

LECTURE 21

$$P = \{ p_i(x) \geq 0 \mid x = 1 \dots m\}$$

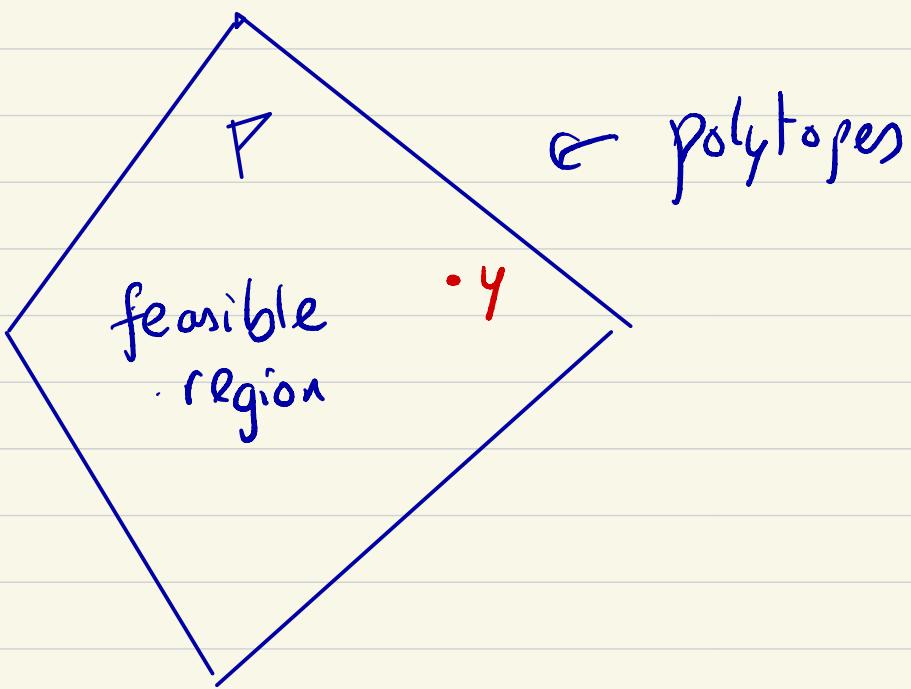
deg-d SoS SDP solution

= "pseudo moments" of a distribution
over solutions to P

DUALITY vs Proofs

LP:

$$\left\{ \langle a_i, x \rangle \leq b_i \mid i=1..m \right\}$$



$y \in$ Feasible region / polytope P

Axioms of $\left\{ \langle a_i, y \rangle \leq b_i \quad \forall i=1..m \right\}$

Is $y_1 + y_2 \geq 7$??

Statement: $\langle c, y \rangle \leq d$??

Suppose $\langle c, y \rangle \leq d$ | $y_1 + y_2 \leq 7$ \forall feasible solutions y

$$2 \underbrace{(\langle a_1, y \rangle \leq b_1)}_{\Downarrow} + 3 \underbrace{(\langle a_2, y \rangle \leq b_2)}_{\Downarrow}$$

$$\underbrace{\langle 2a_1 + 3a_2, y \rangle \leq 2b_1 + 3b_2}_{\text{Nonnegative linear combinations of decisions}}$$

(Nonnegative linear combinations of decisions)
 \Rightarrow Prove new statement.

$$\begin{array}{c|c} \overline{S_1, S_2} & \overline{S_1} \rightarrow \forall a > 0 \\ \overline{S_1 + S_2} & aS_1 \end{array}$$

SOUNDNESS: Every statement that is proved
is actually true

(Weak Duality)

COMPLETENESS: Every true statement
can be proved
(Strong duality for L_P)

$$y_1 + y_2 \leq 7 \quad \text{iff } y \text{ feasible}$$

↑

$\exists c_1, \dots, c_m \geq 0$ such that

$$\sum_i c_i (\langle a_i, y \rangle \leq b_i) = (y_1 + y_2 \leq 7)$$

$$\begin{array}{l} \text{Max } y_1 + y_2 = \langle d, y \rangle \\ \langle a_i, y \rangle \leq b_i; \forall i=1..m \end{array} \left| \begin{array}{l} \text{Min } \sum c_i b_i \\ \text{subject to} \\ \sum c_i \vec{a}_i \geq \vec{d} \end{array} \right.$$

