Homework #3 **MEAM 5460 - Spring 2023**
Assigned: 03/16/2023, Due: 03/22/2023

There is a new VTOL related program in the company and your technical lead engineer wants you to contribute at the Preliminary Design (PD) stage of the program. To get you started she has assigned you the task of developing a tool to predict the variation of thrust and power of a rotor in hover. Specifically, she wants you to develop the tool to be able to showcase total thrust and power of the rotor as well as their respective spanwise distributions. Since you gunning for that promotion, you take it upon yourself to develop a Blade Element Momentum Theory (BEMT) based tool and plan to test by analyzing a 30 ft radius rotor with an ideal twist distribution and constant chord (2 ft) operating at a tip speed of 650 ft/sec (MSL density = 0.002378 sl/ft^3) using a linear thin airfoil assumption (lift curve slope = 2*pi). To keep things simple, you are not including any tip loss corrections (for now) and are considering the variation as a function of input collective (at 75% radial location). As a cross-check you are comparing your results against momentum theory results (rotor "disk" model). Your technical lead engineer prefers data in non-dimensional form (it helps her understand trends better).

hw3_report

March 23, 2023

0.1 MEAM 5460 HW3

0.1.1 Author: Nathaniel Ruhl

0.1.2 GitHub: https://github.com/nruhl25/HoveringVehicleDesign/tree/main/hw3

0.1.3 1) Introduction

In order to predict the performance of a rotor, it is necessary to calculate the inflow induced by the rotor, $\lambda = v_i/\Omega R$ where v_i is the induced velocity at the rotor plane, Ω is angular velocity of the rotor blades, and R is the rotor radius. Blade element momentum theory (BEMT) is a powerful tool to predict the inflow and the performance measures of a rotor in hover, such as the coefficients of thrust and power, C_T and C_P. Moreover, BEMT is versatile enough to account for variations in blade chord, twist, and taper, and it can also make use of a tip loss model, thus providing an extremely useful and realistic tool to compare and analyze different rotor designs.

In this report, I present a tool that I have made using BEMT to calculate the induced inflow ratio λ, distributions of local coefficients of thrust and power c_T and c_P, as well as their total counterparts C_T and C_P. I report and plot both the *induced* and *total* power coefficients, C_P and $C_{P,tot}$, although I only plot the *induced* power coefficient c_P on the blade section level (I do not explicitly use the i subscript in order to be consistent with the software). Since we are using linear airfoil theory, and therefore the linear regime, we can approximate the local airfoil coefficient of drag as $c_{d,0} = 0.01$ and therefore calculate the total profile power, which I will demonstrate in Section 4. The script "hw3_BEMT.py" contains the functions to perform BEMT analysis and the code in this notebook demonstrates how to use the tool. In this version of the BEMT tool, I make a couple simplifying assumptions which are listed below.

1. The rotor is in hover: $\lambda_c = 0$
2. The rotor has an ideal twist distibution specified by the input collective at 75%R: $\theta(r) = 0.75\theta_{75}/r$
3. Constant blade chord (no taper variation)
4. Linear thin airfoil assumption: $c_{l,\alpha} = 2\pi$ lift-curve slope. Constant section coefficient of drag: $c_{d,0} = 0.01$.
5. Tip losses are ignored

In the next Project 1 assignment, I will break assumptions 2-5.

0.1.4 2) The "Rotor" class and baseline test rotor specifications

The analysis below is performed on a rotor blade with the folowing specifications (R and c are similar to a rotor of the chinook):

- $N_b = 3$ # number of blades
- $c = 2.0$ # Chord length, ft
- $v_{tip} = 650$ # ft/sec
- $R = 30$ # ft
- $\sigma = N_b c/\pi R = 0.0636$ # solidity
- $c_{l,\alpha} = 2\pi$ # Lift-curve slope
- $c_{d,0} = 0.01$ # Section drag coefficient

The variables listed above are the default instance variables for a 'Rotor' object. After a rotor object is defined via `rotor = Rotor()`, you can change the properties via `rotor.c=3`, etc… Note that we wish to analyze our rotor at the MSL density of 0.002378 slug/ft^3, but this number does not come into play when strictly using the non-dimensional coefficients.

0.1.5 3) BEMT Equations

BEMT predicts the local coefficiecients of thrust and induced power to be

$$c_T(r) = 4\lambda^2(r)r$$
$$c_P(r) = 4\lambda^3(r)r$$

where

$$\lambda(r) = \frac{\sigma c_{l,\alpha}}{16}\left(\sqrt{1 + \frac{32}{\sigma c_{l,\alpha}}\theta(r)r} - 1\right)$$

and the rotor solidity is $\sigma = \frac{N_b c}{\pi R}$.

The local coefficients are also known as thrust and power "gradients": $c_T = dC_T/dr$ and $c_P = dC_P/dr$.

As stated previously, in this report, we make the simplifying assumptions that

$$\theta(r) = 0.75\theta_{75}/r, \quad c_{l,\alpha} = 2\pi.$$

In this case, BEMT predicts that λ is independent of r:

$$\lambda = \frac{\sigma c_{l,\alpha}}{16}\left(\sqrt{1 + \frac{32}{\sigma c_{l,\alpha}}0.75\theta_{75}} - 1\right)$$

Therefore, the local and total coefficients of thrust and **induced** power can be written as:

$$c_T(r) = 4\lambda^2 r$$
$$c_P(r) = 4\lambda^3 r$$
$$C_T = 4\lambda^2 \int_0^1 r dr = 2\lambda^2$$
$$C_P = 4\lambda^3 \int_0^1 r dr = 2\lambda^3.$$

Using the notation of this report, the total power coefficient is

$$C_{P,tot} = C_P + C_{P,0}$$

where, in the linear regime, the total coefficient of profile power can be written as

$$C_{P,0} = \sigma \int_0^1 c_d r^3 dr = \frac{1}{8}\sigma c_{d,0}.$$

Lastly, we can write the local angle of attack of an airfoil as

$$\alpha(r) = \theta(r) - \phi(r)$$

where $\phi(r)$ is the induced inflow angle. In hover, the inflow angle can be written as:

$$\phi(r) = \frac{\lambda}{r}$$

.

For completeness, recall the equation for the local airfoil coefficient of lift under linear aerodynamics:

$$c_l(r) = c_{l,\alpha}\alpha(r)$$

3a) Comparison to Momentum Theory In order to validate our results, we will compare the BEMT to blade-less rotor-disk momentum theory. In this theory, the induced inflow in hower is equal to $\lambda_h = \sqrt{C_T/2}$, where we will use the value of C_T from BEMT in order to evaluate λ_h.

