LECTURE 12

~~Distance~~ Metric on n-points (X, d)

Ex: Shortest path on graph

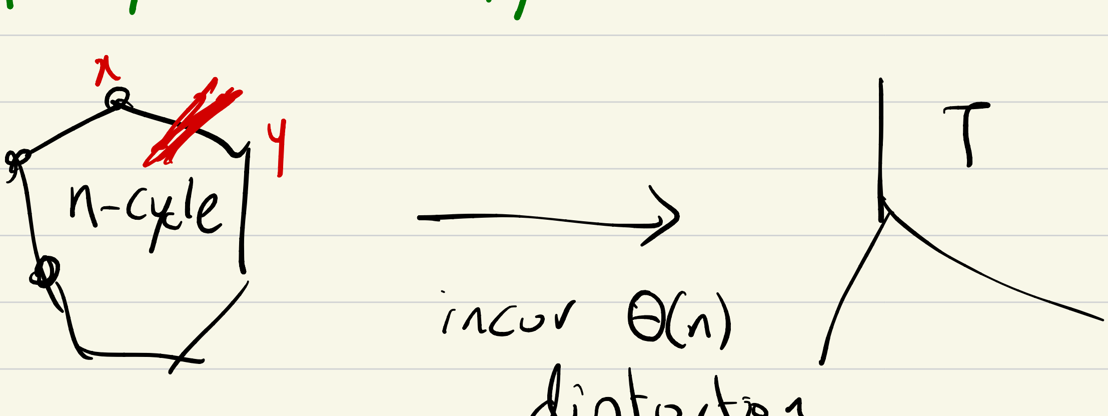
$\Downarrow$ embeddings

Tree Metrics (T, d_T)

(shortest path on a tree)

[Approximation Algos & Online Algos]

$$d_T(x,y) = d(x,y) \quad \text{[IMPOSSIBLE]}$$



Randomized / Probabilistic Tree embeddings

$((G=(V,E), w_e)$

Given (X, d), a randomized tree embedding is a distribution $\mathcal{D}$ on trees metrics (spanning)

such that $\forall x,y \in X$

$$d(x,y) \leq d_T(x,y) \quad \forall T \sim \mathcal{D}$$

AND

$$\mathbb{E}_T \left[d_T(x,y) \right] \leq \alpha \cdot d(x,y)$$

Low stretch spanning trees : Trees are spanning trees of G.

Theorem $\forall (X,d)$ $\exists \mathcal{D}$ over tree metrics with $\alpha = O(\log |X|)$

LOW-DIAMETER DECOMPOSITIONS

Given: (X, d) metric space

Goal: Partition X as $P = X_1 \cup X_2 \ldots X$

s.t. $\boxed{\text{diameter}(X_i) \leq \Delta}$

$\forall x, y \in X$

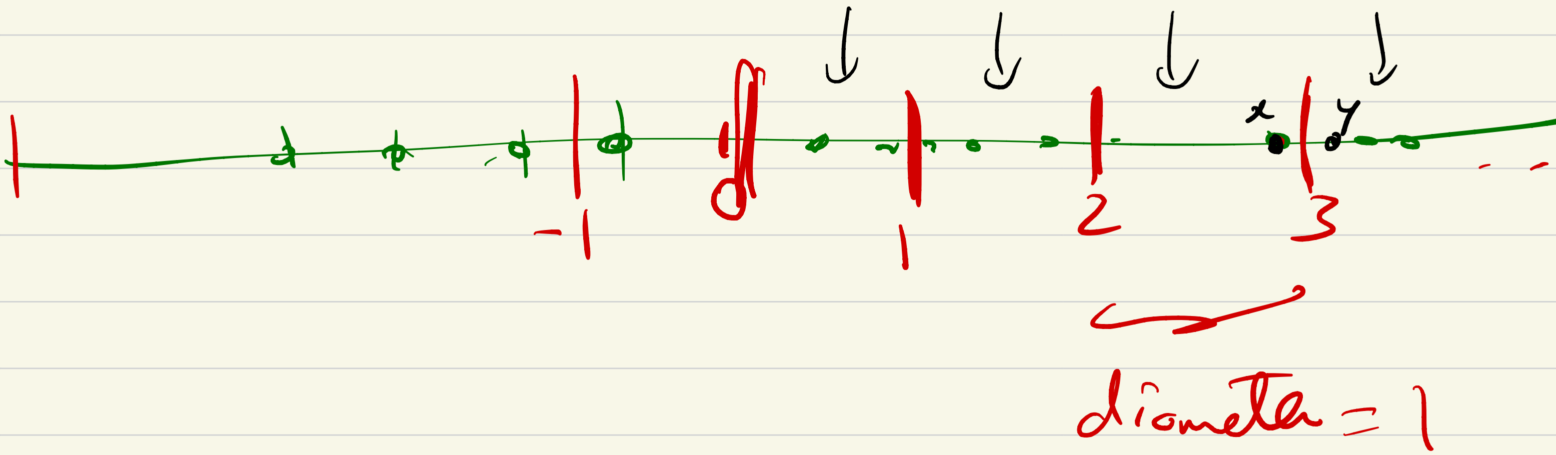
$$\Pr[P(x) \neq P(y)] \leq \beta \cdot \frac{d(x,y)}{\Delta}$$

x & y are separated

$O(\log n)$

$P(x)$ = set in the partition to which x belongs.

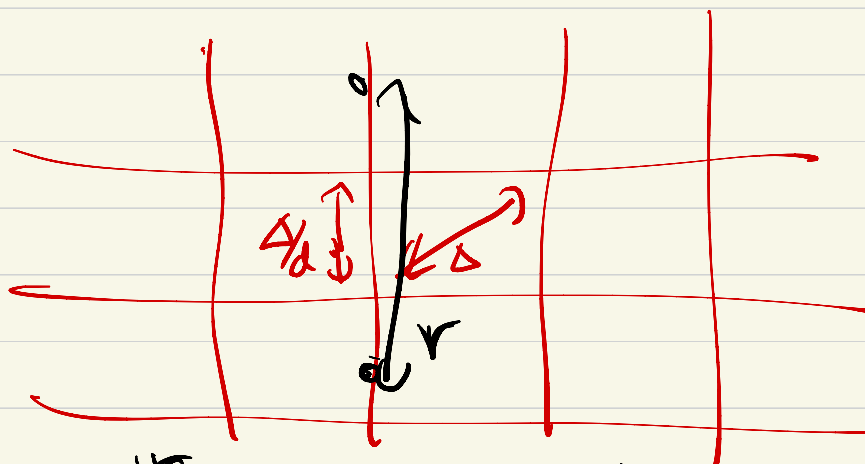
Example $(X, d) = (\mathbb{R}, |\,|)$



$$\Pr[\,P(x) \neq P(y)\,] = 1$$

$$\Pr[\,P(x) \neq P(y)\,] \leq \underset{\substack{\| \\ \rho}}{1} \cdot \frac{|x-y|}{1 = \Delta}$$

$(\mathbb{R}^d, \|\ \|_1)$



$$\Pr(P(x) \neq P(y)) = \frac{r}{\Delta/d} \approx d\left(\frac{r}{\Delta}\right)$$

LOW-DIAMETER DECOMPOSITIONS

INPUT: (X, d) Goal: Δ-bounded partition

- Pick a random radius $R \in [\Delta/4, \Delta/2]$

- Randomly Order (permutation) $X = \{x_1, \dots x_n\}$

set of points within distance Δ.

