

Frequency Domain Granger Causality Test

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This document explains how to perform a frequency domain Granger causality test using an EViews add-in based on [1]. This document is divided in three sections, the first explains the details of the test, the second explains how to perform the test in presence of cointegrated variables and finally some examples using the add-in are showed.

1 Causality test in the frequency domain

The test is developed in a context of VAR models, it will be explained in a bivariate VAR model, nevertheless it can be extended to more variables. Let $z_t = [x_t, y_t]'$ be the vector of endogenous variables observed at time $t = 1, \dots, T$, the vector has a representation of the form:

$$\Theta(L)z_t = e_t$$

Where $\Theta(L) = I - \Theta_1L - \dots - \Theta_pL^p$ having L as the lag operator such as $Lz_t = z_{t-1}$. The error vector is assumed to be white noise with mean zero and covariance matrix $E(e_t e_t') = \Sigma$ defined positive. If the system is stable then the *VAR* will have an *VMA*(∞) with the following structure.

$$z_t = \Phi(L)e_t$$

Where $\Phi(L) = I + \Phi_1L + \Phi_2L^2 + \dots$ and $\Phi(L) = \Theta(L)^{-1}$. Let G be the inferior triangular matrix of the Cholesky decomposition such as $GG' = \Sigma$. Therefore, the representation with orthogonal errors can be expressed as:

$$z_t = GG^{-1}e_t + \Phi_1GG^{-1}e_{t-1} + \dots$$
$$z_t = \Psi(L)\eta_t = \begin{bmatrix} \Psi_{11}(L) & \Psi_{12}(L) \\ \Psi_{21}(L) & \Psi_{22}(L) \end{bmatrix} \begin{bmatrix} \eta_{1t} \\ \eta_{2t} \end{bmatrix}$$

Where $\Psi(L) = \Phi(L)G$, $\eta_t = G^{-1}e_t$ y $\Phi_0 = I$. Therefore, $E(\eta_t \eta_t') = I$. According to [4] the spectral density of x_t is given by:

$$f_x(w) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_k e^{-ikw}$$

Where w is the angular frequency, e^{-ikw} is the Euler equation and γ_k is the autocovariance of order k of x_t . Let $\gamma(L) = \sum_{k=-\infty}^{\infty} \gamma_k L^k$ be the autocovariance generating function (A.G.F), then the spectrum of x_t will be:

$$f_x(w) = \frac{1}{2\pi} \gamma(e^{-iw})$$

On the other hand, if x_t is defined in terms of their moving average representation, then:

$$x_t = \Pi(L)u_t$$

Where $\Pi(L) = 1 + \pi_1L + \dots$. It can be showed that the A.G.F of x_t will be given by $\gamma(L) = \sigma^2 \Pi(L) \Pi(L^{-1})$, therefore the spectrum of x_t will be equal to:

$$f_x(w) = \frac{\sigma^2}{2\pi} |\Pi(e^{-iw})|$$

The above expression is known as the *rational spectrum* or as the spectrum of an *ARMA*(p, q) model. In an *ARMA*(p, q) process the polynomial $\Pi(L)$ is equal to $\frac{B_q(L)}{A_p(L)}$, where $A_p(L)$ and $B_q(L)$ are the lag polynomials corresponding the *AR*(p) and *MA*(q) terms respectively. Therefore, given the *VMA*(∞) representation, the spectrum of x_t corresponds to:

$$f_x(w) = \frac{\sigma^2}{2\pi}(|\Psi_{11}(e^{-iw})|^2 + |\Psi_{12}(e^{-iw})|^2)$$

The hypothesis that y_t does not cause in the Granger sense x_t at frequency w can be proved with the following measure.

$$M_{y \rightarrow x}(w) = \log \left[1 + \frac{|\Psi_{12}(e^{-iw})|^2}{|\Psi_{11}(e^{-iw})|^2} \right]$$

The measure is zero if $|\Psi_{12}(e^{-iw})|^2 = 0$, in which case y_t not cause x_t in the frequency w . Given that $\Psi(L) = \Theta(L)^{-1}G$, the same lag polynomial will be equal to:

$$\Psi(L) = \frac{1}{|\Theta(L)|} \begin{bmatrix} \Theta_{22}(L) - \Theta_{12}(L)g_{21} & -\Theta_{12}(L)g_{22} \\ -\Theta_{21}(L)g_{11} + \Theta_{11}(L)g_{21} & \Theta_{11}(L)g_{22} \end{bmatrix}$$

