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Principal Component Analysis in Quasi Monte Carlo Simulation

Alexander Kreinin, Leonid Merkoulovitch, Dan Rosen and Michael Zerbs

We apply Principal Component Analysis (PCA) to reduce the dimensionality of the risk factor space of a Monte Carlo simulation to compute Value-at-Risk. Dimensionality reduction can result in substantial computational savings when used in conjunction with Quasi Monte Carlo (QMC) methods. We also suggest a new approach, Portfolio PCA, that uses a new criterion for selecting the appropriate principal components based on portfolio sensitivities. In a simple application we demonstrate that the combined strategy of QMC and Portfolio PCA leads to speed-ups of four to twenty-seven times when compared to a standard Monte Carlo simulation.

The art of the Monte Carlo (MC) method contains many recipes that help to make simulation more efficient. For example, there are numerous academic papers on the application of Quasi Monte Carlo (QMC) methods to accelerate option pricing (see Broadie and Glasserman 1997; Boyle 1977; Schoenmakers and Heemink 1997). In a previous paper (Kreinin et al. 1998), we demonstrate that QMC leads to substantial computational time savings in portfolio risk measurement.

Although the advantages of QMC methods are clear for problems of moderate dimensionality, their benefit to high dimensional problems is a topic of active academic debate (Caflisch et al. 1997). The advantage of QMC methods based on low discrepancy sequences arises from a more effective placement of sample points that avoids cluster formation. Low discrepancy sequences usually work better in low dimensional spaces because their discrepancy depends on the dimension of the space. Hence, there is a strong motivation to reduce the dimensionality of the risk factor space before sampling.

In this paper, we apply Principal Component Analysis (PCA) techniques to reduce the dimensionality of the problem for Value-at-Risk (VaR) estimation. We demonstrate the computational savings that arise from applying QMC methods in conjunction with PCA.

Traditionally, PCA is applied to the covariance matrix of the risk factor log-returns. Dimensionality is reduced by selecting those components that explain most of the variance of the risk factors, without regard to the structure of the portfolio. However, it is not difficult to construct a portfolio which has a large position sensitive to risk factors that appear unimportant in the PCA. The error in the VaR estimation of such a portfolio may be very large when these factors are excluded *a priori.*

To counter this drawback, we propose a modification of PCA based on a different criterion to select principal components, which takes into account portfolio sensitivities. We call this approach Portfolio PCA.

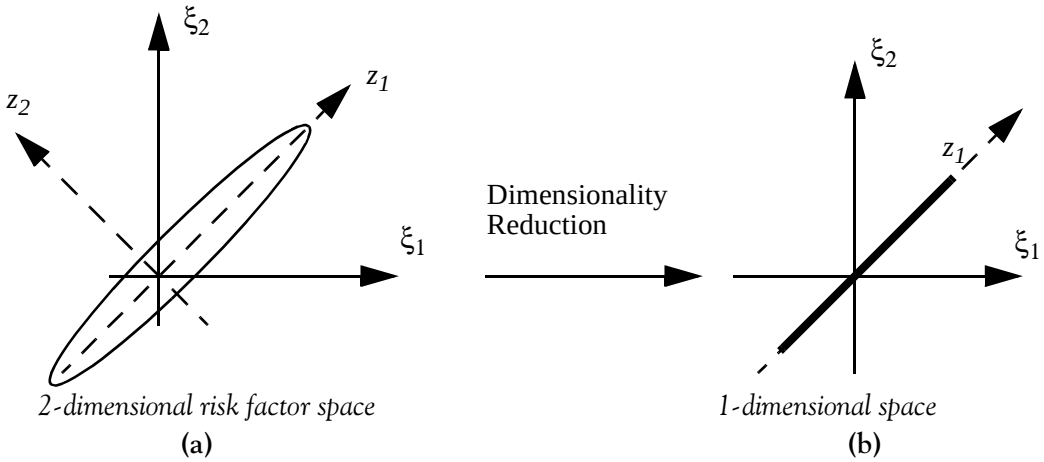


Figure 1: Dimensionality reduction based on PCA

The rest of the paper is organized as follows. In the next section, we describe PCA and discuss the application of this method to measure the risk of the test portfolio. Then we introduce Portfolio PCA and contrast the results of its application to the standard PCA. The last section of the paper contains concluding remarks and directions for future research.

