LECTURE 1

CONSTRAINT SATISFACTION PROBLEMS

1) 3-SAT

Ex: $(x \vee y \vee z) \wedge (\bar{x} \vee \bar{y} \vee \omega) \wedge (\bar{\omega} \vee \bar{y} \vee \bar{z})$

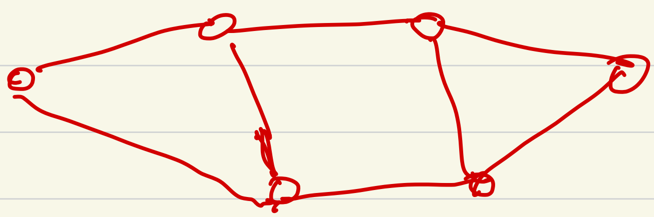
Variables: x, y, z, ω

Domain: $\{0, 1\}$

Constraints: Clauses

2) 3-COLORING

Ex:



Variables: vertices

Domain: $\{R, B, G\}$

Constraints: $\forall$ edge (u, v)
color$(u) \neq$ color(v).

3) 3-LIN ($\mathbb{Z}_{17}$)

Ex: $x+y-z = 9 \pmod{17}$

$x - w + y = 10 \pmod{17}$

$x + 2y + 3z = 3 \pmod{17}$

Variables : x, y, z, w

Domain : $\mathbb{Z}_{17}$

Generally,

A CSP instance consists of

1) Set of variables. V taking values in a constant sized domain D

2) A set of LOCAL constraints on them

$\Updownarrow$

constant # of variables.

TYPE OF CONSTRAINTS

defines the KIND OF CSP.

COMPUTATIONAL PROBLEMS

Satisfiability:

INPUT: CSP Instance I

OUTPUT: Does there $\exists$ a satisfying assignment?

APPROXIMABILITY.

INPUT: CSP INSTANCE I

OUTPUT: An assignment satisfying the maximum # of constraints

a) SATISFIABLE CSPs

b) PROMISE CSPs

COUNTING:

INPUT: CSP Instance I

OUTPUT: How many satisfying assignments are there?

RANDOM CSPS:

→ Given random instance I, is

$\Pr[I \text{ is satisfiable}] \approx 1$ or ≈ 0 ?

→ Solution Space

→ How big is it?

→ What is its geometry?

→ ALGORITHMIC QUESTIONS:

a) Find satisfying assignments for random CSPs?

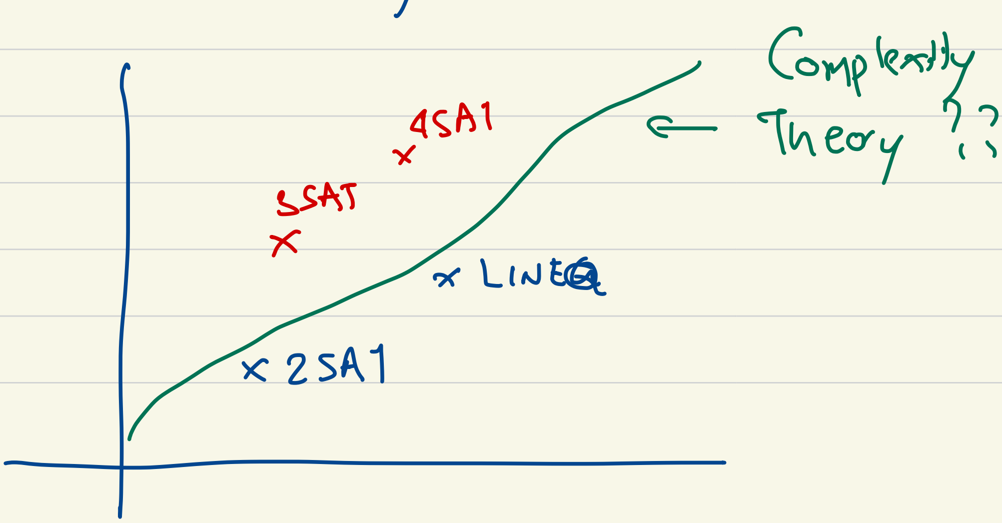
QUANTUM CSPs:

(Quantum local Hamiltonians)

→ How well can we approximate ground state?

