ELECTENG 332

Notes on Control Systems

Dear god help me, not another one...

by

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Module 1:
Basics of Signals and Systems

Learning Outcomes

- Uniqueness of the Exponential Signal
- Concept of Engineering Infinity
- Concept of Complex Frequency
- Classification of Signals: Energy & Power
- Classification of Systems
- What is a Control System
- Classification of a Control System: Open-loop & Closed-loop

1.1 Topic 1: Importance of Exponential Functions

The Exponential function, written as either e^{ax} or e^{at} depending on whether it is $f(t)$ or $f(x)$, has properties that make it mathematically unique.

1. The derivative (rate of change) of the exponential function is the exponential function itself. More generally, this is a function whose rate of change is proportional to the function itself.

$$\frac{\mathrm{d}e^{ax}}{\mathrm{d}x} = ae^{ax} \tag{1}$$

2. The integral of the exponential function is also the exponential function itself.

$$\int e^{ax}\,\mathrm{d}x = \frac{1}{a}e^{ax} \tag{2}$$

1.2 Topic 2: Concept of Engineering Infinity

Consider a signal e^{-at}. The time constant for this signal is $T = \frac{1}{a}$. Theoretically, the signal is meant to decay to zero as time approaches infinity, i.e.

$$\lim_{t\to\infty} e^{-at} = 0 \tag{3}$$

But in practice, this is not the case, as its value will be very, very small after five time constants $5T$. This is the *Concept of Engineering Infinity*.

1.3 Topic 3: Concept of Complex Frequency

Complex frequency is found commonly in electrical engineering. It is often notated as $j\omega$ or $s = \sigma \pm j\omega$. These frequencies always come in pairs, so the use of $\pm$ is implicit to this, as complex numbers have complex conjugates, i.e. $s = \sigma + j\omega$ has the conjugate $s = \sigma - j\omega$.

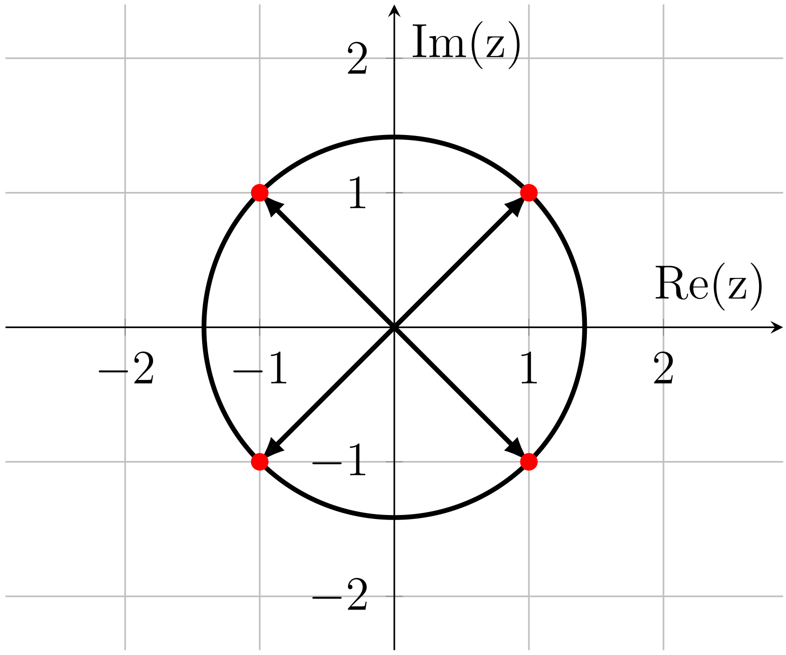

Figure 1.1: Plot of the circle $|z| = \sqrt{2}$

This is also backed up by De Moivre's Formula, which is defined mathematically as:

$$\forall x \in \mathbb{R}, \quad \forall n \in \mathbb{Z}, \tag{4}$$

$$e^{jnx} = \cos(nx) + j\sin(nx) \tag{5}$$

Or more generally for our applications:

$$e^{jx} = \cos(x) + j\sin(x) \tag{6}$$

$$\text{Where } x \in \mathbb{R} \text{ (x is Real)} \tag{7}$$

$$\text{and } j \equiv i = \sqrt{-1}. \tag{8}$$

This means that:

