

LECTURE 18

Maximum flow in Undirected Graphs (unweighted)

eg 1

Laplace Solvers

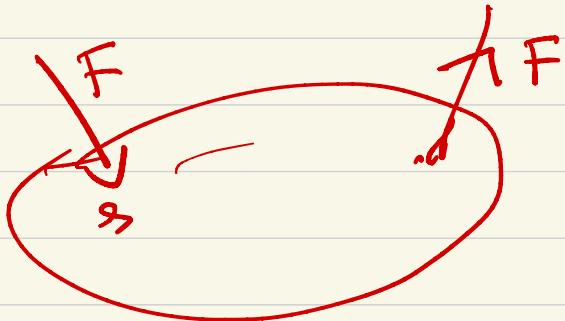
$$Lx = b$$

find x in time $\tilde{O}(m \ln^{1/\epsilon})$

$$\sum_{e \in E} f_e^2 \cdot r_e$$

↑
resistance on edge

$$K = \left\{ f \in \mathbb{R}^E \mid \sum_{i \rightarrow j} f_{ij} = \begin{cases} 0 & \text{for } i \neq s/t \\ F & \text{if } i = s \\ -F & \text{if } i = t \end{cases} \right\}$$



Capacity Constraints

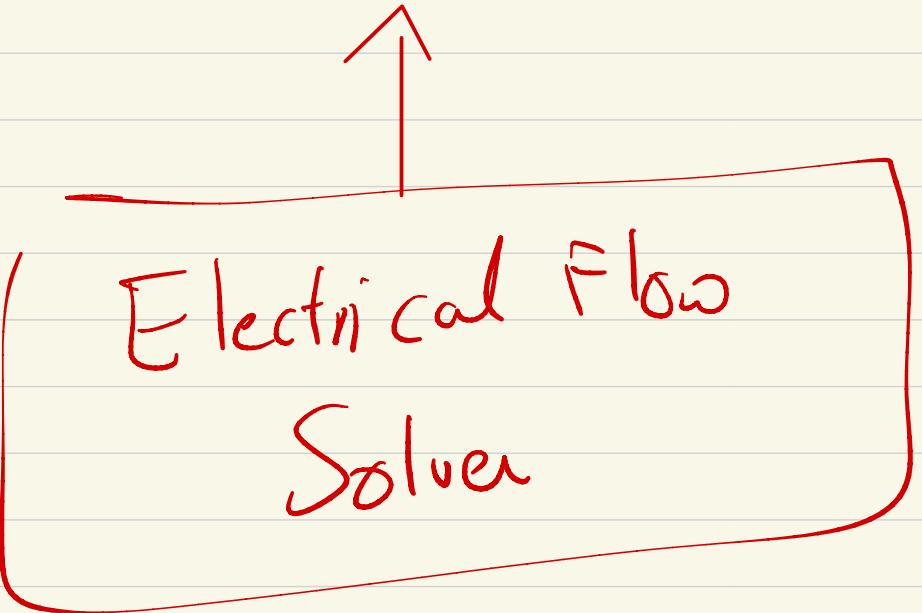
$\forall e \in E \quad f_e \leq 1$

Max Flow = $f \in K$ and
satisfies capacity
constraints.

“Constraints in K ”



