# LECTURE 10

- Proof of Oblivious Subspace Embedding

- Compressed Sensing.

THEOREM: $G \leftarrow \frac{1}{\sqrt{t}}\left(N(0,1)\right)^{t \times n}$ random Gaussian matrix

if $\boxed{t = \Omega\left(\boxed{d} + \frac{\log(1/\delta)}{\varepsilon^2}\right)}$ $\forall$ subspace $S$, $\dim(S) = d$

with prob. $1-\delta$, $\forall u \in S$ $\|Gu\| \approx (1 \pm \varepsilon)\|u\|$.

Proof:

Def: A $\gamma$-net $N$ of ball$(0,1) \subseteq \mathbb{R}^d$

is a set of points such that

$\forall x \in \text{Ball}(0,1)$ $\exists y \in N$ s.t $\|x-y\| \leq \gamma$.

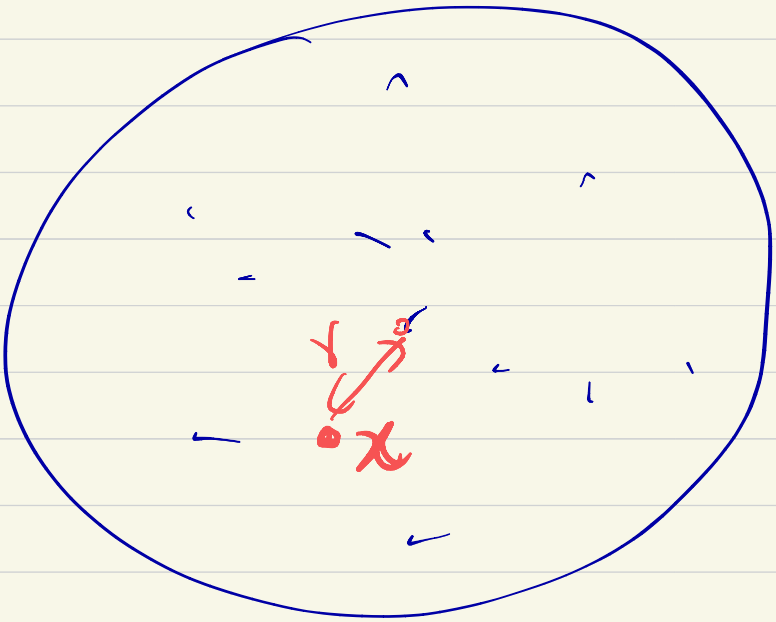
Obs 1: $\exists$ Net $N$ with $|N| \leq \boxed{\left(\frac{1}{\gamma}\right)^d}$

$\gamma = 0.001$

Obs 2: By distributional JL, w.p. $1-\delta$,

$\forall x, y \in N$, $\langle Gx, Gy \rangle \approx \langle x, y \rangle \pm \varepsilon$

Ball(0, 1)



**Claim:** $\forall x \in \text{Ball}(0,1)$ $x$ can be written

$$x = \sum_{i=1}^{\infty} \alpha_i y_i \quad \text{where } |\alpha_i| \leq \frac{1}{2^{i-1}}$$

**Proof:** $x_1 = x$

$y_1 \leftarrow$ closest point in the net $N$ to $x_1$

$$x_1 = \underbrace{y_1}_{\text{net}} + \underbrace{(x_1 - y_1)}_{\leq \gamma}$$

$$= y_1 + \|x_1 - y_1\| \cdot \underbrace{\left(\frac{x_1 - y_1}{\|x_1 - y_1\|}\right)}_{x_2}$$

$y_2 \leftarrow \text{Closest}_{N_2}(x_2)$

$$= y_1 + \|x_1 - y_1\| \left(y_2 + \|x_2 - y_2\| \cdot \frac{(x_2 - y_2)}{\|x_2 - y_2\|}\right)$$

$$= \overset{\alpha_1 = 1}{y_1} + \overbrace{\|x_1 - y_1\|}^{\leq \gamma} y_2 + \overbrace{\|x_1 - y_1\|}^{\leq \gamma} \overbrace{\|x_2 - y_2\|}^{\leq \gamma} x_3$$

