

# LECTURE 23

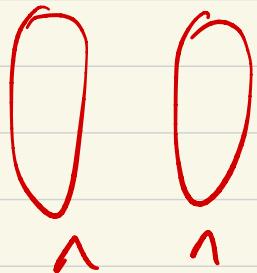
## Linear Programming in Combinatorial Optimization

- Matchings
- Spanning Trees

### Bipartite Matching

Input:  $G = (V_1 \cup V_2, E)$

$\omega: E \rightarrow \mathbb{R}^+$



Goal: Minimum weight matching

$$\min_M \sum_{e \in M} \omega_e$$

Variable:

$$x_e = \begin{cases} 0 & \text{if } e \notin M \\ 1 & \text{if } e \in M \end{cases}$$

$e \in E$

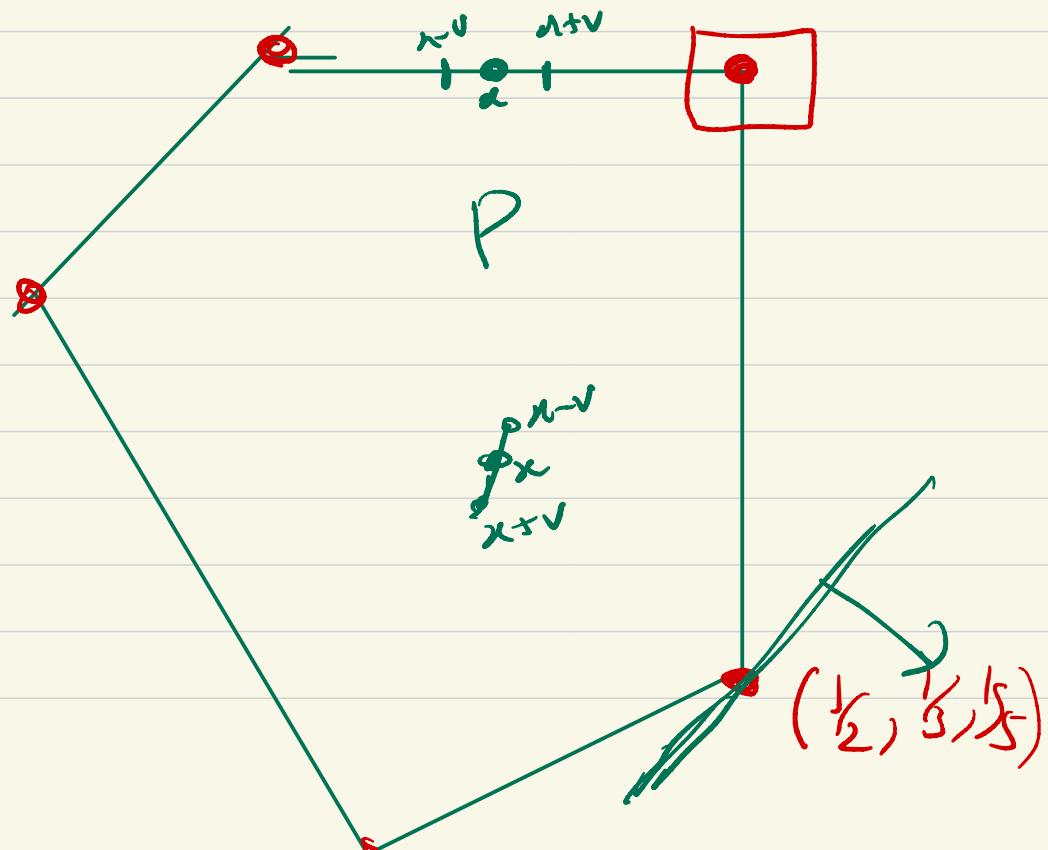
Polytope:

$$\chi(\delta(v)) := \sum_{e \ni v} x_e \leq 1$$

$$\forall v \in V_1 \cup V_2$$

$$x_e \geq 0$$

Feasible region



Defn: Polytope  $P$  is integral if  
all its extreme points  $\in \mathbb{Z}^m$

$\sum_{i=1}^m$

$x \notin$  Extreme point, if  $\exists v \neq 0$

such that  $x-v, x+v \in P$

Every extreme point  $x =$  intersection of  
of a polytope  $P \in \mathbb{R}^n$   $n$ -constraints

$$\begin{array}{l} \text{Linear Program: } \\ \begin{aligned} & \left\{ \begin{array}{l} \langle a_i, x \rangle \leq b_i \\ x_i \geq 0 \end{array} \right. \quad i=1..m \\ & n \end{aligned} \end{array}$$

