ELECTENG 332

Notes on Control Systems

Dear god help me, not another one...

by

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Module 1:
Basics of Signals and Systems

Learning Outcomes

- Uniqueness of the Exponential Signal
- Concept of Engineering Infinity
- Concept of Complex Frequency
- Classification of Signals: Energy & Power
- Classification of Systems
- What is a Control System
- Classification of a Control System: Open-loop & Closed-loop

1.1 Topic 1: Importance of Exponential Functions

The Exponential function, written as either e^{ax} or e^{at} depending on whether it is $f(t)$ or $f(x)$, has properties that make it mathematically unique.

1. The derivative (rate of change) of the exponential function is the exponential function itself. More generally, this is a function whose rate of change is proportional to the function itself.

$$\frac{\mathrm{d}e^{ax}}{\mathrm{d}x} = ae^{ax} \tag{1}$$

2. The integral of the exponential function is also the exponential function itself.

$$\int e^{ax}\,\mathrm{d}x = \frac{1}{a}e^{ax} \tag{2}$$

1.2 Topic 2: Concept of Engineering Infinity

Consider a signal e^{-at}. The time constant for this signal is $T = \frac{1}{a}$. Theoretically, the signal is meant to decay to zero as time approaches infinity, i.e.

$$\lim_{t\to\infty} e^{-at} = 0 \tag{3}$$

But in practice, this is not the case, as its value will be very, very small after five time constants $5T$. This is the ***Concept of Engineering Infinity***. The signal will never reach zero, but it will be so small that it can be considered zero for all practical purposes. This is a very important concept in control systems, as it allows us to simplify our calculations and analysis.

1.3 Topic 3: Concept of Complex Frequency

Complex frequency is found commonly in electrical engineering. It is often notated as $j\omega$ or $s = \sigma \pm j\omega$. These frequencies always come in pairs, so the use of $\pm$ is implicit to this, as complex numbers have complex conjugates, i.e. $s = \sigma + j\omega$ has the conjugate $s = \sigma - j\omega$.

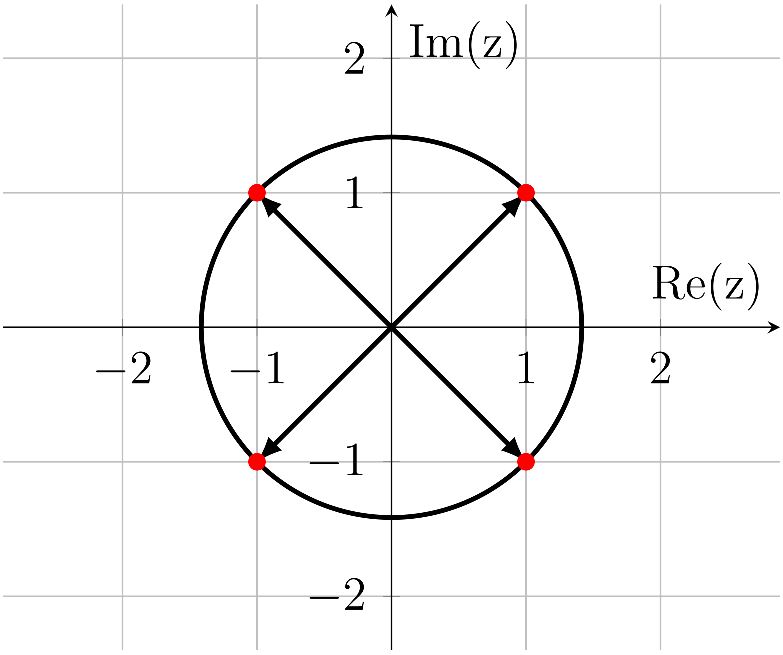

Figure 1.1: Plot of the circle $|z| = \sqrt{2}$

This is also backed up by De Moivre's Formula, which is defined mathematically as:

$$\forall x \in \mathbb{R}, \quad \forall n \in \mathbb{Z}, \tag{4}$$

$$e^{-jnx} = \cos(nx) + j\sin(nx) \tag{5}$$

Or more generally for our applications:

$$e^{jx} = \cos(x) + j\sin(x) \tag{6}$$

$$\text{Where}\, x \in \mathbb{R}\, (x \text{ is Real}) \tag{7}$$

$$\text{and}\, j \equiv i = \sqrt{-1}. \tag{8}$$

This means that:

$\boxed{\textit{A complex frequency } j\omega \textit{ represents a pure sinusoidal signal of frequency } \omega \text{ rad/s}}$

For example, if a signal has a complex frequency $j314\,\text{rad/s}$, then this corresponds to a pure sinusoid of frequency $314\,\text{rad/s}$ (i.e. $50\,\text{Hz}$).
Furthermore:

A complex frequency $s = \sigma + j\omega$ represents an exponentially damped signal of frequency ω rad/s, and decays/amplifies at a rate decided by σ.

