Comprehension and quotient structures in the language of 2-categories

Paul-André Melliès 🅙

CNRS, Institut de Recherche en Informatique Fondamentale (IRIF), Université de Paris, France
http://www.irif.fr/~mellies
mellies@irif.fr

Nicolas Rolland 🅙

Institut de Recherche en Informatique Fondamentale (IRIF), Université de Paris, France
nicolas.rolland@gmail.com

— Abstract

Lawvere observed in his celebrated work on hyperdoctrines that the set-theoretic schema of comprehension can be elegantly expressed in the functorial language of categorical logic, as a comprehension structure on the functor $p : \mathcal{E} \to \mathcal{B}$ defining the hyperdoctrine. In this paper, we formulate and study a strictly ordered hierarchy of three notions of comprehension structure on a given functor $p : \mathcal{E} \to \mathcal{B}$, which we call (i) comprehension structure, (ii) comprehension structure with section, and (iii) comprehension structure with image. Our approach is 2-categorical and we thus formulate the three levels of comprehension structure on a general morphism $p : \mathbf{E} \to \mathbf{B}$ in a 2-category $\mathcal{K}$. This conceptual point of view on comprehension structures enables us to revisit the work by Fumex, Ghani and Johann on the duality between comprehension structures and quotient structures on a given functor $p : \mathcal{E} \to \mathcal{B}$. In particular, we show how to lift the comprehension and quotient structures on a functor $p : \mathcal{E} \to \mathcal{B}$ to the categories of algebras or coalgebras associated to functors $F_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ and $F_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}$ of interest, in order to interpret reasoning by induction and coinduction in the traditional language of categorical logic, formulated in an appropriate 2-categorical way.

1 Introduction

A fundamental duality between comprehension and quotient structures

One fundamental discovery by Lawvere [14] is that the *comprehension schema* of Zermelo set theory [15] can be elegantly expressed in the functorial language of categorical logic, in the following way. Consider the category **Set** of sets and functions, and the category **Pred** of predicates, defined in the following way: its objects are the pairs (A, R) consisting of a set A and of a function $R : A \to \Omega$ to the set $\Omega = \{\text{false}, \text{true}\}$ of booleans, describing a specific predicate R of A ; its morphisms $f : (A, R) \to (B, S)$ are the functions $f : A \to B$ such that $\forall a \in A, Ra \Rightarrow S(fa)$. The functor $p : \mathbf{Pred} \to \mathbf{Set}$ is the forgetful functor which transports every predicate (A, R) to its underlying set A. The comprehension schema enables one to turn every predicate (A, R) into a set $[A, R]$ defined as follows

$$[A, R] := \{ a \in A \mid Ra = \text{true} \}$$

equipped moreover with a function

$$\iota_{A,R} \quad : \quad [A, R] \longrightarrow A$$

which transports every element a of the set $[A, R]$ to itself, seen as element in A. The construction is natural in (A, R) in the sense that it defines a functor

$$[-] \quad : \quad \mathbf{Pred} \longrightarrow \mathbf{Set}$$

together with a natural transformation

$$\iota \; : \; [-] \Longrightarrow p \; : \; \mathbf{Pred} \longrightarrow \mathbf{Set}.$$

Here, naturality means that every morphism $f : (A, R) \to (B, S)$ between predicates induces the commutative diagram below, in the category **Set**.

$$\begin{array}{ccc} [A, R] & \xrightarrow{\;\iota_{A,R}\;} & A \\ {\scriptstyle[A,f]}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ [B, S] & \xrightarrow{\;\iota_{B,S}\;} & B \end{array} \tag{1}$$

More generally, considering this example of the functor $p : \mathbf{Pred} \to \mathbf{Set}$ as typical, it makes sense to formulate the following "minimalist" notion of comprehension structure:

▶ **Definition 1.** *A comprehension structure on a functor* $p : \mathcal{E} \to \mathcal{B}$ *is a pair* $([-], \iota)$ *consisting of a functor* $[-] : \mathcal{E} \to \mathcal{B}$ *and of a natural transformation* $\iota : [-] \Rightarrow p$.

Interestingly, Jacobs provides in [6] a useful and detailed survey of a hierarchy of axiomatic requirements on a functor $p : \mathcal{E} \to \mathcal{B}$ appearing in the literature, from which such a comprehension structure can be derived. In a decreasing order of generality, one finds:

- Jacob's comprehension categories, defined in [6], Def. 4.1, page 181.
- Ehrhard's D-categories [2] called comprehension categories with unit in [6], Def. 4.12.
- Lawvere categories [14] as Jacobs defined them in [6], first paragraph of p. 190.

Our definition just given of a comprehension structure (Def. 1) does not appear as such in the literature, at least in the elementary 2-categorical way we express it here. The reason is that the comprehension pair $([-], \iota)$ can be equivalently formulated as a functor $\mathcal{P} : \mathcal{E} \to \mathcal{B}^{\to}$ to the category of arrows of $\mathcal{B}$, making the diagram below commute:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\;\;\mathcal{P}\;\;} & \mathcal{B}^{\to} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{cod}} & \\ & \mathcal{B} & \end{array} \tag{2}$$

where $\mathrm{cod} : \mathcal{B}^{\to} \to \mathcal{B}$ denotes the codomain functor. The definitions of comprehension category [6] and of Lawvere category [14] are based on this formulation, while the definition of D-categories [2] works in an entirely different way, which we analyze later in this introduction, as well as in §4. One purpose of the present paper is to revisit these three levels definitions from a purely 2-categorical point of view. This search for a clean 2-categorical account of comprehension in categorical logic is motivated by our desire to understand at this level of abstraction a recent observation by Fumex, Ghani and Johann [3, 4], who establish a very nice duality between (a) the *operation of comprehension* which underlies reasoning by induction using initial algebras, and (b) the *operation of quotienting* which underlies

60 reasoning by coinduction using terminal coalgebras. In particular, Fumex introduces in his
61 PhD thesis a notion of tC-opfibration $p : \mathcal{E} \to \mathcal{B}$ (where tC refers to the section functor t
62 and the comprehension functor C of the structure) adapted for induction reasoning, which
63 he then dualizes into a notion of QCE-category $p^{op} : \mathcal{E}^{op} \to \mathcal{B}^{op}$ (where QCE stands for
64 quotient category with equality) adapted for coinduction reasoning, and simply obtained by
65 reversing the orientation of every morphism in $\mathcal{E}$ and $\mathcal{B}$.

In order to understand and to illustrate this idea of quotient structures, consider the
category **Rel** whose objects are pairs (A, R) consisting of a set A and of a binary relation
$R \subseteq A \times A$, and whose morphisms $f : (A, R) \to (B, S)$ are functions $f : A \to B$ such that
$R(a, a') \Rightarrow S(fa, fa')$. As in the previous case, the functor $p : \mathbf{Rel} \to \mathbf{Set}$ is the forgetful
function which transports every binary relation (A, R) to its underlying set A. Every binary
relation (A, R) induces a set A/R defined as the quotient

$$A/R := A/\sim_R$$

of the underlying set A by the equivalence relation $\sim_R$ generated by the binary relation R.
The set A/R comes together with a function

$$\pi_{A,R} \quad : \quad A \longrightarrow A/R$$

which transports every element of A to its equivalence class modulo $\sim_R$. The construction is
natural in (A, R) in the sense that it defines a functor

$$\llbracket - \rrbracket \quad : \quad \mathbf{Rel} \longrightarrow \mathbf{Set}$$

with $\llbracket A, R \rrbracket = A/R$, together with a natural transformation

$$\pi \; : \; p \Longrightarrow \llbracket - \rrbracket \; : \; \mathbf{Rel} \longrightarrow \mathbf{Set}.$$

66 Here, naturality means that every predicate morphism $f : (A, R) \to (B, S)$ induces the
67 commutative diagram below in the category **Set**.

$$
\begin{array}{ccc}
A & \xrightarrow{\;\pi_{A,R}\;} & \llbracket A, R \rrbracket \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \llbracket f \rrbracket} \\
B & \xrightarrow{\;\pi_{B,S}\;} & \llbracket B, S \rrbracket
\end{array}
\tag{3}
$$

69 In the same way as previously, this example leads us to the following definition, obtained by
70 dualizing Def. 1.

71 ▶ **Definition 2.** *A quotient structure on a functor $p : \mathcal{E} \to \mathcal{B}$ is a pair $(\llbracket - \rrbracket, \pi)$ consisting*
72 *of a functor $\llbracket - \rrbracket : \mathcal{E} \to \mathcal{B}$ and of a natural transformation $\pi : p \Rightarrow \llbracket - \rrbracket$.*

73 In the same way as previously, and by duality, a quotient structure is the same thing as a
74 functor $\mathcal{Q} : \mathcal{E} \to \mathcal{B}^{\to}$ to the category of arrows of $\mathcal{B}$, making the diagram below commute:

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\;\;\mathcal{Q}\;\;} & \mathcal{B}^{\to} \\
& {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{dom}} & \\
& \mathcal{B} &
\end{array}
\tag{4}
$$

76 where dom $: \mathcal{B}^{\to} \to \mathcal{B}$ denotes the domain functor.

