Game Theory

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Logistics

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Office Hours: Tuesdays 3-5pm @ 4515 WWPH

 Course website: Canvas for all the material, Gradescope for submissions

Assessments

 4 Quizzes: average 10% released Thursday, due just before Monday's lecture

 2 Problem sets: each one 20% due Friday before Midterm and Final respectively.

• Midterm: 20% on June 1

• Final: 30% on June 22

References

- Introductory references
 - Robert Gibbons, Game Theory for Applied Economists, Princeton University Press, 1992
 - Martin J. Osborne, An Introduction to Game Theory, Oxford University Press, 2004
 - Steven Tadelis, Game Theory: An Introduction, Princeton University Press, 2013

Advanced references

- Drew Fudenberg and Jean Tirole, Game Theory, MIT Press, 1991
- Martin J. Osborne and Ariel Rubinstein, A Course in Game Theory, MIT Press, 1994
- Michael Maschler, Eilon Solan, and Shmuel Zamir, Game Theory, Cambridge University Press, 2013

What is game theory?

- From Maschler, Solan, and Zamir: "methodology of using mathematical tools to model and analyze situations of interactive decision making"
 - Situations involving several decision-makers ("players")
 - Interactions vs. isolated decision making (I care about what others do)
 - Actions change behavior of others, they have an impact on the final outcome
 - Interests of different agents/players are interrelated

What is game theory?

- Collection of models:
 - "all models are wrong but some of them are useful"
 - Useful representations of the world
 - Rely on assumptions which are often not completely realistic
- Applications: economics (e.g., market competition), political science (e.g., voting), biology (e.g., evolutionary biology)

- Think about a very simple game: Rock, Paper, Scissors
- There are two players
- Each player could play either rock, paper, or scissors
- The winner is determined by actions chosen by each player

- Another example: competition between candidates in an election
- They choose how to shape their campaign
 - Which policies to promote
 - What topics talk about
 - Etc
- Outcome of the election will depend on the campaigns used by each candidate (and how citizens react to such campaigns)

- Location decision by competing firms
- Think about Walmart, Target, Whole Foods, Aldi, etc
- Being only grocery store in a particular zone vs. being one more in a very competitive zone
- Key in Starbucks strategy as well

Most sports, card and board games could also work as example

Before the game starts...

 Before properly introducing games, lets briefly introduce the theory of rational choice in single agent decision problems

Decision problems and rational choice

A decision problem is a problem faced by a decision-maker,
i.e, a problem where someone needs to make a choice

How to define a decision problem?

- Formally, a decision problem is defined by
 - 1 A set of available alternatives or actions
 - 2 Outcomes or consequences of such actions
 - **3** Preferences or payoffs over such consequences

 $Actions \Rightarrow Outcomes \Rightarrow Payoff$

- Main underlying assumption: decision-maker chooses the action he/she likes the most
- Rationality comes from consistency on choices, not restrictions on his preferences
- No normative content, focus on positive content: we want to identify what agent likes/want given his choices

- Consistency rely on two further assumptions or axioms:
 - Completeness: for any pair of alternatives, decision-maker could tell which he/she prefers or whether he/she likes both the same.

• Transitivity: his/her choices does not generate cycles

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• Here "A preferred to B" means "I prefer A to B" and we sometimes will denote it as $A \succ B$

A comment on the theory of rational choice

- Such properties are defined over a fixed problem
- They do not impose that you must choose the same today and tomorrow, since these are two different problems!
- Still, they could introduce some structure and limitations
- There are theoretical and empirical developments that deal directly with such failures (outside the scope of this course)

Why do we need rational choice?

- Two properties above guarantee existence of a payoff function that represents the preferences
- That is, we can rank the alternatives/outcomes
- Moreover, we can work with a numeric representation which is easier!
- We will able to use tools from algebra, calculus and optimization to solve the problems we are analyzing

- What to have for breakfast?
 - Actions or alternatives

Outcomes

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```
coffee, toasts, eggs, burrito, hot chocolate, toast+coffee, ...
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Outcomes

 \simeq actions

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Outcomes

$$\simeq$$
 actions

```
toast+coffee \succ hot chocolate \succ coffee \succ eggs ...
```

From decision problems to games

• So far: single agent,

own action \rightarrow outcome

- In a decision problem the environment is fixed: only agent's action is changing
- His action doesn't change anything else
- Everyone else is part of the environment, and doesn't take part in the decision