0.1.6 4) Analysis

This analysis makes use of the modules, classes, and functions imported in the cell block below. It should be noted that I often used upper case greek letters as function definitions, and I also use the notation $c_T(r)$ = `dCT(r)` and $c_P(r)$ = `dCP(r)` in the code. The functions also take inputs of a "Rotor" object, and you can write `Theta?` in a code cell to see information about the $\theta(r)$ function.

Lastly, the two codebloacks below are all that is needed to use this tool. The "rotor" object can be changed dynamically as the user desires (although it is always good to confirm that you are correctly changing the rotor object in the Kernel memory, and re-run the `importlib.reload()` lines if changes are made to the scripts).

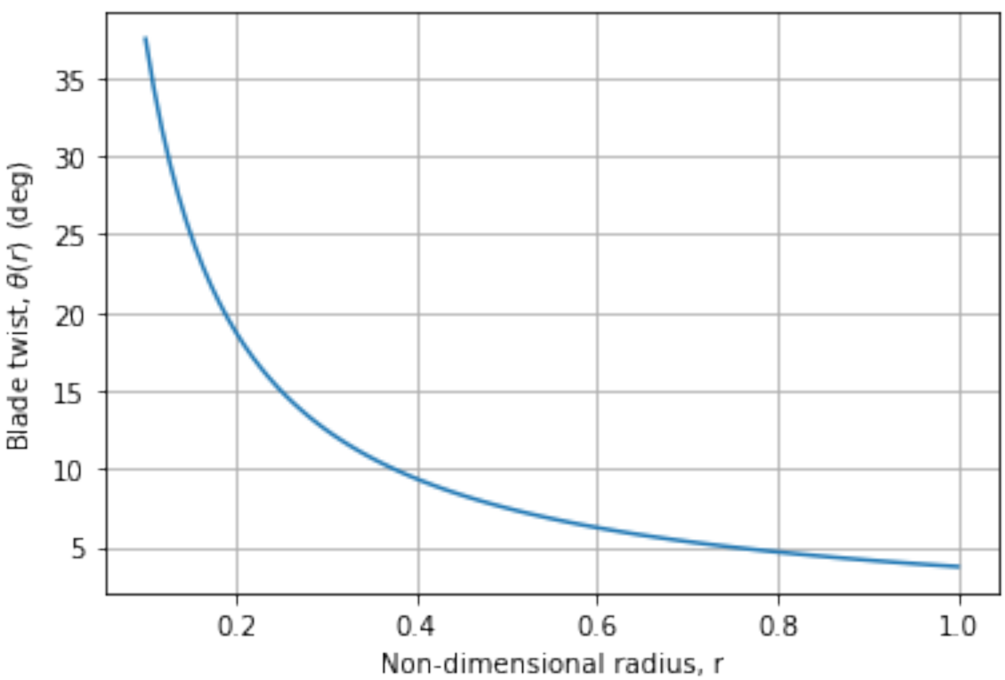
[29]:
```
# import standard libraries
import importlib
import numpy as np
import matplotlib.pyplot as plt

# import local modules (these scripts must be in the working directory)
import Rotor
importlib.reload(Rotor)
from Rotor import Rotor  # Rotor class definition
```

```
import hw3_BEMT   # BEMT tool
importlib.reload(hw3_BEMT)
from hw3_BEMT import Theta, Lambda, Alpha, Phi, dCT, dCP, calc_CT_CP, CP0
```

[30]:
```
rotor = Rotor()
# Here is where you can change rotor properties with rotor.c=3, rotor.Nb=4, etc.
 ↪..
```

The plot below shows the ideal twist distribution of a rotor with $\theta_{75} = 5°$

[31]:
```
rs = np.linspace(0.1, 1, 100)  # Non-dimensional blade radius
theta_75 = np.deg2rad(5)       # Blade twist at 75%R

plt.plot(rs, np.rad2deg(Theta(rs, theta_75)))
plt.ylabel(r"Blade twist, $\theta(r)$ (deg)")
plt.xlabel("Non-dimensional radius, r")
plt.grid()
```



Performance Profiles Let's first consider the profiles of λ, c_T, and c_P as a function of non-dimensional radius. The plots below the expected relationships when the blade has ideal twist: uniform inflow λ, as well as linearly increasing $c_T(r)$ and $c_P(r)$. For the profiles in the first group of plots below, I will consider $\theta_{75} = 5°$. The plot for λ below shows the comparison to momentum theory in the legend, where $\lambda_h = \sqrt{C_T/2}$, and thus the C_T calculated by BEMT is consistent with momentum theory. Furthermore, since the values of C_T and C_P predicted by momentum theory are explicitly related to λ_h, we see the same constitentcy for C_T and C_P between simple momentum theory and BEMT. The legend of the thrust and power profiles shows the total C_T and C_P (area under the c_T and c_P curves).

[32]:
```python
# Code for plotting lmbda, c_T, and c_P profiles

theta_75 = np.deg2rad(5)

CT, CP = calc_CT_CP(theta_75, rotor)
# I originally wrote the above function to do numerical integration, but␣
↪replaced it with the closed form solution

plt.figure(1)
plt.title("Fig 1a: Inflow Distribution across rotor blade")
plt.plot(rs, Lambda(rs, theta_75, rotor),label=f"$\lambda={Lambda(1, theta_75,␣
↪rotor):.6f}$")
plt.plot([],[],label=rf"Momentum theory: $\lambda_h$={np.sqrt(CT/2):.6f}")
plt.grid()
plt.xlabel("Non-dimensional radial position, r")
plt.ylabel(r"Local Inflow Ratio, $\lambda$")
plt.legend()

plt.figure(2)
plt.title("Fig 2a: Thrust Distribution")
plt.plot(rs, dCT(rs, theta_75, rotor), label=fr'$C_T$={CT:.5f}')
plt.grid()
plt.xlabel("Non-dimensional radial position, r")
plt.ylabel("Local Thrust Coefficient, $c_T$")
plt.legend()

plt.figure(3)
plt.title("Fig 3a: Induced Power Distribution")
plt.plot(rs, dCP(rs, theta_75, rotor), label=fr'$C_P$={CP:.5f}')
plt.grid()
plt.xlabel("Non-dimensional radial position, r")
plt.ylabel("Local Coefficient of Induced Power, $c_P$")
plt.legend()

plt.figure(4)
plt.title("Fig 4a: Local angle of attack distribution")
plt.plot(rs, np.rad2deg(Alpha(rs, theta_75, rotor)))
plt.grid()
plt.xlabel("Non-dimensional radial position, r")
plt.ylabel(r"Local angle of attack, $\alpha$ (deg)")
plt.ylim([0,15])
```

