

LECTURE 20

SOP \rightarrow linear program (variables are
inner products between
vectors)

$$\hookrightarrow (\underbrace{\text{Max}_{\mathcal{T}}(\text{ut})}_{\mathcal{T}})$$

Systems of Poly inequalities:

$$P = \left\{ \begin{array}{l} P_1(x) \geq 0 \\ P_2(x) \geq 0 \\ \vdots \\ P_m(x) \geq 0 \end{array} \right\} \text{ over } x = (x_1, \dots, x_n)$$

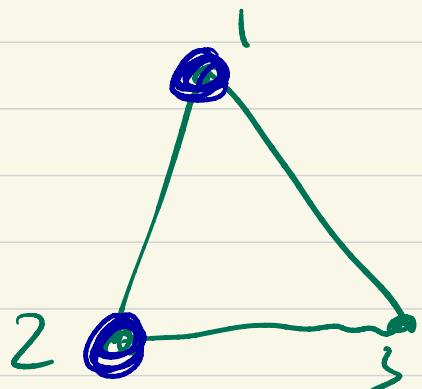
Minimum Vertex Cover (NP-hard)

INPUT: Graph $G = (V, E)$

GOAL: Smallest $S \subseteq V$ such that

every edge $(i, j) \in E$ is covered by S

$i \in S$ or $j \in S$ or both.



$$S = \{1, 2\}$$

$$S = \{2, 3\} \quad \{3, 1\}$$

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases} \quad (\underline{\text{intended}})$$

$$x_i \in \{0, 1\} \iff$$

$$x_i^2 - x_i = 0$$

$$\forall (i, j) \in E \quad i \in S \iff \\ \text{or } j \in S$$

$$(1-x_i)(1-x_j) = 0 \\ \forall (i, j) \in E$$

$$\min \sum x_i$$

$$\sum x_i \leq c$$

P VERTEX COVER

P_{Maxcut}:

$$x_i^2 = 1 \iff \{x_i = \pm 1\}$$

$$\sum_{(i, j) \in E} (x_i - x_j)^2 \geq c$$

$S = \{\text{set of } \pm 1 \text{ assignments}\}$
 with value $\geq c$

$\Delta_S = \begin{matrix} \text{prob distributions} \\ (\text{vector } |S| \text{ entries}) \end{matrix}$

$$P = \{ p_i(x) > 0 \}$$

(let S be set of solutions to P)

Convenifying $S \Rightarrow$ prob. distributions
over S

$$\frac{1}{\prod}$$

very large.

$$S = \text{all } \{\pm 1\}^n \quad \leftarrow \quad \sum_{i=1}^n x_i^2 = 1 \quad \forall i$$

μ is prob. dist over S

$$\Rightarrow \{ \mu(x) \mid \forall x \in S \}$$

$$\mu(x) > 0$$

$$\sum_{x \in S} \mu(x) = 1$$

μ prob dist over $\{0, 1\}^n$

$x \sim \mu$ [sample from μ]

degree 1
x ~ μ $E[x_1]$ $E[x_2]$... $E[x_n]$ }

degree 2
x ~ μ $E[x_i^2]$ $E[x_i x_j]$ }

$$x_i^2 - x_i = 0$$

~~=====~~

$$\Rightarrow (1-x_i)(1-x_j) = 0 \quad \forall (i,j) \in E$$

$$\sum x_i \leq C$$

Goal: Find degree 1 & degree 2 moments of a distribution over solutions to P

P
VERTEX COVER

$$X_i = \underset{x \sim \mathcal{U}}{\mathbb{E}} [x_i]$$

$$X_{ij} = \underset{x \sim \mathcal{U}}{\mathbb{E}} [x_i x_j]$$

$$X_{ii} = \underset{x \sim \mathcal{U}}{\mathbb{E}} [x_i^2]$$

{ intent
~~=~~}

$n + n + \binom{n}{2}$ variables

$x_i \ x_{ii} \ x_{ij}$

$$\underbrace{x_i^2 - x_i}_{} = 0 \Rightarrow E_{x \sim \mu} [x_i^2 - x_i] = 0$$

$$\underbrace{X_{ii} - X_i}_{} = 0$$

$\forall (i, j) \in E$

$$(1 - x_i)(1 - x_j) = 0 \Rightarrow E_{x \sim \mu} [(1 - x_i)(1 - x_j)] = 0$$

$$1 - x_i - x_j + x_{ij} = 0$$

$$\sum x_i \leq 0 \Rightarrow E[\sum x_i - c] \leq 0$$

$$\Rightarrow \underbrace{\sum_i x_i - c}_{\leq 0}$$

$$(x_1 - x_5 + x_7)^2 \geq 0 \Rightarrow E_{x \sim \mu} [(x_1 - x_5 + x_7)^2] \geq 0$$