Principal Component Analysis

Principal Component Analysis is a well known mathematical technique (Press et al. 1996) that is applied, in particular, to Monte Carlo simulation. It is commonly used in risk management applications to reduce the dimensionality of the risk factor space (Loretan 1997).

The idea behind PCA is simple. Consider a set of risk factors and the joint normal distribution of their log-returns. First, PCA obtains a new set of uncorrelated variables that are a linear combination of the log-returns. These variables are called principal components (PC). A Monte Carlo simulation of the log-returns is performed by generating independent random vectors of the principal components, and then using the linear transformation to obtain the correlated log-returns. The dimensionality of the simulation can be reduced by selecting only that subset of PCs that contribute the most to the variance of the log-returns.

This process is depicted for a two-dimensional case in Figure 1. The ellipse in Figure 1a represents a level curve of the joint density of the log-returns, ξ_1 and ξ_2, which are positively correlated. The principal components, z_1 and z_2, can be thought of as random variables varying over two new perpendicular axes for which the ellipse is aligned. In our example, the variable z_1 is more volatile than z_2; therefore, it contributes more to the overall variance of the risk factor space. The essence of the joint distribution of ξ_1 and ξ_2, is captured by the distribution of the first principal component, z_1, as illustrated in Figure 1b.

Let us consider this method in more detail. We assume that the random vector of risk factors in time Δt is described by the equation

$$r(\Delta t) = r^0 \exp(\sqrt{\Delta t}\xi) \qquad (1)$$

where r^0 is the current vector of the risk factors, ξ is the vector of log-returns with normal distribution, $N(0,\mathbf{Q})$, and $\mathbf{Q}$ is an $n \times n$ covariance matrix. The matrix $\mathbf{Q}$ must be symmetric positive semi-definite.

Let $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$ be the eigenvalues of $\mathbf{Q}$, ranked in decreasing order, and let $\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_n$ be the corresponding eigenvectors (since $\mathbf{Q}$ is symmetric positive semi-definite, the eigenvalues

$\lambda_j \geq 0$, $j=1,2,\ldots, n$, and the eigenvectors $\mathbf{U}_j$ can be orthonormalized); i.e., $\boldsymbol{Q}\boldsymbol{U}_j = \lambda_j \boldsymbol{U}_j$. This relation is easily rewritten in matrix form:

$$\boldsymbol{Q}\boldsymbol{U} = \boldsymbol{U}\Lambda \tag{2}$$

where $\Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$

The matrix $\mathbf{U}$ consists of the orthonormal eigenvectors $\mathbf{U}_j$; that is, $\mathbf{U}\mathbf{U}^T = \mathbf{U}^T\mathbf{U} = \mathbf{I}$. Therefore, Equation 2 implies

$$\boldsymbol{Q} = \boldsymbol{U}\Lambda\boldsymbol{U}^{\boldsymbol{T}} \tag{3}$$

Consider the linear transformation

$$\xi = \boldsymbol{U}\sqrt{\Lambda}\eta \tag{4}$$

where

$$\sqrt{\Lambda} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & \ldots & 0 \\ 0 & \sqrt{\lambda_2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \sqrt{\lambda_n} \end{bmatrix}$$

If $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ is a random vector with independent, normally-distributed components (and therefore, with the unit covariance matrix $\boldsymbol{I}$), then the vector, ξ, is a normal random vector that has the covariance matrix

$$Q_\xi = (U\sqrt{\Lambda})I(U\sqrt{\Lambda})^T = U\Lambda U^T = Q$$

Equation 4 can be rewritten as

$$\xi = \eta_1\sqrt{\lambda_1}U_1 + \eta_2\sqrt{\lambda_2}U_2 + \ldots + \eta_n\sqrt{\lambda_n}U_n \tag{5}$$

The random variables $z_j = \sqrt{\lambda_j}\eta_j$, $j=1,\ldots,n$ are called the **principal components** of the random variable ξ, the vector $\mathbf{U}_j$, $j=1,\ldots,n$ is referred to as **the direction of the j-th principal component** $\mathbf{z}_j$, and Equation 5 is the **principal component expansion of** ξ (i.e., a linear combination of the principal components).

Equation 5 plays a central role both in scenario generation and Principal Component Analysis. Based on Equation 5, we generate scenarios using the vector of independent standard normal random variables $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$. If the last several eigenvalues $\lambda_{k+1}, \ldots, \lambda_n$ are small, then the truncated vector

$$\hat{\xi} = \eta_1\sqrt{\lambda_1}U_1 + \eta_2\sqrt{\lambda_2}U_2 + \ldots + \eta_k\sqrt{\lambda_k}U_k \tag{6}$$

that uses only the vector $\hat{\eta} = (\eta_1, \eta_2, \ldots, \eta_k)$ will be a good approximation of the random vector ξ.

