (iii). For a bicategory \mathcal{B} satisfying (iii), ripeness of a pair of maps f, g amounts to: for maps a, b, if $fa \cong fb$, $ga \cong gb$ then $a \cong b$. So Rel(\mathcal{E}) satisfies (i), (ii) by Propositions 5 and 6.

Conversely, suppose \mathscr{B} satisfies the conditions. Since (iii) implies \mathscr{B} and $C\mathscr{B}$ are biequivalent we may assume all invertible 2-cells in \mathscr{B} are identities. Each arrow in \mathscr{B} does have a tabulation by (i) and (ii). It is important to observe that, if (f, ϱ, g) is a tabulation of r, then, in T2, the arrow w is a map when v is (and so u = fw using (iii)). To see this, let (m, σ, n) be a tabulation of w. Since $m \to m^*$, $fnm^* \cong fw \leq u$ implies $fn \leq um$; so fn = um by (iii). The pair of maps m, gn is ripe; for ma = nb, gna = gnb imply fna = uma = umb = fnb, and so we have na = nb (since f, g are ripe), so a = b (since m, n are ripe). By (ii), m, gn tabulate $gnm^* = gw = v$. But 1, v tabulate v. So n is an isomorphism. So $w = nm^*$ is a map.

Let $\mathscr{E} = \mathscr{B}^*$. We shall show that \mathscr{E} is a regular category. To prove R1 take maps $h: A \to C$, $k: B \to C$ and let f, g tabulate k*h. So $g \le k*hf$ implies $kg \le hf$ which means kg = hf by(iii). That f, g provide a pullback for h, k now follows from the last paragraph.

To prove R2, take a span $(u, S, v): A \rightarrow B$ in \mathscr{E} . Let f, g tabulate vu^* ; ripeness means (f, R, g) is a relation in \mathscr{E} . By the second last paragraph there exists a map e with ge = v, fe = u. We claim $ee^* = 1$. To see this, let m, n tabulate ee^* . Then $nm^* = ee^* \le 1$ gives $n \le m$ which, using (iii), gives n = m. Since m, n form a relation in \mathscr{E} (ripeness), this means m is monic. So fm, m form a ripe pair and so tabulate $m(fm)^* = mm^*f^* = ee^*f^* = e(fe)^* = eu^*$. But T2 applies to give $eu^* = f^*$ since $g(eu^*) = vu^* = gf^*$, $f(eu^*) = uu^* \le 1$, and $ff^* \le 1$. So fm, m tabulate f^* . But f, 1 tabulate f^* . So $fm = mm^* = 1$. This means R2 will be proved once we prove that any map with identity counit is strong epic in \mathscr{E} .

Let $e: Y \to X$ be a map in \mathscr{B} with $ee^* = 1$. Take a relation $(f, R, g): A \to B$ in \mathscr{E} and a, b in \mathscr{E} with ae = fc, be = gc. Then $a = aee^* = fce^*$, $b = bee^* = gce^*$. By the third last paragraph, ce^* is a map. So e is a strong epic in \mathscr{E} .

Suppose $e: Y \to X$ is a strong epic in \mathscr{E} . The reflection of the span $(e, Y, e): X \to X$ into the subcategory of relations from X to X is the identity relation (1, X, 1). By

the last two paragraphs this reflection is also given by the tabulation of ee^* . So 1_X , 1_X tabulate ee^* . So $ee^*=1$. Thus strong epics are precisely maps with identity counits.

Now we prove R3. Recall the construction of pullbacks in the proof of R1 above. Suppose further that $h \to h^*$ has identity counit. Then $gg * k^* = gf * h^* = k * hh^* = k^*$. This means that the reflection of the span (kg, R, g) into relations from C to B is (k, B, 1). Thus the underlying map g of the 2-cell $(kg, R, g) \Rightarrow (k, B, 1)$ is strong epic in δ .

Thus \mathscr{E} is a regular category. The homomorphism $\mathscr{B} \to \operatorname{Rel}(\mathscr{E})$, which is the identity on objects and takes each arrow to a tabulating relation, is clearly a biequivalence. \square

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