A 2-categorical classification of comprehension structures

(i) Comprehension structures. In order to understand the duality between comprehension and quotient structures, we find enlightening to take seriously the 2-categorical nature of Def. 1 and 2, and to reformulate them in the following way. Suppose given a 2-category $\mathcal{K}$ such as $\mathcal{K} = \mathbf{Cat}$, the 2-category of categories. We consider the 2-category $\mathcal{K}/\!/\mathcal{K}$ whose objects are the triples $(\mathbf{E}, \mathbf{B}, p : \mathbf{E} \to \mathbf{B})$ consisting of a pair of 0-cells $\mathbf{E}$ and $\mathbf{B}$ and a 1-cell $p : \mathbf{E} \to \mathbf{B}$ of the 2-category $\mathcal{K}$, and whose morphisms

$$(f_{\mathbf{E}}, f_{\mathbf{B}}, \varphi) \;:\; (\mathbf{E}_1, \mathbf{B}_1, p_1 : \mathbf{E}_1 \to \mathbf{B}_1) \longrightarrow (\mathbf{E}_2, \mathbf{B}_2, p_2 : \mathbf{E}_2 \to \mathbf{B}_2) \tag{5}$$

are triples consisting of a pair of 1-cells $f_{\mathbf{B}} : \mathbf{B}_1 \to \mathbf{B}_2$ and $f_{\mathbf{E}} : \mathbf{E}_1 \to \mathbf{E}_2$ and a 2-cell

$$
\begin{array}{ccc}
\mathbf{E}_1 & \xrightarrow{\;f_{\mathbf{E}}\;} & \mathbf{E}_2 \\
{\scriptstyle p_1}\big\downarrow & \xLeftarrow{\;\;\varphi\;\;} & \big\downarrow{\scriptstyle p_2} \\
\mathbf{B}_1 & \xrightarrow[\;f_{\mathbf{B}}\;]{} & \mathbf{B}_2
\end{array}
\qquad
\varphi : p_2 \circ f_{\mathbf{E}} \Longrightarrow f_{\mathbf{B}} \circ p_1
$$

A morphism (5) is called *strict* when the 2-cell φ is the identity. We write in that case $(f_{\mathbf{E}}, f_{\mathbf{B}})$ instead of $(f_{\mathbf{E}}, f_{\mathbf{B}}, \mathrm{id})$. We also write $\mathcal{K}/\mathcal{K}$ for the sub-2-category of $\mathcal{K}/\!/\mathcal{K}$ of strict morphisms, with the same notion of 2-cell. It is essentially immediate that

▶ **Proposition 3.** *A comprehension structure* $([-], \iota)$ *(in the sense of Def. 1) is the same thing as a morphism in* $\mathbf{Cat}/\!/\mathbf{Cat}$ *of the form*

$$(f_{\mathcal{E}}, \mathrm{id}_{\mathcal{B}}, \varphi) \;:\; (\mathcal{E}, \mathcal{B}, p : \mathcal{E} \to \mathcal{B}) \longrightarrow (\mathcal{B}, \mathcal{B}, \mathrm{id}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}) \tag{6}$$

One main contribution of the paper is to revisit in this 2-categorical style the hierarchy of comprehension categories described by Jacobs [6]. To that purpose, we introduce three corresponding levels of comprehension structures, each of them coming with an elementary and concise 2-categorical formulation, as depicted in the figure below:

- comprehension structures, Def. 1 as reformulated in Prop. 3,
- comprehension structures with section, Def. 4 as reformulated in Prop. 6,
- comprehension structures with image, Def. 7 as formulated in Def. 25.

One basic observation is that our minimalist notion of comprehension structure (Def. 1) generalizes Jacobs' notion of comprehension category, by relaxing the assumption that the associated functor $\mathcal{P} : \mathcal{E} \to \mathcal{B}^{\to}$ in (2) transports every p-cartesian map of $\mathcal{E}$ to a *cod*-cartesian map of $\mathcal{B}^{\to}$, that is, to a pullback diagram of the form (1) in the category $\mathcal{B}$. This observation underlies the first layer (in dark green) of our classification below.
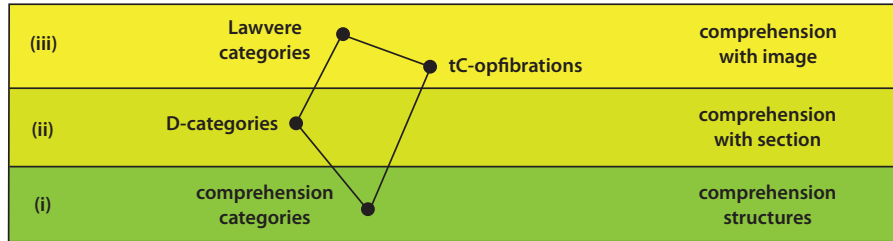
(ii) Comprehension structures with section. We move to the next layer and consider Ehrhard's notion of D-category [2, 6] which is based on a convenient but somewhat mysterious recipe to equip a functor $p : \mathcal{E} \to \mathcal{B}$ with a comprehension structure $([-], \iota)$. The recipe [2, 6] works in two stages: (1) first, one equips the functor p with a section $\star : \mathcal{B} \to \mathcal{E}$, (2) then one requires that the section $\star$ has a right adjoint $[-] : \mathcal{E} \to \mathcal{B}$. Recall that a section $\star : \mathcal{B} \to \mathcal{E}$ is a functor such that $p \circ \star = \mathrm{id}_{\mathcal{B}}$. This leads us to the following definition:

▶ **Definition 4.** *A comprehension structure with section on a functor $p : \mathcal{E} \to \mathcal{B}$ is a section $\star : \mathcal{B} \to \mathcal{E}$ together with a right adjoint functor $[-] : \mathcal{E} \to \mathcal{B}$.*

One astonishing aspect of the definition is that the natural transformation $\iota : [-] \Rightarrow p$ of the associated comprehension structure $([-], \iota)$ is not given explicitly, but derived as the image by the functor $p : \mathcal{E} \to \mathcal{B}$ of the counit $\star \circ [-] \Rightarrow \mathrm{id}_{\mathcal{E}}$ of the adjunction $\star \dashv [-]$. From this follows that the relationship between the natural transformation ι and the two functors $\star, [-]$ is not entirely obvious from a conceptual point of view. We clarify this point by observing here that the original adjunction $\star \dashv [-]$ of Def. 4 living in $\mathcal{K} = \mathbf{Cat}$ is the "emerged part" of a more fundamental adjunction $\star \dashv ([-], \iota)$ living in the 2-category $\mathbf{Cat} /\!/ \mathbf{Cat}$, and where the natural transformation ι is thus integrated. A preliminary observation is that

▶ **Proposition 5.** *A section of the functor $p : \mathcal{E} \to \mathcal{B}$ is the same thing as a strict morphism in $\mathbf{Cat}/\mathbf{Cat}$ of the form*

$$(s_{\mathcal{E}}, \mathrm{id}_{\mathcal{B}}) \quad : \quad (\mathcal{B}, \mathcal{B}, \mathrm{id}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}) \longrightarrow (\mathcal{E}, \mathcal{B}, p : \mathcal{E} \to \mathcal{B}) \tag{7}$$

We will prove in the course of the paper (see §4, Prop. 17) that the adjunction $\star \dashv [-]$ in Def. 4 may be equivalently formulated as an adjunction in $\mathbf{Cat}/\!/\mathbf{Cat}$ between the section $\star$ seen as a strict morphism (7) and the comprehension structure $([-], \iota)$ seen as a morphism (6). This property establishes the secretly 2-categorical nature of the notion (Def. 4) of comprehension structure with section:

▶ **Proposition 6.** *A comprehension structure with section is a comprehension structure (6) right adjoint to a section (7) in the 2-category $\mathbf{Cat}/\!/\mathbf{Cat}$.*

The resulting 2-categorical notion of *comprehension structure with section* (Prop. 6) captures the essence of the notion of *D*-category, and generalizes in in an interesting and useful way to the categories of algebras and coalgebras, see §4 and §8 for a discussion.