The following criterion is usually used for selection of the number of PCs:

Suppose that ε^* *is an admissible proportion of unexplained variance of the risk factor space. Then we select the minimal number, k, of principal components satisfying the inequality*

$$\frac{\lambda_1 + \ldots + \lambda_k}{\lambda_1 + \ldots + \lambda_n} > 1 - \varepsilon^* \tag{7}$$

The principal components with indices $j > k$ have a small effect on the underlying vector of risk factors since the corresponding eigenvalues are small.

This criterion specifies the ratio of the total variances of $\hat{\xi}$ and ξ. Indeed,

$$VaR(\hat{\xi}) = \sum_i VaR(\hat{\xi}_i) = E\|\hat{\xi}\|^2 = E\|\xi^T \cdot \hat{\xi}\|$$

$$= \sum_{j=1}^{k} \lambda_i E\left(\eta_i^2\right)(U_i^T, U_i) = \sum_{i=1}^{k} \lambda_i$$

Thus, the criterion in Inequality 7 represents the fraction of variance covered by the truncated random vector $\hat{\xi}$. Moreover, the error of this approximation is represented by the residual vector

$$\varepsilon = \|\xi - \hat{\xi}\| = \sum_{i=k+1}^{n} \sqrt{\lambda_i \cdot \eta_i \cdot U_i}$$

and we can calculate the expected norm of this error

$$E\|\varepsilon\|^2 = E\left\|\varepsilon^T \cdot \varepsilon\right\| = \sum_{i=k+1}^{n} \lambda_i E\left(\eta_i^2\right)(U_i^T, U_i) = \sum_{i=k+1}^{n} \lambda_i$$

Scenario generation using PCA and the QMC method

We now summarize the process of scenario generation using PCA. First, we notice that from Equation 1 and Equation 5 the model for the risk factor values at time Δt is

$$r(\Delta t) = r^0 exp(\sqrt{\Delta t}(\eta_1\sqrt{\lambda_1}U_1 + \eta_2\sqrt{\lambda_2}U_2 + \ldots + \eta_n\sqrt{\lambda_n}U_n))$$

The steps to generate scenarios are

1. Find the principal component decomposition of the covariance matrix Q of the risk factor log-returns (Equation 3). Modify, if necessary, negative eigenvalues of this matrix (for details see Kreinin and Levin 1998).
2. Given the relative error ε^*, define the dimension, *k*, of the truncated risk factor space using Inequality 7.
3. Generate a sample of points in the *k*-dimensional cube with a uniform distribution based either on pseudo-random numbers or on low discrepancy sequences.
4. Transform the sample from a uniform distribution into *k*-dimensional vectors with a multivariate normal distribution with uncorrelated components using a polynomial approximation of the inverse distribution function (see Devroye 1986). We obtain the sample $\hat{\eta} = (\eta_1, \ldots, \eta_k)$.
5. Apply the linear transformation defined by Equation 6 to obtain the correlated random vectors of log-returns, ξ, in the *n*-dimensional risk factor space.
6. Transform the vector of log-returns into scenarios in the risk factor space using the truncated equation

$$\tilde{r}(\Delta t) = r^0 exp(\sqrt{\Delta t}\xi) \qquad (8)$$

Equation 8 describes the evolution of the risk factors in the space of dimension *n*.

Applying QMC methods with PCA

In this section, we present the results of PCA on a simple fixed income portfolio. This sample portfolio has been previously used in the industry as a benchmark (Marshall and Seigel 1996).

The portfolio contains 14 fixed-rate government bonds in five currencies. The base currency of the portfolio is USD. The bonds have maturities ranging from 182 days to ten years. The mark-to-market value of the portfolio is 357.3 million USD.

The zero curves in each currency are modeled using 16 nodes, except for the JPY curve which has 15 nodes. Four risk factors describe the foreign exchange rate with the USD. The dimensionality of the risk factor space is $16 \times 15 + 4 = 83$. We note that an intrinsic characteristic of such covariance matrices is that the maximum number of independent risk factors is less than 40.

