Robust Data-Driven Safe Control Using Density Functions

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***Abstract*—This letter presents a tractable framework for data-driven synthesis of robustly safe control laws. Given noisy experimental data and some priors about the structure of the system, the goal is to synthesize a state feedback law such that the trajectories of the closed loop system are guaranteed to avoid an unsafe set even in the presence of unknown but bounded disturbances (process noise). The main result of this letter shows that for polynomial dynamics, this problem can be reduced to a tractable convex optimization by combining elements from polynomial optimization and the theorem of alternatives. This optimization provides both a rational control law and a density function safety certificate. These results are illustrated with numerical examples.**

***Index Terms*—Data-driven control, safety, sum-of-squares, robust control.**

I. Introduction

THE GOAL of this letter is to develop a tractable framework for data-driven synthesis of safe control laws that are robust to unmeasurable, polytopic-bounded perturbations during both data collection and execution. Specifically, given experimental data generated by an unknown system and some priors about its structure, the objective is to synthesize a state feedback control law such that the trajectories of the closed loop system starting in a given initial condition set $\mathcal{X}_0$ are guaranteed to avoid an unsafe set $\mathcal{X}_u$, even in the presence of unknown but bounded disturbances. Our main result shows that, for polynomial dynamics, the safe Data Driven Control (DDC) problem can be posed as the feasibility of a Sum of Squares (SOS) program. A substantial reduction in the number of variables involved (and hence computational complexity) is achieved by exploiting the theorem of alternatives, leading to a Semidefinite Program (SDP) that provides both a density-function based control law and a robust safety certificate.

Safety verification and synthesis of safe control laws have been the subject of intense research during the past decade. Level-set methods separate the initial and unsafe set by the 0-contour of a solved function. Barrier functions [1] are a level-set method to certify the safety of trajectories, given that the superlevel sets of the barrier function are invariant. This superlevel invariance can be relaxed through slack (class-$\mathcal{K}$) conditions, while ensuring that the 0-level set is invariant [2], [3]. The level-set certificate of stability may be solved jointly with a safety-guaranteeing control policy $u(\cdot)$ (Control Barrier Function (CBF)). When a barrier function is given, the min-norm controller will ensure safety of trajectories, and can be found through quadratic programming [4]. Robustness of given barrier functions to disturbances may be analyzed using input-to-state stability [5]. Barrier functions and funnels [6], [7], [8] contain bilinearities when jointly synthesizing controllers and barriers. An alternative level-set certificate is Density functions [9], which are based on Dual Lyapunov methods for stability [10]. Controllers and density functions can be simultaneously solved in a convex manner. In some systems, density functions may exist and provide improved performance as compared to barrier functions [11].

We briefly compare against other methods of safety-constrained control. Interval analyses, such as Mixed Monotonicity [12], offer real-time performance at the expense of conservatism in safe generation. Hamilton-Jacobi reachability [13] performs forward and backward reachable set analysis based on level sets of a differential games' value function, whose computation could require solving PDEs or neural net approximations. Reinforcement Learning necessitates training and prior information of safety properties (e.g., Lipschitz bounds on dynamics), and does not generally exploit physical principles and model structure [14]. Learning-based methods in [15], [16] require Lipschitz bounds on error and an ϵ-net discretization.

DDC is a methodology that synthesizes control laws directly from acquired system observations (with some priors) and skips a system-identification/robust-synthesis pipeline [17]. Amongst the vast literature in DDC, the closest approaches related to the present paper are those that pursue a set membership approach, which seeks to find a controller that

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stabilizes the set of all plants compatible with the observed data (the consistency set) [18], [19], [20], [21], [22], [23], [24]. These approaches provide a controller together with a stability certificate, usually in the form of a common Lyapunov function. Further, the methods can be extended to provide worst case performance bounds (e.g., the H_2, H_∞ or L_∞ sense), over the set of data-consistent plants. However, these approaches cannot handle safety constraints beyond those expressed in terms of these norms.

Recent work on DDC under safety constraints includes [25], [26], [27], [28]. The method in [25] performs iterative model predictive control for a discrete-time system by constraining state trajectories to always lie in a sampled safe set (using integer programming). The work in [26] uses contraction methods to form robust adaptive CBFs under a set membership approach, but assumes that the input relation $g(\cdot)$ is known. The approach in [27] uses a disturbance observer to provide robust CBFs by separating known and unknown dynamics. In our setting, we assume only prior knowledge of the system model (polynomial up to a specified degree) and cannot generally provide this separation. The work in [28] uses polynomial matrix inequalities to enforce Nagumo invariance certificates [29] for polynomial systems using data corrupted by L_2-bounded noise. However, the controller is designed to only enforce nominal invariance and a degree of robustness is achieved through a tuning parameter ϵ, not directly related to the perturbation $\boldsymbol{w}$. Further, the computational scaling of the Positive Semidefinite (PSD) matrices in the matrix SOS constraints suffers as the degree increases as compared to scalar SOS constraints.

Our work involves continuous-time dynamics and interpretable (density) certificates of robust safety. To the best of our knowledge, our approach is the first DDC method under safety constraints that simultaneously explicitly considers disturbances both during the data-collection and run-time execution.

Contributions of this letter are,

- A DDC framework for density-based robust safe control.
- Tractable synthesis of robustly safe density functions by exploiting the theorem of alternatives.
- Numerical examples demonstrating robustly safe control on polynomial systems.