For example, the signal $f(t) = e^{-10t}\sin(40\pi t)$ would look like this:

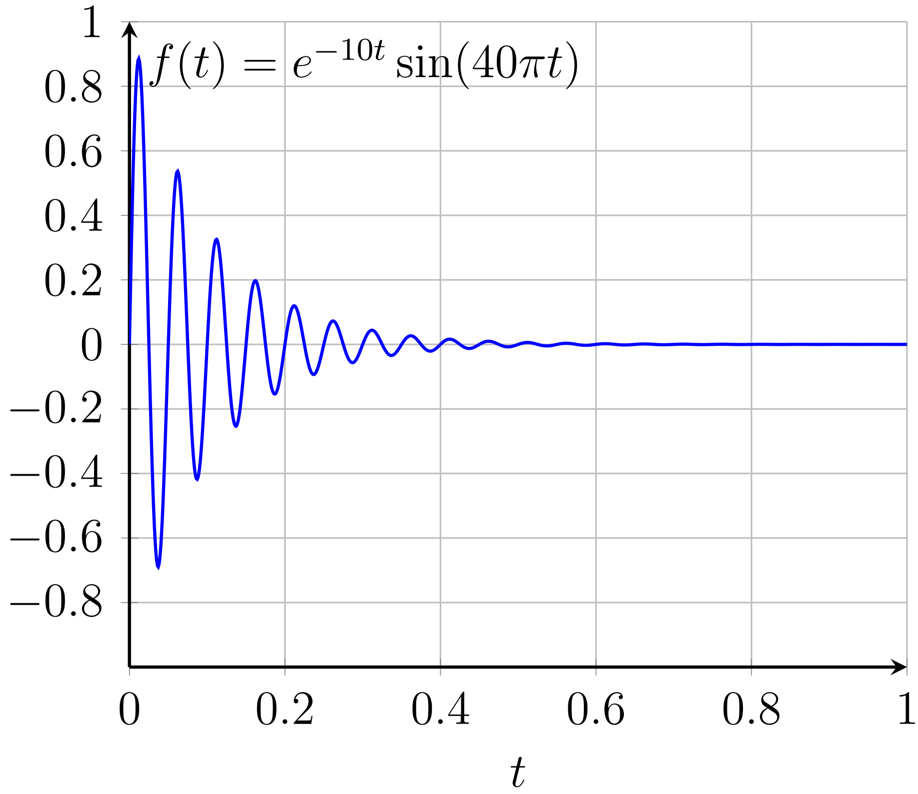

Figure 1.2: Plot of the signal $f(t) = e^{-10t}\sin(40\pi t)$.

1.4 Topic 4: What are Signals?

1.4.1 Introduction

It is difficult to find a unique definition of a signal. However in the context of this course, we give a workable definition which suits most of our purposes as:

A signal conveys information about a physical phenomenon which evolves in time or space.

Examples of such signals include: Voltage, current, speech, television, images from remote space probes, voltages generated by the heart and brain, radar and sonar echoes, seismic vibrations, signals from GPS satellites, signals from human genes, and countless other applications.

1.4.2 Energy & Power Signals

Energy Signals

A signal is said to be an *energy signal* if and only if it has finite energy.

Power Signals

A signal is said to be a *power signal* if and only if the average power of the signal is finite and non-zero.

The instantaneous power $p(t)$ of a signal $x(t)$ is expressed as:

$$p(t) = x^2(t) \tag{9}$$

The total energy of a continuous-time signal $x(t)$ is given by:

$$E = \lim_{T\to\infty} \int_{-T/2}^{T/2} x^2(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} x^2(t)\,\mathrm{d}t \tag{10}$$

For a complex valued signal:

$$E = \int_{-\infty}^{\infty} \left|x(t)\right|^2 \mathrm{d}t \tag{11}$$

Since power equals to the time average of the energy, the average power is given by:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\,\mathrm{d}t = \frac{E}{T} \tag{12}$$

Note that during calculation of energy, we average the power over an infinitely large interval.

A signal with finite energy has zero power and a signal with finite power has infinite energy.

a. A signal can not both be an energy and a power signal. This classification of signals based on power and energy are mutually exclusive.

b. However, a signal can belong to neither of the above two categories.

c. The signals which are both deterministic and non-periodic have finite energy and therefore are energy signals. Most of the signals, in practice, belong to this category.

d. Periodic signals and random signals are essentially power signals.

e. Periodic signals for which the area under $|x(t)|^2$ over one period is finite are power signals.

1.4.3 Examples

Example 1: Unit Step Function

Consider a unit step function defined as:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

Determine whether this is an energy signal or a power signal or neither.
Solution: Let us compute the energy of this signal as:

$$E = \int_{-\infty}^{\infty} [u(t)]^2 \, \mathrm{d}t = \int_{0}^{\infty} [0]^2 \, \mathrm{d}t = \int_{0}^{\infty} [1]^2 \, \mathrm{d}t = \infty \tag{14}$$

Since the energy of this signal is infinite, it cannot be an energy signal. Let us compute the power of this signal as:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [u(t)]^2 \, \mathrm{d}t = \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T/2} [u(t)]^2 \, \mathrm{d}t = \frac{1}{2} \tag{15}$$

The power of this signal is finite. Hence, this is a power signal.

