A Numerical Approach to the Analyses of Composite Cross Sectioned Euler-Bernoulli Beams

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1 Introduction

Analysis of the strength of structural elements play an important role in the design process of buildings in the way that the optimization of these sections can decrease the costs drastically. Furthermore, the implementation of efficient algorithms allows the utilization more realistic models. Analysis of the composite cross sections is a subset of this problem, and a hard one indeed, since the materials involved often exhibit nonlinear behavior. Then the engineer is often left no choice but to make assumptions. This is a bad trade-off, since the efforts to stay on the safest zone often lead to generalizations that would increase the overall cost. It is also redundant, considering the existing computational power.
The purpose of this document is to show that devising and implementing an algorithm that can take into account:

- arbitrary number of materials,
- arbitrary constitutive behavior,
- and arbitrary geometry

is possible.

2 Theory

2.1 Allowable Strain

In the Euler-Bernoulli theory of the beams, the flexural strain is a linear function of the distance with respect to the axis perpendicular to the direction of the moment and coincident with the section itself, y, as seen in Figure 1. In a two dimensional section, this property manifests itself in the form of a two dimensional strain field. The corresponding two dimensional stress field is collinear with this representative strain.
While analyzing a section, the engineer often chooses to define an allowable interval for the strain to prevent the occurrence of undesirable events such as yielding and failure. This is done by defining

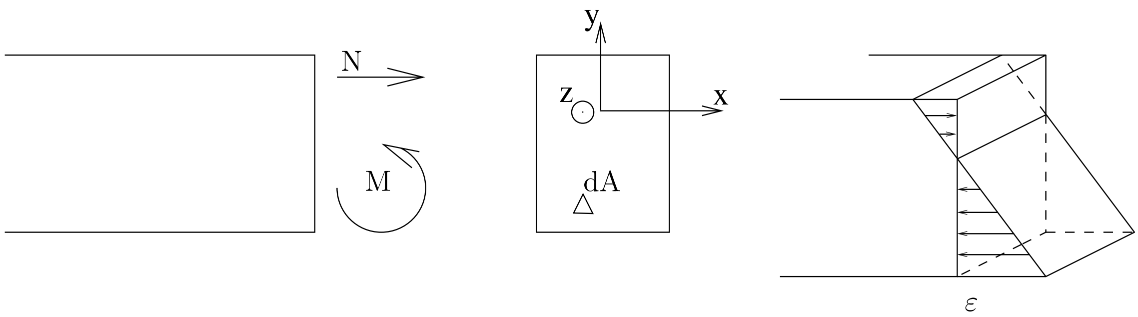

Figure 1: As a result of the applied force F and moment M, a strain field ε is formed through the section. The coordinate system (x, y, z) and the differential area dA is shown on the figure. Note that the assumption of plane sections remaining plane still holds.

ultimate strains for tension and compression, corresponding to the lower and upper limits. Hence for a section with a single material, there exists a bounding box in the 2D space $< y, \varepsilon(y) >$ defined by four variables: 2 extrema for y and 2 extrema for ε. These extrema can be interpreted as the corners of the bounding box. For this simple section, the ultimate strain distributions can be defined as a pair of lines passing through each corner of the bounding box. For each corner, the slope space is divided into allowable and prohibited intervals, defined by the three lines passing through that point: $((P_1 \to P_2), (P_1 \to P_3), (P_1 \to P_4))$. This leads to the definition of the allowable strain distribution D through a point P and two compound intervals A and U representing the allowable and prohibited intervals for the slope.

$$\begin{aligned} D &:= < P, A, U > \\ P &:= (y, \varepsilon) \\ A &:= \{[\kappa_{1i}, \kappa_{1j}], [\kappa_{2i}, \kappa_{2j}], \ldots, [\kappa_{ni}, \kappa_{nj}]\} \\ U &:= \{[\kappa_{1i}, \kappa_{1j}], [\kappa_{2i}, \kappa_{2j}], \ldots, [\kappa_{ni}, \kappa_{nj}]\} \end{aligned} \tag{1}$$

