

## LECTURE 12

Distance

Metric on  $n$ -points

Ex: Shortest path on graph

$(X, d)$



embeddings

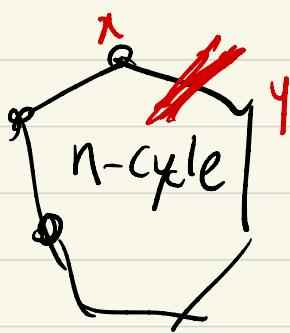
$\left. \begin{array}{l} \text{Approximation} \\ \text{Algs} \\ \& \\ \text{Online Algs} \end{array} \right\}$ 
V/V

Tree Metrics

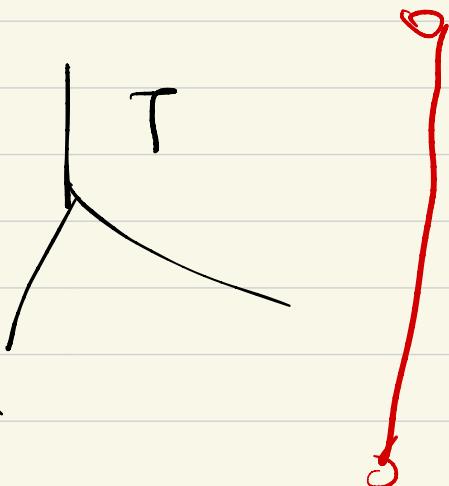
(shortest path on tree)

$(T, d_T)$

$$d_T(u, y) = d(x, y) \quad [\text{IMPOSSIBLE}]$$



incur  $\Theta(n)$   
distortion



# Randomized / Probabilistic Tree Embeddings

$(G_i = (V, E), w_i)$

Given  $(X, d)$ , a randomized tree embedding  
is a distribution  $\mathcal{D}$  on tree metrics  
such that  $\forall x, y \in X$

$$d(x, y) \leq d_T(x, y) \quad \forall T \sim \mathcal{D}$$

AND

$$\mathbb{E}_T [d_T(x, y)] \leq d \cdot d(x, y)$$

Low stretch spanning trees : Trees are  
spanning trees of  $G$ .

Theorem  $\forall (X, d) \exists \mathcal{D}$  over tree metrics  
with  $\alpha = O(\log |X|)$

# LOW-DIAMETER DECOMPOSITIONS

Given:  $(X, d)$  metric space

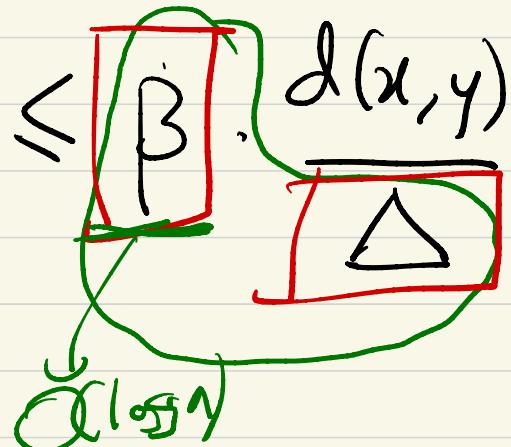
Goal: Partition  $X$  as  $P = X_1 \cup X_2 \cup \dots \cup X_n$