[32]:
```
(0.0, 15.0)
```

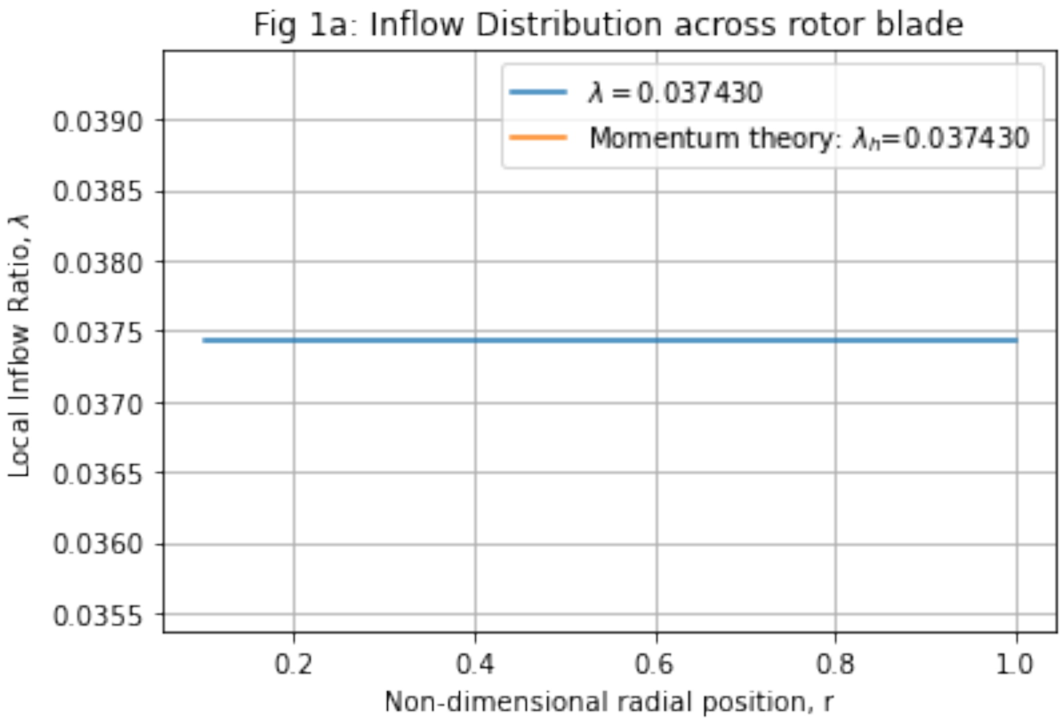


Fig 1a: Inflow Distribution across rotor blade
$\lambda = 0.037430$
Momentum theory: λ_h=0.037430
Local Inflow Ratio, λ
0.0390
0.0385
0.0380
0.0375
0.0370
0.0365
0.0360
0.0355
0.2
0.4
0.6
0.8
1.0
Non-dimensional radial position, r

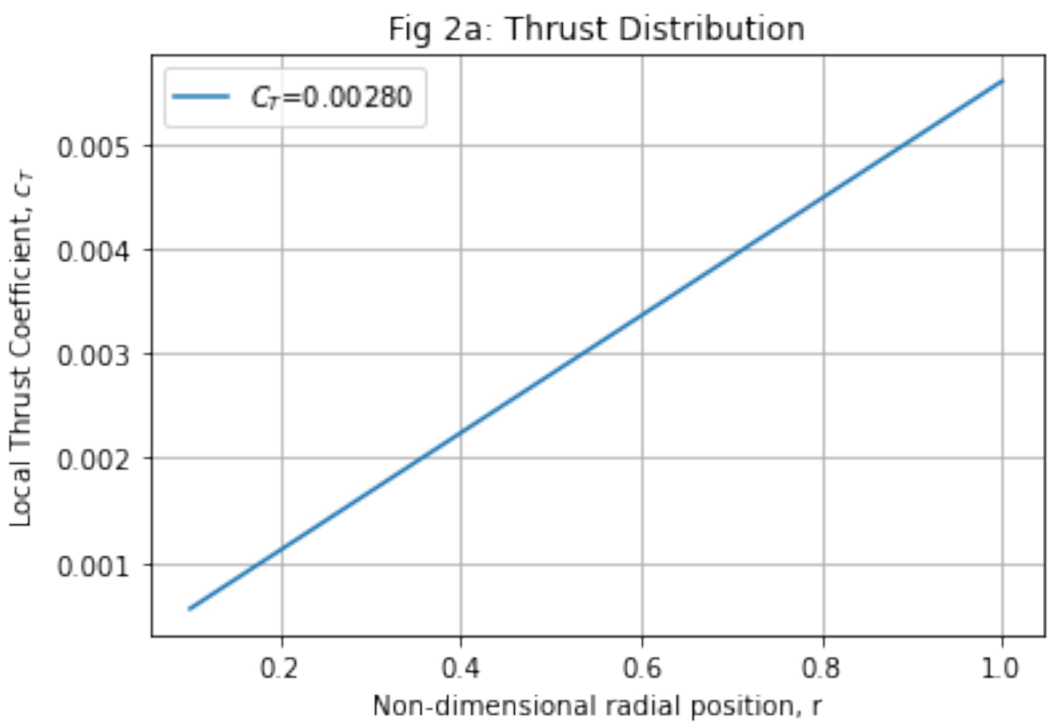


Fig 2a: Thrust Distribution
C_T=0.00280
Local Thrust Coefficient, c_T
0.005
0.004
0.003
0.002
0.001
0.2
0.4
0.6
0.8
1.0
Non-dimensional radial position, r