↓

$$X_{11} + X_{55} + X_{77} - 2X_{15} - 2X_{57} \\ + 2X_{71} \geq 0$$

$$Y \quad p(x) = \underbrace{(p_0 + \sum p_i x_i)^2}_{\geq 0} \geq 0$$

$$\Rightarrow p_0^2 + 2 \sum_i p_0 p_i \cancel{x_i} + \sum_i p_i^2 \cancel{x_{ii}} + 2 \sum_{i \neq j} p_i p_j \cancel{x_{ij}}$$

$$\uparrow \qquad \qquad \qquad \geq 0$$

(Moment Matrices are P.s.d)

$$M_2 = \begin{bmatrix} 1 & x_1 & x_2 & & x \\ x_1 & 1 & X_1 & & x \\ x_2 & X_1 & 1 & & x \\ \vdots & & & \ddots & \vdots \\ x_i & & & -X_{ij} = E[x_i u_j] & x \\ \vdots & & & & \vdots \\ x_n & & & & x \end{bmatrix} \geq 0$$

|

$$\underline{H P = (p_0, p_1, \dots, p_n)}$$

$$P^T M_2 P = E_{x \sim \mu} \left[(p_0 + \sum p_i x_i)^2 \right]$$

$$\geq 0$$

\sum

$$M_2 \geq 0$$

deg 2

SOP Relaxation

$$\begin{cases} X_{ii} - X_i = 0 \\ 1 - X_i - X_j + X_{ij} = 0 \\ \rightarrow \sum_i X_i \leq c \end{cases} \quad \forall i \quad \forall (i,j) \in E$$

$$\begin{bmatrix} 1 & X_1 & \dots & X_n \\ X_1 & & & X_{ij} \\ \vdots & & & \vdots \\ X_n & & & \end{bmatrix} \geq 0$$

Fix $d \in \mathbb{N}$.

deg- $2d$ - Sum-of-Squares SDP relaxation

Variables: Moments up to degree $2d$

$$\mathbb{R}^{2d} \rightarrow \{X_i, X_{ij}, X_{ijk}, X_{ijkl}, \dots, X_{i_1 i_2 \dots i_d}\}$$

Constraints: $p(x) = \sum p_\sigma x_\sigma \geq 0 \in \mathcal{P}$

$$\Downarrow$$

$$\sum p_\sigma X_\sigma \geq 0$$

Moment matrix

$$M_{2d} =$$

$$\begin{bmatrix} 1 & & & & \\ x_1 & & & & \\ x_2 & & \dots & & \\ \vdots & & & & \\ x_n & & & & \\ x_{n+1} & & & & \\ \vdots & & & & \\ x_{n+d} & & & & \end{bmatrix} \xrightarrow{x_\sigma} X_{\sigma\sigma} \geq 0$$

$$\mathbb{E}_{x \sim q}[q^2(x)] \geq 0 \iff \deg(q) \leq d$$

Notational Switch

$\{ \text{deg 2 moments } X_1 \dots X_n \}$
 X_{ij}

$$\left. \begin{aligned} & E \left[\sum p_{ij} X_i X_j + \sum q_{lj} X_j + q_0 \right] \\ & \quad " \\ & \quad \sum p_{ij} X_{ij} + \sum q_{lj} X_j + q_0 \end{aligned} \right\}$$

expectations of degree 2 polynomials

$\rightarrow \tilde{E}$: $\left\{ \begin{array}{l} \text{deg 2} \\ \text{Polynomial} \end{array} \right\} \rightarrow \mathbb{R}$

pseudo expectation
(linear functional)

$$\tilde{E}[p_i(x)] \geq 0 \quad \forall \{p_i(x) \geq 0\}$$

$$\tilde{E}[q^2(x)] \geq 0 \quad \forall q .$$

Degree-2d SoS SDP

Given a poly system $P = \{p_i(u) \geq 0 \mid \forall i\}$

the deg^{2d} SoS SDP relaxation

Find:

\tilde{E} :

$\mathbb{R}^{(2d)}[x_1, \dots, x_n]$

\mathbb{R}

(real
number)

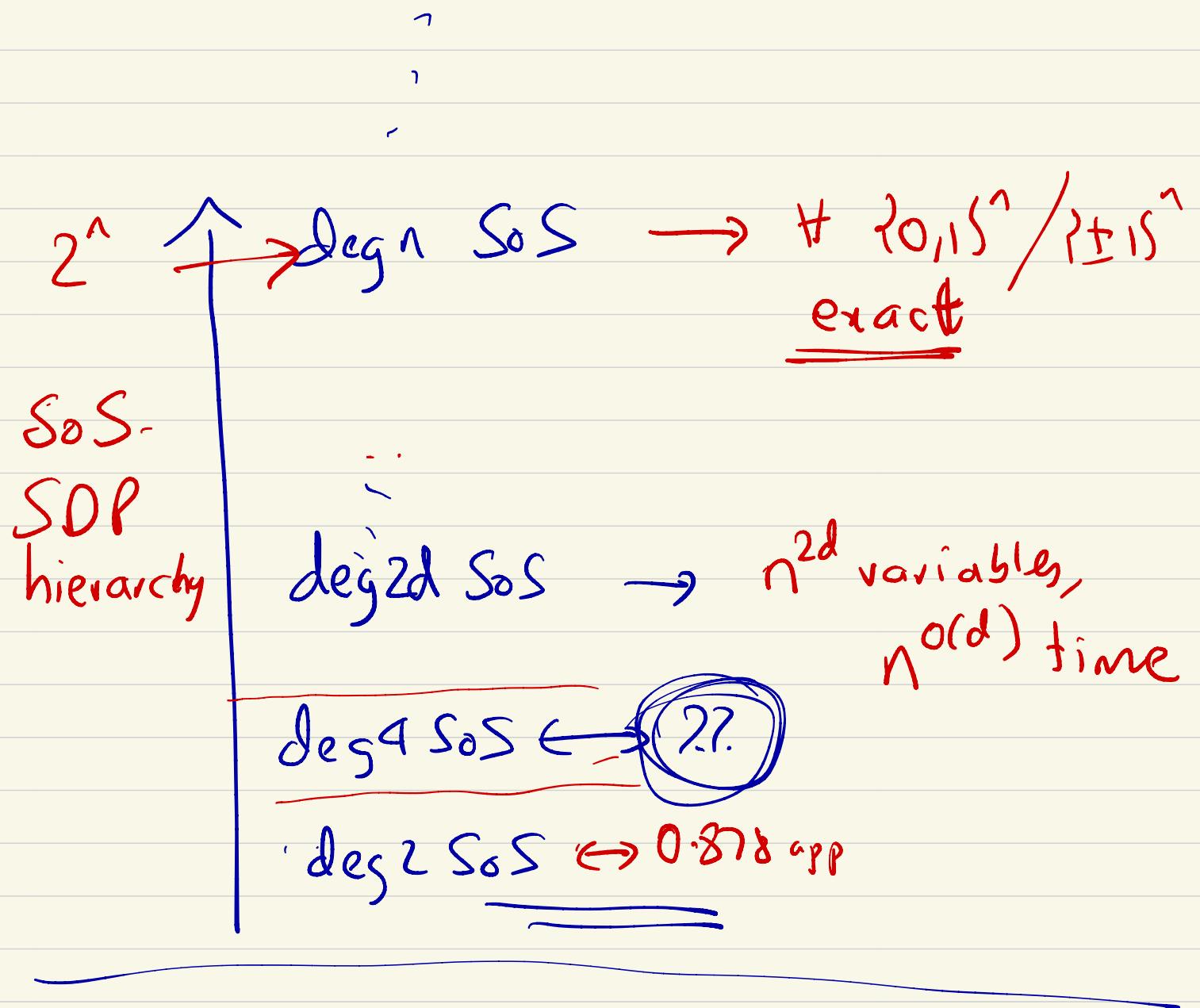
(linear functional) (deg^{2d},
poly)

s.t.