Example 2: Exponential Function

Consider an exponential function defined as:

$$x(t) = e^{-at}u(t), \text{ where } u(t) \text{ is the unit step signal, } a > 0 \tag{16}$$

Classify this signal as an energy, power, or neither.
Solution: Let us compute the energy of this signal as:

$$E = \int_{-\infty}^{\infty} [x(t)]^2 \, \mathrm{d}t = \int_{0}^{\infty} [e^{-at}]^2 \, \mathrm{d}t = \int_{0}^{\infty} e^{-2at} \, \mathrm{d}t = \frac{1}{2a} < \infty \tag{17}$$

Thus, $x(t) = e^{-at}u(t)$ is an energy signal.

Example 3: Ramp Function

Consider a ramp function defined as:

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

Classify this signal as an energy, power, or neither.
Solution: Let us compute the energy of this signal as:

$$E = \int_{-\infty}^{\infty} r(t)^2 \, \mathrm{d}t = \int_{-\infty}^{0} [0]^2 \, \mathrm{d}t = \int_{0}^{\infty} A^2 t^2 \, \mathrm{d}t = A^2 \frac{T^3}{3} \bigg|_0^{\infty} = \infty \tag{19}$$

Since the energy of this signal is infinite, it cannot be an energy signal. Let us compute the power of this signal as:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [r(t)]^2 \, \mathrm{d}t = \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T/2} A^2 t^2 \, \mathrm{d}t = A^2 \lim_{T\to\infty} \frac{1}{T} \frac{T^3}{3} \bigg|_0^{\infty} = \infty \tag{20}$$

The power of this signal is infinite. Hence, this is neither a power nor an energy signal.

1.5 Topic 5: What are Systems?

1.5.1 Introduction

The term *system* is derived from the Greek word *systema*, which means an organised relationship among functioning units or components. It is often used to describe any orderly arrangement of ideas or constructs.

According to the Webster's Dictionary,

> *"A system is an aggregation or assemblage of objects united by some form of regular interaction or interdependence; a group of diverse units so combined by nature or art as to form an integral; whole and to function, operate, or move in unison and often in obedience to some form of control..."*

According to the International Council on Systems Engineering (INCOSE),

> *A system is an arrangement of parts or elements that together exhibit behaviour or meaning that the individual constituents do not.*
>
> The elements or parts, can include people, hardware, software, facilities, policies, and documents; that is, all things required to produce system-level results.

- It is difficult to give a single and precise definition of the term *system*, which will suit to different perspectives of different people.
- In practice, what is meant by "the system" depends on the objectives of a particular study.
- From the control engineering perspective, the system is any interconnection of components to achieve desired objectives. It is characterised by its **inputs**, **outputs**, and the rules of operations or laws. For example:
  a. The laws of operation in electrical systems are Ohm's law, which gives the voltage-current relationships for resistors, capacitors and inductors, and Kirchhoff's laws, which govern the laws of interconnection of various electrical components.
  b. Similarly, in mechanical systems, the laws of operation are Newton's laws. These laws can be used to derive mathematical models of the system.

1.5.2 System as an Operator

A system is defined mathematically as a transformation which maps an input signal $x(t)$ to an output signal $y(t)$ as shown in Figure: 1.3. For a continuous time system, the input-output mapping is expressed as:

$$y(t) = \mathcal{S}[x(t)], \quad \text{where } \mathcal{S} \text{ is an operator.} \tag{21}$$

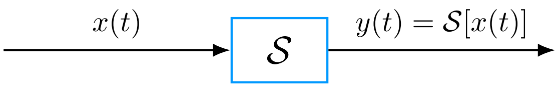

Figure 1.3: System as an Operator

A control system may be defined as an interconnection of components which are configured to provide a desired response.

1.5.3 Classification of Systems

The basis of classifying systems are many. They can be classified according to the following:

a. **The Time Frame:** (*discrete, continuous or hybrid*);

b. **System Complexity:** (*physical, conceptual and esoteric*);

c. **Uncertainties:** (*deterministic and stochastic*);

d. **Nature and type of components:** (*static or dynamic, linear or nonlinear, time-invariant or time variant, lumped or distributed etc*);

- Linear and nonlinear systems;
- Time-invariant and time-variant systems;
- Static (memory less) and dynamic (with memory) systems;
- Causal and Non-causal systems;
- Lumped and distributed parameter systems;
- Deterministic and stochastic systems;
- Continuous and discrete systems;

1.5.4 Linear and Nonlinear Systems

A system is said to be linear provided it satisfies the *principle of superposition* which is the combination of the *additive* and *homogeneity* properties. Otherwise, it is *non-linear*

Principle of Additivity:

Assume the system initially at rest.

- Suppose an input $x_1(t)$ to this system produces an output $y_1(t)$ and an input $x_2(t)$ produces an output $y_2(t)$.