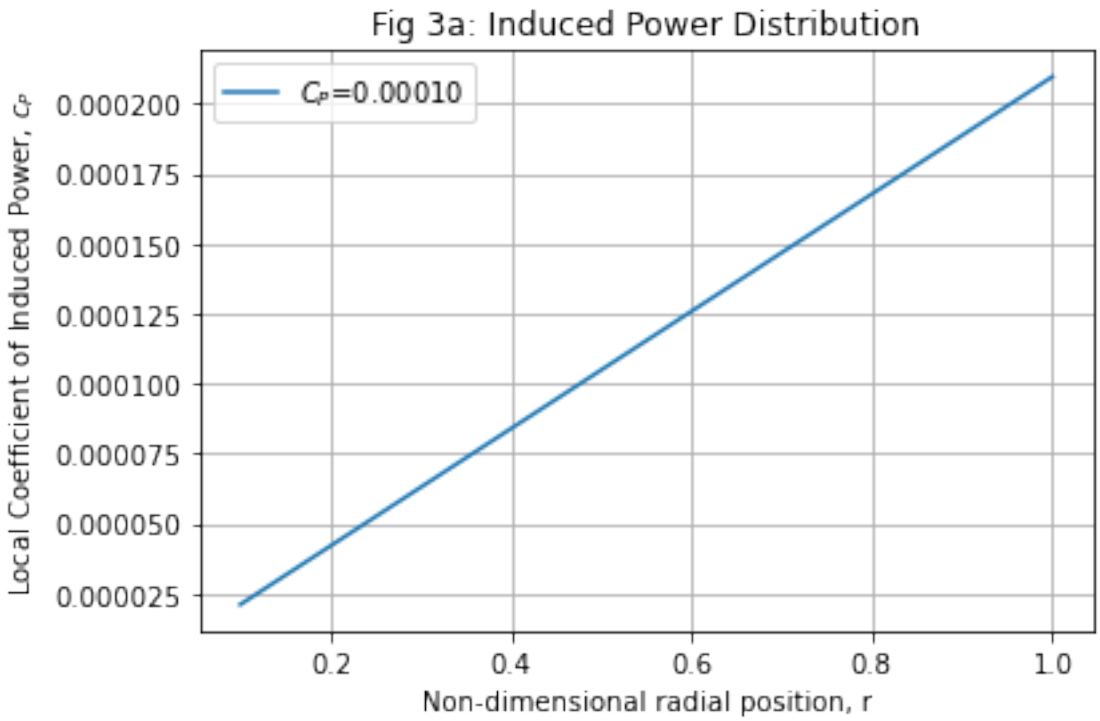


Fig 3a: Induced Power Distribution
C_P=0.00010
Local Coefficient of Induced Power, c_P
0.000200
0.000175
0.000150
0.000125
0.000100
0.000075
0.000050
0.000025
0.2
0.4
0.6
0.8
1.0
Non-dimensional radial position, r

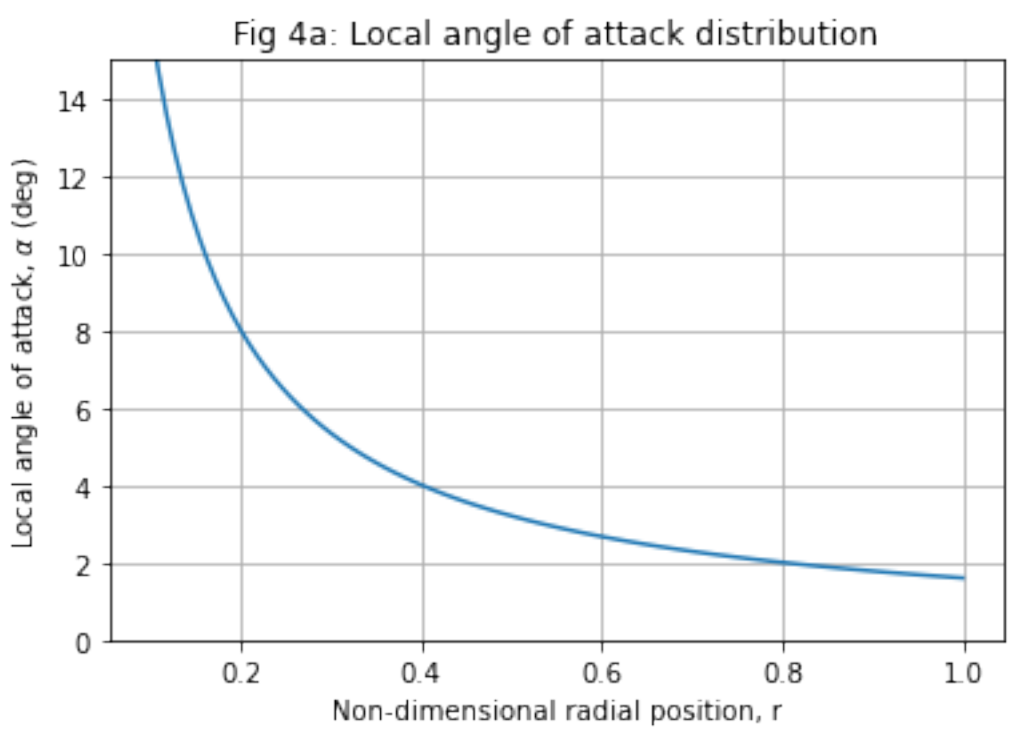


Fig 4a: Local angle of attack distribution
Local angle of attack, α (deg)
14
12
10
8
6
4
2
0
0.2
0.4
0.6
0.8
1.0
Non-dimensional radial position, r

By assuming the airfoil coefficient of drag in the linear regime to be $c_d = c_{d,0} = 0.01$, and making using of the fact the fact that there is unform inflow, we can approximate the profile power coefficient

$$C_{P,0} = \frac{1}{8}\sigma c_{d,0} \approx 0.00008.$$

This is only 20% smaller than the coefficient of induced power C_P at $\theta_{75} = 5°$, and as we will see below, it is even more significant for smaller values of θ_{75}.

I will return to this discussion more closely in the Section "Performance as a function of input collective", where I will make plots of both C_P and $C_{P,tot}$, as well as the relative contribution of the profile power to the total power

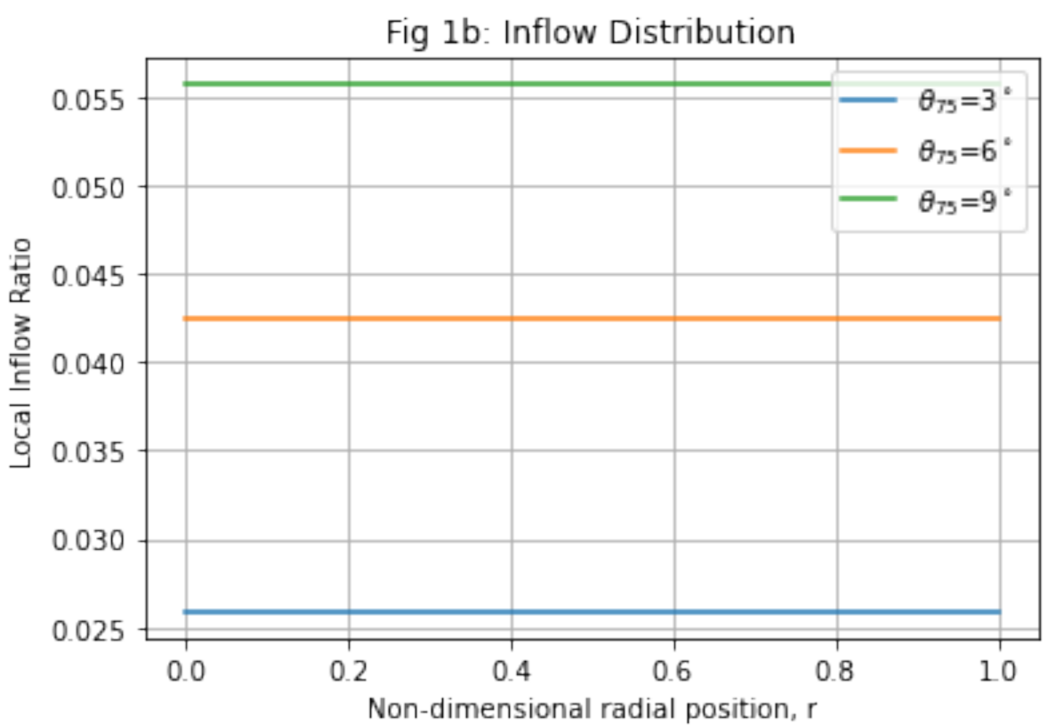
$$P_{ratio} = \frac{C_{P,0}}{C_{P,tot}}.$$