such that  $\boxed{\text{diameter}(X_i) \leq \Delta}$

$\forall x, y \in X$

$$\Pr [P(x) \neq P(y)] \leq \frac{d(x, y)}{\Delta}$$

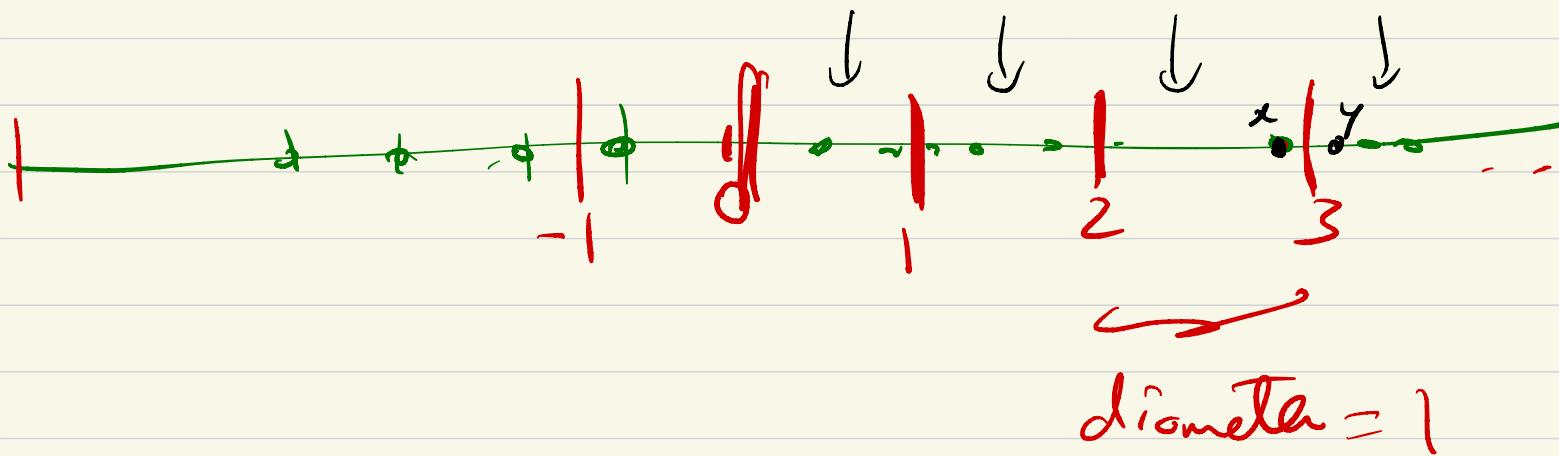
$x$  &  $y$  are separated



$P(x) = \text{set in the partition to which } x \text{ belongs.}$

Example

$$(X, d) = (\mathbb{R}, |||)$$

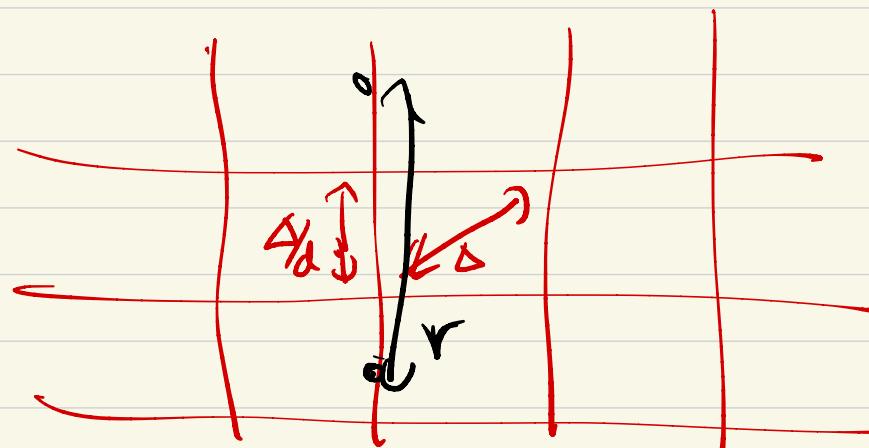


$$\Pr\{ \rho(x) \neq \rho(y) \} = 1$$

$$\Pr\left(\rho(x) \neq \rho(y)\right) \leq \frac{1}{1} \cdot \frac{|x-y|}{1} = \Delta$$

A

$$(\mathbb{R}^d, |||, |||)$$



$$\Pr(\rho(x) \neq \rho(y)) = \frac{\pi r^2}{\Delta_d^d} \approx d \left( \frac{r}{\Delta} \right)$$

