Revenue Maximization with Imperfect Information

Nicolas Pastrian

University of Pittsburgh

October 3, 2021

Motivation

- Classic (bayesian) mechanism design assumes the private information hold by the agents follows a known distribution
- Instead, robust mechanism design assumes too little is known about this distribution,
- Or evaluates a mechanism using a robust criteria (e.g., worst-case, maxmin)
- I depart slightly on the standard approach allowing the designer to consider a set of candidate distributions, and a prior belief over them

Model

We consider a symmetric IPV auction model:

- Single unit need to be sold, reserve value for the seller is zero
- N bidders with valuation $\theta_i \in [\underline{\theta}, \overline{\theta}]$ distributed according to F
- Both θ_i and F are privately known by the bidder only
- Valuations of all bidders follow the same distribution
- There is a finite set of potential distributions $\mathcal{F}$
- $\mu \in \Delta(\mathcal{F})$ denotes the prior belief of the seller over the potential distributions

Model

- We restrict the seller to be able to ask about the valuation only
- A (reduced) direct mechanism is given by
  - An allocation rule
    $$q : [\underline{\theta}, \overline{\theta}]^N \to \Delta$$
  - A transfer rule
    $$t : [\underline{\theta}, \overline{\theta}]^N \to \mathbb{R}^N$$
- Note the revelation principle works over $[\underline{\theta}, \overline{\theta}] \times \mathcal{F}$
- Since $\mathcal{F}$ is finite, using message space $[\underline{\theta}, \overline{\theta}]$ is enough to characterize all mechanisms

Model

- Both designer and bidders are risk neutral
- Given allocation q_i and transfer t_i, bidder i with valuation θ_i obtains utility

$$\theta_i q_i - t_i$$

- For a given mechanism (q, t), we denote bidder's utility from reporting r when his valuation is θ_i and others report θ_{-i} by

$$\tilde{u}_i(r, \theta_i, \theta_{-i}) = \theta_i q_i(r, \theta_{-i}) - t_i(r, \theta_{-i})$$

- And his utility from truthful reporting (given the reports of other bidders) by

$$u_i(\theta) = \tilde{u}_i(\theta_i, \theta)$$

Model

- We will require that the mechanism satisfy
  - (ex-post) Incentive compatibility

$$u_i(\theta) \geq \tilde{u}_i(r, \theta) \quad \forall i, \theta, r$$

  - Individual rationality

$$u_i(\theta) \geq 0 \quad \forall i, \theta$$

Model

- The goal of the designer is maximizing his expected revenue

$$\max_{q,t} \sum_{F} \mu(F) \cdot \left(\int \sum_{i} t_i \, dF \right)$$

subject to IC and IR

Myerson's original result

- Myerson (1981) characterized the solution when F is known, and valuations are distributed independently
- In this case, the optimal mechanism could be implemented as a second price auction with an optimally chosen reserve price (which depends on the distribution F)
- This mechanism is obtained as a virtual valuation maximization
- In our problem the solution will look similar

Comments

- Note that there is correlation among the valuations from the perspective of the designer but not the buyers
- This makes impossible to implement the type of "bets" used in Cremer and McLean (1988) mechanisms
- Hence bidders will retain informational rents in our model
- Also, the current model serves as an intermediate case between independent distributions and correlated distributions

Virtual valuations

- As common in mechanism design problems, the virtual valuations are a key object in the characterization of the optimal mechanism
- Here we need first to define the virtual valuation conditional on the distribution F

$$\varphi(\theta_i, F) = \theta_i - \frac{1 - F(\theta_i)}{f(\theta_i)}$$

- We will also make use of a *weighted virtual valuation*

$$\hat{\varphi}_i(\theta) = \sum_{F \in \mathcal{F}} \left(\frac{\mu(F) f(\theta)}{\sum_{\tilde{F} \in \mathcal{F}} \mu(\tilde{F}) \tilde{f}(\theta)} \right) \varphi(\theta_i, F)$$

- Note conditional virtual valuation doesn't depend on i (due to symmetry), while weighted virtual valuation depends on i and the full profile θ

