# LECTURE 4

- Centroid Algorithm
- Ellipsoid Algorithm

$\left.\right\}$ $\boxed{\text{poly}\left(\log\left(1/\varepsilon\right)\right)}$ for $\varepsilon$-accurate optima

Thm: Linear programming is in P

$$\left\{ \max c^T x \mid Ax \leq b \right\}$$

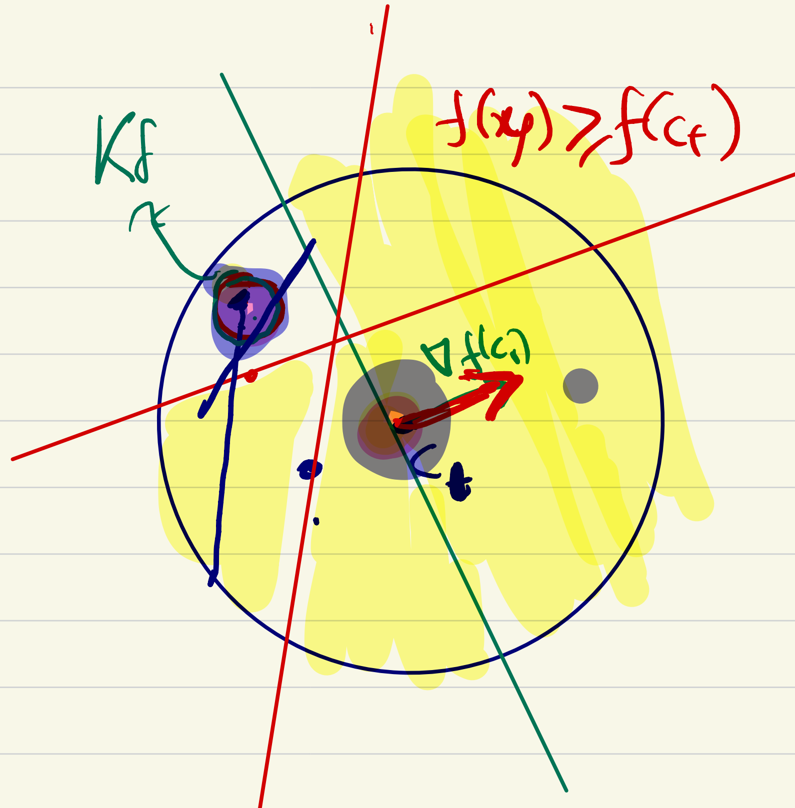
runtime = $\text{poly}(n, m, \langle A\rangle, \langle b\rangle, \langle c\rangle)$

Strongly polynomial time = $\text{poly}(m, n)$ in a model of infinite precision time

[OPEN QUESTION]

$$\boxed{\text{Runtime} = \binom{m}{n}}$$ [Exercise]

# Centroid Algorithm




$$\min_{x \in K} f(x)$$

$K_1 = K$

for $t = 1$ to $T$

$c_t \leftarrow$ centroid of $K_t$

$K_{t+1} \leftarrow K_t \cap \{\langle \nabla f(c_t), y - c_t \rangle < 0\}$

Return $\min\{f(c_1) \dots f(c_T)\}$

$\forall y$

$f(y) \geq f(c_t) + \langle \nabla f(c_t), y - c_t \rangle$

$\{y : \langle \nabla f(c_t), y - c_t \rangle \geq 0\}$

In each step volume decreases by $\left(1 - \frac{1}{e}\right)$

After $T$ steps: $\mathrm{vol}(K_t) < \exp(-T) \cdot \mathrm{Vol}(K)$

1) $x_*$ is the $\underset{x}{\operatorname{argmin}} f(x)$

2) $f(x) \in [-B, B]$

$$K^\delta = \left\{ (1-\lambda)x_* + \lambda x \;\middle|\; \begin{array}{l} x \in K \\ \lambda \in [0, \delta] \end{array} \right\}$$