- If the system is linear, then the application of the input $x_1(t) + x_2(t)$ will produce an output $y_1(t) + y_2(t)$. Thus if

$$x_1(t) \to y_1(t) \quad \text{and} \quad x_2(t) \to y_2(t), \quad \text{then} \quad x_1(t) + x_2(t) \to y_1(t) + y_2(t) \tag{22}$$

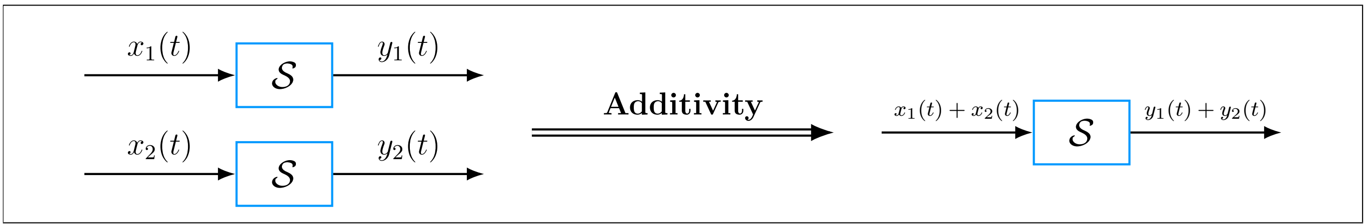

Figure 1.4: Principle of Additivity

Principle of Homogeneity or Scaling:

- Let an input *(cause)* $x(t)$ produce an output *(effect)* $y(t)$. If the system is linear then,

- Scaling the input (*cause*) $x(t)$ by a factor "a" will scale the output (*effect*) $y(t)$ by the same factor.

- Thus, if the input $x(t)$ results in $y(t)$, then the scaled input $ax(t)$ gives the output $ay(t)$ where "a" can either be a real or imaginary number. Thus, for a linear system:

$$x(t) \to y(t) \quad \Rightarrow ax(t) \to ay(t) \tag{23}$$

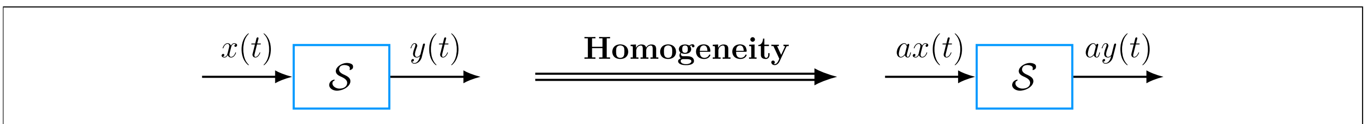

Figure 1.5: Principle of Homogeneity

Principle of Superposition:

This is the combination of both the additive and homogeneity properties for a linear system. (For those definitions, See Equations: 22 and 23).

If the zero state response of a linear system due to a finite N-number of inputs $x_1(t), x_2(t), \ldots, x_N(t)$ equals to $y_1(t), y_2(t), \ldots, y_N(t)$ respectively, then the response of the system to the linear combination of these inputs

$$a_1x_1(t) + a_2x_2(t) + a_3x_3(t) + \ldots \cdots + a_Nx_N(t) \tag{24}$$

is given by the linear combination of the individual outputs i.e.

$$a_1y_1(t) + a_2y_2(t) + a_3y_3(t) + \ldots \cdots + a_Ny_N(t) \tag{25}$$

where $a_1, a_2, a_3, \ldots, a_N$ are arbitrary constants (either real or imaginary). Thus, if

$$x_1(t) \to y_1(t), \quad x_2(t) \to y_2(t), \ldots, x_N(t) \to y_N(t) \tag{26}$$

then, for all $a_1, a_2, \ldots, a_N$

$$a_1x_1(t) + a_2x_2(t) + \ldots a_Nx_N(t) \to a_1y_1(t) + a_2y_2(t) + \ldots a_Ny_N(t) \tag{27}$$

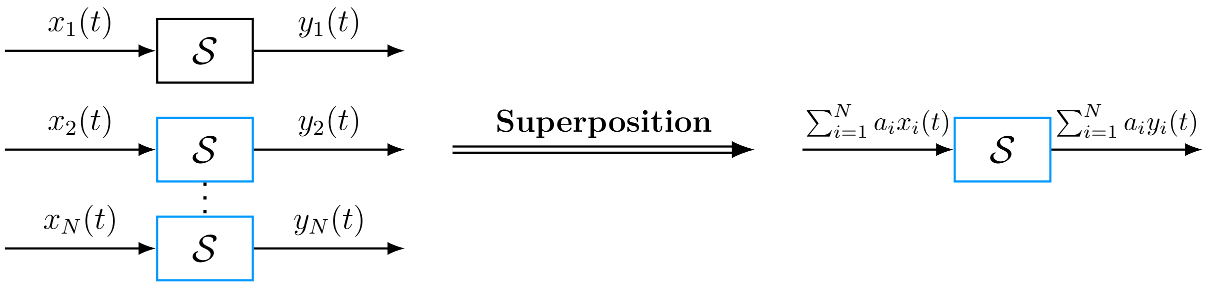

Figure 1.6: Superposition Principle Diagram

In more mathematical terms:

Let $\mathcal{S}$ be a linear system, and let $x_1(t), x_2(t), \ldots, x_N(t)$ be inputs with corresponding outputs $y_1(t), y_2(t), \ldots, y_N(t)$, such that $\mathcal{S}(x_i(t)) = y_i(t)$ for each $i \in \{1, 2, \ldots, N\}$. Then, for any constants $a_1, a_2, \ldots, a_N \in \mathbb{C}$, the system's response to the linear combination of the inputs is given by the corresponding linear combination of the outputs.
Formally:

$$\forall N \in \mathbb{N}, \tag{28}$$

$$\forall \{a_i \in \mathbb{C}, x_i(t) \in \mathbb{C}, y_i(t) \in \mathbb{C}\}_{i=1}^{N}, \tag{29}$$

$$\text{if } \mathcal{S}(x_i(t)) = y_i(t) \text{ for all } i, \tag{30}$$

$$\text{then } \mathcal{S}\left(\sum_{i=1}^{N} a_i x_i(t)\right) = \sum_{i=1}^{N} a_i y_i(t). \tag{31}$$

In this definition:

- $\mathbb{N}$ denotes the set of natural numbers $\{1, 2, \ldots\}$, representing the number of inputs;
- $\mathbb{C}$ denotes the set of complex numbers, allowing for both real and imaginary components in the coefficients, inputs, and outputs;
- The system $\mathcal{S}$ maps each input $x_i(t)$ to its corresponding output $y_i(t)$;
- The statement asserts that the linear combination of inputs yields the linear combination of outputs, which is the core of the superposition principle.

1.5.5 Example: Linear and Nonlinear Systems

Test the linearity/non-linearity of the system whose relation between the output $y(t)$ and input $x(t)$ is expressed as:

$$y(t) = mx(t) + c \tag{32}$$

where m is a constant.
Solution: Let $x_1(t)$ and $x_2(t)$ be the two distinct inputs applied to the system. The outputs corresponding to these inputs are

$$y_1(t) = mx_1(t) + c, \quad \& \quad y_2(t) = mx_2(t) + c \tag{33}$$

Let us apply an input, which is the linear combination of inputs $x_1(t)$ and $x_2(t)$ i.e.

$$x(t) = ax_1(t) + bx_2(t) \tag{34}$$

The output due to this input is

$$y(t) = m[ax_1(t) + bx_2(t)] + c \neq ay_1(t) + by_2(t) \tag{35}$$

Hence, the system is *non-linear*.

1.5.6 Time-invariant and Time-varying Systems

Definition: A system is time invariant of a time shift in the input signal results in an identical time shift in the output signal. For example, in a time-invariant system, if an input $u(t)$ produces an output $y(t)$, then application of a delayed input $u(t-d)$ results in $y(t-d)$. Mathematically, if

$$y(t) = \mathcal{S}[u(t)], \quad \text{then} \tag{36}$$

$$y(t-d) = \mathcal{S}[u(t-d)] \tag{37}$$

Remark 1: If the input-output relation of a system is described by a linear differential equation of the form

$$\tfrac{\mathrm{d}^n y}{\mathrm{d}t^n} + a_1 \tfrac{\mathrm{d}^{n-1} y}{\mathrm{d}t^{n-1}} + \cdots + a_{n-1} \tfrac{\mathrm{d}y}{\mathrm{d}t} + a_n y \tag{38}$$

$$= b_0 \tfrac{\mathrm{d}^m u}{\mathrm{d}t^m} + b_1 \tfrac{\mathrm{d}^{m-1} u}{\mathrm{d}t^{m-1}} + \cdots + b_{m-1} \tfrac{\mathrm{d}u}{\mathrm{d}t} + b_m u \tag{39}$$

then, this system is time invariant if the coefficients $a_i, i = 1, 2, \ldots, n$ and $b_i, i = 0, 1, \ldots, m$ are constants (i.e. if $\forall a_i,\ i \in \mathbb{N}$ (starting at 1) & $\forall b_i,\ i \in \mathbb{N}$ (starting at 0)). If these coefficients are functions of time, then the system is a **linear time-varying** system.

Module 2:
Mathematical Modelling of Dynamic Systems

Learning Outcomes

- What is a model? Why do we need it?
- Different types of models and their significance
- Differential equation models
- Transfer function models
- Impulse response models
- Physical concepts of poles, zeros, and order of a system

2.1 Introduction to Modeling

Common man observes the physical phenomena around him; the *scientist* encodes them into models.

2.2 Definition of a Model

- A model of a system is a tool we use to answer questions about the system without having to do an experiment.
- A collection of mathematical relationships between system/process variables which purports to describe the behavior of a physical system.
- A convenient surrogate of the physical system.

2.3 What is it's main use?

- To investigate system response under various input conditions both rapidly, and inexpensively, without tampering with the actual physical entity.
- Analytically design controllers.

2.4 Basic Concepts: Poles, Zeros, and System Order

Poles and Zeros

- Physical phenomena can be categorized essentially into:
  1. Energy dissipation
  2. Energy absorption or storage
- Familiar models: Resistance, Capacitance, and Inductance
  - Resistance models the phenomenon of energy dissipation.
  - Capacitance and Inductance model the phenomenon of energy absorption or storage.

