

LECTURE 17

Solving Laplacian linear systems

- Laplacian $L = D - A$

of a graph

$$\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} \quad \begin{bmatrix} \downarrow \\ \text{Adjacency mat.} \end{bmatrix}$$

degrees

$$x^T L x = \sum_{(i,j) \in E} (x_i - x_j)^2$$

INPUT:

$$Lx = b$$

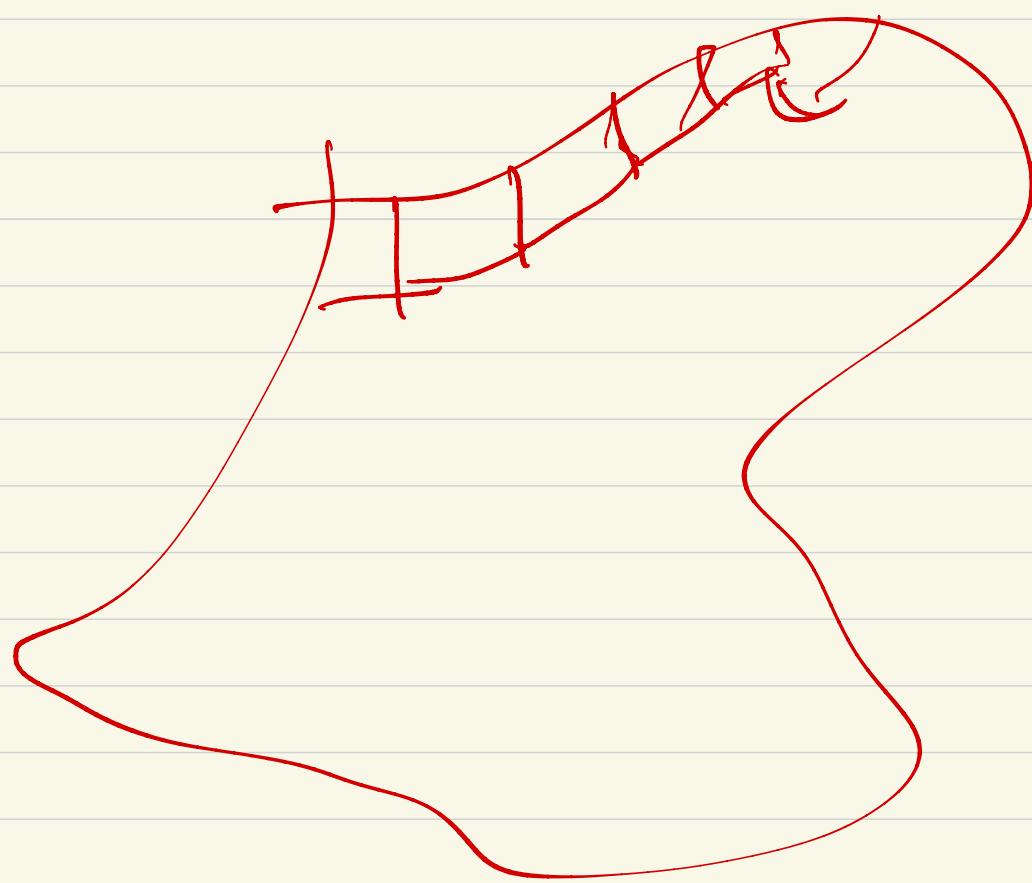
$\left. \begin{array}{l} \text{Gaussian elimination} \\ \text{Gradient descent} \end{array} \right\}$

Goal: Find x

{Spielnotfeng}

00-06

Near linear time
solver $\tilde{\mathcal{O}}(|E|)$



→ Max Flow / Multicommodity Flows / ...



Use Laplacian primitive
(i.e. c , time)