→ $\operatorname{Vol}(K^\delta) = \operatorname{Vol}(K) \cdot \delta^n$

→ Pick $y \in K_\delta$

$y = (1-\lambda)x_* + \lambda x$ for some $x \in K$

$$f(y) \leq (1-\lambda) f(x_*) + \lambda f(x)$$

$$\leq f(x_*) + \lambda \left( f(x) - f(x_*) \right)$$

$$\leq f(x_*) + 2\delta B.$$

**Def:** $K \subseteq \mathbb{R}^n$

$$\text{Centroid}(K) = \frac{\int_{x \in K} x \, dx}{\text{Vol}(K)}$$

$$\text{Vol}(K) = \int_{x \in K} 1 \, dx$$

$\downarrow$

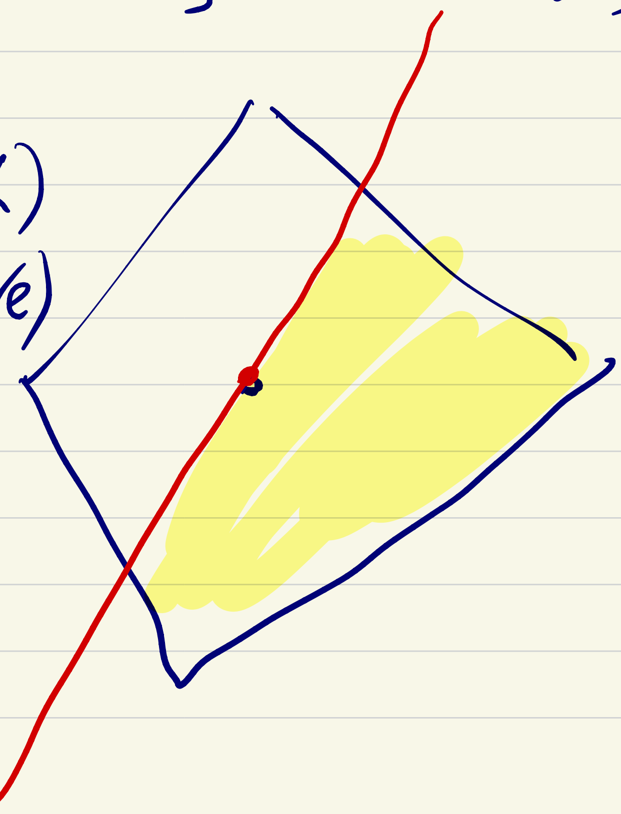
Sampling algorithm

**Thm:** [Grunbaum]

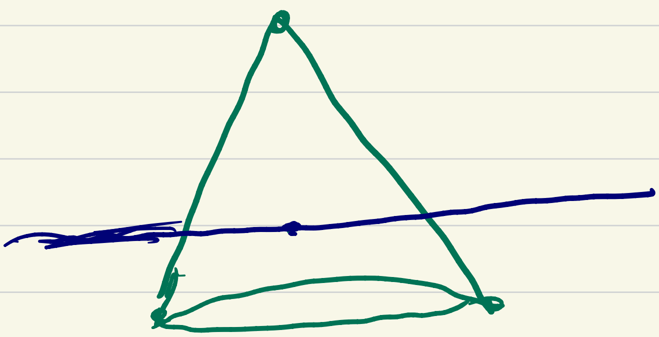
$\forall$ convex set $K$

$\forall$ halfspace $H$ passing through centroid$(K)$

$$\frac{\text{vol}(K)}{e} \leq \text{vol}(K \cap H) \leq \text{vol}(K) \cdot \left(1 - \tfrac{1}{e}\right)$$