(iii) Comprehension structures with image. We move finally to the next layer of our hierarchy, and observe that the functor $p : \mathcal{E} \to \mathcal{B}$ is required to be an opfibration in both notions of Lawvere category and of *tC*-opfibration [6, 3]. From this follows that the functor $p : \mathcal{E} \to \mathcal{B}$ has an image structure, in the sense elaborated in §6 of this paper. This additional image structure on the functor p enables one to construct a functor

$$\mathbf{image} \quad : \quad \mathcal{B}^{\to} \longrightarrow \mathcal{E} \tag{8}$$

from the arrow category $\mathcal{B}^{\to}$ of the basis category $\mathcal{B}$ to the category $\mathcal{E}$. The functor **image** transports every morphism $f : A \to B$ of the basis category $\mathcal{B}$ to an object $\mathbf{image}(f)$ in the fiber category $\mathcal{E}_B$ of the object B, called the image of $f : A \to B$, and satisfying the expected universality property, see §6 for details. By construction, the image functor (8) makes the diagram below commute:

$$\begin{array}{ccc} \mathcal{E} & \xleftarrow{\;\mathbf{image}\;} & \mathcal{B}^{\to} \\ {\scriptstyle p} \searrow & & \swarrow {\scriptstyle \mathrm{cod}} \\ & \mathcal{B} & \end{array} \tag{9}$$

In order to recover a comprehension structure (2), the definition of a Lawvere category requires that the functor **image** has a right adjoint $\mathcal{P} : \mathcal{E} \to \mathcal{B}^{\to}$ in the fibered sense above the category $\mathcal{B}$. This leads us to the following definition.

▶ **Definition 7.** *A comprehension structure with image on a functor $p : \mathcal{E} \to \mathcal{B}$ is a functor* **image** $: \mathcal{B}^{\to} \to \mathcal{E}$ *together with a right adjoint* $\mathcal{P} : \mathcal{E} \to \mathcal{B}^{\to}$ *in the fibered sense above $\mathcal{B}$.*

On the other hand, and somewhat surprisingly, the definition of *tC*-opfibration is apparently weaker, since it only requires that the functor $p : \mathcal{E} \to \mathcal{B}$ has a comprehension structure with section, in the sense of Def. 4. In order to clarify the situation, and to get a clean and harmonious picture, we establish that every *tC*-opfibration has comprehension with image (in the sense of Def 7) using the following statement, which applies in particular to the case of an opfibration $p : \mathcal{E} \to \mathcal{B}$:

▶ **Proposition 8.** *Suppose that the functor $p : \mathcal{E} \to \mathcal{B}$ has an image structure. In that case, every comprehension structure with section (in the sense of Def. 4) defines a comprehension structure with image (in the sense of Def. 7).*

Illustration: inductive reasoning on algebras, coinductive reasoning on coalgebras

Suppose given a functor $p : \mathcal{E} \to \mathcal{B}$ equipped with a comprehension structure with section $\star : \mathcal{B} \to \mathcal{E}$, where the categories $\mathcal{E}$ and $\mathcal{B}$ are moreover equipped with endofunctors $F : \mathcal{B} \to \mathcal{B}$ and $G : \mathcal{E} \to \mathcal{E}$ related by a distributivity law

$$\delta \quad : \quad F \circ p \Longrightarrow p \circ G \quad : \quad \mathcal{E} \longrightarrow \mathcal{B}. \tag{10}$$

One guiding ambition of our 2-categorical account of comprehension structures is to explain by conceptual means the recent characterization by Fumex, Ghani and Johann [3, 4] of the initial G-algebra of $\mathcal{E}$ as the section $\star_A$ of the initial F-algebra μF of the basis category $\mathcal{B}$. To that purpose, we describe in §8 the necessary and sufficient conditions which characterize when the distributivity law (10) on a comprehension structure with section $p : \mathcal{E} \to \mathcal{B}$ induces a comprehension structure with section $\mathbf{Alg}(p) : \mathbf{Alg}_G(\mathcal{E}) \to \mathbf{Alg}_F(\mathcal{B})$ on the associated categories of algebras. In this situation, we obtain a simple conceptual explanation for the forementioned result ([4], Thm 2.10) by Fumex, Ghani and Johann:

▶ **Corollary 9.** *The comprehension structure with section $\star : \mathcal{B} \to \mathcal{E}$ lifts to a comprehension structure with section $\star : \mathbf{Alg}_F(\mathcal{B}) \to \mathbf{Alg}_G(\mathcal{E})$ which is left adjoint to comprehension $\lceil - \rfloor$ and thus transports the initial F-algebra μF to the initial G-algebra $\mu G = \star_{\mu F}$.*

We proceed dually in the case of quotient structures and obtain necessary and sufficient conditions to ensure that

▶ **Corollary 10.** *The quotient structure with section $\star : \mathcal{B} \to \mathcal{E}$ lifts to a quotient structure with section $\star : \mathbf{CoAlg}_F(\mathcal{B}) \to \mathbf{CoAlg}_G(\mathcal{E})$ which is right adjoint to quotient $[\![-]\!]$ and thus transports the terminal F-coalgebra νF to the terminal G-coalgebra $\nu G = \star_{\nu F}$.*

Plan of the paper

After this long and detailed introduction, we recall in §2 the notion of slice 2-category $\mathcal{K}/\!/\mathcal{K}$ and establish in §3 a simple and useful description of the formal adjunctions in this 2-category. This leads us to formulate in §4 our 2-categorical notion of *comprehension with section*. We then formulate in §5 and §6 the notion of path object $(\mathbf{B}^{\to}, \beta)$ of an object $\mathbf{B}$ in any 2-category $\mathcal{K}$, and the related notion of morphism $p : \mathbf{E} \to \mathbf{B}$ with an image structure. This leads us to establish in §7 that a comprehension with image $p : \mathbf{E} \to \mathbf{B}$ is the same thing as a comprehension structure with section $\star : \mathbf{B} \to \mathbf{E}$, whose underlying section comes with an image structure, and thus a morphism **image** $: \mathbf{B}^{\to} \to \mathbf{E}$. We then illustrate in §8 the benefits of our 2-categorical approach with the example of inductive and coinductive reasoning on algebra and coalgebra structures, and finally conclude in §9.

2 Definition of the slice 2-categories $\mathcal{K}//\mathcal{K}$ and $\mathcal{K}/\mathcal{K}$

We explained in the introduction, see (5), how to define the objects and the morphisms of the 2-category $\mathcal{K}//\mathcal{K}$ associated to a 2-category $\mathcal{K}$. For the sake of completeness, we recall now that a 2-cell

$$(\theta_{\mathbf{B}}, \theta_{\mathbf{E}}) \; : \; (f_{\mathbf{E}}, f_{\mathbf{B}}, \varphi) \Longrightarrow (g_{\mathbf{E}}, g_{\mathbf{B}}, \psi) \; : \; (\mathbf{E}_1, \mathbf{B}_1, p_1) \longrightarrow (\mathbf{E}_2, \mathbf{B}_2, p_2)$$

of the 2-category $\mathcal{K}//\mathcal{K}$ is defined as a pair of 2-cells $\theta_B : f_{\mathbf{B}} \Rightarrow g_{\mathbf{B}}$ and $\theta_{\mathbf{E}} : f_{\mathbf{E}} \Rightarrow g_{\mathbf{E}}$ of the original 2-category $\mathcal{K}$, making the two pasting diagrams equal:

It is worth mentioning that, thanks to this carefully chosen definition of 2-cells, there exists a pair of 2-functors

$$\mathcal{K}//\mathcal{K} \xrightarrow[\textbf{target}]{\textbf{source}} \mathcal{K} \tag{11}$$

defined as the expected first and second projections, which transport every object $(\mathbf{E}, \mathbf{B}, p : \mathbf{E} \to \mathbf{B})$ of the 2-category $\mathcal{K}//\mathcal{K}$ to the object

$$\textbf{source}(\mathbf{E}, \mathbf{B}, p : \mathbf{E} \to \mathbf{B}) = \mathbf{E} \qquad \textbf{target}(\mathbf{E}, \mathbf{B}, p : \mathbf{E} \to \mathbf{B}) = \mathbf{B}$$

of the underlying 2-category $\mathcal{K}$. Finally, let us also mention that the 2-category $\mathcal{K}/\mathcal{K}$ of strict morphisms in $\mathcal{K}//\mathcal{K}$ comes exactly with the same notion of 2-cell. In other words, the inclusion 2-functor $\mathcal{K}/\mathcal{K} \to \mathcal{K}//\mathcal{K}$ is locally fully faithful.