M W



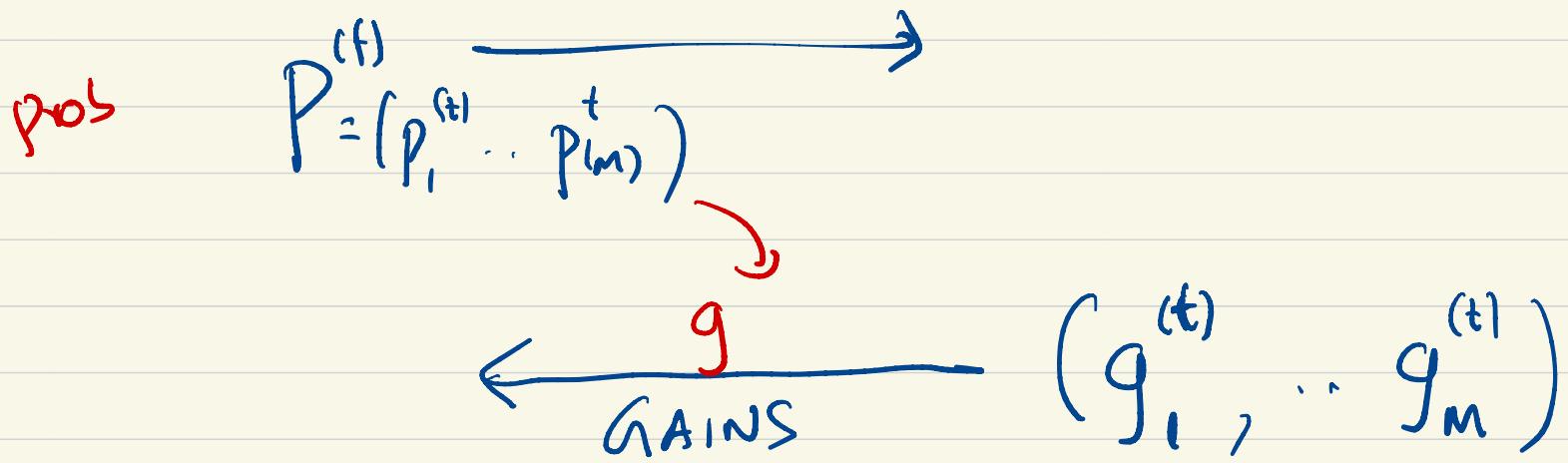
Electrical Flow
Solver

MULTIPLICATIVE WEIGHTS

Experts: 1..m



In round t:



Incur a gain: $\langle P^{(t)}, g^{(t)} \rangle = \sum_i p_i^{(t)} g_i^{(t)}$

THEOREM: If all gains $g_i^{(t)} \in [-\gamma, \gamma]$

forall gain $g^{(1)}, \dots, g^{(T)}$

$$T = O\left(\frac{\gamma \rho \ln m}{\varepsilon^2}\right)$$

$$\frac{1}{T} \sum_{t=1}^T \langle P^t, g^t \rangle \geq \max_i \frac{1}{T} \sum_{t=1}^T g_i^{(t)} - \varepsilon$$

Solving unLP (via multiplicative weights)

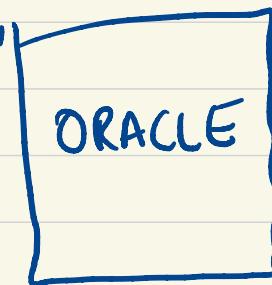
Find $x \in \{ \langle a_i, x \rangle \leq b_i \mid i=1 \dots m \}$

$x \in K \leftarrow$ easy constraints:

Game



Experts = constraints



Prob. dist $p_1^t \dots p_m^t \rightarrow$

over constraints



$x^{(t)} \in K$

$$g_i^{(t)} = \underbrace{\langle a_i, x^{(t)} \rangle - b_i}_{\text{constraint violation}}$$

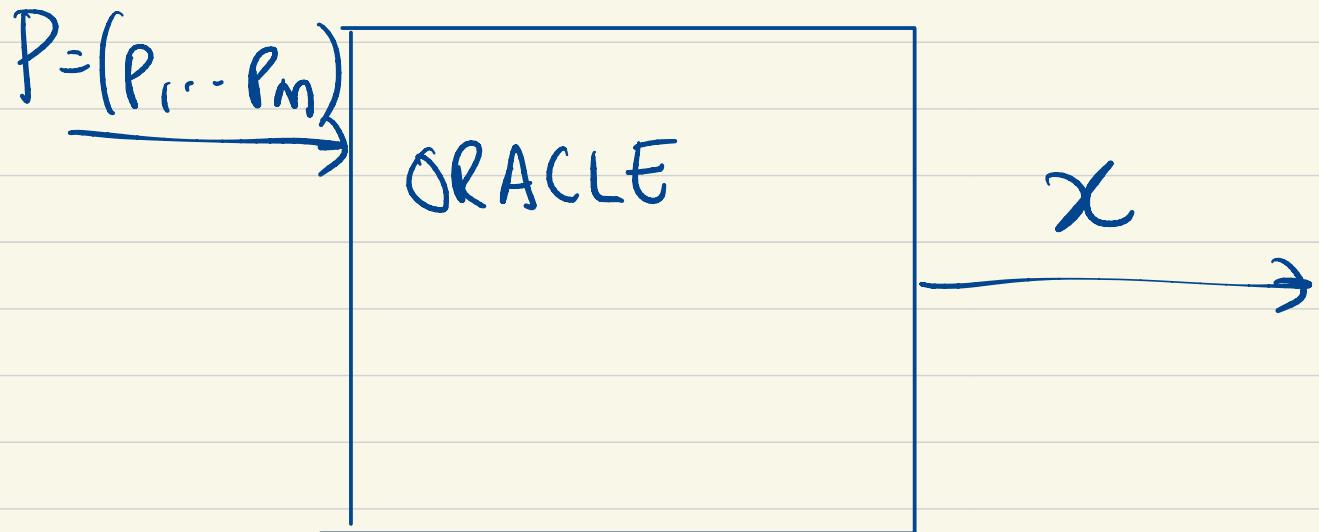
$$\text{Average gain}^{(t)} = \langle p^{(t)}, g^{(t)} \rangle = \sum p_i^{(t)} (\langle a_i, x^{(t)} \rangle - b_i)$$

