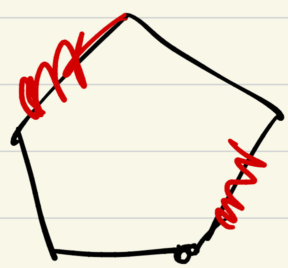
LECTURE 24

Matchings:

- cornerstone of Algorithm
- "P"

Polyhedra / LP



Graph = $G = (V, E)$

Variables: $x_e \quad e \in E$

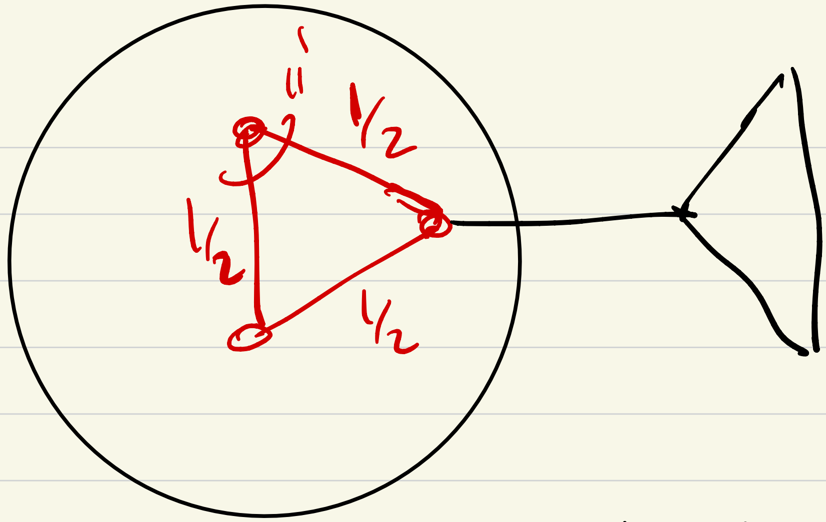
$$x_e = \begin{cases} 1 & \text{if } e \in \text{Matching} \\ 0 & \text{otherwise} \end{cases}$$

Constraints: $x_e \geq 0 \quad \forall e \in E$

$$x(\delta(v)) = \sum_{e \ni v} x_e = \underline{1} \quad \forall v \in V$$

$\forall\ U \subseteq V \quad |U| = \text{odd}$

$$x(\delta(U)) = \sum_{e \text{ leaves } U} x_e \geq 1 \quad \Big\} \text{ exponentially many}$$



U $\qquad |U| = \text{odd}$

(integral)

Polytope P

Constraints: $\quad x_e \geq 0 \quad \forall e \in E$

degree constraints $\left\{ x(\delta(v)) = \sum_{e \ni v} x_e = 1 \right\} \quad \forall v \in V$

$\forall U \subseteq V \quad |U| = \text{odd}$

$x(\delta(U)) = \sum_{e \text{ leaves } U} x_e \geq 1 \Big\}$ odd set

Theorem: All the extreme points of P are perfect matchings!

Proof: Let $G = (V, E)$ be the smallest graph $|V(G)| + |E(G)|$

such that $x \in \mathbb{R}^E$ is an extreme point

that is not a matching!

$$x_e = \begin{cases} 1 & \text{if } e \in M \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow$ $x_e \neq 0 \quad \forall e \in E$

otherwise $x_e = 0$, delete e,

$G \setminus e$ is a smaller counterexample

$\rightarrow$ $\deg(v) \geq 2$, $\because$ if $\deg(v) = 1$

then $x(\delta(v)) = 1 \Rightarrow x_e = 1$

$\Rightarrow e = (u, v) \quad G \setminus \{u, v\}$

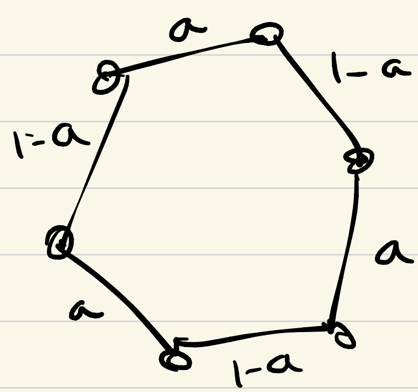
$\rightarrow$ $\deg(v) \geq 2 \Rightarrow |E| \geq |V|$

$\left[\because \sum \deg(v) = 2|E|\right]$

$\rightarrow$ G is connected.