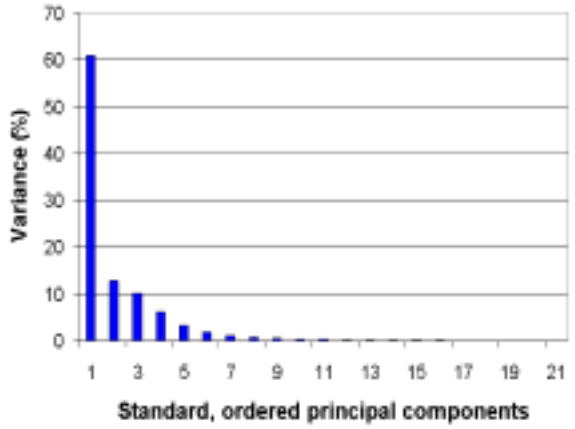
This test portfolio is used in a previous paper (Kreinin et al. 1998) in which we compare the performance of the QMC method to that of the standard MC method. The speed-up of the QMC method with respect to the MC method is defined as the ratio of the number of scenarios required to achieve an estimate of the required accuracy with MC simulation (itself a random variable) and with the QMC method. An accurate estimate of the VaR is obtained by using an essentially infinite number of scenarios.

It is shown that the QMC method generally outperforms the MC method. Speed-up factors of 6.7 and 2.1 are realized when calculating a *VaR(0.95)* estimate with an error of 2% and 5%, respectively. The speed-up factors are greater when a higher accuracy and a higher confidence level are required.

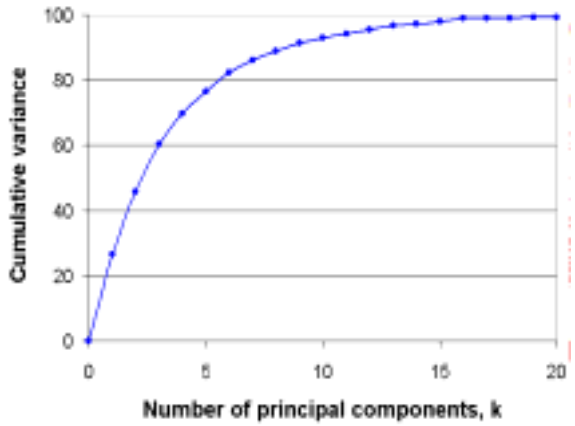
Let us now apply PCA to analyze the risk factor space. Figure 2a presents the fraction of the variance covered by *j*-th principal component,

$\lambda_j / \sum_{i=1}^{n} \lambda_i$, $(j = 1, 2, ..., 20)$ and Figure 2b the cumulative percentage variance $\theta(k) = \sum_{i=1}^{k} \lambda_i / \sum_{i=1}^{n} \lambda_i$, where k is the number of principal components, $k = 1, 2, ..., 20$.

We see, for example, that while the first principal component accounts for 60% of the variance, ten principal components describe 97.8% of the variance, 15 components explain 99% and 16 principal components contribute 99.3%.



(a)



(b)

Figure 2: Variance of the principal components

In Figure 3 we compare the relative error of the *VaR(0.95)* estimate varying the number of principal components and the number of QMC scenarios generated.

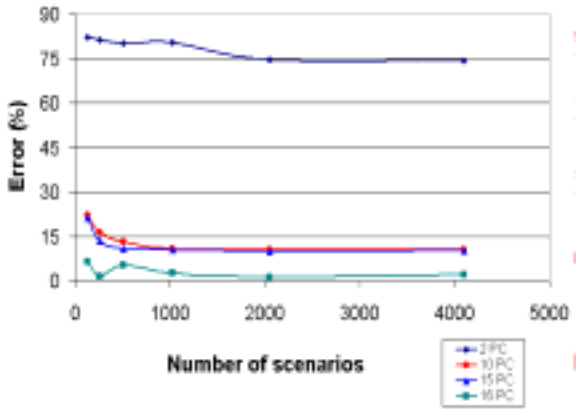


Figure 3: Error of VaR estimation using PCs

The choice of two PCs leads to a very high error regardless of the number of scenarios generated. With ten PCs, the error remains above 10%. Our experiments show that using 15 PCs does not lead to a significant improvement. This is in sharp contrast to Figure 2 where 15 PCs explain almost 100% of the variance of the risk factor space. The errors in the VaR abruptly decrease when the 16th PC is introduced. We can conclude that 16th PC contributes significantly to the portfolio VaR.

