

## LECTURE 19

Semidefinite Programming

LP  
Spectral techniques

↓  
generalisation of linear programming

↳ Very expressive & powerful  
optimization primitive

↳ Python vs Assembly

↳ Linear programming  
+  
Eigen vector / eigenvalue comp

# Semidefinite program

$\text{poly}(n, \log 1/\epsilon)$

~~====~~

L linear program "over" positive semidefinite matrices.

$$\underset{\text{Min}}{\langle C, X \rangle} = \sum C_{ij} X_{ij} \rightarrow f(X_{ij}) =$$

$$\langle A^{(l)}, X \rangle = \sum_{i,j} A_{ij}^{(l)} \cdot X_{ij} \leq b^{(l)} \quad \text{for } l=1..m$$

$$X = \left( X_{ij} \right)_{\substack{i=1..n \\ j=1..n}} \gamma_0 \}$$



Vectors  $v_1 .. v_n \in \mathbb{R}^n$

st

$$\text{Min} \sum C_{ij} \langle v_i, v_j \rangle$$

subject to

$$\sum A_{ij}^{(l)} \langle v_i, v_j \rangle \leq b^{(l)}$$

# Maximum Cut [Goemans - Williamson]

- Matrix Multiplication  
weights

Input: Graph  $G = (V, E)$

Goal: Find a cut  $S \cup S^c = V$  that

maximizes  $|E[S, S^c]|$

↑  
# of crossing edges.

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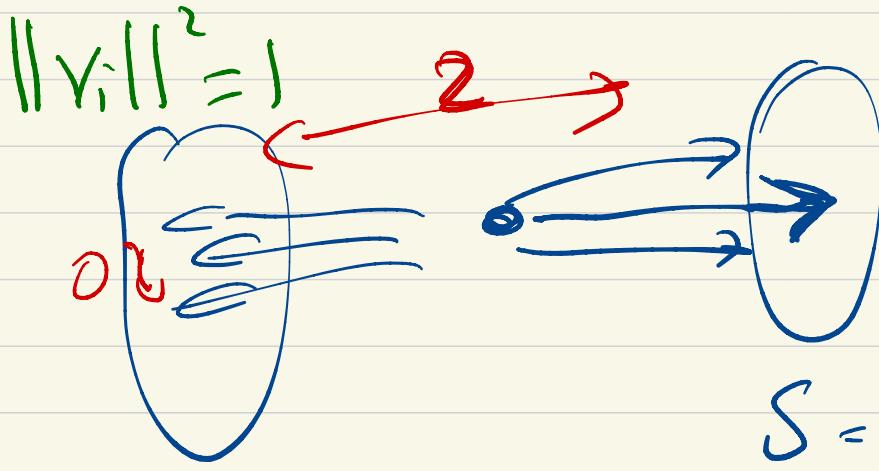
Random Cut  $\geq \frac{1}{2} \cdot \text{MaxCut}(G)$

- subexponential sized LPs don't do  
any better

$$\tilde{O}(|E| \cdot \text{poly}(\frac{1}{\epsilon}))$$

$G \rightarrow \{v_1, \dots, v_n\}$   
 vectors  
 (denote the cut)

Intended solution:  $(S, \bar{S})$  is the optimum  
 max



$$S = +1$$

$$\bar{S} = -1 \text{ vector}$$

$$v_i = \begin{cases} +1 \cdot e_i & \text{if } i \in S \\ -1 \cdot e_i & \text{if } i \notin S \end{cases}$$

$$E[S, \bar{S}] = \frac{1}{4} \sum_{(i,j) \in E} \|v_i - v_j\|^2$$

" 1 if  $i \in S, j \in \bar{S}$   
 0 if  $i, j \in S$   
 $i, j \in \bar{S}$

## GW-SDP Relaxation

$$\text{Max} \quad \frac{1}{4} \sum_{(i,j) \in E} \|v_i - v_j\|^2$$

linear in  $\langle v_i, v_j \rangle$

linear constraints  $\rightarrow \|v_i\|^2 = 1$

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$\|v_i\|^2 = \langle v_i, v_i \rangle$