$$\tilde{E}[p_i(u)] \geq 0 \quad \forall i$$

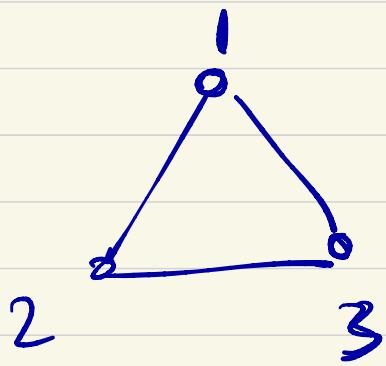
$$\boxed{\tilde{E}[q^2(u)] \geq 0}$$

$$\forall q, \deg(q) \leq d.$$



} Lovasz-Schrijver LP hierachy
 Sherali Adams LP
 ↗ $S(n)$

Vertex Cover



$$x_1 = 1 \quad x_3 = 1 \quad \frac{1}{2}$$

$$\left. \begin{array}{l} x_1 + x_3 \geq 1 \\ x_2 + x_3 \geq 1 \\ x_3 + x_1 \geq 1 \end{array} \right\} \deg 1$$

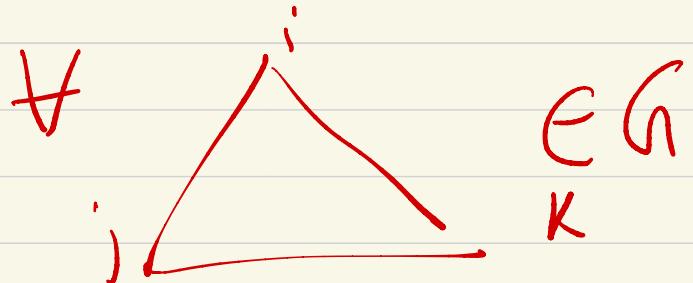
$\left. \begin{array}{l} x_2 + x_3 + x_1 = \frac{1}{2} \\ \Rightarrow x_1 + x_2 + x_3 \leq \frac{3}{2} \end{array} \right\} \text{LP relaxation}$

$$x_2 + x_3 + x_1 = \frac{1}{2} \Rightarrow x_1 + x_2 + x_3 \leq \frac{3}{2}$$

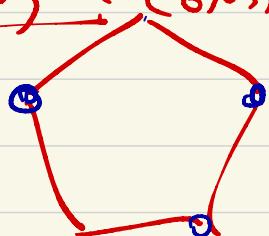
$$\text{OPT (vertex cover)} = 2$$

Triangle Constraint:

$$x_i + x_j + x_k \geq 2$$



Pentagon constraint



$$x_1 + x_2 + x_3 + x_4 + x_5 \geq 3$$

Theorem: Suppose $\{x \in \{0,1\}^n\} \Leftrightarrow \{\sum_{i \in S} x_i = 0\}$

then degree- $2d$ "SOS SDP" captures every "true constraint" on a set of $\underline{2d}$ variables.

Proof: (Moments \Rightarrow Actual local distributions)

Claim: $\forall S \subseteq \{x_1, \dots, x_n\}, |S|=2d$

\exists a prob. dist M_S over $\{0,1\}^S$

such that $\tilde{E}[x_\sigma] = E_{x \sim M_S}[x_\sigma]$
 $\forall \sigma \in S$.

Proof: $E[1[x_S = \alpha]] = \prod_{\substack{i \in S \\ \alpha_i = 0}} (1-x_i) \cdot \prod_{\substack{i \in S \\ \alpha_i = 1}} x_i$

$$\mu_S \geq \Pr[x_S = \alpha] = E[\mathbb{1}_{\{x_S = \alpha\}}]$$

$$\Leftarrow \tilde{E}\left[\prod_{x_i=0} (-x_i) \prod_{x_i=1} x_i \right]$$

$$\Leftrightarrow \tilde{E}[x_\sigma] = E[x_\sigma] \quad \forall \sigma \in S$$

$$\Rightarrow \text{If } \sum_{\sigma \in S} c_\sigma x_\sigma \geq 0 \quad \forall x \in S$$

$$\Downarrow \\ E_{x \sim \mu_S} [\sum c_\sigma x_\sigma] \geq 0$$

$$\Downarrow \\ \tilde{E}[\sum c_\sigma x_\sigma] \geq 0$$

"Proofs" SoS "Algorithms"

$$x_1^2 = 1 \quad x_2^2 = 1 \quad x_3^2 = 1$$

$$x_1 + x_2 \geq 1$$

$$x_2 + x_3 \geq 1$$

$$x_3 + x_1 \geq 1$$

Prove $x_1 + x_2 + x_3 \geq 2$