Below, I will plot the profiles at three different input collectives, θ_{75}

[18]:
```
theta_75_list_deg = np.array([3,6,9])
theta_75_list_rad = np.deg2rad(theta_75_list_deg)

for i, theta_75 in enumerate(theta_75_list_rad):

    CT, CP = calc_CT_CP(theta_75, rotor)

    rs = np.linspace(0.0001, 1, 100)  # Non-dimensional blade radius

    plt.figure(1)
    plt.plot(rs, Lambda(rs, theta_75,␣
 ↪rotor),label=fr"$\theta_{{75}}$={theta_75_list_deg[i]:.0f}$^\circ$")

    plt.figure(2)
    plt.plot(rs, dCT(rs, theta_75, rotor),␣
 ↪label=fr"$\theta_{{75}}$={theta_75_list_deg[i]:.0f}$^\circ$")

    plt.figure(3)
    plt.plot(rs, dCP(rs, theta_75, rotor),␣
 ↪label=fr"$\theta_{{75}}$={theta_75_list_deg[i]:.0f}$^\circ$")

    plt.figure(4)
    plt.plot(rs, np.rad2deg(Alpha(rs, theta_75, rotor)),␣
 ↪label=fr"$\theta_{{75}}$={theta_75_list_deg[i]:.0f}$^\circ$")

plt.figure(1)
plt.title("Fig 1b: Inflow Distribution")
plt.grid()
plt.xlabel("Non-dimensional radial position, r")
plt.ylabel("Local Inflow Ratio")
plt.legend()
```

```
plt.figure(2)
plt.title("Fig 2b: Thrust Distribution")
plt.grid()
plt.xlabel("Non-dimensional radial position, r")
plt.ylabel("Local Thrust Coefficient")
plt.legend()

plt.figure(3)
plt.title("Fig 3b: Induced Power Distribution")
plt.grid()
plt.xlabel("Non-dimensional radial position, r")
plt.ylabel("Local Induced Power Coefficient")
plt.legend()

plt.figure(4)
plt.title("Fig 4b: Local angle of attack across rotor blade")
plt.grid()
plt.xlabel("Non-dimensional radial position, r")
plt.ylabel(r"Local angle of attack, $\alpha$ (deg)")
plt.ylim([0,15])
plt.legend()
```

[18]: <matplotlib.legend.Legend at 0x7f9a90d22520>



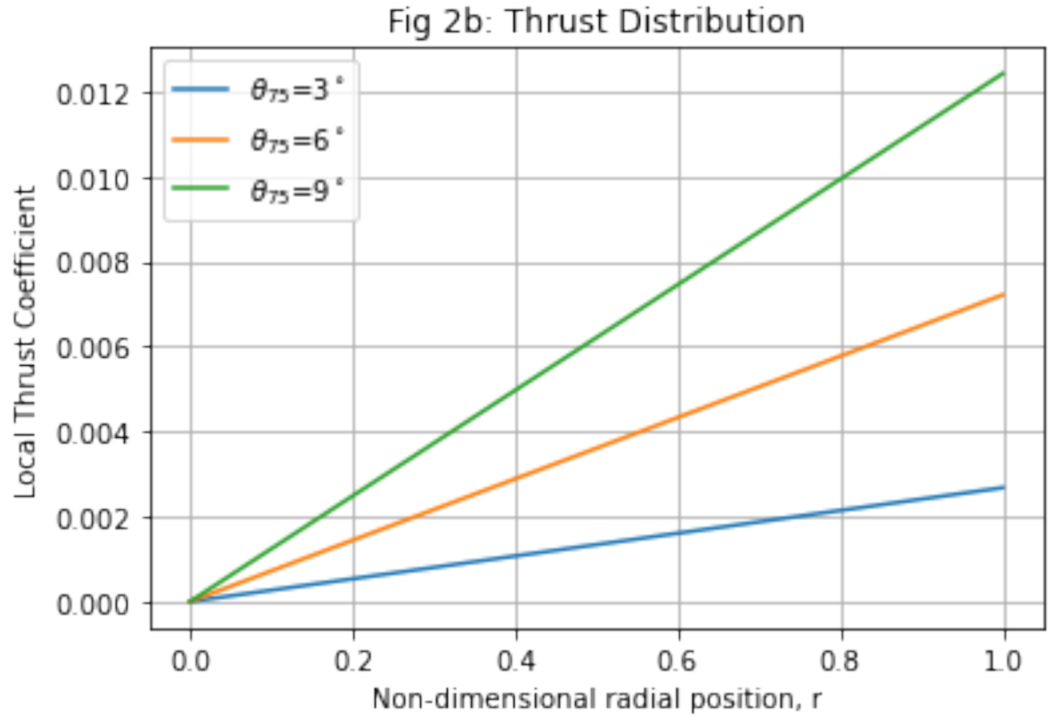


Fig 2b: Thrust Distribution
Local Thrust Coefficient
Non-dimensional radial position, r
θ_{75}=3°
θ_{75}=6°
θ_{75}=9°
0.000
0.002
0.004
0.006
0.008
0.010
0.012
0.0
0.2
0.4
0.6
0.8
1.0