$\forall x \in \text{Ball}(0,1) \qquad \|Gx\| \approx \|x\| \pm \varepsilon$

let $\quad \|Gx\|^2 = \langle Gx, Gx \rangle$

$$= \left\langle G\sum_{i=1}^{\infty} \alpha_i y_i , G\sum_{i=1}^{\infty} \alpha_i y_i \right\rangle$$

$$= \sum_{i,j} \alpha_i \alpha_j \langle \underbrace{Gy_i, Gy_j}_{\text{points in the net}} \rangle$$

$$= \sum_{i,j} \alpha_i \alpha_j \left[ \langle y_i, y_j \rangle \pm \varepsilon \right]$$

$$= \left\langle \underbrace{\sum_i \alpha_i y_i}, \sum_j \alpha_j y_j \right\rangle \pm \varepsilon \left( \sum_{i,j} \alpha_i \alpha_j \right)$$

$$= \langle x, x \rangle + \varepsilon \left[ \left( \sum_i \alpha_i \right)^2 \leq O(1) \right] = O(\varepsilon)$$

# Compressed Sensing

[ Dimension reduction for sparse vectors ]

→ Images / Audio → sparse in some basis

signal $\boldsymbol{x} = (x_1, x_2, \ldots x_n)$

↑ in an appropriate basis

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**Defn:** A vector $x \in \mathbb{R}^n$ is <u>k-sparse</u> if $< k$ co-ordinates of nonzero

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$x$ is $\varepsilon$-approximately $k$-sparse, if $\exists\, S \subseteq [n]$ of $k$ coordinates

$$\|x_{\bar{S}}\|_1 \leq \varepsilon \|x\|$$

## COMPRESSED SENSING

Signal $x = (x_1, \dots, x_n)$ is $k$-sparse $\Longrightarrow$ 1) Use "$\ll n$" measurements

2) Recover true $x$

1) Measurement matrix $\boxed{M} \in \mathbb{R}^{t \times n}$, $t$: $k \log n \ll n$

$\underline{(x_1, \dots, x_n)}$ $\Longrightarrow$ measure co-ordinates $\boxed{M}x$

sparse

$\langle M_1, x \rangle \dots \langle M_t, x \rangle$

2) Given $\boxed{Mx = b}$ recover a $\boxed{k\text{-sparse } x.}$

$\int n$ $\quad k \log n$ equations in $n$ variables

**Recovery:** **Input:** $\underline{M}\, x = b$

Goal: K-sparse solution to the system.

Optimisation:

$$\boxed{\begin{array}{l} \min \|x\|_0 \\ \text{subject to } Mx = b \end{array}}$$