System Order

- The order of a system equals the number of independent energy-storing elements of the system.
- Example:
  - R-Circuit: No. of Energy storing elements = 0 Order = 0
  - R-L Circuit: No. of Energy storing elements = 1 Order = 1
  - R-C Circuit: No. of Energy storing elements = 1 Order = 1
  - R-L-C Circuit: No. of Energy storing elements = 2 Order = 2

2.5 Differential Equation Models of Simple Dynamical Systems

Input-output relationships for various systems such as electrical, mechanical, hydraulic, thermal systems.

Irrespective of their complexity, each of the systems is composed of few simple basic components which include resistors, inductors, capacitors, masses, springs, dampers, cross-sectional areas of fluid tanks, and thermal capacities.

2.6 Modelling of Electrical Systems-1

The basic elements of electrical systems are the resistor, inductor, and capacitor, shown in Figure, and the variables of interest are voltages and currents. The differential equation models of electrical systems will be derived by balancing the voltages and currents by applying the well-known Ohm's law and Kirchhoff's laws.

2.6.1 Resistor

If the resistor, with resistance R, is linear, then the voltage drop $v(t)$ across the resistor is proportional to the current $i(t)$ flowing through the resistor and is expressed as:

$$v(t) = Ri(t)$$

2.6.2 Inductor

At any instant, the voltage drop $v(t)$ across the inductor L is proportional to the rate of change of current and is given by:

$$v(t) = L\frac{di(t)}{dt}$$

2.6.3 Capacitor

The voltage drop $v(t)$ across a capacitor C is proportional to the integral of the current through the capacitor:

$$v(t) = \frac{1}{C}\int i(t)\,dt$$

Alternatively, the current through the capacitor $i(t)$ is proportional to the rate of change of voltage across the capacitor and is given by:

$$i(t) = C\frac{dv(t)}{dt}$$

2.7 Modelling of Electrical Systems-2

Consider the electrical circuit consisting of a resistor and an inductor shown in Fig 4.

Applying Kirchhoff's voltage law to this circuit, we have:

$$Ri(t) + L\frac{di(t)}{dt} = v(t)$$

2.8 Modelling of Electrical Systems-3

Next, let us consider the electrical circuit consisting of a resistor and a capacitor shown in Fig 5.

Applying Kirchhoff's voltage law to this circuit, we have:

$$Ri(t) + \frac{1}{C}\int i(t)\,dt = v(t)$$

or,

$$R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{dv(t)}{dt}$$

2.9 Modelling of Electrical Systems-4

Let us model a circuit where all three passive elements—resistor, inductor, and capacitor—are connected in series (see Fig 6).

Applying Kirchhoff's voltage law to this circuit, we have:

$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt = v(t)$$

or,

$$L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{dv(t)}{dt}$$

2.10 Modelling of Mechanical Systems

2.10.1 Modelling of Mechanical Translational Systems

Mechanical systems can either be translational, rotational, or a combination of both. The basic components of translational systems are mass, spring, and damper. The variables of interest are displacement, velocity, acceleration, and force.

Force-Mass System

Consider a rigid body with mass m. According to Newton's second law, the relationship between the applied force $f(t)$ and its acceleration $a(t)$ is:

$$f(t) = ma(t) = m\frac{d^2x(t)}{dt^2} = m\frac{dv(t)}{dt}$$

where $v(t)$ is the linear velocity and $x(t)$ is the displacement.

Force-Spring System

A spring is an elastic object that stores mechanical (potential) energy. For a linear spring, the force $f(t)$ is directly proportional to the displacement $x(t)$:

$$f(t) = Kx(t) = K\int v(t)\,dt$$

where K is the spring constant (Nm^{-1}) and $v(t)$ is the velocity.

Viscous Damper

Frictional forces oppose the motion between two surfaces and can be modeled by a viscous damper (dashpot). The frictional force $f(t)$ is linearly proportional to the velocity:

$$f(t) = B\frac{dx(t)}{dt} = Bv(t)$$

where B is the viscous frictional coefficient.

Example: Mass-Spring-Damper System

Consider the mass-spring-damper system shown in Figure 8. Applying Newton's law to the free body diagram gives the force equation:

$$m\frac{d^2x(t)}{dt^2} + B\frac{dx(t)}{dt} + Kx(t) = f(t)$$

where $x(t)$ is the displacement, $\frac{dx(t)}{dt}$ is the velocity, and $\frac{d^2x(t)}{dt^2}$ is the acceleration.

2.10.2 Modelling of Mechanical Rotational Systems

When a body rotates about a fixed axis, the resulting motion is termed rotational motion. The basic components of rotational systems are inertia, torsional spring, and rotational damper.

Torque-Inertia System

The inertia of a rotational system is represented by the moment of inertia J. For a body with inertia J, the relationship between the applied torque $T(t)$ and the resulting angular acceleration $\alpha(t)$ is:

$$T(t) = J\alpha(t) = J\frac{d\omega(t)}{dt} = J\frac{d^2\theta(t)}{dt^2}$$

where $\theta(t)$ is the angular displacement and $\omega(t)$ is the angular velocity.

Torsional Spring

For a torsional spring, the torque $T(t)$ is directly proportional to the angular displacement $\theta(t)$:

$$T(t) = K\theta(t) = K\int \omega(t)\,dt$$

where K is the torsional spring constant.

