ELECTENG 332

Notes on Control Systems

Dear god help me, not another one...

by

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1.1 Topic 1: The Importance of the Exponential Function

The Exponential function, written as either e^{ax} or e^{at} depending on whether it is $f(t)$ or $f(x)$, has properties that make it mathematically unique.

1. The derivative (rate of change) of the exponential function is the exponential function itself. More generally, this is a function whose rate of change is proportional to the function itself.

$$\frac{\mathrm{d}e^{ax}}{\mathrm{d}x} = ae^{ax} \tag{1}$$

2. The integral of the exponential function is also the exponential function itself.

$$\int e^{ax}\,\mathrm{d}x = \frac{1}{a}e^{ax} \tag{2}$$

1.2 Topic 2: The Concept of Engineering Infinity

Consider a signal e^{-at}. The time constant for this signal is $T = \frac{1}{a}$. Theoretically, the signal is meant to decay to zero as time approaches infinity, i.e.

$$\lim_{t\to\infty} e^{-at} = 0 \tag{3}$$

But in practice, this is not the case, as its value will be very, very small after five time constants $5T$ (or 5τ). This is the ***Concept of Engineering Infinity***. The signal will never reach zero, but it will be so small that it can be considered zero for all practical purposes. This is a very important concept in control systems, as it allows us to simplify our calculations and analysis.

1.3 Topic 3: The Concept of Complex Frequency

Complex frequency is found commonly in electrical engineering. It is often notated as $j\omega$ or $s = \sigma \pm j\omega$. These frequencies always come in pairs, so the use of $\pm$ is implicit to this, as complex numbers have complex conjugates (normally notated by z^* or $\bar{z}$). i.e. $s = \sigma + j\omega$ has the conjugate $s = \sigma - j\omega$.

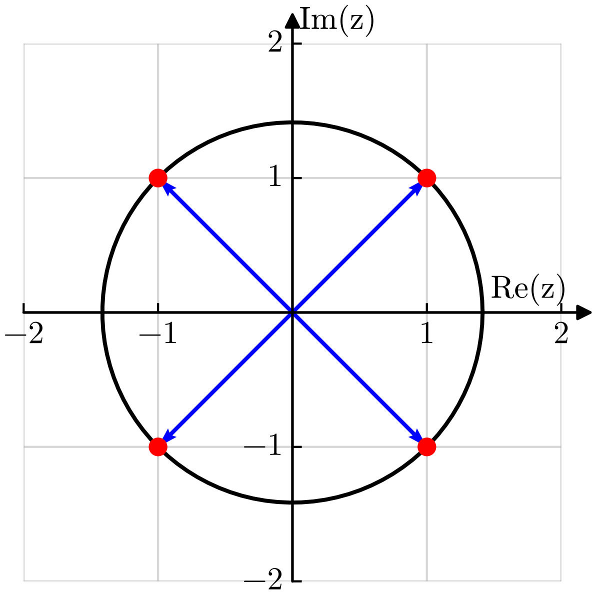


Figure 1.1: Argand diagram of $|z| = \sqrt{2}$

This is also backed up by De Moivre's formula which is defined mathematically as:

$$\forall x \in \mathbb{R}, \quad \forall n \in \mathbb{Z}, \tag{4}$$
$$e^{jnx} = \cos(nx) + j\sin(nx) \tag{5}$$

Or more generally for our applications (this is also known as Euler's formula):

$$e^{jx} = \cos(x) + j\sin(x) \tag{6}$$
$$\text{Where } x \in \mathbb{R} \, (x \text{ is real}) \tag{7}$$
$$\text{and } j \equiv i = \sqrt{-1} \tag{8}$$

This means that:

A complex frequency $j\omega$ represents a pure sinusoidal signal of frequency ω rad/s

For example, if a signal has a complex frequency $j314$ rad/s, then this responds to a pure sinusoid of frequency 314 rad/s (i.e. 50 Hz).

Furthermore:

A complex frequency $s = \sigma + j\omega$ represents an exponentially damped signal of frequency $j\omega$ rad/s, and decays/amplifies at a rate decided by σ.

