

LECTURE 15

$$\frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2} \quad \lambda_2 = \lambda_2\left(\frac{L_d}{d}\right)$$

Path on n -vertices:

$$\lambda_2(L) \approx \Theta\left(\frac{1}{n^2}\right)$$

Cycle on n -vertices:

$$\phi(\text{Path}) = \Theta\left(\frac{1}{n}\right)$$

$$\frac{1}{n^2} \approx \frac{2}{n}$$

Higher Cheeger:

$$\lambda_K = 0 \Rightarrow K - \text{connected components}$$

Thm: [2012] In a graph G , $\exists S_1 \dots S_K$

so that

$$\max_i \phi(S_i) \leq \sqrt{\lambda_K \cdot \text{poly}(K)}$$

$$\lambda_K \approx \varepsilon \Rightarrow$$



Bourgain's embedding
+ LP

Finds a set S
with expansion
 $\phi(S) \leq \phi(h) \cdot \log n$

Cheeger

$\Rightarrow S$ with
Expansion $\phi(S) \leq \sqrt{\phi(h)}$

Expander Graphs:

Def: (Combinatorial Expander)

A d^{"4}-regular graph G such that

$$\phi(G) \geq \Omega(1) = 0.01$$

Given G verify its an expander ??

Def: (Spectral Expander)

A d-regular graph G such that

$$\lambda_2\left(\frac{L_G}{d}\right) \geq \Omega(1) \cdot 10^{-4}$$

[verifiable]

Combinatorial
Expander

\Rightarrow Spectral Expander

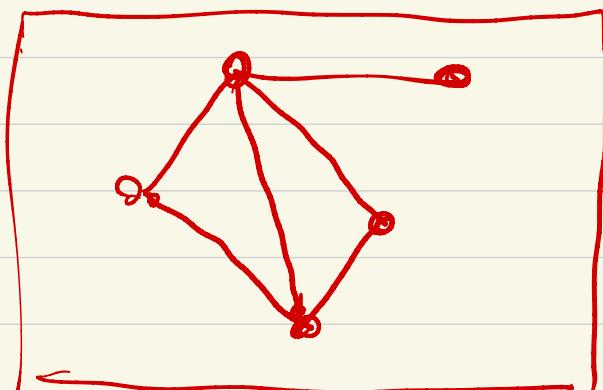
Theorem: If planar graph G

$$\lambda_2\left(\frac{L_G}{d}\right) \leq O\left(\frac{1}{n}\right) \approx \frac{8}{n}$$

Proof: Exhibit an embedding

$$v_1, \dots, v_n \in \mathbb{R} \text{ s.t. } \rightarrow \left\{ \frac{\sum_{(i,j) \in E} \|v_i - v_j\|^2}{d \sum \|v_i\|^2} \right\} \leq O\left(\frac{1}{n}\right)$$

