Mixed strategies and equilibrium existence

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Cournot competition

- N firms competing in quantity
- P(Q)

Bertrand Competition

- Again, two firms competing but this time choosing prices instead of quantities
- Demand function is now

$$D(p) = 100 - p$$

  if $p < 100$, 0 if $p \geq 100$

- Consumers buy from the cheapest firm
- If both firms set the same price, then they split equally between them

Bertrand Competition

- How we setup the problem in this case?

Bertrand Competition

- Let's graph the profits

Bertrand Competition

- Best responses

Bertrand Competition

- Is there an equilibrium?

Bertrand Competition

- Note that we get an equilibrium even though best responses are not always well defined
- Also, two firms is enough to drive prices to marginal costs!
- If firms compete in prices, competition is super strong
- How the outcome changes with respect to quantity competition (Cournot)?

Hotelling location model

- Location competition
- Political competition (Downs model)

Hotelling location model

- Two candidates A and B compete in an election
- Each candidate choose a policy $x_i \in [0, 1]$ to implement if elected (must implement its promise)

Hotelling location model

- Continuum of voters uniformly distributed over $[0, 1]$, and their position determine their policy preference
- They vote for the candidate with policy closer to their preference/position, randomize uniformly if both at same distance
- No voter abstains, and the candidate with more votes wins (if case of draw they flip a coin to determine the winner)

Hotelling location model

- Candidates only care about winning the election, not the policy they implement
- Payoff $= 1$ if wins, $= 0$ if not

Hotelling location model

- Who vote for each candidate?

Hotelling location model

- Payoffs

Hotelling location model

- Best responses

Hotelling location model

- Nash equilibrium

Hotelling location model

- Note that for asymmetric distributions, candidates will choose the median policy (instead of $1/2$)
- This result is known as the *median voter theorem*

Hotelling location model

- What if candidates also care about the policy?
- For example, suppose that each candidate prefers one of the extreme policies

Hotelling location model

- Result still holds since they need to win election first!

Hotelling location model

- Original economic application consider two firms deciding where to locate their stores
- There are some differences if firms also choose prices
- We will (probably) revisit this model when we introduce extensive form games

Matching pennies

- Two players
- Each put a penny on a table simultaneously
- If both heads or both tails, Player 1 wins
- If they are different, Player 2 wins
- This is a classic *zero-sum game*: a game in which one player gains are the other player losses

Matching pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Mixed Strategies

- The way we have defined strategies until now is as certain actions
- What if we allow players to randomize?

Mixed Strategies

- For a fixed set of strategies S_i, we let player i to randomize between his available actions
- We will refer to S_i as the set of pure strategies
- ΔS_i will represent the simplex of S_i, the set of all probability distributions over S_i
- A mixed strategy for i is an element $\sigma_i \in \Delta S_i$, where $\sigma_i = (\sigma_i(s_{i1}), \sigma_i(s_{i2}), \sigma_i(s_{i3}), ..., \sigma_i(s_{im}))$ and $\sigma_i(s_i)$ is the probability that player i plays s_i.

Mixed Strategies

- So, a mixed strategy is just a probability distribution over his pure strategies
- As σ_i is a prob. distribution it must satisfy

$$\sigma_i(s_i) \geq 0 \quad \forall s_i \in S_i$$

$$\sum_{s_i \in S_i} \sigma_i(s_i) = 1$$

Mixed Strategies

- We will allow each player to randomize independently, so the realization of the mixed strategies for each player i will be independent of the realizations of the other players
- A profile of mixed strategies will be given as before by

$$\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$$

- We will also denote by σ the probability distribution over profiles of **pure** strategies.
- Since each player randomize independently, the probability that profile s realizes is given by

$$\sigma(s) = \prod_{i \in N} \sigma_i(s_i) = \sigma_1(s_1) * \sigma_2(s_2) * ... * \sigma_n(s_n)$$

Expected payoffs

- We have extended strategies to consider distributions
- What about payoffs?