This letter has the following structure: Section II reviews preliminaries such as density functions for safety, and SOS polynomials. Section III performs data-driven synthesis of safe controllers using density functions and SOS methods in the case where polytopic-bounded disturbances are present both during data collection and run time execution. Section IV demonstrates the effectiveness of our approach on several example systems. Section V concludes this letter.

II. Preliminaries

A. Notation

$\mathbb{R}^n$	Set of n-tuples of real numbers
$x, \boldsymbol{x}, \boldsymbol{X}$	Scalar, vector, matrix
$\mathbf{1}, \mathbf{0}, \boldsymbol{I}$	Vector/matrix of all 1s, 0s, identity matrix
$\|\boldsymbol{x}\|_\infty$	L_∞-norm of vector $\boldsymbol{x}$
$\boldsymbol{X} \succeq 0$	$\boldsymbol{X}$ is positive semi-definite
$\otimes$	Kronecker product
$\text{vec}(\boldsymbol{X})$	Vectorized matrix along columns: $\text{vec}(\boldsymbol{X}) = [\boldsymbol{X}(:,1)^T, \ldots, \boldsymbol{X}(:,n)^T]^T$
$\rho \in C^d$	ρ has a continuous d^{th} derivative
$\nabla\rho$	Gradient of scalar function ρ
$\nabla \cdot f$	Divergence of vector function f

B. Sum-of-Squares

We briefly review the concept of SOS polynomials and certificates of nonnegativity [30]. A polynomial $p \in \mathbb{R}[\boldsymbol{x}]$ is SOS (and hence nonnegative) if there exist polynomials $\{q_\ell \in \mathbb{R}[\boldsymbol{x}]\}_{\ell=1}^L$ such that $p(\boldsymbol{x}) = \sum_{\ell=1}^L q_\ell(\boldsymbol{x})^2$.

The cone of SOS polynomials is $\Sigma[\boldsymbol{x}]$, and its up-to-degree $2d$ restriction is $\Sigma_d[\boldsymbol{x}]$. The cone $\Sigma_d[\boldsymbol{x}]$ is semidefinite representable as $p(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x})^T \boldsymbol{Q} \boldsymbol{v}(\boldsymbol{x})$ where $\boldsymbol{v}(\boldsymbol{x})$ is the monomial vector up to degree d and $\boldsymbol{Q} \succeq 0$ is the Gram matrix. A sufficient condition for a polynomial p to be nonnegative over the semialgebraic region $\{\boldsymbol{x} \mid h_i(\boldsymbol{x}) \geq 0,\ i = 1 \ldots N_c\}$ is that there exists $\sigma_0, \ldots, \sigma_{N_c} \in \Sigma[\boldsymbol{x}]$ such that $p(\boldsymbol{x}) = \sigma_0 + \sum_{i=1}^{N_c} \sigma_i h_i$ [31].

C. Level-Set-Based Safety Certification

Consider a continuous-time system of the form

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{w}) \tag{1}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ is the state and $\boldsymbol{w}(\cdot) \in \mathcal{W}$ is a disturbance. Further, assume that $\boldsymbol{w}(\cdot)$ is such that the trajectories of (1) are well defined for any initial condition $\boldsymbol{x}_0 \in \mathcal{X}_0$. In the sequel, we will denote these trajectories as $\boldsymbol{x}(t, \boldsymbol{w}, \boldsymbol{x}_0)$.

Definition 1: Given an initial condition set $\mathcal{X}_0 \subseteq \mathbb{R}^n$ and an unsafe set $\mathcal{X}_u \subseteq \mathbb{R}^n$, system (1) is $\mathcal{W}$-robustly safe if, for all t, all initial conditions $\boldsymbol{x}_0 \in \mathcal{X}_0$ and all $\boldsymbol{w}(\cdot) \in \mathcal{W}$, $\boldsymbol{x}(t, \boldsymbol{w}, \boldsymbol{x}_0) \notin \mathcal{X}_u$.

Typically, safety is certified through the use of barrier functions, defined as:

Definition 2: A differentiable $B(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ is a robust barrier function for (1) with respect to $\mathcal{X}_0$ and $\mathcal{X}_u$ if

$$B(\boldsymbol{x}) \leq 0,\ \forall \boldsymbol{x} \in \mathcal{X}_0,\ B(\boldsymbol{x}) > 0,\ \forall \boldsymbol{x} \in \mathcal{X}_u \tag{2}$$

$$\frac{\partial B}{\partial \boldsymbol{x}} f(\boldsymbol{x}, \boldsymbol{w}) < 0,\ \forall \boldsymbol{w} \in \mathcal{W} \quad \text{whenever } B(\boldsymbol{x}) = 0. \tag{3}$$

As shown for instance in [1], existence of a barrier function is a sufficient condition to certify safety. Note however that the conditions above are non-convex, even when $\boldsymbol{w} \equiv 0$, due to the constraint (3). For instance, in the case of polynomial dynamics and semialgebraic $\mathcal{X}_0$ and $\mathcal{X}_u$, if $B(\boldsymbol{x})$ is also polynomial, this constraint can be enforced by introducing a polynomial multiplier $h(\boldsymbol{x})$ and imposing that

$$-\frac{\partial B}{\partial \boldsymbol{x}} f(\boldsymbol{x}, \boldsymbol{w}) + h(\boldsymbol{x}) B(\boldsymbol{x}) \in \Sigma[\boldsymbol{x}] \tag{4}$$

The condition above cannot be written as a single semi-definite optimization due to the multiplication of the coefficients of the two unknown polynomials, h and B. Possible relaxations include choosing a fixed multiplier h, or simply dropping the $B(\boldsymbol{x}) = 0$ quantifier [2]. An alternative, convex approach based on the use of densities was proposed in [9].