$\rightarrow$ $|E| = |V|$, $\deg(v) = 2 \Rightarrow$ G is a cycle

$\Rightarrow$ Statement true for a cycle



$\Downarrow$

$x = a$

$+$

$(1-a)$

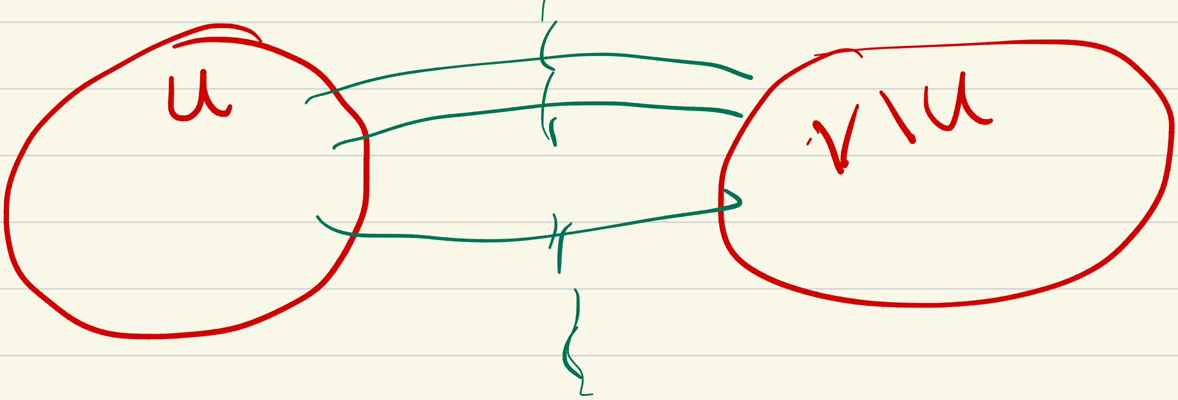
$\rightarrow \quad |E| > |V| \qquad |E| \geq |V| + 1$