**Proof:**



# Ellipsoid Algorithm

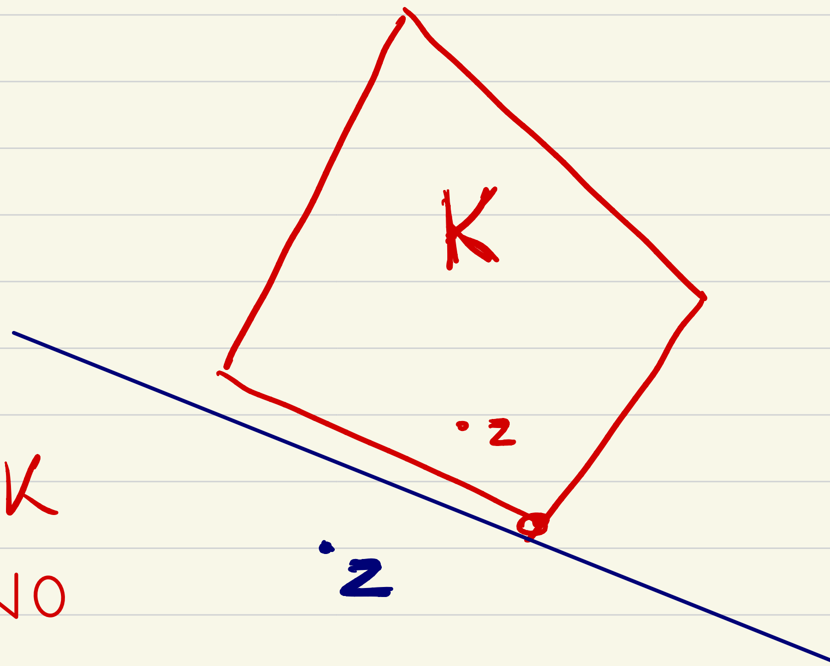
Def: (Separation Oracle)

$\downarrow z$

$\boxed{\text{Separation Oracle for } K}$

$z \in K$ → YES

$z \notin K$ → NO




a hyperplane separating $z$ from $K$

$a \in \mathbb{R}^n$, $b$ s.t.

$\langle a, z \rangle < b$

but $\forall x \in K$

$\langle a, x \rangle \geq b$