Theorem 1 [9]: Given open sets $\mathcal{X}_0$ and $\mathcal{X}_u$, $\dot{x} = f(x)$ is safe if there exists a scalar function $\rho(\boldsymbol{x}) \in C^1$ such that

$$\nabla \cdot [\rho(\boldsymbol{x}) f(\boldsymbol{x})] > 0, \ \ \forall \boldsymbol{x} \in \mathbb{R}^n \tag{5a}$$

$$\rho(\boldsymbol{x}) > 0, \ \ \forall \boldsymbol{x} \in \mathcal{X}_0, \ \ \rho(\boldsymbol{x}) \leq 0, \ \ \forall \boldsymbol{x} \in \mathcal{X}_u \tag{5b}$$

The advantage of this approach is that it leads to a convex problem in ρ. On the other hand, imposing that the divergence condition holds everywhere can be unnecessarily conservative.

The concepts above can be easily extended to the case where the goal is to synthesize a control action that keeps a system robustly safe by introducing the concept of robust CBFs (RCBFs).

Definition 3: A function $B(\boldsymbol{x})$ is an RCBF for the system $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, u, \boldsymbol{w})$ if there exists a control law $u(\boldsymbol{x})$ such that $B(\boldsymbol{x})$ is a robust barrier function for the closed loop dynamics $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, u(\boldsymbol{x}), \boldsymbol{w})$.

In principle, a CBF and associated control law can be found by modifying (4) to

$$-\frac{\partial B}{\partial \boldsymbol{x}} f(\boldsymbol{x}, u(\boldsymbol{x}), \boldsymbol{w}) + h(\boldsymbol{x}) B(\boldsymbol{x}) \in \Sigma[\boldsymbol{x}] \tag{6}$$

Problem (6) is bilinear in the coefficients of B, u even when restricted to polynomial dynamics and control laws and a fixed multiplier h, necessitating the use of relaxations. On the other hand, as shown in [9], the density based formulation can be easily modified to lead to problems that are jointly convex in ρ and $\psi \doteq \rho u$.

III. Data-Driven Safe Control

A. Problem Statement

The goal of this letter is to design a safe control law based on (noisy) experimental measurements for unknown polynomial systems where only minimal a-priori information is available. Specifically, we consider single-input control affine nonlinear systems of the form

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}) + g(\boldsymbol{x}) u(t) + \boldsymbol{w}(t) \tag{7}$$

where $u \in \mathbb{R}$ is the control and the input $\boldsymbol{w}$ satisfying $\forall t \geq 0$: $\boldsymbol{w}(\cdot) \in \mathcal{W}$ represents a bounded disturbance. We further assume that there exists a set W such that $\mathcal{W}$ is the class of signals that can switch arbitrarily quickly within W, and that W admits a polytopic description of the form $W \doteq \{\boldsymbol{w} \colon \boldsymbol{W}\boldsymbol{w} \leq \boldsymbol{d}_{\boldsymbol{w}}\}$. The only information available about the ground-truth dynamics (7) is that they can be expressed as linear combinations of functions $\phi : \mathbb{R}^n \to \mathbb{R}^{d_f}$, $\gamma : \mathbb{R}^n \to \mathbb{R}^{d_g}$ with

$$f(\boldsymbol{x}) = \boldsymbol{F}\boldsymbol{\phi}(\boldsymbol{x}); \ \ g(\boldsymbol{x}) = \boldsymbol{G}\boldsymbol{\gamma}(\boldsymbol{x}) \tag{8}$$

for some unknown system parameter matrices $\boldsymbol{F} \in \mathbb{R}^{n \times d_f}$ and $\boldsymbol{G} \in \mathbb{R}^{n \times d_g}$. Our training data $\mathcal{D} = \{(\dot{\boldsymbol{x}}_s, \boldsymbol{x}_s, u_s)\}_{s=t_1 \ldots t_T}$ consist of T derivative-state-input tuples sampled from the trajectories of (7) under some bounded disturbance $w \in W$, indexed by the observations times $t_1 \ldots t_T$. In this context, the problem under consideration can be formally stated as:

Problem 1: Given a disturbance set description $(\boldsymbol{W}, \boldsymbol{d}_{\boldsymbol{w}})$, T training tuples $\mathcal{D} = \{(\dot{\boldsymbol{x}}_s, \boldsymbol{x}_s, u_s)\}_{s=t_1 \ldots t_T}$, and basic semialgebraic sets $\mathcal{X}_0$, $\mathcal{X}_u$, find a state-feedback control law $u(\boldsymbol{x})$ that renders all closed-loop systems consistent with the observed data and priors $\mathcal{W}$-robustly safe with respect to $\mathcal{X}_0$ and $\mathcal{X}_u$.

B. Model Based Safety

In order to solve Problem 1, in this section we first develop a convex condition, less conservative than (5), that guarantees robust controlled safety of a model of the form (7) assuming that $f(.)$ and $g(.)$ are known.