\exists a feasible point x^*

$\curvearrowright \Sigma \leq \epsilon$

\Rightarrow gains for $x^* < 0$, average gain $< -\epsilon$

\Rightarrow ORACLE picks $x^{(t)}$; average gain $\leq \epsilon$



$$\boxed{\begin{array}{l} 1) x \in K \\ 2) \sum_{i=1}^m P_i ((a_i, x) - b_i) \leq \varepsilon \end{array}}$$

Runtime = $O\left(\frac{P \gamma \ln m}{\varepsilon^2}\right)$

γ (Grains) $\in [-\gamma, \gamma]$

Solving an LP with one constraint.

$P = \underline{\text{maximum violation}} = \max_i \langle x^{(+)}, a_i \rangle - b_i$

$$\gamma =$$

α^* ↗
↓ In each round t

$\varepsilon \geq \text{Gain (MWALGO)}$

$$\varepsilon \geq \frac{1}{T} \sum_{t=1}^T \langle P^{(t)}, g^{(t)} \rangle$$

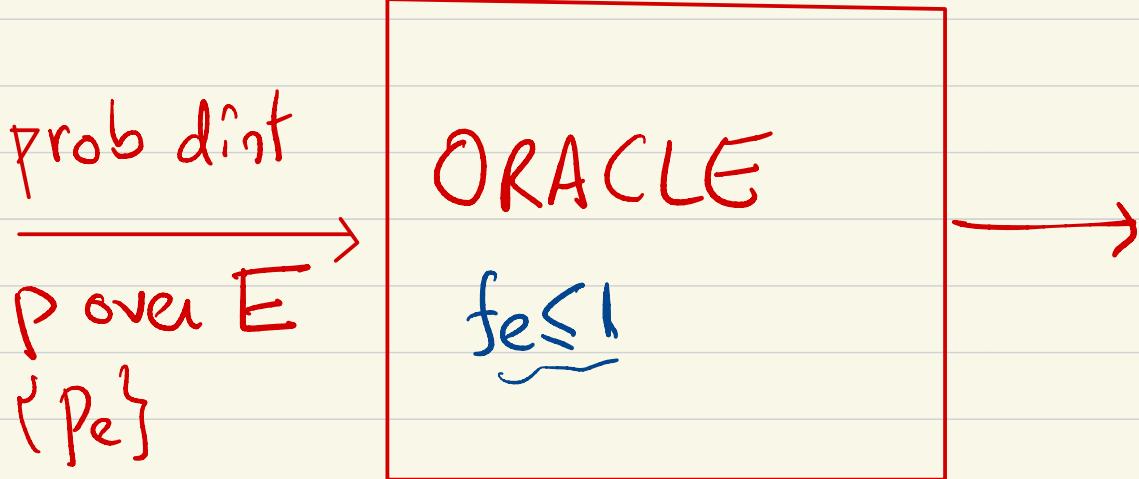
(Theorem)
≥

$$\max_{i=1 \dots m} \frac{1}{T} \sum_{t=1}^T (\langle a_i, x^{(t)} \rangle - b_i) - \varepsilon^*$$

$$\varepsilon \geq \max_i \left(\langle a_i, \underbrace{\left(\frac{1}{T} \sum_{t=1}^T x^{(t)} \right)}_{x^*} \rangle - b_i \right) - \varepsilon$$

$$2\varepsilon \geq \max_i (\langle a_i, x^* \rangle - b_i)$$

x^* is a 2ε feasible solution
=====



Find a fbo f

- 1) $f \in K$
- 2) $\left| \sum_e P_e \cdot f_e - 1 \right| \leq \frac{\epsilon}{m}$