->

Kaczmarz Method

→ Solve: $\{ \langle a_i, x \rangle = b_i \mid i=1\dots n \}$

→ Start at $x_0 \in \mathbb{R}^n$

→ Pick a constraint

$$\langle a_i, x \rangle = b_i$$



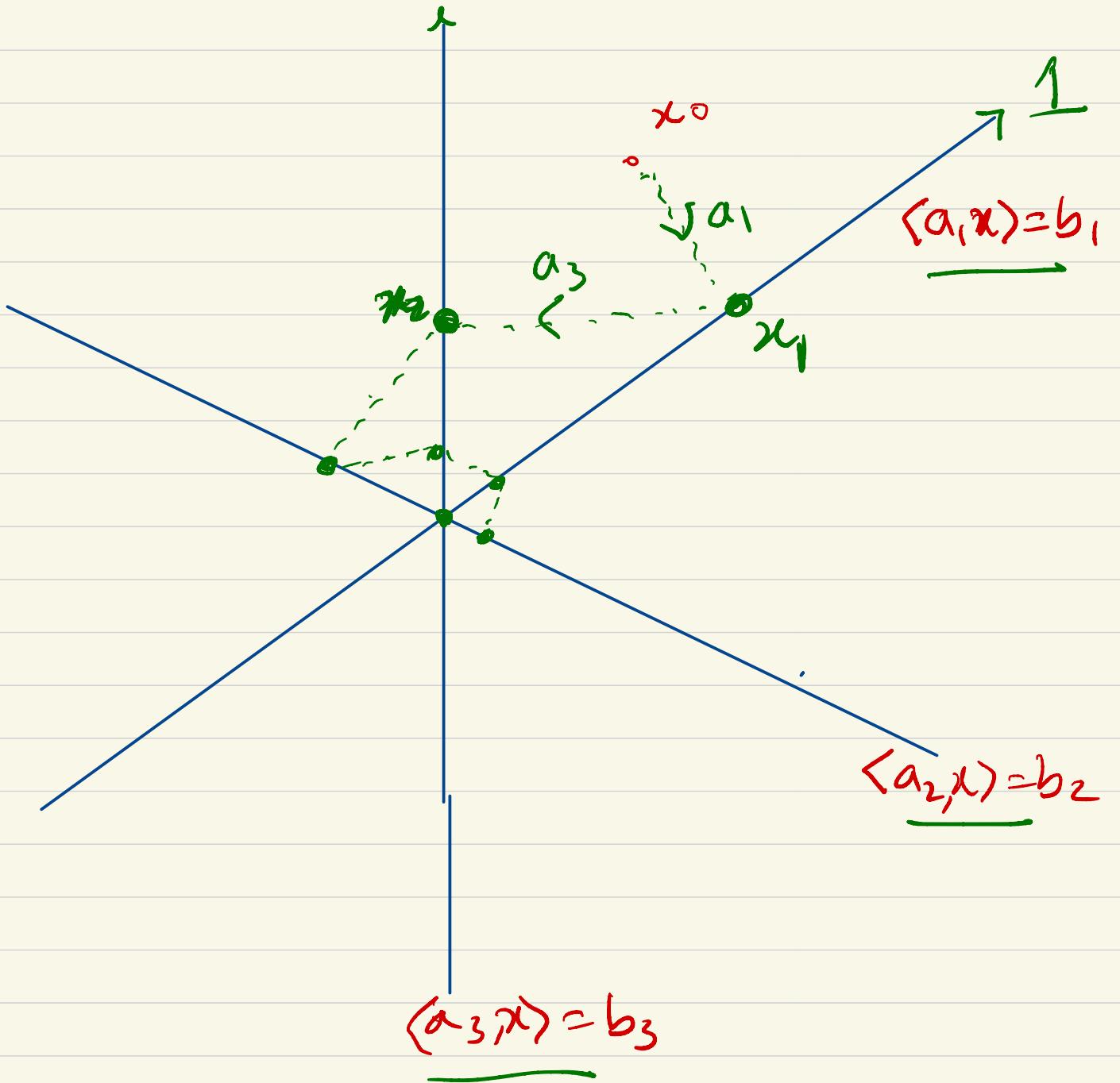
$$x_{t+1} = x_t + (b_i - \langle a_i, x_t \rangle) \cdot \frac{a_i}{\|a_i\|^2}$$



→ $x_{t+1} - x_t \approx \text{multiple of } "a_i"$

$$\rightarrow \langle a_i, x_{t+1} \rangle = \langle a_i, x_t \rangle + (b_i - \langle a_i, x_t \rangle) \frac{\|a_i\|}{\|a_i\|^2}$$

$$= b_i$$



Randomized Kaczmarz

- Pick $\langle a_i, x \rangle$ with prob $\frac{\|a_i\|^2}{\sum_i \|a_i\|^2}$
- $x_{t+1} = \prod_{a_i, a_i \cdot x = b_i} x_t$