Extreme point x has atleast $|E|$ $\geq |V|+1$ tight constraints

$\Rightarrow \exists$ some odd set U ($|U|$ is odd, odd set constraint is tight

$$\boxed{x(\delta(U)) = 1} \leftarrow \text{tight}$$

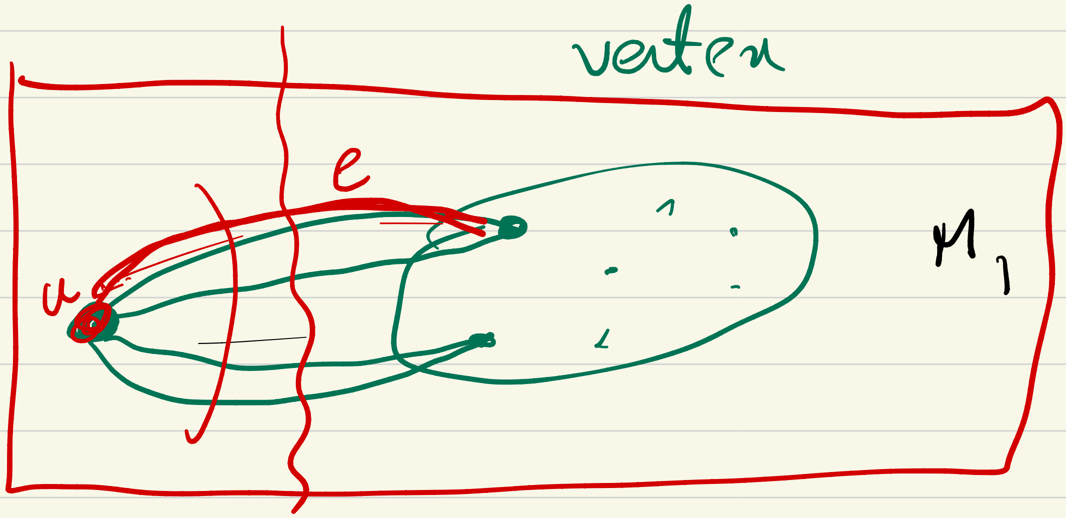
$\sum x_e = 1$



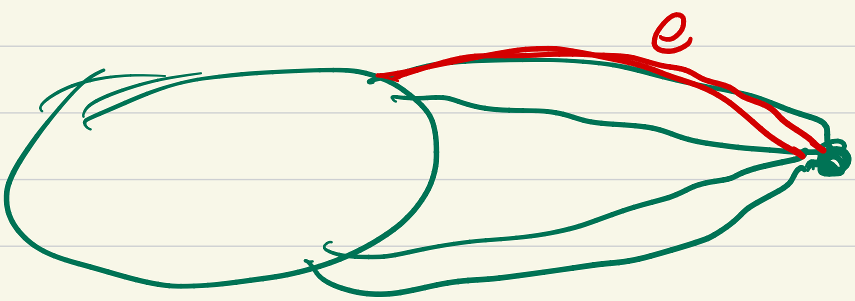
G_1 = Contract U to a single vertex

G_2 = contract $V \setminus u$ to a single vertex

G_1



G_2



M_2

$X^1 = x|$ restricted to G_1

$X^2 = x|$ restricted to G_2

Check that these satisfy LP constraints

red

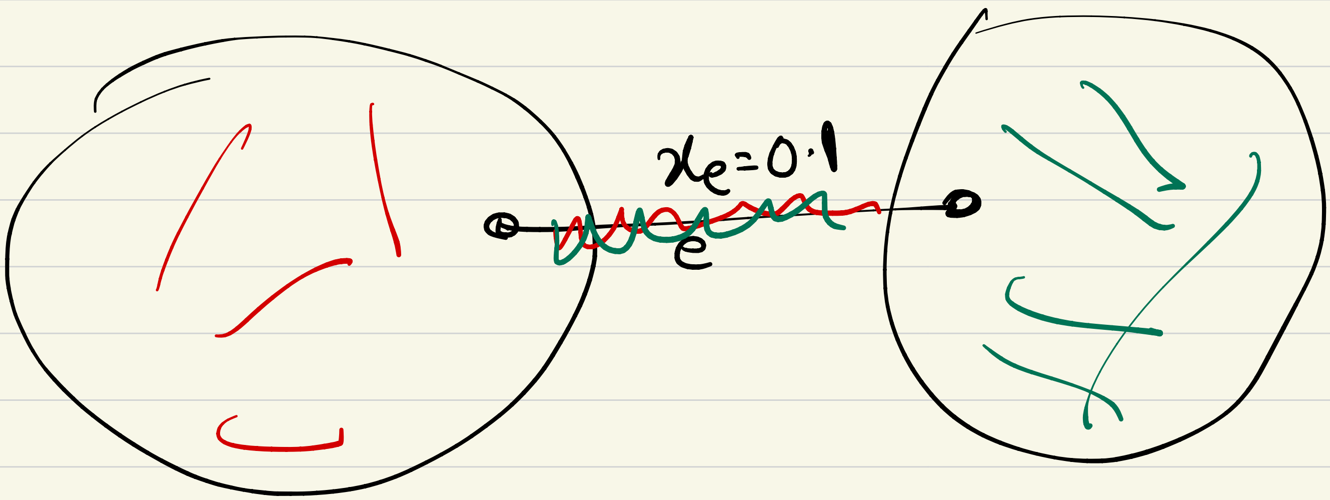
integer matching

$$X^{(1)} = \frac{1}{K} \sum_{M_1 \subseteq G_1} c_{M_1}$$ → matchings

$$X^{(2)} = \frac{1}{K'} \sum_{M_2 \subseteq G_2} c_{M_2}$$

Paste matchings $\underline{\underline{M_1}}$ & $\underline{\underline{M_2}}$