Lemma 1: Assume that the set $\mathcal{X}_u$ has a description:

$$\mathcal{X}_u \doteq \{\boldsymbol{x} \colon h_i(\boldsymbol{x}) \geq 0, \ \ i = 1 \ldots N_c\}.$$

If there exist scalar functions $\rho(\boldsymbol{x}), \psi(\boldsymbol{x}) \in C^1$ such that: (i) $u(\boldsymbol{x}) \doteq \frac{\psi(\boldsymbol{x})}{\rho(\boldsymbol{x})}$ is well defined over the safe region $\rho(\boldsymbol{x}) \geq 0$, (ii) for all $\boldsymbol{w}(\cdot) \in \mathcal{W}$ and initial condition $\boldsymbol{x}_0 \in \mathcal{X}_0$, the trajectories of (7) are well defined, and (iii) the following conditions hold:

$$\nabla \cdot [\rho(\boldsymbol{x})(f(\boldsymbol{x}) + \boldsymbol{w}) + \psi(\boldsymbol{x}) g(\boldsymbol{x})] - \rho(\boldsymbol{x}) h(\boldsymbol{x}) > 0 \tag{9a}$$
$$\forall \boldsymbol{x} \in \mathbb{R}^n \text{ and } \boldsymbol{w} \in W$$

$$\rho(\boldsymbol{x}) \geq 0, \ \ \forall \boldsymbol{x} \in \mathcal{X}_0, \ \ \rho(\boldsymbol{x}) < 0, \ \ \forall \boldsymbol{x} \in \mathcal{X}_u \tag{9b}$$

where $h \doteq \min_i\{h_i(\boldsymbol{x})\}$, then the control law $u(\boldsymbol{x})$ renders the closed loop system robustly safe with respect to $\mathcal{X}_u$.

Proof: Since by assumption $\rho, \psi \in C^1$ and u is well defined when $\rho \geq 0$ by condition (i), (9a) is equivalent to (omit $\boldsymbol{x}$):

$$\frac{\partial \rho}{\partial \boldsymbol{x}}(f + gu + \boldsymbol{w}) + \rho(\nabla \cdot (f + gu) - h) > 0 \tag{10}$$

where we used the fact that $\psi = \rho u$. Hence, for all $\boldsymbol{w} \in W$,

$$\frac{d\rho}{dt} + \rho(\nabla \cdot (f + gu) - h) > 0$$

along the closed loop trajectories, which implies that $\frac{d\rho}{dt} > 0$ when $\rho[\boldsymbol{x}(t)] = 0$. Assume that there exists a trajectory $\boldsymbol{x}(t \mid \boldsymbol{x}_0, \boldsymbol{w}_p(\cdot))$ that starts at $\boldsymbol{x}_0 \in \mathcal{X}_0$ and such that $\boldsymbol{x}(T \mid \boldsymbol{x}_0, \boldsymbol{w}_p(\cdot)) \in \mathcal{X}_u$. By continuity, there exists some $0 < t_1 < T$ and some dt such that $\rho(t_1) = 0$ and $\rho(t) < 0$ for all $t \in [t_1, t_1 + dt]$. However, this contradicts the fact that $\frac{d\rho}{dt}|_{t=t_1} > 0$. ∎

Remark 1: Since $\min_i\{h_i(\boldsymbol{x})\}$ has a semialgebraic representation by [32, Lemma 3], finding polynomial functions ρ and ψ reduces to SOS optimization via standard arguments.

Remark 2: Problem (9) is an infinite-dimensional Linear Program (LP) in the values of (ρ, ψ) at each $\boldsymbol{x}$, possessing both strict and non-strict inequality constraints. When compared against (6), this formulation has two advantages: (i) it avoids using an arbitrary, fixed multiplier $h(\boldsymbol{x})$, and (ii) it leads to jointly convex (in ρ and ψ) optimization problems for safe control synthesis. On the other hand, (9), while retaining the desirable convexity properties of (5), is less conservative: since the second term in (9a) is nonnegative over the safe region, it does not require the first term to be positive everywhere, as is the case with (5). Note that any feasible solution to (5) is also feasible for (9).

C. Safe Data Driven Control

This section presents the main result of this letter: a tractable, convex reformulation of Problem 1. We begin by presenting a tractable characterization of all systems that could have generated the observed data.

Given training data $\mathcal{D} = \{(\dot{\boldsymbol{x}}_s, \boldsymbol{x}_s, u_s)\}_{s=t_1 \ldots t_T}$ and the uncertainty description $(\boldsymbol{W}, \boldsymbol{d}_{\boldsymbol{w}})$, the consistency set $\mathcal{C}$, which

contains all systems that are consistent with the data is defined, under the restrictions $f(\boldsymbol{x}) = \boldsymbol{F}\boldsymbol{\phi}(\boldsymbol{x})$ and $g(\boldsymbol{x}) = \boldsymbol{G}\boldsymbol{\gamma}(\boldsymbol{x})$, as:

$$\mathcal{C} \doteq \left\{f, g\colon \boldsymbol{W}\left[\dot{\boldsymbol{x}}_s - f(\boldsymbol{x}_s) - g(\boldsymbol{x}_s)u_s\right] \leq \boldsymbol{d}_{\boldsymbol{w}},\, s = t_1 \ldots t_T\right\}. \quad (11)$$