A complex frequency $j\omega$ represents a pure sinusoidal signal of frequency ω rad/s

For example, if a signal has a complex frequency $j314$ rad/sec, then this corresponds to a pure sinusoid of frequency 314 rad/sec (i.e. 50 Hz).
Furthermore:

A complex frequency $s = \sigma + j\omega$ represents an exponentially damped signal of frequency ω rad/s*, and decays/amplifies at a rate decided by σ.*

For example, the signal $f(t) = e^{-10t}\sin(40\pi t)$ would look like this:

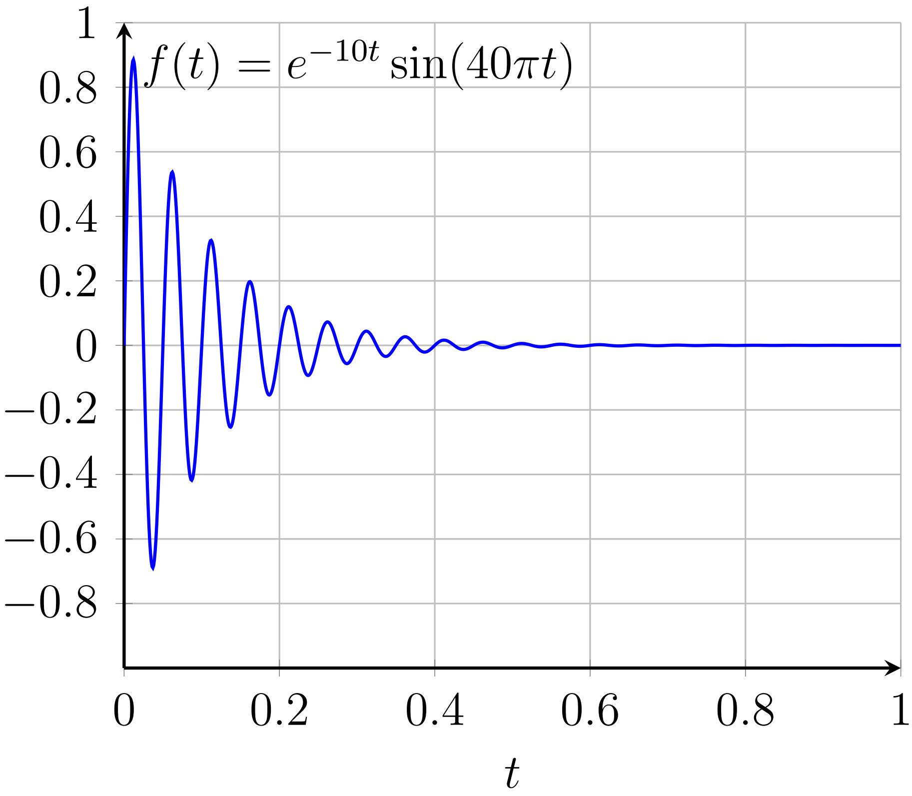

Figure 1.2: Plot of the signal $f(t) = e^{-10t}\sin(40\pi t)$.

1.4 Topic 4: What are Signals?

It is difficult to find a unique definition of a signal. However in the context of this course, we give a workable definition which suits most of our purposes as:

A signal conveys information about a physical phenomenon which evolves in time or space.

Examples of such signals include: Voltage, current, speech, television, images from remote space probes, voltages generated by the heart and brain, radar and sonar echoes, seismic vibrations, signals from GPS satellites, signals from human genes, and countless other applications.

1.4.1 Energy & Power Signals

Energy Signals

A signal is said to be an *energy signal* if and only if it has finite energy.

Power Signals

A signal is said to be a *power signal* if and only if the average power of the signal is finite and non-zero.