to get matching of G



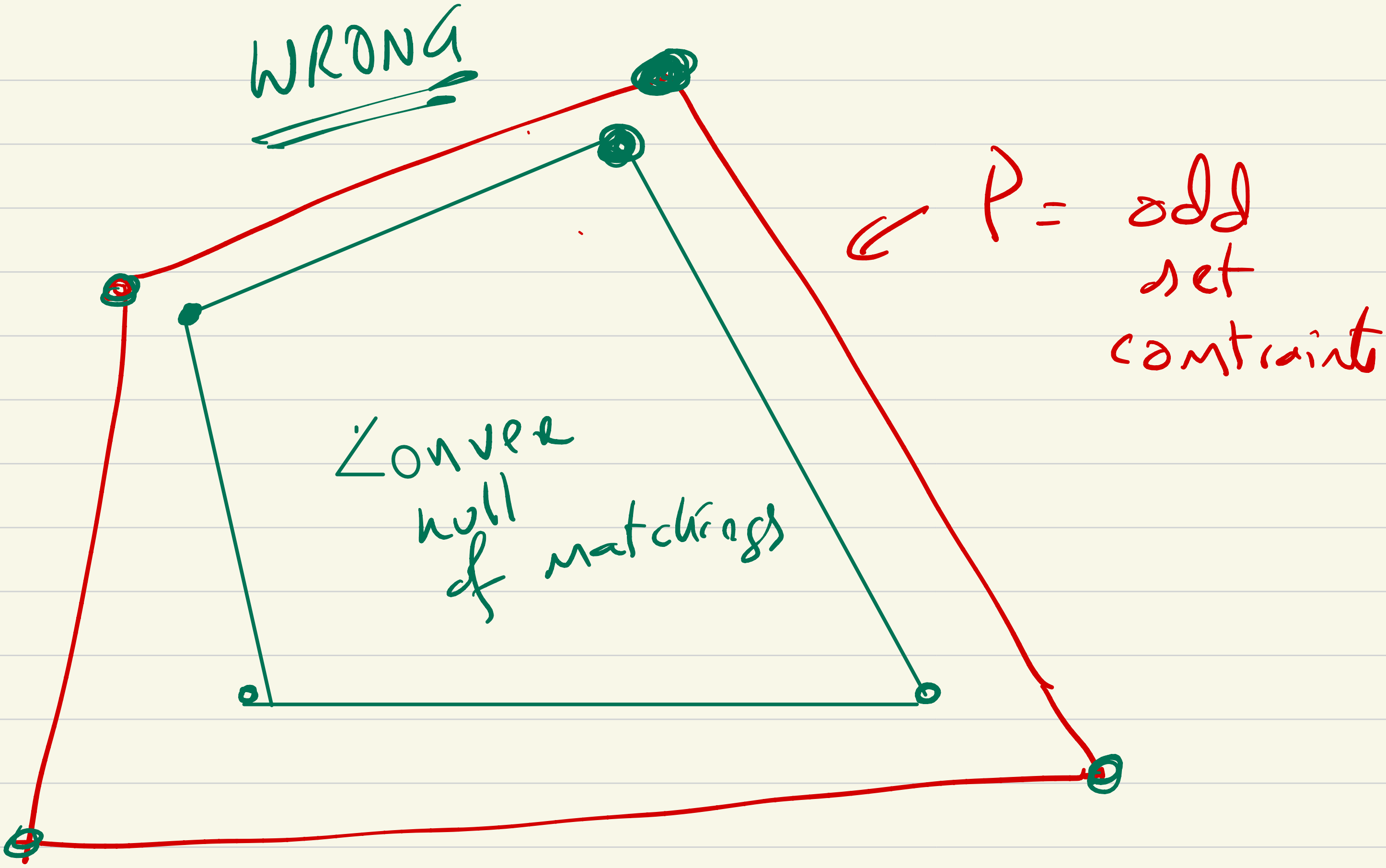
0.1 fraction of M_1

0.1 fraction of M_2

Theorem:

x is an extreme point for G

$\Rightarrow$ x = Convex combination of matchings of G



Convex hull of matchings

$= P$

→ Exponential sized LP / Separation oracle exists

→ Spanning tree = ∃ a LP with poly constraints

→ ∄ a polysized LP

Algebraic Approach

Bipartite Matching: $G = (V_1 \cup V_2, E)$

$\exists$ a perfect matching in G ??

$$B = \begin{array}{c} \\ V_1 \end{array} \overset{\longleftarrow \; V_2 \; \longrightarrow}{\begin{bmatrix} & x_{ij} & \\ & & \\ 0 & & \end{bmatrix}}$$

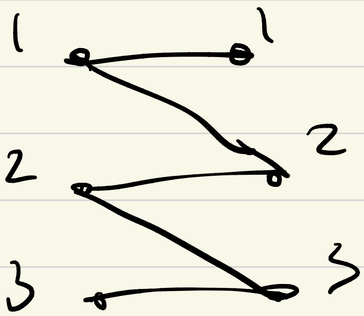
matrix of variables

$$B_{ij} = \begin{cases} x_{ij} & \text{if } (i,j) \in E \\ 0 & \text{if } (i,j) \notin E \end{cases}$$