Thm: Given an LP

$$\{\min c^T x \mid x \in K\}$$

for polytope $K$ with a separation oracle then ellipsoid can solve LP optimally in time poly($n$, bit complexity)

[ Runtime does not depend # of constraints ]

{ exponentially many const }

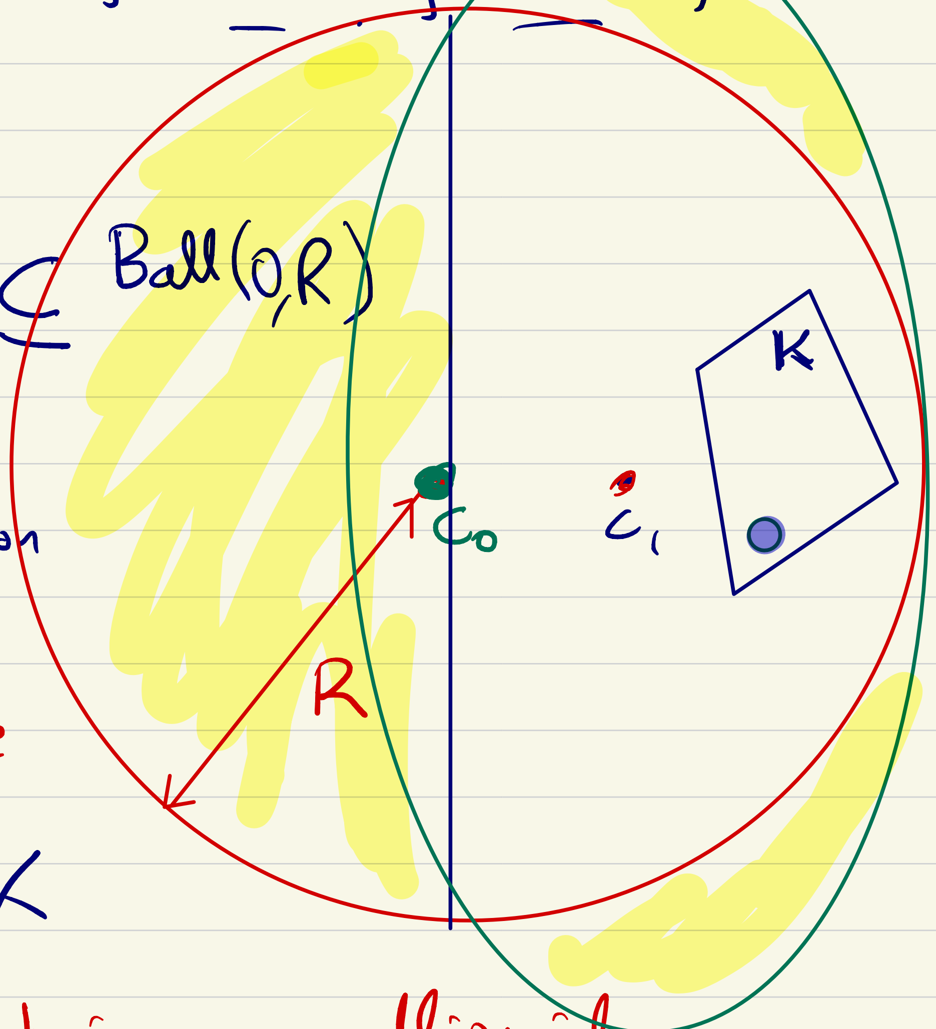
# Ellipsoid Algorithm: for LP feasibility