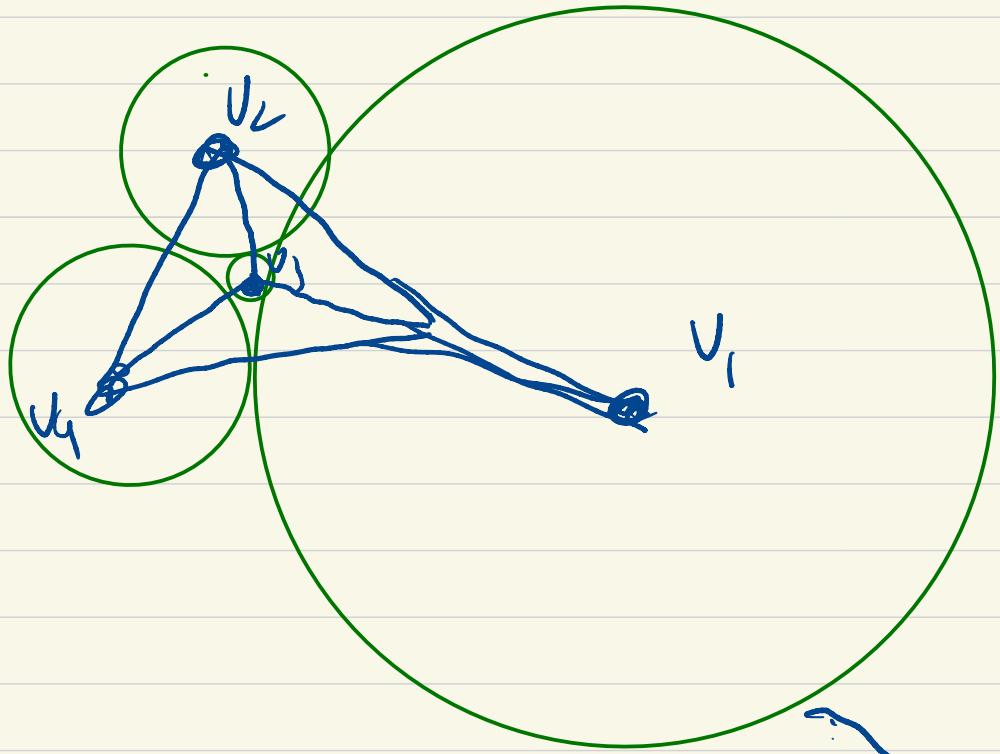
$$\sum_{i=1}^n v_i = 0$$



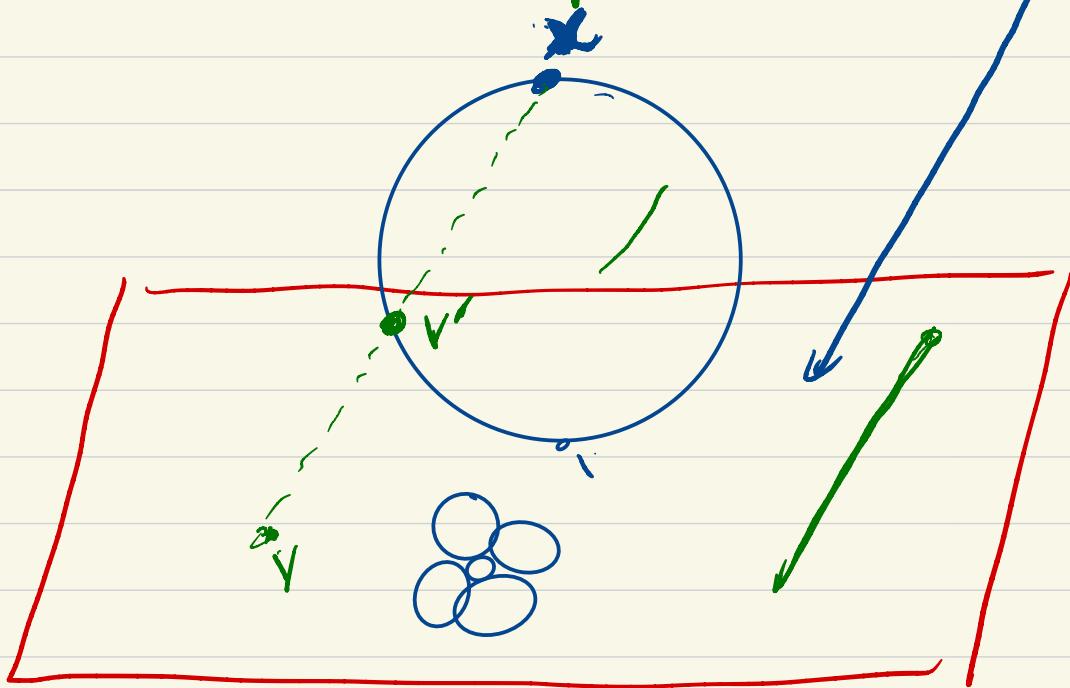
[Kissing Circles]

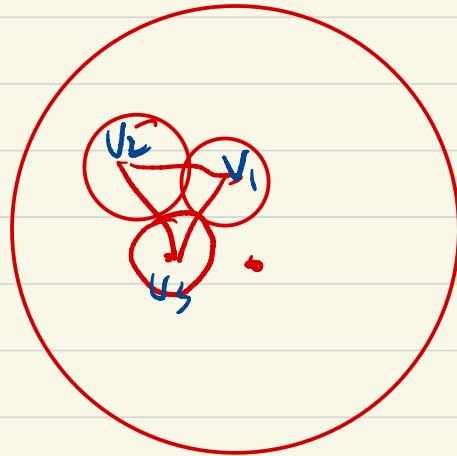
G - planar graph
 \exists disjoint circles
 C_1, \dots, C_n

s.t. $(i,j) \in E$ iff C_i touches C_j



Stereographic projection
maps circles to circles.