Every extreme point  $x_* =$   $n$  linearly  
independent tight constraints

$$(x_1, \dots, x_n) = (0, 0, 0, 0, 0)$$

Thm: Bipartite Matching polytope is  
"integral". [No need to round]

[ Max flow algorithm returns  
integral flows on integral  
capacities ]



Thm: If weights  $\omega: E \rightarrow \mathbb{R}^+$   
[ an algorithm that finds a matching  $M$   
whose cost = LP optimum ]

Graph  $G = (V_1 \cup V_2, E)$ ,

- Solve LP to get an extreme

point  $x \in \mathbb{R}^E$

-  $x_e = 0$  for some  $e \in E$

$G \leftarrow G \setminus e$  and restart

-  $x_e = 1$  for some  $e \in F$ .

$e = (u, v)$  add  $(u, v) \rightarrow F$

$G \leftarrow G \setminus \{(u, v)\}$  and restart.

Every edge  $x_e$  is fractional !!.

$$\rightarrow \left[ x(\delta(v)) = \sum_{e \in v} x_e \leq 1 \right] \quad \forall v \in V, UV_2$$

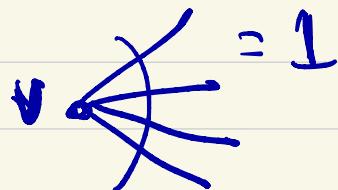
$\{ x_e \geq 0 \} \rightarrow \underline{\text{not tight!}}$

$x$  is an extreme point with  $x_e$  fractional

$\forall e \in E$ .

let  $W = \{v \mid x(\delta(v)) = \sum_{e \in v} x_e = 1\}$

$$\Rightarrow |W| = |E|$$

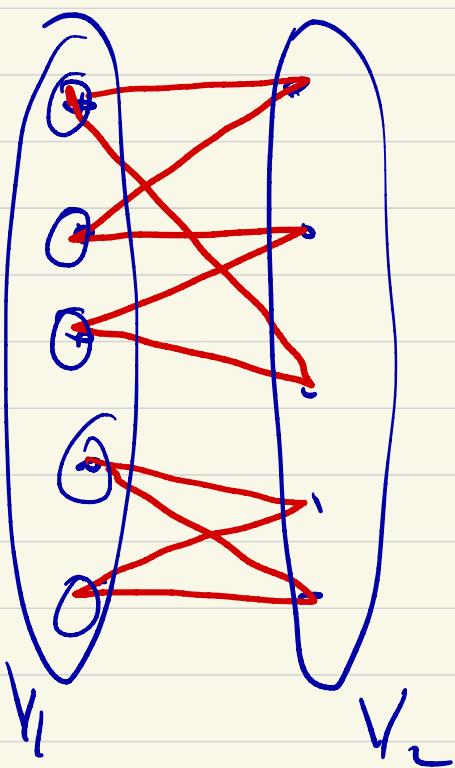


$$\forall v \in W \quad d_v \geq 2$$

$$[\because \sum_{e \in v} x_e = 1]$$

$$\Rightarrow 2|E| = \sum_{v \in V, UV_2} d_v \geq \sum_{v \in W} d_v \geq 2|W|$$

$$\Rightarrow \forall v \in W \quad d_v = 2 \quad \forall v \notin W \quad d_v = 0$$



( $\deg_2$  = <sup>every</sup> <sub>sum of</sub> cycles)

$$\sum_{v \in V_1} x_{\delta(v)} = \sum_{v \in V_2} x_{\delta(v)}$$

Sum of  
all edges

$\Rightarrow \exists 1$  linear dependence !!