Rotational Dashpot

Similar to translational systems, rotational systems can experience frictional forces, which can be modeled by a rotational damper. The torque $T(t)$ due to friction is proportional to the angular velocity:

$$T(t) = B\frac{d\theta(t)}{dt} = B\omega(t)$$

where B is the rotational frictional coefficient.

Example: Simple Rotational System

Consider the rotational system shown below. The body rotates at a speed $\omega(t)$. Applying Newton's law to the free body diagram, the torque equation is:

$$J\frac{d^2\theta(t)}{dt^2} + B\frac{d\theta(t)}{dt} + K\theta(t) = T(t)$$

where $\theta(t)$ is the angular displacement, $\frac{d\theta(t)}{dt}$ is the angular velocity, and $\frac{d^2\theta(t)}{dt^2}$ is the angular acceleration.

2.11 Impulse Response Model of a Linear Time Invariant System

What is the impulse response function of a system?

The response (output) of a system when the input is an impulse.

Why is it so important?

- If we know the impulse response function (model) of a system, i.e., the response of a system to an impulse input, the response of the system to any arbitrary input $u(t)$ can be found by convolving the impulse response function $h(t)$ with the input $u(t)$. - The convolution relation is given as:

$$y(t) = \int_0^t h(\tau)u(t-\tau)d\tau = \int_0^t h(t-\tau)u(\tau)d\tau$$

2.12 Transfer Function Model of Linear Time Invariant System-1

- Note that the impulse response model of Eqn(13) is the time-domain description of the system. - Let us apply the Laplace transform to the impulse response model of Eqn(13). This gives:

$$Y(s) = \mathcal{L}[y(t)] = \int_0^\infty y(t)e^{-st}dt = \int_0^\infty \left[\int_0^t h(t-\tau)u(\tau)d\tau\right] e^{-st}dt$$

- Since $h(t-\tau) = 0$ for $\tau > t$, the upper limit t of the integration in this equation can be set to ∞. Thus:

$$Y(s) = \int_0^\infty \left[\int_0^\infty h(t-\tau)u(\tau)d\tau\right] e^{-st}dt$$

2.13 Transfer Function Model of Linear Time Invariant System-2

- Let us substitute $\theta = t - \tau$ in the equation:

$$Y(s) = \int_0^\infty \left[\int_0^\infty h(\theta)e^{-s\theta}d\theta\right] e^{-s\tau}u(\tau)d\tau$$

- Since $h(\theta) = 0$ for $\theta < 0$, we can express the equation as:

$$Y(s) = \left[\int_0^\infty h(\theta)e^{-s\theta}d\theta\right]\left[\int_0^\infty u(\tau)e^{-s\tau}d\tau\right] = H(s)U(s)$$

where $H(s)$ is the transfer function of the system.

2.14 Transfer Function Model of Linear Time Invariant System-3

- The transfer function $H(s)$ is defined as the Laplace transform of the impulse response:

$$H(s) = \int_0^\infty h(t)e^{-st}dt$$

- The transfer function is a powerful tool in analyzing the system's behavior in the frequency domain.

2.15 Computation of Transfer Function Model from Differential Equation Model

To compute the transfer function from a differential equation:

1. Derive the differential equation model of the system.
2. Select the output $y(t)$ and input $u(t)$.
3. Take the Laplace transform of the differential equation with zero initial conditions.
4. Solve for $\frac{Y(s)}{U(s)}$ to find the transfer function.

2.16 Example-1: Computation of Transfer Function Model

Consider a system whose differential equation model is given by:

$$\ddot{y}(t) + a_1\dot{y}(t) + a_2y(t) = b_0\dot{u}(t) + b_1u(t)$$

Compute the transfer function model of the system considering $y(t)$ as the output and $u(t)$ as the input.

Solution:

Step-1: Take the Laplace transform of the differential equation model with zero initial conditions. This gives:

$$s^2Y(s) + a_1sY(s) + a_2Y(s) = b_0sU(s) + b1U(s)$$

Step-2: Take the ratio of $Y(s)$ to $U(s)$ to get the transfer function $H(s)$ as follows:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0s + b_1}{s^2 + a_1s + a_2}$$

2.17 Example-2: Computation of Transfer Function Model

Consider a system whose differential equation model is given by:

$$\frac{d^3y(t)}{dt^3} + 3\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + y(t) = \frac{d^3u(t)}{dt^3} + 4\frac{d^2u(t)}{dt^2} + 6\frac{du(t)}{dt} + 8u(t)$$

Compute the transfer function model of the system considering $y(t)$ as the output and $u(t)$ as the input.