- $B(x_1, R)$ = ball of radius Δ

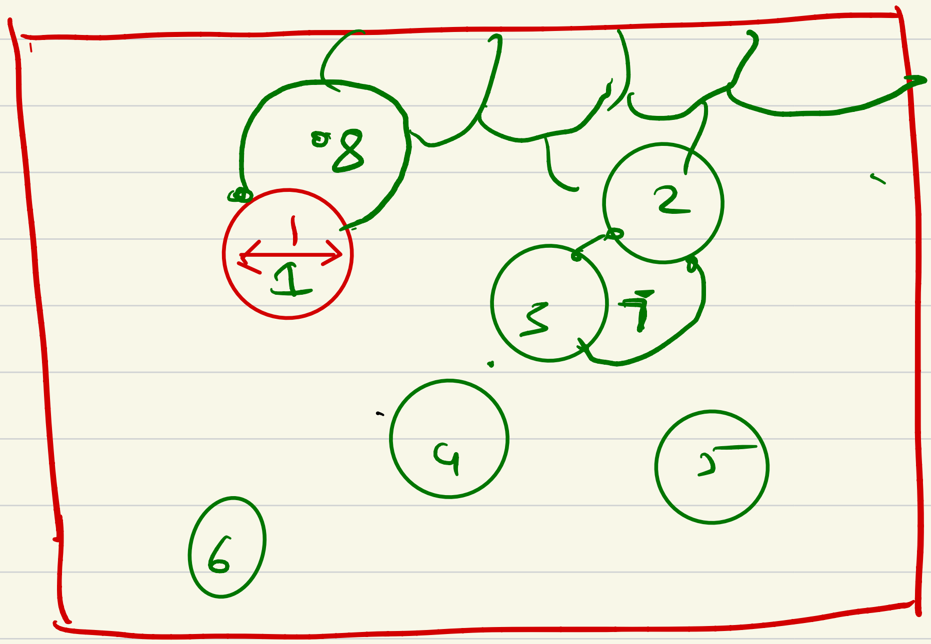
$B(x_2, R) \setminus B(x_1, R)$

$B(x_3, R) \setminus (B(x_1, R) \cup B(x_2, R))$

$\vdots$

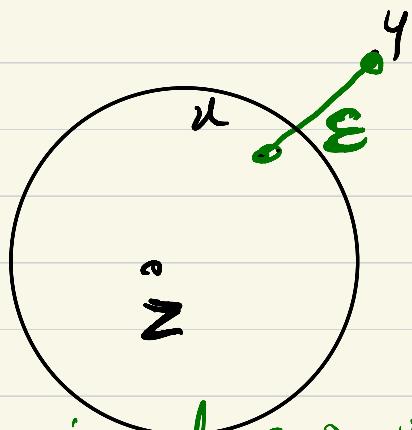
until all points are assigned.

$l_2 = (\mathbb{R}^2, \| \|_2)$



$x, y \in \mathbb{R}^2$

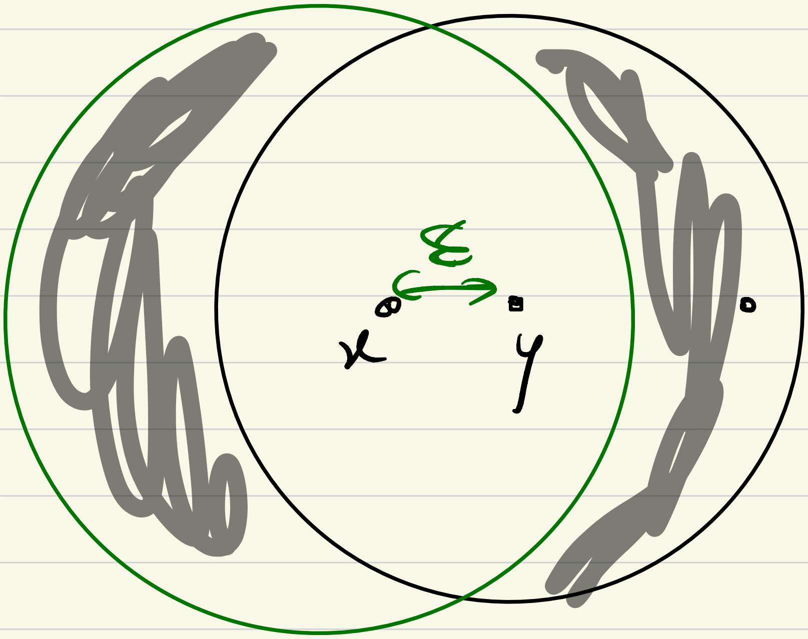
$\Pr[\, P(x) \neq P(y) \,]$



Wlog let x be assigned a partition first

$\|x - z\|_2 \leq 1$ but $\|y - z\|_2 \geq 1$

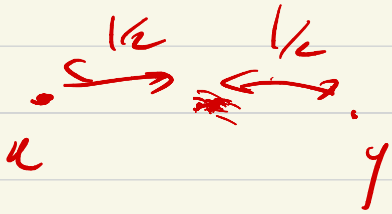
$\Rightarrow$



← 2 lands in

$$\Pr(h(x) \neq h(y)) = \frac{\text{Volume of } B(x,1) \,\triangle\, B(y,1)}{\text{Volume of union of } B(x,1) \cup B(y,1)}$$

$\propto$ Surface area of the ball.