average gain $\leq \epsilon$

Electrical Flow : $\tilde{\Theta}(n)$

$$U_{re} = P_e + \frac{\epsilon}{m}$$

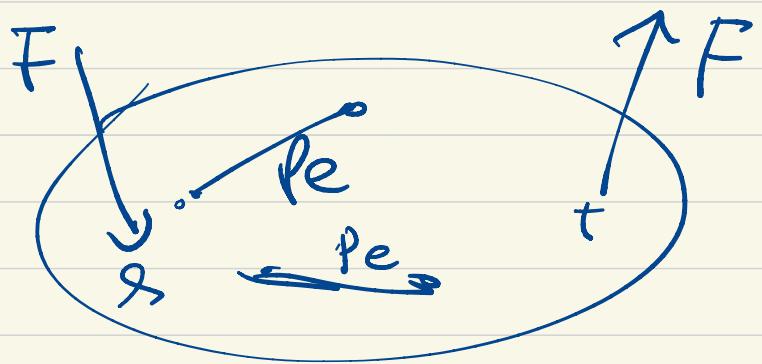
to compute \hat{f} electrical flow

$$\min \sum_{e \in E} p_{fe}$$

$\hat{=}$

$$f \in K$$

edge weights = p_e



1) Treat $\{p_e\}$ as edge weights

2) Compute shortest path P from $s \rightsquigarrow t$

3) Route F units on P .

$$\Rightarrow \underline{\text{len}(P)} \leq \frac{1}{F} \Rightarrow \underline{\sum p_{fe}} \approx \frac{1}{F} \cdot F \leq \underline{1 + \epsilon}$$

$$\rightarrow \underline{\text{len}(P)} > \frac{1}{F}$$

$$\text{if flow } f \quad \underline{\sum p_{fe}} > 1 + \epsilon$$

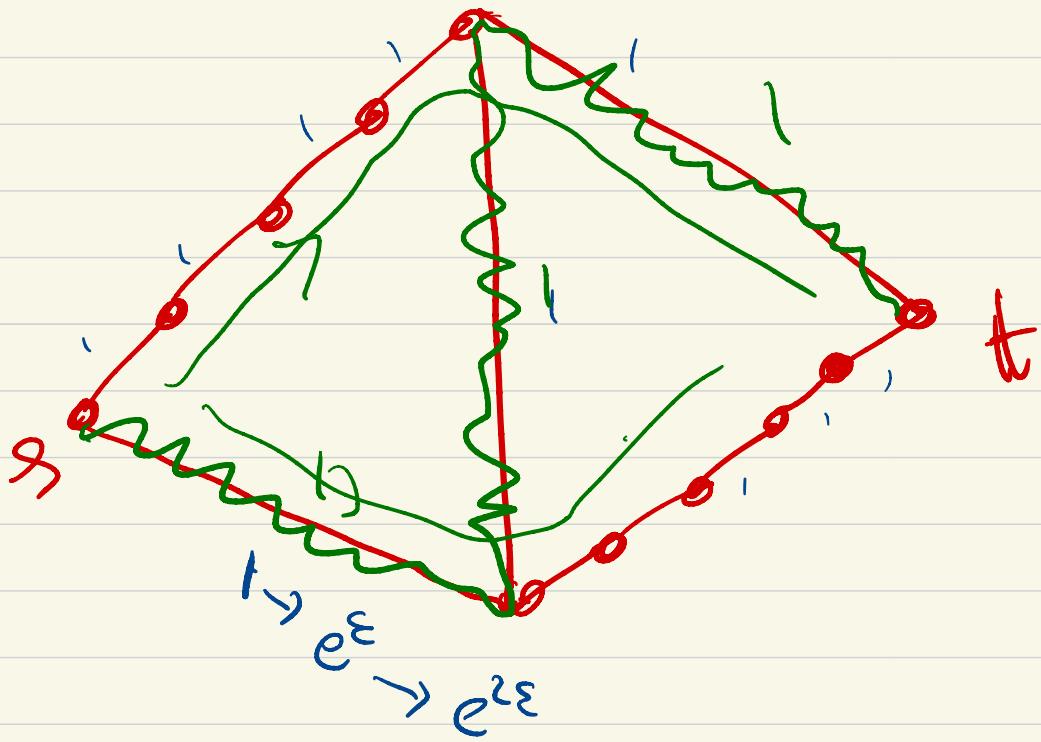
\nexists flow with $\hat{=}$ average violation $\leq \epsilon$

$\Rightarrow \exists$ no flow satisfies constraints

$$W_e^{(t+1)} = W_e^{(t)} \cdot e^{\epsilon(f_e^{(t)} - 1)}$$

$$W_e^{(t)} = e^{\epsilon \left(\sum_{j=1}^t f_e^{(j)} \right) - t}$$

$\approx e^{\epsilon \text{ (Total Flow on that edge)}}$



$\gamma \geq$ min violat.