With 16 PCs, 1,024 scenarios yield an error of less than 3%. If a 5% error is acceptable, the required number of scenarios is only 512. This is an additional acceleration of approximately two times on top of the speed-up achieved by applying standard QMC on a full set of risk factors.

Thus, the results demonstrate three things. First, highly accurate VaR estimates can be achieved while substantially reducing the dimensionality of the problem (from 39 to 16 dimensions). Second, if we use enough PCs, PCA improves the performance of the QMC method.

Finally, it is not clear, *a priori*, how many and which PCs should be selected. In our example, though the first ten PCs explain almost 99% of the risk factor variance, they account for roughly less than 90% of the portfolio VaR. The portfolio turns out to be quite sensitive to the 16th principal component which accounts for almost 10% of the error in the VaR estimation.

However, standard PCA does not identify this principal component as highly significant.

Portfolio Principal Component Analysis

Portfolio Principal Component Analysis is a modification of the standard PCA that is designed to overcome the drawbacks described above. Both methods use the same principal components but they differ in the manner in which the principal components are ranked.

The idea driving Portfolio PCA is to rank the principal components obtained by PCA with respect to their contributions to the portfolio risk. Of course, this requires that we know *a priori* the contribution of each component to portfolio risk. For this purpose we use portfolio sensitivities. More precisely, we rank the PCs with respect to their contribution to a first-order approximation of portfolio variance. Note that portfolio linearization is used exclusively for ranking the principal components; a full simulation that captures the non-linear features of the portfolios is performed to calculate the VaR (a simulation would not be required if the whole portfolio was truly linear).

Let $z = (z_1, \ldots, z_n)^T$ be the vector of the principal components of the random vector, ξ, and let

$$P_z = \frac{\partial P}{\partial z}(0) = \left(\frac{\partial P}{\partial z_1}, \ldots, \frac{\partial P}{\partial z_n}\right)\Big|_{z=0}$$

be the row vector of the portfolio sensitivities with respect to the principal components at point $z = 0$. Then the portfolio differential $dP = P_z \cdot dz$, and, therefore, its variance satisfies the relation:

$$Var(dP) = P_z \Lambda P_z^T = \sum_{j=1}^{n} \left(\frac{\partial P}{\partial z_j}(0)\right)^2 \lambda_j \qquad (9)$$

Thus, the value

$$\mu_j = \left(\frac{\partial P}{\partial z_j}(0)\right)^2 \lambda_j \qquad (10)$$

is the contribution of the j-th principal component to the variance of dP.

To select the Portfolio PCs, we use the following criterion which is based on portfolio variance, instead of the criterion defined by Inequality 7, which is based on risk factor variance. Let us reorder the variances, μ_j:

$$\mu_{i_1} \geq \mu_{i_2} \geq \ldots \geq \mu_{i_n}$$

Suppose that ε^* *is an admissible proportion of the unexplained portfolio variance. Then, to form the truncated space we select the minimal number, k, of the portfolio principal components satisfying the inequality*

$$\frac{\mu_{i_1} + \ldots + \mu_{i_k}}{\mu_1 + \ldots + \mu_n} > 1 - \varepsilon^* \qquad (11)$$

This criterion is more natural for portfolio VaR problems: it represents the ratio of the portfolio variance covered by the first k PCs, according to the reordered variances, and the full portfolio variance given by Equation 9.

The greater the value of μ_j in Inequality 11, the greater its impact on portfolio variance.

Equation 10 describes the portfolio sensitivity to the principal components, not its sensitivity to the original risk factors. Using Equations 1 and 5 we can obtain from Equation 10 an expression for μ_j in terms of the portfolio sensitivities to the original risk factors.

