Solutions to Irodov's Problems in General Physics

CONTENTS

PHYSICAL FUNDAMENTALS OF MECHANICS

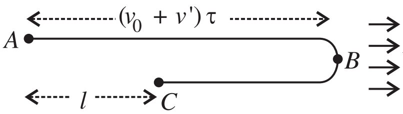
PART 1

1.1 Kinematics

1.1 Let v_0 be the stream velocity and v' the velocity of motorboat with respect to water. The motorboat reached point B while going downstream with velocity $(v_0 + v')$ and then returned with velocity $(v' - v_0)$ and passed the raft at point C. Let t be the time for the raft (which flows with stream with velocity v_0) to move from point A to C, during which the motorboat moves from A to B and then from B to C.

Therefore,

$$t = \frac{l}{v_0} = \tau + \frac{(v_0 + v')\tau - l}{(v' - v_0)}$$



On solving, we get

$$v_0 = \frac{l}{2\tau} = 3 \text{ km/h} \ \text{(on substituting values)}$$

Alternate:

In the frame of reference moving with the stream velocity the raft becomes stationary. The boat travels for time τ and turns and then travels back to reach the raft. As the raft is still at its original position, hence it would take exactly further time τ (the same time interval) for the boat to reach the raft. The boat thus takes time 2τ to reach the raft in the frame of stream. In Newtonian mechanics time interval between two events is frame independent, so 2τ is also the time interval in the frame of river bank. In the mean time the raft has travelled l. Hence the speed v_0 of the river is

$$v_0 = \frac{l}{2\tau} = 3 \text{ km/h}$$

1.2 Let s be the total distance traversed by the point and t_1 the time taken to cover half the distance. Further let $2t$ be the time to cover the rest half of the distance.

Therefore, $$\frac{s}{2} = v_0 t_1 \quad \text{or} \quad t_1 = \frac{s}{2v_0}$$

and $$\frac{s}{2} = (v_1 + v_2)t \quad \text{or} \quad 2t = \frac{s}{v_1 + v_2}$$

Hence the sought average velocity is given by

$$<v> = \frac{s}{t_1 + 2t} = \frac{s}{[s/2v_0] + [s/[(v_1 + v_2)]} = \frac{2v_0(v_1 + v_2)}{v_1 + v_2 + 2v_0}$$

1.3 As the car starts from rest and finally comes to a stop, and the rate of acceleration and deceleration are equal, the distances covered as well as the times taken are same in these phases of motion.

Let Δt be the time for which the car moves uniformly. Then the acceleration/deceleration time is $(\tau - \Delta t)/2$ each. So,

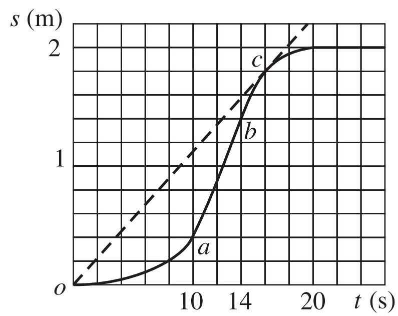
$$<v>\,\tau = 2\left\{\frac{1}{2}\,w\frac{(\tau - \Delta t)^2}{4}\right\} + w\frac{(\tau - \Delta t)}{2}\Delta t$$

or $$\Delta t^2 = \tau^2 - \frac{4 <v> \tau}{w}$$

Hence, $$\Delta t = \tau\sqrt{1 - \frac{4<v>}{w\tau}} = 15\text{ s}$$

1.4 (a) Sought average velocity

$$<v> = \frac{s}{t} = \frac{200\text{ cm}}{20\text{ s}} = 10\text{ cm/s}$$

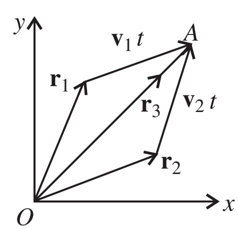


(b) For the maximum velocity, ds/dt should be maximum. From the figure ds/dt is maximum for all points on the line ab thus the sought maximum velocity becomes average velocity for the line ab and is

$$= \frac{100\text{ cm}}{4\text{ s}} = 25\text{ cm/s}$$

(c) Time t_0 should be such that corresponding to it the slope ds/dt should coincide with the chord Oc, to satisfy the relationship $ds/dt = s/t_0$. From figure the tangent at point c passes through the origin and thus corresponding time $t = t_0 = 16$ s.

1.5 Let the particles collide at the point A (see figure), whose position vector is $\mathbf{r}_3$ (say). If t be the time taken by each particle to reach point A, then from triangle law of vector addition:



$$\mathbf{r}_3 = \mathbf{r}_1 + \mathbf{v}_1 t = \mathbf{r}_2 + \mathbf{v}_2 t$$

So, $$\mathbf{r}_1 - \mathbf{r}_2 = (\mathbf{v}_2 - \mathbf{v}_1)t \qquad (1)$$

Therefore, $$t = \frac{|\mathbf{r}_1 - \mathbf{r}_2|}{|\mathbf{v}_2 - \mathbf{v}_1|} \qquad (2)$$

From Eqs. (1) and (2)

$$\mathbf{r}_1 - \mathbf{r}_2 = (\mathbf{v}_2 - \mathbf{v}_1) = \frac{|\mathbf{r}_1 - \mathbf{r}_2|}{|\mathbf{v}_2 - \mathbf{v}_1|}$$

Thus, $$\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{|\mathbf{v}_2 - \mathbf{v}_1|},$$ which is the sought relationship.

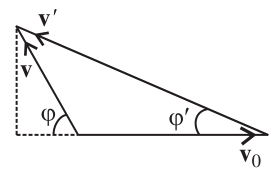
1.6 We have

$$\mathbf{v}' = \mathbf{v} - \mathbf{v}_0$$

In the vector diagram for above relation, let us drop a dotted perpendicular from the tip of $\mathbf{v}$ onto the line of action of $\mathbf{v}_0$. Using the property of right angle triangle,

$$v' = \sqrt{(v\cos\varphi + v_0)^2 + (v\sin\varphi)^2} = \sqrt{v_0^2 + v^2 + 2v_0\, v\cos\varphi}$$

$$= 39.7 \text{ km/h (on substituting values)}$$



and $$\tan\varphi' = \frac{v\sin\varphi}{v\cos\varphi + v_0}$$

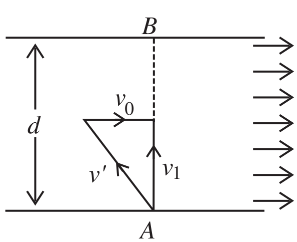
On substitution $$\varphi' = 19.1°$$

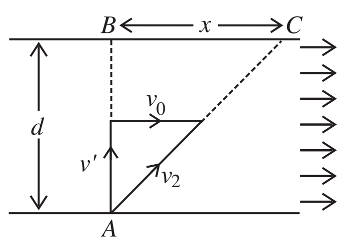
1.7 Let us suppose one of the swimmers (say 1) crosses the river along AB, which is obviously the shortest path. Time taken to cross the river by the swimmer 1

$$t_1 = \frac{d}{\sqrt{v'^2 - v_0^2}} \quad \text{(where } AB = d \text{ is the width of the river)} \qquad (1)$$

For the other swimmer (say 2), who follows the quickest path, the time taken to cross the river,

$$t_2 = \frac{d}{v'} \qquad (2)$$





In the time t_2, drifting of the swimmer 2, becomes

$$x = v_0 t_2 = \frac{v_0}{v'}\, d \quad \text{(using Eq. 2)} \qquad (3)$$

If t_3 be the time for swimmer 2 to walk the distance x to come C to B (see figure), then

$$t_3 = \frac{x}{u} = \frac{v_0 d}{v' u} \quad \text{(using Eq. 3)} \tag{4}$$

According to the problem $t_1 = t_2 + t_3$

or
$$\frac{d}{\sqrt{v'^2 - v_0^2}} = \frac{d}{v'} + \frac{v_0 d}{v' u}$$

$$\Rightarrow \qquad \frac{v_0}{u} = \frac{v'}{v'^2 - v_0^2} - 1$$

Thus,
$$u = \frac{v_0}{(1 - v_0^2/v'^2)^{-1/2} - 1} = 3 \text{ km/h}$$

1.8 Let l be the distance covered by the boat A along the river as well as by the boat B across the river. Let v_0 be the stream velocity and v' the velocity of each boat with respect to water. Therefore time taken by the boat A in its journey

$$t_A = \frac{l}{v' + v_0} + \frac{l}{v' - v_0} = \frac{2lv'}{v'^2 - v_0^2}$$

and for the boat B

$$t_B = \frac{l}{\sqrt{v'^2 - v_0^2}} + \frac{l}{\sqrt{v'^2 - v_0^2}} = \frac{2l}{\sqrt{v'^2 - v_0^2}}$$

Hence,
$$\frac{t_A}{t_B} = \frac{v'}{\sqrt{v'^2 - v_0^2}} = \frac{\eta}{\sqrt{\eta^2 - 1}} \left(\text{where } \eta = \frac{v'}{v_0}\right)$$

On substitution,
$$\frac{t_A}{t_B} = 1.8$$

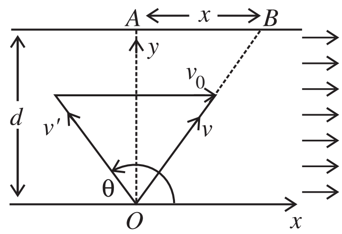
1.9 Let v_0 be the stream velocity and v' the velocity of boat with respect to water. As $v_0/v' = n = 2 > 0$, some drifting of the boat is inevitable.

Let $\mathbf{v}'$ make an angle θ with flow direction (see figure), then the time taken to cross the river

$$t = \frac{d}{v' \sin\theta} \quad \text{(where } d \text{ is the width of the river)}$$

In this time interval, the drifting of the boat

$$\begin{aligned} x &= (v' \cos\theta + v_0) t \\ &= (v' \cos\theta + v_0) \frac{d}{v' \sin\theta} = (\cot\theta + \eta \operatorname{cosec}\theta) d \end{aligned}$$



For $x_{\min}$ (minimum drifting)

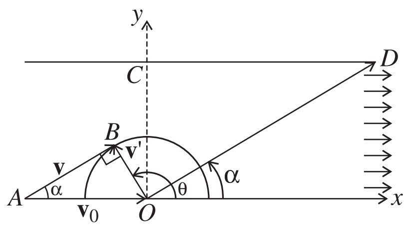
$$\frac{d}{d\theta}(\cot\theta + \eta\operatorname{cosec}\theta) = 0, \text{ which yields}$$

$$\cos\theta = -\frac{1}{n} = -\frac{1}{2}$$

Hence, $$\theta = 120°$$

Alternate:

Let $\mathbf{v}_0$ be the stream velocity, $\mathbf{v}'$ the velocity of boat with respect to water and $\mathbf{v}$ be the resultant velocity of boat, i.e., $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}'$.



The angle θ from the direction of stream at which boat can be rowed lies in the interval 0 to π ($0 \le \theta \le \pi$). Now let us draw the vector diagram of velocity vectors. In the figure, a semicircle has been drawn whose radius is v'. The tip of vector $\mathbf{v}'$ lies on this semicircle. One can observe that for minimum drifting, angle α should be maximum and the line AB representing vector $\mathbf{v}$ becomes tangent to the semicircle so that line OB representing $\mathbf{v}'$ becomes perpendicular to it. The minimum drifting is CD.