# LOW-DIAMETER DECOMPOSITIONS

INPUT:  $(X, d)$  Goal:  $\Delta$ -bounded partition

- Pick a random radius

$$R \in [\Delta/4, \Delta/2]$$

permutation



- Randomly Order  $X = \{x_1, \dots, x_n\}$

set of points within distance  $\Delta$

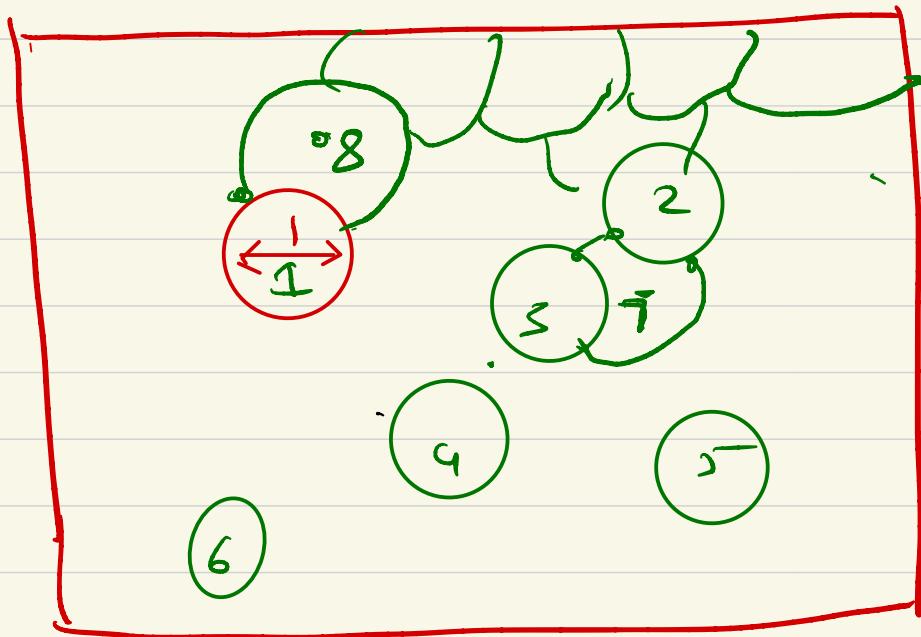
- $B(x_1, R) = \text{ball of radius } \Delta$

$$B(x_2, R) \setminus B(x_1, R)$$

$$B(x_3, R) \setminus (B(x_1, \Delta) \cup B(x_2, R))$$

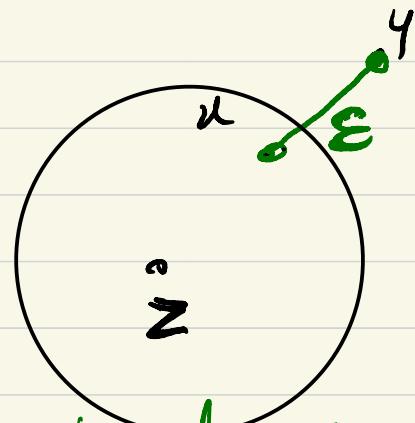
until all points are assigned.