$$\|v_i\|^2 = \langle v_i, v_i \rangle$$

$$\|v_i - v_j\|^2 = \langle v_i - v_j, v_i - v_j \rangle$$


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Any cut  $(S, \bar{S})$

Any  $\{\pm 1\}$  assignment  
to vertices

$\Rightarrow$  feasible  
solution  
to GW-SDP

$$v_i = +$$

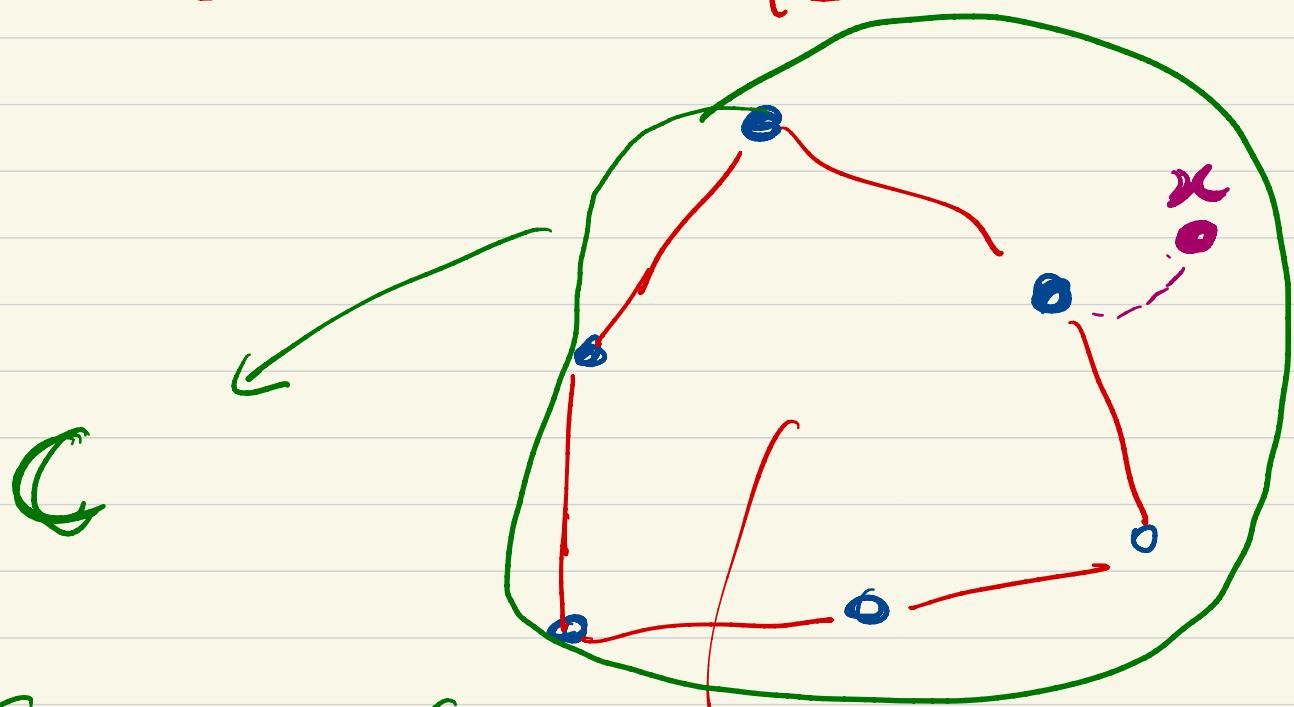
OBSERVATION: If graph  $G$ .

$$\text{OPTIMUM}_{\text{GW-SDP}}(G) \geq \text{MaxCut}(G)$$


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Combinatorial Optimization problem: MaxCut