From the figure $$\sin\alpha = \frac{OB}{AO} = \frac{v'}{v_0} = \frac{1}{n} \tag{1}$$

$$\theta = \frac{\pi}{2} + \alpha \text{ (see figure)}$$

Hence, $$\theta = \sin^{-1}(1/n) + \pi/2 = 120° \text{ (on substituting values)}$$

In the triangle OCD

$$\sin\alpha = \frac{OC}{OD} \tag{2}$$

From Eqs. (1) and (2)

$$\frac{OC}{OD} = \frac{1}{n}$$

or $$OD = n\,(OC)$$

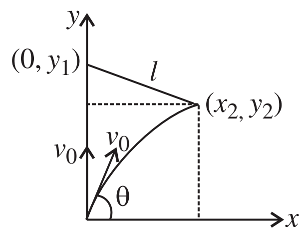
So, $$OC = \sqrt{(OD)^2 - (OC)^2} = OC\sqrt{n^2 - 1}$$

Hence, $$x_{\min} = d\sqrt{n^2 - 1}$$

1.10 Let us suppose one of the bodies, say 1, is thrown upwards and the other body (say 2) at an angle θ to the horizontal. Indicate y-axis vertically upward and x-axis along horizontal. Let after the time interval t, the particles 1 and 2 have coordinates $(0, y_1)$ and (x_2, y_2)

For body 1,

$$y_1 = v_0 t - \frac{1}{2} g t^2$$



and for body 2,

$$x_2 = v_0 \cos\theta \, t \quad \text{and} \quad y_2 = v_0 \sin\theta \, t - \frac{1}{2} g t^2$$

Thus the sought separation between the particles 1 and 2 after the time interval t

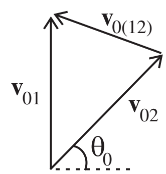
$$l = \sqrt{x_2^2 + (y_1 - y_2)^2} = \sqrt{(v_0 \cos\theta t)^2 + \{v_0 t (1 - \sin\theta)^2\}}$$

$$= v_0 \, t\sqrt{2(1 - \sin\theta)}$$

$$= 22 \text{ m, on putting the values of } v_0, \ t \text{ and } \theta$$

Alternate:

The solution of this problem becomes interesting in the frame attached with one of the bodies. Let the body thrown straight up be 1 and the other body be 2, then for the body 1 in the frame of 2 from the kinematical equation for constant acceleration (since both are moving under constant acceleration) is



$$\mathbf{r}_{12} = \mathbf{r}_{0(12)} + \mathbf{v}_{0(12)} \, t + \frac{1}{2} \mathbf{w}_{12} t^2$$

So, $\quad \mathbf{r}_{12} = \mathbf{v}_{0(12)} \, t$ (because $\mathbf{w}_{12} = 0$ and $\mathbf{r}_{0(12)} = 0$)

or $\quad |\mathbf{r}_{12}| = |\mathbf{v}_{0(12)}| \, t$

But, $\quad |\mathbf{v}_{01}| = |\mathbf{v}_{02}| = v_0$

So, from properties of triangle

$$|\mathbf{v}_{0(12)}| = \sqrt{v_0^2 + v_0^2 - 2 v_0 v_0 \cos(\pi/2 - \theta_0)}$$

Hence, the sought distance

$$|\mathbf{r}_{12}| = v_0 t \sqrt{2\,(1 - \sin\theta_0)} = 22 \text{ m}$$

1.11 Let the velocities of the particles (say $\mathbf{v}_1'$ and $\mathbf{v}_2'$) become mutually perpendicular after time t. Then their velocities become

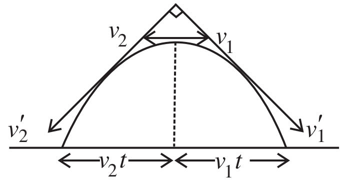
$$\mathbf{v}_1' = \mathbf{v}_1 + \mathbf{g}t; \ \mathbf{v}_2' = \mathbf{v}_2 + \mathbf{g}\,t$$

As $\quad \mathbf{v}_1' \perp \mathbf{v}_2'$ so, $\mathbf{v}_1' \cdot \mathbf{v}_2' = 0$

or $\quad (\mathbf{v}_1 + \mathbf{g}t) \cdot (\mathbf{v}_2 + \mathbf{g}t) = 0$

or $\quad -v_1 v_2 + g^2 t^2 = 0$

Hence, $\quad t = \dfrac{\sqrt{v_1 v_2}}{g}$



In the frame attached with 2 for the particle 1

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t + \frac{1}{2}\mathbf{w}\, t^2$$

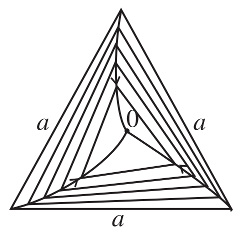
Both the particles are initially at the same position and have same acceleration **g,** so $\mathbf{r}_0 = 0$, $\mathbf{w} = 0$, and $\mathbf{v}_0 = |\mathbf{v}_1 - \mathbf{v}_2|$. Thus the sought distance is

$$|\mathbf{r}| = |\mathbf{v}_0| t = (v_1 + v_2)\, t$$

$$= \frac{v_1 + v_2}{g}\sqrt{v_1 v_2} \text{ (using value of } t)$$

$$= 2.5 \text{ m, on putting the values of } v_1, v_2 \text{ and } g$$

1.12 From the symmetry of the problem all the three points are always located at the vertices of equilateral triangles of varying side length and finally meet at the centroid of the initial equilateral triangle. Symmetry of the problem tells us that the velocity with which point 1 approaches 2 is equal to the velocity by which 2 approaches 3 and 3 approaches 1.

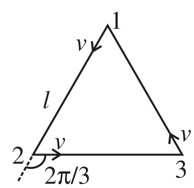


Method 1:

The rate at which 1 approaches 2 or the separation between 1 and 2 decreases with time is the projection of $\mathbf{v}_{12}$ towards $\mathbf{r}_{21}$. At an arbitrary instant when the equilateral triangle has edge length $l < a$, the velocity with which 1 approaches 2 becomes

$$\frac{-dl}{dt} = v - v\cos\left(\frac{2\pi}{3}\right)$$

On integrating: $\quad \displaystyle\int_a^0 -dl = \frac{3v}{2}\int_0^t dt$ (where t is the sought time)



or $\quad a = \dfrac{3}{2}vt \quad$ so, $\quad t = \dfrac{2a}{3v}$

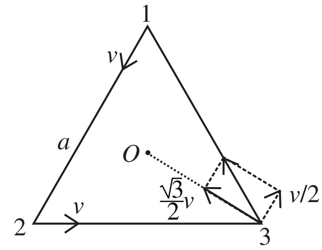
Method 2:

If we concentrate on the motion of any one point say 3, one can observe that at all the time its velocity makes an angle 30° with $3O$ and is equal to $v\cos 30° = \sqrt{3}/2\, v$.

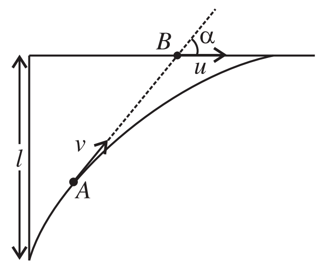
Initially $3O$ equals $a/\sqrt{3}$, so the sought time

$$t = \frac{a/\sqrt{3}}{(\sqrt{3}/2)v} = \frac{2a}{3v}$$

(In the figure, velocity of point 3 has been resolved into two rectangular components, one pointing towards the centre of the triangle and other perpendicular to it.)



1.13 Let us locate the points A and B at an arbitrary instant of time (see figure).



If A and B are separated by the distance s at this moment, then the points converge or point A approaches B with velocity $-ds/dt = v - u\cos\alpha$ where angle α varies with time. On integrating,

$$-\int_l^0 ds = \int_0^T (v - u\cos\alpha)\,dt \quad \text{(where } T \text{ is the sought time)}$$

or
$$l = \int_0^T (v - u\cos\alpha)\,dt$$

$$= vT - u\int_0^T \cos\alpha\,dt \tag{1}$$

As both A and B cover the same distance in x-direction during the sought time interval, so the other condition which is required, can be obtained by the equation,

$$uT = \int_0^T v\cos\alpha\,dt \tag{2}$$

Solving Eqs. (1) and (2), we get

$$T = \frac{ul}{v^2 - u^2}$$

One can see that if $u = v$, or $u < v$, point A cannot catch B.

1.14 In the reference frame fixed to the train, the distance between the two events is obviously equal to l. Suppose the train starts moving at time $t = 0$ in the positive direction and take the origin ($x = 0$) at the head-light of the train at $t = 0$. Then the coordinate of first event in the earth's frame is

$$x_1 = \frac{1}{2}wt^2$$

and similarly the coordinate of the second event is

$$x_2 = \frac{1}{2} w(t + \tau)^2 - l$$

The distance between the two events is obviously

$$x_1 - x_2 = l - w\tau(t + \tau/2) = 0.242 \text{ km}$$

in the reference frame fixed on the Earth.

For the two events to occur at the same point in the reference frame K, moving with constant velocity V relative to the earth, the distance traveled by the frame in the time interval τ must be equal to the above distance.

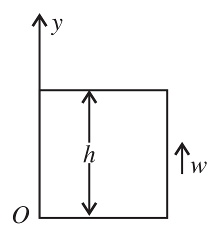
Thus, $$V\tau = l - w\tau(t + \tau/2)$$

So, $$V = \frac{l}{\tau} - w(t + \tau/2) = 4.03 \text{ m/s}$$

The frame K must clearly be moving in a direction opposite to the train so that if (for example) the origin of the frame coincides with the point x_1 on the earth at time t, it coincides with the point x_2 at time $t + \tau$.

1.15 (a) As time interval between two events is reference independent in Newtonian mechanics, so it is better to solve the problem in the elevator's frame having the observer at its floor. (see figure). Let us denote separation between floor and ceiling by $h = 2.7$ m and the acceleration of the elevator by $w = 1.2$ m/s^2.

From the kinematical formula



$$y = y_0 + v_{0y} t + \frac{1}{2} w_y t^2$$

Here, $$y = 0,\ y_0 = +h,\ v_{0y} = 0$$

and $$w_y = w_{\text{bolt}(y)} - w_{\text{ele}(y)}$$

$$= (-g) - (w) = -(g + w)$$

So, $$0 = h + \frac{1}{2}[-(g + w)]t^2$$

or $$t = \sqrt{\frac{2h}{g + w}} = 0.7 \text{ s} \quad \text{(on substituting values)}$$

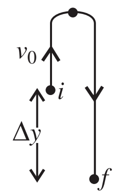
(b) At the moment the bolt loses contact with the elevator, it has already acquired the velocity equal to elevator, given by;

$$v_0 = (1.2)\,(2) = 2.4 \text{ m/s}$$

In the reference frame attached with the elevator shaft (ground) and pointing the y-axis upward, we have for the displacement of the bolt,

$$\Delta y = v_{0y} t + \frac{1}{2} w_y t^2$$

$$= v_0 t + \frac{1}{2}(-g)t^2$$



or
$$\Delta y = (2.4)(0.7) + \frac{1}{2}(-9.8)(0.7)^2$$

$$= -0.7 \text{ m}$$

Hence the bolt comes down or displaces downward relative to the point, when it loses contact with the elevator by the amount 0.7 m (see figure).

Obviously the total distance covered by the bolt during its free fall time

$$s = |\Delta y| + 2\left(\frac{v_0^2}{2g}\right) = 0.7 \text{ m} + \frac{(2.4)^2}{(9.8)} \text{ m} = 1.3 \text{ m}$$

1.16 After time $t > 0$, the separation between the particles

$$l = \sqrt{(l_1 - v_1 t)^2 + (l_2 - v_2 t)^2}$$

For l and hence l^2 to be minimum

$$\frac{dl^2}{dt} = 0$$

or
$$\frac{d}{dt}\{(l_1 + v_1 t)^2 + (l_2 - v_2 t)^2\} = 0$$

which gives
$$t = \frac{(v_1 l_1 - v_2 l_2)}{\sqrt{v_1^2 + v_2^2}}$$

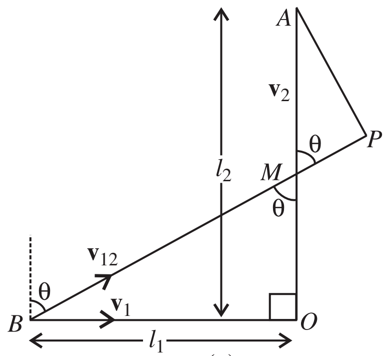
On using the value of t, in the expression of l

$$l_{\text{min}} = \frac{|v_1 l_2 - v_2 l_1|}{\sqrt{v_1^2 + v_2^2}}.$$

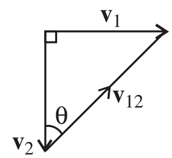
Alternate:

Let the particles 1 and 2 be at points B and A respectively at $t = 0$, at the distances l_1 and l_2 from intersection point O.