3 Formal adjunctions in the 2-category $\mathcal{K}//\mathcal{K}$

As explain in the introduction in the case $\mathcal{K} = \mathbf{Cat}$, one main observation of the paper is that the notion of *comprehension structure with section* (Def. 4) can be elegantly expressed as a specific form of adjunction living in the 2-category $\mathcal{K}//\mathcal{K}$ (Prop. 6). As a warm up exercise, we study the notion of formal adjunction in $\mathcal{K}//\mathcal{K}$ in the sense of Street [13] and relate it in full generality to the notion of formal adjunction in the original 2-category $\mathcal{K}$. Suppose given a pair of morphisms

$$\begin{aligned} L = (L_{\mathbf{E}}, L_{\mathbf{B}}, \varphi) &\; : \; (\mathbf{E}_1, \mathbf{B}_1, p_1 : \mathbf{E}_1 \to \mathbf{B}_1) \longrightarrow (\mathbf{E}_2, \mathbf{B}_2, p_2 : \mathbf{E}_2 \to \mathbf{B}_2) \\ R = (R_{\mathbf{E}}, R_{\mathbf{B}}, \psi) &\; : \; (\mathbf{E}_2, \mathbf{B}_2, p_2 : \mathbf{E}_2 \to \mathbf{B}_2) \longrightarrow (\mathbf{E}_1, \mathbf{B}_1, p_1 : \mathbf{E}_1 \to \mathbf{B}_1) \end{aligned} \tag{12}$$

living in the 2-category $\mathcal{K}//\mathcal{K}$, and thus depicted as below in the underlying 2-category $\mathcal{K}$:

By definition, a *formal adjunction* between L and R in the 2-category $\mathcal{K}//\mathcal{K}$ is defined as a pair of 2-cells

$$(\eta_{\mathbf{B}}, \eta_{\mathbf{E}}) \; : \; (\mathrm{id}, \mathrm{id}, \mathrm{id}) \Rightarrow (R_{\mathbf{E}} \circ L_{\mathbf{E}}, R_{\mathbf{B}} \circ L_{\mathbf{B}}, (R_{\mathbf{B}} \circ \varphi)(\psi \circ L_{\mathbf{E}})) \; : \; (\mathbf{E}_1, \mathbf{B}_1, p_1) \to (\mathbf{E}_1, \mathbf{B}_1, p_1)$$

$$(\varepsilon_{\mathbf{B}}, \varepsilon_{\mathbf{E}}) \; : \; (L_{\mathbf{E}} \circ R_{\mathbf{E}}, L_{\mathbf{B}} \circ R_{\mathbf{B}}, (L_{\mathbf{B}} \circ \psi)(\varphi \circ R_{\mathbf{E}})) \Longrightarrow (\mathrm{id}, \mathrm{id}, \mathrm{id}) \; : \; (\mathbf{E}_2, \mathbf{B}_2, p_2) \longrightarrow (\mathbf{E}_2, \mathbf{B}_2, p_2)$$

in the 2-category $\mathcal{K}//\mathcal{K}$, satisfying the triangular equations, see [13, 10] for details. One nice consequence of this definition by generators and relations is that the resulting notion of formal adjunction is preserved by 2-functors. Every formal adjunction $L \dashv R$ in $\mathcal{K}//\mathcal{K}$ is thus transported by the 2-functors (11) into a pair of formal adjunctions $L_{\mathbf{B}} \dashv R_{\mathbf{B}}$ and $L_{\mathbf{E}} \dashv R_{\mathbf{E}}$ in the underlying 2-category $\mathcal{K}$. From this follows that the 2-cell $\psi : p_1 \circ R_{\mathbf{E}} \Rightarrow R_{\mathbf{B}} \circ p_2$ in $\mathcal{K}$ induces a 2-cell $\widetilde{\psi} : L_{\mathbf{B}} \circ p_1 \Rightarrow p_2 \circ L_{\mathbf{E}}$ called the *mate* of ψ, of the form below:

$$
\begin{array}{ccc}
\mathbf{E}_1 & \xrightarrow{\;\; L_{\mathbf{E}} \;\;} & \mathbf{E}_2 \\
{\scriptstyle p_1}\downarrow & \overset{\widetilde{\psi}}{=\!=\!=\!\Longrightarrow} & \downarrow{\scriptstyle p_2} \\
\mathbf{B}_1 & \xrightarrow[\;\; L_{\mathbf{B}} \;\;]{} & \mathbf{B}_2
\end{array}
$$

Suppose given two morphisms L and R in the 2-category $\mathcal{K}//\mathcal{K}$ as in (12). In that case,

▶ **Proposition 11.** *A formal adjunction $L \dashv R$ in the 2-category $\mathcal{K}//\mathcal{K}$ is the same thing as a pair of formal adjunctions $L_{\mathbf{B}} \dashv R_{\mathbf{B}}$ and $L_{\mathbf{E}} \dashv R_{\mathbf{E}}$ in the 2-category $\mathcal{K}$, such that the induced mate $\widetilde{\psi}$ of the 2-cell ψ is the inverse of the 2-cell φ in the 2-category $\mathcal{K}$.*

From this follows easily that

▶ **Proposition 12.** *A pair of formal adjunctions $L_{\mathbf{B}} \dashv R_{\mathbf{B}}$ and $L_{\mathbf{E}} \dashv R_{\mathbf{E}}$ in the 2-category $\mathcal{K}$ lifts to a formal adjunction $(L_{\mathbf{B}}, L_{\mathbf{E}}, \varphi) \dashv (R_{\mathbf{B}}, R_{\mathbf{E}}, \psi)$ in the 2-category $\mathcal{K}//\mathcal{K}$ precisely when the 2-cell φ is invertible in $\mathcal{K}$ and the 2-cell ψ coincides with the mate of φ^{-1}.*

It should be mentioned that a similar observation is made by Kelly [9] (Prop. 1.3) on the 2-category of D-algebras derived from a 2-monad D on the 2-category $\mathcal{K}$, see also [8], Section 3.5. It should be also noted that the characterization of formal adjunctions in $\mathcal{K}//\mathcal{K}$ is also very similar to the description of formal adjunctions in the 2-category of monoidal categories and lax monoidal functors, see for instance [10].

4 Comprehension structures with section

In this section, we suppose given a morphism $p : \mathbf{E} \to \mathbf{B}$ in the 2-category $\mathcal{K}$ and establish (Prop. 17) that a *comprehension structure with section* on a morphism $p : \mathbf{E} \to \mathbf{B}$ originally defined (Def. 16) as an adjunction $\star \dashv [-]$ in the 2-category $\mathcal{K}$, can be in fact lifted (and thus equivalently defined) as a specific form of adjunction $\star \dashv ([-], \iota)$ in the 2-category $\mathcal{K}//\mathcal{K}$. As expected, the proof of Prop. 17 relies on the characterization of formal adjunctions in $\mathcal{K}//\mathcal{K}$ just established in the previous section, see Prop. 12. The result provides a general 2-categorical formulation of the proposition (Prop. 6) stated in the introduction for the particular case $\mathcal{K} = \mathbf{Cat}$. In order to perform our construction at this general 2-categorical level of abstraction, we start by defining a *comprehension structure* on a morphism $p : \mathbf{E} \to \mathbf{B}$ in the 2-category $\mathcal{K}$, in a way which generalizes what we did in the introduction (see Def. 1) in the specific case $\mathcal{K} = \mathbf{Cat}$.