Claim: Suppose $d(x,y) = r$ then

$$\Pr[P(x) = P(y)] \geq \exp\left[-\frac{8r}{\Delta} \log \frac{|B(x,\Delta)|}{|B(x,\Delta/8)|}\right]$$

Corollary:

$$\Pr[P(x) \neq P(y)] = 1 - \exp(\quad)$$

$e^{-x} \geq 1-x$

$$\leq \frac{r}{\Delta} 8 \log \frac{|B(x,\Delta)|}{|B(x,\Delta/8)|}$$

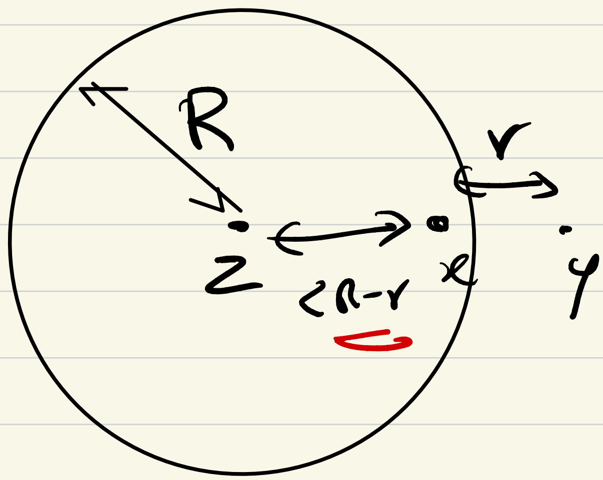
$$\leq (8 \log n) \cdot \boxed{\frac{r}{\Delta}}$$

$\boxed{(8\log n)} \boxed{\frac{d(x,y)}{\Delta}}$

center z captures x

Proof:

$$\Pr[P(x)=P(y)] \geq \Pr[d(x,z) < R-r]$$



Volume intersection / Volume of union

$$\geq \mathbb{E}_R\left[\frac{|B(x, R-r)|}{|B(x, R+r)|}\right]$$

$$= \mathbb{E}_R\left[\exp\left(-\log\frac{|B(x,R+r)|}{|B(x,R-r)|}\right)\right]$$

By convexity of e^x $\quad E[e^z] \geq e^{E[z]}$

$$\geq \exp\left[\mathop{\mathbb{E}}_{R} \left[-\log \frac{|B(x, R+r)|}{|B(x, R-r)|} \right] \right]$$

$$\geq \exp\left\{ \frac{\displaystyle\int_{\Delta/4}^{\Delta/2} -\log \frac{|B(x, R+r)|}{|B(x, R-r)|} \, dR}{\Delta/4} \right\}$$