Polynomial System

$$P = \left\{ \begin{array}{l} P_1(x) = 0 \\ P_2(x) = 0 \\ \vdots \\ P_m(x) = 0 \end{array} \right\} \quad \text{Axioms}$$

$\forall x$ satisfying P , $\underline{\underline{q(x) = 0}}$

true statement

$$q(x) = \sum_{i=1}^m r_i \cdot p_i(x) \quad (\text{certificate})$$

Hilbert Nullstellenatz

$$\exists \left\{ q(x) = \sum_{i=1}^m r_i(x) \cdot p_i(x) \right\}^{\deg d}$$

$\Rightarrow q(x) = 0$ whenever x satisfies P

Polynomial System with Inequalities

$$P = \left\{ P_i(u) \geq 0 \mid i = 1 \dots m \right\}$$

Does $P \Rightarrow q(u) > 0$??

Suppose $P \Rightarrow q(u) \geq 0$, how can you prove it ??

$$q(x) = \left(\sum_{i=1}^N g_i^2(x) \right) + \sum_{i=1}^m p_i(x) \cdot \left(\sum_{j=1}^r y_{ij}^2(x) \right)$$

SOS polynomial SoSpoly

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Defn: Given a poly system $P = \{p_i(u) \geq 0\}$

an SOS proof that $P \Rightarrow (q(u) \geq 0)$

is

$$q(u) = s_0(u) + \sum_{i=1}^m p_i(u) s_i(u)$$

(sum-of-squares)

where : i) s_0, s_1, \dots, s_m are SOS polynomials

2) max degree $\left\{ \deg(s_0), \dots, \deg(p_i(x)s_i(x)) \right\}$

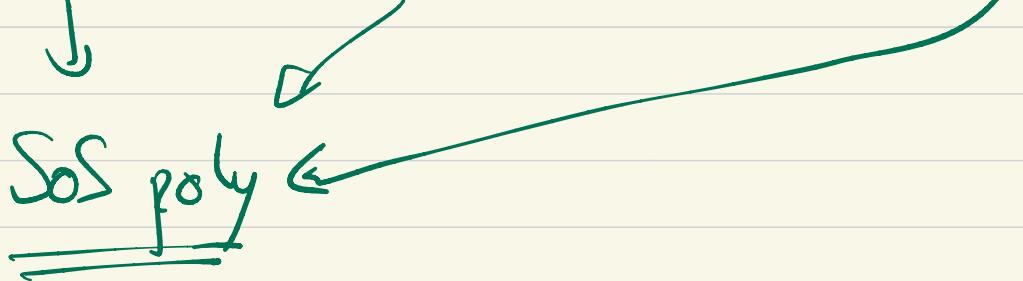
Positivstellensatz: There always exists

an SOS proof that $P \Rightarrow (q(u) \geq 0)$

$$g(n) \left(1 + R(n) \right) = g_0(n) + \sum p_i(n) g_i(n)$$

\downarrow

SOS poly



Max Cut SDP

$$\left\{ \begin{array}{l} \text{Max } \langle L, X \rangle = \sum_{ij} L_{ij} X_{ij} \\ X_{ii} \leq 1 \quad \forall i=1..n \end{array} \right.$$

$$v^T X v \geq 0 \quad \leftarrow \boxed{X \succeq 0} \quad (X \text{ is p.s.d})$$

$\forall v \in \mathbb{R}^n$

$$\text{Max } \langle L, X \rangle$$

$$\text{s.t. } X_{ii} \leq 1 \quad \left. \begin{array}{l} \forall i=1..n \end{array} \right\} \alpha_i$$

$$\left. \begin{array}{l} \langle v v^T, X \rangle \geq 0 \end{array} \right\} \forall v \in \mathbb{R}^n \} \beta_v$$

$$(B - \langle L, X \rangle) = \sum_{i=1}^n \alpha_i (1 - \underbrace{X_{ii}}_{\geq 0})$$

