

LECTURE 8

Matrices

↳ "rank"

$$\text{rank 1 matrix} = \begin{bmatrix} v \\ u \end{bmatrix}^T = (uv^T)$$

$$\boxed{(uv^T)_{ij} = u_i v_j}$$

Def'n : A matrix M is rank K

$$M = \sum_{i=1}^K M_i = \underbrace{\sum_{i=1}^K u_i v_i^T}_{\text{rank}(M_i) = 1}$$

$\& M \neq \sum_{i=1}^{K+1} M_i \quad \text{rank}(M_i) = 1$

=

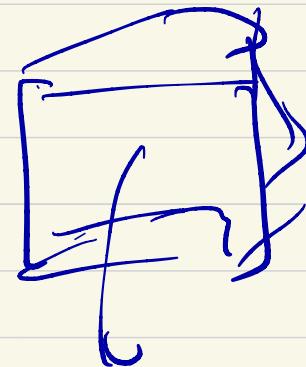
TENSORS:

$$u \otimes v = uv^T$$

Notation:

$$u \otimes v \otimes w$$

$$u, v, w \in \mathbb{R}^n$$



$$u_i v_j w_k$$

Rank 1 tensor: $u \otimes v \otimes w$ for $u, v, w \in \mathbb{R}^n$

Defn: Rank of a tensor T is the smallest K

such that $T = \sum_{i=1}^K \underbrace{u_i \otimes v_i \otimes w_i}_{k \cdot (3n) \text{ numbers}}$

Maximum

Rank of $[n] \times [n] \times [n]$ tensor

$$\overbrace{n^3}$$

$$\geq \underline{\underline{\Omega(n^2)}}$$

Rank:

1) Rank depends on R/C values

2) For Matrices:

A sequence of rank K matrices

$M_1, M_2, \dots, M_t, \dots$

$\lim_{t \rightarrow \infty} M_t$ is also a rank K matrix

\exists tensors T s.t $\text{rank}(T) \gg \text{large}$

but $H \in \mathcal{J}^T$ $\text{rank}(T) \ll \text{small}$

$$\|T - H\| < \varepsilon \quad \underline{\underline{=}}$$

3) NP-hard to compute.

Eigen decomposition

real symmetric
Matrix $M =$

$$\sum \lambda_i v_i v_i^T$$

where each v_i is an eigenvector
with eigen value λ_i

2) v_i are orthogonal to each other

Suppose a tensor $T = \sum_{i=1}^n a_i^{\otimes 3} = a_i \underbrace{\otimes a_i \otimes a_i}$

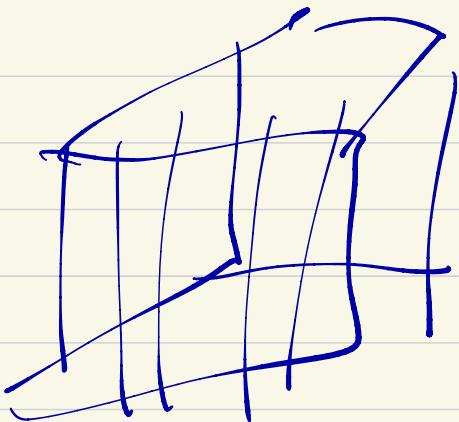
where $\{a_i\}$ are orthogonal vectors

Recover: $\{a_i\}$

$$T = \sum_{i=1}^n a_i \otimes a_i \otimes a_i$$

Pick a random vector
 $g \in \mathbb{R}^n$

$$[T[g, :, :]] = \sum_{i=1}^n \langle a_i, g \rangle \cdot a_i \otimes a_i$$



$$T[g, \cdot, h] = \sum_{i=1}^n \langle a_i, g \rangle \cdot a_i \cdot \langle a_i, h \rangle$$

$$\left(\sum_{i=1}^n \langle a_i, g \rangle \cdot \underbrace{a_i \otimes a_i^T}_{a_i a_i^T} \right) a_i = \text{circle with } a_i \text{ inside}$$

- eigen vectors $\{a_i\}$

- eigenvalues $(a_i) \underbrace{\langle a_i, g \rangle}_{j \neq i} \rightarrow \text{random numbers}$

$$\sum_{j=1}^n \langle a_j, g \rangle (a_j \otimes a_j^T) \cdot a_i =$$

$$\begin{cases} \langle a_i, g \rangle a_i & j=i \\ 0 & j \neq i \end{cases}$$

$A, B, C \in \mathbb{R}^{n \times n}$

$T \in \mathbb{R}^{(n) \times (n) \times (n)}$

$(A \circ B \circ C) T$

$$T[x, x, x] = \sum_{\parallel} T_{ijk} x_i x_j x_k$$

$$\sum a_i^{\otimes 3} \underbrace{\sum_{i=1}^n \langle a_i, x \rangle^3}_{\parallel}$$

Theorem: $T[x, x, x]$

1) Local Maxima on the unit ball

are a_1, \dots, a_n

Jenrich's Algorithm

INPUT : $T = \sum_{i=1}^{\text{rank } n} u_i \otimes v_i \otimes w_i$

- 1) $\{v_1 \dots v_r\}$ are linearly independent
- 2) $\{w_1 \dots w_r\}$ are linearly independent
- 3) u_1, \dots, u_r are distinct.