# of linearly independent constraints  $\leq |W| - 1$

$\Rightarrow$  Contradiction

# Spanning Trees (Bounded-Degree)

Variables:  $x_e = \begin{cases} 1 & \text{if } e \in \text{Tree} \\ 0 & \text{otherwise} \end{cases}$

Constraints:

$$\sum_{e \in E(S)} x_e \leq |S| - 1 \quad \forall \text{ sets } S \subseteq V$$

(exponentially many constraints)

$$\sum_{e \in E} x_e = |V| - 1$$

$$(|W|) \left( \left( \sum_{e \ni v} x_e \leq b_v \right) \right) \quad \forall v \in W$$

$$LP(G, \underline{W}, \underline{b})$$

Thm: Spanning tree polytope is integral.

Ex:

$$\left\{ \begin{array}{l} \deg(s) \leq 1 \rightarrow 3 \\ \deg(t) \leq 1 \rightarrow 3 \\ \deg(u) \leq 2 \text{ } \forall u \in V \end{array} \right.$$



Hamiltonian path

Thm: ∃ a poly time alg to compute a tree  $T$  1)  $\text{cost}(T) \leq L P\text{-cost}$

$$2) \forall v \deg_T(v) \leq b_v + 2$$

~~( $b_v + 1$ )~~

While not done

- Solve & compute  $x \in$  extremal optima.  
 $\text{LP}(G, W, b)$

- 1)  $x_e = 0 \Rightarrow$  restart

$$\xrightarrow{\hspace{1cm}} \text{LP}(G|e, W, b)$$

- 2) Suppose vertex  $v$  has degree 1

then  $(u, v)$  is the edge

-  $F \leftarrow F \cup \{u, v\}$

-  $G \leftarrow G \setminus v$

-  $b'_u \leftarrow b_u - 1$  (Restart)  
 $(G \setminus v, W, b')$

Cass  $\rightarrow \exists$  a vertex  $v$  w/  $\deg(v) \leq 3$

Drop " $\sum_{e \in v} x_e \leq b_v$ "  
Restart

$\Rightarrow \underline{|E|}$  tight constraints

$\forall v \in W \quad \deg(v) \geq 4$

$\forall v \in V \setminus W \quad \deg(v) \geq 2$

$$|E| \geq \frac{1}{2} \sum_r \deg(r)$$

$$= \frac{1}{2} [ 4|W| + 2(n - |W|) ]$$

$$\geq \underline{\underline{n + |W|}}$$

$\Rightarrow$  We need  $\geq \underline{\underline{n + |W|}}$  tight constraints

$\leq |W|$  tight degree constraints

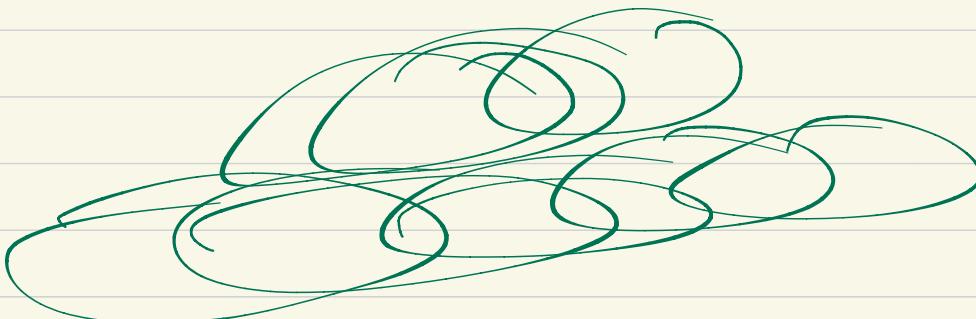
$\leq (n-1) \leq n-1 + |W|$  tight linearly independent (Laminar family)