(Note that in the small deformation theory, this slope represents the curvature of the beam in that section.) Therefore there are four D for each corner of the bounding box. κ is the slope of the line passing through two points, and for two points P and Q, it is calculated as:

$$\kappa = ||P - Q|| \tag{2}$$

2.1.1 Multiple Materials

The situation gets a little bit more complicated when there are more than one materials to account for. First of all, the number of D is now equal to the number of materials $\times 4$. Furthermore, the intervals for the slopes may or may intersect, rendering the derivation of a single analytic formula impossible. Hence, Algorithm 1 is proposed for finding the allowable compound interval for every D out of a material B. All B can be grouped into S, the ultimate superset of materials. Note that the case reflected on this paper is a specific case of a bounding rectangle, however the algorithm can be modified to become valid for an arbitrary polygon.

$$\begin{aligned} B &= \{D_1, D_2, \ldots, D_n\} \\ S &= \{B_1, B_2, \ldots, B_n\} \end{aligned} \tag{3}$$

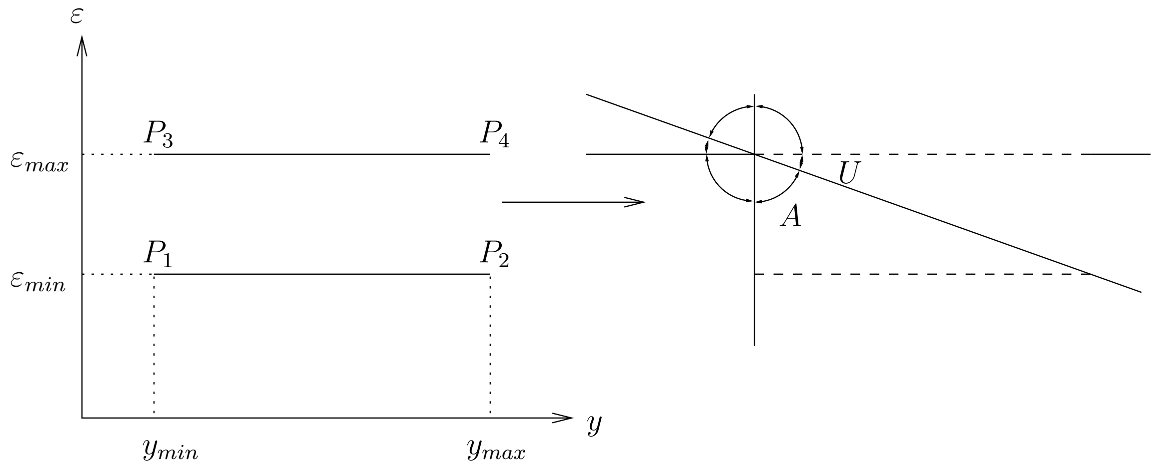

Figure 2: For a point P in a bounding box there is always an allowable interval A and a prohibited interval U of slopes. Additionally, the slope has to be either negative or positive, e.g. the slope always have to be negative for the point P_3 in order not to have any strain surpassing the ultimate values through the section.

2.2 Obtaining the external force F and moment M from the derived strain distribution

In order to support equilibrium, the integrated stresses through the cross section must be equal to the external force N applied to the beam. Hence:

$$\int_A \sigma \, \mathrm{d}A - N = 0 \tag{4}$$

The numerical approach in the title implies an approximation to the real solution. Hence the 2D domain can be discretized into elements representing the differential area $\mathrm{d}A$. Then the strains can be calculated at the centroids of these elements. The corresponding stresses at the centroids can be obtained by using the constitutive equations. Then the stresses can be integrated over the domain to obtain F:

$$\sum_i \sigma(\varepsilon_i) \; \Delta A_i - F = 0 \tag{5}$$

Obtaining the moment M requires a similar approach. Moment exerted by a differential area $\mathrm{d}A$ is multiplied with the stress σ and moment arm a. The arm length is measured as the distance of the differential area from the neutral axis. Neutral axis is in the same direction with the moment vector and coincident with the centroid.

$$\int_A a \, \sigma \, \mathrm{d}A - M = 0 \tag{6}$$

This integral can also be expressed in a discretized form as in above:

$$\sum_i a_i \; \sigma(\varepsilon_i) \; \Delta A_i - M = 0 \tag{7}$$

Algorithm 1 Finding the allowable compound interval for slopes, A, for each point D, for each material $B := \{D_1, D_2, D_3, D_4\}$ in the superset S.