Hence the element $\Psi_{12}(L) = \frac{-\Theta_{12}(L)g_{22}}{|\Theta(L)|}$. In this way y_t not cause x_t in the frequency w if the following condition is fulfilled.

$$\Theta_{12}(e^{-iw}) = \left| \sum_{k=1}^p \theta_{12,k} \cos(kw) - i \sum_{k=1}^p \theta_{12,k} \sin(kw) \right| = 0$$

The condition is fulfilled if the two sums are jointly equal to zero. The hypothesis that $M_{y \rightarrow x}(w) = 0$ is equivalent to prove $H_0 : R(w)\beta$ where:

$$\begin{aligned} \beta &= [\theta_{12,1}, \theta_{12,2}, \dots, \theta_{12,p}] \\ R(w) &= \begin{bmatrix} \cos(w) & \cos(2w) & \cdots & \cos(pw) \\ \sin(w) & \sin(2w) & \cdots & \sin(pw) \end{bmatrix} \end{aligned}$$

In the add-in the hypothesis is tested using the following statistical:

$$F[2, T - k] = \frac{(Rb - q)'(R[s^2(X'X)^{-1}R']^{-1}(Rb - q))}{2}$$

The null hypothesis correspond to $H_0 : Rb = q$. Where R is a matrix of restrictions of size $2xk$, k is the number of coefficients estimated per equation in the *VAR*, b is the vector of estimated coefficients of the respective equation, q is a $2x1$ zero vector, s^2 is the estimation of the variance error of the corresponding equation and X is a matrix $T \times k$ with the observations of the independent variables in the model. The statistical only applies to $w \in (0, \pi)$, the extremes of the interval are not included given that in these cases $|R| = 0$, therefore the inverse of the matrix can not be computed and the statistical can not be calculated. On the other hand, to calculate the test it is required that the number of restrictions be strictly lower than the number of coefficients estimated per equation and per variable in the *VAR*, that is $2 < p$, hence the test can only be performed in systems with lags greater than two, see [2].

2 Cointegrated variables

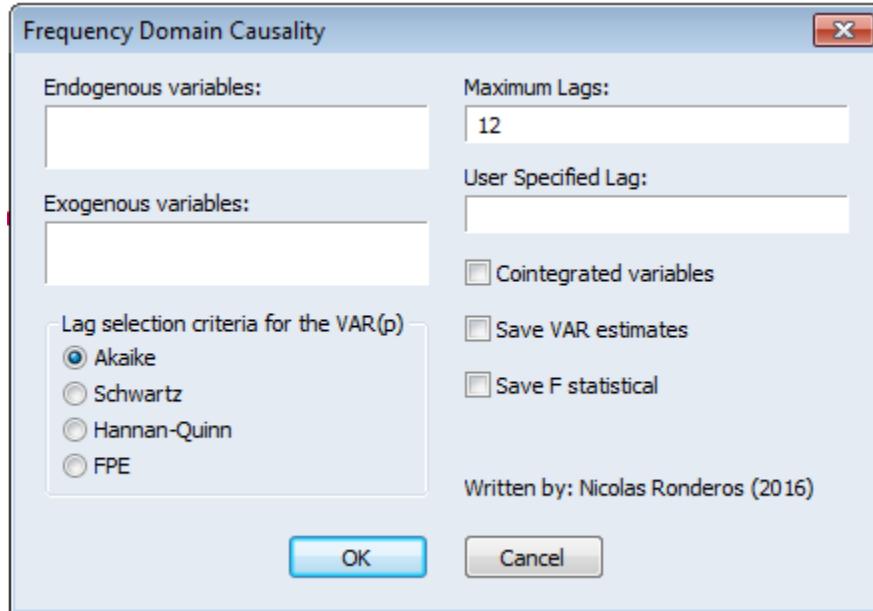
In presence of cointegrating relationships the causality test must be performed in the following model.

$$\Delta z_t = \tilde{\Theta}(L)z_{t-1} + e_t$$

Where $\tilde{\Theta}(L) = \Theta_1 - I + \Theta_2 L + \dots + \Theta_p L^p$. This procedure is valid when all the variables are integrated of the same order. In the case where the variable are integrated of different orders the F statistic does not longer have an standard distribution. Using the results of [3] the causality test can be performed in the levels of the series using the *VAR* augmented in one lag i.e. *VAR*($p + 1$), nevertheless the test is only performed on the coefficients corresponding to the p lags.

3 Examples using the add-in

The add-in could be used via *Global command*, that is just type the *speccaus* word in the command bar and the following window will appear.