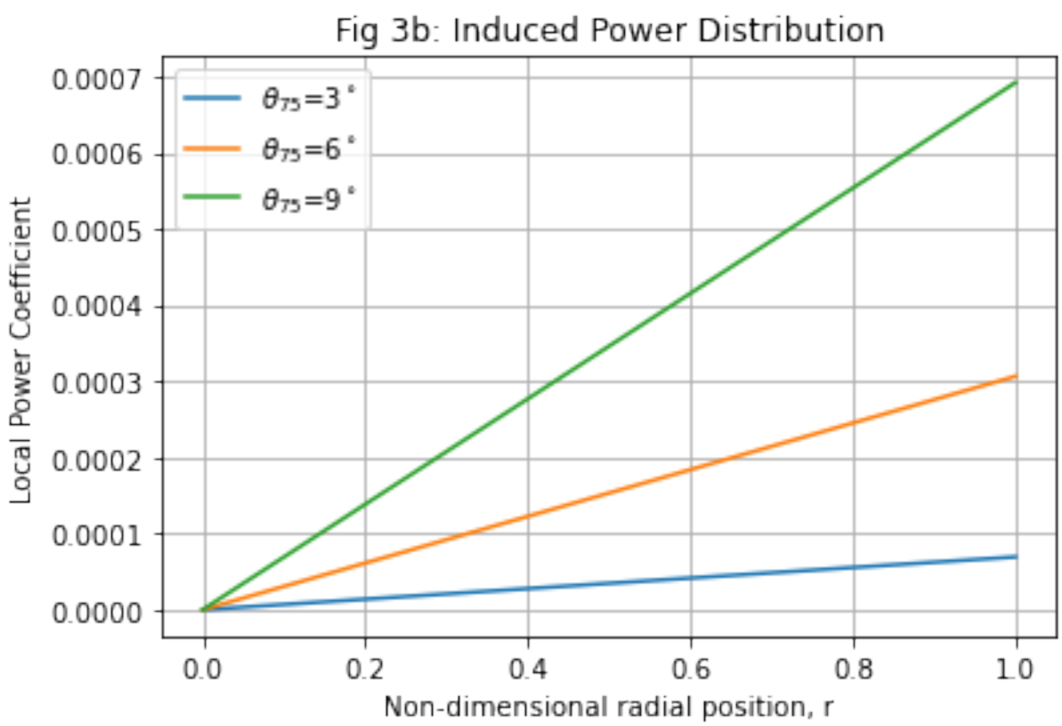
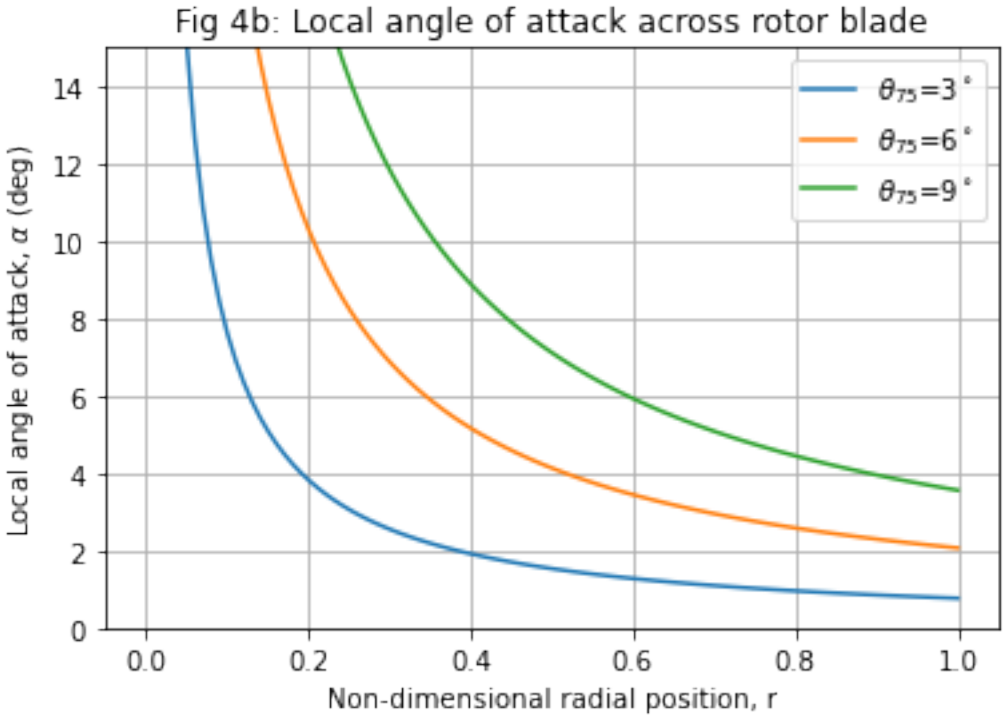


Fig 3b: Induced Power Distribution
Local Power Coefficient
Non-dimensional radial position, r
θ_{75}=3°
θ_{75}=6°
θ_{75}=9°
0.0000
0.0001
0.0002
0.0003
0.0004
0.0005
0.0006
0.0007
0.0
0.2
0.4
0.6
0.8
1.0

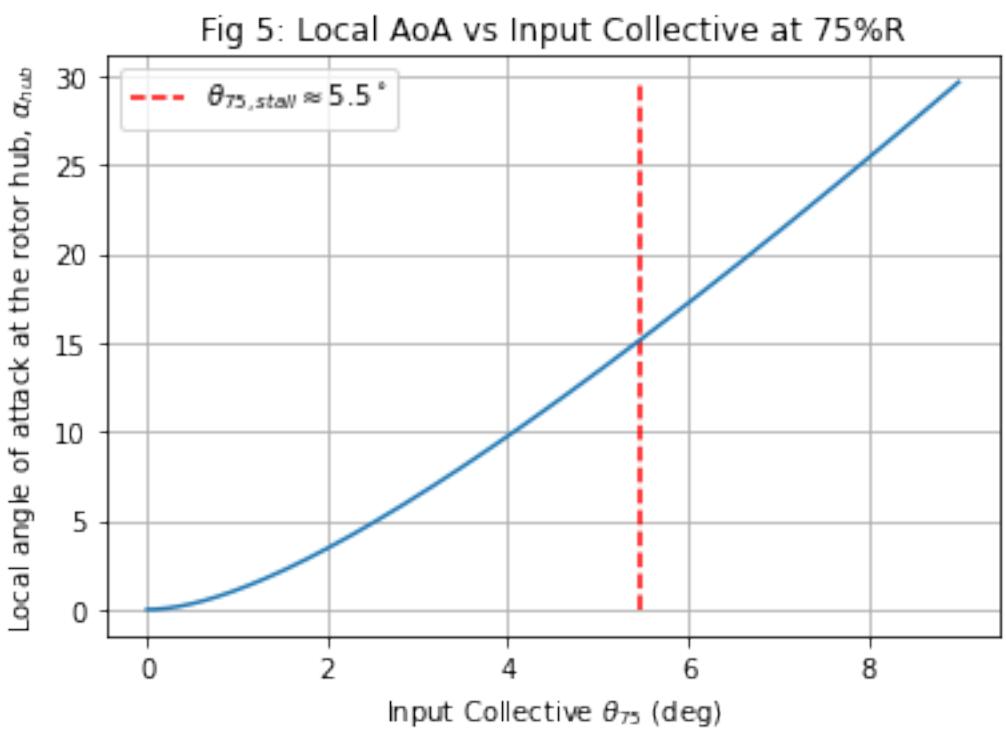


Performance as a Function of Input Collective The next step of the analysis is to calculate the total performance coefficients C_T and C_P as a function of input collective θ_{75}. In order to better understand these plots, I wanted to approximate the maximum value for input collective, which I indicate with a red dashed line in the plots below. For the ideal twist distribution (with uniform inflow), the local airfoils on the rotor blade near the hub have the largest angles of attack, and are at risk of stalling (figure 4 above) when the input collective θ_{75} is too large. For this reason, I identified the approximate θ_{75} at which the local AoA at $r_{hub} = 0.12$ is greater than $15°$. Therefore, to the right of the red dashed lines in each plots below, there is a risk of stall at the base of the rotor blades. This also helps us better determine the maximum blade performance, especially since the performance coefficients will increase monotonically with input collective, as we will see.