Exploiting the property of the Kronecker product

$$\mathrm{vec}(\boldsymbol{PXQ}) = (\boldsymbol{Q}^T \otimes \boldsymbol{P})\mathrm{vec}(\boldsymbol{X}),$$

with $\boldsymbol{f} = \mathrm{vec}(\boldsymbol{F}^T)$, $\boldsymbol{g} = \mathrm{vec}(\boldsymbol{G}^T)$ leads to an equivalent representation of (11)

$$\mathcal{C} = \left\{\boldsymbol{f}, \boldsymbol{g}\colon \begin{bmatrix}\boldsymbol{A} & \boldsymbol{B}\end{bmatrix}\begin{bmatrix}\boldsymbol{f}\\ \boldsymbol{g}\end{bmatrix} \leq \boldsymbol{\xi} - \boldsymbol{1}\otimes \boldsymbol{d}_w\right\} \quad (12)$$

using the matrix blocks (with $f(\boldsymbol{x}_s) = \mathrm{vec}(\boldsymbol{\phi}(\boldsymbol{x}_s)^T\boldsymbol{F}^T)$)

$$\boldsymbol{A} \doteq \begin{bmatrix}\boldsymbol{W}\otimes\boldsymbol{\phi}^T(t_1)\\ \vdots \\ \boldsymbol{W}\otimes\boldsymbol{\phi}^T(t_T)\end{bmatrix}, \boldsymbol{B} \doteq \begin{bmatrix}\boldsymbol{W}\otimes u_{t_1}\boldsymbol{\gamma}^T(t_1)\\ \vdots \\ \boldsymbol{W}\otimes u_{t_T}\boldsymbol{\gamma}^T(t_T)\end{bmatrix}, \boldsymbol{\xi} \doteq \begin{bmatrix}\boldsymbol{W}\dot{\boldsymbol{x}}(t_1)\\ \vdots \\ \boldsymbol{W}\dot{\boldsymbol{x}}(t_T)\end{bmatrix} \quad (13)$$

Combining this description with the polytopic description of the disturbances leads to an augmented consistency set describing the set of all possible plants and disturbances:

$$\mathcal{P}_1 \doteq \left\{f, g, \boldsymbol{w}_p\colon \begin{bmatrix}\boldsymbol{A} & \boldsymbol{B} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{W}\end{bmatrix}\begin{bmatrix}\boldsymbol{f}\\ \boldsymbol{g}\\ \boldsymbol{w}\end{bmatrix} \leq \begin{bmatrix}\boldsymbol{\xi} - \boldsymbol{1}\otimes\boldsymbol{d}_w\\ \boldsymbol{d}_w\end{bmatrix}\right\} \quad (14)$$

It follows that a pair $(\rho(\boldsymbol{x}), \psi(\boldsymbol{x}))$ solves Problem 1 if

$$\nabla\cdot[\rho(\boldsymbol{x})f(\boldsymbol{x}) + \psi(\boldsymbol{x})g(\boldsymbol{x}) + \rho(\boldsymbol{x})\boldsymbol{w}] - \rho(\boldsymbol{x})h(\boldsymbol{x}) > 0 \quad (15)$$

holds for all $\boldsymbol{x}$ and all $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{w}) \in \mathcal{P}_1$. In principle, this condition can be reduced to an SOS optimization over the coefficients of ρ, ψ by a straight application of Putinar's Positivstellensatz [31]. However, this approach quickly becomes intractable. As we show next, computational complexity can be substantially reduced by exploiting duality.

For a given pair (ρ, ψ), consider the set of all systems of the form (7) that are rendered safe by the control action $u = \frac{\psi}{\rho}$, along with the corresponding admissible perturbations, that is, the set of all $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{w})$ such that (15) holds for all $\boldsymbol{x} \in \mathbb{R}^n$. For each $\boldsymbol{x}$, this set is a polytope of the form:

$$\mathcal{P}_2 \doteq \left\{\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{w}\colon -\begin{bmatrix}(\nabla\cdot[\rho(\boldsymbol{I}_n\otimes\boldsymbol{\phi}^T)])^T\\ (\nabla\cdot[\psi(\boldsymbol{I}_n\otimes\boldsymbol{\gamma}^T)])^T\\ (\nabla\rho)^T\end{bmatrix}^T\begin{bmatrix}\boldsymbol{f}\\ \boldsymbol{g}\\ \boldsymbol{w}\end{bmatrix} < -\rho h\right\} \quad (16)$$

where the divergence operator is applied column-wise to the matrix. The term $\nabla\cdot[\rho(\boldsymbol{x})(\boldsymbol{I}_n\otimes\boldsymbol{\phi}(\boldsymbol{x})^T)]\boldsymbol{f}$ may be interpreted as $\nabla\cdot\mathrm{vec}(\rho(\boldsymbol{x})\phi(\boldsymbol{x})^T\boldsymbol{F}^T) = \nabla\cdot[\rho(\boldsymbol{x})f(\boldsymbol{x})]$.

It follows that (15) holds for all admissible disturbances $\boldsymbol{w} \in W$ ($\boldsymbol{w}(\cdot) \in \mathcal{W}$) and all plants in the consistency $\mathcal{C}$ set if and only if $\mathcal{P}_1 \subseteq \mathcal{P}_2$. This inclusion can be enforced through duality as follows:

Lemma 2: Assume that the data and priors are consistent (e.g., $\mathcal{C} \neq \emptyset$). Then $\mathcal{P}_1 \subseteq \mathcal{P}_2$ if there exists a vector function $\boldsymbol{y}(\boldsymbol{x}) \geq 0, \boldsymbol{y}(\boldsymbol{x}) \in \mathbb{R}^{2nT+2n}$ such that the following functional set of affine constraints is feasible:

$$\boldsymbol{y}^T(\boldsymbol{x})\boldsymbol{N} = \boldsymbol{r}(\boldsymbol{x}) \text{ and } \boldsymbol{y}^T(\boldsymbol{x})\boldsymbol{e} < -\rho(\boldsymbol{x})h(\boldsymbol{x}) \quad (17)$$

where

$$\boldsymbol{N} \doteq \begin{bmatrix}\boldsymbol{A} & \boldsymbol{B} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{W}\end{bmatrix},\ \boldsymbol{e} \doteq \begin{bmatrix}\boldsymbol{\xi} - \boldsymbol{1}\otimes\boldsymbol{d}_w\\ \boldsymbol{d}_w\end{bmatrix},$$
$$\boldsymbol{r}(\boldsymbol{x}) \doteq -\begin{bmatrix}\nabla\cdot[\rho(\boldsymbol{I}_n\otimes\boldsymbol{\phi}^T)] & \nabla\cdot[\psi(\boldsymbol{I}_n\otimes\boldsymbol{\gamma}^T)] & \nabla\rho\end{bmatrix} \quad (18)$$

Proof: From [33, Sec. 5.8.1] it follows that the systems of inequalities

$$\begin{bmatrix}\boldsymbol{N}\\ -\boldsymbol{r}\end{bmatrix}\begin{bmatrix}\boldsymbol{f}\\ \boldsymbol{g}\\ \boldsymbol{w}\end{bmatrix} \leq \begin{bmatrix}\boldsymbol{e}\\ \rho h\end{bmatrix} \text{ and } \begin{matrix}\boldsymbol{y}^T\boldsymbol{N} - \mu\boldsymbol{r} = 0\\ \boldsymbol{y}^T\boldsymbol{e} + \mu\rho h < 0\\ \boldsymbol{y} \geq 0,\ \mu \geq 0\end{matrix} \quad (19)$$

are (weak) alternatives. Thus, feasibility of the right set of inequalities in (19), implies that the left inequalities are infeasible. Further, since $\mathcal{C} \neq \emptyset$ and $\mu > 0$, we can take $\mu = 1$ without loss of generality. Hence, if (17) holds, a triple $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{w}) \in \mathcal{P}_1$ if and only if $\begin{bmatrix}\boldsymbol{f}^T & \boldsymbol{g}^T & \boldsymbol{w}^T\end{bmatrix}\boldsymbol{r}^T < -\rho h$, that is $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{w}) \in \mathcal{P}_2$. ■

Remark 3: If $\mathcal{P}_1$ is compact, then (19) are strong alternatives and (17) are necessary and sufficient for $\mathcal{P}_1 \subseteq \mathcal{P}_2$.

Remark 4: Proceeding as in [19, Th. 2], it can be shown that if $\boldsymbol{\phi}(\boldsymbol{x}), \boldsymbol{\gamma}(\boldsymbol{x})$ are continuous functions, then $\boldsymbol{y}(\boldsymbol{x})$ can be chosen to be continuous.

The observations above lead to our main result:

Theorem 2: A sufficient condition for the existence of a state-feedback control law $u(\boldsymbol{x})$ such that all systems in the consistency set $\mathcal{C}$ are rendered robustly safe, is that there exists a continuous vector function $\boldsymbol{y}(\boldsymbol{x}) \geq 0$ and functions $\rho \in C^1$, $\psi \in C^1$ such that

$$\boldsymbol{y}^T(\boldsymbol{x})\boldsymbol{N} = \boldsymbol{r}(\boldsymbol{x}),\ \forall \boldsymbol{x} \in \mathbb{R}^n \quad (20a)$$
$$\boldsymbol{y}^T(\boldsymbol{x})\boldsymbol{e} < -\rho(\boldsymbol{x})h(\boldsymbol{x}),\ \forall \boldsymbol{x} \in \mathbb{R}^n \quad (20b)$$
$$|\psi(\boldsymbol{x})| \leq -\rho(\boldsymbol{x})h(\boldsymbol{x}),\ \forall \boldsymbol{x} \in \mathbb{R}^n \quad (20c)$$
$$\rho(\boldsymbol{x}) \geq 0,\ \forall \boldsymbol{x} \in \mathcal{X}_0 \quad (20d)$$
$$\rho(\boldsymbol{x}) < 0,\ \forall \boldsymbol{x} \in \mathcal{X}_u \quad (20e)$$

The corresponding control law is given by $u(\boldsymbol{x}) = \frac{\psi(\boldsymbol{x})}{\rho(\boldsymbol{x})}$.

Proof: The proof follows from the fact that from Lemma 2, (20a) and (20b) guarantee that (15) holds for all plants in $\mathcal{C}$ and all admissible disturbances $\boldsymbol{w}(\cdot) \in \mathcal{W}$. Hence the conditions in Lemma 1 hold for all plants that could have generated the observed data. ■

Remark 5: Constraint (20c) is a convex tightening of the condition that $\psi = 0$ when $\rho = 0$ in the safe zone $\rho(\boldsymbol{x}) \geq 0$. This ensures satisfaction of Assumption (i) in Lemma 1.

D. Sum-of-Squares Safety Program

In order to solve the infinite-dimensional Problem (20) in a tractable manner, we restrict the variables $\rho, \psi, \boldsymbol{y}$ to be polynomials. Under this polynomial restriction, the extracted controller $u(\boldsymbol{x}) = \psi(\boldsymbol{x})/\rho(\boldsymbol{x})$ is then a rational function.