Let us fix the inertial frame with the particle 2. Now the particle 1 moves relative to this reference frame with a relative velocity $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$, and its trajectory is the straight line BP. Obviously, the minimum distance between the particles is equal to the length of the perpendicular AP dropped from point A on to the straight line BP (Fig. (a)).



(a)



(b)

From Fig. (b), $v_{12} = \sqrt{v_1^2 + v_2^2}$, $\tan\theta = \dfrac{v_1}{v_2}$,

So, $$\cos\theta = \frac{v_2}{\sqrt{v_1^2 + v_2^2}} \quad \text{and} \quad \sin\theta = \frac{v_1}{\sqrt{v_1^2 + v_2^2}} \qquad (1)$$

From Fig. (a), the shortest distance

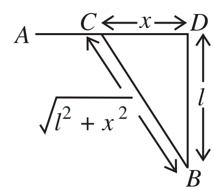
$$AP = AM\sin\theta = (OA - OM)\sin\theta = (l_2 - l_1\cot\theta)\sin\theta$$

or $$AP = \left(l_2 - l_1\frac{v_2}{v_1}\right)\frac{v_1}{\sqrt{v_1^2 + v_2^2}} = \frac{|v_1 l_2 - v_2 l_1|}{\sqrt{v_1^2 + v_2^2}} \qquad \text{(using Eq. 1)}$$

The sought time $$t = \frac{BP}{|\mathbf{v}_{12}|} = \frac{BM + MP}{|\mathbf{v}_{12}|} = \frac{l_1\operatorname{cosec}\theta + (l_2 - l_1\cot\theta)\cos\theta}{\sqrt{v_1^2 + v_2^2}}$$

$$= \frac{l_2\cos\theta + l_1\sin\theta}{\sqrt{v_1^2 + v_2^2}} = \frac{l_2 v_2 + l_1 v_1}{v_1^2 + v_2^2} \qquad \text{(using Eq. 1)}$$

1.17 Let the car turns off the highway at a distance x from the point D. So, $CD = x$, and if the speed of the car in the field is v, then the time taken by the car to cover the distance $AC = (AD - x)$ on the highway



$$t_1 = \frac{AD - x}{\eta v} \qquad (1)$$

and the time taken to travel the distance CB in the field

$$t_2 = \frac{\sqrt{l^2 + x^2}}{v} \qquad (2)$$

So, the total time elapsed to move the car from point A to B

$$t = t_1 + t_2 = \frac{AD - x}{\eta v} + \frac{\sqrt{l^2 + x^2}}{v}$$

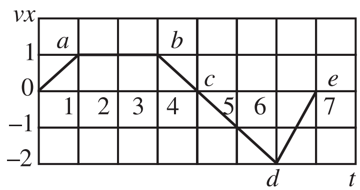
For t to be minimum

$$\frac{dt}{dx} = 0 \quad \text{or} \quad \frac{1}{v}\left[-\frac{1}{\eta} + \frac{x}{\sqrt{l^2 + x^2}}\right] = 0$$

or

$$\eta^2 x^2 = l^2 + x^2 \quad \text{or} \quad x = \frac{l}{\sqrt{\eta^2 - 1}}$$

1.18 From question $v_x(0) = 0$, so the motion starts from origin, thus from the plot of $v_x(t)$ we have

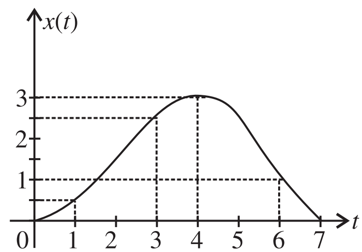


$$v_x = \begin{cases} t;\ 0 \le t \le 1 \\ 1;\ 1 < t \le 3 \\ 4 - t;\ 3 < t \le 6 \\ 2t - 14;\ 6 < t \le 7 \end{cases}$$

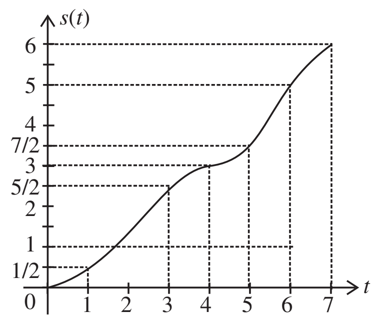
and $v_x = 0$ for $t > 7$.

As at $t = 0$, $x = 0$,

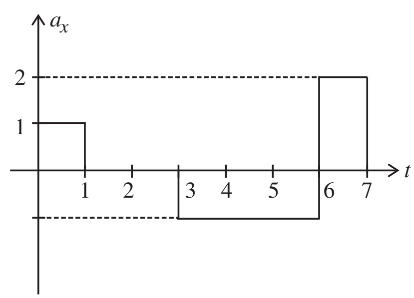
So, $$x(t) = \int_0^t v_x dt = \begin{cases} \dfrac{t^2}{2};\ 0 \le t \le 1 \\ t - \dfrac{1}{2};\ 1 < t \le 3 \\ 4t - \dfrac{t^2}{2} + 5;\ 3 < t \le 6 \\ t^2 - 14t + 49;\ 6 < t \le 7 \end{cases}$$



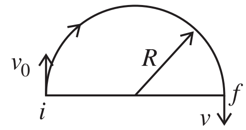
Now, $$s(t) = \int_0^t |v_x| dt = \begin{cases} \dfrac{t^2}{2};\ 0 \le t \le 1 \\ t - \dfrac{1}{2};\ 1 < t \le 3 \\ 4t - \dfrac{t^2}{2} - 5;\ 3 < t \le 4 \\ \dfrac{t^2}{2} - 4t + 11;\ 4 < t \le 6 \\ 14t - t^2 - 43;\ 6 < t \le 7 \end{cases}$$



Now, $$a_x = \frac{dv_x}{dt} = \begin{cases} 1;\ 0 \le t < 1 \\ 0;\ 1 < t \le 3 \\ -1;\ 3 < t \le 6 \\ 2;\ 6 < t \le 7 \end{cases}$$



1.19 (a) Mean velocity



$$< v > = \frac{\text{Total distance covered}}{\text{Time elapsed}}$$

$$= \frac{s}{t} = \frac{\pi R}{\tau} = 50 \text{ cm/s} \tag{1}$$

(b) Modulus of mean velocity vector $|<\mathbf{v}>| = \dfrac{|\Delta \mathbf{r}|}{\Delta t} = \dfrac{2R}{\tau} = 32$ cm/s (2)

(c) Let the point moves from i to f along the half circle (see figure) and v_0 and v be the speed at the points respectively.

We have
$$\frac{dv}{dt} = w_t$$

or $\quad v = v_0 + w_t t$ (as w_t is constant, according to the problem)

So,
$$<v> = \frac{\int_0^t (v_0 + w_t t)dt}{\int_0^t dt} = \frac{v_0 + (v_0 + w_t t)}{2} = \frac{v_0 + v}{2} \tag{3}$$

So, from Eqs. (1) and (3)
$$\frac{v_0 + v}{2} = \frac{\pi R}{\tau} \tag{4}$$

Now, the modulus of the mean vector of total acceleration
$$|<\mathbf{w}>| = \frac{|\Delta \mathbf{v}|}{\Delta t} = \frac{|\mathbf{v} - \mathbf{v}_0|}{\tau} = \frac{v_0 + v}{\tau} \text{ (see figure)} \tag{5}$$

Using Eq. (4) in Eq. (5), we get
$$|<\mathbf{w}>| = \frac{2\pi R}{\tau^2} = 10 \text{ cm/s}^2 \text{ (on substituting values)}$$

1.20 (a) We have
$$\mathbf{r} = \mathbf{a}t(1 - \alpha t)$$

So,
$$\mathbf{v} = \frac{\mathbf{dr}}{dt} = \mathbf{a}(1 - 2\alpha t)$$

and
$$\mathbf{w} = \frac{d\mathbf{v}}{dt} = -2\alpha \mathbf{a}$$

(b) From the equation
$$\mathbf{r} = \mathbf{a}t(1 - \alpha t)$$
$$\mathbf{r} = 0, \text{ at } t = 0 \text{ and also at } t = \Delta t = \frac{1}{\alpha}$$

So, the sought time $\Delta t = \dfrac{1}{\alpha}$

As $\quad \mathbf{v} = \mathbf{a}(1 - 2\alpha t)$

So, $$v = |\mathbf{v}| = \begin{cases} a(1 - 2\alpha t) & \text{for } t \le \dfrac{1}{2\alpha} \\ a(2\alpha t - 1) & \text{for } t > \dfrac{1}{2\alpha} \end{cases}$$

Hence, the sought distance

$$s = \int v\, dt = \int_0^{1/2\alpha} a(1 - 2\alpha\, t)\, dt + \int_{1/2\alpha}^{1/\alpha} a(2\alpha t - 1)\, dt$$

Simplifying, we get, $s = \dfrac{a}{2\alpha}$

1.21 (a) As the particle leaves the origin at $t = 0$

So, $$\Delta x = x = \int v_x\, dt \tag{1}$$

As $$\mathbf{v} = \mathbf{v}_0\left(1 - \frac{t}{\tau}\right)$$

(where $\mathbf{v}_0$ is directed towards the +ve x-axis).

So, $$v_x = v_0\left(1 - \frac{t}{\tau}\right) \tag{2}$$

From Eqs. (1) and (2),

$$x = \int_0^t v_0\left(1 - \frac{t}{\tau}\right)dt = v_0 t\left(1 - \frac{t}{2\tau}\right) \tag{3}$$

Hence x coordinate of the particle at $t = 6$ s

$$x = 10 \times 6\left(1 - \frac{6}{2 \times 5}\right) = 24 \text{ cm} = 0.24 \text{ m}$$

Similarly at $t = 10$ s

$$x = 10 \times 10\left(1 - \frac{10}{2 \times 5}\right) = 0$$

and at $t = 20$ s

$$x = 10 \times 20\left(1 - \frac{20}{2 \times 5}\right) = -200 \text{ cm} = -2\,\text{m}$$

(b) At the moment the particle is at a distance of 10 cm from the origin, $x = \pm 10$ cm. Putting $x = +10$ in Eq. (3)

$$10 = 10t\left(1 - \frac{t}{10}\right) \quad \text{or} \quad t^2 - 10t + 10 = 0$$

So,
$$t = \frac{10 \pm \sqrt{100 - 40}}{2} = 5 \pm \sqrt{15} \text{ s}$$

Now putting $x = -10$ in Eq. (3)

$$-10 = 10\left(1 - \frac{t}{10}\right)$$

On solving, $t = 5 \pm \sqrt{35}$ s

As t cannot be negative, so,

$$t = (5 + \sqrt{35}) \text{ s}$$

Hence the particle is at a distance of 10 cm from the origin at three moments of time:

$$t = 5 \pm \sqrt{15} \text{ s}, 5 + \sqrt{35} \text{ s}$$

(c) We have
$$\mathbf{v} = \mathbf{v}_0\left(1 - \frac{t}{\tau}\right)$$

So,
$$v = |\mathbf{v}| = \begin{cases} v_0\left(1 - \dfrac{t}{\tau}\right) & \text{for } t \le \tau \\ v_0\left(\dfrac{t}{\tau} - 1\right) & \text{for } t > \tau \end{cases}$$

So,
$$s = \int_0^t v_0\left(1 - \frac{t}{\tau}\right) dt \quad \text{for} \quad t \le \tau = v_0 t(1 - t/2\tau) \tag{4}$$

and
$$s = \int_0^t v_0\left(1 - \frac{t}{\tau}\right) dt + \int_\tau^t v_0\left(\frac{t}{\tau} - 1\right) dt \quad \text{for} \quad t > \tau$$

$$= v_0\tau[1 + (1 - t/\tau)^2]/2 \quad \text{for} \quad t > \tau \tag{5}$$

$$s = \int_0^4 v_0\left(1 - \frac{t}{\tau}\right) dt = \int_0^4 10\left(1 - \frac{t}{5}\right) dt = 24 \text{ cm}$$

For $t = 8$ s

$$s = \int_0^5 10\left(1 - \frac{t}{5}\right) dt + \int_5^8 10\left(\frac{t}{5} - 1\right) dt$$

On integrating and simplifying, we get

$$s = 34 \text{ cm}$$

On the basis of Eqs. (3) and (4), $x(t)$ and $s(t)$ plots can be drawn as shown in the answer sheet.