The user just have to type the endogenous and exogenous variables of the *VAR*, select the information criteria to select the order of lags, the maximum order of lags to be considered, if the endogenous are cointegrated and if the user wants to save the estimates *VAR*. It is important to mention that if the variables are cointegrated the endogenous variables must be in levels, not in differences, the add-in performs the respective transformations to the data. The add-in could also be used via command line, with the causality word followed by the options between brackets. In table 1 it is showed the add-in options.

Speccaus(Options)

<i>Options</i>	<i>Command</i>
<i>Endogenous variables in the VAR</i>	<i>Variables=list of variables</i>
<i>Maximum order of lags to be included in the VAR</i>	<i>Maxlags= Integer (Default 12)</i>
<i>Exogenous variables in the VAR</i>	<i>Exogenous=list of variables</i>
<i>User specified lags to be included in the VAR</i>	<i>Slag=Integer</i>
<i>Akaike information criteria of lag selection</i>	<i>AIC (Default)</i>
<i>Schwarz information criteria for lag selection</i>	<i>SC</i>
<i>Hannan-Quinn information criteria for lag selection</i>	<i>HQ</i>
<i>Final prediction error criteria for lag selection</i>	<i>FPE</i>
<i>Allow for cointegrated variables</i>	<i>C</i>
<i>Save the estimates of the F statistical</i>	<i>Save the estimates of the F statistical</i>
<i>Keep the estimates of the VAR</i>	<i>E</i>

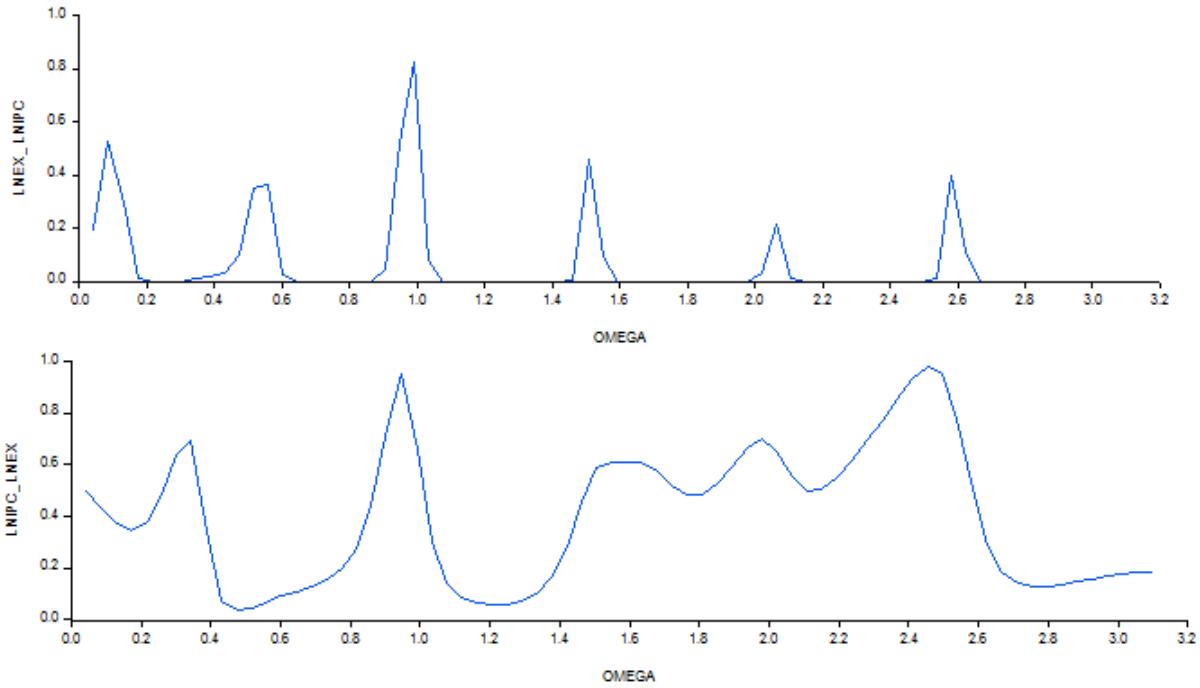
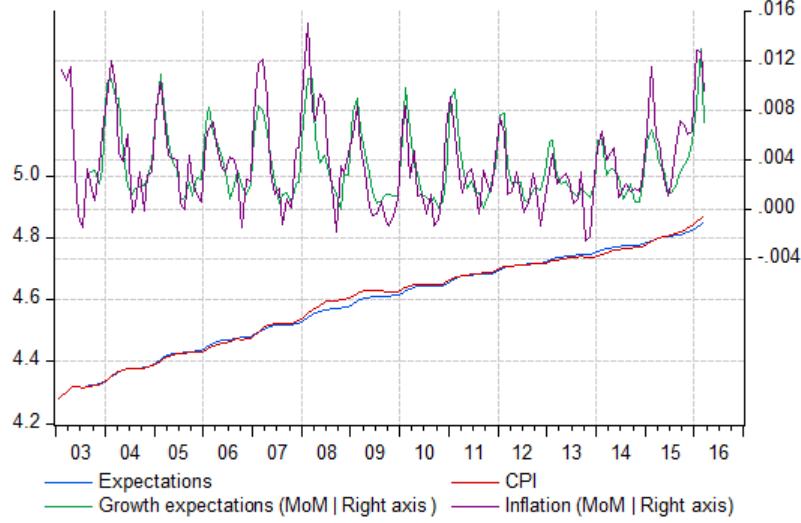
Table 1: Command line