Assumptions

1. **Regularity:** For all $F \in \mathcal{F}$, density $f(\theta_i) > 0$ for all θ_i, and $\varphi(\theta_i, F)$ increasing in θ_i for all θ_i
2. **LR ordering:** Distributions in $\mathcal{F}$ could be ordered according to their likelihood ratio

Optimal mechanism

Proposition

The optimal mechanism is characterized by

$$q_i(\theta) = \begin{cases} 1 & \text{if } \theta_i > \max_{j \neq i} \theta_j \text{ and } \hat{\varphi}_i(\theta) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$t_i(\theta) = \begin{cases} \max_{j \neq i} \theta_j & \text{if } \theta_i > \max_{j \neq i} > r(\theta_{-i})\theta_j \text{ and } \hat{\varphi}_i(\theta) \geq 0 \\ r(\theta_{-i}) & \text{if } \theta_i > r(\theta_{-i}) > \max_{j \neq i} \theta_j \text{ and } \hat{\varphi}_i(\theta) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $r(\theta_{-i}) = \inf\{r \in [\underline{\theta}, \overline{\theta}] : \hat{\varphi}_i(r, \theta_{-i}) = 0\}$

Optimal mechanism

- The optimal mechanism could be interpreted as a second price auction with a changing reserve price

Optimal mechanism

- The optimal mechanism could be interpreted as a second price auction with a changing reserve price
- This reserve price is set according to the updated belief of the seller after observing the valuations of the bidders

Sketch of proof

First, we rewrite seller's problem as a virtual valuation maximization

- Using the envelope theorem, we can rewrite seller's problem as

$$\max_q \sum_F \mu(F) \int \left(\sum_i \theta_i q_i(\theta) - \int_{\underline{\theta}}^{\overline{\theta}} q_i(x, \theta_{-i}) \, dx \right) \, dF$$

subject to

$$q_i(\theta) \text{ increasing in } \theta_i \quad \forall i, \theta$$

Sketch of proof

- Furthermore, we can make use of the definition of the weighted virtual valuation to get

$$\max_{q} \sum_{F} \mu(F) \left(\int \varphi_i(\theta) q_i(\theta)\, dF \right)$$

subject to

$$q_i(\theta) \text{ increasing in } \theta_i \quad \forall i, \theta$$

Sketch of proof

- Now, consider the relaxed problem which ignores the monotonicity condition over q_i
- Given θ, the proposed mechanism maximizes the weighted virtual valuation point-wise, hence solves the relaxed problem
- It remains to show that q_i is increasing
- It could be shown that under regularity $+$ LR ordering, q_i is indeed increasing

Bayesian Incentive Compatibility

- We characterized the mechanisms under EPIC
- Under BIC, revelation principle fails hence proposed mechanism not guaranteed to be optimal
- It remains optimal if we require bidders to report truthfully their valuation regardless of the true distribution
- This could be relevant if designer also has an indirect interest on identifying the actual distribution of valuations

Alternative assumption

- The following sufficient condition could replace *LR ordering* to guarantee optimality of the proposed mechanism
  3. **Convexity:** For all $F \in \mathcal{F}$,

$$\frac{\partial \log f(\theta_i)}{\partial \theta_i} \geq \frac{-2}{\theta_i}$$

Limitations of the optimal mechanism

- However, the mechanism proposed above is not really implementable
- It requires bidders which will never win the auction to report truthfully
- Hence a more reasonable approach would be study auctions which requires revelation only from types that has a chance of winning the auction, or auctions with a fixed reserve price

Related literature

- Myerson (1981) and Cremer and McLean (1985) characterize the optimal mechanism for a known distribution under independent values and correlated values respectively
- Robust approach to mechanism design, e.g. Bergemann and Morris (2005), Carroll (2017), Hart and Nisan (2017), and survey by Carroll (2019)
- Sampling from distributions, e.g. Fu et al. (2021) for correlated distributions

Concluding remarks

- We characterize the optimal mechanisms in an environment where the true distribution is unknown to the seller
- The optimal mechanism is obtained as the solution to a properly defined virtual valuation maximization
- It could be interpreted as a second price auction with a changing reserve price

Future directions

- Modify set of distributions $\mathcal{F}$
- Ambiguity aversion
- General auctions with reserve price
- Optimal mechanism under BIC