

## LECTURE 14

### SPECTRAL GRAPH THEORY

↳ Understanding graphs via eigenvalues & eigenvectors.

$$G = (V, E)$$

Laplacian  $L_G(x) = \sum_{(i,j) \in E} (x_i - x_j)^2 \geq 0$

$$L_G(x) = x^T L_G x$$

||

$$D - A'$$

$D \equiv$

$$\begin{bmatrix} d_1 & 0 \\ d_2 & \ddots \\ 0 & d_n \end{bmatrix}$$

Degree matrix

Adjacency

## Observation:

1)  $h_{ii} \neq 0$  real symmetric

$$0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$$

$v_1 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

$$v_1 = \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}}_{\text{const vector}}$$

$$x^T L_A x = \sum_{(i,j) \in E} (x_i - x_j)^2 = 0$$

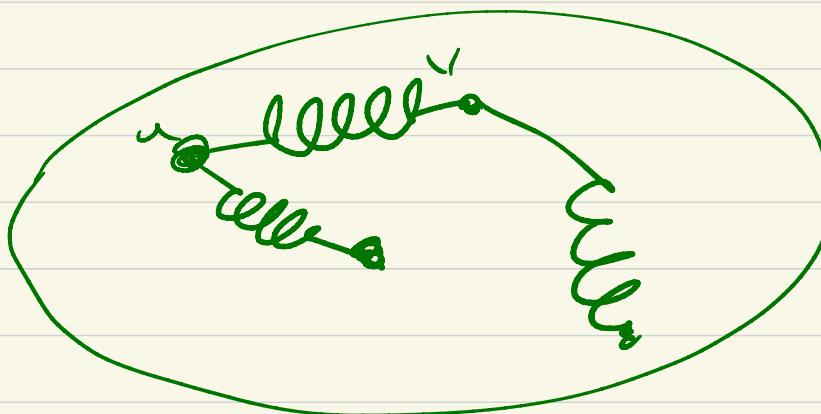
[Courant-Finsch] Let  $M$  be a real symmetric matrix  $\lambda_1 \leq \lambda_2 \dots \leq \lambda_n$

$$\lambda_K = \min_{\substack{V \subseteq \mathbb{R}^n \\ \dim(V)=K}} \left[ \max_{x \in V} \frac{x^T M x}{x^T x} \right]$$

Exercise: Prove it

$$\lambda_K = \min_{\substack{V \in \mathbb{R}^n \\ \dim(V)=k}} \max_{x \in V}$$

$$\sum_{(i,j) \in E} (x_i - x_j)^2$$

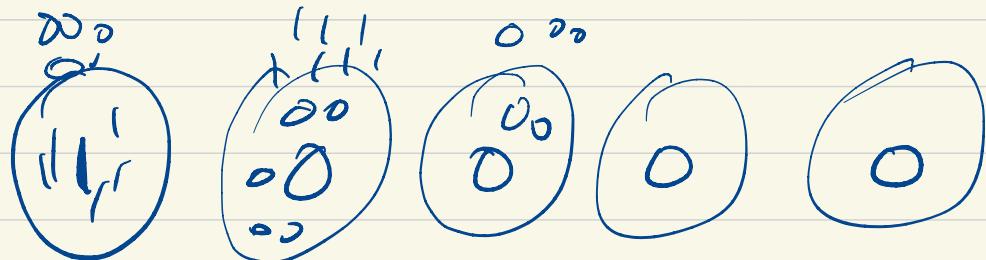


$$(v_2, v_3) : V \rightarrow \mathbb{R}^2$$

Lemma: G has  $\geq k$  connected components  $\Updownarrow \lambda_k = 0$

Proof: G has  $k$ -connected components

$$S_1 \cup S_2 \cup \dots \cup S_k$$



$$W_i \perp W_j$$

$$W_1 = \frac{1}{|S_1|} \sum_{i \in S_1} w_i$$

$$W_2 = \frac{1}{|S_2|} \sum_{i \in S_2} w_i$$

$$W_k = \frac{1}{|S_k|} \sum_{i \in S_k} w_i$$

$$W_a^T \perp W_{\bar{a}}^T = \sum_{(i,j)} (w_{ai} - w_{aj})^2 = 0$$

$\text{Span}\{\omega_1, \dots, \omega_k\}$        $\omega_i \perp \omega_j$

$$\forall i \quad \frac{\omega_i^\top L \omega_i}{\omega_i^\top \omega_i} = 0$$

$$\Rightarrow \lambda_k = 0$$

$$\lambda_k = 0 \quad \lambda_1 = 0 \quad \lambda_2 = 0 \dots \lambda_{k-1} = 0$$

