Game Theory

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Logistics

- Instructor: Nicolas Pastrian
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  - Office Hours: Tuesdays 3-5pm @ 4515 WWPH
- Course website: Canvas for all the material, Gradescope for submissions
- Assessments
  - 4 Quizzes: average 10% released Thursday, due just before Monday's lecture
  - 2 Problem sets: each one 20% due Friday before Midterm and Final respectively.
  - Midterm: 20% on June 1
  - Final: 30% on June 22

References

- Introductory references
  - Robert Gibbons, *Game Theory for Applied Economists*, Princeton University Press, 1992
  - Martin J. Osborne, *An Introduction to Game Theory*, Oxford University Press, 2004
  - Steven Tadelis, *Game Theory: An Introduction*, Princeton University Press, 2013
- Advanced references
  - Drew Fudenberg and Jean Tirole, *Game Theory*, MIT Press, 1991
  - Martin J. Osborne and Ariel Rubinstein, *A Course in Game Theory*, MIT Press, 1994
  - Michael Maschler, Eilon Solan, and Shmuel Zamir, *Game Theory*, Cambridge University Press, 2013

What is game theory?

- From Maschler, Solan, and Zamir: "*methodology of using mathematical tools to model and analyze situations of interactive decision making*"
  - Situations involving several decision-makers ("players")
  - Interactions vs. isolated decision making (I care about what others do)
  - Actions change behavior of others, they have an impact on the final outcome
  - Interests of different agents/players are interrelated

What is game theory?

- Collection of models:
  - "all models are wrong but some of them are useful"
  - Useful representations of the world
  - Rely on assumptions which are often not completely realistic
- Applications: economics (e.g., market competition), political science (e.g., voting), biology (e.g., evolutionary biology)

Some examples

- Think about a very simple game: Rock, Paper, Scissors
- There are two players
- Each player could play either rock, paper, or scissors
- The winner is determined by actions chosen by each player

Some examples

- Another example: competition between candidates in an election
- They choose how to shape their campaign
  - Which policies to promote
  - What topics talk about
  - Etc
- Outcome of the election will depend on the campaigns used by each candidate (and how citizens react to such campaigns)

Some examples

- Location decision by competing firms
- Think about Walmart, Target, Whole Foods, Aldi, etc
- Being only grocery store in a particular zone vs. being one more in a very competitive zone
- Key in Starbucks strategy as well

Some examples

- Most sports, card and board games could also work as example

Before the game starts...

- Before properly introducing games, lets briefly introduce the theory of rational choice in single agent decision problems

Decision problems and rational choice

- A decision problem is a problem faced by a decision-maker, i.e, a problem where someone needs to make a choice

How to define a decision problem?

- Formally, a decision problem is defined by
  1. A set of available alternatives or actions
  2. Outcomes or consequences of such actions
  3. Preferences or payoffs over such consequences

$$\textit{Actions} \Rightarrow \textit{Outcomes} \Rightarrow \textit{Payoff}$$

Theory of rational choice

- Main underlying assumption: decision-maker chooses the action he/she likes the most
- Rationality comes from consistency on choices, not restrictions on his preferences
- No normative content, focus on positive content: we want to identify what agent likes/want given his choices

Theory of rational choice

- Consistency rely on two further assumptions or axioms:
    - Completeness: for any pair of alternatives, decision-maker could tell which he/she prefers or whether he/she likes both the same.
    - Transitivity: his/her choices does not generate cycles

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  - Here "A preferred to B" means "I prefer A to B" and we sometimes will denote it as $A \succ B$

A comment on the theory of rational choice

- Such properties are defined over a fixed problem
- They do not impose that you must choose the same today and tomorrow, since these are two different problems!
- Still, they could introduce some structure and limitations
- There are theoretical and empirical developments that deal directly with such failures (outside the scope of this course)

Why do we need rational choice?

- Two properties above guarantee existence of a payoff function that represents the preferences
- That is, we can rank the alternatives/outcomes
- Moreover, we can work with a numeric representation which is easier!
- We will able to use tools from algebra, calculus and optimization to solve the problems we are analyzing

Examples

- What to have for breakfast?
  - Actions or alternatives
  - Outcomes
  - Preferences

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    toast+coffee $\succ$ hot chocolate $\succ$ coffee $\succ$ eggs ...