▶ **Definition 13.** *A comprehension structure on the morphism $p : \mathbf{E} \to \mathbf{B}$ is a pair $([-], \iota)$ consisting of a morphism $[-] : \mathbf{E} \to \mathbf{B}$ and of a 2-cell $\iota : [-] \Rightarrow p$ in the 2-category $\mathcal{K}$.*

We carry on as we did in the introduction (see Prop. 3) and observe that

▶ **Proposition 14.** *A comprehension structure* $([-], \iota)$ *on* $p : \mathbf{E} \to \mathbf{B}$ *is the same thing as a morphism in the 2-category* $\mathcal{K}//\mathcal{K}$ *of the form*

$$(f_{\mathbf{E}}, \mathrm{id}_{\mathbf{B}}, \varphi) \quad : \quad (\mathbf{E}, \mathbf{B}, p : \mathbf{E} \to \mathbf{B}) \longrightarrow (\mathbf{B}, \mathbf{B}, \mathrm{id}_{\mathbf{B}} : \mathbf{B} \to \mathbf{B}) \tag{13}$$

We proceed in just the same way as we did in the introduction with Prop. 5, and characterize a section $\star : \mathbf{B} \to \mathbf{E}$ of the morphism $p : \mathbf{E} \to \mathbf{B}$ as a specific form of strict morphism:

▶ **Proposition 15.** *A section* $\star : \mathbf{B} \to \mathbf{E}$ *of the morphism* $p : \mathbf{E} \to \mathbf{B}$ *in* $\mathcal{K}$ *is the same thing as a strict morphism in* $\mathcal{K}//\mathcal{K}$ *of the form*

$$(s_{\mathbf{E}}, \mathrm{id}_{\mathbf{B}}) \quad : \quad (\mathbf{B}, \mathbf{B}, \mathrm{id}_{\mathbf{B}} : \mathbf{B} \to \mathbf{B}) \longrightarrow (\mathbf{E}, \mathbf{B}, p : \mathbf{E} \to \mathbf{B}) \tag{14}$$

We are now ready to give our general 2-categorical definition of *comprehension structure with section* on the morphism $p : \mathbf{E} \to \mathbf{B}$ in the 2-category $\mathcal{K}$.

▶ **Definition 16.** *A comprehension structure with section on* $p : \mathbf{E} \to \mathbf{B}$ *is a section* $\star : \mathbf{B} \to \mathbf{E}$ *together with a right adjoint* $[-] : \mathbf{E} \to \mathbf{B}$ *in the 2-category* $\mathcal{K}$.

We then take advantage of Prop. 12 in order to establish that:

▶ **Proposition 17.** *A comprehension structure with section is a comprehension structure (13) right adjoint to a section (14) in the 2-category* $\mathcal{K}//\mathcal{K}$.

Proof. Suppose given a section $\star : \mathbf{B} \to \mathbf{E}$ of the morphism $p : \mathbf{E} \to \mathbf{B}$ in the 2-category $\mathcal{K}$, described (Prop. 15) as a strict morphism

$$L = (\star, \mathrm{id}_{\mathbf{B}}, \mathrm{id}) \quad : \quad (\mathbf{B}, \mathbf{B}, \mathrm{id}_{\mathbf{B}} : \mathbf{B} \to \mathbf{B}) \longrightarrow (\mathbf{E}, \mathbf{B}, p : \mathbf{E} \to \mathbf{B})$$

in the 2-category $\mathcal{K}/\mathcal{K}$. Consider moreover a morphism of the form

$$R = ([-], \mathrm{id}_{\mathbf{B}}, \iota) \quad : \quad (\mathbf{E}, \mathbf{B}, p : \mathbf{E} \to \mathbf{B}) \longrightarrow (\mathbf{B}, \mathbf{B}, \mathrm{id}_{\mathbf{B}} : \mathbf{B} \to \mathbf{B})$$

in the 2-category $\mathcal{K}//\mathcal{K}$. The morphisms L and R of the 2-category $\mathcal{K}//\mathcal{K}$ can be depicted as follows in the underlying 2-category $\mathcal{K}$:

Now, suppose that $\star$ and $[-]$ define a comprehension structure with section, in the sense of Def 16. By definition, this means that there is an adjunction $\star \dashv [-]$. By Prop. 12, the adjunction $\star \dashv [-]$ lifts to an adjunction $L \dashv R$ between the morphisms L and R in the 2-category $\mathcal{K}//\mathcal{K}$ precisely when the 2-cell $\iota : [-] \Rightarrow p$ in the 2-category $\mathcal{K}$ is the mate defined as $\iota = p \circ \varepsilon$, of the identity 2-cell $\mathrm{id} : \mathrm{id}_{\mathbf{B}} \Rightarrow p \circ \star$. Here, the 2-cell $\varepsilon : [-] \circ \star \Rightarrow \mathrm{id}$ denotes the counit of the adjunction $\star \dashv [-]$ in the 2-category $\mathcal{K}$. This establishes one direction of the proof, while the other direction is immediate. ◀

Note that the definition of the 2-cell $\iota : [-] \Rightarrow p$ as the mate $\iota : [-] \Rightarrow p$ of the identity 2-cell $\mathrm{id} : \mathrm{id}_{\mathbf{B}} \Rightarrow p \circ \star$ provides a conceptual explanation for the definition of $\iota : [-] \Rightarrow p$ as the image by p of the counit ε of the counit of the adjunction $\star \dashv [-]$.

5   Path objects in a 2-category

We introduce the notion of *path object* on an object $\mathbf{B}$ in a 2-category $\mathcal{K}$. This construction, which generalizes the usual construction of the arrow category $\mathcal{B}^{\rightarrow}$ of a category $\mathcal{B}$ when $\mathcal{K} = \mathbf{Cat}$, will play an important role in §8 when we apply our constructions to inductive and coinductive reasoning on initial algebras and terminal algebras. Every object $\mathbf{B}$ in a 2-category $\mathcal{K}$ induces a 2-functor

$$\mathcal{K}(-, \mathbf{B})^{\rightarrow} \; : \; \mathcal{K} \longrightarrow \mathbf{Cat} \tag{15}$$

which transports every object $X \in \mathcal{K}$ to the arrow category $\mathcal{K}(\mathbf{X}, \mathbf{B})^{\rightarrow}$ of the hom-category $\mathcal{K}(\mathbf{X}, \mathbf{B})$ between $\mathbf{X}$ and $\mathbf{B}$.

▶ **Definition 18.** *A path object for the object* $\mathbf{B}$ *in the 2-category* $\mathcal{K}$ *is a pair* $(\mathbf{B}^{\rightarrow}, \beta)$ *consisting of an object* $\mathbf{B}^{\rightarrow}$ *and of a family of isomorphisms*

$$\beta_{\mathbf{X}} \; : \; \mathcal{K}(\mathbf{X}, \mathbf{B}^{\rightarrow}) \cong \mathcal{K}(\mathbf{X}, \mathbf{B})^{\rightarrow}$$

2-natural in the object X. *Terminology: one says in that case that the pair* $(\mathbf{B}^{\rightarrow}, \beta)$ *defines a representation of the 2-functor (15).*

Note that every path object $(\mathbf{B}^{\rightarrow}, \beta)$ comes equipped with three morphisms and a 2-cell

$$\mathbf{cod}, \mathbf{dom} : \mathbf{B}^{\rightarrow} \to \mathbf{B} \qquad \mathbf{id} : \mathbf{B} \to \mathbf{B}^{\rightarrow} \qquad \mathbf{hom} : \mathbf{cod} \Rightarrow \mathbf{dom} : \mathbf{B}^{\rightarrow} \to \mathbf{B}$$

satisfying $\mathbf{dom} \circ \mathbf{id} = \mathrm{id}_{\mathbf{B}} = \mathbf{cod} \circ \mathbf{id}$ and that the 2-cell $\mathbf{hom} \circ \mathbf{id}$ coincides with the identity 2-cell on $\mathrm{id}_{\mathbf{B}}$. A remarkable property is that

▶ **Proposition 19.** *The three morphisms* $\mathbf{cod}$, $\mathbf{id}$ *and* $\mathbf{dom}$ *are related by a pair of formal adjunctions* $\mathbf{cod} \dashv \mathbf{id}$ *and* $\mathbf{id} \dashv \mathbf{dom}$ *in the 2-category* $\mathcal{K}$.