The Hannan-Quinn and the Final Prediction Error lag selection criteria are not available with cointegrated variables. Two examples will be showed, with and without cointegrated variables.

3.1 Non-Cointegrated variables

The example with no cointegration calculates the causality test with data of the Consumer Price Index (CPI) and the expectations of the CPI from Colombia¹, which are integrated of order 1. In the following graph the logarithm of CPI, the logarithm of the index of expectations and their differences are showed.

¹The CPI is taken from DANE and the expectations of the CPI are taken from Banco de la Republica de Colombia



Given that the two variables are not cointegrated the test can be performed in a *VAR* with the differences of the logarithms of the variables or a *VAR* with the levels (logarithms) of the variables. The causality test with the *VAR* in differences was calculated with the two variables in differences as the endogenous variables, with 24 as the maximum order of lags, with the AIC criteria to select the best model and we keep the estimates of the *VAR*. The instructions can be executed typing on the command line `speccaus(variables=dlnipc dlnex,maxlags=24,e)`. On the other hand, the causality test with *VAR* with the variables in levels was calculated with the logarithms of the variables as the endogenous, to apply the methodology outlined in [3] the maxlag parameter were set in 14, given that the different information criteria select 14 as the optimal lag, and the 15 redundant augmentation lag of the two variables were introduced as an exogenous variables. The instructions can be executed typing

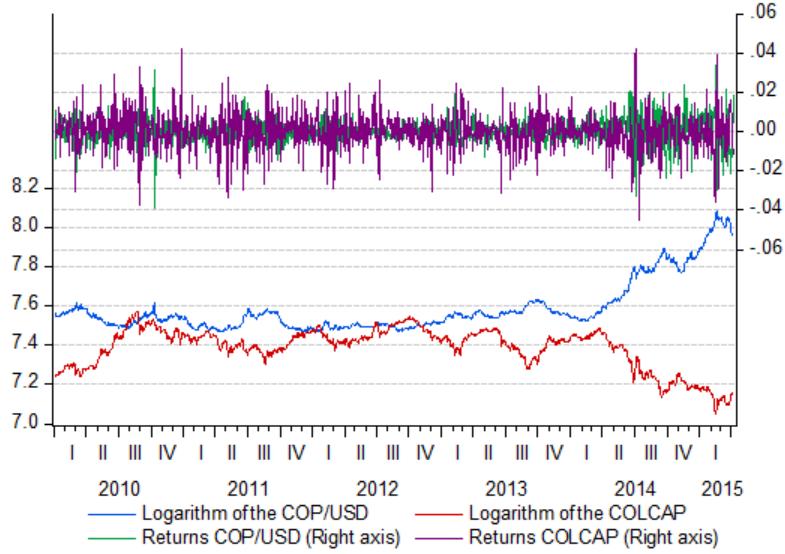
`speccaus(variables=lnipc lnex, exogenous=lnipc(-15) lnex(-15), maxlags=14, e)`. The results of the frequency domain causality test are quite similar if the levels or the differences of the variables are used, in the following graph is showed the results using the variables in levels.

The top graph corresponds to the hypothesis that the logarithm of the expectations does not cause the logarithm of the CPI at frequency w , it can be seen that the expectations cause the CPI at frequency 0.5, which corresponds to cycle with frequency of 12 months, this contrasts with the Granger causality test in the time domain which states that the expectations does not cause the CPI. On the other hand it can be seen that the CPI cause the expectations only at the non-seasonal frequency.