From decision problems to games

- So far: single agent,

$$\text{own action} \;\rightarrow\; \text{outcome}$$

- In a decision problem the environment is fixed: only agent's action is changing
- His action doesn't change anything else
- Everyone else is part of the environment, and doesn't take part in the decision

From decision problems to games

- In games: multiple agents,

$$\text{actions (own + others)} \;\rightarrow\; \text{outcome}$$

- Now the actions of all players determine the outcome
- We assume now each player is also a rational decision-maker trying to pursuit its own interest
- This is the strategic interaction part

Introducing games

- We will start with the simplest basic form of games: static games of complete information

Static games of complete information

- Game: several agents, each with his own decision problem, payoffs depending on the decisions of all players
- Static: players take actions simultaneously, in a single point in the game (without knowing what other players have chosen)

Static games of complete information

- Complete information: all actions available, all outcomes, all preferences are known by every player
  - We will further require details of the game to be *common knowledge*, i.e., everyone knows the details, and everyone knows everyone knows the details, and everyone...
  - This will allow us to use strategic reasoning to "solve" the game
  - Each player is able to figure out what the other players will do, and understands that the other players will also be doing the same.

Rock, Paper, Scissors again

- Let's try to formalize the definition of a game using this simple game again

Rock, Paper, Scissors again

- Two players, let's call them John and Sally

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  - $(R, P) \Rightarrow$ Sally wins
  - $(R, S) \Rightarrow$ John wins
  - $(R, R) \Rightarrow$ draw
  - ...

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- Actions available to each player: R, P, S
- Winner is determined by the profile of actions chosen
  - $(R, P) \Rightarrow$ Sally wins
  - $(R, S) \Rightarrow$ John wins
  - $(R, R) \Rightarrow$ draw
  - ...
- Note we can represent profiles of actions as vectors in which each entry represents an action performed by a player

Math recap and some notation

- $x \in Y$ means element x belongs or is in set Y
- Similarly, $x \notin Y$ means x is not in the set Y
- For example π is a real number so $\pi \in \mathbb{R}$ where $\mathbb{R}$ is the commonly used symbol for denoting the set of real numbers
- But π is not a rational number, so $\pi \notin \mathbb{Q}$

Math recap and some notation

- $\sum$ is used for sums
- For example the sum $1 + 2 + 3 + 4$ could be written as $\sum_{i=1}^{4} i$
- Similarly, $x_1 + x_2 + x_3 + x_4 = \sum_{i=1}^{4} x_i$

Math recap and some notation

- $f : X \rightarrow Y$ represents a function f associating/mapping elements on set X to element in set Y
- For example, the function $f(x) = x^2$ maps real numbers into real numbers

Normal-form games

- A normal-form game is defined by
  1. A set of players $N = \{1, 2, 3, ..., n\}$
  2. A set of strategies S_i for each player i
  3. A payoff function u_i for each player i
- For now, lets think about strategies just as actions
- We usually denote by $S = S_1 \times S_2 \times S_3 \times ... \times S_n$ the set of profiles of strategies

Actions as outcomes

- Note that we get rid of the outcome part from the decision problem
- If there is a clearly defined outcome for each profile of actions, then we can define the payoffs directly as a function of the action profile:

$$u_i : S \to \mathbb{R}$$

- In a game, the outcomes will be represented by the combination of actions of all players, and the payoff of each player will depend on the full profile of actions
- Hence, the full description of a game G will be given by

$$G = \{N, S, (u_i)_{i \in N}\}$$

Studying problems using game theory

- We could think of the whole process of study a problem with the lens of game theory as a 3 steps procedure:
  1. Modeling: formally describe the "world" we will be working with (using math): $G = \{N, S, (u_i)_{i \in N}\}$
     - Who are the relevant players?
     - What are their available choices?
     - What will they obtain from each action profile?
  2. Solving: given the game, *what is the prediction? what we should expect as the final outcome of this interaction?*
  3. Interpretation and analysis:
     - What does the prediction means? (in the "real world")
     - Why we obtain such prediction?
     - What if we change X feature?
     - What would be the impact of policy/regulation Y?