Solution:

Step-1: Take the Laplace transform of the differential equation model with zero initial conditions. This gives:

$$s^3Y(s) + 3s^2Y(s) + 5sY(s) + Y(s) = s^3U(s) + 4s^2U(s) + 6sU(s) + 8U(s)$$

Step-2: Take the ratio of $Y(s)$ to $U(s)$ to get the transfer function $G(s)$ as follows:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s^3 + 4s^2 + 6s + 8}{s^3 + 3s^2 + 5s + 1}$$

2.18 Example-3: Computation of Transfer Function Model

Consider a general n-th order linear time-invariant system whose differential equation model is given by:

$$a_0\frac{d^ny(t)}{dt^n} + a_1\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_{n-1}\frac{dy(t)}{dt} + a_ny(t) = b_0\frac{d^mu(t)}{dt^m} + b_1\frac{d^{m-1}u(t)}{dt^{m-1}} + \cdots + b_{m-1}\frac{du(t)}{dt} + b_mu(t)$$

Compute the transfer function model of the system considering $y(t)$ as the output and $u(t)$ as the input.

Solution:

Step-1: Take the Laplace transform of both sides of the differential equation model with zero initial conditions. This gives:

$$a_0s^nY(s) + a_1s^{n-1}Y(s) + \cdots + a_nY(s) = b_0s^mU(s) + b_1s^{m-1}U(s) + \cdots + b_mU(s)$$

Step-2: Take the ratio of $Y(s)$ to $U(s)$ to get the transfer function $G(s)$ as follows:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_0s^m + b_1s^{m-1} + \cdots + b_m}{a_0s^n + a_1s^{n-1} + \cdots + a_n}$$

2.19 Conversion of Transfer Function Model to Differential Equation Model

To convert a transfer function back to a differential equation model:

1. Express the transfer function in algebraic form.

2. Take the inverse Laplace transform of the expression to obtain the differential equation.

2.20 Transfer Function Model of R-L Circuit

Consider the RL circuit shown in the figure below:

The dynamics of the system is given by:

$$Ri(t) + L\frac{di(t)}{dt} = v_i(t)$$

Taking the Laplace transform of gives:

$$RI(s) + LsI(s) = V_i(s) \quad \text{or} \quad I(s)[R + Ls] = V_i(s) \quad \text{or} \quad I(s) = \frac{V_i(s)}{R + Ls}$$

The voltage across the inductor is considered as the output. Thus:

$$v_o(t) = L\frac{di(t)}{dt} \quad \Rightarrow \quad V_o(s) = LsI(s)$$

Substituting the value of $I(s)$ from in gives:

$$V_o(s) = \frac{LsV_i(s)}{R + Ls} \quad \text{or} \quad \frac{V_o(s)}{V_i(s)} = \frac{Ls}{R + Ls} = \frac{s}{s + \frac{R}{L}}$$

2.21 Transfer Function Model of R-C Circuit

Consider the RC circuit shown in the figure below:

The dynamics of the system is given by:

$$Ri(t) + \frac{1}{C}\int i(t)dt = v_i(t)$$

Taking the Laplace transform of gives:

$$RI(s) + \frac{I(s)}{Cs} = V_i(s) \quad \text{or} \quad I(s)[RCs + 1] = CsV_i(s) \quad \text{or} \quad I(s) = \frac{CsV_i(s)}{RCs + 1}$$

The voltage across the resistor is considered as the output. Thus:

$$v_o(t) = Ri(t) \quad \Rightarrow \quad V_o(s) = RI(s)$$

Substituting the value of $I(s)$ from Eqn(24) in Eqn(25) gives:

$$V_o(s) = \frac{RCsV_i(s)}{RCs+1} \quad \text{or} \quad \frac{V_o(s)}{V_i(s)} = \frac{RCs}{RCs+1} = \frac{s}{s+\frac{1}{RC}}$$

2.22 Example-1: Converting Transfer Function Model to Differential Equation Model

Write the differential equation model of a system whose transfer function model is given by:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2+5s+6}$$

Solution:

Let us express it as an algebraic equation:

$$s^2Y(s) + 5sY(s) + 6Y(s) = U(s)$$

Substituting the terms $s^2Y(s)$ by $\ddot{y}(t)$, $sY(s)$ by $\dot{y}(t)$, $Y(s)$ by $y(t)$, and $U(s)$ by $u(t)$ gives:

$$\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = u(t)$$

2.23 Example-2: Converting Transfer Function Model to Differential Equation Model

Write the differential equation model of a system whose transfer function model is given by:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{5(s+2)(s+4)}{(s+1)(s+3)(s+5)}$$

Solution:

Now express it as an algebraic equation:

$$s^3Y(s) + 9s^2Y(s) + 23sY(s) + 15Y(s) = 5s^2U(s) + 30sU(s) + 40U(s)$$

Substituting the terms $s^3Y(s)$ by $\frac{d^3y(t)}{dt^3}$, $s^2Y(s)$ by $\frac{d^2y(t)}{dt^2}$, $sY(s)$ by $\frac{dy(t)}{dt}$, and so on gives:

$$\frac{d^3y(t)}{dt^3} + 9\frac{d^2y(t)}{dt^2} + 23\frac{dy(t)}{dt} + 15y(t) = 5\frac{d^2u(t)}{dt^2} + 30\frac{du(t)}{dt} + 40u(t)$$

Module 3: Block Diagrams & Feedback Systems Overview

Module 4:
Time Domain Analysis of Linear Systems

Module 5:
Stability Analysis of Linear Systems