Let $\mathcal{X}_0 = \{\boldsymbol{x}\colon k(\boldsymbol{x}) \geq 0\}$ and $\mathcal{X}_u = \{\boldsymbol{x}\colon h(\boldsymbol{x}) \geq 0\}$ denote the initial condition and unsafe sets, respectively. Algorithm 1 is SOS-based finite-degree tightening of (20) for robustly safe control. Successful execution of Algorithm 1 is sufficient for finding a robustly safe control law.

Algorithm 1: Data-Driven Safe Control Design

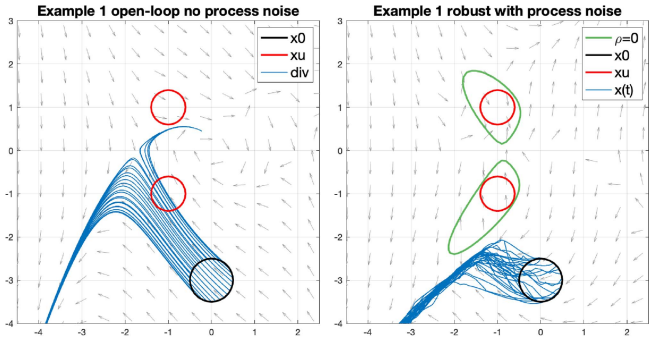
Input: sample data $\mathcal{D}$, and degrees d_f, d_g, d_ρ, d_ψ

Let $2d_1 \geq \max\{d_f + d_\rho, d_g + d_\psi\}$, $2d_2 \geq \max\{d_\rho, d_\psi\}$

Solve: the feasibility problem with $c_1,\ c_2 > 0$

$$\text{coeff}_{\boldsymbol{x}}(\boldsymbol{y}^T\boldsymbol{N} - \boldsymbol{r}) = 0 \qquad (A.1)$$
$$-\rho h - \boldsymbol{y}^T\boldsymbol{e} - c_1,\ \forall i : \boldsymbol{y}_i \in \Sigma_{d_1}[\boldsymbol{x}] \qquad (A.2)$$
$$-\rho h - \psi,\ -\rho h + \psi \in \Sigma_{d_2}[\boldsymbol{x}] \qquad (A.3)$$
$$\rho - s_1 k,\ -\rho - s_2 h - c_2 \in \Sigma_{d_2}[\boldsymbol{x}] \qquad (A.4)$$
$$s_1,\ s_2 \in \Sigma_{d_2}[\boldsymbol{x}] \qquad (A.5)$$

Output: the safe control law $u = \psi/\rho$ or a certificate of infeasibility at degree (d_1, d_2)



(a) open-loop (b) robust closed-loop

Fig. 1. Flow (21) simulations for Example 1.

E. Computational Complexity Analysis

A straightforward application of Putinar's Positivstellensatz to solve (15) requires considering polynomials in the indeterminates $(\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{w})$ with a total dimension $d_p = d_f + d_g + 2n$. Thus, for an SOS relaxation of order d_r, the total number of variables (hence the maximal size of Gram matrices) in the optimization is $\binom{d_r+d_p}{d_r}$. In contrast, by exploiting duality, Algorithm 1 only requires Gram matrices of maximal size $\binom{2+d_r}{d_r}$. In the case where (f, g, ρ, ψ) are all defined by degree 2 polynomials ($d_f = d_g = 6$), the maximal Gram matrix size $d_r = 3$ drops from $\binom{19}{3} = 969$ to $\binom{5}{3} = 10$.

IV. Numerical Examples

The proposed algorithm is tested on a pair of examples. Both experiments are implemented in MATLAB 2020b with Yalmip [34] and solved by Mosek [35]. Code to generate experiments and plots is publicly available at https://github.com/J-mzz/ddc-safety.

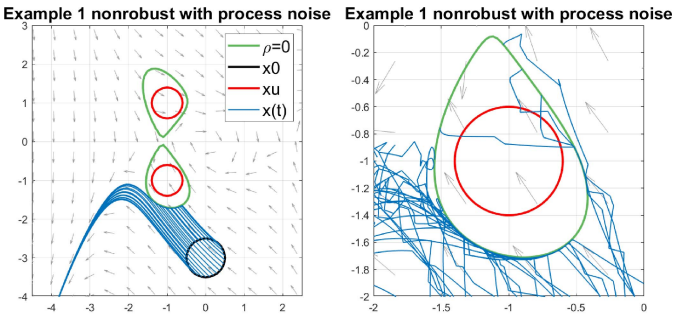
Example 1: Consider the Flow system [9] with

$$f = \left[x_2;\quad -x_1 + \tfrac{1}{3}x_1^3 - x_2\right],\ g = \left[0;\ \ 1\right] \qquad (21)$$

The initial and unsafe sets are the (union of) disks:

$$\mathcal{X}_0 = \{\boldsymbol{x} \mid\ 0.25 - x_1^2 - (x_2+3)^2 \geq 0\},$$
$$\mathcal{X}_u = \{\boldsymbol{x} \mid\ h_1(\boldsymbol{x}) = 0.16 - (x_1+1)^2 - (x_2+1)^2 \geq 0,$$
$$\text{OR } h_2(\boldsymbol{x}) = 0.16 - (x_1+1)^2 - (x_2-1)^2 \geq 0\}$$

Results of the control design for Example 1 are shown in Fig. 1 and 2. In each figure, 30 trajectories (blue curves) start from within the initial set $\mathcal{X}_0$ (black circle). The unsafe set $\mathcal{X}_u$ is the pair of red disks, implemented as $h(\boldsymbol{x}) = -h_1(\boldsymbol{x})h_2(\boldsymbol{x}) \geq 0$. Some of the open-loop trajectories in Fig. 1(a) enter the unsafe set $\mathcal{X}_u$ when starting in $\mathcal{X}_0$.