$$l_2 = (\mathbb{R}^2, \|\cdot\|_2)$$



$$x, y \in \mathbb{R}^2$$

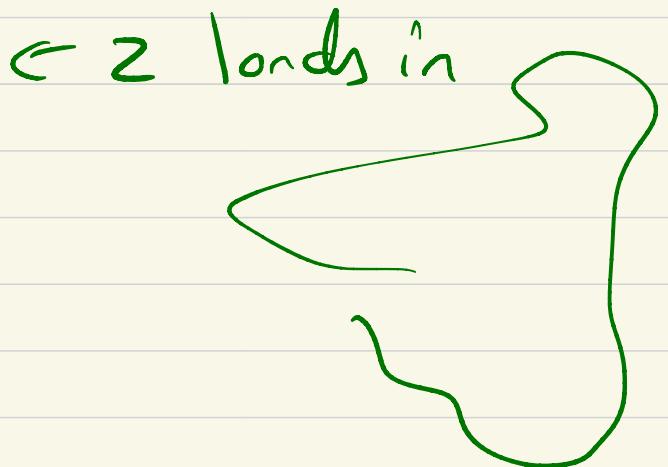
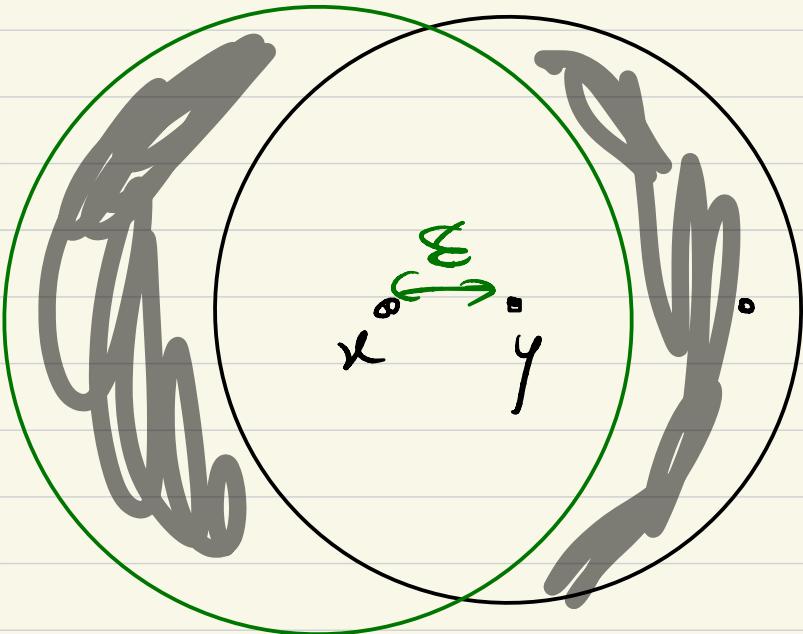
$$\Pr [f(x) \neq f(y)]$$



Wlog let  $x$  be assigned a position  
for  $\Delta$

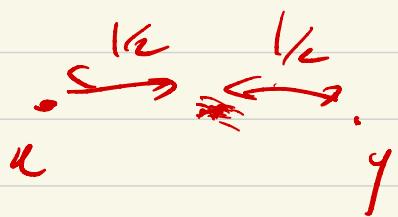
$$\|x - z\|_2 \leq 1 \quad \text{but} \quad \|y - z\|_2 \geq 1$$

$\Rightarrow$



$$P_1(A(u) \neq A(y)) = \frac{\text{Volume of } B(x, 1) \Delta B(y, 1)}{\text{Volume of union of } B(u, 1) \cup B(y, 1)}$$

$\propto$  Surface area of the ball.



Claim: Suppose  $d(x, y) = r$  then

$$\Pr[\rho(x) = \rho(y)] \geq \exp\left[-\frac{8r}{\Delta} - \log \frac{|\mathcal{B}(x, \Delta)|}{|\mathcal{B}(x, \Delta/8)|}\right]$$

Corollary:

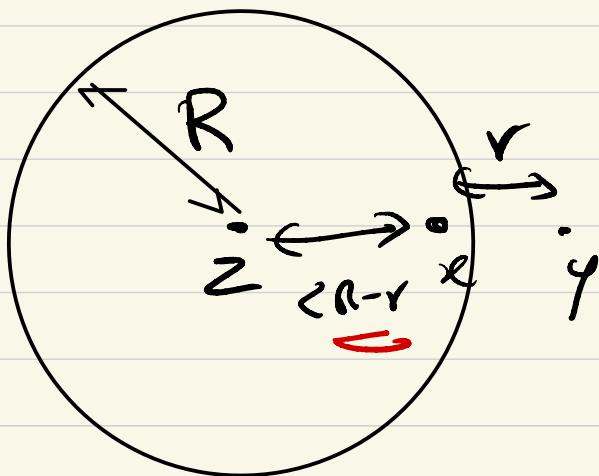
$$\Pr[\rho(x) \neq \rho(y)] = 1 - \exp(-)$$

$$\leq \frac{r}{\Delta} \cdot \frac{8 \log |\mathcal{B}(x, \Delta)|}{|\mathcal{B}(x, \Delta/8)|}$$

$$\leq \underline{(8 \log n)} \cdot \overline{r} \cdot \overline{\frac{8 \log n}{|\mathcal{B}(y, \Delta)|}} \cdot \overline{\frac{d(x, y)}{\Delta}}$$

Proof:  $\overset{\text{center}}{z}$  captures  $x$

$$\Pr[\rho(x) = \rho(y)] \geq \Pr[d(x, z) < R - r]$$



Volume of intersection  
Volume of union

$$\Pr[x \in B(x, R-r)] = \frac{|B(x, R-r)|}{|B(x, R+r)|}$$

$$= \mathbb{E}_R \left[ \exp \left( -\log \frac{|B(x, R+r)|}{|B(x, R-r)|} \right) \right]$$

By convexity of  $e^x$   $E[e^x] \geq e^{E[x]}$

$$\geq \exp \left[ \underset{R}{\mathbb{E}} \left[ - \log \frac{|B(x, R+r)|}{|B(x, R-r)|} \right] \right]$$

$$\geq \exp \left[ \int_{\Delta/4}^{\Delta/2} - \log \frac{|B(x, R+r)|}{|B(x, R-r)|} dR \right]$$