From decision problems to games

• In games: multiple agents,

actions (own
$$+$$
 others) \rightarrow outcome

- Now the actions of all players determine the outcome
- We assume now each player is also a rational decision-maker trying to pursuit its own interest
- This is the strategic interaction part

Introducing games

• We will start with the simplest basic form of games: static games of complete information

Static games of complete information

- Game: several agents, each with his own decision problem, payoffs depending on the decisions of all players
- Static: players take actions simultaneously, in a single point in the game (without knowing what other players have chosen)

Static games of complete information

- Complete information: all actions available, all outcomes, all preferences are known by every player
 - We will further require details of the game to be common knowledge, i.e., everyone knows the details, and everyone knows everyone knows the details, and everyone...
 - This will allow us to use strategic reasoning to "solve" the game
 - Each player is able to figure out what the other players will do, and understands that the other players will also be doing the same.

• Let's try to formalize the definition of a game using this simple game again

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 - $(R,R) \Rightarrow \text{draw}$
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 - ...
- Note we can represent profiles of actions as vectors in which each entry represents an action performed by a player

Math recap and some notation

- $x \in Y$ means element x belongs or is in set Y
- Similarly, $x \notin Y$ means x is not in the set Y
- For example π is a real number so $\pi \in \mathbb{R}$ where \mathbb{R} is the commonly used symbol for denoting the set of real numbers
- But π is not a rational number, so $\pi \not\in \mathbb{Q}$

Math recap and some notation

- \sum is used for sums
- For example the sum 1 + 2 + 3 + 4 could be written as $\sum_{i=1}^{n} i^{i}$

• Similarly,
$$x_1 + x_2 + x_3 + x_4 = \sum_{i=1}^{4} x_i$$

Math recap and some notation

- $f: X \to Y$ represents a function f associating/mapping elements on set X to element in set Y
- For example, the function $f(x) = x^2$ maps real numbers into real numbers

Normal-form games

- A normal-form game is defined by
 - **1** A set of players $N = \{1, 2, 3, ..., n\}$
 - **2** A set of strategies S_i for each player i
 - **3** A payoff function u_i for each player i
- For now, lets think about strategies just as actions
- We usually denote by $S = S_1 \times S_2 \times S_3 \times ... \times S_n$ the set of profiles of strategies

Actions as outcomes

- Note that we get rid of the outcome part from the decision problem
- If there is a clearly defined outcome for each profile of actions, then we can define the payoffs directly as a function of the action profile:

$$u_i:S\to\mathbb{R}$$

- In a game, the outcomes will be represented by the combination of actions of all players, and the payoff of each player will depend on the full profile of actions
- Hence, the full description of a game G will be given by

$$G = \{N, S, (u_i)_{i \in N}\}$$

Studying problems using game theory

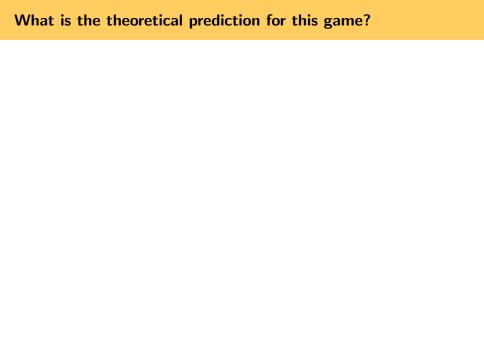
- We could think of the whole process of study a problem with the lens of game theory as a 3 steps procedure:
 - Modeling: formally describe the "world" we will be working with (using math): $G = \{N, S, (u_i)_{i \in N}\}$
 - Who are the relevant players?
 - What are their available choices?
 - What will they obtain from each action profile?
 - ② Solving: given the game, what is the prediction? what we should expect as the final outcome of this interaction?
 - Interpretation and analysis:
 - What does the prediction means? (in the "real world")
 - Why we obtain such prediction?
 - What if we change X feature?
 - What would be the impact of policy/regulation *Y*?

Guessing game

- Choose a number (non-negative integer) between 0 and 100, and send it to me via private message
- The winner will be the closest number to two thirds of the average of the numbers reported
- If there is a tie, an actual winner will be randomly chosen from the set of potential winners

Results

Formal description of the game



• Each player reports a number x between 0 and 100

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- Which means $\left(\frac{2}{3}\right) \overline{x} \leq \left(\frac{2}{3}\right) 67 \approx 45...$
- If we continue with this process we will end up with $x \le 0$
- So, the theoretical prediction is that everyone should be reporting zero!