Going back to the definition (Def. 16) of a comprehension structure with section, this establishes that

▶ **Proposition 20.** *Every path object* $(\mathbf{B}^{\rightarrow}, \beta)$ *defines a comprehension structure with section* $\mathbf{id} : \mathbf{B} \to \mathbf{B}^{\rightarrow}$ *on the morphism* $\mathbf{cod} : \mathbf{B}^{\rightarrow} \to \mathbf{B}$ *in the 2-category* $\mathcal{K}$.

Note that the comprehension structure $([-], \iota)$ constructed in Prop. 17 using a 2-categorical mate in $\mathcal{K}$ is provided in that case by the pair $(\mathbf{dom}, \mathbf{hom})$ with 2-cell $\mathbf{hom} : \mathbf{dom} \Rightarrow \mathbf{cod}$.

6   Functors with image structure

In this section, we introduce the notion of *functor* $p : \mathcal{E} \to \mathcal{B}$ *with image structure* which weakens (and thus generalizes) the usual notion of Grothendieck opfibration $p : \mathcal{E} \to \mathcal{B}$.

▶ **Definition 21** (Image structure)**.** *An image structure on a functor* $p : \mathcal{E} \to \mathcal{B}$ *is a pair* $(\star, \lambda)$ *consisting of a section* $\star$ *defined as a functor* $\star : \mathcal{B} \to \mathcal{E}$ *satisfying the equation* $p \circ \star = \mathrm{id}_{\mathcal{B}}$, *together with a family* λ *of opcartesian morphisms*

$$u \; : \; A \longrightarrow B \qquad \vDash \qquad \lambda_u \; : \; \star_A \longrightarrow \exists_u[\star_A] \tag{16}$$

indexed by the morphisms $u : A \to B$ *of the basis category* $\mathcal{B}$. *We will also suppose for convenience that the opcartesian morphism*

$$\mathrm{id}_A \; : \; A \longrightarrow A \qquad \vDash \qquad \lambda_u \; : \; \star_A \longrightarrow \exists_{\mathrm{id}_A}[\star_A]$$

coincides with the identity morphism on $\star_A$.

Here, we follow the fibered philosophy of refinement type systems [11], and write

$$u \;:\; A \longrightarrow B \qquad \vDash \qquad f \;:\; R \longrightarrow S$$

when a morphism $f : R \to S$ in the category $\mathcal{E}$ has image $p(f) = u : A \to B$ in the category $\mathcal{B}$. The intuition is that the morphism $f : R \to S$ is "above" the morphism $u : A \to B$, and "dependent" of it. Accordingly, we write $\mathcal{E}_{u:A\to B}(R,S)$ for the set of such morphisms $f : R \to S$ such that $p(f) = u$. Let us recall what universal property is required of the morphism (16) in order to make it opcartesian. By precomposition in the category $\mathcal{E}$, every morphism

$$v \;:\; B \longrightarrow B' \qquad \vDash \qquad h \;:\; \exists_u[\star_A] \longrightarrow S'$$

induces a morphism

$$v \circ u \;:\; A \longrightarrow B' \qquad \vDash \qquad h \circ \lambda_u \;:\; \star_A \longrightarrow S'$$

The fact that the morphism (16) is opcartesian simply means that the operation is reversible, and thus induces a bijection

$$\mathcal{E}_{v\circ u:A\to C}(\star_A, S') \quad \cong \quad \mathcal{E}_{v:B\to C}(\exists_u[\star_A], S')$$

for every morphism $v : B \to B'$ and every object S' in the fiber of B'.

▶ **Proposition 22.** *For every morphism $u : A \to B$ of the category $\mathcal{B}$, every functor $p : \mathcal{E} \to \mathcal{B}$ with an image structure $(\star, \lambda)$ comes equipped with a family of morphisms*

$$v \;:\; B \longrightarrow B' \qquad \vDash \qquad v \triangleright \;:\; \exists_u[\star_A] \longrightarrow \exists_{v\circ u}[\star_A] \tag{17}$$

indexed by the morphisms $v : B \to B'$ of the category $\mathcal{B}$, and a family of morphisms

$$\mathrm{id}_B \;:\; B \longrightarrow B \qquad \vDash \qquad \triangleleft w \;:\; \exists_{u\circ w}[\star_{A'}] \longrightarrow \exists_u[\star_A] \tag{18}$$

indexed by the morphisms $w : A' \to A$ of the category $\mathcal{B}$. These morphisms make a series of diagrams commute. First of all, the three coherence diagrams below commute

$$\tag{19}$$

for every path $A' \xrightarrow{w} A \xrightarrow{u} B \xrightarrow{v} B'$ in the category $\mathcal{B}$. Then, the functorial nature of (17) and (18) is ensured by the fact that the diagrams below commute

$$\tag{20}$$

for every pair of paths $A \xrightarrow{u} B \xrightarrow{v} B' \xrightarrow{v'} B''$ and $A'' \xrightarrow{w'} A' \xrightarrow{w} A \xrightarrow{u} B$ of the category $\mathcal{B}$, and moreover, that the morphisms

$$\mathrm{id}_B \;:\; B \longrightarrow B \qquad \vDash \qquad (id_B)\triangleright \;:\; \exists_u[\star_A] \longrightarrow \exists_u[\star_A]$$
$$\mathrm{id}_B \;:\; B \longrightarrow B \qquad \vDash \qquad \triangleleft(id_A) \;:\; \exists_u[\star_A] \longrightarrow \exists_u[\star_A]$$
$$\tag{21}$$

coincide with the identity, for every morphism $u : A \to B$ of the basis category $\mathcal{B}$.

Proof. The two morphisms (17) and (18) are defined by the universal property of the family of opcartesian morphisms λ defining the image structure, as the unique morphisms $v\triangleright$ and $\triangleleft w$ making the two diagrams commute in (19-ab). The three coherence properties (19-c) (20) and (21) follow easily from the definition of the two morphisms (17) and (18). ◀

We deduce from the statement (Prop. 22) just established that

▶ **Corollary 23.** *Every functor* $p : \mathcal{E} \to \mathcal{B}$ *with an image structure comes with a functor*

$$\mathbf{image} \ : \ \mathcal{B}^{\to} \longrightarrow \mathcal{E}$$

*called the **image functor** associated to the image structure.*

Proof. The image functor transports every object $u : A \to B$ of the arrow category $\mathcal{B}^{\to}$ to the object $\exists_u[\star_A]$ defined by the image structure, and every morphism

$$(v, w) \quad : \quad (A, B, u : A \to B) \longrightarrow (A', B', u' : A' \to B')$$

to the composite morphism below in the category $\mathcal{E}$

$$v \ : \ B \longrightarrow B' \quad \vDash \quad \exists_u[\star_A] \xrightarrow{v\triangleright} \exists_{v\circ u}[\star_A] \xrightarrow{\mathrm{id}} \exists_{u'\circ w}[\star_A] \xrightarrow{\triangleleft w} \exists_{u'}[\star_{A'}]$$

The functoriality of **image** follows from the coherence properties (19-c) (20) and (21) established in Prop. 22. ◀

The resulting image functor **image** $: \mathcal{B}^{\to} \to \mathcal{E}$ extends the section $\star : \mathcal{B} \to \mathcal{E}$ in the slice category $\mathcal{K}/B$, in the expected sense that the diagram below commutes:

$$(22)$$

In particular, as explained in the introduction, the diagram (9) commutes by definition of the image functor. Note that every Grothendieck opfibration $p : \mathcal{E} \to \mathcal{B}$ with a section $\star : \mathcal{B} \to \mathcal{E}$ comes equipped with an image structure, which is canonical when the opfibration is cloven.

7 Comprehension structures with image

In order to work in full generality, and to include the case of the 2-categories of algebras and coalgebras treated in §8, we find convenient to generalize our definition Def. 21 of image structure for a functor $p : \mathcal{E} \to \mathcal{B}$ in the specific case $\mathcal{K} = \mathbf{Cat}$ to any morphism $p : \mathbf{E} \to \mathbf{B}$ in a 2-category $\mathcal{K}$.