## Setup:

$\mathrm{Ball}(c,r) \subseteq$ Convex Set $K$ $\subseteq$ $\mathrm{Ball}(0,R)$

↓

feasible region of LP




2) Separation Oracle for $K$

Goal: Find $x \in K$

**Algorithm:** maintains an ellipsoid containing $K$

$$(\text{Ellipsoid}_0, \text{Centre}_0) \leftarrow (\mathrm{Ball}(0,R), 0)$$

- Separation Oracle $\leftarrow$ Is $c_t \in K$?
  - YES, return $c_t$
  - NO, get a halfspace $H$, find $E_{t+1} \leftarrow$ contains $E_t \cap H$

$$\boxed{\frac{\mathrm{Vol}(E_{t+1})}{\mathrm{Vol}(E_t)} \leq e^{-1/2(n+1)} \approx \left(1-\frac{1}{n}\right)}$$

$$\mathrm{Vol}(E_T) \leq e^{-T/n+1} \cdot \mathrm{Vol}(\mathrm{Ball}(0,R))$$

$$\Downarrow$$

$$\mathrm{Vol}(K) \geq r^n$$

$$\Downarrow$$

At some $t$, $c_t \in K$

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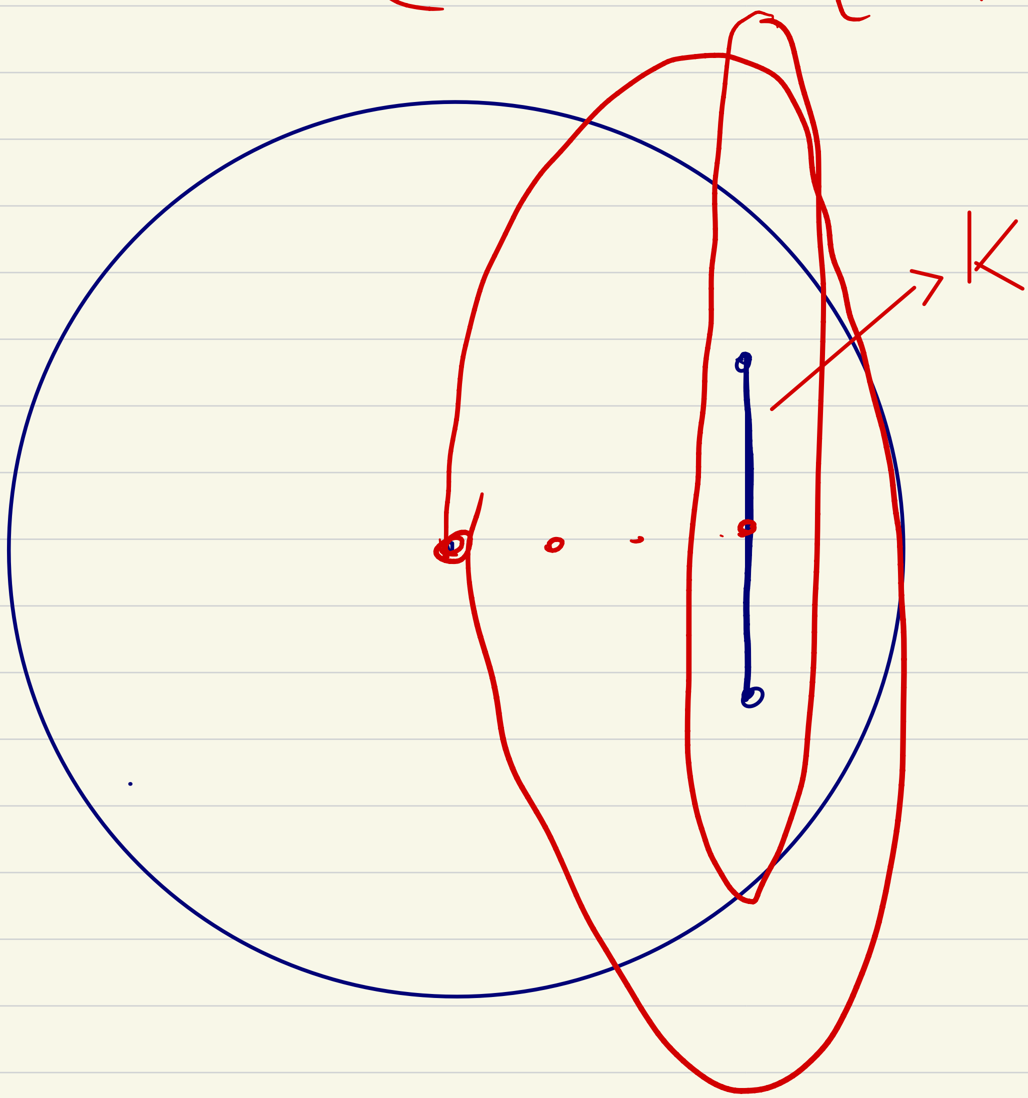
$\min_{a \in K} f(x) \iff$ Feasiblity $K \cap \{f(x) \leq \theta\}$

$$Ax = b \iff Ax \in [b-\varepsilon, b+\varepsilon]$$



## Ellipsoid

$$\text{Ball}(0,1) = \{x \mid \sum x_i^2 \le 1\}$$

Ellipsoid: $\left(\begin{array}{c}\text{invertible linear transformation} \\ L : \mathbb{R}^n \to \mathbb{R}^n\end{array}\right) \cdot \text{Ball}(0,R)$