$\omega_k$                    $\omega_1$                    $\omega_{k-1}$

$\Rightarrow$   $k$ -connected components

$$v_i \rightarrow (\omega_{1i}, \omega_{2i}, \dots, \omega_{ki}) \in \mathbb{R}^k$$



# CHEEGER INEQUALITY

$\lambda_1 = 0 \Leftrightarrow \text{Graph } G$

If  $\lambda_2 = \epsilon$  then  $G$  has two

components sparsely connected to each other.

$$\phi(S) = \frac{E[S, V \setminus S]}{\text{Vol}(S)} \in [0, 1]$$

$$\text{Vol}(S) = |S|$$

$$\boxed{\phi(G) = \min_{\substack{|S| \leq n/2 \\ S \subseteq V}} \phi(S)}$$

Normalized Laplacian: For a d-regular graph

$$\tilde{L}_a = \frac{1}{d} \cdot L_a$$

[0, 2]  
(0, . . . , 2d)

Cheeger's Inequality

$$\frac{\lambda_2(\tilde{L}_a)}{2} \leq \phi(a) \leq \sqrt{2\lambda_2(\tilde{L}_a)}$$

easy  
normalised  
Laplacian

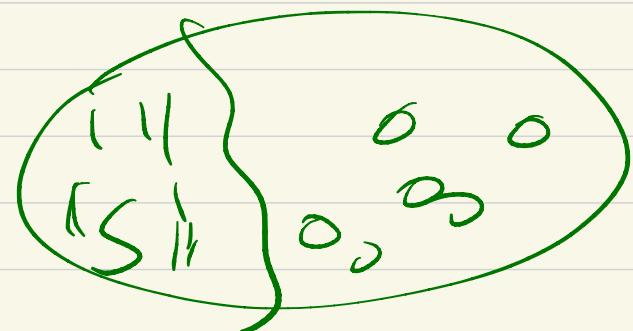
$$\text{Thm: } \phi(G) \geq \boxed{\frac{\lambda_2(\tilde{L}_G)}{2}}$$

$$\text{Let } S \subseteq V \quad \phi(G) = \frac{E(S, V \setminus S)}{d|S|}$$

$$\underline{1}_S \rightarrow \underline{1}_S^T \tilde{L}_G \underline{1}_S = \frac{1}{d} \sum_{(i,j) \in E} [l_S(i) - \underline{1}_S(j)]^2$$

↙

$$= \frac{E[S, V \setminus S]}{d}$$



$$\underline{1}_S^T \underline{1}_S = \sum_i [1_S(i)]^2 = |S|$$

$$\frac{\underline{1}_S^T \tilde{L}_G \underline{1}_S}{\underline{1}_S^T \underline{1}_S} = \frac{E[S, V \setminus S]}{d|S|} = \phi(S)$$

$$|S| = 8 \cdot n$$

$$\boxed{w = \mathbf{1}_S - 8\mathbf{\bar{1}}} \quad \text{all } \mathbf{v} \text{ vector}$$

$$\begin{aligned}\langle w, \mathbf{1} \rangle &= \underbrace{\langle \mathbf{1}_S, \mathbf{1} \rangle}_{n} - 8 \underbrace{\langle \mathbf{\bar{1}}, \mathbf{1} \rangle}_{0} \\ &= |S| - 8n = 0\end{aligned}$$

$$\frac{w^T \tilde{L}_G w}{w^T w} \leq 2 \frac{E[S, V \setminus S]}{d(|S|)} = 2 \phi(S) =$$

