Extensive Form Games

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Normal form games

- We have studied the simplest form of game so far
- However for several games, normal form is not enough
- Think about the game of chess, poker or any card game
- Sequences of actions are quite relevant
- Normal form of games is not suitable to capture the interactions in such games
- We will need to introduce the extensive form of games

Games over time

- We will need to identify
  - Players
  - Actions available for each player, each time she is called to play
  - An order of play
  - Outcomes

The extensive form

- A set of players N
- A history $h \in H$ is a sequence of actions performed by the players during the game
- The set of sequence or histories H such that
  - The empty sequence $\varnothing$ is in H
  - For $L < K$, if $(a^1, a^2, ...a^L, ..., a^K) \in H$ then $(a^1, a^2, ..., a^L) \in H$
- A history h is terminal if there is no a such that $(h, a) \in H$
- We denote the set of terminal histories by Z

The extensive form

- An order of moves or player function

$$P : H \backslash Z \to N$$

which determines which player $P(h) \in N$ is called to play after a non-terminal history h

- For h such that $P(h) = i$, we can define the set of available actions for player i at h as

$$A(h) = \{a : (h, a) \in H\}$$

- Payoffs are defined over terminal histories

$$u_i : Z \to \mathbb{R}$$

for each player i

The extensive form

- So, an extensive form game is completely defined by

$$\Gamma = \{N, H, P, (u_i)_{i \in N}\}$$

Finite games vs. finite horizon

- Two notions of finiteness in this context
- Finite horizon: if all possible histories in a game have finite length, i.e., if a game has a clear end
- Finite games: further requires both the set of players and the set of potential histories to be finite

Game trees

- Formally, (game) trees are a particular type of directed graphs
- A set of nodes X with a special node x_0 called the *root* or *origin* node
- A set of edges or precedence relation over nodes such that there is a unique path from x_0 to any other node x
  - These edges or precedence relation will determine whether x precedes x' or not
  - Nodes which doesn't precede any node are called terminal nodes

Game trees and extensive form

- We can alternatively define an extensive form game using a game tree
  - Nodes $\rightarrow$ Histories
  - $x_0 \rightarrow \varnothing$
  - Terminal nodes $\rightarrow$ Terminal histories
  - Edge linking x with $x' \iff$ there is an action a such that $h_{x'} = (h_x, a)$
  - And we can link the payoffs with terminal nodes instead of terminal histories

Extensive form BoS with perfect information

- Consider the BoS again, but suppose that your friend will arrive first to the restaurant
- There he ask for a phone and call you to let you know in which restaurant he is right now

Extensive form BoS with perfect information

Perfect information

- The previous definition could only handle games with perfect information
- Games in which each player knows exactly the point in the game in which he is whenever he is called to play
- This naturally excludes most card games!
- We will look at the more general case of games with imperfect information in the next class

Strategies

- Let's start with pure strategies
- Pure strategies in an extensive form games represent a complete plan of action for a player i
- That is, which action to perform at any possible history in which she is called to play
- Note this plan should include even histories which are never reached!

Extensive form BoS with perfect information

Mixed strategies

- Mixed strategies in an extensive form game will be probability distributions over pure strategies
- I.e., probability distributions over full plan of actions
- Usually, is easier to work with a different type of strategies which involve randomization called *behavioral strategies*

Behavioral strategies

- A behavioral strategy for player i is a collection of independent probability distributions over the actions available to him at each history she is called to play.

No loss in behavioral instead of mixed strategies

- There is no loss in using behavioral instead of mixed strategies in games of perfect information (and perfect recall)
- For each behavioral strategy we can find an equivalent mixed strategy and vice versa

Extensive form games in normal form

- We can represent extensive form games in normal form
- However, we will be losing relevant information

Entry deterrence

- Two firms, I and E
  - Firm I is an incumbent already in an industry or market
  - Firm E is an entrant deciding whether to enter the same industry or not
  - First, E chooses whether to enter or not
  - If E enters, then I could choose either to fight or not
  - If E enters and I fights, then E gets -1 as payoff while I gets a payoff of 1
  - If E enters and I doesn't fight, then E gets 2 as payoff while I gets a payoff of 3
  - If E doesn't enter then he gets 0 and I gets 5

Entry deterrence

Credible threats

- Note that in the entry deterrence game, I's threat of fighting is not credible
- After E, I will be better off by not fighting
- Hence Nash equilibrium is not enough to get an appealing prediction