A.

$$\underline{x_{t+1} - x_*} = (\underline{x_t - x_*}) - \langle a_i, x_t - x_* \rangle \frac{a_i}{\|a_i\|^2}$$

$$= \left[I - \underline{(\bar{a}_i)(\bar{a}_i)^T} \right] \underline{(x_t - x_*)}$$

where $\bar{a}_i = \frac{a_i}{\|a_i\|}$ Using $\underline{q_i, x^* = b_i}$

$$\underbrace{\mathbb{E}_i \|(x_{t+1} - x_*)\|^2}_{=} = \mathbb{E}_i \left\| \underbrace{(\mathbf{I} - \bar{a}_i \bar{a}_i^\top)(x_t - x_*)}_{\cdot} \right\|^2$$

$$= \mathbb{E}_i \|x_t - x_*\|^2 - \mathbb{E}_i \langle \bar{a}_i, x_t - x_* \rangle^2$$

$$= \|x_t - x_*\|^2 - \sum_i \frac{\|\bar{a}_i\|^2}{\left(\|A\|_{F_V}^2 \right)} \cdot \underbrace{\langle \bar{a}_i, x_t - x_* \rangle^2}_{\sum_{i=1}^m \|\bar{a}_i\|^2}$$

$$= \|x_t - x_*\|^2 - \frac{1}{\|A\|_{F_V}^2} \cdot \underbrace{\sum_i \langle \bar{a}_i, x_t - x_* \rangle^2}_{\|A(x_t - x_*)\|^2}$$

$$= \underbrace{\|x_t - x_*\|^2}_{=} \left[1 - \frac{1}{\|A\|_{F_V}^2} \cdot \frac{\|A(x_t - x_*)\|^2}{\|x_t - x_*\|^2} \right] \underbrace{\qquad}_{\chi(A)}$$

$$\kappa^2(A) = \frac{1}{\|(A)\|_{F_1}^2}, \quad \inf_z \frac{\|Az\|^2}{\|z\|^2}$$

$$E[(x_{t+1} - x_*)^2] = \|x_t - x_*\|^2 (1 - \kappa^2(A))$$

Converge within ϵ : in time

$$O\left(\frac{1}{\kappa^2(A)} \cdot \log\left(\frac{\|x_0 - x_*\|^2}{\epsilon}\right)\right)$$

Graph - network of resistors of resistance $\underline{1}$.