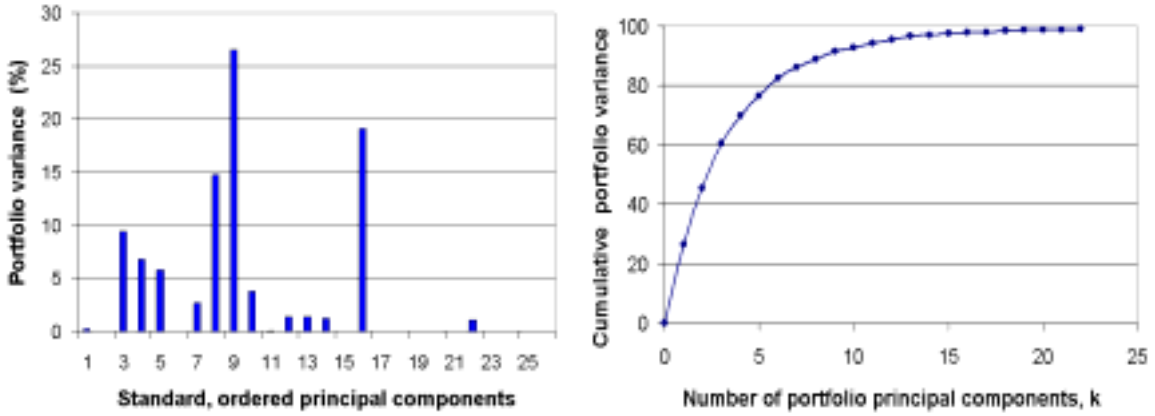
The portfolio sensitivities satisfy the relation

$$P_z(0) = \frac{\partial P}{\partial z}(0) = \frac{\partial P}{\partial r}(r^0) \cdot \frac{\partial r}{\partial \xi}(0) \cdot \frac{\partial \xi}{\partial z}(0)$$

But $\frac{\partial r}{\partial \xi}(0) = diag(r^0)\sqrt{\Delta t}$ (see Equation 1) and $\frac{\partial \xi}{\partial z}(0) = U$ (see Equation 5). Therefore:

$$P_z(0) = \frac{\partial P}{\partial r}(r^0) \cdot diag(r^0) \cdot \sqrt{\Delta t} \cdot U$$

$$= \sqrt{\Delta t} \cdot \left(\frac{\partial P}{\partial r_1} r_1, \ldots, \frac{\partial P}{\partial r_n} r_n\right)\Big|_{r=r^0} \cdot U_j$$



(a) (b)

Figure 4: Portfolio variance

and, from Equation 10 we derive

$$\mu_j = \left(\frac{\partial P}{\partial z_j}(0)\right)^2 \lambda_j = \left(\sqrt{\Delta t} \cdot \left(\frac{\partial P}{\partial r_1} r_1, \ldots, \frac{\partial P}{\partial r_n} r_n\right)\Big|_{r = r^0} \cdot U_j\right)^2 \lambda_j$$

Denote $V^0 = \left(\frac{\partial P}{\partial r_1} r_1, \ldots, \frac{\partial P}{\partial r_n} r_n\right)\Big|_{r = r^0}$. Finally, we obtain

$$\mu_j = \Delta t \left(V^0 U_j\right)^2 \lambda_j$$

Applying Portfolio PCA with QMC methods

In this section, we analyze the test portfolio introduced above using Portfolio PCA.

First, we find the contributions of each component to the portfolio variance. Figure 4a presents these contributions, expressed as a percent of total portfolio variance (the components in Figure 4a have the same order as those in Figure 2a). The most important is the ninth component: it covers 27% of the portfolio variance. However, this component accounts for only 1% of the variance of the risk factors. The 16th component is the second most important component, accounting for almost 20% of the portfolio variance. This is consistent with the simulation results presented in Figure 3. On the other hand, the first, second and sixth principal components have comparably small contributions to the portfolio variance.

We now rank the PCs according to their contribution to portfolio variance. Figure 4b displays the cumulative percentage of variance covered by the first *k* Portfolio PCs. We show that Figure 4b more realistically reflects the PCs required for VaR estimation.

Figure 5 shows the relative error of the *VaR(0.95)* estimation as a function of the number of scenarios for various experiments, based on different numbers of Portfolio PCs.

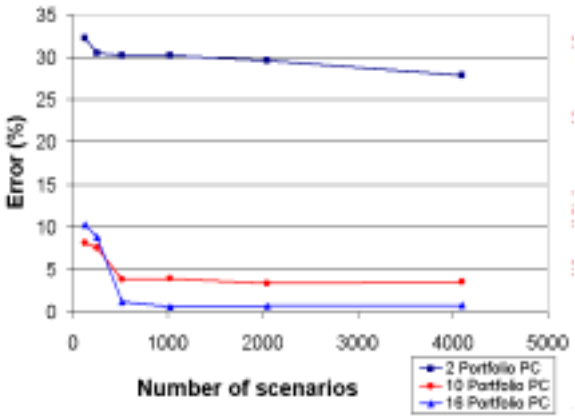


Figure 5: Error of VaR estimation using Portfolio PCs

Comparing Figures 3 and 5, it is clear that the Portfolio PCA results in smaller errors, as expected. For example, if two Portfolio PCs are used, the error is about 27% as opposed to 72% in the case of the standard PCs. With ten portfolio PCs the error decreases from 10% to almost 4%.