$$d(x,y) = r < \Delta/8$$

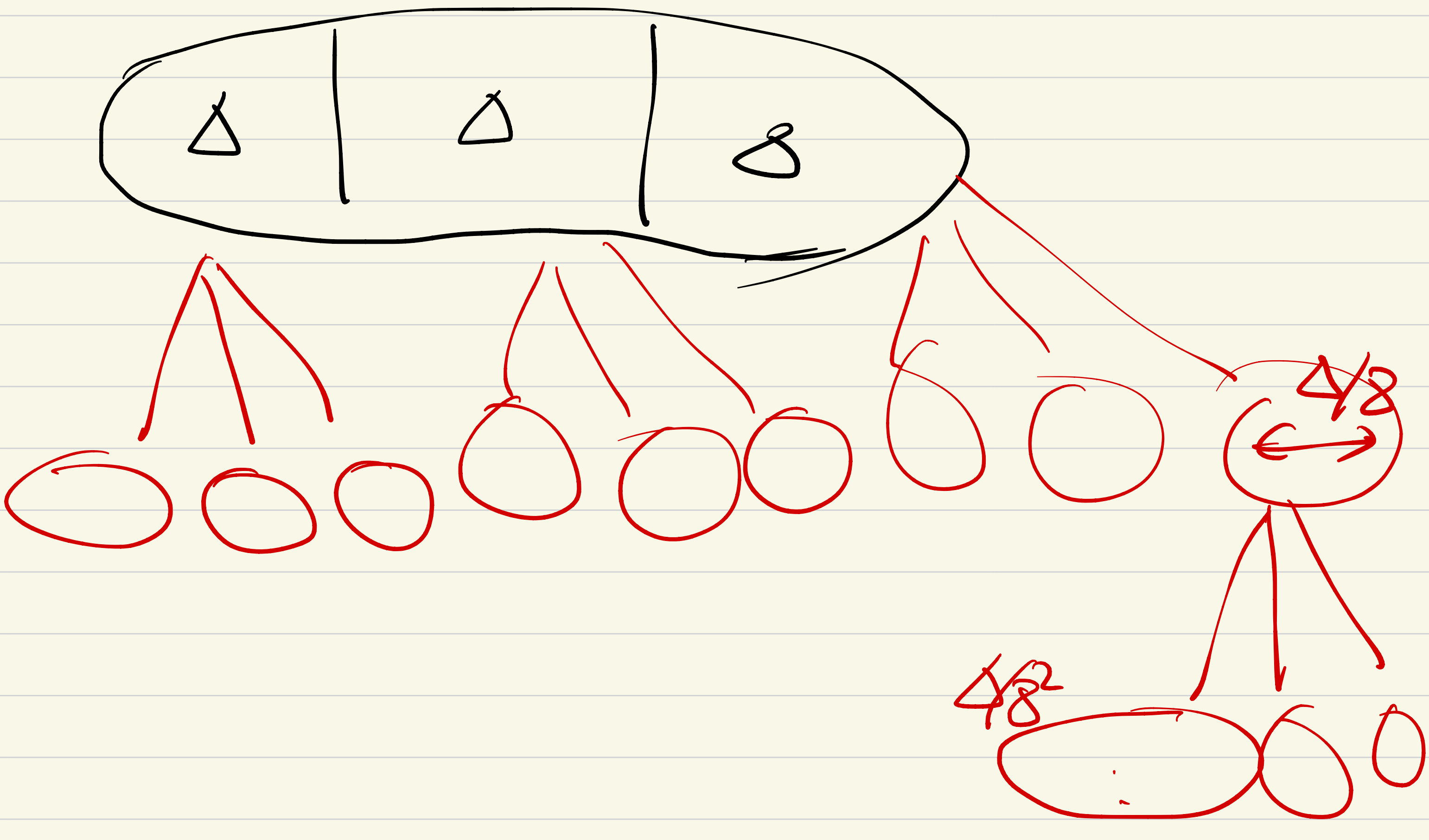
$$R+r < \Delta/2 + \Delta/8 < \Delta$$

$$R-r \geq \Delta/4 - \Delta/8 \geq \Delta/8$$

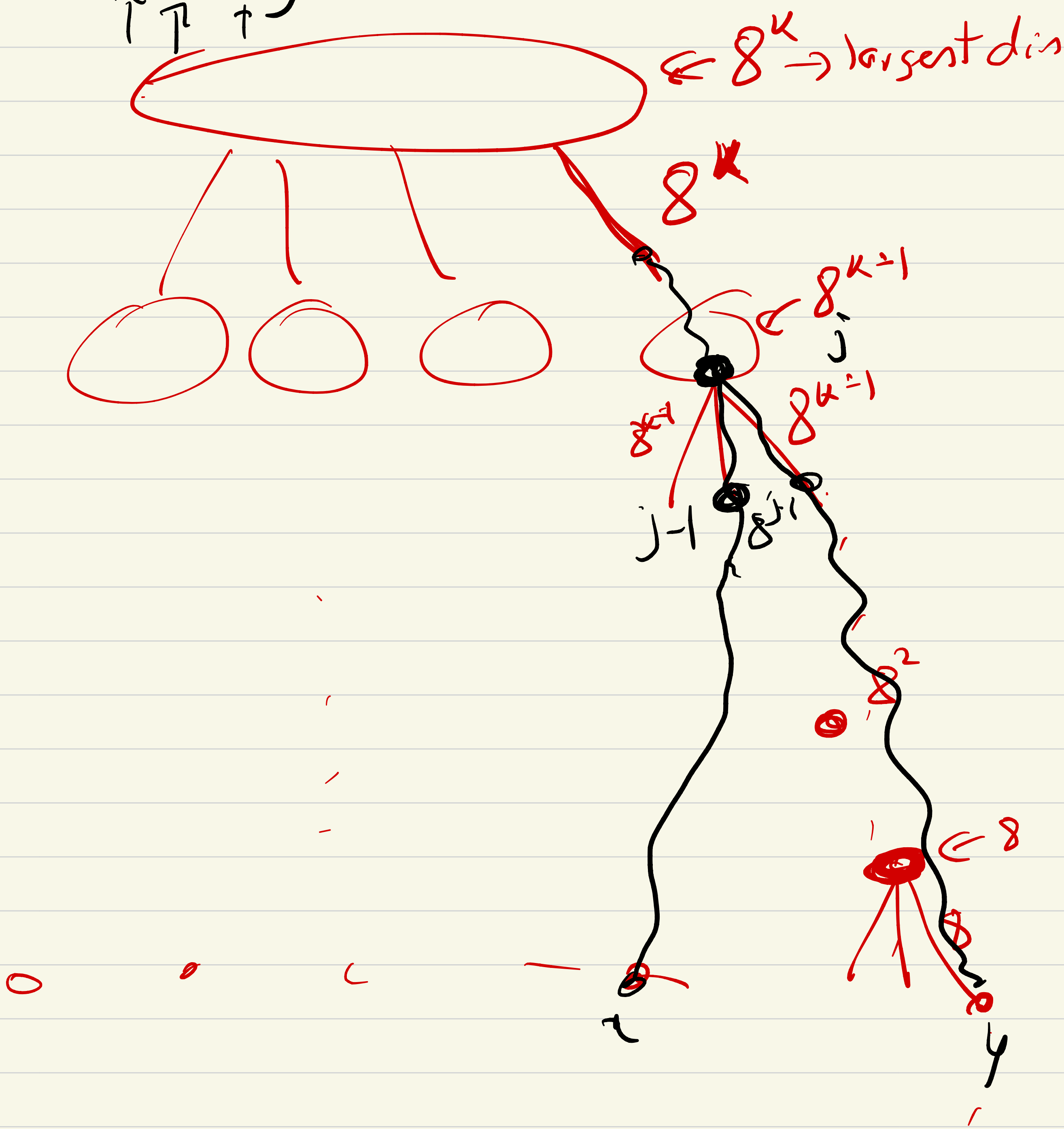
$$\geq \exp\left(\frac{\displaystyle\int_{\Delta/4}^{\Delta/2} -\log \frac{|B(x, \Delta)|}{|B(x, \Delta/8)|} \, dR}{\Delta/4} \right)$$

$$\geq \exp\left(8r/\Delta \log \frac{|B(x,\Delta)|}{|B(x,\Delta/8)|} \right)$$

Heirarchical Tree Decomposition



[FRT)

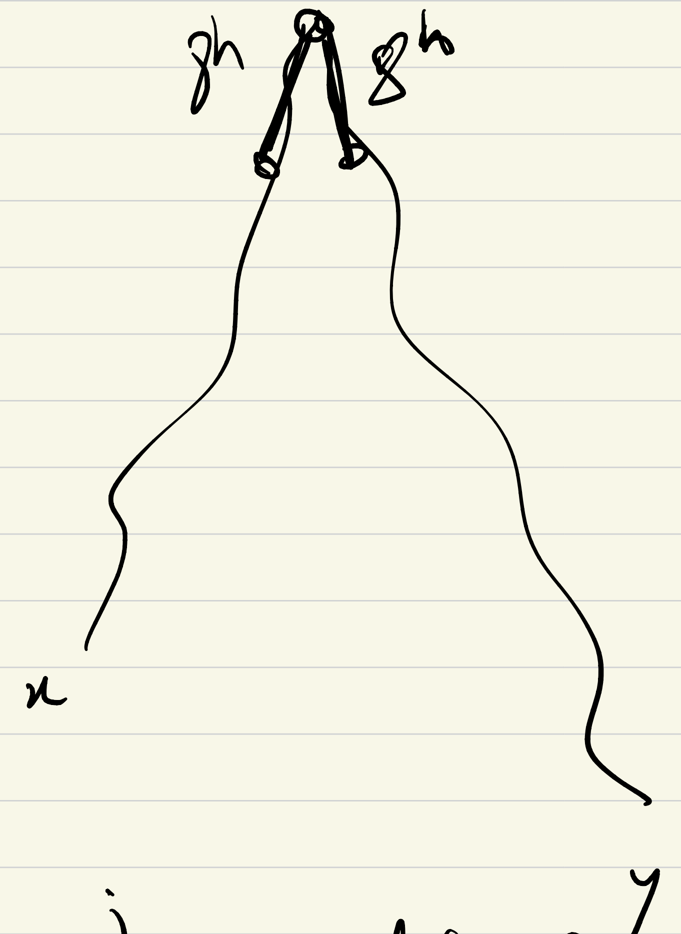


$8^{j-1} \leq d(x,y) \leq 8^j$

lca(x, y) is of height at least j

let h = height of lca(x, y) $\quad h \geq j$

$$d_T(x,y) \geq 8^h + 8^{h-1} + $$



$$\geq 8^j \geq d(x,y)$$

$8^{j-1} \le d(x,y) \le 8^{j}$

$$E\left[d_T(x,y)\right] \le 8^{j} + \sum_{t=j+1}^{\infty} 8^{t} \cdot \Pr\left[P_t(x) \ne P_t(y)\right]$$

$$\le 8^{j} + \sum_{j+1}^{\infty} \cancel{8^{t}} \cdot 8 \, \frac{d(x,y)}{\cancel{8^{t}}} \log \frac{|B(x,8^{t})|}{|B(x,8^{t-1})|}$$