Let

$$T = \left\{ S \subseteq V \mid \sum_{e \in E(S)} x_e = |S|-1 \right\}$$

Fact:

$$A, B \in T \Rightarrow A \cup B, A \cap B \in T$$



B



A ∪ B

Lemma:  $\exists$  a laminar family  $L$ ,  $\text{Span}(L) = \text{Span}(T)$

$$\Rightarrow \dim(\text{Span}(T)) \leq |V|-1$$

Laminar family  $\mathcal{L}$ :

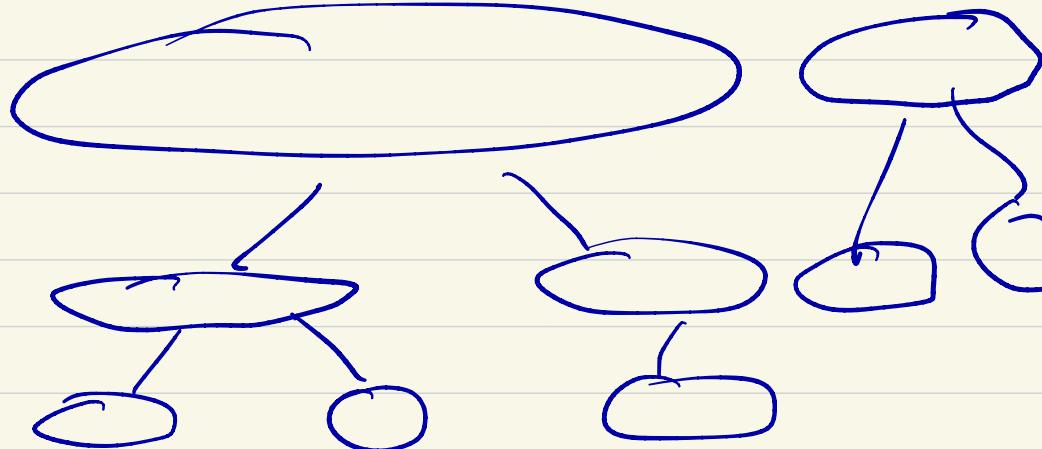
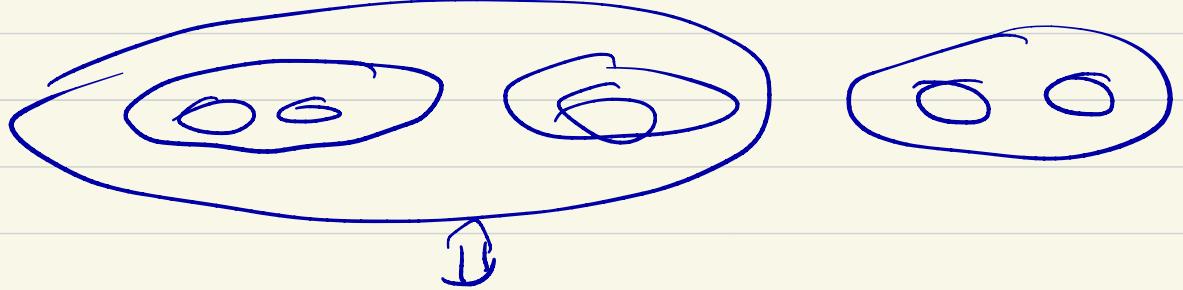
$\mathcal{L} =$  Collection of sets s.t  
no pairs cross. =