for each $B_i \in S$ **do**
    for each $D_i \in B_i$ **do**
        $D_i.A = D_i.A \ \cup$
$\{\ [\kappa(D_i.P, B_i.D_1.P),\ \kappa(D_i.P, B_i.D_3.P)] \ \cup\ [\kappa(D_i.P, B_i.D_2.P),\ \kappa(D_i.P, B_i.D_4.P)]\}$
        $D_i.U = D_i.U \ \cup$
$\{\ [\kappa(D_i.P, B_i.D_1.P),\ \kappa(D_i.P, B_i.D_2.P)] \ \cup\ [\kappa(D_i.P, B_i.D_3.P),\ \kappa(D_i.P, B_i.D_4.P)]\}$
        for each $B_j \in S$ **do**
            $D_i.A = D_i.A \ \cap$
$\{[\kappa(D_i.P, B_j.D_1.P),\ \kappa(D_i.P, B_j.D_3.P)] \ \cup\ [\kappa(D_i.P, B_j.D_2.P),\ \kappa(D_i.P, B_j.D_4.P)]\}$
            $D_i.U = D_i.U \ \cup$
$\{[\kappa(D_i.P, B_j.D_1.P),\ \kappa(D_i.P, B_j.D_2.P)] \ \cup\ [\kappa(D_i.P, B_j.D_3.P),\ \kappa(D_i.P, B_j.D_4.P)]\}$
        end for
        $D_i.A = D_i.A \setminus D_i.U$
        if $D_i = D_1$ **or** $D_i = D_4$ **then**
            $D_i.A = D_i.A \setminus \{[-\infty, 0]\}$
        else if $D_i = D_2$ **or** $D_i = D_3$ **then**
            $D_i.A = D_i.A \setminus \{[0, +\infty]\}$
        end if
    end for
end for

2.3 Dimensions of Analyses

The difficulty in the analysis of composite cross sections arises from the existence of a few variables that are functions of the geometrical properties and are also implicitly related to each other. Both the moment and the normal force are functions of the stress distribution, which is a function of the strain distribution, which is a function of the beam curvature and strain offset. The three main variables that the engineer is interested in is the curvature ϕ, the moment M, and the normal force N. Desired responses for different structural elements are acquired by decreasing the dimensions of the system, namely taking one variable constant while pivoting the others. Below are some examples.

2.3.1 Moment – Curvature Behavior

The moment–curvature response of a beam is defined as the section response where the normal force is constant, hence the moment–curvature behavior.

There are two dimensions to the distribution of strain (as seen in Figure 3: curvature and the strain offset. The strain offset s is defined as the distance of the point that intersects the allowable strain interval from the strain limits.

$$F(\phi, s) = constant \tag{8}$$

The goal is to iterate through a range of strain offsets, while obtaining the solution of the system, the curvature. A root finding algorithm such as the bisection method can be used to find the different roots for varying strain offsets. The algorithm does not run on linear time, and takes longer than obtaining the interaction diagram.

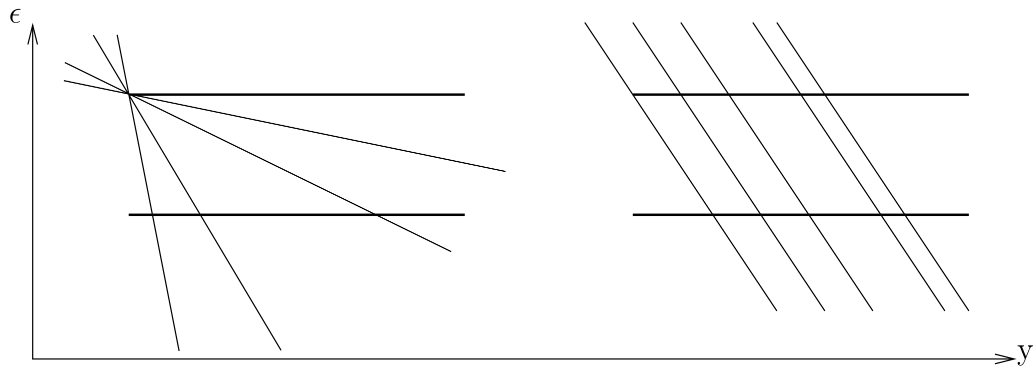

Figure 3: Different dimensions of the strain distribution. Curvature on the left, strain offset on the right.