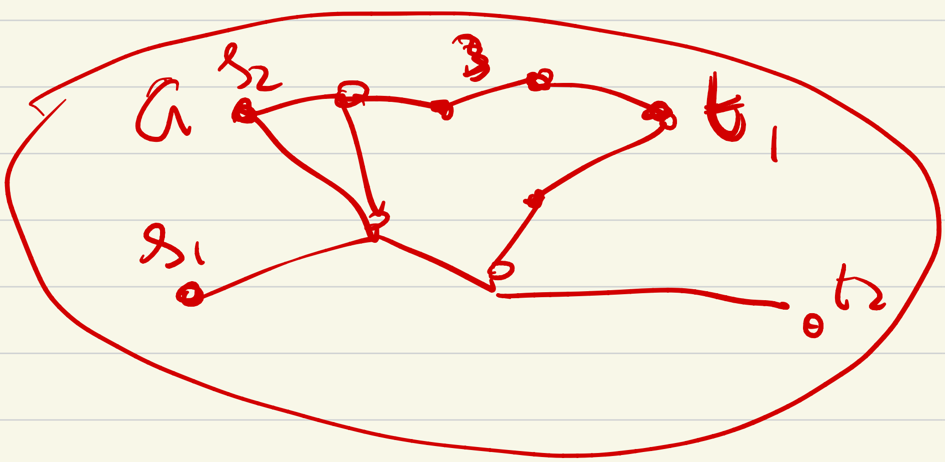
$$\le 8^{j} + 8d(x,y) \cdot \sum_{j+1}^{\infty} \underbrace{\log \frac{|B(x,8^{t})|}{|B(x,8^{t-1})|}}_{\log|B(x,8^{t})| - \log|B(x,8^{t-1})|}$$

$$\le 8^{j} + 8d(x,y) \cdot \log n \simeq O(\log n)\, d(x,y)$$

Low-Stretch Spanning Trees:

Buy at Bulk Network Design

→ design a graph that can support a flow



$s_1 \to t_1$ support a flow of value f_1

$s_2 \to t_2$ f_2

$s_k \to t_2$ f_k

To buy capacity c on an edge e

$\downarrow$

Cost(c)

$$\text{Min} \sum_e \text{cost}(c_e)\,\text{dist}(e)$$

Find c_e such that

supports $s_1 \sim t_1$

$\vdots$

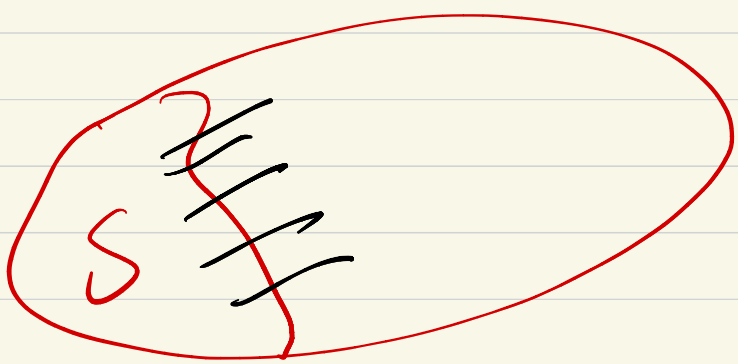
$s_k \sim t_k$

LECTURE 13

SPARSEST CUT / Graph Expansion
– Metric embeddings

GRAPH EXPANSION

$G = (V, E)$ unweighted / d-regular (degree $= d$)



$$\phi(S) = \frac{E[S, \bar{S}]}{\mathrm{Vol}(S)}$$

$\mathrm{Vol}(S) =$ total # of edges incident on S

$= d\,|S|$

$$\phi(S) = \frac{E[S, \bar{S}]}{d|S|}$$

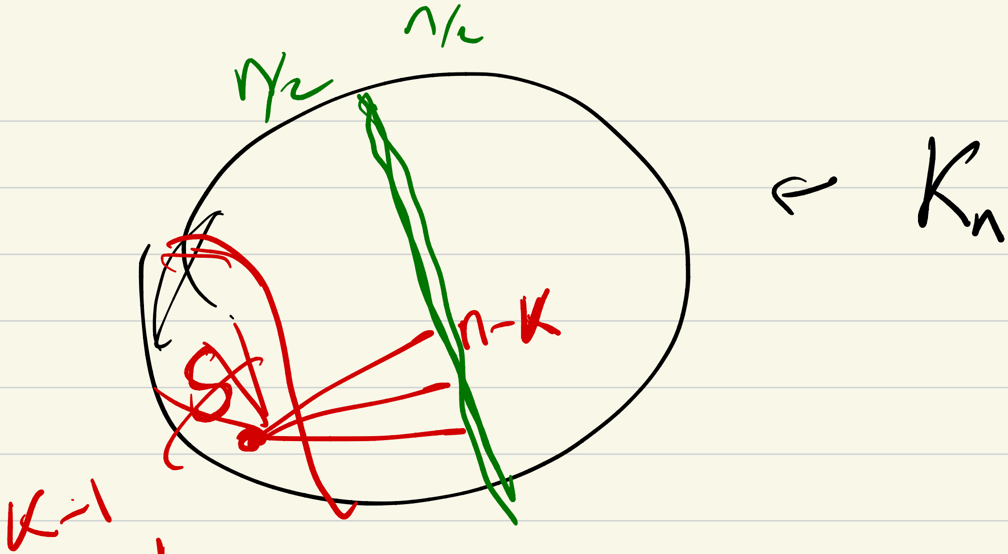
$\leftarrow$ vdone

$$= \Pr\begin{bmatrix} \text{pick a random vertex } u \in S \\ \text{take a random } (u,v) \text{ edge} \\ \text{leave the set } v \notin S \end{bmatrix}$$

$\in [0,1]$

conductance

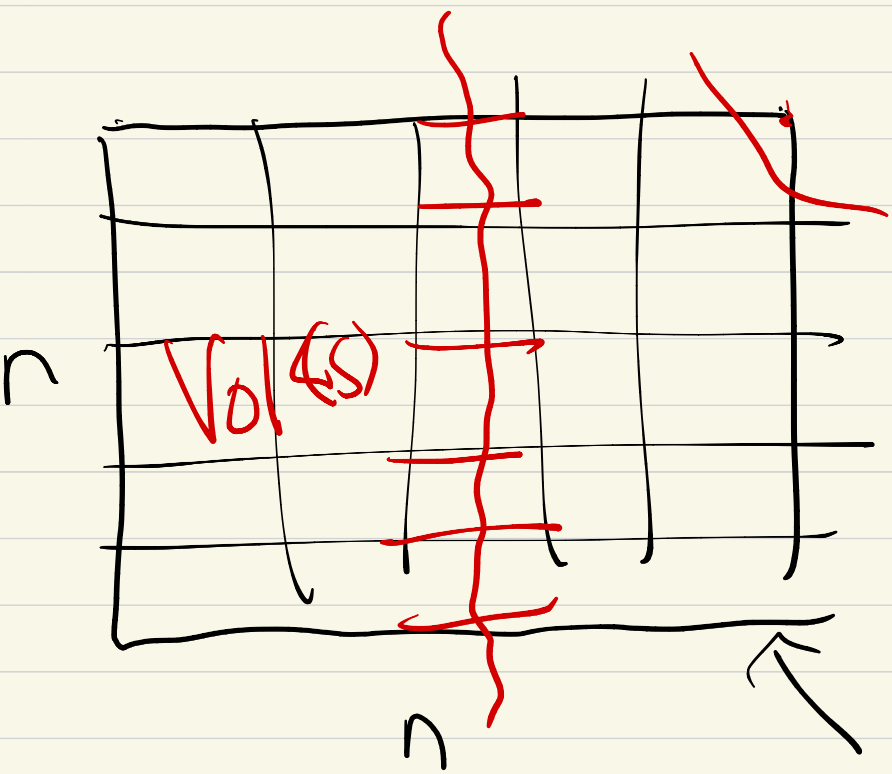
$$\phi(G) = \min_{|S| \leq |V|/2} \phi(S)$$