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 - Not perfectly but it captures the incentives involved and the "direction" toward which the reports should move to (i.e., you should report smaller numbers!)

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- The process we used to obtain a prediction is called *Iterative* elimination of strictly dominated strategies
- This is one of the solution concepts we will study in this course

Solution concepts

- We have developed the tools to formally describe a simple game
- Now we want to get a prediction for such games
- We need to define what are called a solution or equilibrium concepts

Finite Games and Matrix Representation

• A class of important games are *finite* games

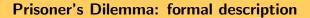
Definition (finite games)

A game G is finite if the set of players N is finite, and the set of strategies S_i for each player i is also finite.

 Small finite games could be represented in a matrix form which allows to analyze them easily

Classic game: Prisoner's Dilemma

- There are two suspects (Bonnie and Clyde, i.e., the "prisoners") of multiple bank robberies
- They are being interrogated in two different rooms, and are not allowed to communicate with each other
- Each suspect could either confess or not
- If they both confess, then they will convicted for all the crimes
- If neither of them confess, the will receive a reduced sentence due to lack of evidence to convict them for all crimes
- If only one of them confesses, then that suspect will receive a very low sentence as a reward for his/her cooperation, while the other suspect will receive a very severe sentence



Prisoner's Dilemma: matrix representation

We can represent this game in a matrix as follows

	С	NC
С	-10,-10	0,-20
NC	-20,0	-5,-5

- Here, we represent the first player (Bonnie) as the "row" player, and the second player (Clyde) as the "column" player
- Each row represent one available strategy for Bonnie
- Each column represent one available strategy for Clyde
- Payoffs are ordered so the first number in each cell represents the payoff of Bonnie and the second the payoff of Clyde

Prisoner's Dilemma: outcome

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Prisoner's Dilemma: outcome

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Prisoner's Dilemma: discussion

- Note that players in this game ends up in a "inefficient" (from their perspective) outcome: there is another outcome which both strictly prefer (NC,NC) but are unable to coordinate in such outcome when both are looking for their own private interest!
- The type of solution we obtained here is called *equilibrium in* strictly dominant strategies

Prisoner's Dilemma: applications

- The reason why Prisoner's Dilemma is a popular game in game theory and economics is because several interactions could be modeled as Prisoner's Dilemma games
- The tension captured by the Prisoner's Dilemma is present in several different situations
- Examples: simple duopoly, joint effort, tragedy of the commons

Prisoner's Dilemma: simple duopoly

- Two firms selling the same good
- They could either charge a high price (H) or a low price (L)

	Н	L
Н	7000,7000	0,8000
L	8000,0	4000,4000

Prisoner's Dilemma: joint project

 You and your friend working on a joint project and deciding how much effort to put in the project

	Н	L
Н	50,50	0,100
L	100,0	20,20

Dominated strategies

- Some games have strategies/actions that are more appealing than others
- For example, in the guessing game we should not expect numbers above 67

Dominated strategies

Definition

We say an strategy s'_i is **strictly dominated** by strategy s_i for player i if for any profile of strategies for all other players, $s_{-i} \in S_{-i}$

$$u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$$

 That is, s_i always give a higher payoff than s'_i regardless of what other players are playing

About dominated strategies

- A rational player should never use **strictly** dominated strategies
- \bullet We can define "weakly dominated" strategies by replacing > by \geq in the previous definition

Dominant strategies

• A related concept is *strictly dominant strategy*

Definition:

 s_i is a **strictly dominant strategy** for i if every other strategy s'_i of i is strictly dominated by s_i

• s_i gives higher payoff than any other action for each potential profile of actions played by other players

Equilibrium in strictly dominant strategies

We are ready to define our first solution concept

Definition

The profile of strategies $s^* \in S$ is an **equilibrium in strictly dominant strategies** (or strict dominant strategy equilibrium) if $s_i^* \in S_i$ is a strictly dominant strategy for all $i \in N$

 Note that an equilibrium is characterized by the profile of strategies, NOT THE PAYOFFS!

Equilibrium in strictly dominant strategies

- Advantages: easy to compute, doesn't rely on common knowledge of rationality
- Issue: not all games have dominant strategies!