2.3.2 Columns – Interaction Diagrams

For column behavior, ultimate cases for the section are required, hence the strain offsets are constant, and the strain distribution always intersects one of the limits, as seen in Figure 2. Moment and force values can be calculated for different ranges of curvatures in linear time, rendering it a fast and efficient algorithm for obtaining column behavior.

3 Examples

3.1 CE 481 Project

You will write a moment curvature analysis program for T-beams. The program should adopt the layer by layer approach and find neutral axis depth to establish force equilibrium for each imposed top fiber concrete strain as we discussed in class. The program should allow the user to input geometry of the section, number of concrete layers, number of steel layers. You can use Hognestad parabola as your concrete model and trilinear steel model (Parameters of these models should be inputs of the program).

1. *Outline the input and output data structure of your program.*
2. *Verify your program by comparing the response of the beam that we solved in class and homework problem with your program's results. Also check the ultimate capacity by hand calculations and compare. Comment on the number of layers required for your analysis.*

For the beam section shown below, conduct the following parametric studies.

Concrete Strength, f_c [MPa]	Tension Reinforcement [%]	Compression Reinforcement [%]	Strain at onset of strain hardening
30	0.5	0.25	0.005
40	1.	0.5	0.0075
50	1.5	0.75	0.01

Implementation

We implemented a comprehensive library capable of discretizing the domain and outputting section responses by utilizing single functions. It is possible to implement any time independent constitutive

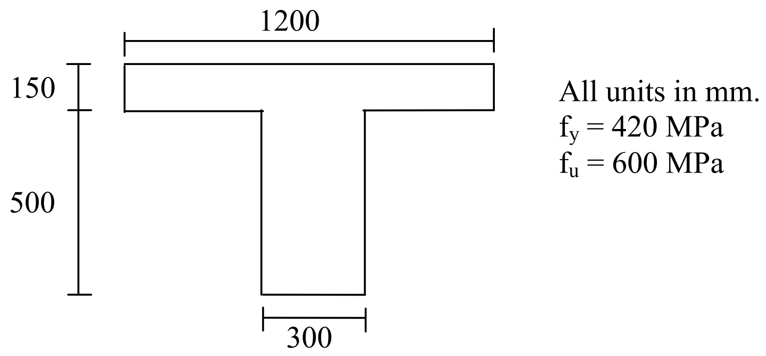

Figure 4: Section geometry.

behavior by building from a base material class.

Shewchuk's `Triangle` was incorporated into the program for discretization. The section is first defined as a planar straight line graph (PSLG), transferred into `Triangle` with special data structures, processed, and then retrieved from `Triangle`'s output data structures.

We also added a simple parser to input parameters. The parser accepts single line commands with multiple arguments in order to set the program parameters.

The program can accept input files. Below are the files for the three cases of this example. Some commands are:

- `rein_layer width height diameter number`, where `width` is the width of the reinforcement layer, `height` is the height of the layer in the section where origin is the section bottom, `diameter` is the reinforcement diameter and `number` is the number of reinforcements in that layer.
- Rest of the commands only have single arguments and speak for themselves. The units are:

```
rein_layer       width [mm] height [mm] diamater [mm] number -

flange_width     [mm]
beam_width       [mm]
beam_height      [mm]
flange_thickness [mm]

concrete_str     [MPa] concrete maximum strength
concrete_eps_0    -    concrete epsilon_0
concrete_eps_u    -    concrete ultimate strain

steel_sig_y      [MPa] yield strength
steel_young      [MPa] young's modulus
steel_sig_ult    [MPa] ultimate strength
steel_eps_ult    -     ultimate strain
steel_eps_sh     -     strain at onset of
                       strain hardening
```