▶ **Definition 24** (Image structure). *An image structure on a morphism* $p : \mathbf{E} \to \mathbf{B}$ *in a 2-category* $\mathcal{K}$ *is a section* $\star : \mathbf{B} \to \mathbf{E}$ *equipped with a family* λ *of 2-cells*

$$\lambda_u \quad : \quad \star_a \Longrightarrow \exists_u[\star_a] \quad : \quad X \longrightarrow \mathbf{E} \tag{23}$$

326 *indexed by the objects X, the morphisms $a, b : X \to B$ and the 2-cells $u : a \Rightarrow b$ of the*
327 *2-category $\mathcal{K}$, where the morphism $\star_a$ is defined as the composite $\star \circ a : X \to \mathbf{E}$. One*
328 *requires moreover that each 2-cell λ_u defines an opcartesian morphism*

329
$$u \ : \ a \Longrightarrow b \quad \vDash \quad \lambda_u \ : \ \star_a \Longrightarrow \exists_u [\star_a] \tag{24}$$

with respect to the postcomposition functor

$$\mathcal{K}(\mathbf{X}, p) \quad : \quad \mathcal{K}(\mathbf{X}, \mathbf{E}) \longrightarrow \mathcal{K}(\mathbf{X}, \mathbf{B})$$

330 *above the morphism $u : a \Rightarrow b$ in the category $\mathcal{K}(\mathbf{X}, \mathbf{B})$. We also ask for convenience that*
331 *$\lambda_u : \star_a \Rightarrow \exists_u [\star_a]$ coincides with the identity 2-cell when $u : a \Rightarrow a$ is the identity 2-cell.*

332 Note that every morphism $p : \mathbf{E} \to \mathbf{B}$ with an image structure $(\star, \lambda)$ to an object $\mathbf{B}$
333 equipped with a path-object $(\mathbf{B}^{\to}, \beta)$ in the 2-category $\mathcal{K}$ comes equipped with a morphism
334 $\mathbf{image} : \mathbf{B}^{\to} \to \mathbf{E}$ defined as $\mathbf{image} = \exists_{\mathbf{hom}}[\star_{\mathbf{dom}}]$, and thus satisfying the equality:

335

$$\tag{25}$$

336 Moreover, the resulting image morphism makes the counterpart of diagram (22) commute in
337 the 2-category $\mathcal{K}$, for the same reason as in the specific case of the 2-category $\mathcal{K} = \mathbf{Cat}$. We
338 are now in the position of defining the notion of *comprehension structure with image* at that
339 2-categorical level of generality.

340 ▶ **Definition 25.** *Suppose given an object $\mathbf{B}$ equipped with a path-object $(\mathbf{B}^{\to}, \beta)$. A compre-*
341 *hension structure with image on a morphism $p : \mathbf{E} \to \mathbf{B}$ is a morphism $\mathbf{image} : B^{\to} \to E$*
342 *together with a right adjoint $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\to}$ in the slice category $\mathcal{K}/B$.*

343 A comprehension structure with image on $p : \mathbf{E} \to \mathbf{B}$ in the sense of Def. 25 comes equipped
344 with a pair of adjunctions $\mathbf{id} : \mathbf{B} \leftrightarrows \mathbf{B}^{\to} : \mathbf{dom}$ and $\mathbf{image} : \mathbf{B}^{\to} \leftrightarrows \mathbf{E} : \mathcal{P}$. From that, one
345 easily deduces that

346 ▶ **Proposition 26.** *Every comprehension structure with image (Def. 25) induces a compre-*
347 *hension structure with section defined as $\star = \mathbf{image} \circ \mathbf{id} : \mathbf{B} \to \mathbf{E}$ (Def. 16), where the right*
348 *adjoint functor $[-] : \mathbf{E} \to \mathbf{B}$ is defined as the composite $[-] = \mathbf{dom} \circ \mathcal{P}$.*

349 We establish now the converse property which extends to every 2-category $\mathcal{K}$ the property
350 stated in the introduction (Prop. 8) in the specific case of $\mathcal{K} = \mathbf{Cat}$. The statement
351 extends [3], lemma 2.2.10 by relaxing the assumption that $p : \mathbf{E} \to \mathbf{B}$ is a bifibration.

352 ▶ **Proposition 27.** *Suppose that $p : \mathbf{E} \to \mathbf{B}$ has an image structure in the 2-category $\mathcal{K}$ (in*
353 *the sense of Def. 24). In that case, every comprehension structure with section (in the sense*
354 *of Def. 16) defines a comprehension structure with image (in the sense of Def. 25).*

Proof. Suppose that $p : \mathbf{E} \to \mathbf{B}$ has an image structure and at the same time a comprehension
structure with section $\star : \mathbf{E} \to \mathbf{B}$ in the 2-category $\mathcal{K}$. The 2-cell $\iota : [-] \to p : \mathbf{E} \to \mathbf{B}$
mentioned in Prop. 12 defines a morphism in the 2-category $\mathcal{K}(\mathbf{E}, \mathbf{B})$, and thus an object

in the 2-category $\mathcal{K}(\mathbf{E}, \mathbf{B})^{\rightarrow}$. By definition of the path object $(\mathbf{B}^{\rightarrow}, \beta)$ in Def. 18, the 2-cell $\iota : [-] \to p : \mathbf{E} \to \mathbf{B}$ induces an object of the 2-category $\mathcal{K}(\mathbf{E}, \mathbf{B}^{\rightarrow})$, and thus a morphism noted $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\rightarrow}$ and characterized by the equation

$$\mathbf{hom} \circ \mathcal{P} \;\; = \;\; \iota \;\; : \;\; [-] \Longrightarrow p \;\; : \;\; \mathbf{E} \longrightarrow \mathbf{B}$$

We want to show that this morphism $\mathcal{P} : \mathbf{E} \to \mathbf{B}^{\rightarrow}$ is right adjoint to the morphism $\mathbf{image} : \mathbf{B}^{\rightarrow} \to \mathbf{E}$ in the 2-category $\mathcal{K}$. To that purpose, we consider an object $\mathbf{X}$ of the 2-category $\mathcal{K}$ and a pair of morphisms $u : \mathbf{X} \to \mathbf{B}^{\rightarrow}$ and $S : \mathbf{X} \to \mathbf{E}$, and we exhibit a one-to-one relationship (see [10], Section 5.11) between the 2-cells φ and ψ of the form:

(26)

The key observation is that a 2-cell ψ of that form is the same thing as a 2-cell (ψ_1, ψ_2) in the 2-category $\mathcal{K}//\mathcal{K}$ between the composite morphisms:

It follows from the existence of the adjunction in $\mathcal{K}//\mathcal{K}$ established in Prop. 17 that there is a one-to-one relationship between the pairs of 2-cells (ψ_1, ψ_2) in the 2-category $\mathcal{K}$ of the form above, and the pairs (φ_1, φ_2) of 2-cells in the 2-category $\mathcal{K}$ defining a 2-cell (φ_1, φ_2) in the 2-category $\mathcal{K}//\mathcal{K}$ between the composite morphisms:

The definition of the morphism $\mathbf{image} : \mathbf{B}^{\rightarrow} \to \mathcal{E}$ and the cartesianity of the 2-cell $\lambda_{\mathbf{hom}}$ in (25) with respect to the functor $\mathcal{K}(\mathbf{X}, p) : \mathcal{K}(\mathbf{X}, \mathbf{E}) \to \mathcal{K}(\mathbf{X}, \mathbf{B})$ implies that there is a one-to-one relationship between the pairs of 2-cells (φ_1, φ_2) in the 2-category $\mathcal{K}//\mathcal{K}$ above, and the 2-cells φ of the form (26) in the 2-category $\mathcal{K}$. The end of the proof is easy.   ◀