$$\Rightarrow 2 \cdot \phi(S) \geq \lambda_2$$

$$\boxed{\phi(S) \geq \lambda_2 / 2}$$

$$\phi(s) \leq \sqrt{2\lambda_2(L)}$$

Let  $x \in \mathbb{R}^n$   $\frac{x^T L_a x}{x^T x} = \lambda_2(L_a)$

Goal: Recover a set  $S \subseteq V$   $\phi(s) \leq \sqrt{2\lambda_2}$

→ Construct  $y \in \mathbb{R}^n$  by centering  $x \in \mathbb{R}^n$

$$y = x + \alpha \cdot \vec{1}$$

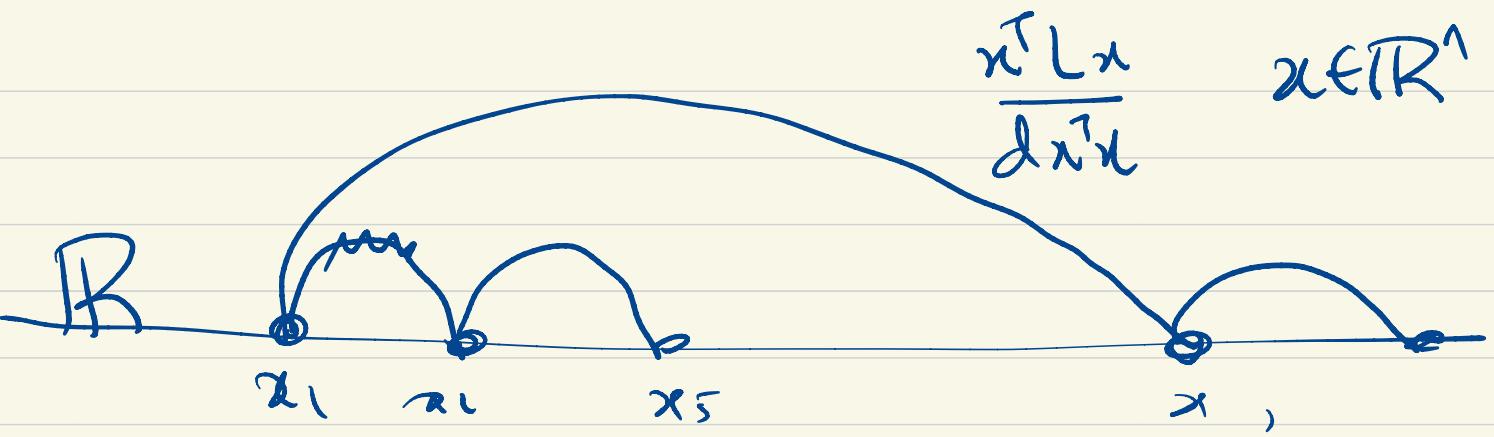
so  $n/2$  vertices have  $y_i \leq 0$   
 $n/2$  have  $y_i > 0$