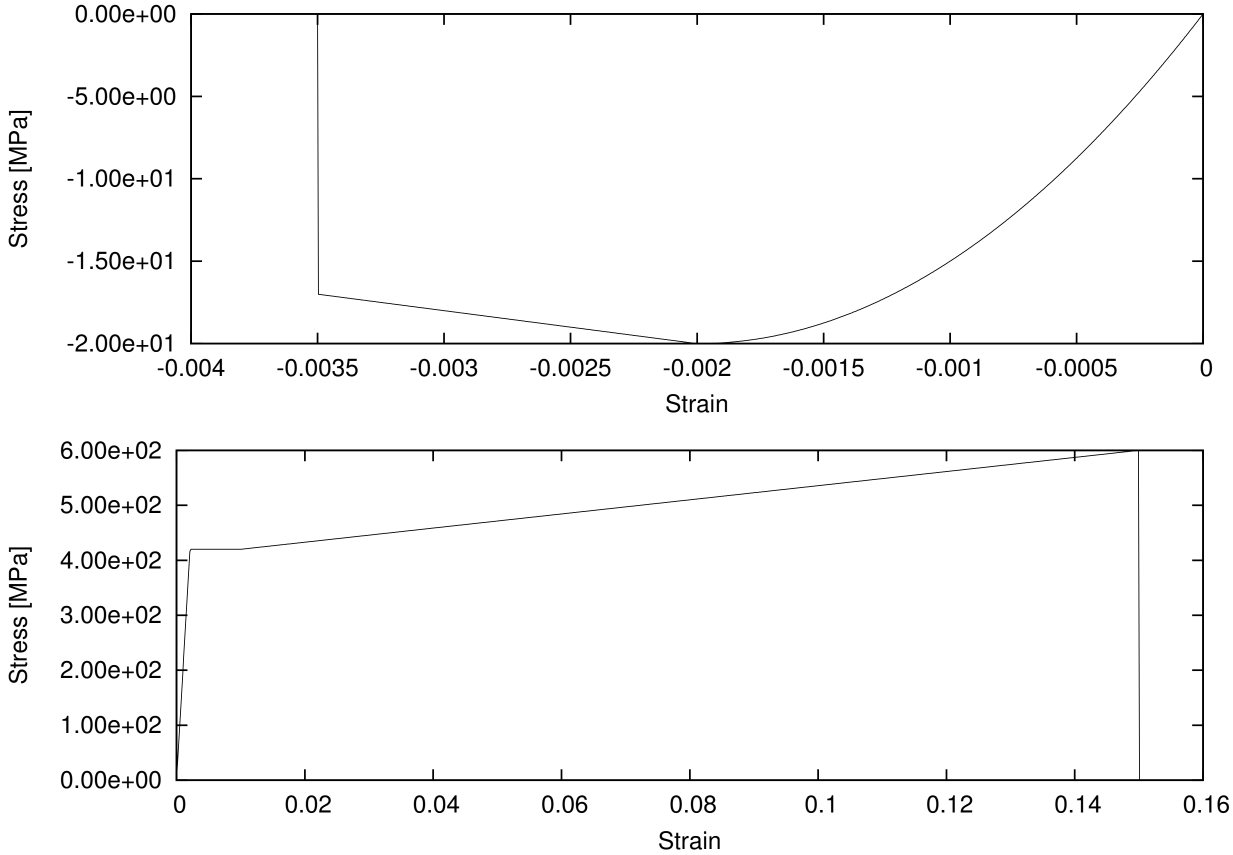

Figure 5: Constitutive behaviors for the materials. On the top, the Hognestad model for concrete (inversed due to sign convention) and on the bottom, the trilinear model for the steel.

```
file i1.in:
rein_layer 180 70 18 4
rein_layer 180 120 18 3
rein_layer 900 570 12 7

flange_width  1200
beam_width 300
beam_height 650
flange_thickness 150

concrete_str 30
concrete_eps_0 0.002
concrete_eps_u 0.0035

steel_sig_y 420
steel_young 2e5
steel_sig_ult 600
steel_eps_ult 0.15
steel_eps_sh 0.005
```

```
file i2.in:
rein_layer 180 70 26 3
rein_layer 180 120 26 3
rein_layer 900 570 16 8

flange_width  1200
beam_width 300
beam_height 650
flange_thickness 150

concrete_str 40
concrete_eps_0 0.002
concrete_eps_u 0.0035

steel_sig_y 420
steel_young 2e5
steel_sig_ult 600
steel_eps_ult 0.15
steel_eps_sh 0.0075
```

```
file i3.in:
rein_layer 180 70 26 5
rein_layer 180 120 26 4
rein_layer 900 570 22 7

flange_width  1200
beam_width 300
beam_height 650
flange_thickness 150

concrete_str 50
concrete_eps_0 0.002
concrete_eps_u 0.0035

steel_sig_y 420
steel_young 2e5
steel_sig_ult 600
steel_eps_ult 0.15
steel_eps_sh 0.01
```