$A \cap B$  cross

||

$A \cap B \neq \emptyset$  but  
 $A \not\subseteq B$   $B \not\subseteq A$

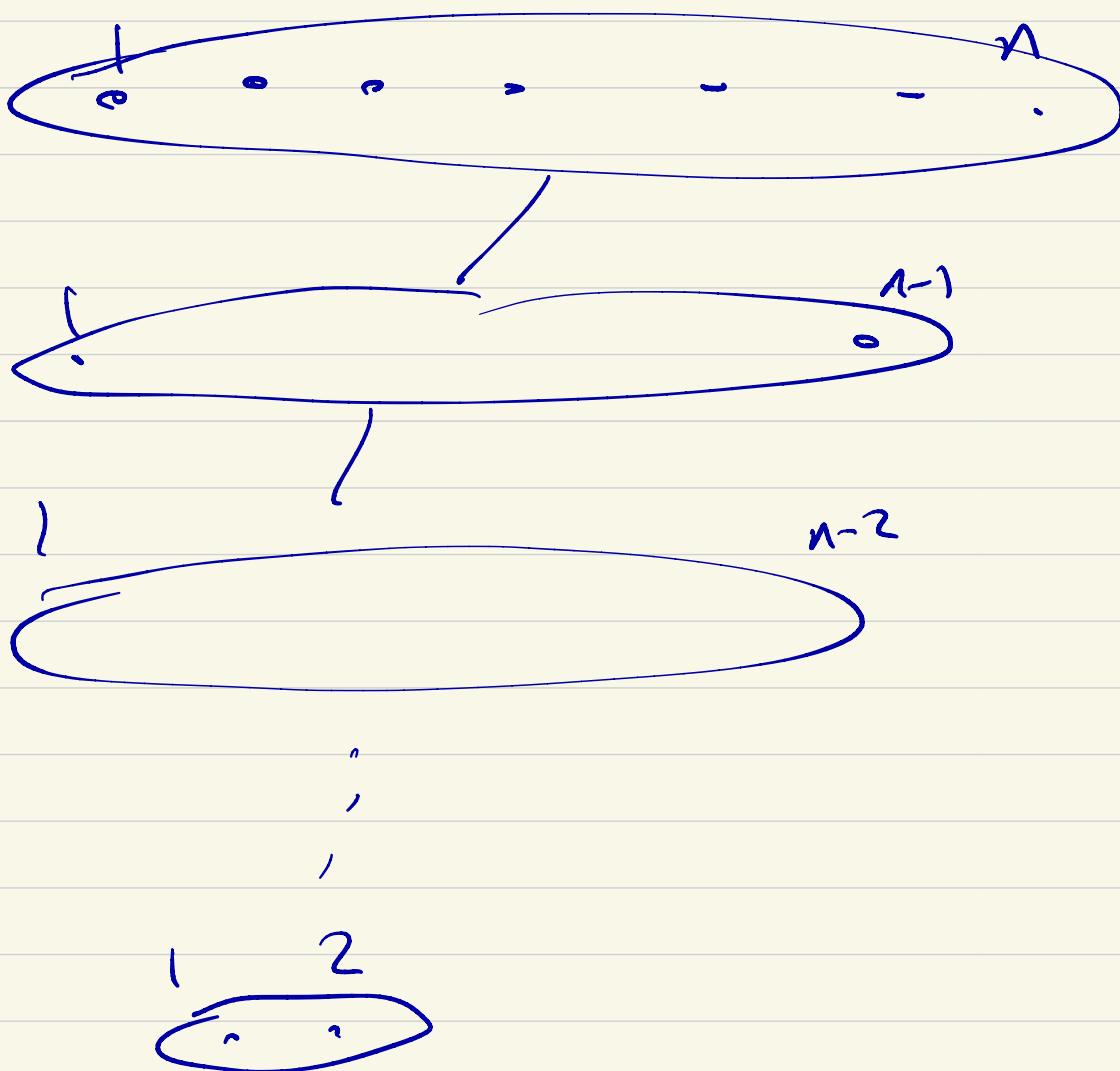


Lemma: A laminar family  $L$ , can have

$$|L| \leq 2n-1$$

A laminar family  $L$  with no singleton sets

$$|L| \leq n-1$$



LP for Vertex Cover

$$H_{e=(u,v)}: \underbrace{x_u + x_v \geq 1}_{\text{tight}} \quad \forall e = (u,v)$$

not tight  $\times$   $\left\{ \begin{array}{l} 0 \leq x_u \leq 1 \\ 0 \leq x_v \leq 1 \end{array} \right\}$

$$\text{Min } \sum w_u x_u$$

Iterative

Solve LP (extreme point)

1) Suppose  $x_v \geq 1/2$  for  $v \in V$

add it to  $v$  to vertex cover

(Repeat)

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~~tight~~ tight constraints