It is important to note that the labels of the graphs will be of the form $var_i_var_j$ were var_i is the dependent variable and var_j is the independent variable. Also, after the user calculates any causality test a table will appear in the workfile, which contains the results results of the causality test for each frequency.

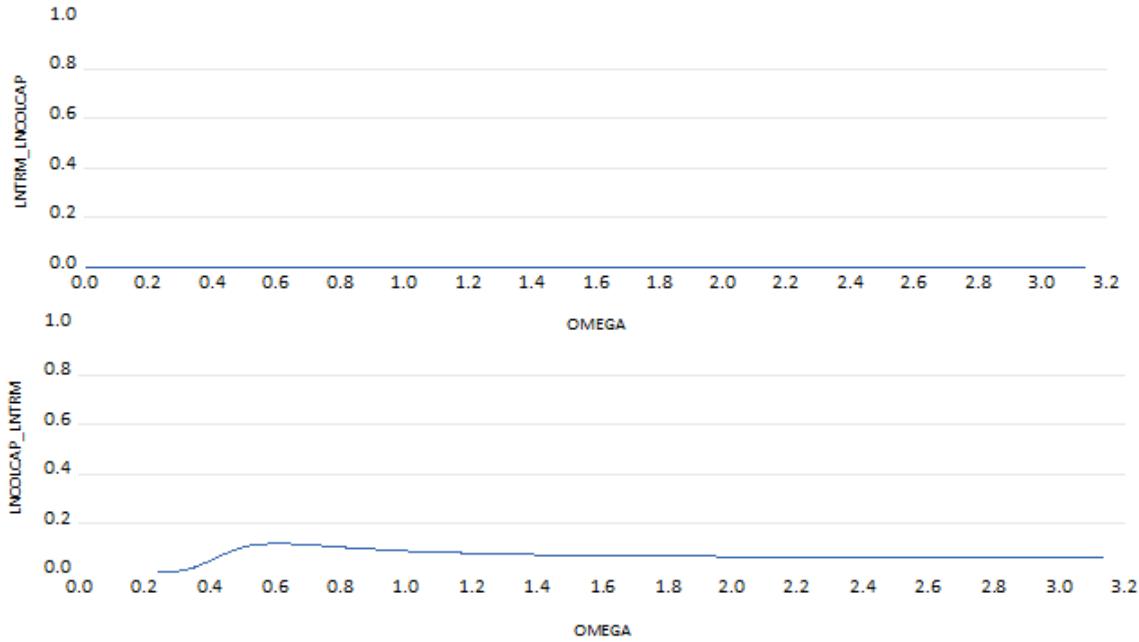
3.2 Cointegrated variables

The example with cointegrated varables involves the logarithm of the COP/USD floating exchange rate and the logarithm of the COLCAP stock index of Colombia². In the following graph both variables are showed, in the graph it can be seen that the existence of a long-run relationship it is highly likely. The Phillips-Outliaris, the trace and maximum eigenvalue cointegration tests were performed, the find evidence of one cointegration relationship.



The causality test in the frequency domain was calculated with the logarithms of the two variables as the endogenous, with 36 lags as the maximum lags to calculate the information criteria, the AIC criteria as the lag selection method, with the cointegrated variables option activated and with the option to keep the *VAR* in the workfile on. The instructions can be executed typing `speccaus(variables=lntrm lncolcap,maxlags=36,c,e)`. The results are showed in the following graph, it can be seen that the exchange rate cause the low frequency components of the COLCAP, this is consistent with the cointegration testing, given that cointegration implies Granger causality in zero frequency. Also it can be seen that the COLCAP index cause the exchange rate at all frequencies, that is the COLCAP helps to predict the long-run and the short-run components of the exchange rate.

²The exchange were rate taken from Banco de la Republica and the COLCAP index were taken from Bolsa de Valores de Colombia.



References

- [1] Jörg Breitung and Bertrand Candelon. Testing for short-and long-run causality: A frequency-domain approach. *Journal of Econometrics*, 132(2):363–378, 2006.
- [2] William H Greene. *Econometric analysis*. Pearson Education India, 2003.
- [3] Hiro Y Toda and Taku Yamamoto. Statistical inference in vector autoregressions with possibly integrated processes. *Journal of econometrics*, 66(1):225–250, 1995.
- [4] William Wu-Shyong Wei. *Time series analysis*. Addison-Wesley publ Reading, 1994.