ELECTENG 332

Notes on Control Systems

Dear god help me, not another one...

by

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Module 1: Basics of Signals and Systems

Learning Outcomes

- Uniqueness of the Exponential Signal
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- What is a Control System
- Classification of a Control System: Open-loop & Closed-loop

1.1 Topic 1: Importance of Exponential Functions

The Exponential function, written as either e^{ax} or e^{at} depending on whether it is $f(t)$ or $f(x)$, has properties that make it mathematically unique.

1. The derivative (rate of change) of the exponential function is the exponential function itself. More generally, this is a function whose rate of change is proportional to the function itself.

$$\frac{\mathrm{d}e^{ax}}{\mathrm{d}x} = ae^{ax}$$

2. The integral of the exponential function is also the exponential function itself.

$$\int e^{ax}\,\mathrm{d}x = \frac{1}{a}e^{ax}$$

1.2 Topic 2: Concept of Engineering Infinity

Consider a signal e^{-at}. The time constant for this signal is $T = \frac{1}{a}$. Theoretically, the signal is meant to decay to zero as time approaches infinity, i.e.

$$\lim_{t\to\infty} e^{-at} = 0$$

But in practice, this is not the case, as its value will be very, very small after five time constants $5T$. This is the *Concept of Engineering Infinity*.

1.3 Topic 3: Concept of Complex Frequency

Complex frequency is found commonly in electrical engineering. It is often notated as $j\omega$ or $s = \sigma \pm j\omega$. These frequencies always come in pairs, so the use of $\pm$ is implicit to this, as complex numbers have complex conjugates, i.e. $s = \sigma + j\omega$ has the conjugate $s = \sigma - j\omega$.

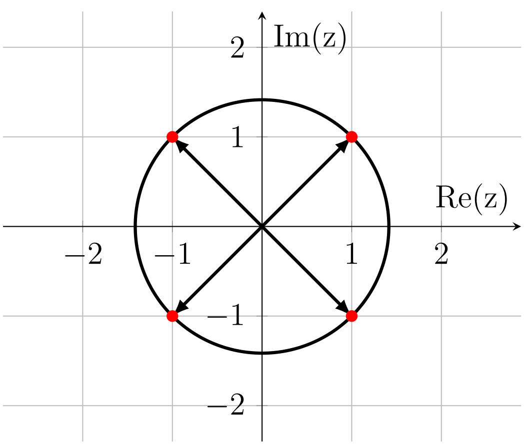

Figure 1.1: Plot of the circle $|z| = \sqrt{2}$

This is also backed up by De Moivre's Formula, which is defined mathematically as:

$$\forall x \in \mathbb{R}, \quad \forall n \in \mathbb{Z},$$
$$e^{jnx} = \cos(nx) + j\sin(nx)$$

Or more generally for our applications:

$$e^{jx} = \cos(x) + j\sin(x)$$
$$\text{Where } x \in \mathbb{R} \ (x \text{ is Real})$$
$$\text{and } j \equiv i = \sqrt{-1}.$$

This means that:

A complex frequency $j\omega$ represents a pure sinusoidal signal of frequency ω rad/s

Figure 1.2: Complex Sinusoidal Signal

For example, if a signal has a complex frequency $j314$ rad/sec, then this corresponds to a pure sinusoid of frequency 314 rad/sec (i.e. 50 Hz). Furthermore:

A complex frequency $s = \sigma + j\omega$ represents an exponentially damped signal of frequency ω rad/s, *and decays/amplifies at a rate decided by σ.*

Figure 1.3: Exponentially Damped Signal

For example, the signal $e^{-10t}\sin(40\pi t)$ would look like this:

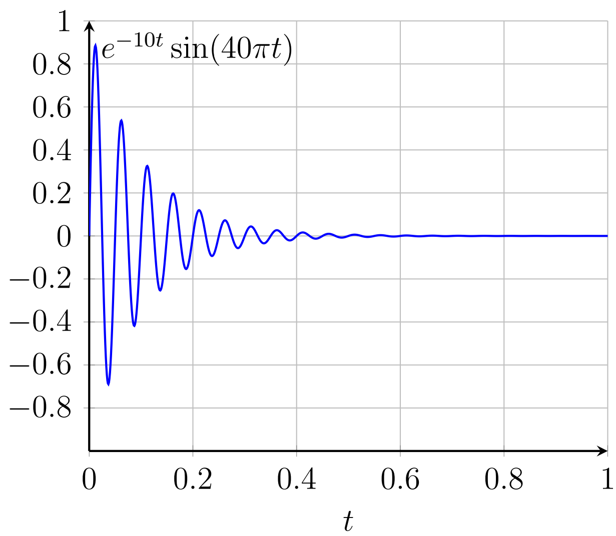

Figure 1.4: Plot of the signal $e^{-10t}\sin(40\pi t)$.

1.4 Topic 4: What are Signals?

It is difficult to find a unique definition of a signal. However in the context of this course, we give a workable definition which suits most of our purposes as:

A signal conveys information about a physical phenomenon which evolves in time or space.

Figure 1.5: Definition of a Signal

Examples of such signals include: Voltage, current, speech, television, images from remote space probes, voltages generated by the heart and brain, radar and sonar echoes, seismic vibrations, signals from GPS satellites, signals from human genes, and countless other applications.

1.4.1 Energy & Power Signals

Energy Signals

A signal is said to be an *energy signal* if and only if it has finite energy.

Figure 1.6: Energy Signals

Power Signals

A signal is said to be a *power signal* if and only if the average power of the signal is finite and non-zero.

Figure 1.7: Power Signals

The instantaneous power $p(t)$ of a signal $x(t)$ is expressed as:

$$p(t) = x^2(t)$$

The total energy of a continuous-time signal $x(t)$ is given by:

$$E = \lim_{T\to\infty} \int_{-T/2}^{T/2} x^2(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} x^2(t)\,\mathrm{d}t$$

For a complex valued signal:

$$E = \int_{-\infty}^{\infty} \left|x(t)\right|^2\,\mathrm{d}t$$

Since power equals to the time average of the energy, the average power is given by:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\,\mathrm{d}t = \frac{E}{T}$$

Note that during calculation of energy, we average the power over an infinitely large interval.

A signal with finite energy has zero power and a signal with finite power has infinite energy.

Figure 1.8: Relationship between Energy and Power Signals

a. A signal can not both be an energy and a power signal. This classification of signals based on power and energy are mutually exclusive.

b. However, a signal can belong to neither of the above two categories.

c. The signals which are both deterministic and non-periodic have finite energy and therefore are energy signals. Most of the signals, in practice, belong to this category.

d. Periodic signals and random signals are essentially power signals.

e. Periodic signals for which the area under $\left|x(t)\right|^2$ over one period is finite are power signals.

Figure 1.9: Further Notes on Energy and Power Signals

1.4.2 Examples

Example 1: Unit Step Function

Consider a unit step function defined as:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Determine whether this is an energy signal or a power signal or neither.
Solution: Let us compute the energy of this signal as:

$$E = \int_{-\infty}^{\infty} [u(t)]^2 \, \mathrm{d}t = \int_{0}^{\infty} [0]^2 \, \mathrm{d}t = \int_{0}^{\infty} [1]^2 \, \mathrm{d}t = \infty$$

Since the energy of this signal is infinite, it cannot be an energy signal. Let us compute the power of this signal as:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [u(t)]^2 \, \mathrm{d}t = \lim_{T\to\infty} \frac{1}{T} \int_{0}^{T/2} [u(t)]^2 \, \mathrm{d}t = \frac{1}{2}$$

The power of this signal is finite. Hence, this is a power signal.