[38]:
```
# Code for plotting Performance as a function of input collective
r_hub = 0.12  # non-dimensional radius of the hub
N_thetas = 200
theta75_deg = np.linspace(0,9,N_thetas)
theta75_rad = np.deg2rad(theta75_deg)

CT_list = np.zeros(N_thetas)
CP_list = np.zeros(N_thetas)
lambda_list = np.zeros(N_thetas)
alpha_hub_list = np.zeros(N_thetas)
```

```
for i in range(N_thetas):
    lambda_list[i] = Lambda(1, theta75_rad[i], rotor)  # note: r=1 is arbitray␣
 ↪since lambda=const(r) for ideal twist
    CT_list[i], CP_list[i] = calc_CT_CP(theta75_rad[i], rotor)
    alpha_hub_list[i] = np.rad2deg(Alpha(r_hub, theta75_rad[i], rotor))

# Determine approximate index of stall: where AoA > 15 deg at r_hub
stall_indx = np.where(alpha_hub_list>15)[0][0]
theta75_stall = theta75_deg[stall_indx]

plt.figure(5)
plt.title('Fig 5: Local AoA vs Input Collective at 75%R')
plt.plot(theta75_deg, alpha_hub_list)
plt.vlines(x=theta75_stall, ymin=min(alpha_hub_list), ymax=max(alpha_hub_list),␣
 ↪color='r', linestyle='--',label=rf"$\theta_{{75,stall}}␣
 ↪\approx${theta75_stall:.1f}$^\circ$")
plt.xlabel(r"Input Collective $\theta_{75}$ (deg)")
plt.ylabel(r"Local angle of attack at the rotor hub, $\alpha_{hub}$")
plt.grid()
plt.legend()
```

[38]: <matplotlib.legend.Legend at 0x7f9a81475e50>



The plots below show λ, C_T, and C_P, each as a function of θ_{75}.

[39]:
```
plt.figure(6)
plt.title("Fig 6: Inflow Ratio as a Function of Input Collective")
plt.plot(theta75_deg, lambda_list)
plt.vlines(x=theta75_stall, ymin=min(lambda_list), ymax=max(lambda_list),␣
↪color='r', linestyle='--', label=rf"$\theta_{{75,stall}}␣
↪\approx${theta75_stall:.1f}$^\circ$")
plt.xlabel(r"Input Collective $\theta_{75}$ (deg)")
plt.ylabel(r"Inflow ratio, $\lambda$")
plt.grid()
plt.legend()

plt.figure(7)
plt.title("Fig 7: Coefficient of Thrust as a Function of Input Collective")
plt.plot(theta75_deg, CT_list)
plt.vlines(x=theta75_stall, ymin=min(CT_list), ymax=max(CT_list), color='r',␣
↪linestyle='--', label=rf"$\theta_{{75,stall}} \approx${theta75_stall:.
↪1f}$^\circ$")
plt.xlabel(r"Input Collective $\theta_{75}$ (deg)")
plt.ylabel("Coefficient of Thrust")
plt.grid()
plt.legend()

plt.figure(8)
plt.title("Fig 8: Coefficient of Induced Power as a Function of Input␣
↪Collective")
plt.plot(theta75_deg, CP_list)
plt.vlines(x=theta75_stall, ymin=min(CP_list), ymax=max(CP_list), color='r',␣
↪linestyle='--', label=rf"$\theta_{{75,stall}} \approx${theta75_stall:.
↪1f}$^\circ$")
plt.xlabel(r"Input Collective $\theta_{75}$ (deg)")
plt.ylabel("Coefficient of Induced Power")
plt.grid()
plt.legend()
```

[39]:
```
<matplotlib.legend.Legend at 0x7f9a81859ee0>
```

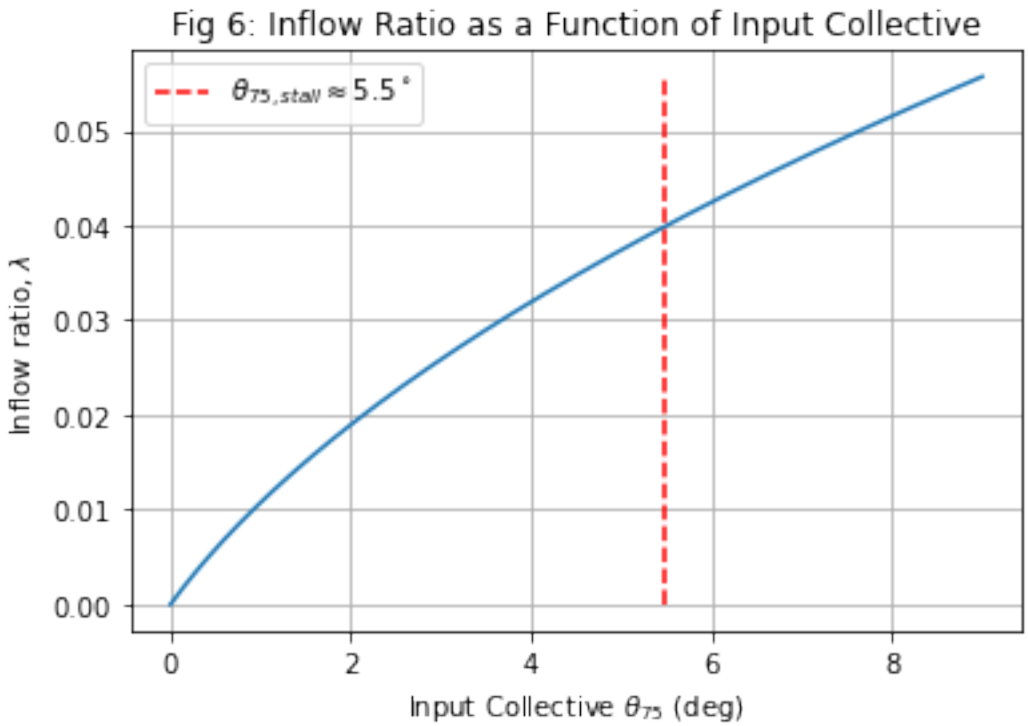


Fig 6: Inflow Ratio as a Function of Input Collective
$\theta_{75,stall} \approx 5.5^\circ$
0.05
0.04
0.03
0.02
0.01
0.00
Inflow ratio, λ
0
2
4
6
8
Input Collective θ_{75} (deg)