Finally, using 16 Portfolio PCs, we can guarantee that the error stays below 2% with 512 scenarios (half as many scenarios as required with the standard PCs).

Recall that the speed-up of the QMC method with respect to the MC method is defined as the ratio of the number of scenarios required to achieve an estimate of the required accuracy with MC simulation and with the QMC method, and that speed-up factors of 6.7 and 2.1 are realized when calculating a *VaR(0.95)* estimate with an error of 2% and 5%, respectively.

Table 1 presents the speed-ups obtained with QMC methods in conjunction with PCA and Portfolio PCA for *VaR(0.95)* at 2% and 5% error levels in the estimation.

Method	Error 2%	Error 5%
QMC, 40PC	6.7	2.1
QMC, 16PC	6.7	4.2
QMC, 16 Portfolio PC	26.8	4.2

Table 1: Speed-ups compared to standard MC

If 16 PCs are selected for VaR estimation with 2% error, there is no additional acceleration on top of QMC with 40 PCs. The number of scenarios required is reduced by half if the error of the VaR estimate is 5%, leading to a speed-up of a factor of two. There is no additional improvement realized by using Portfolio PC. However, if 16 Portfolio PC are selected, then VaR estimation at 2% error can be performed in one quarter the time of QMC with 40 PCs.

These results demonstrate two points. First, the use of Portfolio PCA yields additional improvements in computational efficiency over the standard PCA. Second, the technique clearly dominates standard PCA, and can be used as a more reliable guideline for choosing the number of components for simulation.

Figure 6 shows the error of the *VaR(0.95)* estimate as a function of the number of PCs, for both standard and Portfolio PCs (with appropriate ranking of PCs). Using ten Portfolio PCs, the VaR error is in the order of 5% as predicted by Figure 4b.

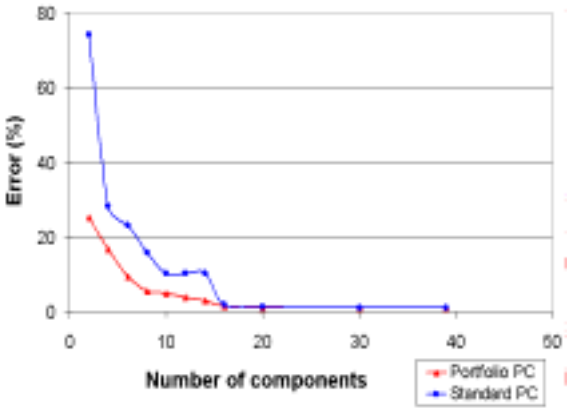


Figure 6: Error of VaR(0.95); Portfolio PC vs. standard PC

Concluding remarks

Dimensionality reduction plays an important role in Monte Carlo simulation of portfolio risk. However, standard application of Principal Component Analysis does not provide a reliable solution to this problem. Indeed, as we have demonstrated, it is difficult to know *a priori,* how many principal components to select to guarantee that the VaR will be found with the required accuracy.

To solve this problem, we have introduced the Portfolio Principal Components method. This method uses information about the portfolio to select the principal components for scenario generation.This approach is based on ranking the PCs by their contribution to the portfolio variance. By selecting enough Portfolio PCs, we guarantee that the portfolio variance is explained with the desired accuracy.

We demonstrate that the dimensionality reduction based on Portfolio PCA results in substantial computational savings when used in conjunction with QMC methods. This was expected since the convergence of QMC methods depends on the dimensionality of the problem.

The Portfolio PCA criterion is based on a linearization of the portfolio. Although

simulation of the portfolio captures the non-linear features, the method as presented may still cause problems for portfolios with strong non-linearities and small linear contributions (e.g., when a part of the portfolio is perfectly delta hedged and has significant gamma contributions). In such cases, it is possible that the Portfolio PCA method will exclude an important principal component from the simulation. Generally, a straightforward inspection based on some qualitative knowledge of the portfolio can indicate whether components should be included in the simulation. Furthermore, formal extensions of the Portfolio PCA method where the contribution of a principal component to the portfolio variance is adjusted by higher order derivatives (e.g., gamma) are possible. This is the subject of future work.

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