Boundary current
 b_i units
of current
in to vertex

$$\sum b_i = 0$$

2) Currents: $f_{uv} = \text{current on edge } (u, v)$

$$f \in \mathbb{R}^E$$

3) Voltages = $v = (v_1, \dots, v_n) \in \mathbb{R}^n$

$b_i = \sum_{i \rightarrow j} (v_i - v_j) / r$

$\Leftrightarrow Lx = b$

Find voltages (v_1, \dots, v_n) / \mathbf{S}_0
↑ linear time

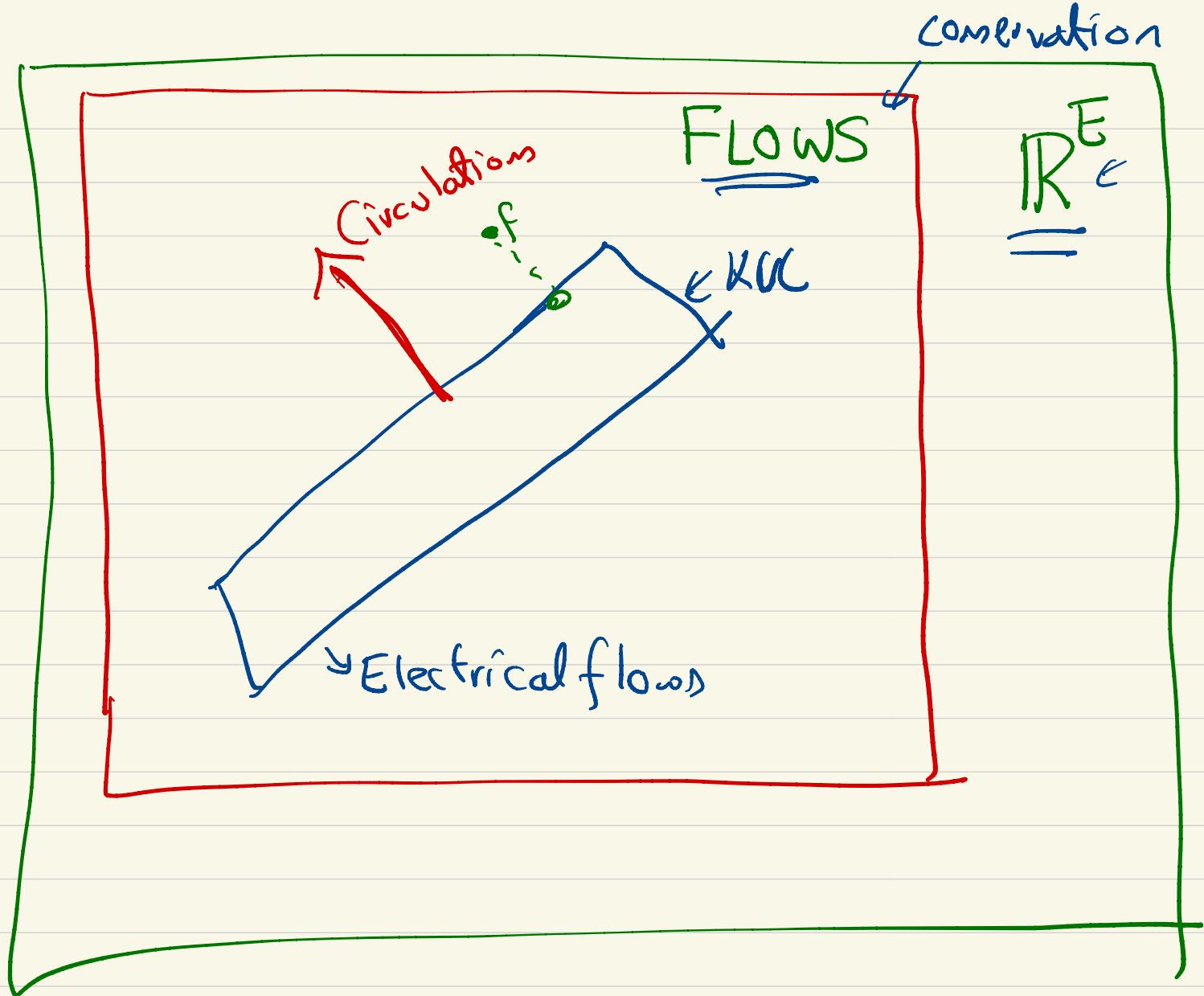
Find currents $\underline{\underline{\mathbf{f}}}$ $\{ f_{ij} \mid i \rightarrow j \}$

$$\underline{\underline{\mathbf{V}}} = \underline{\underline{\mathbf{b}}} \quad |$$

instead solve for $\mathbf{f} \in \mathbb{R}^E$

- Conservation / Kirchoff current flow

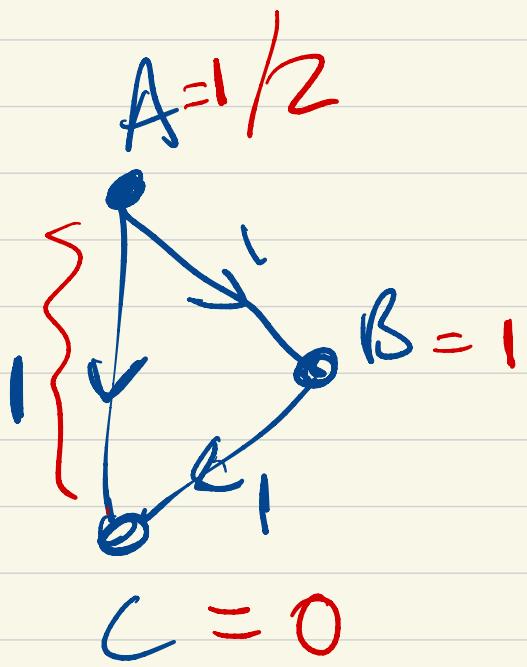
- $\sum_{\substack{i,j \in \text{cycle} \\ c}} f_{ij} = 0$ / Kirchoff voltage law