$$+ \sum_v \beta_v \underbrace{\langle v v^T, X \rangle}_{\geq 0}$$

$$(B - \langle L, X \rangle) = \sum_{i=1}^n \alpha_i (1 - \underbrace{x_{ii}}_{\geq 0})$$

$$+ \sum_v \beta_v \underbrace{\langle w^T, X \rangle}_{\geq 0}$$

$$= \sum_{i=1}^n \alpha_i (1 - \underbrace{x_{ii}}_{\geq 0})$$

$$+ \left\langle \sum_v \beta_v v v^T, \underbrace{X}_{\geq 0} \right\rangle$$

Dual SDP:

$$\text{Find } \alpha_1, \dots, \alpha_n, \geq 0$$

and $Y \in \mathbb{R}^{n \times n}$ \leftarrow prod matrix

such that

$$(B - \langle L, X \rangle) = \sum_{i=1}^n \alpha_i (1 - x_{ii}) + \langle Y, X \rangle$$

$$(B - \langle L, X \rangle) = \sum_{i=1}^n \alpha_i \left(1 - \underbrace{x_{ii}}_{\geq 0} \right)$$

$$+ \sum_v \beta_v \underbrace{\langle w^T, X \rangle}_{\geq 0}$$

$$\langle L, X \rangle \Leftrightarrow \tilde{E} \sum_{(i,j) \in E} (x_i - x_j)^2$$

$$\tilde{E}[u_i^2] = x_{ii} \Leftrightarrow x_i^2$$

$$\langle VV^T, X \rangle = V^T X V = \sum_{i,j} v_i v_j \frac{X_{ij}}{\|V\|}$$

$$= \sum_{i,j} v_i v_j \tilde{E}[u_i u_j]$$

$$= \tilde{E}\left[\left(\sum_i v_i x_i\right)^2\right]$$

$$(B - \tilde{E} \left[\sum_{(i,j) \in E} (x_i - x_j)^2 \right])$$

$$= \sum_i \underbrace{\alpha_i}_{\geq 0} \left(1 - \tilde{E}[x_i^2] \right)$$

$$+ \sum_v \underbrace{\beta_v}_{\geq 0} \tilde{E} \left[\left(\sum_i v_i x_i \right)^2 \right]$$

$$B - \sum_{(i,j) \in E} (x_i - x_j)^2 = \sum_i \underbrace{\alpha_i}_{\geq 0} \underbrace{(1 - x_i^2)}_{\geq 0} \geq 0$$

$$+ \sum_v \underbrace{\beta_v}_{\geq 0} \left(\sum_i v_i x_i \right)^2 \geq 0$$

Dual of deg d SoSSDP

if finds a deg d SOS proof

Defn: [pseudo expectation]

A deg d "pseudo expectation"

$$\tilde{E}: \left\{ \begin{array}{c} \text{deg } d \\ \text{poly} \end{array} \right\} \rightarrow \mathbb{R}$$

(linear
functional)

for a poly system $\{ p_i(u) \geq 0 | i=1..m \}$

$$1) \quad \tilde{E}[p_i(u) \cdot g^2(u)] \geq 0 \quad \forall i = 1..m$$

$$2) \quad \tilde{E}[g^2(u)] \geq 0$$

$$3) \quad \tilde{E}[1] = 1$$

Deg d SoSSDP \Rightarrow deg d pseudo expectation

(Weak Duality) :

$$P \Leftrightarrow q(u) \geq 0$$

admits a

proof

deg d SoS

such as

$$q(u) = g_0(u) + \sum g_i(u) p_i(u)$$

then for every deg-d pseudo expectation

\tilde{E}

$$\tilde{E}[q(u)] \geq 0$$

Robust Linear Regression

1) Poly system \underline{P} for linear regres.

2) Solve deg d SoS SDP for \underline{P}

$$\tilde{E}[x_i u_j] \dots \tilde{E}[$$

3) Prove using low deg SoS proof

that "any solution to \underline{P}
is close to true answer"

\Downarrow
 \tilde{E} values are also
close to the answer

Def: A $X \succcurlyeq 0$ is p.s.d iff

1) $\forall v \in \mathbb{R}^n \quad v^\top X v \geq 0$

2) $X \succcurlyeq 0 \iff X = \sum c_i v_i v_i^\top$