$$|S| = k < n/2$$

$$\phi(S) = \frac{n-k}{n-1} \geq 1/2$$

highest expansion



$$\frac{n}{\Omega(n^2)} \leq \boxed{\frac{1}{n}}$$

deg 4

Complete graphs $\rightarrow$ most well connected graph

$\wr \; \binom{n}{2}$

Expander Graph $\quad G=(V,E) \quad \deg(v) < O(1)$

3

st $\quad \phi(G) \geq \Omega(1)$

Thm: Random 3-regular graph is an expander with high probability

Construction: Vertices = $\{0,1 \ldots p-1\}$ for a prime p

Edges = $x \to x+1 \pmod p$, $x \to x-1 \pmod p$, $x \to 1/x \pmod p$

$$\boxed{x-\frac{1}{2}=\frac{1}{x}}$$

$$\boxed{x^2+1=\frac{1}{x}}$$

$x^2+x-1 \pmod p$

Given a graph $G \to$ approximate $\phi(G)$??

$|S| \leq n/2 \qquad |\bar{S}| \in [n/2, n]$

$$\phi(G) = \frac{E[S,\bar{S}]}{d\,|S|}$$

$\leftarrow$ pairs of vertices

$\leftarrow$ vertices

$$\psi(G) = n \cdot \left[\frac{E[S,\bar{S}]}{|S|\cdot|\bar{S}|}\right]$$

$\leftarrow$ $\dfrac{\#\text{edges } (S,\bar{S}) \text{ in } G}{\#\text{edges } (S,\bar{S}) \text{ in } K_n \text{ complete}}$

$$\phi(G)/\psi(G) \in (1/2, 1)$$

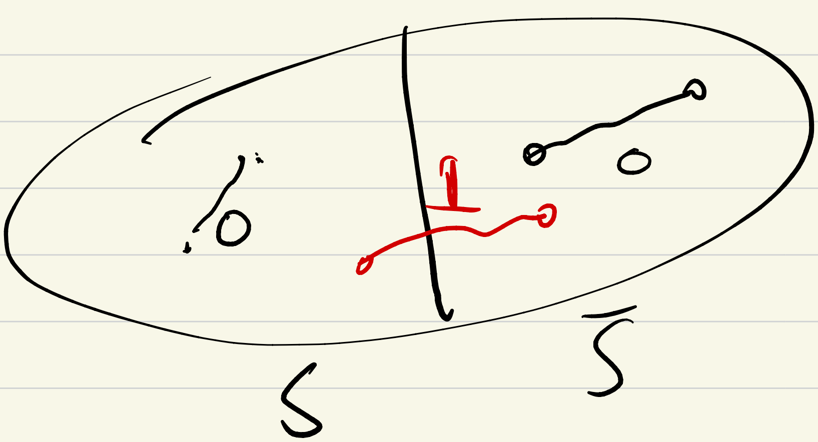
$$\Psi(G) = \min_{S} \frac{\#\text{ of edges } E(S,\bar{S}) \text{ in } G}{\#\text{ of edges from } S \text{ to } \bar{S} \text{ in } K_n}$$

"Cuts" $(S, \bar{S})$

Cut Metric: $S \subseteq V$

$$\mathbb{1}_S(v) = \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{otherwise} \end{cases}$$

$$\text{dist}_S(u,v) = |\mathbb{1}_S(u) - \mathbb{1}_S(v)|$$



$$\phi(a) = \min_{S} \left\{ \frac{\lambda \sum_{(u,v)\in E} \text{dist}_S(u,v)}{\lambda \sum_{u,v \in \binom{V}{2}} \text{dist}_S(u,v)} \right.$$

$$\text{dist}_S(u,v) = |\overset{\{0,1\}}{\mathbb{1}_S(u)} - \mathbb{1}_S(v)|$$

Linear Program: for $\phi(a)$

Minimise $\sum_{(u,v)\in E} d(u,v)$ (Numerator)

Subject: $d : V \times V \to \mathbb{R}^+$

$$\begin{cases} d(u,u) = 0 \\ d(u,v) = d(v,u) \\ d(u,v) + d(v,w) \geq d(u,w) \end{cases}$$

(denominator) $\sum_{u,v \in V} d(u,v) \geq 1$

$d : V \times V \to \mathbb{R}^+$

$\downarrow$ $(\log n)$-distortion

ℓ_1 metric on $\mathbb{R}^k$

$\downarrow$ $\|$ convex combination of cuts

Cut metric / Subset S

Thm: Every n-point metric $d: V \times V \to \mathbb{R}^+$ embeds into ℓ_1 with $O(\log n)$ distortion.

Proof:

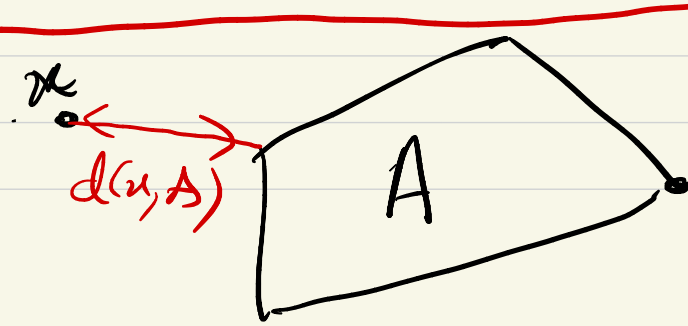
Frechet Embedding:

Pick $\boxed{A} \subseteq V$

$$\boxed{f_A : x \in V \longrightarrow d(x,A) \in \mathbb{R}}$$

$$f : V \longrightarrow \mathbb{R}$$

$$\boxed{d(x,A) = \min_{y \in A} d(x,y)}$$



$\forall A \subseteq V$

Property 1: Frechet embeddings are contract.