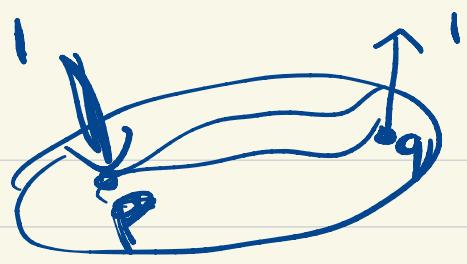
KVL: *directed* cycle $C = u_1 \rightarrow u_2 \dots \rightarrow u_l$

$$\sum_i f_{u_i \rightarrow u_{i+1}} = 0$$

Circulations = Flows with no boundary current



Input: $b \in \mathbb{R}^n$



1) Find some flow F^0 s.t

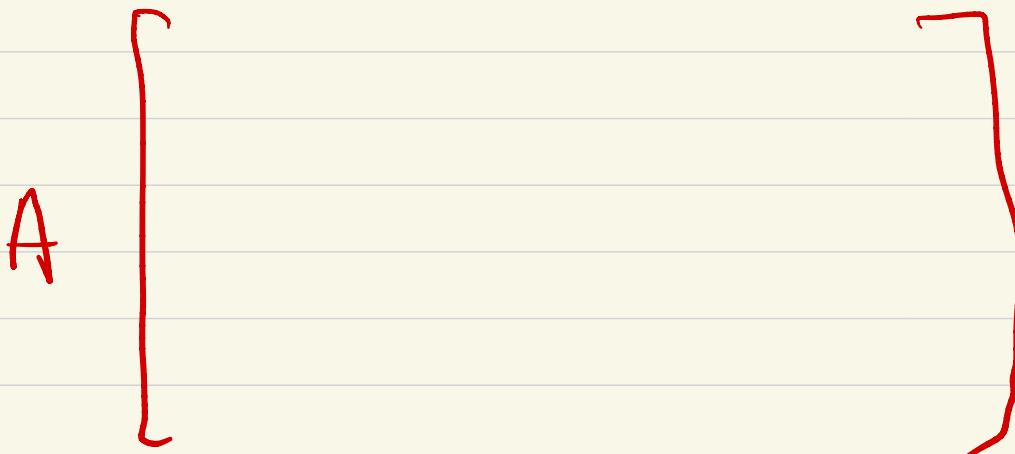
$$b_i = \sum_{i \rightarrow j} F_{ij}^0$$

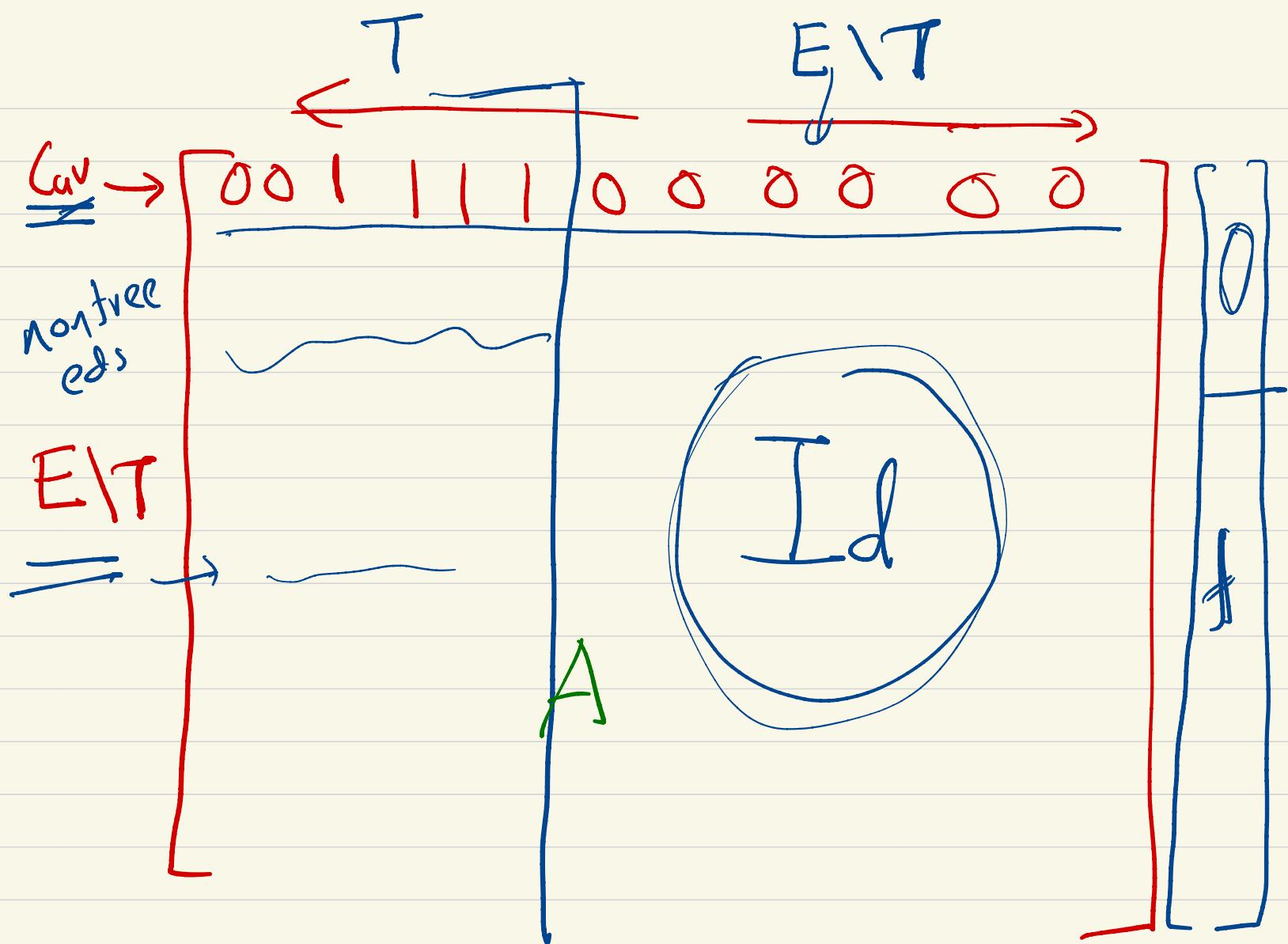
2) Project $F^{(0)}$ in to electrical flows