1.22 (a) As particle is in unidirectional motion, it is directed along the x-axis all the time. At $t = 0$, $x = 0$.

So, $\Delta x = x = s$ and $w = \frac{dv}{dt} = \frac{v\,dv}{ds} = \frac{1}{2}\frac{d}{ds}(v^2)$

According to the problem $v = \alpha\sqrt{x} = \alpha\sqrt{s}$

So, $$v^2 = \alpha^2 s$$

and $$w = \frac{1}{2}\frac{d}{ds}(v^2) = \frac{1}{2}\frac{d}{ds}(\alpha^2 s) = \frac{\alpha^2}{2} \tag{1}$$

As, $$w = \frac{dv}{dt} = \frac{\alpha^2}{2} \quad \text{so,} \quad dv = \frac{\alpha^2}{2}\,dt$$

On integrating, $$v = \frac{\alpha^2}{2}t = v(t) \tag{2}$$

(b) Let t be the time to cover first s m of the path. From Eq. (2)

$$s = \int v\,dt = \int_0^t \frac{\alpha^2}{2}t\,dt = \frac{\alpha^2}{2}\frac{t^2}{2}$$

Hence, $$t = \frac{2}{\alpha}\sqrt{s} \tag{3}$$

The mean velocity of particle

$$<v> = \frac{\int v(t)\,dt}{\int dt} = \frac{\int_0^{2\sqrt{s}/\alpha} \frac{\alpha^2}{2}t\,dt}{2\sqrt{s}/\alpha} = \frac{\alpha\sqrt{s}}{2}$$

1.23 According to the problem

$$-\frac{v\,dv}{ds} = a\sqrt{v} \text{ (as } v \text{ decreases with time)}$$

or $$-\int_{v_0}^{0} \sqrt{v}\,dv = \alpha \int_0^s ds$$

On integrating we get, $$s = \frac{2}{3a}v_0^{3/2}$$

Again according to the problem

$$-\frac{dv}{dt} = a\sqrt{v} \quad \text{or} \quad -\frac{dv}{\sqrt{v}} = a\,dt$$

or
$$-\int_{v_0}^{0} \frac{dv}{\sqrt{v}} = a\int_0^t dt$$

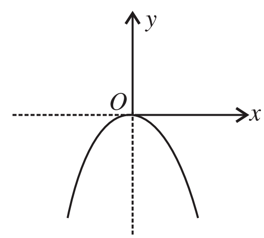
Thus,
$$t = \frac{2\sqrt{v_0}}{a}$$

1.24 (a) As
$$\mathbf{r} = at\,\mathbf{i} - bt^2\,\mathbf{j}$$

So,
$$x = at,\ y = -bt^2$$

and therefore
$$y = \frac{-bx^2}{a^2}$$

which is equation of a parabola, whose graph is shown in the figure.



(b) As
$$\mathbf{r} = at\,\mathbf{i} - bt^2\,\mathbf{j}$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = a\,\mathbf{i} - 2b\mathbf{j} \tag{1}$$

So,
$$v = \sqrt{a^2 + (-2bt)^2} = \sqrt{a^2 + 4b^2t^2}$$

Differentiating Eq. (1) with respect to time, we get

$$\mathbf{w} = \frac{d\mathbf{v}}{dt} = -2b\mathbf{j}$$

So,
$$|\mathbf{w}| = w = 2b$$

(c)
$$\cos\alpha = \frac{\mathbf{v}\cdot\mathbf{w}}{vw} = \frac{(a\mathbf{i} - 2bt\mathbf{j})\cdot(-2b\mathbf{j})}{(\sqrt{a^2 + 4b^2\,t^2})\,2b}$$

or
$$\cos\alpha = \frac{2bt}{\sqrt{a^2 + 4b^2t^2}}$$

So,
$$\tan\alpha = \frac{a}{2\,bt} \quad \text{or} \quad \alpha = \tan^{-1}\left(\frac{a}{2bt}\right)$$

(d) The mean velocity vector

$$<\mathbf{v}> = \frac{\int \mathbf{v}dt}{\int dt} = \frac{\int_0^t (a\mathbf{i} - 2bt\mathbf{j})dt}{t} = a\mathbf{i} - bt\mathbf{j}$$

Hence,
$$|<\mathbf{v}>| = \sqrt{a^2 + (-bt)^2} = \sqrt{a^2 + b^2t^2}$$

1.25 (a) We have

$$x = at \quad \text{and} \quad y = at(1 - \alpha t) \tag{1}$$

Hence, $y(x)$ becomes

$$y = \frac{ax}{a}\left(1 - \frac{\alpha x}{a}\right) = x - \frac{\alpha}{a}x^2 \quad \text{(parabola)}$$

(b) Differentiating Eq. (1) we get

$$v_x = a \quad \text{and} \quad v_y = a(1 - 2\alpha t) \tag{2}$$

So, $$v = \sqrt{v_x^2 + v_y^2} = a\sqrt{1 + (1 - 2\alpha t)^2}$$

Differentiating Eq. (2) with respect to time

$$w_x = 0 \quad \text{and} \quad w_y = -2a\alpha$$

So, $$w = \sqrt{w_x^2 + w_y^2} = 2a\alpha \tag{3}$$

(c) From Eqs. (2) and (3) we have $\mathbf{v} = a\mathbf{i} + a(1 - 2\alpha t)\mathbf{j}$ and $\mathbf{w} = 2a\alpha\mathbf{j}$.

So at $t - t_0$, $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\mathbf{v}\cdot\mathbf{w}}{vw} = \frac{-a(1 - 2\alpha\, t_0)2a\alpha}{a\sqrt{1 + (1 - 2\alpha t_0)^2}\, 2a\alpha}$

On simplifying, $$1 - 2\alpha t_0 = \pm 1$$

As $$t_0 \neq 0,\ t_0 = \frac{1}{\alpha}$$

1.26 Differentiating motion law: $x = a\sin\omega t$, $y = a(1 - \cos\omega t)$, with respect to time, $v_x = a\omega\cos\omega t$ and $v_y = a\omega\sin\omega t$.

So, $$\mathbf{v} = a\omega\cos\omega t\mathbf{i} + a\omega\sin\omega t\mathbf{j} \tag{1}$$

and $$v = a\omega = \text{constant} \tag{2}$$

Differentiating Eq. (1) with respect to time

$$\mathbf{w} = \frac{d\mathbf{v}}{dt} = -a\omega^2\sin\omega t\,\mathbf{i} + a\omega^2\cos\omega t\mathbf{j} \tag{3}$$

(a) The distance s traversed by the point during the time τ is given by

$$s = \int_0^\tau v\,dt = \int_0^\tau a\omega\,dt = a\omega\tau \quad \text{(using Eq. 2)}$$

(b) Taking scalar product of $\mathbf{v}$ and $\mathbf{w}$

We get, $\mathbf{v}\cdot\mathbf{w} = (a\omega\cos\omega t\mathbf{i} + a\omega\sin\omega t\mathbf{j})\cdot(a\omega^2\sin\omega t(-\mathbf{i}) + a\omega^2\cos\omega t\mathbf{j})$

So, $$\mathbf{v}\cdot\mathbf{w} = -a^2\omega^2 \sin\omega t\cos\omega t + a^2\omega^3 \sin\omega t\cos\omega t = 0$$

Thus, $\mathbf{v}\perp\mathbf{w}$, i.e., the angle between velocity vector and acceleration vector equals $\pi/2$.

1.27 According to the problem

$$\mathbf{w} = w(-\mathbf{j})$$

So, $$w_x = \frac{dv_x}{dt} = 0 \quad \text{and} \quad w_y = \frac{dv_y}{dt} = -w \tag{1}$$

Differentiating equation of trajectory, $y = ax - bx^2$, with respect to time

$$\frac{dy}{dt} = \frac{a\,dx}{dt} - 2bx\frac{dx}{dt} \tag{2}$$

So, $$\left.\frac{dy}{dt}\right|_{x=0} = \left.\frac{a\,dx}{dt}\right|_{x=0}$$

Again differentiating with respect to time

$$\frac{d^2y}{dt^2} = \frac{ad^2x}{dt^2} - 2b\left(\frac{dx}{dt}\right)^2 - 2b\,x\frac{d^2x}{dt^2}$$

or $$-w = a(0) - 2b\left(\frac{dx}{dt}\right)^2 - 2bx(0) \quad \text{(using Eq. 1)}$$

or $$\frac{dx}{dt} = \sqrt{\frac{w}{2b}} \quad \text{(using Eq. 1)} \tag{3}$$

Using Eq. (3) in Eq. (2) $$\left.\frac{dy}{dt}\right|_{x=0} = a\sqrt{\frac{w}{2b}} \tag{4}$$

Hence, the velocity of the particle at the origin

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2_{x=0} + \left(\frac{dy}{dt}\right)^2_{x=0}} = \sqrt{\frac{w}{2b} + a^2\frac{w}{2b}} \quad \text{(using Eqs. 3 and 4)}$$

Hence, $$v = \sqrt{\frac{w}{2b}(1 + a^2)}$$

1.28 As the body is under gravity of constant acceleration $\mathbf{g}$, its velocity vector and displacement vectors are:

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{g}t \tag{1}$$

and $$\Delta\mathbf{r} = \mathbf{r} = \mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2 \quad (\text{as } \mathbf{r} = 0 \text{ at } t = 0) \tag{2}$$

So, $<\mathbf{v}>$ over the first t seconds

$$<\mathbf{v}> = \frac{\Delta \mathbf{r}}{\Delta t} = \frac{\mathbf{r}}{t} = \mathbf{v}_0 + \frac{1}{2}\mathbf{g}t \tag{3}$$

Hence from Eq. (3), $<\mathbf{v}>$ over the first τ seconds

$$<\mathbf{v}> = \mathbf{v}_0 + \frac{\mathbf{g}}{2}\tau \tag{4}$$

For evaluating t, take

$$\mathbf{v}\cdot\mathbf{v} = (\mathbf{v}_0 + \mathbf{g}t)\cdot(\mathbf{v}_0 + \mathbf{g}t) = v_0^2 + 2(\mathbf{v}_0 \cdot \mathrm{g})t + g^2 t^2$$

or $$v^2 = v_0^2 + 2(\mathbf{v}_0 \cdot \mathbf{g})\, t + g^2 t^2$$

But we have $v = v_0$ at $t = 0$ and also at $t = \tau$ (also from energy conservation).

Hence using this property in Eq. (5),

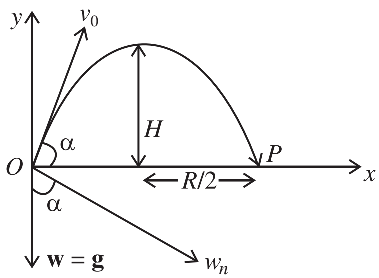
$$v_0^2 = v_0^2 + 2(\mathbf{v}_0 \cdot \mathbf{g})\tau + g^2\tau^2$$

As $$\tau \neq 0, \text{ so, } \tau = -\frac{2(\mathbf{v}_0 \cdot \mathbf{g})}{g^2}$$

Putting this value of τ in Eq. (4), the average velocity over the time of flight

$$<\mathbf{v}> = \mathbf{v}_0 - \mathbf{g}\,\frac{(\mathbf{v}_0 \cdot \mathbf{g})}{g^2}$$

1.29 The body thrown in air with velocity v_0 at an angle α from the horizontal lands at point P on the Earth's surface at same horizontal level (see figure). The point of projection is taken as origin, so $\Delta x = x$ and $\Delta y = y$.