$x, y \in V$ / $A \subseteq V$

$$|f(x) - f(y)| \leq d(x,y)$$

$$\| $$

$$|d(x, \overset{z}{A}) - d(y, A)| \leq d(x,y)$$

($\overset{\times}{\text{M}}$etric Space)

$$|d(x,z) - d(y,z)| \leq d(x,y)$$

(triangle inequality)

Bourgain's embedding

$(V, d) \to \ell_2$

$\to \ell_1$

$\forall p :$

ℓ_p

- For $t = 1 \ldots \log n$

$$A_t = \left\{ \forall\, y \in V \text{ include } y \text{ w.p. } \frac{1}{2^t} \right\}$$

- $F : x \to \big(d(x, A_1), d(x, A_2), \ldots, d(x, A_t) \big]$

$\longleftrightarrow$ $\log n$ - dimensional vector

$\in \mathbb{R}^{\log n}$

With prob 1,

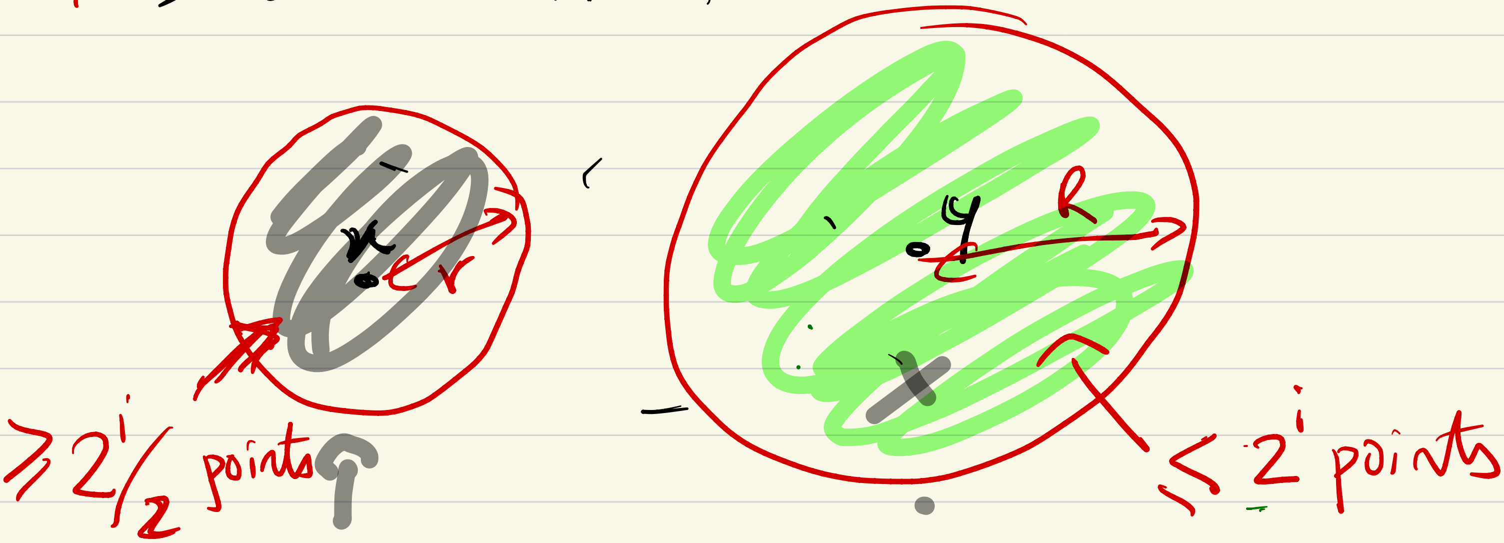
Claim: $\|F(x)-F(y)\|_1 \lesssim (\log n)\cdot d(x,y)$

$$\|F(x)-F(y)\|_1 = \sum_{t=1}^{\log n} \underbrace{|d(x,A_t)-d(y,A_t)|}_{\lesssim d(x,y)}$$

$$\lesssim (\log n)\, d(x,y)$$

Claim: $\boxed{\mathbb{E}\left[\|F(x)-F(y)\|_1\right]} \geq \Omega(1)\cdot d(x,y)$

$A_i \Rightarrow$ picks every point w.p. $1/2^i$



Suppose, $\Pr\left[A_i \cap B(y, R) = \phi\right] = \left(1 - \frac{1}{2^i}\right)$

but $\quad A_i \cap B(x, r) \neq \phi$

$$d(y, A_i) \geq R$$

& $\quad d(x, A_i) \leq r$

$$\Rightarrow \left| d(y, A_i) - d(x, A_i) \right| \geq R - r$$

With const prob

$$\Pr\left[\underset{=\phi}{A_i \cap B(y,R)}\right] = \left(1 - \frac{1}{2^i}\right)^{|B(y,R)|}$$

$$\geq \left(1 - \frac{1}{2^i}\right)^{2^i} \gtrsim \frac{1}{4}$$

$$\Pr\left[\underset{\neq\phi}{A_i \cap B(x,r)}\right] = 1 - \left(1 - \frac{1}{2^i}\right)^{|B(x,r)|}$$

$$\geq 1 - \left(1 - \frac{1}{2^i}\right)^{2^i/2}$$

$$\gtrsim \frac{1}{4}$$

Claim: Suppose $|B(x,r)| \geq 2^i/2$

$|B^\circ(y,R)| \leq 2^i$

$B(x,R) \cap B(y,R) = \phi$

then with const prob:

$$|d(x,A_i) - d(y,A_i)| \geq R-r$$

$$\Rightarrow \mathbb{E}\left[|d(x,A_i) - d(y,A_i)|\right] \geq \Omega(1)\cdot(R-r)$$

