



**ENGINEERING**  
DEPARTMENT OF ELECTRICAL,  
COMPUTER, AND SOFTWARE ENGINEERING

# ELECTENG 332

Notes on Control Systems

*Dear god help me, not another one...*

by

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August 10th, 2024

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# Module 1:

## Basics of Signals and Systems

### Learning Outcomes

- ▶ Uniqueness of the Exponential Signal
- ▶ Concept of Engineering Infinity
- ▶ Concept of Complex Frequency
- ▶ Classification of Signals: Energy & Power
- ▶ Classification of Systems
- ▶ What is a Control System
- ▶ Classification of a Control System: Open-loop & Closed-loop

## 1.1 Topic 1: Importance of Exponential Functions

The Exponential function, written as either  $e^{ax}$  or  $e^{at}$  depending on whether it is  $f(t)$  or  $f(x)$ , has properties that make it mathematically unique.

1. The derivative (rate of change) of the exponential function is the exponential function itself. More generally, this is a function whose rate of change is proportional to the function itself.

$$\frac{de^{ax}}{dx} = ae^{ax} \quad (1)$$

2. The integral of the exponential function is also the exponential function itself.

$$\int e^{ax} dx = \frac{1}{a}e^{ax} \quad (2)$$

## 1.2 Topic 2: Concept of Engineering Infinity

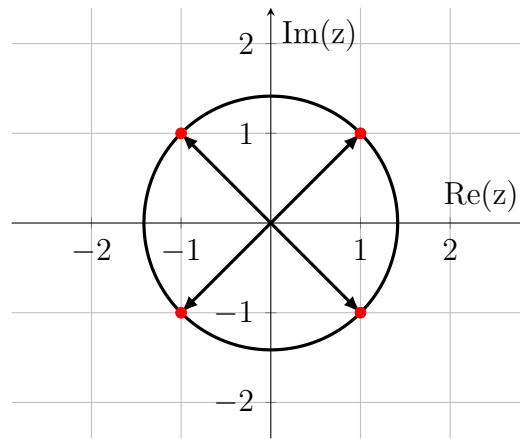
Consider a signal  $e^{-at}$ . The time constant for this signal is  $T = \frac{1}{a}$ . Theoretically, the signal is meant to decay to zero as time approaches infinity, i.e.

$$\lim_{t \rightarrow \infty} e^{-at} = 0 \quad (3)$$

But in practice, this is not the case, as its value will be very, very small after five time constants  $5T$ . This is the *Concept of Engineering Infinity*.

## 1.3 Topic 3: Concept of Complex Frequency

Complex frequency is found commonly in electrical engineering. It is often notated as  $j\omega$  or  $s = \sigma \pm j\omega$ . These frequencies always come in pairs, so the use of  $\pm$  is implicit to this, as complex numbers have complex conjugates, i.e.  $s = \sigma + j\omega$  has the conjugate  $s = \sigma - j\omega$ .

Figure 1.1: Plot of the circle  $|z| = \sqrt{2}$ 

This is also backed up by De Moivre's Formula, which is defined mathematically as:

$$\forall x \in \mathbb{R}, \quad \forall n \in \mathbb{Z}, \quad (4)$$

$$e^{jnx} = \cos(nx) + j \sin(nx) \quad (5)$$

Or more generally for our applications:

$$e^{jx} = \cos(x) + j \sin(x) \quad (6)$$

$$\text{Where } x \in \mathbb{R} \text{ (} x \text{ is Real)} \quad (7)$$

$$\text{and } j \equiv i = \sqrt{-1}. \quad (8)$$

This means that:

*A complex frequency  $j\omega$  represents a pure sinusoidal signal of frequency  $\omega$  rad/s*

Figure 1.2: Complex Sinusoidal Signal

For example, if a signal has a complex frequency  $j314$  rad/sec, then this corresponds to a pure sinusoid of frequency 314 rad/sec (i.e. 50 Hz). Furthermore:

*A complex frequency  $s = \sigma + j\omega$  represents an exponentially damped signal of frequency  $\omega$  rad/s, and decays/amplifies at a rate decided by  $\sigma$ .*

Figure 1.3: Exponentially Damped Signal

For example, the signal  $f(t) = e^{-10t} \sin(40\pi t)$  would look like this:

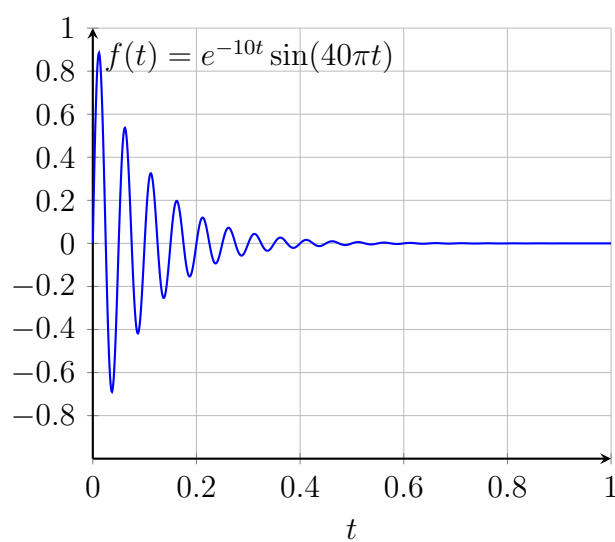


Figure 1.4: Plot of the signal  $f(t) = e^{-10t} \sin(40\pi t)$ .