→ Scale  $z = \beta \cdot y$  so that

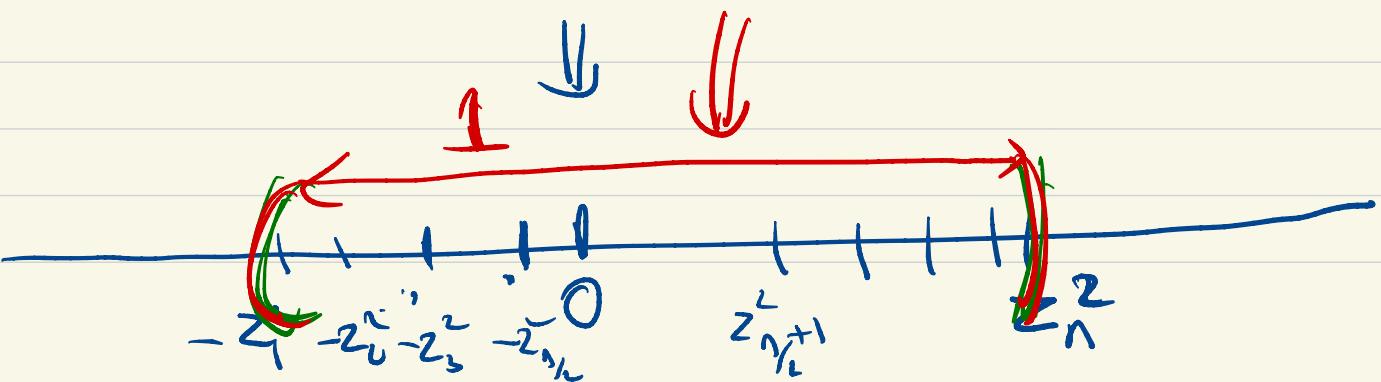
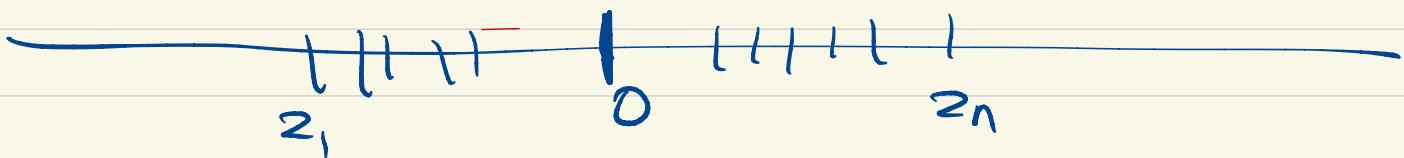
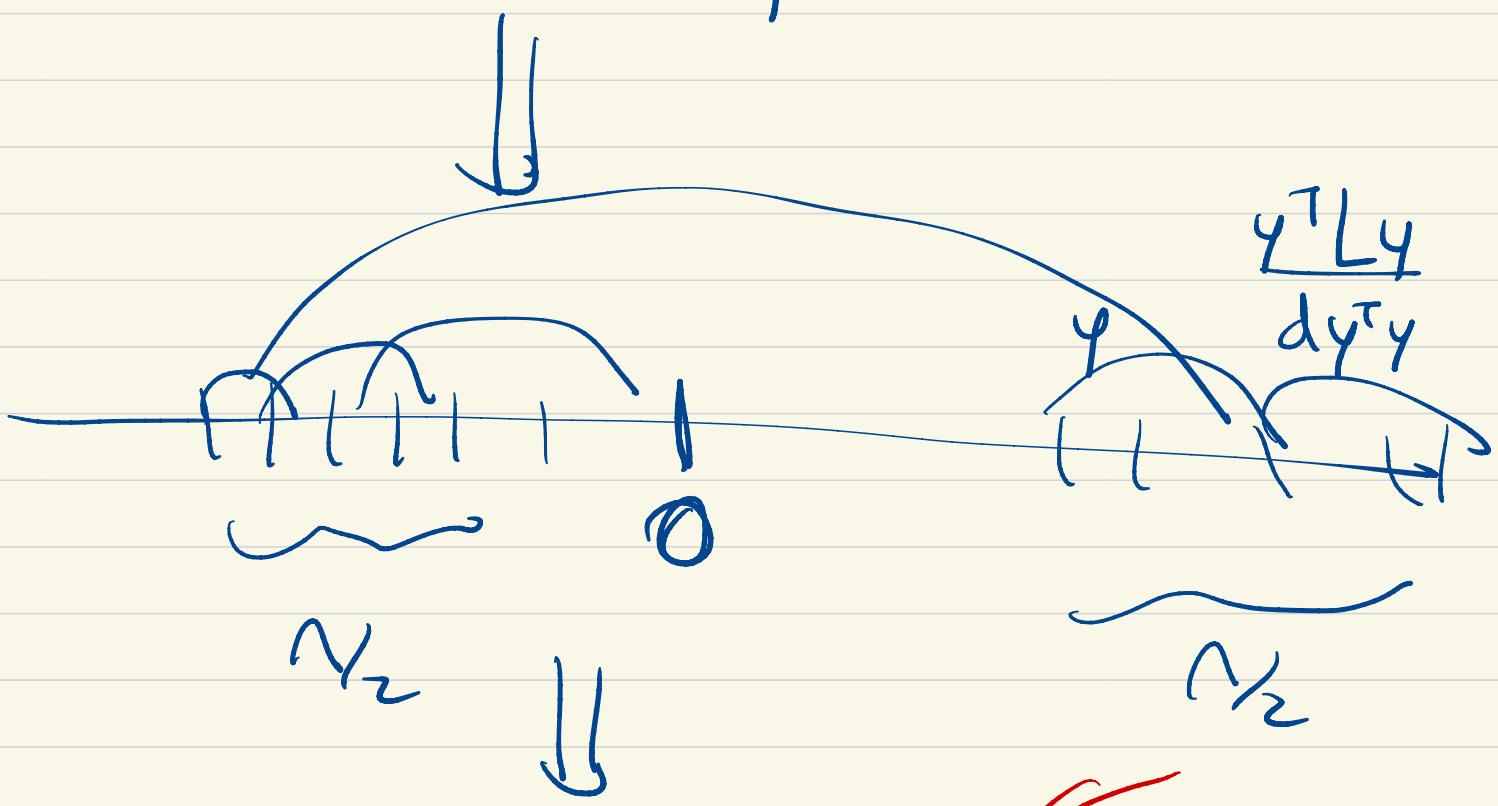
$$z_1^2 + z_n^2 = 1$$