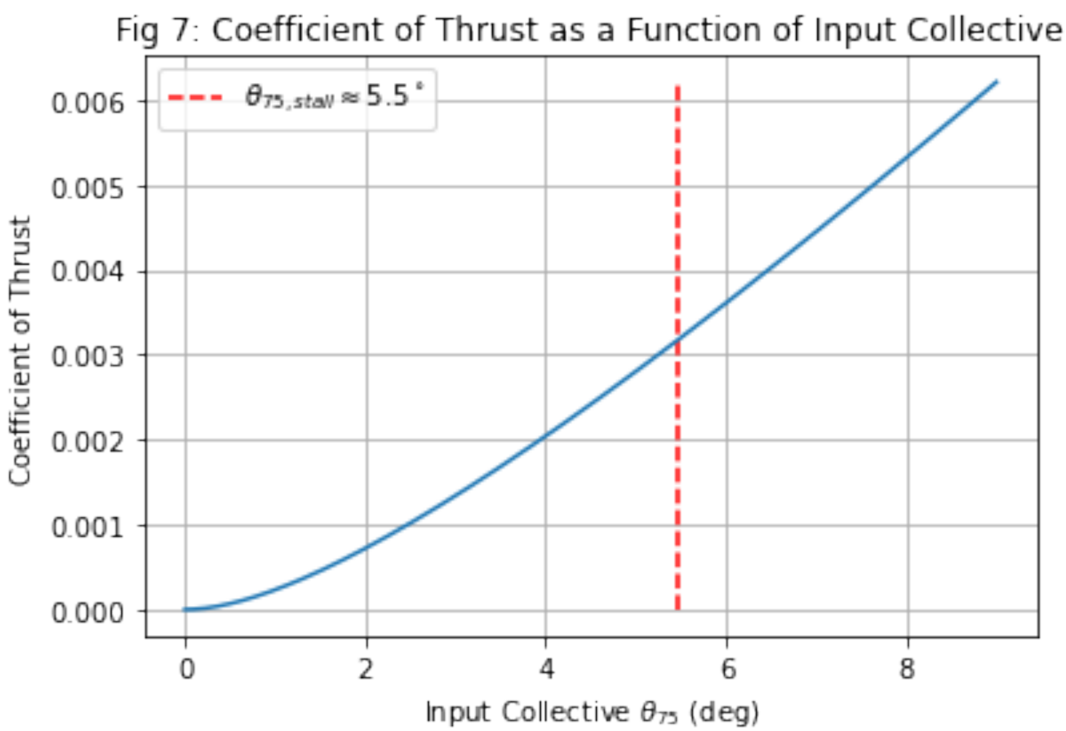
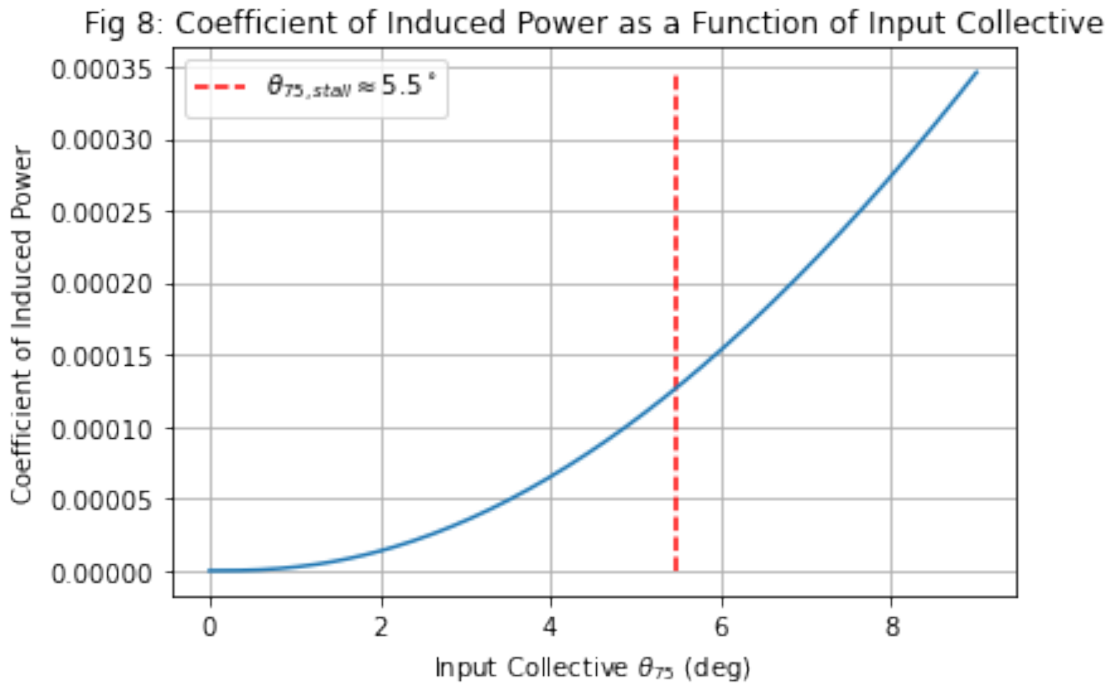


Fig 7: Coefficient of Thrust as a Function of Input Collective
$\theta_{75,stall} \approx 5.5^\circ$
0.006
0.005
0.004
0.003
0.002
0.001
0.000
Coefficient of Thrust
0
2
4
6
8
Input Collective θ_{75} (deg)



Plots of the total coefficient of power $C_{P,tot}$ and the ratio $C_{P,0}/C_P$ are shown below. As stated previously, the profile power contribution is importrant at smaller collectives, θ_{75}.

[40]:
```
plt.figure()
plt.title("Fig 9: Total Coefficient of Power as a Function of Input Collective")
plt.plot(theta75_deg, CP_list+CP0(rotor))
plt.vlines(x=theta75_stall, ymin=min(CP_list+CP0(rotor)),␣
 ↪ymax=max(CP_list+CP0(rotor)), color='r', linestyle='--',␣
 ↪label=rf"$\theta_{{75,stall}} \approx${theta75_stall:.1f}$^\circ$")
plt.xlabel(r"Input Collective $\theta_{75}$ (deg)")
plt.ylabel(r"Total Coefficient of Power, $C_{P,tot}$")
plt.grid()
plt.legend()

plt.figure()
CP_ratio = CP0(rotor)/(CP_list + CP0(rotor))
plt.title("Fig 10: Contribution of profile power to total power as a Function␣
 ↪of Input Collective")
plt.plot(theta75_deg, CP_ratio)
plt.vlines(x=theta75_stall, ymin=min(CP_ratio), ymax=max(CP_ratio), color='r',␣
 ↪linestyle='--', label=rf"$\theta_{{75,stall}} \approx${theta75_stall:.
 ↪1f}$^\circ$")
plt.xlabel(r"Input Collective $\theta_{75}$ (deg)")
plt.ylabel(r"Power ratio $C_{P,0}/C_{P,tot}$")
```

```
plt.grid()
plt.legend()
```

[40]: <matplotlib.legend.Legend at 0x7f9a818bc2e0>