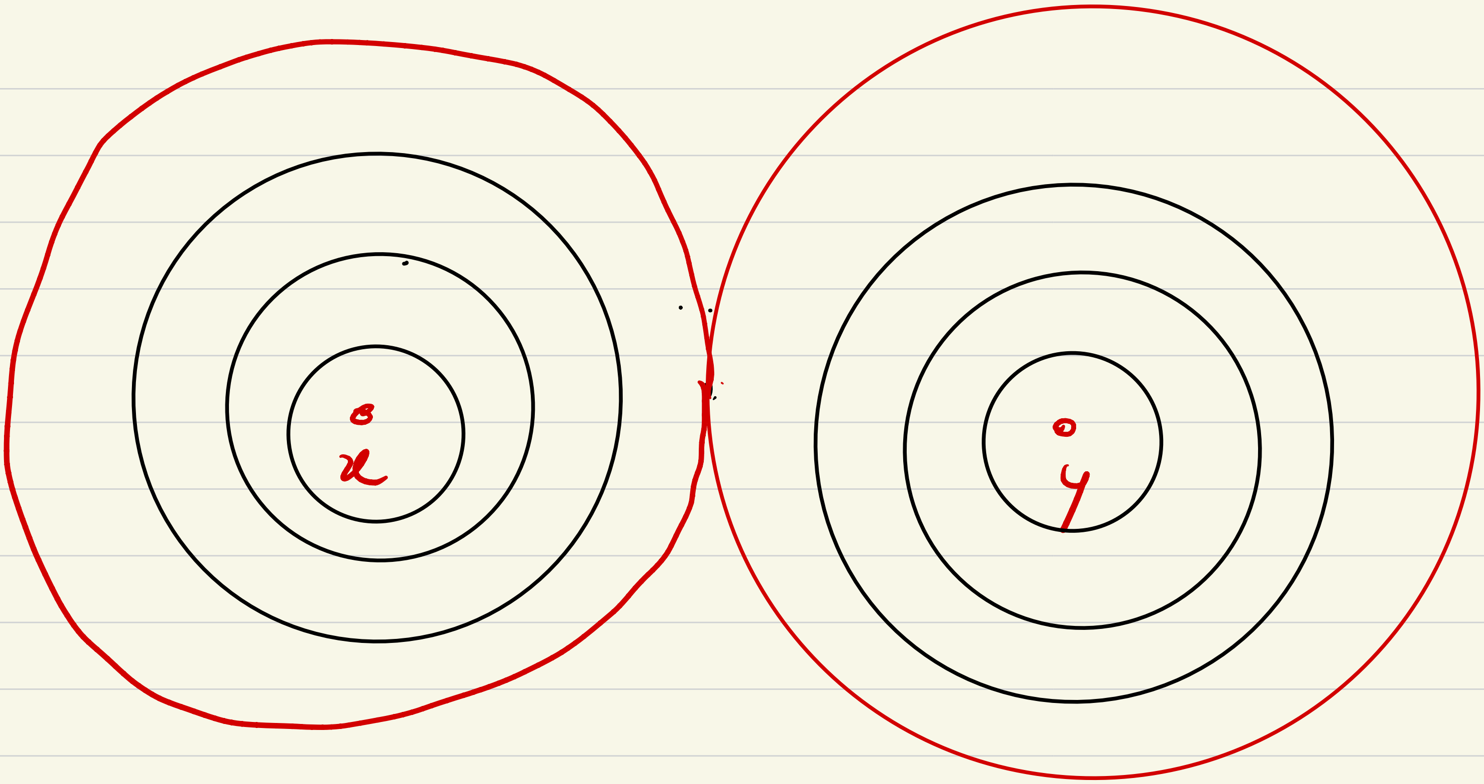
Pick $F_1, F_2 \ldots F_{100\log n} : V \to \mathbb{R}^{\log n}$

$$\overset{\log n^{2}}{\longleftrightarrow}$$

$$\hat{F} = \frac{1}{K}(F_1, F_2, \ldots F_{\underbrace{100\log n}_{K}})$$

$\forall x,y$

$$\mathbb{E}\left[\|F_i(x) - F_i(y)\|_1\right] \leq (\log n)\, d(x,y)$$



$$\Delta_i = \min_r \{ \underbrace{|B(x,r)|}_{\geq 2^i},\ |B(y,r)| \geq 2^i \}$$

unless $r > d(x,y)/2$

then define $\Delta_i = d(x,y)/2$

just before

$R = \Delta_i \boxed{-\varepsilon}$ ← one of the balls has $< 2^i$ points

$r = \Delta_{i-1}$ both have $\geq 2^{i-1}$ points.

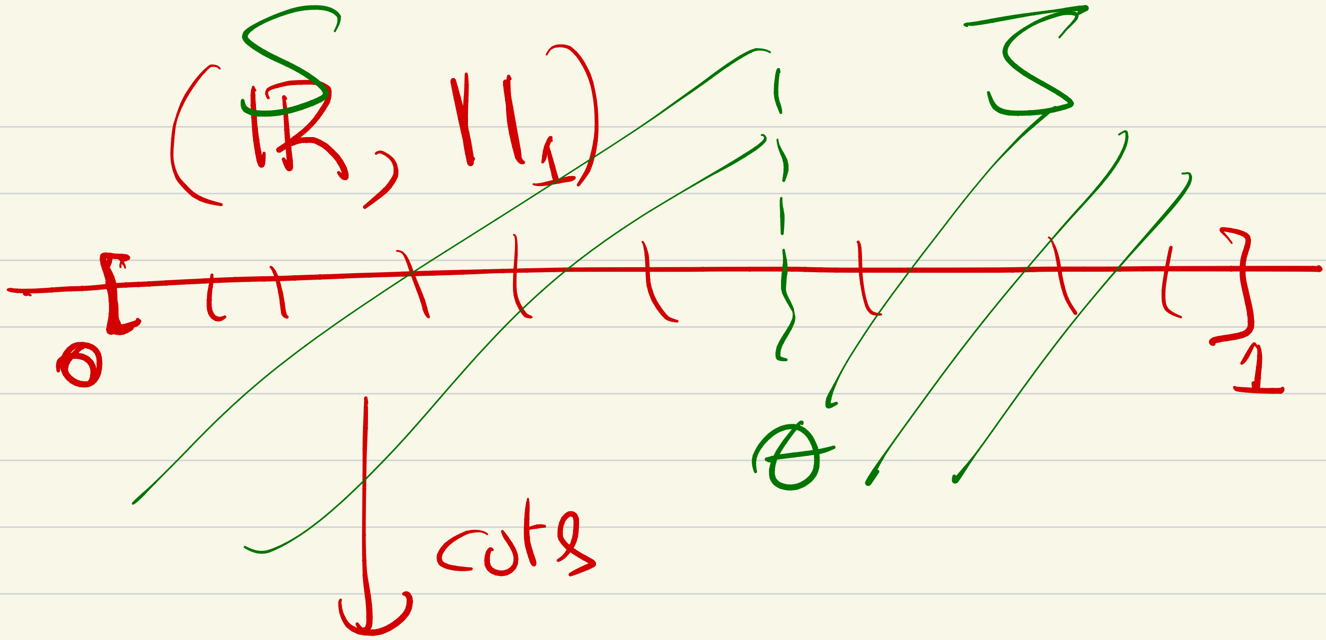
By our claim

$$E\left[\,|d(x,A_i) - d(y,A_i)|\,\right] \geq \Omega(1) \cdot (\Delta_i - \Delta_{i-1})$$

Adding over all i

$$E\left[\,\|F(x) - F(y)\|_1\right] \geq \Omega(1) \cdot \left(d(x,y)\right)$$

$$\frac{d(x,y)}{2} = \sum_j \Delta_j - \Delta_{j-1}$$



S: Pick a random $\theta \in [0,1]$

$$\mathop{\mathbb{E}}_{\theta}\left[\left|\mathbb{1}_S(u) - \mathbb{1}_S(v)\right|\right]$$

$$= |F(u) - F(v)|_1$$

$(\mathbb{R}^k, |\,|_1) \longrightarrow$ Cut Metric

$\parallel$

$\sum (\mathbb{R}, |\,|_1)$