Nash Equilibrium and IESDS

Nicolas Pastrian

May 17, 2022

University of Pittsburgh

Summary

- Game: $G = \{N, S, (u_i)\}$
  - A set of players N
  - A set of strategies S_i for each player i
  - A payoff function $u_i : S \to \mathbb{R}$ for each player i
- Dominated strategy: s_i' is strictly dominated by s_i if $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$ for $s_i \in S_{-i}$
- Dominant strategy: s_i is a strictly dominant strategy if all other strategies of player i are dominated by s_i
- Equilibrium in Dominant Strategies: s^* is a equilibrium in strictly dominant strategies if s_i^* is a strictly dominant strategy for all players i

About Equilibrium in Strictly Dominant Strategies

- An equilibrium in strictly dominant strategies makes use of the assumption of rationality but not of the common knowledge of rationality
- So, simple to compute and doesn't require strong assumptions on the sophistication of players
- Issue: not all games have dominant strategies!

Prisoner's Dilemma: a game with equilibrium in strictly dominant strategies

	C	NC
C	$-10, -10$	$0, -20$
NC	$-20, 0$	$5, 5$

Stag Hunt game

- Two hunters can choose to hunt a stag or a hare
- Stag provides a tasty and large meal, while the hare is less fulfilling
- Each hunter decides which animal to hunt separately (no communication)
- Hunting a stag is difficult and will be successful only if both hunters do so
- The reward of hunting a stag is bigger than the reward of hunting a hare

Stag Hunt game

	Stag	Hare
Stag	$100, 100$	$0, 10$
Hare	$10, 0$	$10, 10$

Battle of the Sexes

- You and your friend to meet for dinner
- You usually eat either in Hemingway's or Union Grill
- Your friend left his phone at home so there is no way to reach him
- There is a problem: you didn't decided in which restaurant you will meet!
- While you like Hemingway's more, your friend likes Union Grill more.
- However, both of you prefer to eat together than alone, so you are trying to coordinate in a restaurant without communication

Battle of the Sexes

	H	U
H	$2, 1$	$0, 0$
U	$0, 0$	$1, 2$

Battle of the Sexes

- Same idea could be applied in other situations
  - Hillman Library or Wesley W. Posvar Hall?
  - Opera or Ballet?
  - Original one: football or ballet?

Rock, Paper, Scissors

	R	P	S
R	$0, 0$	$-1, 1$	$1, -1$
P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

Iterative Elimination of Strictly Dominated Strategies (IESDS)

- We have seen how to get a prediction using this idea in the guessing game
- Formally, the Iterative Elimination of Strictly Dominated Strategies is an algorithm that exploits common knowledge of rationality to potentially obtain a prediction from a game

Iterative Elimination of Strictly Dominated Strategies (IESDS)

- Consider a game $G = \{N, S, (u_i)\}$.
- The algorithm goes as follows:
  - Step 1: Define $S_i^0 = S_i$ for each i, let $G^0 = G$, and set $k = 0$. Move to Step 2.
  - Step 2: Are there players for whom there are strategies $s_i \in S_i^k$ that are strictly dominated in the game G^k? If yes, got to Step 3. If not, stop.
  - Step 3: For all the players $i \in N$, remove any strategies $s_i \in S_i^k$ that are strictly dominated (in game G^k). Let $S_i^{k+1} \subseteq S_i^k$ be the new set of strategies for player i (after removing strictly dominated strategies). Define a new game $G^{k+1} = \{N, S^{k+1}, (u_i)\}$. Set $k = k + 1$ and return to Step 2.

Iterative Elimination of Strictly Dominated Strategies (IESDS)

- Sometimes the strategies surviving the process of IESDS are called **iterative-elimination equilibrium**

IESDS example

	L	C	R
U	4,3	5,1	6,2
M	2,1	8,4	3,6
D	3,0	9,6	2,8

IESDS example

About IESDS

- Order of elimination doesn't change the result
- Equilibrium in strictly dominant strategies survives elimination
- Issue: sometimes no strategy is eliminated!

Beyond elimination and dominant strategies

- Both equilibrium in strictly dominant strategies and IESDS are based on eliminating actions that players would never play.
- But there are several games in which there are neither dominant nor dominated strategies, and instead some actions could be the best choice in some cases while perform really bad in others

Best Response

- What is the best plan of action for i if the other players are choosing s_{-i}?