Expected payoffs

- We will assume that players evaluate those mix strategies using expected payoffs
- That is, starting from a payoff function u_i over profiles of pure strategies for player i, we extend his payoff function over profiles of mixed strategies σ using its expected payoff, that is

$$
\begin{aligned}
u_i(\sigma) &= \sum_{s \in S} \sigma(s) u_i(s) \\
&= \sum_{s \in S} \left(\sigma_i(s_i) \prod_{j \neq i} \sigma_j(s_j) \right) u_i(s_i, s_{-i}) \\
&= \sum_{s_i \in S_i} \left(\sigma_i(s_i) \left(\sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i})\ u_i(s_i, s_{-i}) \right) \right)
\end{aligned}
$$

Expected payoffs

Best responses again

- The definition of best responses remains the same but defined over the relevant set of strategies
- That is, if we allow for mixed strategies, then the best response correspondence for player i is a function

$$BR_i : \underset{j \neq i}{\times} \Delta S_j \rightarrow 2^{\Delta S_i}$$

where

$$\underset{j \neq i}{\times} \Delta S_j = \Delta S_1 \times ... \times \Delta S_{i-1} \times \Delta S_{i+1}$$

and $2^{\Delta S_i}$ is the power set of ΔS_i (set of all subsets)

Best responses again

Definition

The best-response correspondence of player i is defined by

$$BR_i(\sigma_{-i}) = \{\sigma_i \in \Delta S_i | u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}),\ \forall \sigma_i' \in \Delta S_i\}$$

Nash equilibrium in mixed strategies

- The notion of Nash equilibrium considering mixed strategies is the same as before but with mixed instead of pure strategies
- We reproduce them here accounting for mixed strategies explicitly

Definition

The mixed strategy profile σ^* is a **Nash equilibrium in mixed strategies** if σ_i^* is a best response to σ_{-i}^* for each $i \in N$, that is,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*) \quad \forall \sigma_i' \in \Delta S_i$$

for all $i \in N$

Nash equilibrium in mixed strategies

Definition

The mixed strategy profile σ^* is a **Nash equilibrium in mixed strategies** if $\sigma_i^* \in BR_i(\sigma_{-i}^*)$ for all $i \in N$

Nash equilibrium in mixed strategies

- Note that pure strategies correspond to a particular case of mixed strategies: degenerate mixed strategies where only a single strategy have positive probability
- Hence, NE in pure strategies are contained in NE in mixed strategies

Matching pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Matching pennies

Indifference principle

- The following result is useful to find equilibrium in mixed strategies

Proposition

If σ is a Nash equilibrium in mixed strategies, then for all players i,

$$u_i(s_i, \sigma_{-i}) = u_i(s_i', \sigma_{-i}) = u_i(\sigma_i, \sigma_{-i})$$

for any $s_i, s_i' \in S_i$ such that $\sigma_i(s_i), \sigma_i(s_i') > 0$

Matching pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Rock, Paper, Scissors

	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

Prisoner's Dilemma

	C	NC
C	$-10, -10$	$0, -20$
NC	$-20, 0$	$-5, -5$

Multiplicity

- Obviously, taking into account mixed strategies doesn't help us in terms of multiplicity
- Even more equilibriums could arise if we consider also mixed strategies

BoS revisited

	H	U
H	$2, 1$	$0, 0$
U	$0, 0$	$1, 2$

Mixed strategies and dominance

- We can extend dominance to consider mixed strategies
- Note that if we introduce mixed strategies to a game new dominated strategies could arise

Mixed strategies and IESDS

- We can also revisit the definition of iterated elimination considering mixed strategies
- However, we will not working with in this course

Mixed strategies and infinite pure strategies

- We can define mixed strategies over infinite sets of pure strategies
- There, a mixed strategy will again just a distribution over such pure strategies
- For example, in the Cournot game:

$$S_i = [0, \infty)$$

and a mixed strategy could be represented by $F_i : S_i \to [0, 1]$ such that $F_i(x) = \Pr(s_i \leq x)$ is the cdf of s_i

Properties of NE

- As outcome of repeated interactions
- As self-fulfilling agreement
- As recommendations
- As beliefs consistent with what people is playing
- As a reduced model

Nash's Existence Theorem

- If we allow for mixed strategies, then we can guarantee existence of a NE!

Nash's Theorem

Any finite game has a Nash equilibrium in mixed strategies

Nash's Existence Theorem

- We will not cover the proof since it goes beyond the scope of this course
- The usual proof relies on the existence of a fixed point in the best responses correspondences
- You can check any of the references (or ask me) for more details on this theorem