Space of solutions = Discrete set  
 $S$  =  $\{ \pm 1 \}^n$



$$C \supseteq \text{Convex Hull}(S)$$

$$\text{Convex Hull}(S)$$

$\xrightarrow{\text{optimize over } C}$

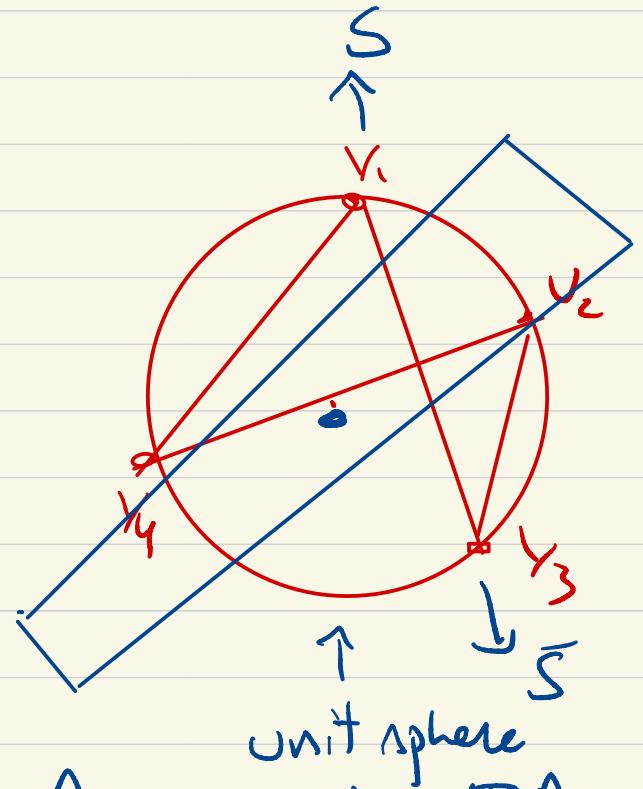
- Solve the relaxation

- Round the solution to relaxation

→ Solve AwSOp(a)

$$\rightarrow \{v_1, \dots, v_n\} \quad |||v_i|||=1$$

Total Squared Distance =  $\frac{1}{4} \sum_{(i,j) \in E} \|v_i - v_j\|^2$

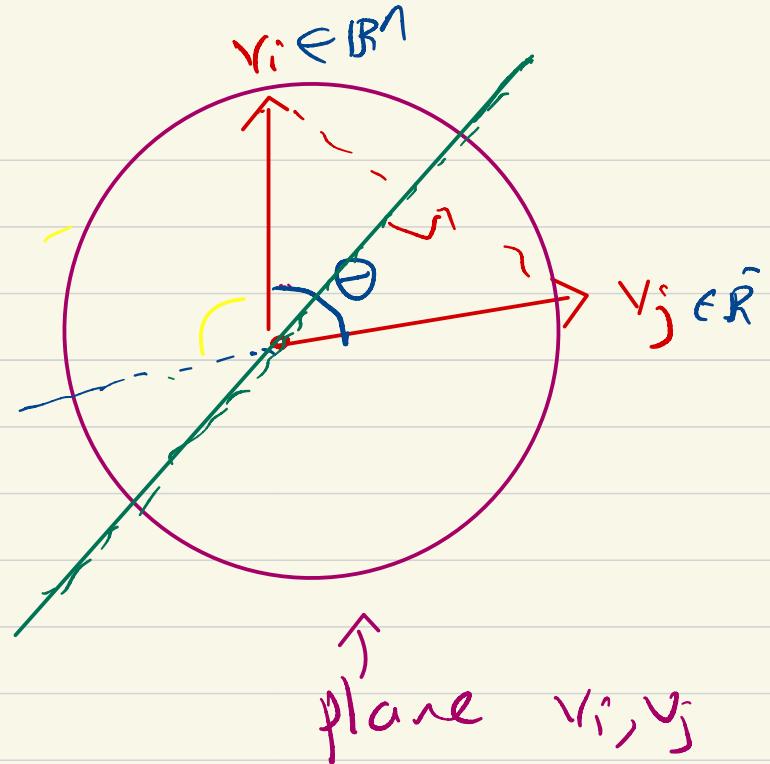


→ Cut the unit sphere by a random half space

- Pick a random  $g \in \mathbb{R}^n$

$$x_i = \begin{cases} +1 & \text{if } \langle g, y_i \rangle > 0 \\ -1 & \text{otherwise} \end{cases} \quad i \in \bar{S}$$