$$d(x, y) = r < \Delta/8$$

$$R+r < \Delta/2 + \Delta/8 < \Delta$$

$$R-r > \Delta/4 - \Delta/8 > \Delta/8$$

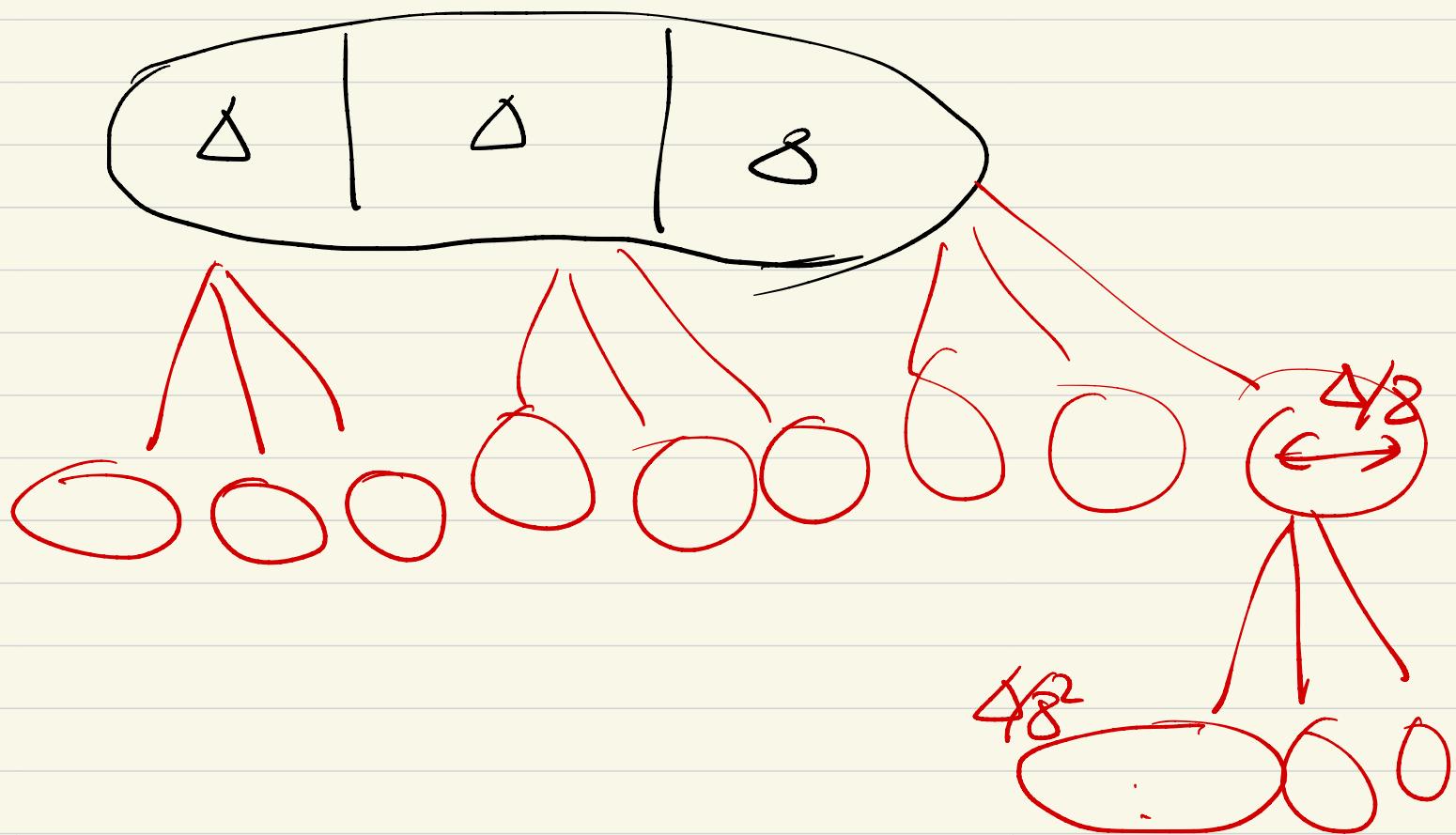
$$\geq \exp \left[ \int_{\Delta/4}^{\Delta/2} - \log \frac{|B(x, \Delta)|}{|B(x, \Delta/8)|} dR \right]$$

$$\geq \exp \left( \frac{8r/\Delta}{D(n,\Delta/3)} \log \frac{|B(n,\Delta)|}{|D(n,\Delta/3)|} \right)$$