For example, the signal $f(t) = e^{-10t}\sin(40\pi t)$ would look like this:

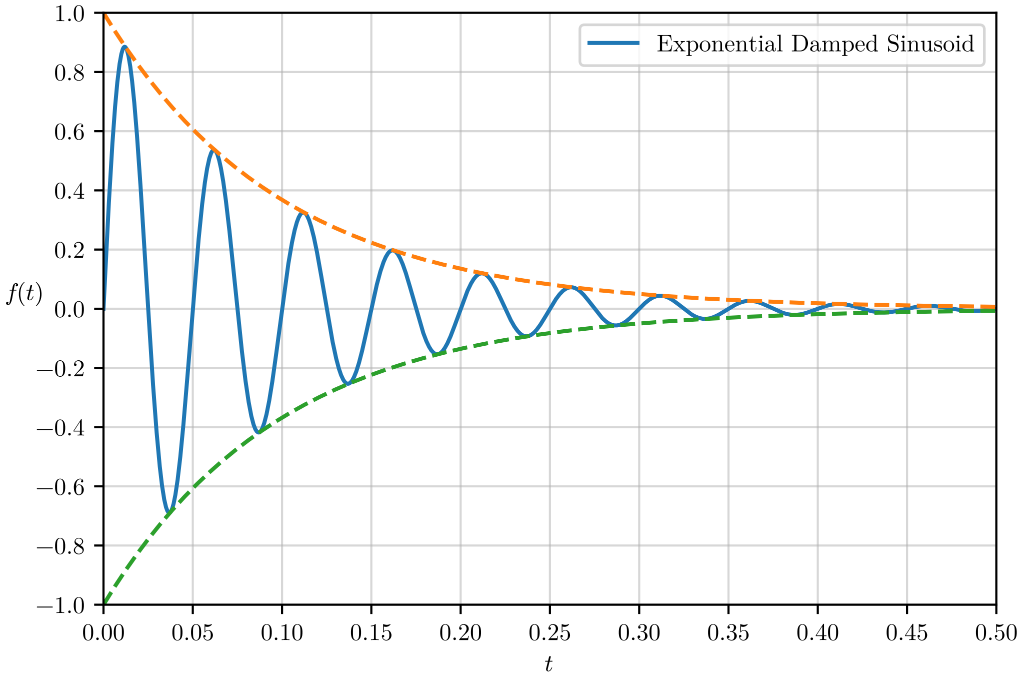


Figure 1.2: Exponentially damped sinusoid signal, $f(t) = e^{-10t}\sin(40\pi t)$

1.4 Topic 4: What are Signals?

1.4.1 Introduction

It is difficult to find a unique definition of a signal. However in the context of this course, we give a workable definition which suits most of out purposes as:

A Signal conveys information about a physical phenomenon which evolves in time or space.

Examples of such signals include: Voltage, Current, Speech, Television, Images from remote space probes, Voltages generated by the heart and brain, Radar and Sonar echoes, Seismic vibrations, Signals from GPS satellites, Signals from human genes, and countless other applications.

1.4.2 Energy and Power Signals

Energy Signals

A signal is said to be an *energy signal* if and only if it has finite energy.

Power Signals

A signal is said to be a *power signal* if and only if the average power of the signal is finite and non-zero.

Instantaneous Power

The instantaneous power $p(t)$ of a signal $x(t)$ is expressed as:

$$p(t) = x^2(t) \tag{9}$$

Continuous-Time Signal Energy

The total energy of a continuous-time signal $x(t)$ is given by:

$$E = \lim_{T\to\infty} \int_{-T/2}^{T/2} x^2(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} x^2(t)\,\mathrm{d}t \tag{10}$$

Complex Valued Signal Energy

For a complex valued signal:

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,\mathrm{d}t \tag{11}$$

Average Power

Since power equals to the time average of the energy, the average power is given by:

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) \, \mathrm{d}t = \frac{E}{T} \tag{12}$$

Note that during calculation of energy, we average the power over an indefinitely large interval.

A signal with finite energy has zero power and a signal with finite power has infinite energy.

Furthermore, some additional concepts of note:

a. A signal can not both be an energy and a power signal. This classification of signals based on power and energy are mutually exclusive.

b. However, a signal can belong to neither of the above two categories.

c. The signals which are both deterministic and non-periodic have finite energy and therefore are energy signals. Most of the signals, in practice, belong to this category.

d. Periodic signals and random signals are essentially power signals.

e. Periodic signals for which the area under $|x(t)|^2$ over one period is finite are power signals.

1.4.3 Examples

1.4.3.1 Example 1: Unit Step Function

Consider a unit step function defined as:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

Determine whether this is an energy signal or a power signal or neither.