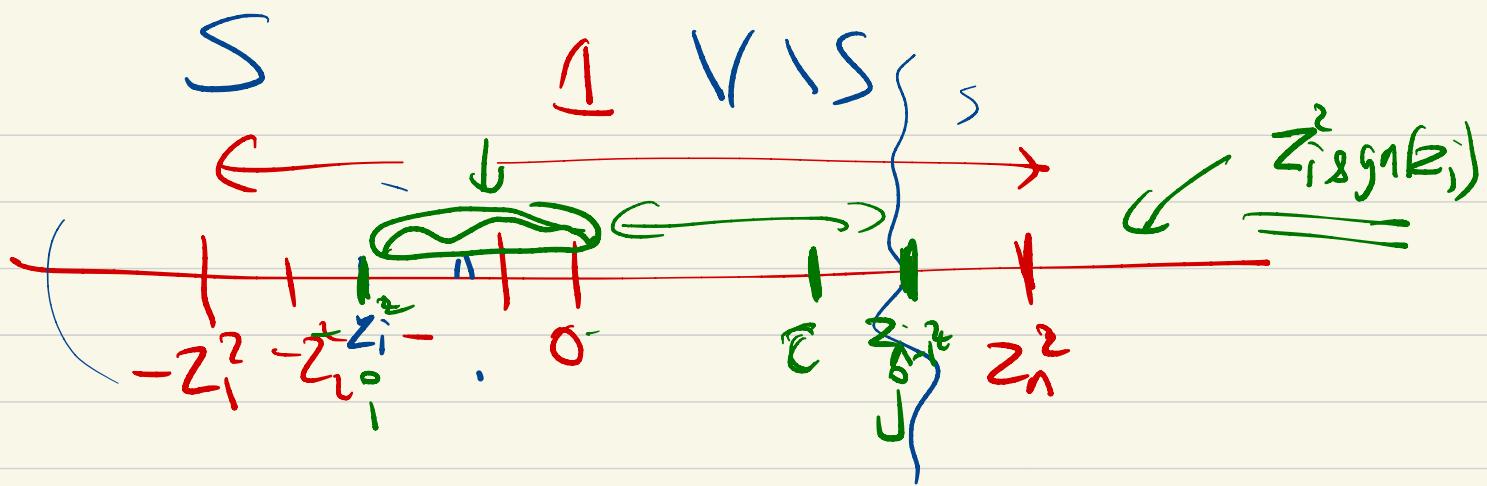
$$\frac{z^T L_a z}{z^T z} = \lambda_2(L_a)$$

→ Pick a random threshold  $t \in [-z_1^2, z_n^2]$



$$y = x + \alpha \cdot \vec{I}$$





Claim:  $\Pr[i \in S] = z_i^2$

Proof:  $i \in S$  if and only if

$\epsilon$  falls between  $i$  and  $0$

$$\boxed{\mathbb{E}[|S|] = \sum_{i=1}^n z_i^2} \rightarrow (2)$$

Claim:  $\Pr[(i, j) \text{ in cut}] = |z_i^2 \operatorname{sgn}(z_i) - z_j^2 \operatorname{sgn}(z_j)|$

Proof:  $(i, j)$  is cut if  $\epsilon$  falls between  $i$  &  $j$

$$\left| \sum_i^2 \text{sgn}(z_i) - z_j^2 \text{sgn}(z_j) \right| \leq \|z_i - z_j\| \cdot (|z_i| + |z_j|)$$

$$E[\text{E}(S)] = \sum_{(i,j) \in E} \Pr[(i,j) \text{ in cut}]$$

$$\leq \sum_{(i,j) \in E} |z_i - z_j| \cdot (|z_i| + |z_j|)$$

(Cauchy Schwartz)  $\rightarrow (1)$

$$\leq \left( \sum_{(i,j) \in E} |z_i - z_j|^2 \right)^{1/2} \left( \sum_{(i,j) \in E} (|z_i| + |z_j|)^2 \right)^{1/2}$$

$$\leq (z^T z)^{1/2} \left[ 2 \sum_{(i,j) \in E} (\underline{z_i^2 + z_j^2}) \right]^{1/2}$$

$$\begin{aligned} & \overbrace{(a+b)^2}^{\uparrow} \\ & \leq 2a^2 + 2b^2 \end{aligned}$$

$$E[E(S \setminus V(S))]$$

$$\leq (z^\top L z)^{1/2} \left[ 2 \cdot d \sum z_i^2 \right]^{1/2}$$

$$E(S) = \sum z_i^2$$

$$\frac{E[E(S \setminus V(S))]}{E[d|S|]} = \frac{(z^\top L z)^{1/2}}{\frac{\sqrt{2d \sum z_i^2}}{(d \sum z_i^2)}}$$

$$\leq \sqrt{2 \left( \frac{z^\top L z}{d \sum z_i^2} \right)}$$

$$= \sqrt{2 \lambda_2(\tilde{L}_n)}$$