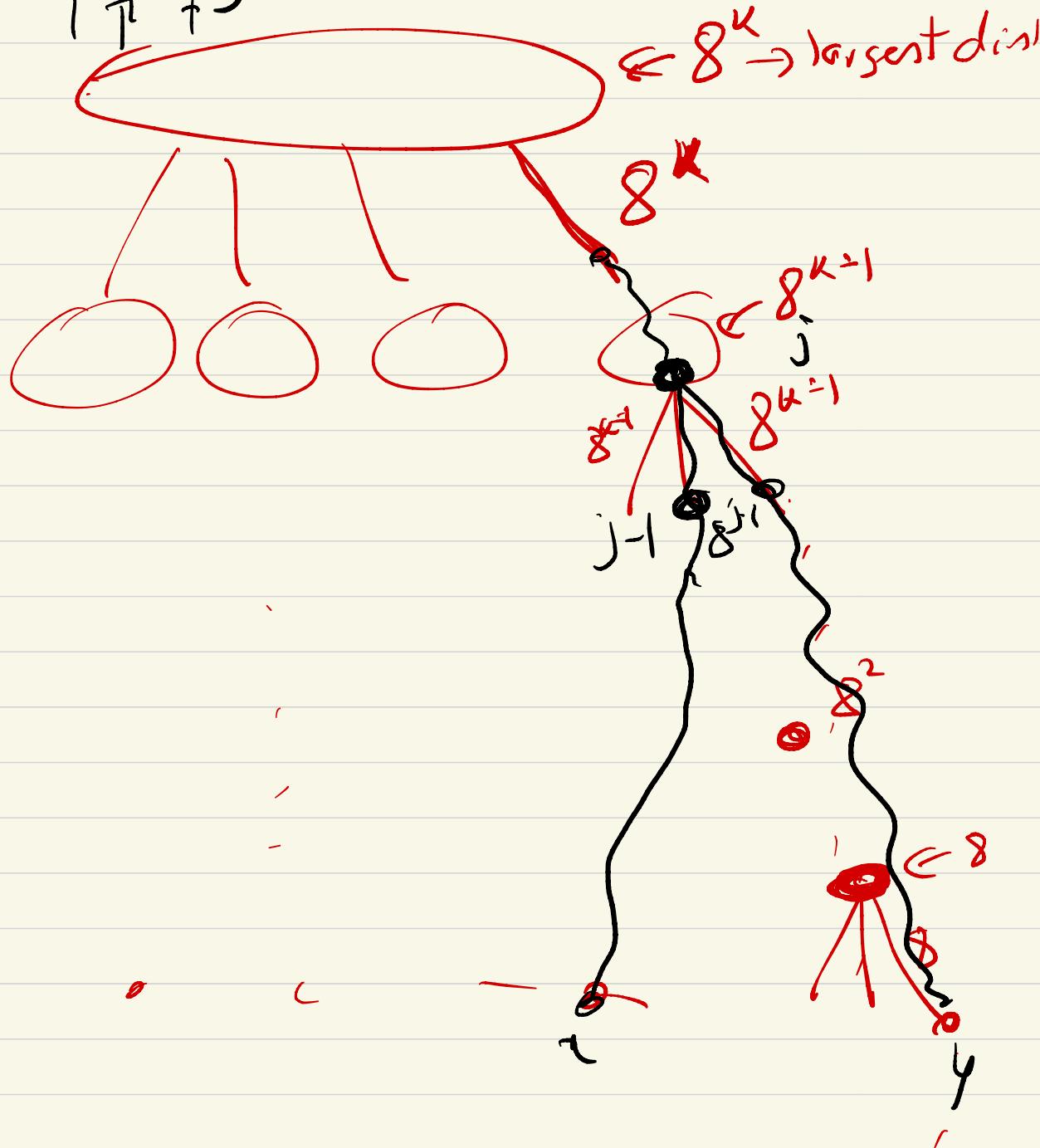
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## Hierarchical Tree Decomposition