(a) From the equation $$\Delta y = v_{0y}t + \frac{1}{2}w_y t^2$$

or $$0 = v_0 \sin\alpha\tau - \frac{1}{2}g\tau^2$$

As $\tau \neq 0$, so, time of motion $\tau = \dfrac{2v_0 \sin\alpha}{g}$

(b) At the maximum height of ascent, $v_y = 0$

so, from the equation $$v_y^2 = v_{0y}^2 + 2w_y\,\Delta y$$
$$0 = (v_0 \sin\alpha)^2 - 2gH$$

Hence maximum height $H = \dfrac{v_0^2 \sin^2\alpha}{2g}$

During the time of motion the net horizontal displacement or horizontal range, will be obtained by the equation

$$\Delta x = v_{0x}t + \frac{1}{2}w_x\tau^2$$

or $$R = v_0 \cos\alpha\tau - \frac{1}{2}(0)\,\tau^2 = v_0 \cos\alpha\,\tau = \frac{v_0^2 \sin 2\alpha}{g}$$

when $R = H$

$$\frac{v_0^2 \sin 2\alpha}{g} = \frac{v_0^2 \sin^2\alpha}{2g} \quad \text{or} \quad \tan\alpha = 4, \quad \text{so,} \quad \alpha = \tan^{-1} 4$$

(c) For the body, $x(t)$ and $y(t)$ are

$$x = v_0 \cos\alpha t \tag{1}$$

and $$y = v_0 \sin\alpha t - \frac{1}{2} g t^2 \tag{2}$$

Hence putting the value of t from Eq. (1) into Eq. (2) we get

$$y = v_0 \sin\alpha\left(\frac{x}{v_0 \cos\alpha}\right) - \frac{1}{2} g \left(\frac{x}{v_0 \cos\alpha}\right)^2 = x\tan\alpha - \frac{g x^2}{2\, v_0^2 \cos^2\alpha}$$

Note: *One can use the formula of curvature radius of a trajectory $y(x)$, to solve part (d),*

$$R = \frac{[1 + (dy/dx)^2]^{3/2}}{|\, d^2y/dx^2 \,|}$$

1.30 We have, $$v_x = v_0 \cos\alpha,\ v_y = v_0 \sin\alpha - gt$$

Thus, $$v^2 = v_x^2 + v_y^2 = v_0^2 - 2gt\, v_0 \sin\alpha + g^2 t^2$$

Now, $$w_t = \frac{dv_t}{dt} = \frac{1}{2v_t}\frac{d}{dt}(v_t)^2 = \frac{1}{v}(g^2 t - g v_0 \sin\alpha)$$

$$= -\frac{g}{v}(v_0 \sin\alpha - gt) = -g\frac{v_y}{v_t}$$

Hence, $$|w_t| = g\frac{|v_y|}{v}$$

Now, $$w_n = \sqrt{w^2 - w_t^2} = \sqrt{g^2 - g^2\frac{v_y^2}{v_t^2}}$$

or $$w_n = g\frac{v_x}{v_t}\left[\text{where } v_x = \sqrt{v_t^2 - v_y^2} = \sqrt{v^2 - v_y^2}\right]$$

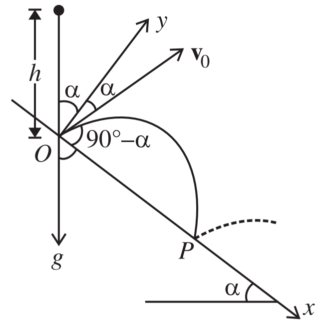
$$w_v = w_t = -g\frac{v_y}{v}$$

On the basis of obtained expressions or facts the sought plots can be drawn as shown in the figure of answer sheet.

1.31 The ball strikes the inclined plane (Ox) at point O (origin) with velocity

$$v_0 = \sqrt{2gh} \tag{1}$$

As the ball classically rebounds, it recalls with same velocity v_0, at the same angle α from the normal on y axis (see figure). Let the ball strike the incline second time at P, which is at a distance l (say) from the point O, along the incline. From the equation



$$y = v_{0y}\,t + \frac{1}{2}\,w_y t^2$$

$$0 = v_0 \cos\alpha\tau - \frac{1}{2}\,g\cos\alpha\tau^2$$

where τ is the time of motion of ball in air while moving from O to P.

As $\qquad \tau \neq 0, \quad \text{so,} \quad \tau = \dfrac{2v_0}{g} \qquad (2)$

Now from the equation.

$$x = v_{0x}t + \frac{1}{2}\,w_x t^2$$

$$l = v_0 \sin\alpha\tau + \frac{1}{2} g\sin\alpha\tau^2$$

So, $\qquad l = v_0 \sin\alpha\left(\dfrac{2v_0}{g}\right) + \dfrac{1}{2}\,g\sin\alpha\left(\dfrac{2v_0}{g}\right) = \dfrac{4v_0^2\sin\alpha}{g} \quad$ (using Eq. 2)

Hence the sought distance,

$$l = \frac{4(2gh)\sin\alpha}{g} = 8h\sin\alpha \quad \text{(using Eq. 1)}$$

1.32 Total time of motion

$$\tau = \frac{2v_0\sin\alpha}{g} \quad \text{or} \quad \sin\alpha = \frac{\tau g}{2v_0} = \frac{9.8\tau}{2\times 240} \tag{1}$$

and horizontal range

$$R = v_0\cos\alpha\tau \quad \text{or} \quad \cos\alpha = \frac{R}{v_0\tau} = \frac{5100}{240\,\tau} = \frac{85}{4\tau} \tag{2}$$

From Eqs. (1) and (2) $\qquad \dfrac{(9.8)^2\tau^2}{(480)^2} + \dfrac{(85)^2}{(4\tau^2)^2} = 1$

On simplifying $\qquad \tau^4 - 2400\tau^2 + 1083750 = 1$

Solving for τ^2, we get

$$\tau^2 = \frac{2400 \pm \sqrt{1425000}}{2} = \frac{2400 \pm 1194}{2}$$

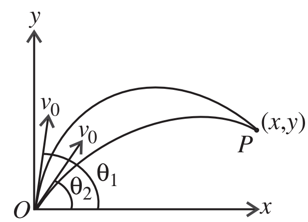
Thus, $\tau = 42.39 \text{ s} = 0.71 \text{ min}$

and $\tau = 24.55 \text{ s} = 0.41 \text{ min}$ (depending on the angle α)

1.33 Let the shells collide at the point $P(x, y)$. If the first shell takes t s to collide with second and Δt be the time interval between the firings, then

$$x = v_0 \cos\theta_1 \, t = v_0 \cos\theta_2 \, (t - \Delta t) \qquad (1)$$

and $$y = v_0 \sin\theta_2 \, (t - \Delta t) - \frac{1}{2} g \, (t - \Delta t)^2 \qquad (2)$$



From Eq. (1) $$t = \frac{\Delta t \cos\theta_2}{\cos\theta_2 - \cos\theta_1} \qquad (3)$$

From Eqs. (2) and (3)

$$\Delta t = \frac{2v_0 \sin(\theta_1 - \theta_2)}{g(\cos\theta_2 + \cos\theta_1)} \text{ as } \Delta t \neq 0$$

$$= 11 \text{ s (on substituting values)}$$

1.34 According to the problem

(a) $$\frac{dy}{dt} = v_0 \quad \text{or} \quad dy = v_0 \, dt$$

Integrating $$\int_0^y dy = v_0 \int_0^t dt \quad \text{or} \quad y = v_0 t \qquad (1)$$

Also, we have $$\frac{dx}{dt} = ay \quad \text{or} \quad dx = ay \, dt = av_0 t \, dt \quad \text{(using Eq. 1)}$$

So, $$\int_0^x dx = av_0 \int_0^t t \, dt, \quad \text{or} \quad x = \frac{1}{2} av_0 t^2 = \frac{1}{2} \frac{ay^2}{v_0} \quad \text{(using Eq. 1)}$$

(b) According to the problem

$$v_y = v_0 \text{ and } v_x = ay \qquad (2)$$

So, $$v = \sqrt{v_x^2 + v_y^2} = \sqrt{v_0^2 + a^2 y^2}$$

Therefore, $$w_t = \frac{dv}{dt} = \frac{a^2 y}{\sqrt{v_0^2 + ay^2}} \frac{dy}{dt} = \frac{a^2 y}{\sqrt{1 + (ay/v_0)^2}}$$

Differentiating Eq. (2) with respect to time.

$$\frac{dv_y}{dt} = w_y = 0 \quad \text{and} \quad \frac{dv_x}{dt} = w_x = a\frac{dy}{dt} = av_0$$

So, $$w = |w_x| = av_0$$

Hence, $$w_n = \sqrt{w^2 - w_t^2} = \sqrt{a^2v_0^2 - \frac{a^4y^2}{1 + (ay/v_0)^2}} = \frac{av_0}{\sqrt{1 + (ay/v_0)^2}}$$

1.35 (a) The velocity vector of the particle

$$\mathbf{v} = a\mathbf{i} + bx\mathbf{j}$$

So, $$\frac{dx}{dt} = a \quad \text{and} \quad \frac{dy}{dt} = bx \qquad (1)$$

From Eq. (1) $$\int_0^x dx = a\int_0^t dt \quad \text{or} \quad x = at \qquad (2)$$

and $$dy = bx\,dt = bat\,dt$$

Integrating $$\int_0^y dy = ab\int_0^t t\,dt \quad \text{or} \quad y = \frac{1}{2}abt^2 \qquad (3)$$

From Eqs. (2) and (3), we get,

$$y = \frac{b}{2a}(x^2) \qquad (4)$$

(b) The curvature radius of trajectory $y(x)$ is

$$R = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|} \qquad (5)$$

Let us differentiate the path equation $y = \frac{b}{2a}x^2$ with respect to x,

$$\frac{dy}{dx} = \frac{b}{a}x \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{b}{a} \qquad (6)$$

From Eqs. (5) and (6), the sought curvature radius is

$$R = \frac{a}{b}\left[1 + \left(\frac{b}{a}x\right)^2\right]^{3/2}$$

1.36 In accordance with the problem

$$w_t = \mathbf{a} \cdot \boldsymbol{\tau}$$

But, $$w_t = \frac{v\,dv}{ds} \quad \text{or} \quad v\,dv = w_t\,ds$$

So, $$v\,dv = (\mathbf{a} \cdot \boldsymbol{\tau})\,ds = \mathbf{a} \cdot ds\,\boldsymbol{\tau} = \mathbf{a} \cdot d\mathbf{r}$$

or $$v\,dv = a\mathbf{i} \cdot d\mathbf{r} = a\,dx \text{ (because } \mathbf{a} \text{ is directed towards the } x\text{-axis)}$$

So, $$\int_0^v v\,dv = a\int_0^x dx$$

Hence, $$v^2 = 2ax \quad \text{or} \quad v = \sqrt{2ax}$$

1.37 The velocity of the particle $v = at$.

So, $$\frac{dv}{dt} = w_t = a \tag{1}$$

and $$w_n = \frac{v^2}{R} = \frac{a^2t^2}{R} \quad \text{(using } v = at\text{)} \tag{2}$$

From $$s = \int v\,dt$$

$$2\pi R\eta = \int_0^t v\,dt = \int_0^t at\,dt$$

So, $$\frac{4\pi\eta}{a} = \frac{t^2}{R} \tag{3}$$

From Eqs. (2) and (3), $w_n = 4\pi a\eta$

Hence, $$w = \sqrt{w_t^2 + w_n^2}$$

$$= \sqrt{a^2 + (4\pi a\eta)^2} = a\sqrt{1 + 16\pi^2\eta^2} = 0.8 \text{ m/s}^2$$

