CS 270 Algorithms Spring 2021

Lecture 16: Effective resistance, commute time, solving Laplacian systems

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16.1 Electrical Networks

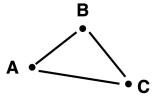
We have already studied spring networks and random walks. Today we will study electrical networks. An electrical network is an undirected graph where each edge is a resistor. Though these resistances can be arbitrary, for simplicity we will consider the case where R=1 for all resistors.

16.1.1 Electrical flows

Given an electrical network, we can define an electrical flow where we assign a directed current through each edge. This electrical flow must satisfy three laws of physics:

- 1. **Kirchoff's current law**: Current/flow into a node (vertex) equals current/flow out. This is also true for flow networks generally.
- 2. Ohm's law: Voltage across an edge is equal to current \times resistance (V = IR).
- 3. Kirchoff's voltage law: Voltage difference across a closed path (cycle) is 0.

The effect of Kirchoff's voltage law is that the flow is very constrained. For instance, if we have a network of nodes $\{A, B, C\}$ that are all connected to each other, with A having the highest voltage and C the lowest, the path $A \to C$ must have the same voltage drop as $A \to B \to C$, meaning $A \to B$ and $B \to C$ would have half the current as $A \to C$.



16.1.2 Converting between current and voltage

Consider now the matrix B, which converts the voltages at vertices to currents on edges. $I_{A\to B}=\frac{V_A-V_B}{R_{AB}}=V_A-V_B$, because of our assumption that all resistances are 1.

If m is the number of edges and n is the number of vertices, B is an $m \times n$ matrix where each row represents an edge $A \to B$ and thus has its A-th column set to 1 and B-th column set to -1.

Then Bv = f where v is a matrix of vertex voltages and f is the matrix of edge currents. It follows that the set of valid electrical flows (edge currents) is the column span of B since it must be constructable via this equation. The set of electrical flows is a subset of the subspace of all valid flows:

$$colspan(B) \subseteq space of flows \subseteq R^E$$

16.1.3 Incoming flows

Suppose we instead fix net flows b_i for each node i (so $b_i > 0$ for a current source, $b_i < 0$ for a current sink, and $b_i = 0$ for neither). It must be that $\sum_i b_i = 0$ due to flow conservation.

To find the internal flows in the network, we can reuse the B matrix discussed earlier:

$$b = B^T f$$

where b is the vector of incoming flows to each node and B^T is the transpose of the B matrix from the earlier part (Bv = f). That is because a row of B^T corresponds to a vertex, and each column of B_i^T is 1 if the edge represented by that column starts at vertex i and -1 if it ends at that vertex. Thus, multiplying the edge currents by B^T yields the sum of outgoing currents from each vertex minus the sum of incoming currents.

16.1.4 Circulations

Flows orthogonal to the subspace of electrical flows are called circulations. These are flows where no current enters or leaves the network.

A circulation f must satisfy $f \perp \text{colspan}(B)$, implying $B^T f = 0 = b$, so the boundary currents are also 0.

Currents in the real world are electrical flows: they start with a node that has outgoing net current and go to a node that has incoming net current. They are not circulations.

16.1.5 Minimum energy flow

The energy of a flow is $\sum f_{u\to w}^2 = ||f||^2$.

The flow h that minimizes energy while satisfying $B^T h = b$ is actually an electrical flow.

Proof

We write the flow h as the sum of an electrical flow h_e and a circulation h_c . These two components can

be found by projecting h onto the subspace of electrical flows (the column space of B). Then we can write $B^T h = B^T h_e + B^T h_c$, but $B^T h_c = 0$, so $B^T h_e = B^T h$.

So any flow can be projected onto the electrical flow subspace to get an electrical flow that satisfies the constraint and would have lower energy (since it's a projection, by Pythagorean theorem and definition of energy as the 2-norm). The minimum energy flow thus must be an electrical flow.

16.2 Effective Resistances

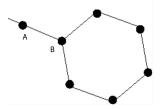
Given a circuit graph with nodes A and B, the effective resistance between A and B $R_{eff}(A, B)$ is the single-resistor equivalent of all the edges between A and B. There are standard ways to calculate this using formulas for resistors in series and parallel.

It can also be computed by driving one unit of current into A and out of B, and measuring the resulting potential difference between A and B, or, equivalently, the energy of a single unit of charge entering A / leaving B.

An obvious corollary of this definition is that adding edges to a network can only decrease the effective resistance between two nodes. Electrical networks are energy-conserving ("current chooses the path of least resistance"), so adding possible paths via new edges will never increase R_{eff} . This also implies that vertices with an edge between them can have an effective resistance of at most one.

16.3 Hitting Times & Effective Resistances

Hitting time $(u \to v)$ is the expected time a random walk from u takes to reach v. H_{uv} is not necessarily equal to H_{vu} .



For instance, in the example above, H_{AB} is small, while H_{BA} is much larger.

Theorem 16.1. For any electrical network, $H_{uv} + H_{vu} = 2|E|R_{eff}(u,v)$, where $H_{uv} + H_{vu}$ are referred to as the "commute times."