$\det(B)$ = polynomial in $\{x_{ij}\}$ variables.

$$\det(B) = \sum_{\substack{\pi\text{ - permutations}\\ \text{of } \{1..n\}}} \overset{\pm 1}{\mathrm{sgn}(\pi)} \prod_{i=1}^{n} B_{i,\pi(i)}$$

$$= \sum_{\substack{\pi\text{ - permutation}\\ \pi \text{ is a perfect matching}}} \mathrm{sgn}(\pi) \cdot \prod_{i=1}^{n} x_{i,\pi(i)}$$



$$\det(B) = x_{11} \cdot x_{22} \cdot x_{33}$$

$$\boxed{\det(B) \neq 0 \text{ if and only if } G \text{ has a perfect matching}}$$

ALG:

$(i,j) \in E$

Assign $x_{ij} \leftarrow$ random numbers $\underbrace{\{0 \ldots n^3\}}$

Compute $\boxed{\det(B)[x_{ij}]}$

$$\begin{pmatrix} & & \\ & 0 & r^* \\ & & \end{pmatrix}$$

efficiently

→ In n^3 time using Gaussian eliminati.

→ same time as matrix multiplication

Check if $\det[B] \neq 0$ ...

Theorem: ALG is correct w.p. $\geq 1 - 1/n^2$

Schwarz-Zippel Lemma (modulo $\mathbb{Z}_p$)

$P[x_1 .. x_n] \leftarrow$ non-zero polynomial

Suppose $x_i \leftarrow$ random val from S

$$\Pr\left[\; P(x_1 .. x_n) = 0 \;\right] \leq \frac{\text{degree}(P)}{|S|}$$

[degree(P) = maximum degree of a monomial in P]

General Graphs: $G = (V, E)$

$$T = \begin{array}{cc} & n \\ n & \begin{bmatrix} & +x_{ij} \\ -x_{ij} & \end{bmatrix} \end{array}$$

$$T_{ij} = \begin{cases} 0 & (i,j) \notin E \\ +x_{ij} & i < j \\ -x_{ij} & i > j \end{cases}$$

Claim: $\det(T)$ is a nonzero polynomial

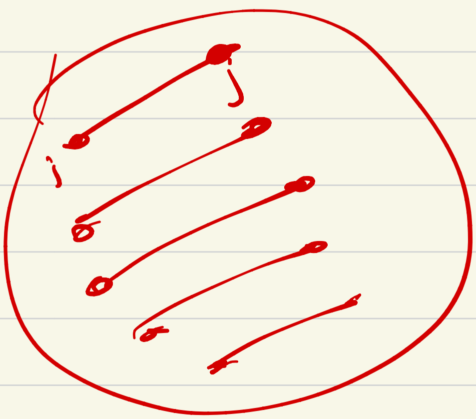
$$\Updownarrow$$

G has a perfect matching.

Proof: M is a perfect matching

$$\det(T) = \boxed{\sum_{\pi} \operatorname{sgn}(\pi) \cdot \prod_{i=1}^{n} T_{i,\pi(i)}}$$

Consider π:



$$(i,j) \in M \Rightarrow \pi(i) = j$$
$$\pi(j) = i$$

$$\prod_{i=1}^{n} T_{i,\pi(i)} = \boxed{\pm \prod_{(a,b)\in M} x_{ab}^2}$$

$$x_{ab} \to T_{a,\pi(a)}$$
$$T_{b,\pi(b)}$$

$\prod_{(a,b)\in M} x_{ab}^2 \neq$ any other permutation!

$\det(T) \neq 0$ as a polynomial!!