Sequential rationality

Definition

Given s_{-i}, we say that s_i is sequentially rational if i is playing a best response to s_{-i} at each history in which she is called to play

- That is, we want players to choose the best action available when they are called to act

Backward induction

- Consider the following algorithm (for finite horizon games)
    - Pick a terminal node z, move to the (unique) node which precedes z.
    - What is the best action for the player moving in such node?
    - Move to the node preceding the previous node
    - Given the action taken in the previous step as given, what is the optimal action for the player moving in the current node?
    - Repeat until the root is reached
    - Repeat for every terminal node
- The outcome of this algorithm will be a sequentially rational profile of strategies!

Entry deterrence

Subgames

- Note that starting from each node visited in the previous algorithm we can define a (smaller) game
  - We have a set of players
  - An order of play
  - A set of histories
  - Some terminal histories
  - And payoffs over such terminal histories
- We refer to these smaller games as *subgames*

Subgames

- More formally, a subgame of an extensive form game $\Gamma = \{N, H, P, (u_i)\}$ starting at history h is an extensive form game $\Gamma|_h = \{N, H|_h, P|_h, (u_i|_h)\}$ such that
  - The set of players is N
  - The set of histories $H|_h$ contains all histories h' such that $(h, h') \in H$
  - Terminal histories $z|_h$ are such that $z(h, z|_h) \in Z$
  - The order of play $P|_h$ is such that $P|_h(h') = P(h, h')$
  - Payoffs $u_i|h$ satisfy

$$u_i|_h(z|_h) = u_i(h, z|_h)$$

Subgames

- Very scary notation to say that a subgame starts at history h, picks everything after it and considers player and payoff functions restricted to histories following h

Subgame Perfect Equilibrium (SPE)

- Given a strategy s_i and a history h such that $P(h) = i$, we denote by $s_i(h)$ the action taken by player i after h
- Similarly, given a profile of strategies s and a non-terminal history h, we denote by $s(h)$ the action taken after history h by player $P(h)$
- Given a profile of strategies s, we define $s|_h$ as the profile of strategies restricted to the subgame $\Gamma|_h$, that is $s|_h(h') = s(h, h')$ for all h' such that $(h, h') \in H \backslash Z$

Subgame Perfect Equilibrium (SPE)

Subgame Perfect Equilibrium

A profile of strategies s^* is a subgame perfect (Nash) equilibrium of Γ if for every subgame of $\Gamma|_h$, $s^*|_h$ is a Nash equilibrium of $\Gamma|_h$

- Note that NE in every subgame coincides with requiring s^* being sequentially rational!
- The profile of strategies obtained using backward induction is indeed a SPE

Centipede game

- Two players move alternately in a game with 4 rounds
- In each round 2 coins are added to the pile
- At each round, a player could either take the pile or pass
- If she takes the pile the game ends, otherwise the game moves to the next round
- The game starts with a pile of 2 coins and player 1 moving.
- At round 4, if player 2 pass then 2 coins are added to the pile but the game ends with each player receiving half of the pile

Centipede game

Chain Store game

- A chain store (CS) has branches in K cities
- In each city there is a potential competitor k
- They play a sequence of entry deterrence games in which in each stage a new city is reached and a single competitor decides to enter the market or not
- The payoffs of each player are analogous to the ones in the original entry deterrence game

Chain Store game

Chain Store game

Stackelberg Competition

- Same setting as in Cournot but now firms move sequentially
- We call the first mover the leader and the second mover the follower
- What is the equilibrium in this game?

Stackelberg Competition

Stackelberg Competition

Stackelberg Competition

Stackelberg Competition

Existence of SPE

- For finite extensive form games with perfect information, we can guarantee existence of SPE even considering only pure strategies

Theorem

Finite extensive form games with perfect information have a SPE

One-shot deviation principle

- Looking for deviations in extensive form games could be cumbersome
- The following result greatly simplifies the procedure to check whether a profile of strategies is a SPE or not

One-shot deviation principle

s^* is a SPE if and only if for all players $i \in N$ there not exist a history h and action $a' \in A(h)$ such that $P(h) = i$

$$u_i(s_i', s_{-i}^*) > u_i(s^*)$$

where $s_i'(h) = a'$ and $s_i(h') = s_i^*(h')$ for all $h' \neq h$ such that $P(h') = i$