Goal: $u_1 \dots u_r \quad v_1 \dots v_r \quad w_1 \dots w_r$

Alg: Pick a random $g \in \mathbb{R}^n$

$$M_g = \sum_{i=1}^r \langle u_i, g \rangle \cdot v_i \otimes w_i \equiv \underline{\underline{VD_g W}}$$

Pick $h \in \mathbb{R}^n$

$$M_h = \sum \langle u_i, h \rangle \cdot v_i \otimes w_i \equiv \underline{\underline{VD_h W}}$$

$$V = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$$

$$D_g = \begin{bmatrix} \langle u_i, g \rangle & 0 \\ \vdots & \ddots \langle u_i, g \rangle \\ 0 & \vdots \end{bmatrix}$$

$$W = \begin{bmatrix} \overbrace{\quad}^{\omega_1} \\ \overbrace{\quad}^{\omega_2} \\ \overbrace{\quad}^{\omega_3} \\ \vdots \\ \overbrace{\quad}^{\omega_n} \end{bmatrix}$$

$$M_g = V D_g W$$

$$M_h = V D_h W$$

$$M_g M_h^{-1} = (V D_g W) (W^{-1} D_h^{-1} V^{-1})$$

$$= \boxed{V (D_g D_h^{-1}) V^{-1}}$$

diagonal

Observation $M = PDP^{-1}$ for
some invertible P then
(eigenvectors of $M = \text{columns of } P$)

→
eigenvectors of $M_g M_n^{-1}$

= columns of V

v_1, \dots, v_n

eigenvectors $(M_g)^T - (M_n^T)^{-1}$

= rows of W

Solve a linear system to recover U .

Independent Component Analysis

Input

Samples

$$\underline{y} = A\underline{x} + b$$

Unknown invertible
linear transformation

$\uparrow \in \mathbb{R}^n$
random vector with
 $\{+1\}$ values
~~values~~

Unknown
shift

Goal: Recover A, b up to permutation
 \Downarrow recover \underline{x} from \underline{y}

Cocktail Party Problem

$$\rightarrow \begin{array}{l} y = Ax + b \\ \hline y_1 \dots y_T \end{array} \quad \left| \begin{array}{l} E[y] = E[Ax] + E[b] \\ 0 = b \end{array} \right.$$

[Centering] $\hat{y}_i = y_i - \bar{y}_i$

= ↑
removing the mean

After "centering" ($b=0$)

$$y = Ax \quad / \quad \underline{\underline{E[y]=0}}$$

[Whitening]

$$\text{Cov}(y) = E[(y - E[y])(y - E[y])^T]$$

$$= E[yy^T] = E[Axx^TA^T]$$

$$= A E[x x^T] A^T$$

$\text{Cov}(y)$	$= A A^T$
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$$E[xx^T] = Id$$

$E(x_i x_j)$
 random \pm values

$$E(x_i x_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

From AA^T impossible to recover A ,

$$AA^T = (\underline{AU})(AU)^T$$

for a rotation
matrix U

$$\underline{\underline{U^T U = Id}}$$

Estimate

$$M_4 = E \left(\underbrace{y \otimes y \otimes y \otimes y}_{\text{random 4 tensor}} \right) = T$$

$$\begin{aligned} [T]_{a,b,c,d} &= E[y_a y_b] \cdot E[y_c y_d] \\ &+ \overbrace{E[y_b y_c]}^{\uparrow} \cdot \overbrace{E[y_a y_d]}^{\uparrow} \\ &+ \overbrace{E[y_a y_c]}^{\uparrow} \cdot \overbrace{E[y_b y_d]}^{\uparrow} \end{aligned}$$

estimate

$$M_4 = \sum_{i=1}^n k_i \cdot \underbrace{(A_i \otimes A_i \otimes A_i \otimes A_i)}_{\text{columns of } A}$$

$(E[x_i^4] - 3)$

$\parallel -3 = -2$

(recover A_i by Jenrich's algo)

If $k_i \neq 0$

$$E[x_i^4] = 3 \Leftrightarrow \text{algorithm fails}$$

If x is Gaussian

↳ \Rightarrow impossible to recover A .
↳ $E[x_i^4] = 3$

$$E[x_i] = 0 \quad E[x_i^2] = 1 \quad E[x_i^4] = 3$$



x_i is Gaussian