$\det(T) \neq 0$ as a polynomial

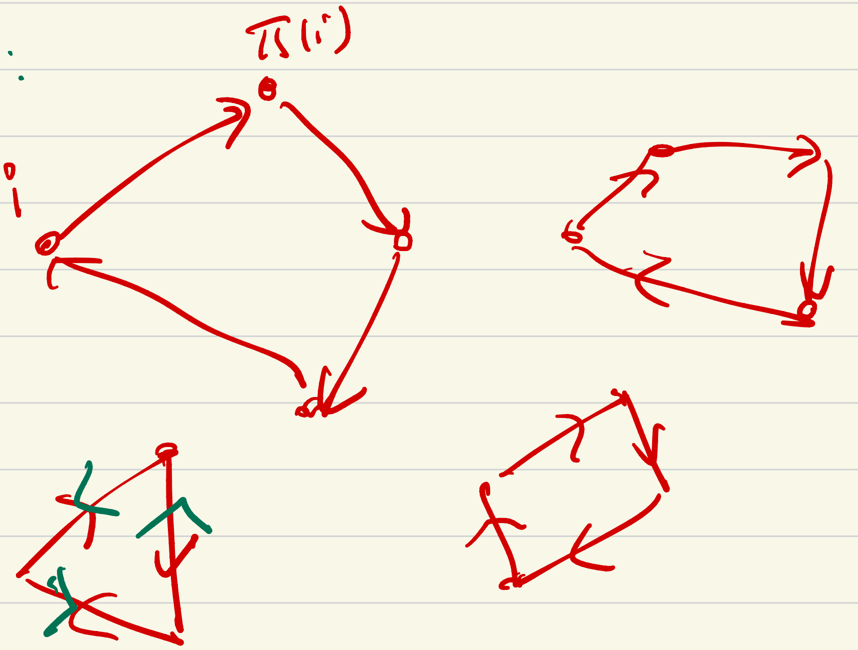
$$\Downarrow$$

$\exists$ a matching

$\prod_{i=1}^{n} T_{i,\pi(i)}$ is nonzero, &

not cancelled!

π :



Every permutation is a union of cycles!!

If π has an odd cycle.

$\pi^R \leftarrow$ permutation with reverse odd cycle

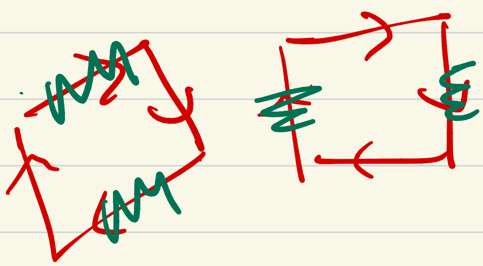
$$\begin{array}{ccc} \operatorname{sgn}(\pi) & \prod & T_{i,\pi(i)} \\ \| & & \| \\ \operatorname{sgn}(\pi^R) & \prod & T_{i,\pi^R(i)} \end{array} = $$

are negations of each other!!

In $\det(T)$ terms π and π^R cancel each other!!

$\Rightarrow$ π must have only even cycles!!

$\Downarrow$



$\exists$ a perfect matching

Red-Black Matchings [No deterministic]

Input: 1) Bipartite Graph

$$G = (V_1 \cup V_2, E)$$

each edge $e \in E$ is either red / black

2) $k < n$

Goal: Find a perfect matching with exactly k red edges.

[No deterministic algo known!!]

$$B = \begin{bmatrix} \quad \end{bmatrix}$$

$$B_{ij} = \begin{cases} 0 & \text{if } (i,j) \notin E \\ x_{ij} & \text{if } (i,j) \in \text{Black edge} \\ \underline{y \cdot x_{ij}} & \text{if } (i,j) \in \underline{\text{Red edge}} \end{cases}$$

$$\det(B) = \sum_{\pi - \text{permutation}} \operatorname{sgn}(\pi) \cdot \prod_{i=1}^{n} B_{i,\pi(i)}$$

$$\underbrace{y^{\#\text{red edges}} \prod_{(a,b) \in M} x_{ab}}$$

poly in $\{x_{ij}\}$ & y

$$\det(B) = \sum_{j=0}^{n} y^j \cdot q_j(x)$$

$\exists$ a K-ved matching M

if and only if

coefficient $y^K \neq 0$ in $\det(B)$

Fix $x_{ij} \in \alpha_{ij}$ (random)

$$\det(B)_\alpha = \sum y^j q_j(\alpha)$$

By evaluating $\det(B)_\alpha$ at n values of y,

interpolate & compute $q_j(\alpha)$

Check if $q_j(\alpha) = 0$??

Polynomial Identity Testing (PIT)

Input: $P[x]$ = a formula / circuit

Goal: Check if $P[x] \equiv 0$

Randomized algorithm $\Leftarrow$ substitute random values

No deterministic algorithm known