Analysis Fix  $(i, j) \in E$



$$\Pr[(i, j) \text{ is cut}] = \frac{\angle v_i, v_j}{\pi}$$

$$= \frac{\theta}{\pi} = \frac{\arccos(\langle v_i, v_j \rangle)}{\pi}$$

$$\underline{\mathbb{E}[|E(S, \bar{S})|]} = \mathbb{E}\left[\sum_{(i, j) \in E} \mathbf{1}_{[(i, j) \text{ is cut}]}\right]$$

maxcut(G)

$$= \sum_{(i, j) \in E} \Pr[(i, j) \text{ is cut}]$$

$$= \sum_{(i, j) \in E} \frac{\arccos(\langle v_i, v_j \rangle)}{\pi}$$

maxcut(G)

$$\left( \mathbb{E} [|\mathbb{E}(S, \bar{S})|] \stackrel{\text{[Prove it.]}}{\geq} \alpha \cdot \text{GWSDF}(a) \right)$$

=

V1 [Relaxation]

$\alpha \cdot \text{MaxCut}(a)$

$$\sum_{(i,j) \in E} \left( \frac{\arccos(\langle v_i, v_j \rangle)}{\pi} \right) \geq \alpha \sum_{(i,j) \in E} \frac{\left( \frac{1}{4} \|v_i - v_j\|^2 \right)}{\|v_i - v_j\|}$$

$$\frac{\theta}{\pi}$$

$$\sin^2 \theta/2$$

$$\text{let } \alpha_{GW} = \min_{\theta \in (0, \pi)} \frac{\theta/\pi}{\sin^2 \theta/2} \geq 0.878\dots$$

$$\Rightarrow E[|E(S, \bar{S})|] \geq \alpha_{\text{GW}} = \text{GWSDP}(a)$$

$\approx$   
0.878...

//

$$\alpha_{\text{GW}} = \overbrace{\text{MaxCut}(a)}$$

Max

$$x^T L x = \frac{1}{4} \sum (x_i - \bar{x})^2$$

$$\text{Max } \text{Cof}(L) = \text{Max}_{\substack{x \in \{-1, 1\}^n \\ \sum x_i^2 = n}} x^T L x$$

Largest eigenvalue =  $\text{Max}_{\substack{x \in \mathbb{R}^n \\ \|x\|^2 = n}} x^T L x$

$$\begin{matrix} x \in \mathbb{R}^n \\ \|x\|^2 = n \end{matrix}$$

↑

$$x \in \mathbb{R}^n \quad \|x\|^2 = n \\ \sum x_i^2 = n$$

$\|V_i\|^2 \leq 1$

$$(\sqrt{n}, 0, 0, \dots, 0)$$

$$L = \begin{pmatrix} & & & & \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & H & 0 \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & & & & \end{pmatrix} \begin{matrix} \downarrow \sqrt{n} \\ \uparrow -\sqrt{n} \end{matrix}$$

# Max-Betweenness Problem

→ Find a permutation  $\{O_1, \dots, O_n\}$

s.t.:

- $m$  betweenness constraints.
- } 1)  $O_5$  is between  $O_6 \& O_{10}$
- } 2)  $O_1$  is between  $O_9 \& O_{12}$
- ⋮  
⋮

→ Guaranteed:  $\exists$  some permutation satisfying all the betweenness constraints

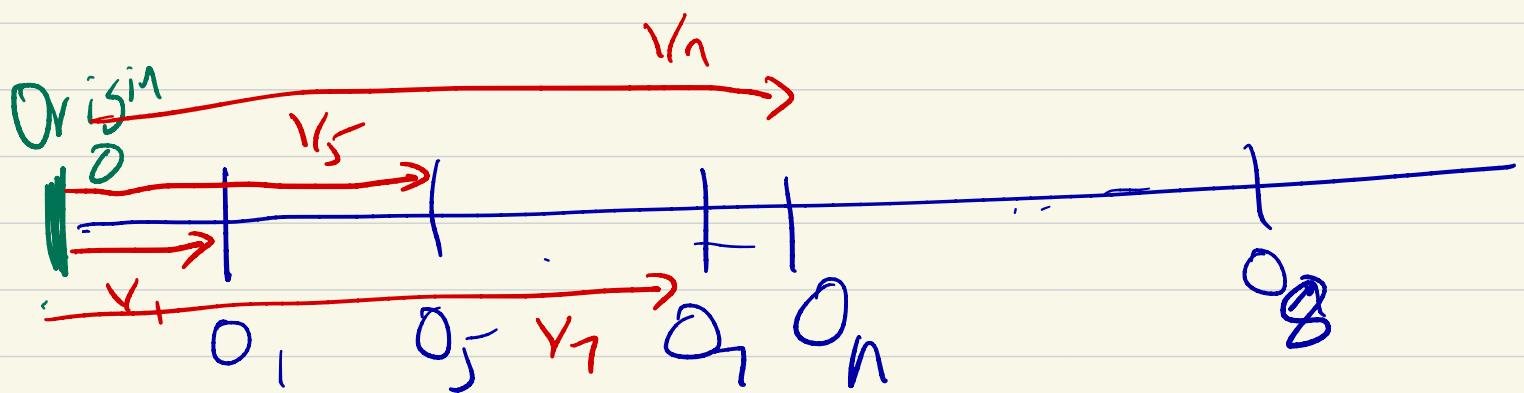
→ Goal: Satisfy as many as possible !!