$$\frac{\sum_{(i,j) \in E} \|v_i - v_j\|^2}{d \sum_i \|v_i\|^2}$$

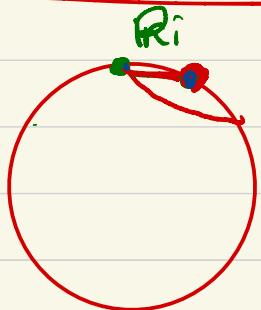
At vertex i $v_i \in \mathbb{R}^3$

Denominator: $\|v_i\| = 1 \Rightarrow d \sum \|v_i\|^2 = dn$

$\forall (i,j) \in E$ triangle inequality

$$\|v_i - v_j\|_2 \leq \underline{R_i + R_j}$$

↑
radius of the cap



$$\|v_i - v_j\|^2 \leq (R_i + R_j)^2 \leq 2(R_i^2 + R_j^2)$$

$$\sum_{(i,j) \in E} \|v_i - v_j\|^2 \leq 2 \sum_{(i,j) \in E} (R_i^2 + R_j^2) = 2d \sum_i R_i^2$$

$$\text{Area of sphere} = 4\pi$$

$$\nabla \sum_{i=1}^n (\text{Area of cap}) = \sum_i \underline{\pi R_i^2}$$

$$\Rightarrow \sum_i R_i^2 \leq 4$$

$$\frac{\sum_{(i,j) \in E} \|v_i - v_j\|^2}{d \sum_i \|v_i\|^2} \approx \frac{(2d) \cdot 4}{dn} \approx \sqrt{\frac{8}{n}}$$

$$\sum_{i=1}^n y_i = 0 \quad ??$$

Brewer's Fixed pt

RANDOM WALKS

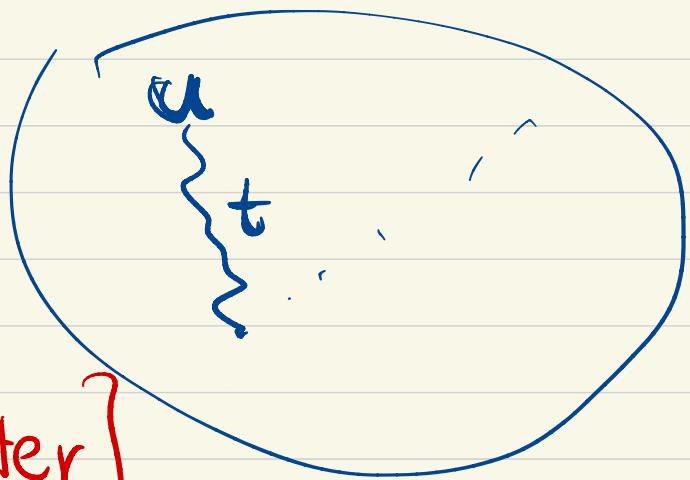
a - d -regular graph

RWalk: At each step u

Pick $u \sim v$ w.p $1/d$

move to v.

Suppose we start at



$P_t[v] = \Pr[\text{walk is at } v \text{ after } t \text{ steps}]$

$P_t \in \mathbb{R}_+^n$, $\sum_{v \in V} P_t[v] = 1$

$P_0[x] = \begin{cases} 1 & \text{if } v = u \\ 0 & \text{otherwise} \end{cases}$

$$\Pr \left[\text{at } v \text{ in time } t \right] = \sum_{(\omega, v) \in E} \Pr \left[\text{at } v \text{ in } t-1 \right] \cdot \frac{1}{d}$$