i) Pick a spanning tree T ✓

$\mathcal{C} = \{ \text{set of cycles formed by adding edges to } T \}$

$$\forall c \in \mathcal{C} \quad \sum_{e \in c} f_e = 0 \quad \begin{cases} m - (n-1) \\ = m - n + 1 \end{cases} \text{ cont}$$





$$A f = 0$$

$$\sum_{e \in \mathcal{E}} f_e = 0$$

$$\|x^T A\| \geq \|x\|$$

$$\approx \|x\|$$

$$k(A) = \frac{1}{\|A\|_{F_1}^2} \cdot \boxed{\inf_f \frac{\|Af\|^2}{\|f\|^2}}$$

Claim:

$$\inf_f \frac{\|Af\|^2}{\|f\|^2} \geq 1 \quad | \quad \|Af\| \geq \|f\|$$

Proof! Id sits inside A.

$$\begin{aligned}
 \|A\|_{F_1}^2 &= \sum_i \|a_i\|^2 \\
 &= \sum_{(u,v) \in E \setminus T} c_{uv} \quad \text{|| } \frac{d_T(u,v)}{d_A(u,v)} \\
 &= \sum_{(u,v) \in E \setminus T} (1 + d_T(u,v)) \\
 &\leq m + \sum_{(u,v) \in E \setminus T} \text{stretch}_T(u,v)
 \end{aligned}$$

Typical edge $\approx \text{stretch}(u_0) \approx O(\text{polylog} n)$

Thm: \exists spanning trees s.t. & (construct
in nearlinear
time)