1.38 (a) According to the problem

$$|w_t| = |w_n|$$

For $v(t)$, $$\frac{-dv}{dt} = \frac{v^2}{R}$$

Integrating this equation from $v_0 \leq v \leq v$ and $0 \leq t \leq t$, we get

$$-\int_{v_0}^{v} \frac{dv}{v^2} = \frac{1}{R}\int_0^t dt \quad \text{or} \quad v = \frac{v_0}{\left(1 + \frac{v_0 t}{R}\right)} \tag{1}$$

Now for $v\,(s)$, $$-\frac{v\,dv}{ds} = \frac{v^2}{R}$$

Integrating this equation from $v_0 \le v \le v$ and $0 \le s \le s$, we get

$$\int_{v_0}^{v} \frac{dv}{v} = -\frac{1}{R}\int_0^s ds \quad \text{or} \quad \ln\frac{v}{v_0} = -\frac{s}{R}$$

Hence, $$v = v_0 e^{-s/R} \tag{2}$$

(b) The normal acceleration of the point

$$w_n = \frac{v^2}{R} = \frac{v_0^2 e^{-2s/R}}{R} \quad \text{(using Eq. 2)}$$

In accordance with the problem

$$|w_t| = |w_n| \quad \text{and} \quad w_t\mathbf{t} \perp w_n\mathbf{n}$$

So, $$w = \sqrt{2}w_n = \sqrt{2}\,\frac{v_0^2}{R}e^{-2s/R} = \sqrt{2}\,\frac{v^2}{R}$$

1.39 From the equation $$v = a\sqrt{s}$$

$$w_t = \frac{dv}{dt} = \frac{a}{2\sqrt{s}}\frac{ds}{dt} = \frac{a}{2\sqrt{s}}\,a\sqrt{s} = \frac{a^2}{2}$$

and $$w_n = \frac{v^2}{R} = \frac{a^2 s}{R}$$

As w_t is a positive constant, the speed of the particle increases with time, and the tangential acceleration vector and velocity vector coincides in direction.
Hence the angle between **v** and **w** is equal to angle between $\mathbf{w}_t\,\mathbf{t}$ and **w** and α can be found by the formula

$$\tan\alpha = \frac{|w_n|}{|w_t|} = \frac{a^2 s/R}{a^2/2} = \frac{2s}{R}$$

1.40 Differentiating the equation $l = a\sin\omega t$

$$\frac{dl}{dt} = v = a\omega\cos\omega t$$

So, $$w_t = \frac{dv}{dt} = -a\omega^2\sin\omega t \tag{1}$$

and $$w_n = \frac{v^2}{R} = \frac{a^2\omega^2\cos^2\omega t}{R} \tag{2}$$

(a) At the point $l = 0$, $\sin\omega t = 0$ and $\cos\omega t = \pm 1$ so, $\omega t = 0, \pi$, etc.

Hence, $$w = w_n = \frac{a^2\omega^2}{R} = 2.6 \text{ m/s}^2 \quad \text{(on substituting values)}$$

Similarly at $l = \pm a$, $\sin\omega t = \pm 1$ and $\cos\omega t = 0$, so, $w_n = 0$.

Hence, $$w = |w_t| = a\omega^2 = 3.2 \text{ m/s}^2 \quad \text{(on substituting values)}$$

(b) Using $\sin\omega t = l/a$ in Eqs. (1) and (2) we get

$$w_t = -\omega^2 l$$

and $$w_n = \frac{\omega^2}{R}(a^2 - l^2)$$

Therefore, $$w_t = \sqrt{\omega^4 l^2 + \frac{\omega^4}{R^2}(a^2 - l^2)^2} = f(l)$$

On differentiating with respect to l and putting

$$\frac{dw_t}{dl} = 0$$

we get, $$l^2 = l_{\text{m}}^2 = a^2 - \frac{R^2}{2} = a^2\left(1 - \frac{R^2}{2a^2}\right)$$

So, $$l_m = \pm a\sqrt{1 - \frac{R^2}{2a^2}} = \pm 0.37 \text{ m} \quad \text{(on substituting values)}$$

The corresponding $w_{\min}$ at $l = l_m$ is

$$w_{\min} = \sqrt{\omega^4\left(a^2 - \frac{R^2}{2}\right) + \frac{\omega^4}{R^2}\left(\frac{R^2}{2}\right)^2}$$

$$= a\omega^2\sqrt{1 - \frac{R^2}{2a^2} + \frac{R^2}{4a^2}}$$

$$= a\omega^2\sqrt{1 - \left(\frac{R}{2a}\right)^2} = 2.5 \text{ m/s}^2 \quad \text{(on substituting values)}$$

1.41 As $w_t = a$ and at $t = 0$, the point is at rest.

So, $v(t)$ and $s(t)$ are $$v = at \quad \text{and} \quad s = \frac{1}{2}at^2 \tag{1}$$

Let R be the curvature radius, then

$$w_n = \frac{v^2}{R} = \frac{a^2t^2}{R} = \frac{2as}{R} \quad \text{(using Eq. 1)}$$

But according to the problem

$$w_n = bt^4$$

So, $bt^4 = \dfrac{a^2t^2}{R}$ or $R = \dfrac{a^2}{bt^2} = \dfrac{a^2}{2bs}$ (using Eq. 1) (2)

Therefore, $w = \sqrt{w_t^2 + w_n^2} = \sqrt{a^2 + (2as/R)^2} = \sqrt{a^2 + (4bs^2/a^2)^2}$ (using Eq. 2)

Hence, $w = a\sqrt{1 + (4bs^2/a^3)^2}$

1.42 (a) Let us differentiate twice the path equation $y(x)$ with respect to time,

$$\frac{dy}{dt} = 2ax\frac{dx}{dt}; \quad \frac{d^2y}{dt^2} = 2a\left[\left(\frac{dx}{dt}\right)^2 + x\frac{d^2x}{dt^2}\right]$$

Since the particle moves uniformly, its acceleration at all points of the path is normal and at the point $x = 0$ it coincides with the direction of derivative d^2y/dt^2. Keeping in mind that at the point $x = 0$, $\left|\dfrac{dx}{dt}\right| = v$,

we get $w = \left|\dfrac{d^2y}{dt^2}\right|_{x=0} = 2av^2 = w_n$

So, $w_n = 2av^2 = \dfrac{v^2}{R}$ or $R = \dfrac{1}{2a}$

Note: *We can also calculate it from the formula of problem 1.35(b).*

(b) Differentiating the equation of the trajectory with respect to time we see that

$$b^2x\frac{dx}{dt} + a^2y\frac{dy}{dt} = 0 \tag{1}$$

which implies that the vector $(b^2x\mathbf{i} + a^2y\mathbf{j})$ is normal to the velocity vector $\mathbf{v} = \dfrac{dx}{dt}\mathbf{i} + \dfrac{dy}{dt}\mathbf{j}$ which, of course, is along the tangent. Thus the former vector is along the normal and the normal component of acceleration is clearly

$$w_n = \frac{b^2x\dfrac{d^2x}{dt^2} + a^2y\dfrac{d^2y}{dt^2}}{(b^4x^2 + a^4y^2)^{1/2}} \quad \left(\text{on using } w_n = \frac{\mathbf{w}\cdot\mathbf{n}}{|\mathbf{n}|}\right)$$

At $x = 0$, $y = \pm b$, so at $x = 0$

$$w_n = \pm\left.\frac{d^2y}{dt^2}\right|_{x=0}$$

Differentiating Eq. (1)

$$b^2\left(\frac{dx}{dt}\right)^2 + b^2x\left(\frac{d^2x}{dt^2}\right) + a^2\left(\frac{dy}{dt}\right)^2 + a^2y\left(\frac{d^2y}{dt}\right) = 0$$

Also from Eq. (1) $\quad \dfrac{dy}{dt} = 0$ at $x = 0$

So, $\quad \left(\dfrac{dx}{dt}\right) = \pm v$ (since tangential velocity is constant $= v$)

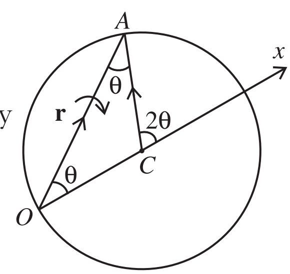
Thus, $\quad \left(\dfrac{d^2y}{dt^2}\right) = \pm\dfrac{b}{a^2}v^2$

and $\quad |w_n| = \dfrac{bv^2}{a^2} = \dfrac{v^2}{R}$

This gives $\quad R = \dfrac{a^2}{b}$

1.43 Angular velocity of point A, with respect to centre C of the circle or turning rate of line CA taking the line OCX as reference line becomes

$$\omega_C = -\frac{d\,(2\theta)}{dt} = 2\left(\frac{-d\theta}{dt}\right) = 2\omega$$



because angular speed of line OA is $\omega = -d\theta/dt$.

The turning rate of line CA is also the turning rate of velocity vector of point A, which is given by v_A/R.

So, $v_A = \omega_C R = (2\omega)R = 0.4$ m/s (on substituting values)

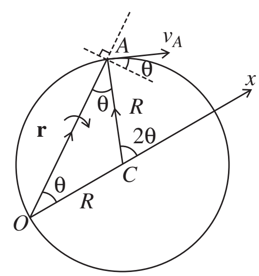
Alternate:

Angular speed of a point relative to origin $= \dfrac{\text{transverse velocity}}{\text{magnitude of position vector}}$

So, angular speed of point A relative to origin O

$$= \frac{\text{component velocity of point } A \text{ perpendicular to } OA}{\text{Length of line } OA}$$

$$\omega = \frac{v_A \cos\theta}{r}$$



But from sine property of triangle OAC, $r = 2R\cos\theta$.

So, $\omega = \dfrac{v_A}{2R}$ or $v_A = 2\omega R$

From both the methods the obtained speed v_A is constant, so the tangential acceleration of particle A is zero and

$$w = w_n = \frac{v_A^2}{R} = \frac{(2\omega R)^2}{R} = 4\omega^2 R = 0.32 \text{ m/s}^2 \text{ (on substituting values)}$$

1.44 Differentiating $\varphi(t)$ with respect to time

$$\frac{d\varphi}{dt} = \omega_z = 2at \qquad (1)$$

For fixed axis rotation, the speed of the point A

$$v = \omega R = 2atR \quad \text{or} \quad R = \frac{v}{2at} \qquad (2)$$

Differentiating with respect to time

$$w_t = \frac{dv}{dt} = 2aR = \frac{v}{t} \quad \text{(using Eq. 1)}$$

But, $$w_n = \frac{v^2}{R} = \frac{v^2}{v/2at} = 2atv \quad \text{(using Eq. 2)}$$

So, $$w = \sqrt{w_t^2 + w_n^2} = \sqrt{(v/t)^2 + (2atv)^2} = \frac{v}{t}\sqrt{1 + 4a^2t^4}$$

$$= 0.7 \text{ m/s}^2 \quad \text{(on substituting values)}$$

1.45 The shell acquires a constant angular acceleration at the same time as it accelerates linearly.

$$l = \frac{1}{2} wt^2 \quad \text{and} \quad 2\pi n = \frac{1}{2}\beta t^2$$

So, $$\frac{w}{l} = \frac{\beta}{2\pi n}$$

(where w = linear acceleration and β = angular acceleration).

Then, $$\omega = \sqrt{2\beta\, 2\pi n} = \sqrt{2\frac{w}{l}(2\pi n)^2}$$

But, $$v^2 = 2wl$$

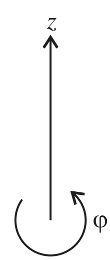
Hence, $$\omega = \frac{2\pi nv}{l} = 2.0 \times 10^3 \text{ rad/s} \quad \text{(on substituting values)}$$

1.46 Let us take the rotation axis as z-axis whose positive direction is associated with the positive direction of the co-ordinate φ, the rotation angle, in accordance with the right-hand screw rule (see figure).