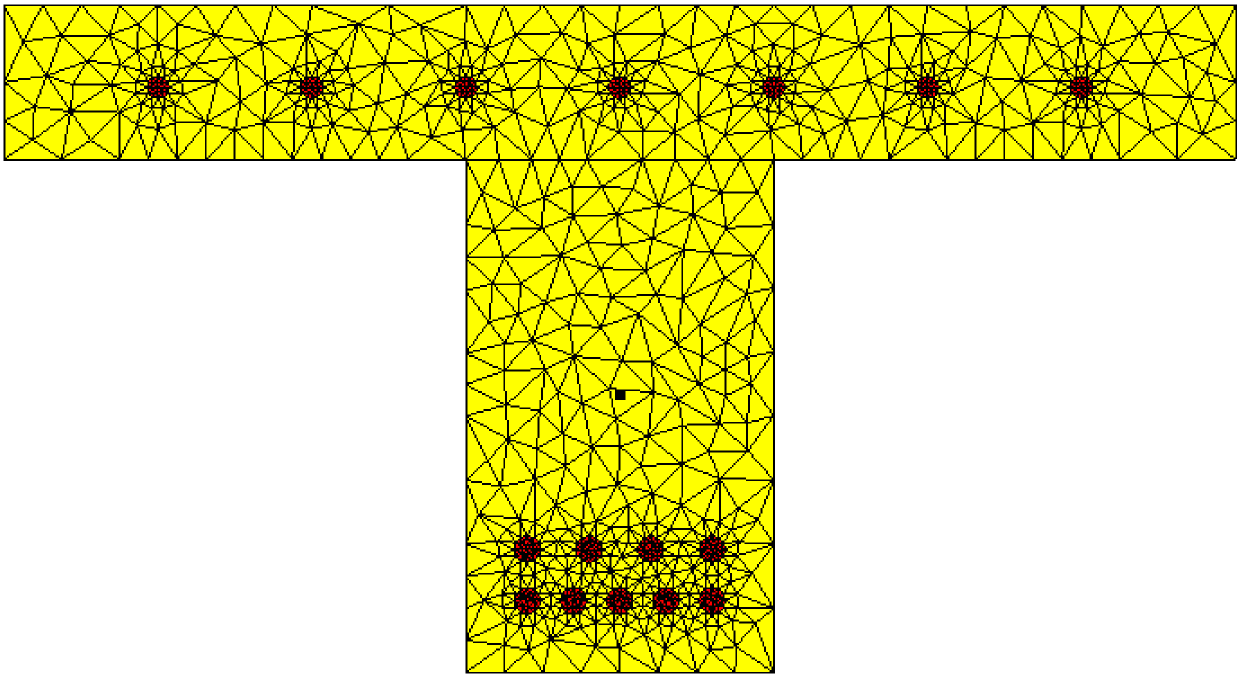

Figure 6: Geometry of the section for the second case.

Results

The resulting moment–curvature diagrams can be observed in Figure 7. We did not use a slice (regular grid) approach, hence we cannot comment on the number of layers. But here are some of the discretization statistics: An average of 4900 triangular elements and 1700 nodes for concrete, 300 elements and 200 nodes for steel. The elements of the nodes are more than enough for a crude sectional analysis. We have also observed that the fineness is critical only for post-failure roots of

Equation 8. Otherwise for the pre-failure roots, the number of elements have little effect after a threshold number of elements.