Proof. Take the network, add $d_i = \text{degree}(i)$ incoming current at each node and, at some node v send outgoing 2|E| current. (This ensures that the total incoming and outgoing flow are equal.) Then, by Kirchoff's Current Law,

$$d_u = \sum_{u \to w} f(u, w) = \sum_{u \to w} P_u - P_w = \sum_{u \to w} P_{uw} = \sum_{u \to w} (P_{uv} - P_{wv}) = d_u P_{uv} - \sum_{u \to w} P_{wv}$$

(Note that the fourth step of this simplification holds because potential is always relative to some ground; thus, definitionally, $P_{uw} = P_{uv} - P_{wv}$).

Rearranging this equation, we get:

$$P_{uv} = 1 + \frac{1}{d_u} \sum_{u \to w} P_{wv}$$

Note that

$$H_{u\to v} = E[T_{u\to v}]$$

or the expected time it takes to get from u to v. Then using the linearity of expectations along with basic first-step analysis on a graph, we get a very similar equation for hitting times:

$$H_{uv} = 1 + \frac{1}{d_u} \sum_{u \to w} H_{wv}$$

.

(This equation follows intuitively; we first take one step from u and then land at any of the d_u neighbors with uniform probability. Suppose we reach the neighboring node w. Then by definition, the expected time to get to v from w is H_{wv} ; we simply average all of the possible neighbors to derive the equation above).

Lemma 16.2. This equation implies that $P_{uv} = H_{uv}$, as we solve for these values by using the exact same system of equations.

Now, having related hitting times to potential differences between nodes, we proceed with the proof.

Let's call the flow network we've created for this proof A. We know from the lemma that $H_{u\to v}=P_{uv}^{(A)}$. Now create a new network C with a two-step modification: First, change the exit node from v to u, ensuring that 2|E| units of flow are exiting from u. (Do not change the incoming flows to each node.) Then reverse all flows. In this alternate network, because we reversed flows, $H_{v\to u}=-P_{vu}^{(C)}=P_{uv}^{(C)}$. Then

$$H_{u \to v} + H_{v \to u} = P_{uv}^{(A)} + P_{uv}^{(C)} = P_{uv}^{(A+C)}$$

,

where the last step holds because flow networks form a subspace. But from our definitions of A and C, A+C is simply the network with 2|E| units of flow entering at u and exiting at v. This (by definition) is just 2|E| times the effective resistance of $u \to v$, allowing us to conclude that

$$H_{uv} + H_{vu} = 2|E|R_{eff}(u,v)$$

Thus, we prove the theorem.

Corollary 16.3. Effective resistances satisfy the triangle inequality.

Proof.

$$R_{eff}(u,v) = \frac{H_{uv} + H_{vu}}{2|E|} \le \frac{H_{uw} + H_{wv} + H_{vw} + H_{wu}}{2|E|} = R_{eff}(u,w) + R_{eff}(w,v)$$

where the inequality on the hitting times holds because travelling through w is only one way to get from u to v.

Corollary 16.4. The expected time for a random walk to reach ("cover") every node on a graph is at most 2|E|(|V|-1)

Proof. As mentioned earlier, for any edge (u, v) on a graph, $R_{eff}(u, v) \leq 1$. Then using the theorem,

$$H_{uv} + H_{vu} \le 2|E|$$

This implies that it takes at most 2|E| steps (in expectation) to travel from u to v and back. Let us fix an ordering of the nodes $\{1, 2, ...N\}$ such that (1, 2), (2, 3), ...(N-1, N) are edges in the graph. Then we simply consider a walk that goes from 1 to 2 and back, 2 to 3 and back, etc., giving us the bound from the corollary on the time needed to cover all nodes.

16.4 Spanning Trees

Consider U_G , the uniform distribution of spanning trees over a graph G. Because spanning trees necessarily contain exactly one path between any two nodes in a graph, they become useful for analyzing flow networks.

Theorem 16.5. Given a set of incoming currents b, suppose we pick a uniformly random spanning tree T and route them in the tree, to get a flow $F_T(b)$. Then, the electrical flow F in the graph G with incoming currents b, is equal to $E_T[F_T(b)]$

We will not cover the proof of this theorem in class.

Note that we use the uniform distribution here because we are assuming that all edges have equal resistance; we would need to use a different distribution in the case of differently weighted edges.

Corollary 16.6. For edge
$$(u, v)$$
 in the graph, $R_{eff}(u, v) = \mathbb{P}_{T \sim U_G}[(u, v) \in T]$

Proof. When computing the expected flow over a single edge, note that the flow will be 1 if the edge belongs to the tree and zero otherwise, since a tree has exactly one path between any two nodes. The corollary follows directly from here: In the case where all edges have unit-resistance, this equivalently gives us an equation for P(u, v), the voltage (or potential difference) between nodes, which in turn is equal to the effective resistance along the edge.

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(Intuitively, an edge that is always included in the graph will necessarily have an effective resistance of one, while an edge that has many possible "alternate paths" (and is thus often omitted from a spanning tree) will have lower effective resistance.) \Box