The instantaneous power $p(t)$ of a signal $x(t)$ is expressed as:

$$p(t) = x^2(t) \tag{9}$$

The total energy of a continuous-time signal $x(t)$ is given by:

$$E = \lim_{T\to\infty} \int_{-T/2}^{T/2} x^2(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} x^2(t)\,\mathrm{d}t \tag{10}$$

For a complex valued signal:

$$E = \int_{-\infty}^{\infty} |x(t)|^2\,\mathrm{d}t \tag{11}$$

Since power equals to the time average of the energy, the average power is given by:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\,\mathrm{d}t = \frac{E}{T} \tag{12}$$

Note that during calculation of energy, we average the power over an infinitely large interval.

A signal with finite energy has zero power and a signal with finite power has infinite energy.

a. A signal can not both be an energy and a power signal. This classification of signals based on power and energy are mutually exclusive.

b. However, a signal can belong to neither of the above two categories.

c. The signals which are both deterministic and non-periodic have finite energy and therefore are energy signals. Most of the signals, in practice, belong to this category.

d. Periodic signals and random signals are essentially power signals.

e. Periodic signals for which the area under $|x(t)|^2$ over one period is finite are power signals.

1.4.2 Examples

Example 1: Unit Step Function

Consider a unit step function defined as:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

Determine whether this is an energy signal or a power signal or neither.
Solution: Let us compute the energy of this signal as:

$$E = \int_{-\infty}^{\infty} [u(t)]^2 \, \mathrm{d}t = \int_{0}^{\infty} [0]^2 \, \mathrm{d}t = \int_{0}^{\infty} [1]^2 \, \mathrm{d}t = \infty \tag{14}$$

Since the energy of this signal is infinite, it cannot be an energy signal. Let us compute the power of this signal as:

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} [u(t)]^2 \, \mathrm{d}t = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T/2} [u(t)]^2 \, \mathrm{d}t = \frac{1}{2} \tag{15}$$

The power of this signal is finite. Hence, this is a power signal.

Example 2: Exponential Function

Consider an exponential function defined as:

$$x(t) = e^{-at}u(t), \text{ where } u(t) \text{ is the unit step signal, } a > 0 \tag{16}$$

Classify this signal as an energy, power, or neither.
Solution: Let us compute the energy of this signal as:

$$E = \int_{-\infty}^{\infty} [x(t)]^2 \, \mathrm{d}t = \int_{0}^{\infty} [e^{-at}]^2 \, \mathrm{d}t = \int_{0}^{\infty} e^{-2at} \, \mathrm{d}t = \frac{1}{2a} < \infty \tag{17}$$

Thus, $x(t) = e^{-at}u(t)$ is an energy signal.

Example 3: Ramp Function

Consider a ramp function defined as:

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

Classify this signal as an energy, power, or neither.
Solution: Let us compute the energy of this signal as:

$$E = \int_{-\infty}^{\infty} r(t)^2 \, \mathrm{d}t = \int_{-\infty}^{0} [0]^2 \, \mathrm{d}t = \int_{0}^{\infty} A^2 t^2 \, \mathrm{d}t = A^2 \frac{T^3}{3} \bigg|_0^{\infty} = \infty \tag{19}$$

Since the energy of this signal is infinite, it cannot be an energy signal. Let us compute the power of this signal as:

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} [r(t)]^2 \, \mathrm{d}t = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T/2} A^2 t^2 \, \mathrm{d}t = A^2 \lim_{T \to \infty} \frac{1}{T} \frac{T^3}{3} \bigg|_0^{\infty} = \infty \tag{20}$$

The power of this signal is infinite. Hence, this is neither a power nor an energy signal.

1.5 Topic 5: What are Systems?

The term *system* is derived from the Greek word *systema*, which means an organised relationship among functioning units or components. It is often used to describe any orderly arrangement of ideas or constructs.
According to the Webster's Dictionary,

> *"A system is an aggregation or assemblage of objects united by some form of regular interaction or interdependence; a group of diverse units so combined by nature or art as to form an integral; whole and to function, operate, or move in unison and often in obedience to some form of control..."*

According to the International Council on Systems Engineering (INCOSE),

> *A system is an arrangement of parts or elements that together exhibit behaviour or meaning that the individual constituents do not.*
> The elements or parts, can include people, hardware, software, facilities, policies, and documents; that is, all things required to produce system-level results.