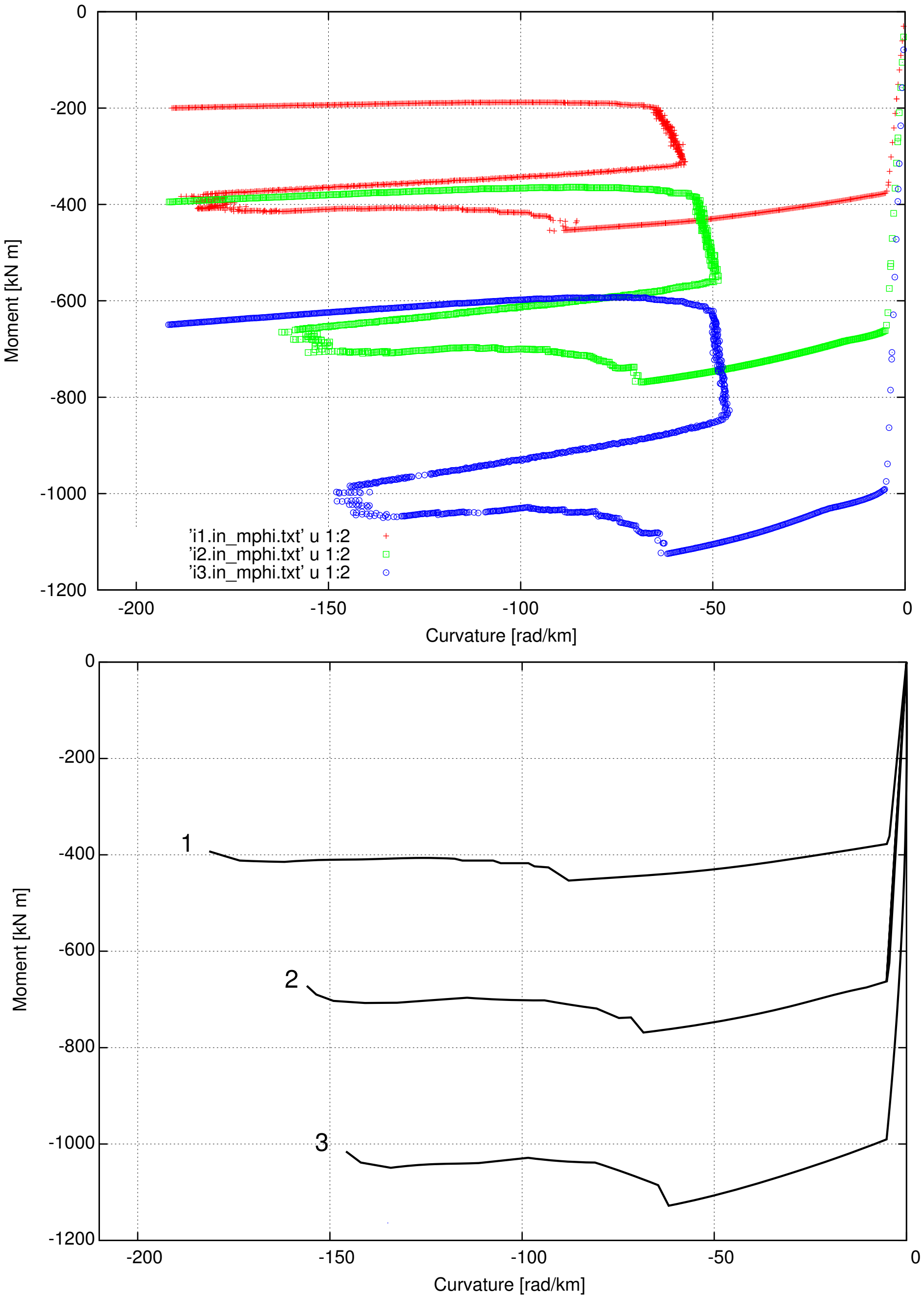

Figure 7: Results of the moment curvature analysis for three cases. The figure on the top shows the overlapped data for all cases. The fact that it is a little encumbered is due to the byproduct roots. The process is comprehensive in the way that every possible root (curvature) that provides equilibrium is accounted for. Hence the moment-curvature behavior of the beam continues after the point of failure. The desired response, shown in the bottom graph, be obtained by interpreting the points.

3.2 Comparison with previous cases

In order to validate the results, we have decided to compare previous analytical results (homework 2) with the output of this program. The parameters are as follows:

```
rein_layer 180 50 28 3
rein_layer 180 113 28 2

flange_width  1800
beam_width 300
beam_height 600
flange_thickness 150

concrete_str 25
concrete_eps_0 0.002
concrete_eps_u 0.0035

steel_sig_y 420
steel_young 2e5
steel_sig_ult 600
steel_eps_ult 0.05
steel_eps_sh 0.008
```

	Cracking	Yielding	Ultimate
Curvature ϕ [rad/km]	0.31	6.1	77.8
Moment M [kN m]	103.6	623.7	808.3

Table 1: Results of the analytical solution

It can be observed from Figure 8 that the output from the program is quite close to the analytical results.