## 1.4 Topic 4: What are Signals?

It is difficult to find a unique definition of a signal. However in the context of this course, we give a workable definition which suits most of our purposes as:

*A signal conveys information about a physical phenomenon which evolves in time or space.*

Figure 1.5: Definition of a Signal

Examples of such signals include: Voltage, current, speech, television, images from remote space probes, voltages generated by the heart and brain, radar and sonar echoes, seismic vibrations, signals from GPS satellites, signals from human genes, and countless other applications.

### 1.4.1 Energy & Power Signals

#### Energy Signals

A signal is said to be an *energy signal* if and only if it has finite energy.

Figure 1.6: Energy Signals

#### Power Signals

A signal is said to be a *power signal* if and only if the average power of the signal is finite and non-zero.

Figure 1.7: Power Signals

The instantaneous power  $p(t)$  of a signal  $x(t)$  is expressed as:

$$p(t) = x^2(t) \quad (9)$$

The total energy of a continuous-time signal  $x(t)$  is given by:

$$E = \lim_{T \rightarrow \infty} \int_{-T/2}^{T/2} x^2(t) dt = \int_{-\infty}^{\infty} x^2(t) dt \quad (10)$$

For a complex valued signal:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad (11)$$

Since power equals to the time average of the energy, the average power is given by:

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) dt = \frac{E}{T} \quad (12)$$

Note that during calculation of energy, we average the power over an infinitely large interval.

A signal with finite energy has zero power and a signal with finite power has infinite energy.

Figure 1.8: Relationship between Energy and Power Signals

- a. A signal **can not both be an energy and a power signal**. This classification of signals based on power and energy are **mutually exclusive**.
- b. However, **a signal can belong to neither of the above two categories**.
- c. The signals which are both deterministic and non-periodic have finite energy and therefore are energy signals. **Most of the signals, in practice, belong to this category**.
- d. **Periodic signals and random signals** are essentially **power signals**.
- e. Periodic signals for which the area under  $|x(t)|^2$  over one period is finite are power signals.

Figure 1.9: Further Notes on Energy and Power Signals

## 1.4.2 Examples

### Example 1: Unit Step Function

Consider a unit step function defined as:

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

Determine whether this is an energy signal or a power signal or neither.

**Solution:** Let us compute the energy of this signal as:

$$E = \int_{-\infty}^{\infty} [u(t)]^2 dt = \int_0^{\infty} [0]^2 dt = \int_0^{\infty} [1]^2 dt = \infty \quad (14)$$

Since the energy of this signal is infinite, it cannot be an energy signal. Let us compute the power of this signal as:

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} [u(t)]^2 dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{T/2} [u(t)]^2 dt = \frac{1}{2} \quad (15)$$

The power of this signal is finite. Hence, **this is a power signal**.

### Example 2: Exponential Function

Consider an exponential function defined as:

$$x(t) = e^{-at}u(t), \text{ where } u(t) \text{ is the unit step signal, } a > 0 \quad (16)$$



Classify this signal as an energy, power, or neither.

**Solution:** Let us compute the energy of this signal as:

$$E = \int_{-\infty}^{\infty} [x(t)]^2 dt = \int_0^{\infty} [e^{-at}]^2 dt = \int_0^{\infty} e^{-2at} dt = \frac{1}{2a} < \infty \quad (17)$$

Thus,  $x(t) = e^{-at}u(t)$  is an **energy signal**.

### Example 3: Ramp Function

Consider a ramp function defined as:

$$r(t) = \begin{cases} At & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (18)$$

Classify this signal as an energy, power, or neither.

**Solution:** Let us compute the energy of this signal as:

$$E = \int_{-\infty}^{\infty} r(t)^2 dt = \int_{-\infty}^0 [0]^2 dt = \int_0^{\infty} A^2 t^2 dt = A^2 \frac{T^3}{3} \Big|_0^{\infty} = \infty \quad (19)$$

Since the energy of this signal is infinite, it cannot be an energy signal. Let us compute the power of this signal as:

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} [r(t)]^2 dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^{T/2} A^2 t^2 dt = A^2 \lim_{T \rightarrow \infty} \frac{1}{T} \frac{T^3}{3} \Big|_0^{\infty} = \infty \quad (20)$$

The power of this signal is infinite. Hence, this is **neither a power nor an energy signal**.

## 1.5 Topic 5: What are Systems?

The term *system* is derived from the Greek word *systema*, which means an organised relationship among functioning units or components. It is often used to describe any orderly arrangement of ideas or constructs.