$$P_t = \left(\frac{1}{d} A \right) \cdot P_{t-1}$$

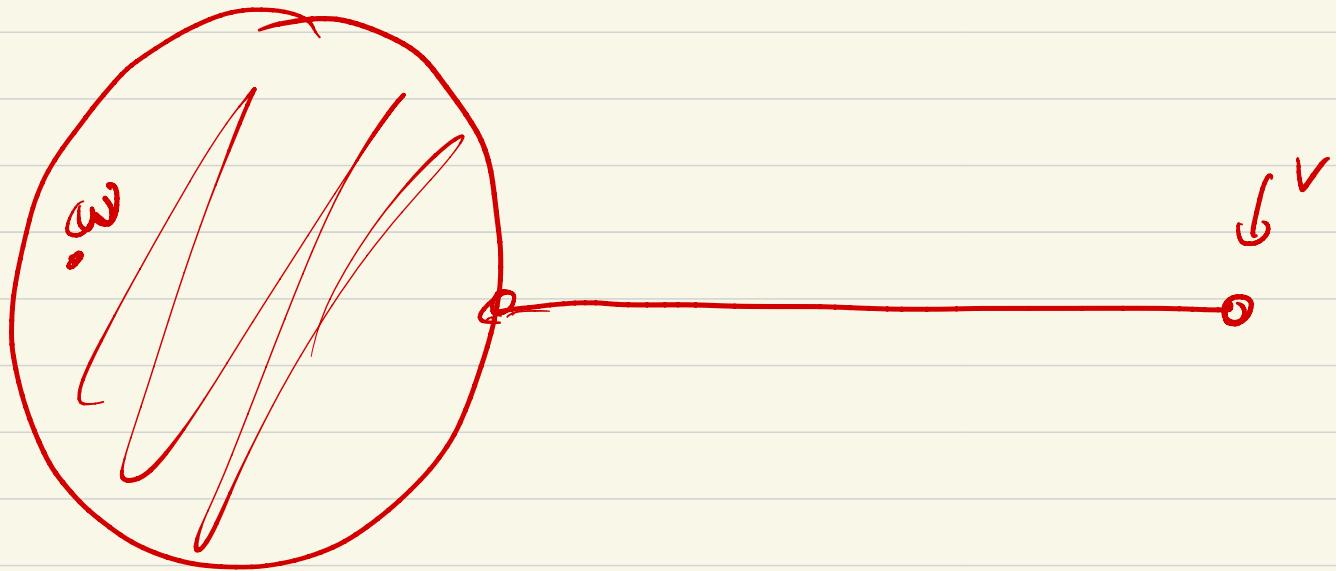
\tilde{A}

$$P_t = \tilde{A} P_{t-1} = (\tilde{A})^2 P_{t-2}$$

$$P_t = (\tilde{A})^t P_0$$

Σ -mixing time: # of steps after which

$$\| P_t - \frac{1}{n} \vec{1} \|_2 \leq \epsilon$$



$$P_t = \frac{1}{d} A \cdot P_{t-1}$$

$$\left(P_t - \frac{1}{n} \vec{I} \right) = \frac{1}{d} A \left(P_{t-1} - \frac{1}{n} \vec{I} \right)$$

$$\left[\frac{A \vec{I}}{d} = \vec{I} \right]$$

$\xrightarrow{\alpha}$

$$\boxed{\left(P_t - \frac{1}{n} \vec{I} \right) = (\tilde{A})^t \left(P_0 - \frac{1}{n} \vec{I} \right)}$$

Fact: $v \in \mathbb{R}^n$ M ↗ real symmetric matrix

$$Mv = \sum_i \lambda_i^t \underbrace{\langle v, e_i \rangle}_{\text{i-th eigen value}} \vec{e}_i$$

↑
i-th eigen vector

$$\left(P_t - \frac{1}{n} \vec{I} \right) = \sum_{i \in S} (\lambda_i^t \cdot c_i) \vec{e}_i$$

$\downarrow 0.9$

$$d_i = \lambda_i(\tilde{A})$$

If $\lambda_i(\tilde{A}) < 0.9$

$$\boxed{\left\| \left(P_0 - \frac{1}{n} \vec{I} \right) \right\|_2 \leq 1}$$

$$\boxed{\left\| P_t - \frac{1}{n} \vec{I} \right\|_2 \leq \left(\max_i |\lambda_i|^t \right)}$$

ϵ -mixing time:

(connected graph)

$$\frac{\log \frac{1}{\epsilon}}{\log \max_i |\lambda_i(A_d)|}$$

$$L = dI - A$$

$$\lambda_i(L) = d - \lambda_i(A)$$

$$\lambda_1(L) \leq \lambda_2(L)$$

||

0

||

0

$$\leq \lambda_n(L)$$

~~λ_n~~
2

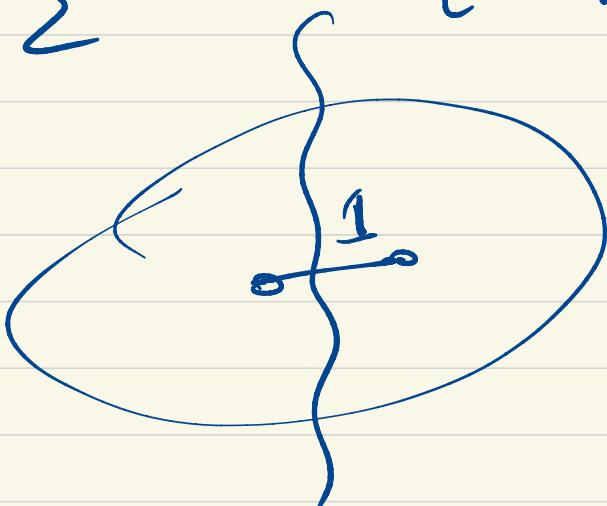
$$\frac{\log(\frac{1}{\epsilon})}{\min\{\log(1 - |\lambda_2|), \log(1 - |\lambda_n|)\}}$$