Definition

The strategy $s_i \in S_i$ is a **best response** to the profile of strategies s_{-i} for player i if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \qquad \forall s_i' \in S_i$$

- Note that best responses are defined for each strategy profile of the other players
- Even for a fixed strategy profile s_{-i} it could be not unique

Best Response Correspondence

- Formally, the best response for player i forms a function from S_{-i} to subsets of S_i
- We refer to this set valued functions as *correspondences*
- For each $s_{-i} \in S_{-i}$, we denote by $BR_i(s_{-i})$ the set of best responses to s_{-i}

Definition

The best-response correspondence of player i is defined by

$$BR_i(s_{-i}) = \{s_i \in S_i | u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}), \forall s_i' \in S_i\}$$

Prisoner's Dilemma

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Hare	10, 0	10, 10

Rock, Paper, Scissors

	R	P	S
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P	$1, -1$	$0, 0$	$-1, 1$
S	$-1, 1$	$1, -1$	$0, 0$

Nash Equilibrium (NE)

- Given some beliefs over what the other players will be playing, player i could identify what are the best actions to take
- If those beliefs coincide with the actual play of the other player, we obtain the most important solution concept in game theory: **Nash Equilibrium**

Nash Equilibrium (NE)

Definition

The strategy profile $s^* \in S$ is a **Nash equilibrium** if s_i^* is a best response to s_{-i}^* for all $i \in N$, that is,

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i', s_{-i}^*) \quad \forall s_i' \in S_i$$

for all $i \in N$

Nash Equilibrium (NE)

Definition

The strategy profile $s^* \in S$ is a **Nash equilibrium** if $s_i^* \in BR_i(s_{-i}^*)$ for all $i \in N$

Nash Equilibrium (NE)

- Two ways to compute/check for NE
  - Check for potential deviations
  - Use best responses (fixed point)

Prisoner's Dilemma

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Stag Hunt game

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Battle of the Sexes

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NE, strictly dominated strategies and IESDS

- Strictly dominated strategies cannot be part of a NE
- Moreover, all NE always survive IESDS

NE and dominant strategies

- If some player has a dominant strategy, he must always use it in a NE

Theorem

If s^* is an equilibrium in dominant strategies then s^* is a NE

About NE

- Using NE we get predictions for even more games!
- However, it makes even more intensive use of the common knowledge of rationality assumption
- Issue: we could have too many predictions!!!

Cournot Duopoly

- Two identical firms (players 1 and 2) compete in a market producing the same good.
- For simplicity, we assume that firms could produce any non-negative amount (fractions are allowed)
- Each firm decides how many units of the good to produce: $q_i \geq 0$
- The cost of producing each unit is $c \geq 0$ for both firms

Cournot Duopoly

- The price at which the product could be sold depends of the total amount of the good sold in the market

$$P(Q) = \begin{cases} a - Q & \text{if } Q \leq a \\ 0 & \text{otherwise} \end{cases}$$

  where $Q = q_1 + q_2$, and a is the maximum price at which someone is willing to buy the good.

- $P(Q)$ represents the inverse demand curve in this market

Cournot Duopoly

- Each firm payoff is given by its profits (revenue - cost)

$$\pi_i(q_1, q_2) = P(q_1 + q_2) * q_i - c * q_1$$

- What would be the outcome in this game?

Cournot Duopoly

Cournot Duopoly

Cournot Duopoly

Cournot Duopoly

Cournot Duopoly

NE with continuous strategies

- Note that Cournot Duopoly is not a finite game
- Due to the characteristics of this game, we can directly apply tools from calculus and optimization to solve it
- Using this we could see that the existence of well behaved solutions to the maximization problems could guarantee existence of a NE!

Multiplicity of equilibria

- As we have seen before, sometimes NE gives too many predictions
- One potential solution is to focus only on equilibria with certain properties, or introduce further requirements that reduce the number of equilibriums

Weakly Dominated Strategies and Weakly Dominant Strategies

Definition

We say an strategy s_i' is **weakly dominated** by strategy s_i for player i if for any profile of strategies for all other players, $s_{-i} \in S_{-i}$

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

Definition:

s_i is a **weakly dominant strategy** for i if every other strategy s_i' of i is weakly dominated by s_i

NE using weakly dominant strategies

Theorem

If every player has a weakly dominant strategy s_i^*, then $s^* = (s_1^*, s_2^*, ..., s_n^*)$ is a Nash equilibrium

Strict NE

Definition

A **strict Nash equilibrium** is a strategy profile $s^* \in S$ such that for every player $i \in N$ and all $s_i \neq s_i^*$,

$$v_i(s^*) > v_i(s_i, s_{-i}^*)$$

Strict NE

Theorem

Weakly dominated strategies cannot be part of a strict Nash equilibrium

Voting game

- A odd number of players n vote in an election with two candidates a and b (no one abstains)
- There are two types of voters: type A which strictly prefer candidate a, and type B voters which strictly prefer candidate b.
- There are k voters of type A, and $n - k$ voters of type B
- The candidate with more votes wins the election

Voting game

Voting game: another equilibrium

NE in weakly undominated strategies

Definition

A profile of strategies $s^* \in S$ is a **Nash equilibrium in weakly undominated strategies** if (i) s^* is a Nash equilibrium, and (ii) for all players $i \in N$ s_i^* is not a weakly dominated strategy

- In the voting, only the first type of equilibrium satisfy this condition