8   Illustration: inductive reasoning on functor algebras and dually, coinductive reasoning on functor coalgebras

Suppose given a functor $p : \mathcal{E} \to \mathcal{B}$ between two categories $\mathcal{B}$ and $\mathcal{E}$ equipped with endofunctors $F : \mathcal{B} \to \mathcal{B}$ and $G : \mathcal{E} \to \mathcal{E}$ and a distributivity law of the form (10). A well-known result by Beck [1] states that the distributivity law $\delta : F \circ p \Rightarrow p \circ G$ describes one specific lifting of the functor $p : \mathcal{E} \to \mathcal{B}$ to a functor $p' : \mathbf{Alg}_G(\mathcal{E}) \to \mathbf{Alg}_F(\mathcal{B})$ between the underlying categories of algebras, in such a way that the diagram below commutes:

where U denotes in both cases the forgetful functor. One main reason for working at a 2-categorical level as we do in the present paper is to provide us with a simple and elegant recipe to characterize in just the same spirit inherited from Beck [1] when a comprehension structure with section $\star : \mathcal{B} \to \mathcal{E}$ on the functor $p : \mathcal{E} \to \mathcal{B}$ lifts to a comprehension structure with section $\star : \mathbf{Alg}_F(\mathcal{B}) \to \mathbf{Alg}_G(\mathcal{E})$ on the functor $p : \mathbf{Alg}_G(\mathcal{E}) \to \mathbf{Alg}_F(\mathcal{B})$. To that purpose, we consider the 2-category $\mathbf{Endo(Cat)}$ with objects the categories equipped with endofunctors, and with morphisms the functors equipped with a distributivity law à la Beck. Note that $\mathbf{Endo(Cat)}$ may be defined as the full sub-2-category of $\mathbf{Cat//Cat}$ whose objects are of the form $(\mathcal{C}, \mathcal{C}, G : \mathcal{C} \to \mathcal{C})$. This leads us the question of characterizing when a comprehension structure with section $\star : \mathcal{B} \to \mathcal{E}$ on the functor $p : \mathcal{E} \to \mathcal{B}$ lifts to a *comprehension structure with section* in the 2-category $\mathbf{Endo(Cat)}$, where this definition should be understood in the 2-categorical sense of §4, Def. 16. We establish that

▶ **Proposition 28.** *Suppose given a comprehension structure with section* $([-], \star)$ *in* $\mathbf{Cat}$ *on a functor* $p : \mathcal{E} \to \mathcal{B}$ *between categories* $\mathcal{B}$ *and* $\mathcal{E}$ *equipped with endofunctors* $S : \mathcal{B} \to \mathcal{B}$ *and* $T : \mathcal{E} \to \mathcal{E}$. *There is a one-to-one correspondence between the liftings to* $\mathbf{Endo(Cat)}$ *of the comprehension structure with section* $([-], \star)$ *and the pairs of distributivity laws*

$$\delta \ : \ F \circ p \Longrightarrow p \circ G \ : \ \mathcal{E} \longrightarrow \mathcal{B} \qquad\qquad \sigma \ : \ G \circ \star \Longrightarrow \star \circ F \ : \ \mathcal{B} \longrightarrow \mathcal{E}$$

such that (1) the composite natural transformation

$$F \xrightarrow{\ equal\ } F \circ p \circ \star \xrightarrow{\ \delta \circ \star\ } p \circ G \circ \star \xrightarrow{\ p \circ \sigma\ } p \circ \star \circ F \xrightarrow{\ equal\ } F$$

is the identity and (2) the natural transformation σ is reversible.

It is worth mentioning that, in this situation, the comprehension functor $[-] : \mathcal{E} \to \mathcal{B}$ lifts as the pair $([-], \tilde{\sigma}) : (\mathcal{E}, G) \to (\mathcal{B}, F)$ where the distributivity law $\tilde{\sigma} : F \circ [-] \Rightarrow [-] \circ G$ is defined as the mate of the inverse $\sigma^{-1} : \star \circ F \Rightarrow G \circ \star$. We obtain that in this situation

▶ **Corollary 29.** *The comprehension structure with section* $\star : \mathcal{B} \to \mathcal{E}$ *lifts to a comprehension structure with section* $(\star, \sigma) : (\mathcal{B}, F) \to (\mathcal{E}, G)$ *which is left adjoint to comprehension* $([-], \tilde{\sigma}) : (\mathcal{E}, G) \to (\mathcal{B}, F)$ *and thus transports the initial F-algebra μF to the initial G-algebra $\mu G = \star_{\mu F}$.*

The result extends the main soundness theorem established by Fumex, Ghani and Johann in [4], Thm 2.10. It establishes in their terminology (see [3], Def. 4.3.1) that G defines an induction scheme for μF in p. The approach translates immediately by duality to the case of quotient structures, and provides in just the same way necessary and sufficient conditions to be in the situation of Corollary 10 and to characterize the terminal F-coalgebra as $\nu F = \star_{\nu F}$.

9   Conclusion

Our main purpose and achievement in this paper is to exhibit the 2-categorical structures secretly at work in the 1-categorical approach to comprehension structures traditionally found in categorical logic. Our work was motivated by the fibered approach to induction on algebras and coinduction on coalgebras recently developed by Fumex, Ghani and Johann [4, 3]. We understand our 2-categorical approach and statements (Cor. 9, 10 and 29) as providing the clean conceptual foundations underlying their soundness theorems. For lack of space, we did not treat here the proof-theoretical aspects of our 2-categorical description of comprehension structures. A natural direction would be to start from the recent multicategorical approach to induction [7] developed in the fibered style of Melliès and Zeilberger's refinement systems [11]. We leave that for future work.

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A   Four alternative notions of comprehension structures

For the sake of completeness, we give the list below of four well-recognized notions of comprehension structures appearing in the literature.

Jacobs comprehension categories.

The notion of comprehension category was introduced by Jacobs ([6], p. 181, and [5], chapter 10.4, page 613). A comprehension category is a Grothendieck fibration $p : \mathcal{E} \to \mathcal{B}$ equipped with a functor $comp : \mathcal{E} \to \mathcal{B}^{\to}$ such that the composite functor $cod \circ \cdot comp : \mathcal{E} \to \mathcal{B}$ coincides with the functor $p : \mathcal{E} \to \mathcal{B}$, where $cod : \mathbf{B}^{\to} \to \mathbf{B}$ denotes the codomain functor. The functor $comp$ is also required to transport every p-cartesian morphism of $\mathcal{E}$ to a cod-cartesian morphism of $\mathcal{B}^{\to}$, which may be equivalently defined as a *pullback square* in the category $\mathcal{B}$.

Ehrhard D-categories.

The notion was introduced by Ehrhard [2] under the name of D-category. Ehrhard's D-categories are also called *comprehension category with units* by Jacobs [6] def. 4.12, and *Ehrhard comprehension category* by Moss [12], p 22. A D-category is defined as a Grothendieck fibration $p : \mathcal{E} \to \mathcal{B}$ equipped with a terminal object functor $s : \mathcal{B} \to \mathcal{E}$ which has a right adjoint noted $comp : \mathcal{E} \to \mathcal{B}$. Here, by terminal object functor $s : \mathcal{B} \to \mathcal{E}$, one means a section of the functor $p : \mathcal{E} \to \mathcal{B}$ which transports every object A of the basis category $\mathcal{B}$ to a terminal object of the fiber $\mathcal{E}_A$ of the object A with respect to the functor $p : \mathcal{E} \to \mathcal{B}$. Note in particular that the terminal object functor $s : \mathcal{B} \to \mathcal{E}$ is fully faithful and right adjoint to the functor $p : \mathcal{E} \to \mathcal{B}$.

Fumex tC-opfibrations.

The notion of tC-opfibration was introduced by Fumex in his PhD thesis, ([3] p. 38, def. 2.2.2.) A tC-category is a Grothendieck opfibration $p : \mathcal{E} \to \mathcal{B}$ with a fully faithful section $\star : \mathcal{B} \to \mathcal{E}$. The section $\star : \mathcal{B} \to \mathcal{E}$ is moreover required to have a right adjoint noted $comp : \mathcal{E} \to \mathcal{B}$. Note that, given an object A of the basis category $\mathcal{B}$, one does not require that the object s_A is terminal in the fiber $\mathcal{E}_A$ of the object A. This is one main difference with Ehrhard's notion of D-category.

Lawvere categories.

The notion of Lawvere category was introduced by Jacobs in [6], p 190, as a way to reflect the work by Lawvere [14] on hyperdoctrines in categorical logic. A Lawvere category is defined as a Grothendieck bifibration $p : \mathbf{E} \to \mathbf{B}$ with a terminal object in each fiber, defining a functor $\star : \mathbf{B} \to \mathbf{E}$, and such that the (ordinary) functor

$$f \mapsto \Sigma_f \star_{(dom f)} : \mathbf{B}^{\to} \to \mathbf{E}$$

induced by the left fibration structure has a right adjoint $comp : \mathbf{E} \to \mathbf{B}^{\to}$, verifying $cod \circ comp = p$, and such that the unit and counit are vertical (their image by cod and p is the identity). Note that every Lawvere category is a tC-opfibration in the sense of Fumex [3].