$$\boxed{Q = (LL^T)^{-1}}$$

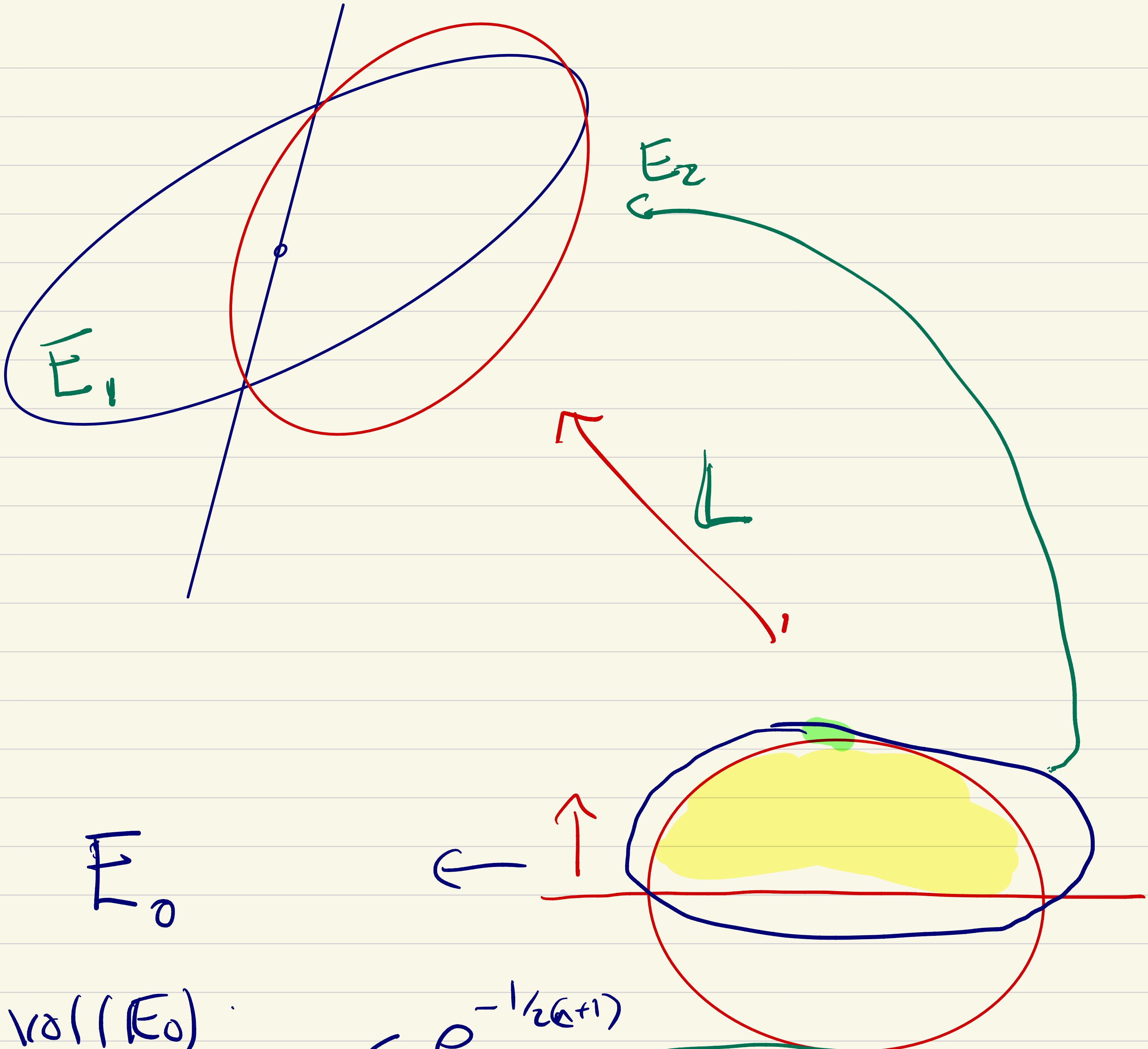
Ellipsoid: $E(\boxed{c, Q})$ [ Q is a positive semidefinite ]

$$= \{x \mid \underline{(x-c)^T Q (x-c) \le 1}\}$$

Thm: $A \subseteq \mathbb{R}^n$

$$\text{Vol}(L(A)) = \boxed{|\det(L)|} \cdot \text{Vol}(A)$$

$$\text{Vol}(E = L \cdot \text{Ball}(0,1)) = |\det(L)| \cdot \text{Vol}(\text{Ball})$$




$$\frac{Vol(E_0)}{Vol(Ball(0,1))} \leq e^{-1/2(n+1)}$$

$$\boxed{\begin{array}{c} Ball(0,1) \\ H = \{x \mid Ball(0,1),\ x_1 > 0\} \end{array}}$$

$$\| $$

$$\frac{Vol(LE_0)}{Vol(L \cdot Ball(0,1))}$$