$$\sum_{(u,v) \in E} \text{stretch}(u,v) \approx O(m \log^2 n)$$

$$\Rightarrow \|A\|_{F_1}^2 = O(m \log^2 n)$$

$$\Rightarrow K(A)^2 = \mathcal{O}\left(\frac{1}{m \log^2 n}\right)$$

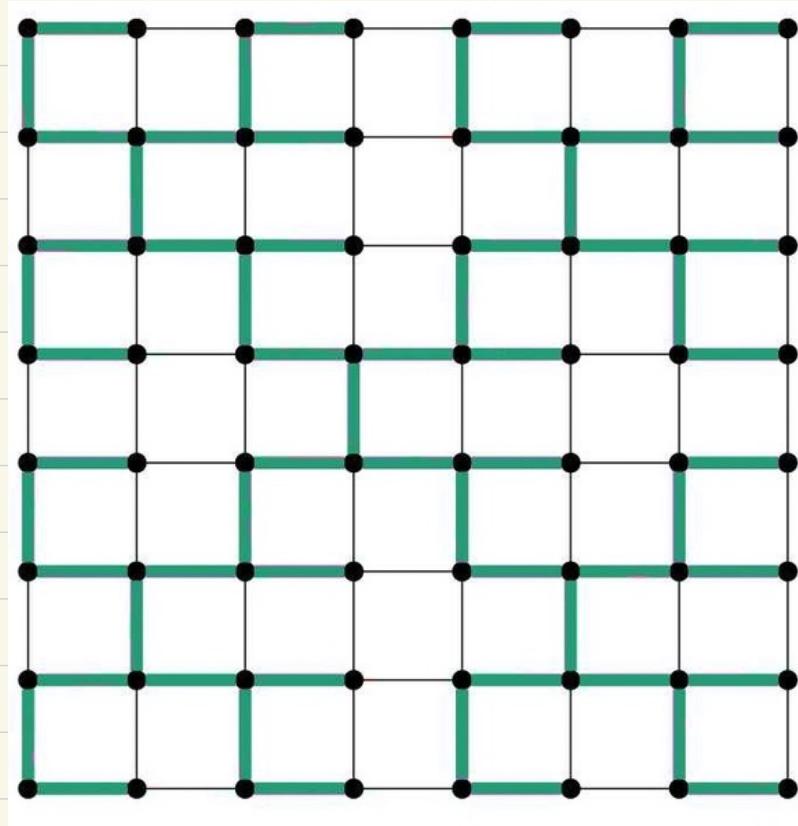
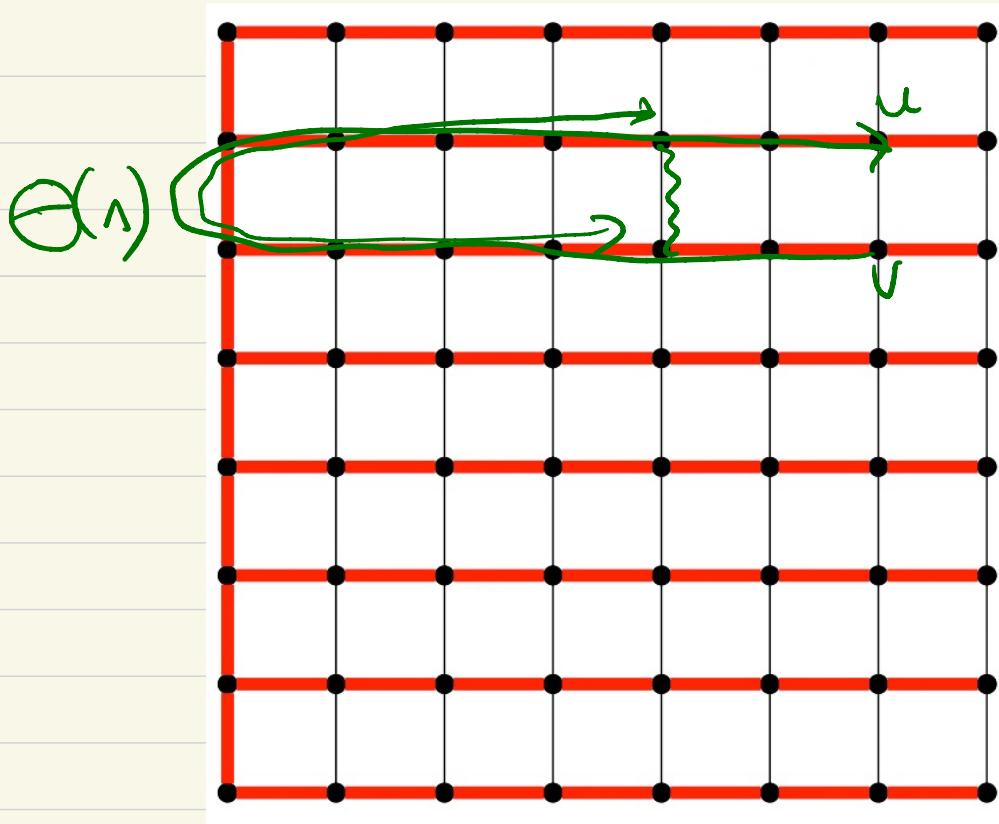
\Rightarrow runtime: $O(m \log^2 n)$ iterations