$$POD: O_i \longrightarrow P(i)$$

$\in \{1, \dots, n\}$

$$O_i \text{ between } O_j \text{ & } O_k \iff P(j) < P(i) < P(k)$$

$$P(k) < P(i) < P(j)$$



$$v_j - v_i \quad \& \quad v_k - v_i$$

$$O_j \quad O_i \quad O_k$$

$\leftarrow \quad \longrightarrow$

point in  
opposite directions

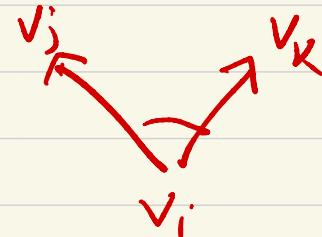
Find  $v_1 \dots v_n$  such that  
 $\|v_i\| = 1$  between  $0 \leq v_i \leq 0_k$

$\Downarrow$

$$\Rightarrow \left\langle \underbrace{v_j - v_i}_{\text{angle}}, \underbrace{v_k - v_i}_{\text{angle}} \right\rangle < 0$$

angle between

is obtuse



1) Solve SDP:  $v_1 \dots v_n \in \mathbb{R}^n$

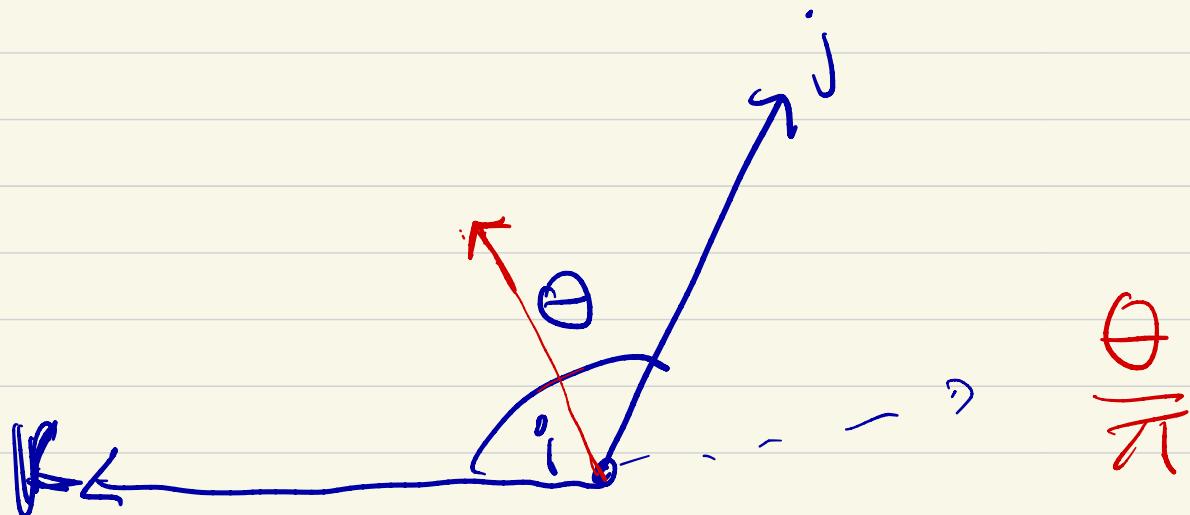
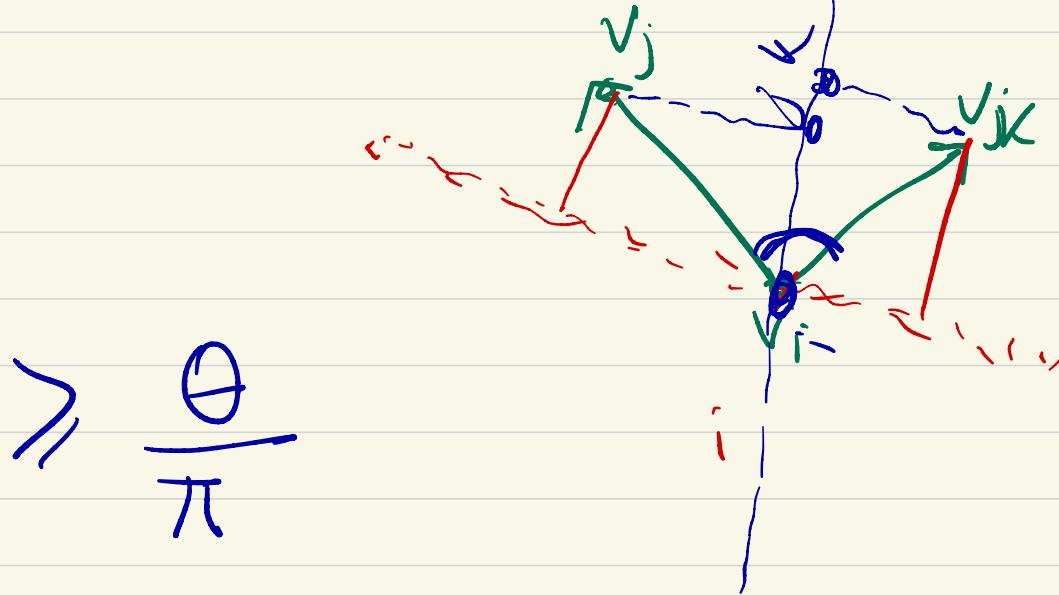
2) Pick random  $g \in \mathbb{R}^n$

$$x_i = \langle v_i, g \rangle \in \mathbb{R} \quad \text{project on } \underbrace{g}_{\in \mathbb{R}^n}$$