(a) no process noise (b) with process noise

Fig. 2. Safe controllers synthesized without process noise may be unsafe when process noise is applied.

The prior knowledge of the system model is that f is a two-dimensional cubic polynomial vector with $f(\boldsymbol{0}) = \boldsymbol{0}$ and that g is a two-dimensional constant vector, where the cubic polynomials in f and the constant terms in g are both unknown. 80 datapoints were collected and used to design a robustly safe controller under a disturbance with $\|\boldsymbol{w}\|_\infty \leq 2$, yielding a polytope $\mathcal{P}_2$ from (14) with 22 dimensions ($\dim \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{w} = 18, 2, 2$) and 324 faces (91 of the faces $\mathcal{P}_2$ are nonredundant [36]). Algorithm 1 was used to find $\rho, \psi \in \mathbb{R}[\boldsymbol{x}]_{\leq 4}$, yielding 99 Gram matrices of maximal size $\binom{6}{4} = 15$ and the rational control law $u = \psi/\rho$. Fig. 1(b) plots trajectories associated with this safe control law, and also features the $\rho = 0$ level set in green.

Fig. 2 highlights the importance of robustness in execution as well as in data-collection. The controller in Fig. 2 was computed with the same noisy training data as in Fig. 1 but assuming no run time disturbances. Fig. 2(a) shows that the control is safe under disturbance-free trajectory execution. Fig. 2(b) is zoomed into the lower red disk, and demonstrates that some controlled trajectories pass through the $\rho = 0$ contour and enter $\mathcal{X}_u$ when a disturbance with $\|\boldsymbol{w}\|_\infty \leq 2$ is applied in execution (trajectories are terminated when $u \geq 10^4$, which is caused by numerical issues and stiffness near the $\rho = 0$ contour).

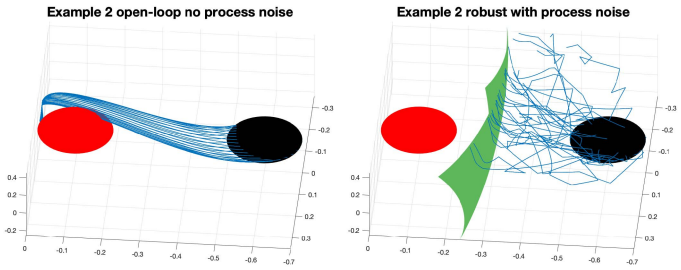
To summarize this example, $\rho \geq 0$ is an invariant set for all consistent systems under a disturbance $\boldsymbol{w}$ when the robust controller is applied. The level set $\rho = 0$ separates initial set $\mathcal{X}_0$ and unsafe set $\mathcal{X}_u$. Uncontrolled (Fig. 1(a)) and non-robustly-safe (Fig. 2(b)) trajectories may enter $\mathcal{X}_u$.

Example 2: Consider the Twist system with [37]:

$$f = \begin{bmatrix} -2.5x_1 + x_2 - 0.5x_3 + 2x_1^3 + 2x_3^3 \\ -x_1 + 1.5x_2 + 0.5x_3 - 2x_2^3 - 2x_3^3 \\ 1.5x_1 + 2.5x_2 - 2x_3 - 2x_1^3 - 2x_2^3 \end{bmatrix}, g = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad (22)$$

The initial and unsafe sets are the spheres:

$$\mathcal{X}_0 = \{\boldsymbol{x} \mid 0.01 - (x_1+0.5)^2 - x_2^2 - x_3^2 \geq 0\},$$
$$\mathcal{X}_u = \{\boldsymbol{x} \mid 0.01 - (x_1+0.1)^2 - x_2^2 - x_3^2 \geq 0\}$$



(a) open-loop (b) robust closed-loop

Fig. 3. Twist (22) simulations for Example 2.

Results for Example 2 are shown in Fig. 3 with the initial set $\mathcal{X}_0$ (black sphere), the unsafe set $\mathcal{X}_u$ (red sphere) and 30 trajectories (blue curves). The open-loop system is unsafe as shown in Fig. 3(a). A prior knowledge of the system model is that f is a three-dimensional cubic polynomial vector with $f(\mathbf{0}) = \mathbf{0}$ and that g is a three-dimensional constant vector. 80 datapoints were collected and used to design a robust safe controller under a disturbance with $\|\boldsymbol{w}\|_\infty \leq 1$, yielding a polytope $\mathcal{P}_2$ with 63 dimensions ($\dim[\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{w}] = 38, 3, 3$) and 304 faces (all nonredundant). Using Algorithm 1 to find $\rho, \psi \in \mathbb{R}[\boldsymbol{x}]_{\leq 4}$ yields a rational control law $u = \psi/\rho$. Fig. 3(b) features the $\rho = 0$ level set surface in green.

V. Conclusion

This letter uses density functions to find provably safe controllers for systems whose data-observations and executions are both corrupted by L_∞-bounded noise. The output of Algorithm 1 (if successful) is a rational controller u, along with a density certificate ρ that guarantees robust safety of all trajectories starting in the initial set. Future work involves steering safe trajectories to a destination set, adding performance objectives, and extension to other noise and disturbance models (e.g., L_2 or semidefinite bounded signals).

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