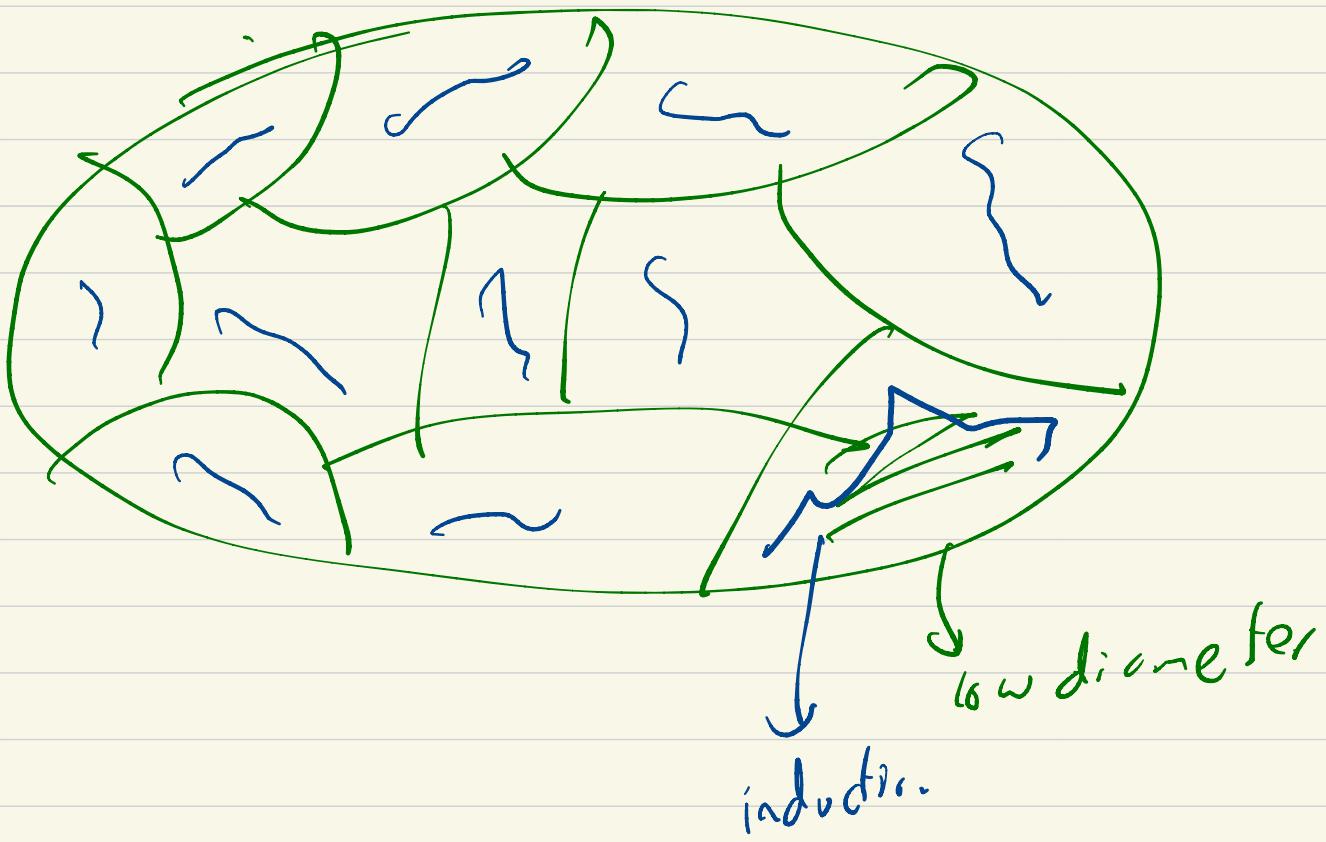


Hedge $e \in E \setminus T$
 $(u, v) \in E \setminus T$

Jacycle $C_{uv} =$ unique cycle
 in $T \cup \{uv\}$

$$|C_{uv}| = \text{dint}_T(u, v)$$





$$\min_{\|x\|=1} \left(\frac{\|Lx\|}{\|x\|} \right)$$

\leftarrow smallest eigenvalue