Example 2: Exponential Function

Consider an exponential function defined as:

$$x(t) = e^{-at}u(t), \text{ where } u(t) \text{ is the unit step signal, } a > 0$$

Classify this signal as an energy, power, or neither.
Solution: Let us compute the energy of this signal as:

$$E = \int_{-\infty}^{\infty} [x(t)]^2 \, \mathrm{d}t = \int_{0}^{\infty} [e^{-at}]^2 \, \mathrm{d}t = \int_{0}^{\infty} e^{-2at} \, \mathrm{d}t = \frac{1}{2a} < \infty$$

Thus, $x(t) = e^{-at}u(t)$ is an energy signal.

Example 3: Ramp Function

Consider a ramp function defined as:

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Classify this signal as an energy, power, or neither.
Solution: Let us compute the energy of this signal as:

$$E = \int_{-\infty}^{\infty} r(t)^2 \, \mathrm{d}t = \int_{-\infty}^{0} [0]^2 \, \mathrm{d}t = \int_{0}^{\infty} A^2 t^2 \, \mathrm{d}t = A^2 \frac{T^3}{3} \bigg|_0^{\infty} = \infty$$

Since the energy of this signal is infinite, it cannot be an energy signal. Let us compute the power of this signal as:

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} [r(t)]^2 \, \mathrm{d}t = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T/2} A^2 t^2 \, \mathrm{d}t = A^2 \lim_{T \to \infty} \frac{1}{T} \frac{T^3}{3} \bigg|_0^{\infty} = \infty$$

The power of this signal is infinite. Hence, this is neither a power nor an energy signal.

1.5 Topic 5: What are Systems?

The term *system* is derived from the Greek word *systema*, which means an organised relationship among functioning units or components. It is often used to describe any orderly arrangement of ideas or constructs.
According to the Webster's Dictionary,

> *"A system is an aggregation or assemblage of objects united by some form of regular interaction or interdependence; a group of diverse units so combined by nature or art as to form an integral; whole and to function, operate, or move in unison and often in obedience to some form of control..."*

Figure 1.10: Dictionary Definition of a System

According to the International Council on Systems Engineering (INCOSE),

> *A system is an arrangement of parts or elements that together exhibit behaviour or meaning that the individual constituents do not.*
> The elements or parts, can include people, hardware, software, facilities, policies, and documents; that is, all things required to produce system-level results.

Figure 1.11: Definition of a System

It is difficult to give a single and precise definition of the term *system*, which will suit to different perspectives of different people. In practice, what is meant by "the system" depends on the objectives of a particular study.
From the control engineering perspective, the system is any interconnection of components to achieve desired objectives. It is characterised by its **inputs**, **outputs**, and the rules of operations or laws. For example:

a. The laws of operation in electrical systems are Ohm's law, which gives the voltage-current relationships for resistors, capacitors and inductors, and Kirchhoff's laws, which govern the laws of interconnection of various electrical components.

b. Similarly, in mechanical systems, the laws of operation are Newton's laws. These laws can be used to derive mathematical models of the system.

1.5.1 System as an Operator

A system is defined mathematically as a transformation which maps an input signal $x(t)$ to an output signal $y(t)$ as shown in Figure: 1.13. For a continuous time system, the input-output mapping is expressed as:

$$y(t) = \mathcal{S}[x(t)], \quad \text{where } \mathcal{S} \text{ is an operator.}$$

Figure 1.12: System as an Operator Definition

$$x(t) \xrightarrow{\text{Input}} \boxed{\mathcal{S}} \xrightarrow{\text{Output}} y(t) = \mathcal{S}[x(t)]$$

Figure 1.13: System as an Operator

A control system may be defined as an interconnection of components which are configured to provide a desired response.