(a) Differentiating $\varphi(t)$ with respect to time twice, we get

$$\frac{d\varphi}{dt} = a - 3bt^2 = \omega_z \qquad (1)$$

$$\frac{d^2\varphi}{dt^2} = \frac{d\omega_z}{dt} = \beta_z = -6bt \qquad (2)$$



From Eq. (1) the solid body comes to stop at

$$\Delta t = t = \sqrt{\frac{a}{3b}}$$

The angular velocity $\omega = a - 3\ bt^2$, for $0 \le t \le \sqrt{a/3b}$

So, $$<\omega> = \frac{\int \omega\, dt}{\int dt} = \frac{\int_0^{\sqrt{a/3b}} (a - 3bt^2)\, dt}{\int_0^{\sqrt{a/3b}} dt}$$

$$= [at - bt^3]_0^{\sqrt{a/3b}} / \sqrt{a/3b} = 2a/3 = 4 \text{ rad/s}$$

Similarly, $\beta = |\beta_z| = 6bt$ for all values of t.

So, $$<\beta> = \frac{\int \beta\, dt}{\int dt} = \frac{\int_0^{\sqrt{a/3b}} 6\ bt\ dt}{\int_0^{\sqrt{a/3b}} dt} = \sqrt{3\ ab} = 6 \text{ rad/s}^2$$

(b) From Eq. (2), $$\beta_z = -6bt$$

So, $$(\beta_z)_{t=\sqrt{a/3b}} = -6b\sqrt{\frac{a}{3b}} = -2\sqrt{3ab}$$

Hence, $$\beta = |(\beta_z)_{t=\sqrt{a/3b}}| = 2\sqrt{3ab} = 12 \text{ rad/s}^2$$

1.47 Angle α is related with $|w_t|$ and w_n by means of the formula

$$\tan\alpha = \frac{w_n}{|w_t|} \quad \text{where} \quad w_n = \omega^2 R \quad \text{and} \quad |w_t| = \beta R \tag{1}$$

where R is the radius of the circle which an arbitrary point of the body circumscribes. From the given equation $\beta = d\omega/dt = at$ (here $\beta = d\omega/dt$ as β is positive for all values of t)

Integrating $$\int_0^{\omega} d\omega = a\int_0^{t} t dt \quad \text{or} \quad \omega = \frac{1}{2}at^2$$

So, $$w_n = \omega^2 R = \left(\frac{at^2}{2}\right)^2 R = \frac{a^2t^4}{4}R$$

and $$|w_t| = \beta R = at\,R$$

Putting the values of $|w_t|$ and w_n in Eq. (1), we get

$$\tan\alpha = \frac{a^2t^4R/4}{atR} = \frac{at^3}{4} \quad \text{or} \quad t = \left[\left(\frac{4}{a}\right)\tan\alpha\right]^{1/3}$$

$$t = 7\text{s} \quad \text{(on substituting values)}$$

1.48 In accordance with the problem, $\beta_x < 0$.

Thus $$-\frac{d\omega}{dt} = k\sqrt{\omega} \quad \text{(where } k \text{ is proportionality constant)}$$

or $$-\int_{\omega_0}^{\omega}\frac{d\omega}{\sqrt{\omega}} = k\int_0^t dt \quad \text{or} \quad \sqrt{\omega} = \sqrt{\omega_0} - \frac{kt}{2} \tag{1}$$

When $\omega = 0$, total time of rotation $t = \tau = \dfrac{2\sqrt{\omega_0}}{k}$

Average angular velocity $$<\omega> = \frac{\int \omega\,dt}{\int dt} = \frac{\int_0^{2\sqrt{\omega_0/k}}\left(\omega_0 + \frac{k^2t^2}{4} - kt\sqrt{\omega_0}\right)dt}{2\sqrt{\omega_0}/k}$$

Hence, $$<\omega> = \frac{\left[\omega_0 t + \frac{k^2t^3}{12} - \frac{k}{2}\sqrt{\omega_0}\,t^2\right]_0^{2\sqrt{\omega_0/k}}}{2\sqrt{\omega_0}/k} = \frac{\omega_0}{3}$$

1.49 (a) We have $$\omega = \omega_0 - a\varphi = \frac{d\varphi}{dt}$$

Integrating this Eq. within its limit for $(\varphi)\ t$

$$\int_0^{\varphi}\frac{d\varphi}{\omega_0 - a\,\varphi} = \int_0^t dt \quad \text{or } \ln\frac{\omega_0 - k\varphi}{\omega_0} = -kt$$

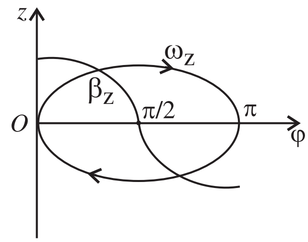
Hence, $$\varphi = \frac{\omega_0}{k}(1 - e^{-kt}) \tag{1}$$

(b) From the equation, $\omega = \omega_0 - k\varphi$ and Eq. (1) or by differentiating Eq. (1)

$$\omega = \omega_0 e^{-kt}$$

1.50 Let us choose the positive direction of z-axis (stationary rotation axis) along the vector β_0. In accordance with the equation

$$\frac{d\omega_z}{dt} = \beta_z \quad \text{or} \quad \omega_z \frac{d\omega_z}{d\varphi} = \beta_z$$



or $$\omega_z d\omega_z = \beta_z d\varphi = \beta \cos \varphi \, d\varphi$$

Integrating this equation within its limit for $\omega_z(\varphi)$

or $$\int_0^{\omega_t} \omega_z \, d\omega_z = \beta_0 \sin \varphi$$

Hence, $$\omega_z = \pm\sqrt{2\beta_0 \sin\varphi}$$

The plot $\omega_z(\varphi)$ is shown in the figure. It can be seen that as the angle φ grows, the vector $\boldsymbol{\omega}$ first increases, coinciding with the direction of the vector $\boldsymbol{\beta}_0(\omega_z > 0)$, reaches the maximum at $\varphi = \varphi/2$, then starts decreasing and finally turns into zero at $\varphi = \pi$. After that the body starts rotating in the opposite direction in a similar fashion ($\omega_z < 0$). As a result, the body will oscillate about the position $\varphi = \varphi/2$ with an amplitude equal to $\pi/2$.

1.51 A rotating disk moves along the x-axis, in plane motion in $x-y$ plane. For the calculation of velocity only, plane motion of a solid can be imagined to be in pure rotation about a point (say I) at a certain instant known as instantaneous centre of rotation. The axis is directed along $\boldsymbol{\omega}$ of the solid which passes through the point I at that instant and is known as instantaneous axis of rotation.

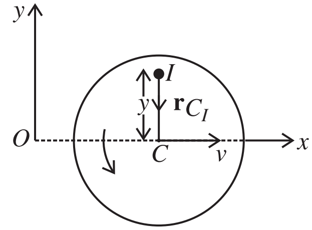
Therefore the velocity vector of an arbitrary point (P) of the solid can be represented as:

$$\mathbf{v}_P = \boldsymbol{\omega} \times \mathbf{r}_{PI} = \omega \times \boldsymbol{\rho}_{PI} \tag{1}$$

(where $\boldsymbol{\rho}_{PI}$ is normal location of point P relative to instantaneous rotation axis passing through point I).

So instantaneous rotation axis I is at the perpendicular distance $\rho_{PI} = v_P/\omega$ from point P. On the basis of Eq. (1) for the centre of mass (C.M.) of the disk, velocity is

$$\mathbf{v}_C = \boldsymbol{\omega} \times \boldsymbol{\rho}_{CI} \tag{2}$$



According to the problem $\mathbf{v}_C \uparrow\uparrow \mathbf{i}$ and $\boldsymbol{\omega} \uparrow\uparrow \mathbf{k}$ so to satisfy the Eq.(2), $\boldsymbol{\rho}_{CI}$ is directed towards $(-\mathbf{j})$. Hence point I is at a distance $\boldsymbol{\rho}_{CI} = y$, above the centre of the disk along y-axis. Using all these facts in Eq. (2), we get

$$v_C = \omega y \quad \text{or} \quad y = \frac{v_C}{\omega} \tag{3}$$

(a) From the angular kinematical equation

$$\omega_z = \omega_{0z} + \beta_z t \quad (4)$$
$$\omega = \beta t$$

On the other hand

$$x = vt,\ t = x/v \ \text{(where } x \text{ is the } x \text{ coordinate of the C.M.)} \quad (5)$$

From Eqs. (4) and (5),

$$\omega = \frac{\beta x}{v}$$

Using this value of ω in Eq. (3) we get

$$y = \frac{v_C}{\omega} = \frac{v}{\beta x/v} = \frac{v^2}{\beta x} \quad \text{(hyperbola)}$$

(b) As centre C moves with constant acceleration w, with zero initial velocity

So, $$x = \frac{1}{2} wt^2 \quad \text{and} \quad v_c = wt$$

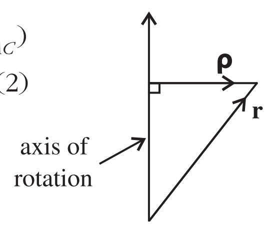
Therefore, $$v_C = w\sqrt{\frac{2x}{w}} = \sqrt{2xw}$$

Hence, $$y = \frac{v_C}{\omega} = \frac{\sqrt{2wx}}{\omega} \quad \text{(parabola)}$$

1.52 (a) The general plane motion of a solid can be imagined as the combination of translation with C.M. and rotation about C.M.

So, $$\mathbf{v}_A = \mathbf{v}_C + \mathbf{v}_{AC} = \mathbf{v}_C + \boldsymbol{\omega} \times \mathbf{r}_{AC} = \mathbf{v}_C + \boldsymbol{\omega} \times \boldsymbol{\rho}_{AC} \quad (1)$$

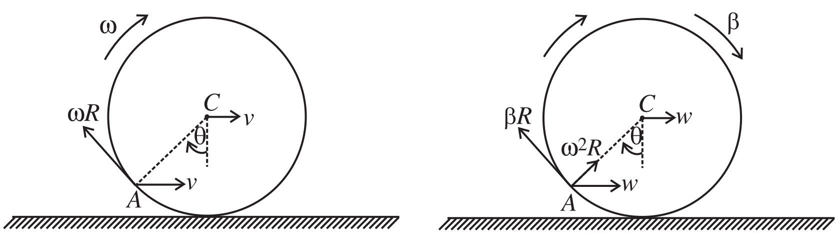
and $$\mathbf{w}_A = \mathbf{w}_C + \mathbf{w}_{AC} = \mathbf{w}_C + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{AC}) + (\boldsymbol{\beta} \times \mathbf{r}_{AC})$$
$$= \mathbf{w}_C + \omega^2(-\boldsymbol{\rho}_{AC}) + (\boldsymbol{\beta} \times \boldsymbol{\rho}_{AC}) \quad (2)$$



where $\boldsymbol{\rho}$ is the component of $\mathbf{r}$ normal to the axis of rotation and directed away from it. In this problem $\rho_{AC} = r_{AC} = R$ and $v_C = v$.

Let the point A touch the horizontal surface at $t = 0$, further let us locate the point A at time t, when it makes an angle θ from vertical (see figure).