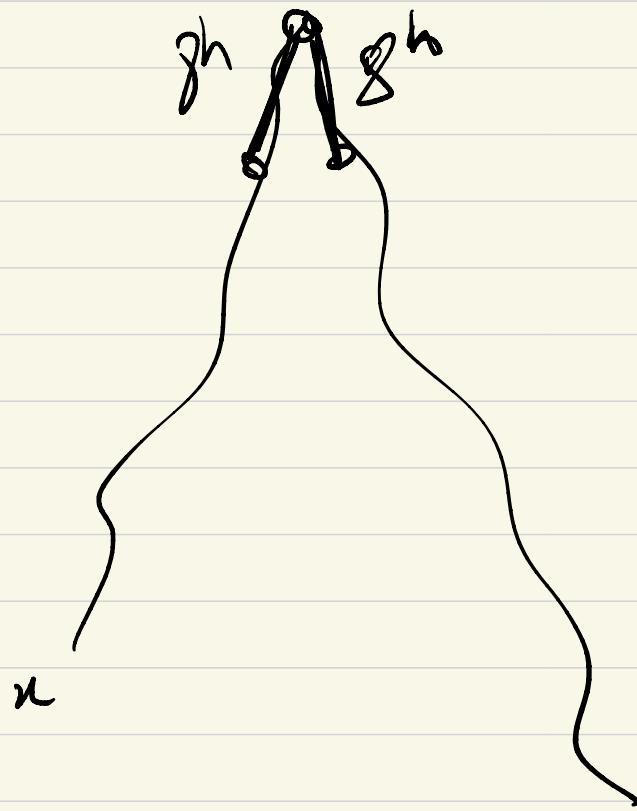
$$[FRT]_{\overbrace{TP}^+})$$



$$s^{j-1} \leq d(x, y) \leq s^j$$

$\text{lca}(x, y)$  is of height at least  $j$

$$\underline{d_T(x,y)} \geq 8^h + 8^{h-1} +$$



$$\geq 8^j \geq \underline{\overbrace{d(x,y)}}$$

$$8^j \leq d(x, y) \leq 8^j$$

$$\mathbb{E}[d_t(x, y)] \leq 8^j + \sum_{t=j+1}^{\infty} 8^t \Pr\left[P_t^{(n)} \neq f_t^{(y)}\right]$$

$$\leq 8^j + \sum_{t=1}^{\infty} 8^t \cdot \frac{8}{8^t} d(x, y) \frac{\log |\mathcal{B}(x, 8^t)|}{|\mathcal{B}(x, 8^{t-1})|}$$

$$\leq 8^j + 8d(x, y) \cdot \sum_{t=1}^{\infty} \frac{\log |\mathcal{B}(x, 8^t)|}{|\mathcal{B}(x, 8^{t-1})|}$$

//

$$\log |\mathcal{B}(x, 8^t)| - \log |\mathcal{B}(x, 8^{t-1})|$$

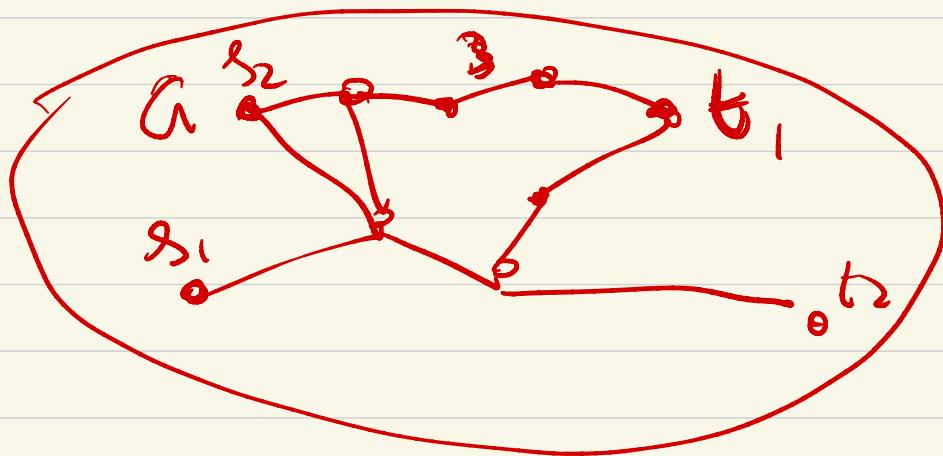
$$\leq 8^j + 8d(x, y) \cdot \log n \approx \underline{\underline{O(\log n) d(x, y)}}$$

# Low-Stretch Spanning Trees:

Buy at Bulk Network Design



design a graph  
that can support flows



$s_1 \rightarrow t_1$  support a flow of value  $f_1$

$s_2 \rightarrow t_2$   $f_2$

$s_k \rightarrow t_2$   $f_k$

To buy capacity  $c$  on on edge  $e$



Cost( $c$ )

$$\text{Min} \sum_e \text{cost}(c_e) \cdot \text{dist}(e)$$

Find  $c_e$  such that

Supports  $s_1 \sim t_1$

$\vdots$   
 $s_k \sim t_k$