Permutation = increasing ordering of  $\{x_i\}$

$$\Pr \left[ \theta_i \text{ is between } \theta_j \text{ & } \theta_k \right] \geq \frac{1}{2}$$

∴ angle in radians



$$\theta = \pi$$

Defn: A matrix  $M \succ 0$  is same as  
real-symmetric

1) All eigenvalues  $> 0$

2)  $x^T M x > 0 \quad \forall x \in \mathbb{R}^n$

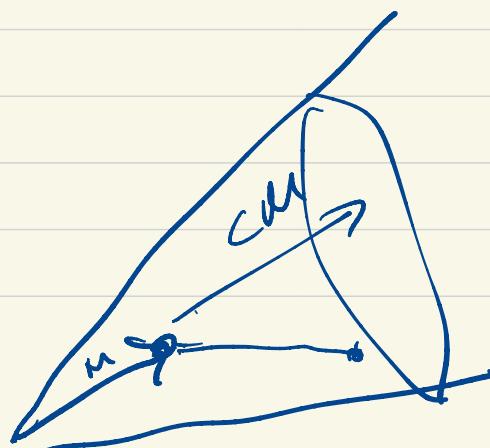
3)  $M = VV^T$  for  $V \in \mathbb{R}^{n \times n}$

4)  $\exists c_{ij}$  s.t.  $x^T M x = \sum_i \left( \sum_j c_{ij} x_j \right)^2$

5)  $\exists v_1 \dots v_n \in \mathbb{R}^n$  s.t.  
 $M_{ij} = \langle v_i, v_j \rangle$

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$S = \{M \mid M \succ 0\}$  is a convex cone  
 $\in \mathbb{R}^{n \times n}$



$$\langle A, B \rangle = \sum_{i,j} A_{ij} B_{ij}$$

$$= \text{Tr}(AB)$$