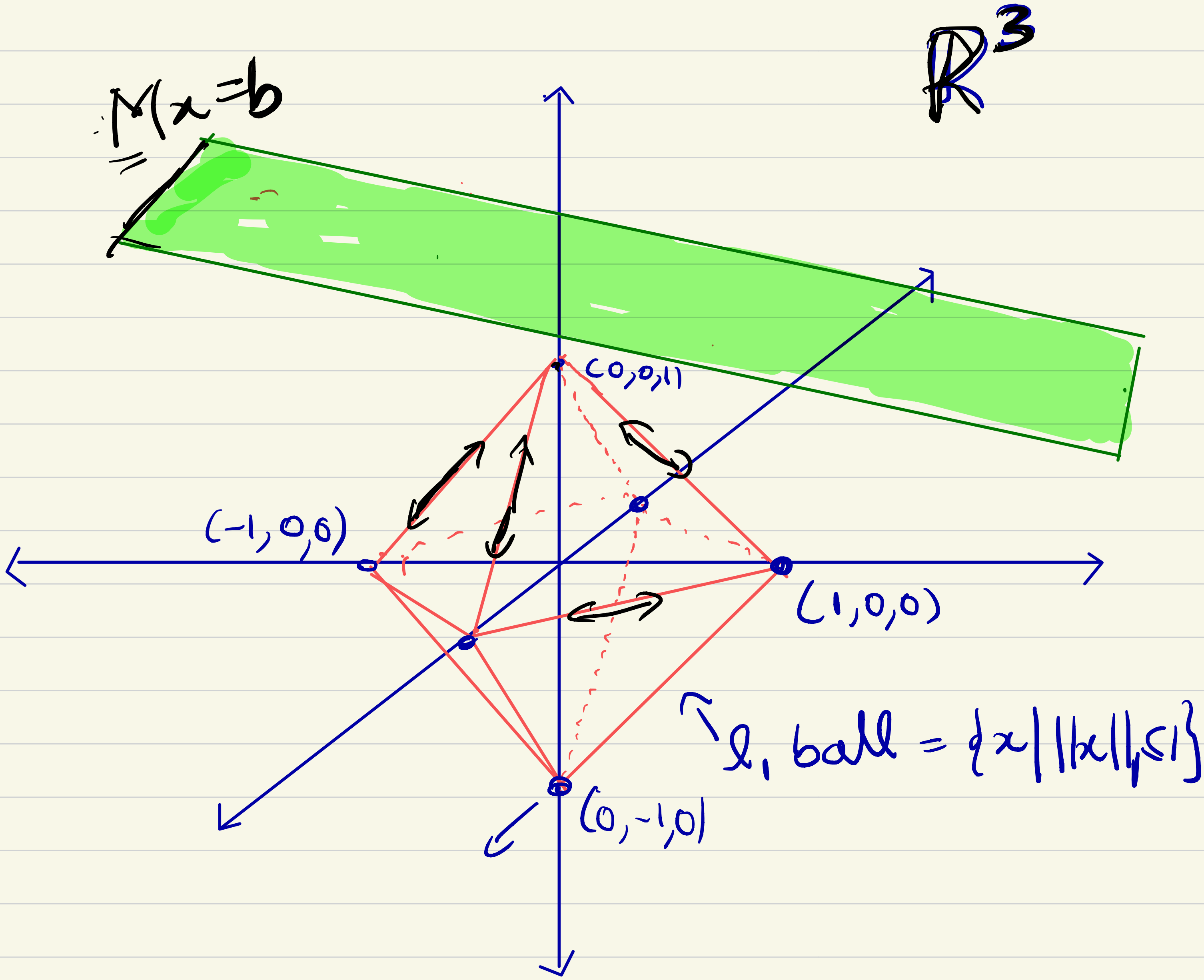
← # of nonzero co-ordinates of $x$

↑

$\underline{\text{not}}$ convex / smooth

$$\|x\|_0 \xleftrightarrow{\text{replace}} \|x\|_1$$

$$\boxed{\begin{array}{l} \min \underline{\|x\|_1} = \sum_i |x_i| \\ \text{subj to } \underline{Mx = b} \end{array}}$$




"low dimensional facets are sparse"

Minimising L1 norm $\iff$ typically gives sparse vectors

$$M = \frac{1}{\sqrt{t}} N(0,1)^{t \times n} \Leftrightarrow \text{random}\{\pm 1\}$$

(preserving lengths of sparse vectors)

[Restricted Isometry Property]

A matrix $M \in \mathbb{R}^{t \times n}$ satisfies $(k, \varepsilon)$ RIP

if $\forall$ $k$-sparse $x$, $\|Mx\| \approx (1 \pm \varepsilon)\|x\|$.

Set of all $k$-sparse vectors

$$= \bigcup_{\substack{\text{sets } S \\ S \subseteq [n] \\ |S| = k}} \Big[ \underbrace{\{ x \mid x_i = 0 \ \forall i \notin S \}}_{V_S} \Big]$$

$\binom{n}{k}$

Pick $t = \left( \frac{k + \log\binom{n}{k} + \log(1/\delta)}{\varepsilon^2} \right)$ w.p. $1-\delta$ M is RIP

$$t = O\left(\frac{\underset{\sim}{k} + \log(k)}{\varepsilon^2}\right)$$

**LEMMA:** Suppose $M$ is $(3K, 0.01)$-RIP

then $\forall$ $K$-sparse $x \in \mathbb{R}^n$

Basis Pursuit / $l_1$ minimization recover $x$

exactly from $Mx$.