Connected Graph (d -regular)

$$\sqrt{2\lambda_2} \geq \phi(G)$$

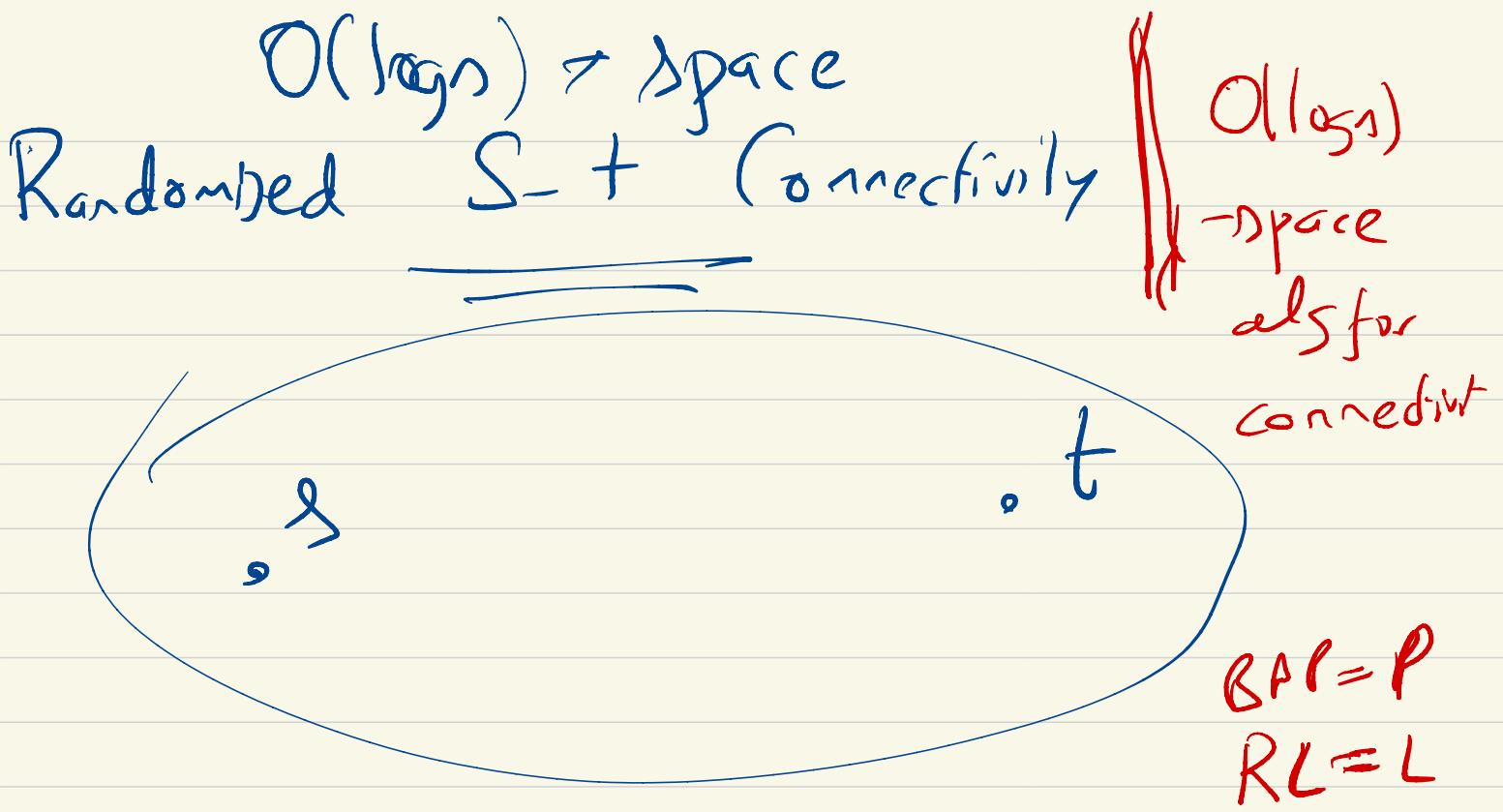
$$\lambda_2(\tilde{L}) \geq (\phi(G))^2 \geq O\left(\frac{1}{dn}\right)$$

$$\lambda_2(\tilde{A}) \leq 1 - \frac{1}{d^2 n^2}$$



$$\phi(G) \geq \frac{1}{dn}$$

$$\Rightarrow \|P_t - \frac{1}{n}I\|_2 \leq \text{in } t = \text{poly}(n) \approx n^3$$



s and t connected ??

Start from s

Run a random walk $\frac{n^5}{\underline{}}$ steps

and if you see t

$s \sim t$ connected

else $s \sim t$ not connected