Solution: Let us compute the energy of this signal as:

$$E = \int_{-\infty}^{\infty} [u(t)]^2 \, \mathrm{d}t = \int_{0}^{\infty} [0]^2 \, \mathrm{d}t = \int_{0}^{\infty} [1]^2 \, \mathrm{d}t = \infty \tag{14}$$

Since the energy of this signal is infinite, it cannot be an energy signal. Let us compute the power of this signal as:

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} [u(t)]^2 \, \mathrm{d}t = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T/2} [u(t)]^2 \, \mathrm{d}t = \frac{1}{2} \tag{15}$$

The power of this signal is finite. Hence, this is a power signal.

1.4.3.2 Example 2: Exponential Function

Consider an exponential function defined as:

$$x(t) = e^{-at}u(t), \text{ where } u(t) \text{ is the unit step signal, } a > 0 \tag{16}$$

Classify this signal as an energy, power, or neither.

Solution: Let us compute the energy of this signal as:

$$E = \int_{-\infty}^{\infty} [x(t)]^2 \, \mathrm{d}t = \int_{0}^{\infty} \left[e^{-at}\right]^2 \mathrm{d}t = \int_{0}^{\infty} e^{-2at} \, \mathrm{d}t = \frac{1}{2a} < \infty \tag{17}$$

Thus, $x(t) = e^{-at}u(t)$ is an energy signal.

1.4.3.3 Example 3: Ramp Function:

Consider a ramp function defined as:

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

Classify this signal as an energy, power, or neither.

Solution: Let us compute the energy of this signal as:

$$E = \int_{-\infty}^{\infty} r(t)^2 \, \mathrm{d}t = \int_{-\infty}^{0} [0]^2 \, \mathrm{d}t = \int_{0}^{\infty} A^2 t^2 \, \mathrm{d}t = A^2 \frac{T^3}{3}\bigg|_0^{\infty} = \infty \tag{19}$$

Since the energy of this signal is infinite, it cannot be an energy signal. Let us compute the power of this signal as:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [r(t)]^2 \, \mathrm{d}t = \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T/2} A^2 t^2 \, \mathrm{d}t = A^2 \lim_{T\to\infty} \frac{1}{T}\frac{T^3}{3}\bigg|_0^{\infty} = \infty \tag{20}$$

The power of this signal is infinite. Hence, this is neither a power nor an energy signal..

1.5 Topic 5: What are Systems?

1.5.1 Introduction

The term *system* is derived from the Greek word *systema*, which means an organised relationship among functioning units or components. It is often used to describe any orderly arrangement of ideas or constructs.

According to the Webster's Dictionary,

> *"A system is an aggregation or assemblage of objects united by some form of regular interaction or interdependence; a group of diverse units so combined by nature or art as to form an integral; whole and to function, operate, or move in unison and often in obedience to some form of control..."*

According to the International Council on Systems Engineering (INCOSE),

> *A system is an arrangement of parts or elements that together exhibit behaviour or meaning that the individual constituents do not.*
>
> The elements or parts, can include people, hardware, software, facilities, policies, and documents; that is, all things required to produce system-level results.

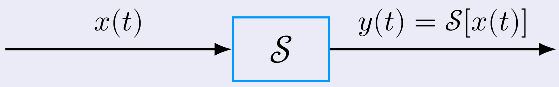
- It is difficult to give a single and precise definition of the term *system*, which will suit to different perspectives of different people.
- In practice, what is meant by "the system" depends on the objectives of a particular study.
- From the control engineering perspective, the system is any interconnection of components to achieve desired objectives. It is characterised by its **inputs**, **outputs**, and the rules of operations or laws. For example:
  a. The laws of operation in electrical systems are Ohm's law, which gives the voltage-current relationships for resistors, capacitors and inductors, and Kirchhoff's laws, which govern the laws of interconnection of various electrical components.
  b. Similarly, in mechanical systems, the laws of operation are Newton's laws. These laws can be used to derive mathematical models of the system.

1.5.2 The System as an Operator

The System Operator

A system is defined mathematically as a transformation which maps an input signal $x(t)$ to an output signal $y(t)$.For a continuous time system, the input-output mapping is expressed as:

$$y(t) = \mathcal{S}[x(t)], \quad \text{where } \mathcal{S} \text{ is an operator.} \tag{21}$$



1.5.3 Classification of Systems