META QUESTIONS

Why is 2-SAT easy but 3-SAT NP-complete?



1) Given the description of a CSP Γ is it in P or NP-complete?

2) Given a CSP Γ, what is the best approximation ratio for it?

Logistics:

→ Course webpage:

→ a) Scribe 1 lecture

b) Review scribing 1 lecture.

→ Class Participation: ASK QUESTIONS and INTERRUPT!

LECTURE PLAN:

→ Polymorphisms & Dichotomy Conjecture
- 2 lectures

→ PCP Theorem: Statement & Proof
- 3 lectures.

CONSTRAINT SATISFACTION PROBLEMS (CSP)

Def: A CSP Γ is specified by

1) A constant sised alphabet D
2) A finite set of relations $R_1, R_2 \ldots R_L$ on D

Ex: $D = \{0,1\}$ $R_1 = \{(0,0), (0,1), (1,0)\}$ $R_2 = \{(0,0), (0,1), (1,1)\}$ $R_3 = \{(0,1), (1,0), (1,1)\}$ $R_4 = \{(0,0), (1,0), (1,1)\}$

An instance of CSP Γ consists of

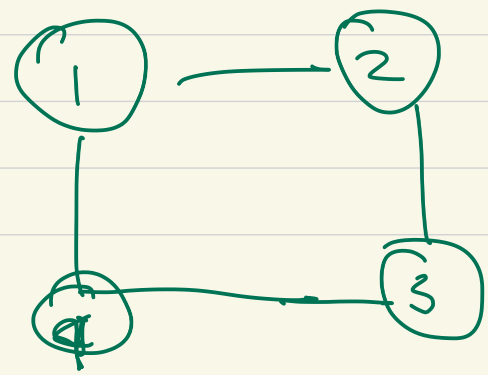
1) A set of variables over alphabet D

$V = \{v_1, \ldots v_n\}$

2) Local constraints $C_1, \ldots, C_m$ on V
where each constraint $C_i := R_j$ applied on a set of variables.

2SAT:

Example:
(Special Case) Γ has only one binary symmetric relation $\Leftrightarrow$ graph H

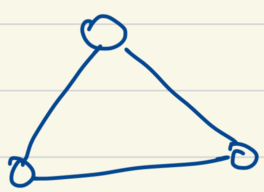
$\Gamma =$ 

Instance $I \equiv$ Graph G on n vertices.

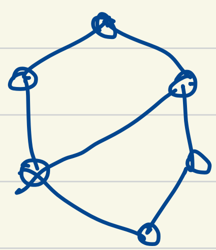
I is satisfiable

$\iff \exists$ a map $f: I \to H$ so that edges map to edges?

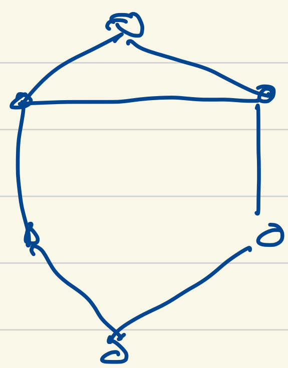
What is complexity of satisfiability Γ_H?



NP complete



P



NP-complete

[Hell-Nesteril]

CSP Γ_H is in P if and only if H is bipartite.

Schaefer's Theorem:

Among all boolean CSPs the following are in P

1) 2-SAT $\longrightarrow (x \vee y) \wedge (\bar{x} \vee z) \wedge (x \vee \bar{z})$

2) Horn SAT $\longrightarrow (a \wedge b \Rightarrow c)\ (d \wedge a \Rightarrow b)$
$(c \wedge b \Rightarrow d)$

3) Dual Horn SAT $\rightarrow (\bar{a} \wedge \bar{b} \Rightarrow \bar{c})\ (\bar{d} \wedge \bar{a} \Rightarrow \bar{b})$ ...