Figure 1.14: System Interconnection Definition

1.5.2 Classification of Systems

The basis of classifying systems are many. They can be classified according to the following:

a. **The Time Frame:** (*discrete, continuous or hybrid*);

b. **System Complexity:** (*physical, conceptual and esoteric*);

c. **Uncertainties:** (*deterministic and stochastic*);

d. **Nature and type of components:** (*static or dynamic, linear or nonlinear, time-invariant or time variant, lumped or distributed etc*);

- Linear and nonlinear systems;
- Time-invariant and time-variant systems;
- Static (memory less) and dynamic (with memory) systems;
- Causal and Non-causal systems;
- Lumped and distributed parameter systems;
- Deterministic and stochastic systems;
- Continuous and discrete systems;

Figure 1.15: System Classification Types

1.5.3 Linear and Nonlinear Systems

A system is said to be linear provided it satisfies the *principle of superposition* which is the combination of the *additive* and *homogeneity* properties. Otherwise, it is *non-linear*

Figure 1.16: Linear & Non-linear System Definition

Additive Property: Assume the system initially at rest.

- Suppose an input $x_1(t)$ to this system produces an output $y_1(t)$ and an input $x_2(t)$ produces an output $y_2(t)$.

- If the system is linear, then the application of the input $x_1(t) + x_2(t)$ will produce an output $y_1(t) + y_2(t)$. Thus if

$$x_1(t) \to y_1(t) \quad \text{and} \quad x_2(t) \to y_2(t), \quad \text{then} \quad x_1(t) + x_2(t) \to y_1(t) + y_2(t)$$

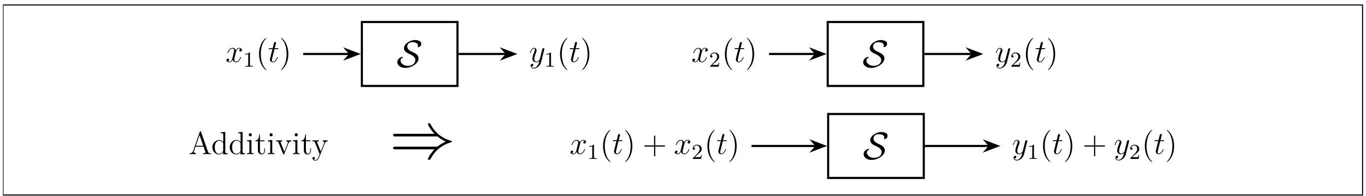

Figure 1.17: Additive Property

Principle of Homogeneity or Scaling:

- Let an input *(cause)* $x(t)$ produce an output *(effect)* $y(t)$. If the system is linear then,

- Scaling the input (*cause*) $x(t)$ by a factor "a" will scale the output (*effect*) $y(t)$ by the same factor.

- Thus, if the input $x(t)$ results in $y(t)$, then the scaled input $ax(t)$ gives the output $ay(t)$ where "a" can either be a real or imaginary number. Thus, for a linear system:

$$x(t) \to y(t) \quad \Rightarrow ax(t) \to ay(t)$$

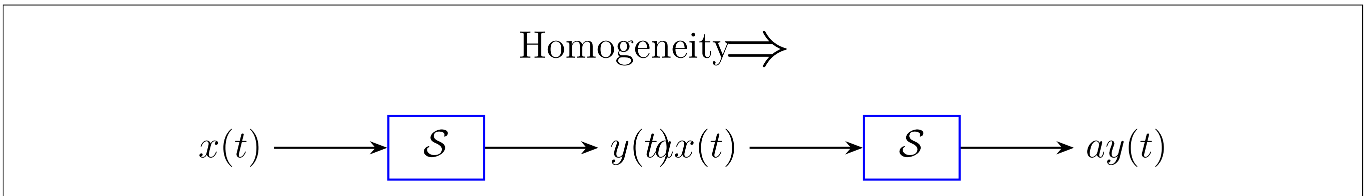

Figure 1.18: Homogeneity Property

Module 2:
Mathematical Modelling of Dynamic Systems

Module 3:
Block Diagrams & Feedback Systems Overview

Module 4:
Time Domain Analysis of Linear Systems

Module 5:
Stability Analysis of Linear Systems