Guessing game

- Choose a number (non-negative integer) between 0 and 100, and send it to me via private message
- The winner will be the closest number to two thirds of the average of the numbers reported
- If there is a tie, an actual winner will be randomly chosen from the set of potential winners

Results

Formal description of the game

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- Which means $\left(\frac{2}{3}\right)\overline{x} \leq \left(\frac{2}{3}\right)67 \approx 45...$
- If we continue with this process we will end up with $x \leq 0$
- So, the theoretical prediction is that everyone should be reporting zero!

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- The process we used to obtain a prediction is called *Iterative elimination of strictly dominated strategies*
- This is one of the solution concepts we will study in this course

Solution concepts

- We have developed the tools to formally describe a simple game
- Now we want to get a prediction for such games
- We need to define what are called a solution or equilibrium concepts

Finite Games and Matrix Representation

- A class of important games are *finite* games

Definition (finite games)

A game G is finite if the set of players N is finite, and the set of strategies S_i for each player i is also finite.

- Small finite games could be represented in a matrix form which allows to analyze them easily

Classic game: Prisoner's Dilemma

- There are two suspects (Bonnie and Clyde, i.e., the "prisoners") of multiple bank robberies
- They are being interrogated in two different rooms, and are not allowed to communicate with each other
- Each suspect could either confess or not
- If they both confess, then they will convicted for all the crimes
- If neither of them confess, the will receive a reduced sentence due to lack of evidence to convict them for all crimes
- If only one of them confesses, then that suspect will receive a very low sentence as a reward for his/her cooperation, while the other suspect will receive a very severe sentence

Prisoner's Dilemma: formal description

Prisoner's Dilemma: matrix representation

- We can represent this game in a matrix as follows

	C	NC
C	-10,-10	0,-20
NC	-20,0	-5,-5

- Here, we represent the first player (Bonnie) as the "row" player, and the second player (Clyde) as the "column" player
- Each row represent one available strategy for Bonnie
- Each column represent one available strategy for Clyde
- Payoffs are ordered so the first number in each cell represents the payoff of Bonnie and the second the payoff of Clyde

Prisoner's Dilemma: outcome

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Prisoner's Dilemma: outcome

	C	NC
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Prisoner's Dilemma: discussion

- Note that players in this game ends up in a "inefficient" (from their perspective) outcome: there is another outcome which both strictly prefer (NC,NC) but are unable to coordinate in such outcome when both are looking for their own private interest!
- The type of solution we obtained here is called *equilibrium in strictly dominant strategies*

Prisoner's Dilemma: applications

- The reason why Prisoner's Dilemma is a popular game in game theory and economics is because several interactions could be modeled as Prisoner's Dilemma games
- The tension captured by the Prisoner's Dilemma is present in several different situations
- Examples: simple duopoly, joint effort, tragedy of the commons

Prisoner's Dilemma: simple duopoly

- Two firms selling the same good
- They could either charge a high price (H) or a low price (L)

	H	L
H	7000,7000	0,8000
L	8000,0	4000,4000

Prisoner's Dilemma: joint project

- You and your friend working on a joint project and deciding how much effort to put in the project

	H	L
H	50,50	0,100
L	100,0	20,20

Dominated strategies

- Some games have strategies/actions that are more appealing than others
- For example, in the guessing game we should not expect numbers above 67

Dominated strategies

Definition

We say an strategy s_i' is **strictly dominated** by strategy s_i for player i if for any profile of strategies for all other players, $s_{-i} \in S_{-i}$

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

- That is, s_i always give a higher payoff than s_i' regardless of what other players are playing

About dominated strategies

- A rational player should never use **strictly** dominated strategies
- We can define "weakly dominated" strategies by replacing $>$ by $\geq$ in the previous definition

Dominant strategies

- A related concept is *strictly dominant strategy*

Definition:

s_i is a **strictly dominant strategy** for i if every other strategy s_i' of i is strictly dominated by s_i

- s_i gives higher payoff than any other action for each potential profile of actions played by other players

Equilibrium in strictly dominant strategies

- We are ready to define our first solution concept

Definition

The profile of strategies $s^* \in S$ is an **equilibrium in strictly dominant strategies** (or strict dominant strategy equilibrium) if $s_i^* \in S_i$ is a strictly dominant strategy for all $i \in N$

- Note that an equilibrium is characterized by the **profile of strategies**, NOT THE PAYOFFS!

Equilibrium in strictly dominant strategies

- Advantages: easy to compute, doesn't rely on common knowledge of rationality
- Issue: not all games have dominant strategies!