4) Linear Equations.

5) Trivial CSP

the rest are NP-complete.

What makes these CSPs easy??

THEY ALL HAVE "POLYMORPHISMS"

Operations combining solutions

Polymorphism: (by example)

Linear systems over $\mathbb{Z}_2$

$$Ax = b \qquad x \in \mathbb{F}_2^n$$
$$A \in \mathbb{Z}_2^{n \times m}$$

Suppose x, y, z are solutions

$$\left.\begin{array}{l} Ax = b \\ Ay = b \\ Az = b \end{array}\right\} \Longrightarrow A(\underbrace{x - y + z}_{\text{new solution.}}) = b$$

More concretely:

$$h : \mathbb{Z}_2^3 \to \mathbb{Z}_2$$
$$h(a, b, c) = a - b + c$$

such that

Given ANY instance of LINEQ $Ax = b$

ANY 3 solutions x, y, z

then
$$
\begin{array}{lllll}
x = & x_1 & x_2 & \dots & x_n \\
y = & y_1 & y_2 & & y_n \\
z = & z_1 & z_2 & & z_n
\end{array}
$$

$$W \stackrel{def}{=} h(x_1, y_1, z_1), h(x_2, y_2, z_2) \quad h(x_n, y_n, z_n)$$

then W is a solution to same linear system

$\Rightarrow$ 3 bit function $h(a, b, c) = a - b + c$

is a polymorphism for LINEQ.

2SAT: Polymorphism $Maj(x_1, x_2, x_3)$

Horn SAT : $AND(x, y, z)$

Dual Horn SAT : $OR(x, y, z)$

$$
\begin{tikzcd}
x \arrow[r, bend left, "x \Rightarrow y"] & y \arrow[r, bend left, "\bar{y} \Rightarrow z"] \arrow[rr, bend right, "\overline{\cancel{x}y} \Rightarrow \bar{w}"'] & z \arrow[r, bend left, "\bar{z} \Rightarrow \bar{w}"] & w
\end{tikzcd}
$$

x	0	1	1	0
y	1	1	1	0
z	0	0	1	0

$w = (0 \quad 0 \quad 1 \quad 0)$

Polymorphism: Given a CSP $\Gamma = (D, \{R_1 \ldots R_k\})$

a function $h : D^\ell \to D$ is a polymorphism for Γ if

$\forall$ relation $R_j \subseteq D^t$

$\forall\ a_1 \ldots a_\ell \in R_j$

$$(a_1^{(1)} \quad a_1^{(2)} \quad \cdots \quad a_1^{(t)}) \in R_j$$

$$(a_2^{(1)} \quad a_2^{(2)} \quad \quad a_2^{(t)}) \in R_j$$

$$\vdots$$

$$(a_\ell^{(1)} \quad a_\ell^{(2)} \quad \quad a_\ell^{(t)}) \in R_j$$

$$h(a_1^{(1)}, a_2^{(1)} \ldots a_\ell^{(1)}) \quad \quad h(a_1^{(t)}, a_2^{(t)} \ldots a_\ell^{(t)}) \in R_j$$

TRIVIAL POLYMORPHISMS (Dictators!!)

$$h(x^{(1)} \dots x^{(t)}) = x^{(i)} \quad \text{for some } i.$$

[Algebraic Dichotomy]

A CSP Γ is in P if it admits

non trivial polymorphisms

otherwise it is NP-complete!

[Hardness]

No non-trivial polymorphisms

$\Longrightarrow$ CSP Γ is NP-complete

[Algorithm]

$\exists$ non-trivial polymorphisms

$\Longrightarrow$ CSP Γ is in P