- Retrieve the companion source code from: `http://github.com/nrs/kesitfalan`
- Direct questions and issues at `onursolmaz@gmail.com`

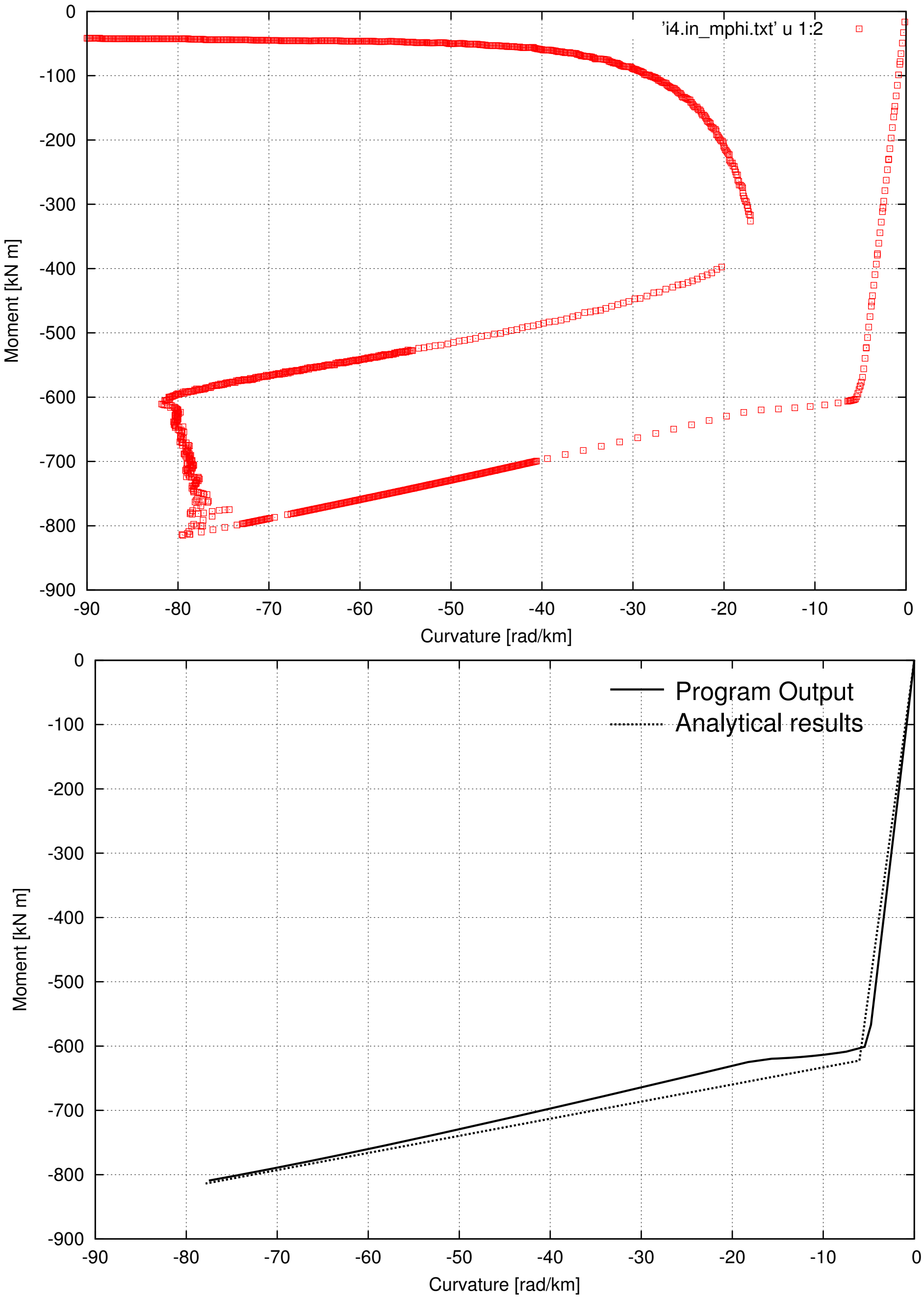

Figure 8: Response of the parameters of homework 2