$$= \min (a_i x_j - b_i)$$

$$= \min (f_e - 1)$$

> -1

$$\rightarrow \boxed{\gamma = -1}$$

$$P = \max_e (f_e - 1) \approx \underline{F-1} \\ \approx F$$

$F \leq m \leftarrow$ all edge capacities = 1

$$\boxed{P = m}$$

Runtime: $O\left(\frac{m^2}{\epsilon^2}\right)$

Electrical Flow Oracle

Suppose f^* is the optimal max flow

$$\begin{aligned} \text{Energy}(f^*) &= \sum_e \underbrace{(f_e^*)^2}_{\lambda_e} \cdot r_e \\ &\leq \sum_e (1)^2 \cdot \left[p_e + \frac{\epsilon}{m} \right] \\ &\leq \sum_e p_e + \frac{\epsilon}{m} \cdot m \\ &\stackrel{=} {=} 1 + \epsilon \end{aligned}$$

$$\text{Energy}(\hat{f}) \leq \text{Energy}(f^*) \leq 1 + \epsilon$$

$$\sum_e \hat{f}_e^2 \cdot r_e$$

$$\sum \hat{f}_e^2 r_e \leq 1 + \epsilon$$

"

$$\sum \hat{f}_e^2 \left(p_e + \frac{\epsilon}{m} \right) \leq 1 + \epsilon$$

\Rightarrow

Width / Max violation

$$\max_e \hat{f}_e$$

H edges $\hat{f}_e^2 \cdot \frac{\epsilon}{m} \leq 1 + \epsilon$

Width $p = \sqrt{\frac{m}{\epsilon}}$

$$\hat{f}_e \leq \sqrt{\frac{m}{\epsilon}}$$

$$\sum p_e f_e \leq \left(\sum_e p_e \right)^{1/2} \left(\sum p_e f_e^2 \right)^{1/2}$$

(Cauchy-Schwarz)

$$\leq (1) \cdot (\leq 1 + \epsilon)$$

Total energy

$$\leq \underline{1 + \epsilon}$$

Runtime: $\boxed{O\left(\frac{\ln m}{\epsilon^2}\right)}$ iterations

$$O\left(\frac{1 \cdot \sqrt{m} \cdot \ln m}{\epsilon^2}\right)$$

$$\tilde{O}(\sqrt{m}) \text{ iterations}$$

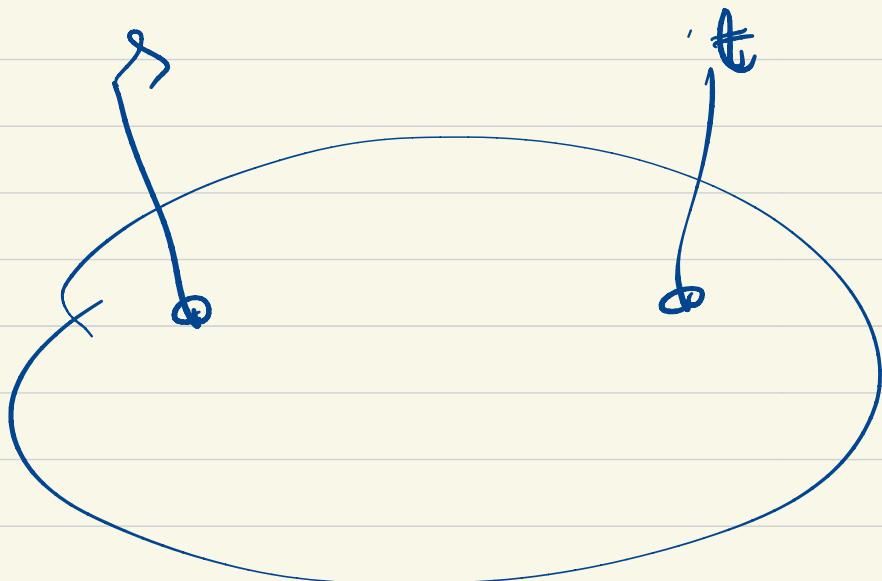
$$\begin{aligned} \text{Total} &= \tilde{O}(\sqrt{m}), \tilde{O}(n) \\ &= \tilde{O}(m^{3/2}) \text{ runtime} \end{aligned}$$

$O\left(\frac{m^{4/3}}{\epsilon^2}\right)$ algorithm
→ Undirected graphs
→ approximate

→ Near linear time - ('approx maxflow
in undirected graphs')

$$\rightarrow P = \max_e \hat{f}_e < m^{1/3}$$

if $\exists e \in E \quad \hat{f}_e > m^{1/3}$



$$R_{\text{eff}}(s, t) \geq \frac{1}{m^2}$$

$$\min_{x \in \mathbb{R}^n} \|Bx\|_\infty$$