**Proof:** $Mx = b$ $x$ is $K$-sparse

$$x' = \underset{Mx=b}{\operatorname{argmin}} \|x\|_1$$

$\Delta = x' - x$

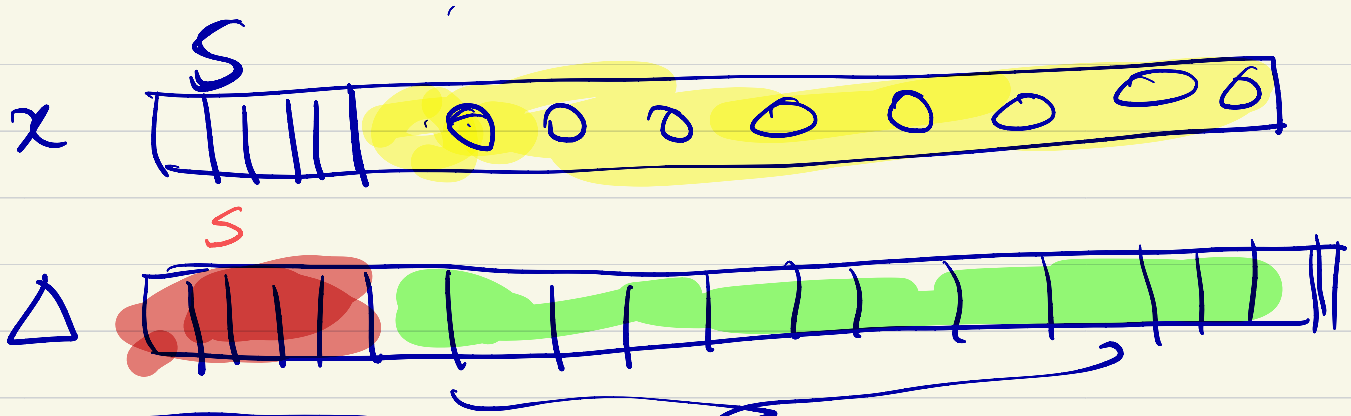
1) $M\Delta = Mx' - Mx = b - b = 0$

2) $\|x'\|_1 = \|x + \Delta\|_1 \leq \|x\|_1$

3) $\|\Delta\|_1 = 1$

let

$S = \{i \mid x_i \neq 0\}$, $|S| \leq K$.




$\Delta_{\bar S}$

$$\boxed{\|x+\Delta\|_1 < \|x\|_1}$$

$\|\Delta_{\bar S}\|_1$ adds to $\|x\|_1$ in $\|x+\Delta\|_1$

$\Rightarrow$ $\Delta_S$ reduces the $\ell_1$ norm inside S by $\geq \|\Delta_{\bar S}\|_1$

$$\Rightarrow \Big| \; \|\Delta_S\|_1 \geq \|\Delta_{\bar S}\|_1$$

$$\Rightarrow \|\Delta_S\|_1 \geq \tfrac{1}{2} \|\Delta\| = \tfrac{1}{2} \cdot 1 = \underline{\tfrac{1}{2}}$$

$\|\Delta_{\bar S}\|_1 \leq 1/2$

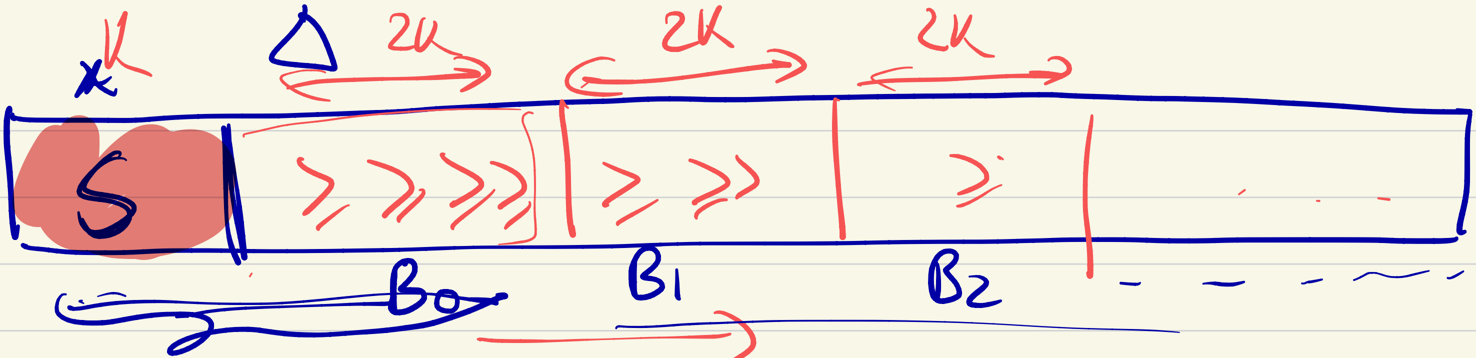
$\Delta_S$ is "heavy" and $\Delta_{\bar S}$ is $k$-sparse

$$\|M\Delta_S\| \approx \|\Delta_S\|$$

$M\Delta_S$ is also heavy

$$\underline{\underline{M\Delta = 0}} = \underbrace{M\Delta_S}_{\text{heavy}} + \underbrace{(M\Delta_{\bar S})}_{\text{light}}$$

Contradiction.