It is difficult to give a single and precise definition of the term *system*, which will suit to different perspectives of different people. In practice, what is meant by "the system" depends on the objectives of a particular study.
From the control engineering perspective, the system is any interconnection of components to achieve desired objectives. It is characterised by its **inputs**, **outputs**, and the rules of operations or laws. For example:

a. The laws of operation in electrical systems are Ohm's law, which gives the voltage-current relationships for resistors, capacitors and inductors, and Kirchhoff's laws, which govern the laws of interconnection of various electrical components.

b. Similarly, in mechanical systems, the laws of operation are Newton's laws. These laws can be used to derive mathematical models of the system.

1.5.1 System as an Operator

> A system is defined mathematically as a transformation which maps an input signal $x(t)$ to an output signal $y(t)$ as shown in Figure: 1.3. For a continuous time system, the input-output mapping is expressed as:
>
> $$y(t) = \mathcal{S}[x(t)], \quad \text{where } \mathcal{S} \text{ is an operator.} \tag{21}$$

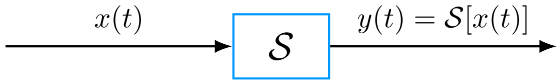

Figure 1.3: System as an Operator

A control system may be defined as an interconnection of components which are configured to provide a desired response.

1.5.2 Classification of Systems

The basis of classifying systems are many. They can be classified according to the following:

a. **The Time Frame:** (*discrete, continuous or hybrid*);

b. **System Complexity:** (*physical, conceptual and esoteric*);

c. **Uncertainties:** (*deterministic and stochastic*);

d. **Nature and type of components:** (*static or dynamic, linear or nonlinear, time-invariant or time variant, lumped or distributed etc*);

- Linear and nonlinear systems;
- Time-invariant and time-variant systems;
- Static (memory less) and dynamic (with memory) systems;
- Causal and Non-causal systems;
- Lumped and distributed parameter systems;
- Deterministic and stochastic systems;
- Continuous and discrete systems;

1.5.3 Linear and Nonlinear Systems

A system is said to be linear provided it satisfies the *principle of superposition* which is the combination of the *additive* and *homogeneity* properties. Otherwise, it is *non-linear*

Principle of Additivity:

Assume the system initially at rest.

- Suppose an input $x_1(t)$ to this system produces an output $y_1(t)$ and an input $x_2(t)$ produces an output $y_2(t)$.

- If the system is linear, then the application of the input $x_1(t) + x_2(t)$ will produce an output $y_1(t) + y_2(t)$. Thus if

$$x_1(t) \to y_1(t) \quad \text{and} \quad x_2(t) \to y_2(t), \quad \text{then} \quad x_1(t) + x_2(t) \to y_1(t) + y_2(t) \tag{22}$$

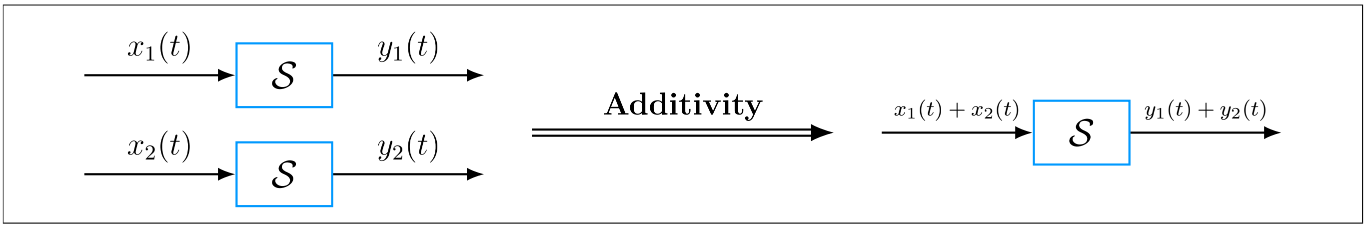

Figure 1.4: Principle of Additivity

Principle of Homogeneity or Scaling:

- Let an input *(cause)* $x(t)$ produce an output *(effect)* $y(t)$. If the system is linear then,
- Scaling the input (*cause*) $x(t)$ by a factor "a" will scale the output (*effect*) $y(t)$ by the same factor.
- Thus, if the input $x(t)$ results in $y(t)$, then the scaled input $ax(t)$ gives the output $ay(t)$ where "a" can either be a real or imaginary number. Thus, for a linear system:

$$x(t) \to y(t) \quad \Rightarrow ax(t) \to ay(t) \tag{23}$$

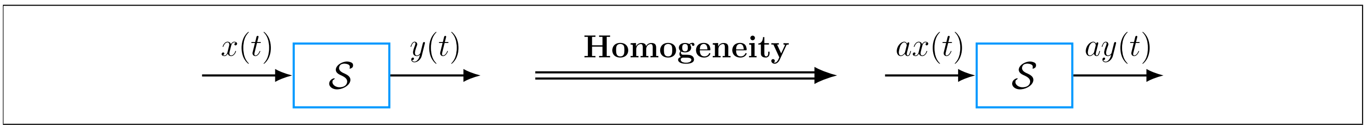

Figure 1.5: Principle of Homogeneity

Principle of Superposition:

This is the combination of both the additive and homogeneity properties for a linear system. (For those definitions, See Equations: 22 and 23).

If the zero state response of a linear system due to a finite N-number of inputs $x_1(t), x_2(t), \ldots, x_N(t)$ equals to $y_1(t), y_2(t), \ldots, y_N(t)$ respectively, then the response of the system to the linear combination of these inputs

$$a_1x_1(t) + a_2x_2(t) + a_3x_3(t) + \ldots \cdots + a_Nx_N(t) \tag{24}$$

is given by the linear combination of the individual outputs i.e.

$$a_1y_1(t) + a_2y_2(t) + a_3y_3(t) + \ldots \cdots + a_Ny_N(t) \tag{25}$$

where $a_1, a_2, a_3, \ldots, a_N$ are arbitrary constants (either real or imaginary). Thus, if

$$x_1(t) \to y_1(t), \quad x_2(t) \to y_2(t), \ldots, x_N(t) \to y_N(t) \tag{26}$$

then, for all $a_1, a_2, \ldots, a_N$

$$a_1x_1(t) + a_2x_2(t) + \ldots a_Nx_N(t) \to a_1y_1(t) + a_2y_2(t) + \ldots a_Ny_N(t) \tag{27}$$

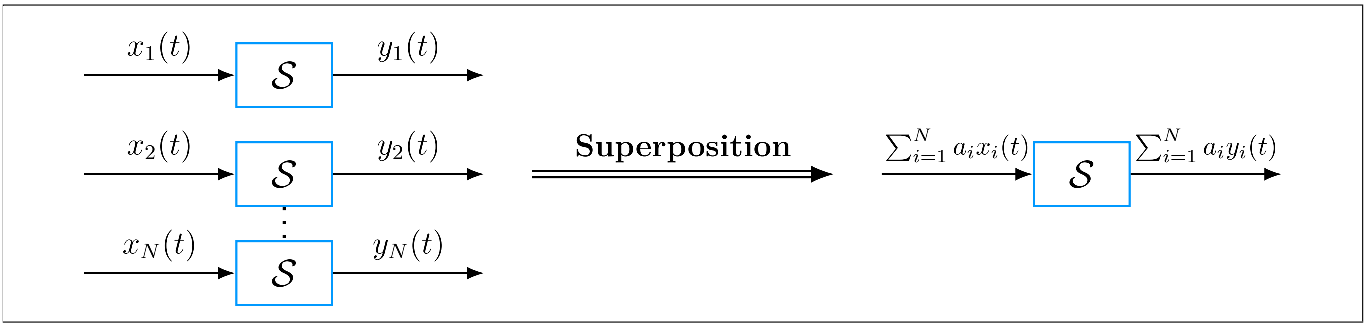

Figure 1.6: Superposition Principle Diagram

In more mathematical terms:

Let $\mathcal{S}$ be a linear system, and let $x_1(t), x_2(t), \ldots, x_N(t)$ be inputs with corresponding outputs $y_1(t), y_2(t), \ldots, y_N(t)$, such that $\mathcal{S}(x_i(t)) = y_i(t)$ for each $i \in \{1, 2, \ldots, N\}$. Then, for any constants $a_1, a_2, \ldots, a_N \in \mathbb{C}$, the system's response to the linear combination of the inputs is given by the corresponding linear combination of the outputs.
Formally:

$$\forall N \in \mathbb{N}, \tag{28}$$

$$\forall \{a_i \in \mathbb{C}, x_i(t) \in \mathbb{C}, y_i(t) \in \mathbb{C}\}_{i=1}^{N}, \tag{29}$$

$$\text{if } \mathcal{S}(x_i(t)) = y_i(t) \text{ for all } i, \tag{30}$$

$$\text{then } \mathcal{S}\left(\sum_{i=1}^{N} a_i x_i(t)\right) = \sum_{i=1}^{N} a_i y_i(t). \tag{31}$$

In this definition:

- $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, \ldots\}$, representing the number of inputs;
- $\mathbb{C}$ denotes the set of complex numbers, allowing for both real and imaginary components in the coefficients, inputs, and outputs;
- The system $\mathcal{S}$ maps each input $x_i(t)$ to its corresponding output $y_i(t)$;
- The statement asserts that the linear combination of inputs yields the linear combination of outputs, which is the core of the superposition principle.

1.5.4 Example: Linear and Nonlinear Systems

Test the linearity/non-linearity of the system whose relation between the output $y(t)$ and input $x(t)$ is expressed as:

$$y(t) = mx(t) + c \tag{32}$$

where m is a constant.
Solution: Let $x_1(t)$ and $x_2(t)$ be the two distinct inputs applied to the system. The outputs corresponding to these inputs are

$$y_1(t) = mx_1(t) + c, \quad \& \quad y_2(t) = mx_2(t) + c \tag{33}$$

Let us apply an input, which is the linear combination of inputs $x_1(t)$ and $x_2(t)$ i.e.

$$x(t) = ax_1(t) + bx_2(t) \tag{34}$$

The output due to this input is

$$y(t) = m[ax_1(t) + bx_2(t)] + c \neq ay_1(t) + by_2(t) \tag{35}$$

Hence, the system is *non-linear*.

1.5.5 Time-invariant and Time-varying Systems

Definition: A system is time invariant of a time shift in the input signal results in an identical time shift in the output signal. For example, in a time-invariant system, if an input $u(t)$ produces an output $y(t)$, then application of a delayed input $u(t-d)$ results in $y(t-d)$. Mathematically, if

$$y(t) = \mathcal{S}[u(t)], \quad \text{then} \tag{36}$$

$$y(t-d) = \mathcal{S}[u(t-d)] \tag{37}$$

Remark 1: If the input-output relation of a system is described by a linear differential equation of the form

$$\frac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_1 \frac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_{n-1} \frac{\mathrm{d}y}{\mathrm{d}t} + a_n y \tag{38}$$

$$= b_0 \frac{\mathrm{d}^m u}{\mathrm{d}t^m} + b_1 \frac{\mathrm{d}^{m-1} u}{\mathrm{d}t^{m-1}} + \cdots + b_{m-1} \frac{\mathrm{d}u}{\mathrm{d}t} + b_m u \tag{39}$$

then, this system is time invariant if the coefficients $a_i, i = 1, 2, \ldots, n$ and $b_i, i = 0, 1, \ldots, m$ are constants (i.e. if $\forall a_i,\ i \in \mathbb{N}$ (starting at 1) & $\forall b_i,\ i \in \mathbb{N}$ (starting at 0)). If these coefficients are functions of time, then the system is a **linear time-varying** system.

Module 2: Mathematical Modelling of Dynamic Systems

Module 3:
Block Diagrams & Feedback Systems Overview

Module 4:
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Stability Analysis of Linear Systems