According to the Webster's Dictionary,

*“A system is an aggregation or assemblage of objects united by some form of regular interaction or interdependence; a group of diverse units so combined by nature or art as to form an integral; whole and to function, operate, or move in unison and often in obedience to some form of control...”*

Figure 1.10: Dictionary Definition of a System

According to the International Council on Systems Engineering (INCOSE),

*A system is an arrangement of parts or elements that together exhibit behaviour or meaning that the individual constituents do not.*

The elements or parts, can include people, hardware, software, facilities, policies, and documents; that is, all things required to produce system-level results.

Figure 1.11: Definition of a System

It is difficult to give a single and precise definition of the term *system*, which will suit to different perspectives of different people. In practice, what is meant by “the system” depends on the objectives of a particular study.

From the control engineering perspective, **the system is any interconnection of components to achieve desired objectives**. It is characterised by its **inputs**, **outputs**, and the rules of operations or laws. For example:

- a. The laws of operation in electrical systems are Ohm's law, which gives the voltage-current relationships for resistors, capacitors and inductors, and Kirchhoff's laws, which govern the laws of interconnection of various electrical components.
- b. Similarly, in mechanical systems, the laws of operation are Newton's laws. These laws can be used to derive mathematical models of the system.

### 1.5.1 System as an Operator

A system is defined mathematically as a transformation which maps an input signal  $x(t)$  to an output signal  $y(t)$  as shown in Figure: 1.13. For a continuous time system, the input-output mapping is expressed as:

$$y(t) = \mathcal{S}[x(t)], \quad \text{where } \mathcal{S} \text{ is an operator.} \quad (21)$$

Figure 1.12: System as an Operator Definition



Figure 1.13: System as an Operator

A control system may be defined as an interconnection of components which are configured to provide a desired response.

Figure 1.14: System Interconnection Definition

### 1.5.2 Classification of Systems

The basis of classifying systems are many. They can be classified according to the following:

- a. **The Time Frame:** (*discrete, continuous or hybrid*);
- b. **System Complexity:** (*physical, conceptual and esoteric*);
- c. **Uncertainties:** (*deterministic and stochastic*);
- d. **Nature and type of components:** (*static or dynamic, linear or nonlinear, time-invariant or time variant, lumped or distributed etc*);

- Linear and nonlinear systems;
- Time-invariant and time-variant systems;
- Static (memory less) and dynamic (with memory) systems;
- Causal and Non-causal systems;
- Lumped and distributed parameter systems;
- Deterministic and stochastic systems;
- Continuous and discrete systems;

Figure 1.15: System Classification Types

### 1.5.3 Linear and Nonlinear Systems

A system is said to be linear provided it satisfies the *principle of superposition* which is the combination of the *additive* and *homogeneity* properties. Otherwise, it is *non-linear*

Figure 1.16: Linear & Non-linear System Definition

#### Principle of Additivity:

Assume the system initially at rest.

- Suppose an input  $x_1(t)$  to this system produces an output  $y_1(t)$  and an input  $x_2(t)$  produces an output  $y_2(t)$ .
- If the system is linear, then the application of the input  $x_1(t) + x_2(t)$  will produce an output  $y_1(t) + y_2(t)$ . Thus if

$$x_1(t) \rightarrow y_1(t) \quad \text{and} \quad x_2(t) \rightarrow y_2(t), \quad \text{then} \quad x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t) \quad (22)$$

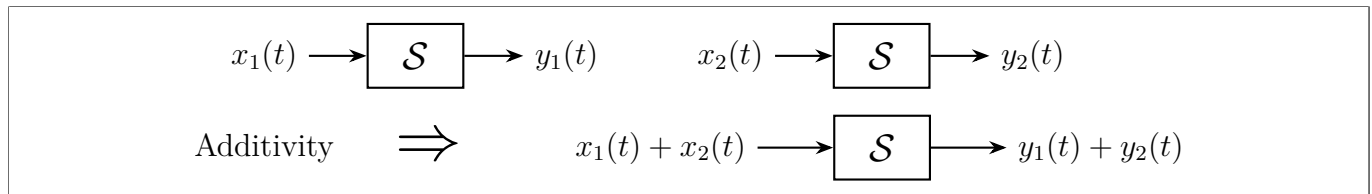


Figure 1.17: Additive Property

#### Principle of Homogeneity or Scaling:

- Let an input (*cause*)  $x(t)$  produce an output (*effect*)  $y(t)$ . If the system is linear then,
- Scaling the input (*cause*)  $x(t)$  by a factor “ $a$ ” will scale the output (*effect*)  $y(t)$  by the same factor.
- Thus, if the input  $x(t)$  results in  $y(t)$ , then the scaled input  $ax(t)$  gives the output  $ay(t)$  where “ $a$ ” can either be a real or imaginary number. Thus, for a linear system:

$$x(t) \rightarrow y(t) \quad \Rightarrow \quad ax(t) \rightarrow ay(t) \quad (23)$$



Figure 1.18: Homogeneity Property

**Principle of Superposition:**

## Module 2:

# Mathematical Modelling of Dynamic Systems

## Module 3:

# Block Diagrams & Feedback Systems Overview

## Module 4:

# Time Domain Analysis of Linear Systems



## Module 5:

# Stability Analysis of Linear Systems