1) sorted in decreasing order

2) chunks of size 2K

Obs1: $\|M\Delta_{S\cup B_0}\|_2 \geq (1-\varepsilon)\|\Delta_{S\cup B_0}\|_2$

3K-span

$\geq \|\Delta_S\|_2(1-\varepsilon)$

$\|\Delta_S\|_1 \geq 1/2 \Rightarrow$ $\geq \frac{1}{\sqrt{2K}}(1-\varepsilon)$

[Lemma: For $v \in \mathbb{R}^\ell$ $\|v\|_2 \geq \frac{\|v\|_1}{\sqrt{\ell}}$]

Claim:

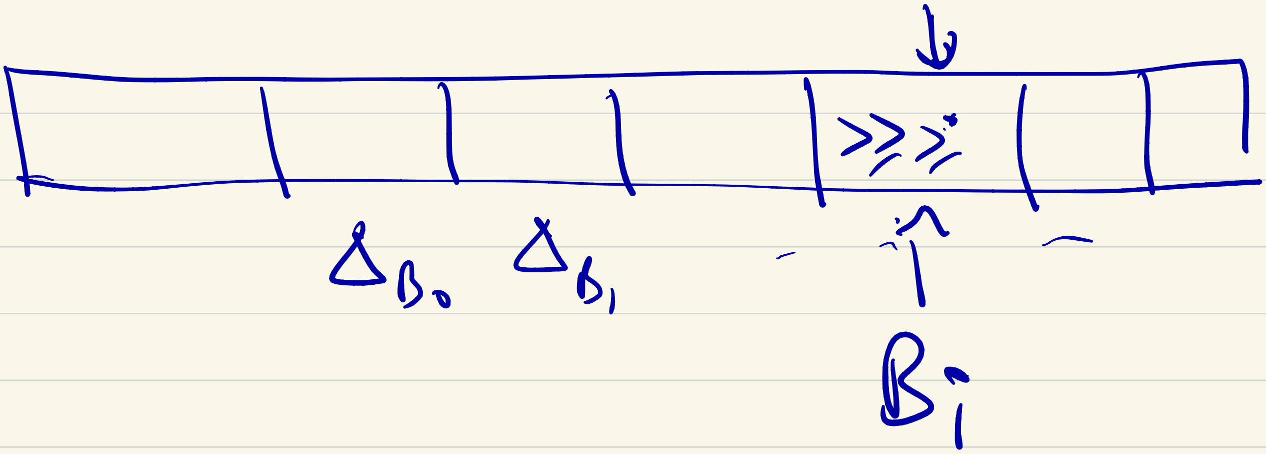
$$\sum_{j\geq 1} \|\Delta_{B_j}\|_2 \leq \frac{0.4}{\sqrt{K}} \approx \frac{1}{2\sqrt{2K}}$$

Assume claim

$$\underline{\underline{M\Delta}} = \underbrace{\left(M\Delta_{S_0 \cup S_0}\right)}_{\substack{\geq \\ \frac{1}{\sqrt{2K}}(1-\varepsilon)}} + \underbrace{\left(\sum_{j\geq 1} M\Delta_{B_j}\right)}_{\substack{\wedge \\ \frac{0.4}{\sqrt{K}}(1+\varepsilon)}}$$

$=$ Too Long $+$ Too short to cancel $= 0$

a contradiction




Any co-ordinate in $B_i$

$$\le \frac{\|\Delta_{B_{i-1}}\|_1}{2K}$$

$$\Rightarrow \|\Delta_{B_i}\|_2 \le \left(2K \cdot \left(\frac{\|\Delta_{B_{i-1}}\|_1}{2K}\right)^2\right)^{1/2}$$

$$\sum_{i=1}^{\infty} \|\Delta_{B_i}\|_2 \le \sum_{i=1}^{\infty} \frac{\|\Delta_{B_{i-1}}\|_1}{\sqrt{2K}} = \frac{\|\Delta_S\|_1}{\sqrt{2K}} = \frac{1}{2\sqrt{2K}}$$