On the basis of Eqs. (1) and (2) the pictorial diagrams for velocity and acceleration are as follows:



As the rolling is without slipping along a line, so, $v_c = \omega R$ and $w_C = \beta R$.
According to the problem $v_C = v$ (constant), so $\omega = v/R$, $w_C = 0$ and $\beta = 0$. Using these facts, $\mathbf{w}_A = v^2/R = 2.0\ \text{m/s}^2$ and the vector $\mathbf{w}$, is directed toward centre C of the wheel:

$$\begin{aligned} v_A &= \sqrt{v^2 + (\omega R)^2 + 2v(\omega R)\cos(\pi - \theta)} \\ &= \sqrt{v^2 + v^2 + 2v^2\cos(\pi - \theta)} = v\sqrt{2(1 - \cos\theta)} \\ &= 2v\sin(\theta/2) = 2v\sin(\omega t/2) \end{aligned}$$

Hence, distance covered by the point A during time interval $2\pi/\omega$

$$s = \int v_A dt = \int_0^{2\pi/\omega} 2\ v\sin(\omega t/2)dt$$

$$= \frac{8v}{\omega} = 8R = 4.0\ \text{m} \quad \text{(on substituting values)}$$

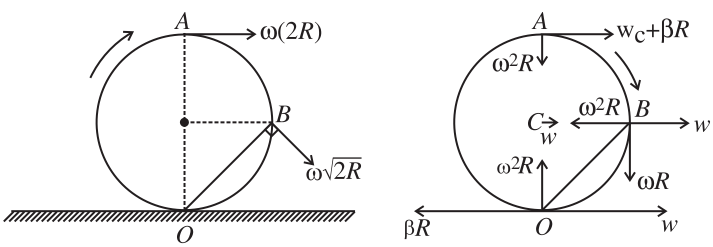
Note: *One can easily find v_A, assuming the body to rotate about the instantaneous centre of rotation of zero velocity (not of zero acceleration), which is the contact point of the rolling body in this case.*

1.53 As the ball rolls without slipping on the rigid surface along a line so, contact point O has zero velocity, which is possible when the ball rotates in clock wise sense if it's centre moves towards right and such that $v_C = \omega R$, and $w_C = w = \beta R$. As motion starts at $t = 0$ with constant acceleration w so at time t, the velocity of mass centre C becomes $v_C = wt$ and $\omega = v_C/R = wt/R$.

(a) The contact point O becomes instantaneous centre of rotation, thus, the velocity of any arbitrary point P (say) of the ball can be obtained by the relation

$$\mathbf{v}_P = \boldsymbol{\omega} \times \boldsymbol{\rho}_{PO} \tag{1}$$

Using Eq. (1), the pictorial diagram to find the velocities of the points A and B of the ball is shown below



Hence, $\quad v_A = 2v_C = 2wt = 10\ \text{cm/s}$

and $\quad v_B = \sqrt{2}\ v_C = \sqrt{2}\ wt = 7.1\ \text{cm/s}$

(b) One can write for the any arbitrary point P

$$\mathbf{w}_P = \mathbf{w}_C + \mathbf{w}_{PC} = \mathbf{w}_C + \boldsymbol{\omega}^2 (-\boldsymbol{\rho}_{PC}) + (\boldsymbol{\beta} \times \boldsymbol{\rho}_{PC}) \tag{2}$$

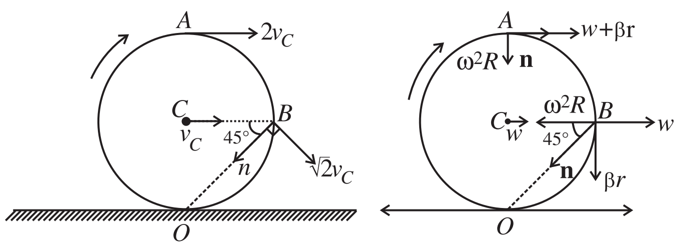
Taking into account Eq. (2), one can easily get the acceleration of the points O, A and B on using the pictorial diagram.

So, $$w_0 = w^2t^2/R = 2.5 \text{ cm/s}^2$$

$$w_A = \sqrt{4w^2 + \frac{w^4t^4}{R^2}} = 2w\sqrt{1 + \left(\frac{wt^2}{2R}\right)^2} = 5.6 \text{ cm/s}^2$$

and $$w_B = \sqrt{\left(w - \frac{w^2t^2}{R}\right)^2 + w^2} = 2.5 \text{ m/s}^2$$

1.54 In the frame of horizontal plane the direction of velocity at a point is along the tangent of the curve at that point, so normal acceleration at that point is perpendicular to the direction of velocity vector and directed inwards. We make the pictorial diagram as in solution of problem 1.53.



As an arbitrary point of the cylinder follows a curve, its normal acceleration and radius of curvature are related by the well-known equation

$$w_n = \frac{v^2}{R}$$

So, for point A, $$w_{A(n)} = \frac{v_A^2}{R_A}$$

or $$\omega^2 r = \frac{(2v_C)^2}{R_A}$$

which gives $$R_A = 4r \quad (\text{on using } v_C = \omega r)$$

Similarly for point B, $$w_{B(n)} = \frac{v_B^2}{R_B}$$

or $$\omega^2 r\cos 45^\circ = \frac{(\sqrt{2}v_C)^2}{R_B}$$

$$R_B = 2\sqrt{2}\,\frac{v_C^2}{\omega^2 r} = 2\sqrt{2}r \text{ (again on using } v_C = \omega r)$$

1.55 The angular velocity is a vector as infinitesimal rotation commute. As for relative linear velocity, the relative angular velocity of the body 1 with respect to the body 2 is clearly

$$\boldsymbol{\omega}_{12} = \boldsymbol{\omega}_1 - \boldsymbol{\omega}_2$$

As $\boldsymbol{\omega}_1 \perp \boldsymbol{\omega}_2$, so, $|\boldsymbol{\omega}_{12}| = \sqrt{\omega_1^2 + \omega_2^2} = 5$ rad/s (on substituting values)

If frame K' rotates with angular velocity $\boldsymbol{\omega}$ with respect to frame K, the relation between the time derivatives of any vector $\mathbf{a}$, seen from different frames is:

$$\left.\frac{d\mathbf{a}}{dt}\right|_K = \left.\frac{d\mathbf{a}}{dt}\right|_{K'} + \boldsymbol{\omega}\times\mathbf{a}$$

If frame attached with the intersection point of two axes (about which solids are rotating) is K and the frame attached with rotating solid with angular velocity $\boldsymbol{\omega}_2$ is K'

Then, $$\left(\frac{d\boldsymbol{\omega}_1}{dt}\right)_K = \left(\frac{d\boldsymbol{\omega}_1}{dt}\right)_{K'} + \boldsymbol{\omega}_2 \times \boldsymbol{\omega}_1$$

However, $\left(\frac{d\boldsymbol{\omega}_1}{dt}\right)_K = 0$ (as the first body rotates with constant angular velocity in space)

and $\left(\frac{d\boldsymbol{\omega}_1}{dt}\right)_{K'} = \boldsymbol{\beta}_{12}$ (the sought angular velocity)

Hence, $$\boldsymbol{\beta}_{12} = \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2$$

So, $$|\boldsymbol{\beta}_{12}| = \omega_1\omega_2 = 12 \text{ rad/s}^2 \text{ (on substituting values)}$$

1.56 (a) We have $$\boldsymbol{\omega} = at\mathbf{i} + bt^2\mathbf{j} \tag{1}$$

So, $\omega = \sqrt{(at)^2 + (bt^2)^2}$, thus, $\omega|_{t\,=\,10\text{s}} = 7.81$ rad/s

Differentiating Eq. (1) with respect to time

$$\boldsymbol{\beta} = \frac{d\boldsymbol{\omega}}{dt} = a\mathbf{i} + 2bt\mathbf{j} \tag{2}$$

So, $$\boldsymbol{\beta} = \sqrt{a^2 + (2bt)^2}$$

and $$\boldsymbol{\beta}\big|_{t=10\text{ s}} = 1.3 \text{ rad/s}^2$$

(b) $$\cos\alpha = \frac{\boldsymbol{\omega}\cdot\boldsymbol{\beta}}{\omega\beta} = \frac{(at\mathbf{i} + bt^2\mathbf{j})\cdot(a\mathbf{i} + 2bt\mathbf{j})}{\sqrt{(at)^2 + (bt^2)^2}\sqrt{a^2 + (2bt)^2}}$$

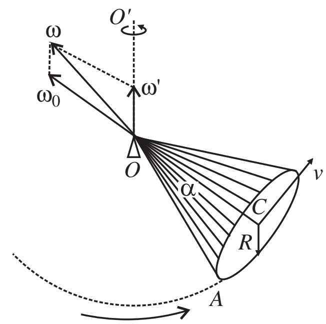
Putting the values of (a) and (b) and taking $t = 10$ s, we get

$$\alpha = 17°$$

1.57 (a) Let the axis of the cone (OC) rotate in anticlockwise sense with constant angular velocity $\boldsymbol{\omega}'$ and the cone itself about it's own axis (OC) in clockwise sense with angular velocity $\boldsymbol{\omega}_0$. Then the resultant angular velocity of the cone

$$\boldsymbol{\omega} = \boldsymbol{\omega} + \boldsymbol{\omega}_0$$

As the rolling is pure the magnitudes of the vectors $\boldsymbol{\omega}'$ and $\boldsymbol{\omega}_0$ can be easily found from the figure as



$$\omega' = \frac{v}{R\cot\alpha} \quad \text{and} \quad \omega_0 = \frac{v}{R}$$

As $\boldsymbol{\omega} \perp \boldsymbol{\omega}_0$, hence

$$\omega = \sqrt{\omega'^2 + \omega_0^2}\sqrt{\left(\frac{v}{R\cot\alpha}\right)^2 + \left(\frac{v}{R}\right)^2}$$

$$= \frac{v}{R\cos\alpha} = 2.32 \text{ rad/s}$$

(b) Vector of angular acceleration

$$\boldsymbol{\beta} = \frac{d\boldsymbol{\omega}}{dt} = \frac{d(\boldsymbol{\omega}' + \boldsymbol{\omega}_0)}{dt} = \frac{d\boldsymbol{\omega}_0}{dt} \quad (\text{as } \boldsymbol{\omega}' = \text{constant})$$

If any vector a (say) rotates with angular velocity $\boldsymbol{\omega}$ keeping its values constant then $d\mathbf{a}/dt = \boldsymbol{\omega}\times\mathbf{a}$. The vector $\boldsymbol{\omega}_0$ keeping its magnitude constant rotates about the OO' axis with the angular velocity $\boldsymbol{\omega}'$. So $d\boldsymbol{\omega}_0/dt = (\boldsymbol{\omega}'\times\boldsymbol{\omega}_0)$. Hence, $\boldsymbol{\beta} = \boldsymbol{\omega}'\times\boldsymbol{\omega}_0$ The magnitude of the vector $\boldsymbol{\beta}$ is equal to $\beta = \omega'\,\omega_0$ (as $\boldsymbol{\omega}'\perp\boldsymbol{\omega}_0$).

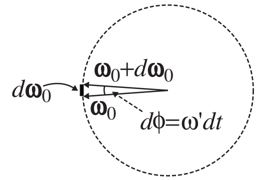
So, $$\beta = \frac{v}{R\cot\alpha}\left(\frac{v}{R}\right)$$

$$= \frac{v^2}{R^2}\tan\alpha = 2.3 \text{ rad/s}^2, \text{ on putting the values of } v, R \text{ and } \alpha$$

Alternate:

Vector $\boldsymbol{\omega}_0$ is turning keeping the magnitude constant. To find $d\boldsymbol{\omega}_0$, let us make a vector diagram as shown in the figure. The circle shown in the figure has the radius ω_0 and the tail of $\boldsymbol{\omega}_0$ and of $\boldsymbol{\omega}_0 + d\boldsymbol{\omega}_0$ coincides at the centre of the circle.

Let $\boldsymbol{\omega}_0$ turn by the small angle $d\varphi$ in the differential time interval dt, so $d\varphi = \omega' dt$. Thus $|d\boldsymbol{\omega}_0| \approx$ arc length $= \omega_0 \, d\varphi = \omega_0 \, \omega' dt$. As $\boldsymbol{\omega}_0$ is along radial line and $d\boldsymbol{\omega}_0$ is along the tangent in the turning sense of $\boldsymbol{\omega}_0$ so $d\boldsymbol{\omega}_0 \perp \boldsymbol{\omega}_0$. In vector form



$$d\boldsymbol{\omega}_0 = (\boldsymbol{\omega}' \times \boldsymbol{\omega}_0)\, dt$$

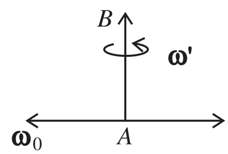
Hence, $$\boldsymbol{\beta} = \frac{d\boldsymbol{\omega}_0}{dt} = (\boldsymbol{\omega}' \times \boldsymbol{\omega}_0)$$

1.58 The axis AB acquired the angular velocity

$$\boldsymbol{\omega}' = \boldsymbol{\beta}_0 t$$

Using the facts of the solution of problem 1.57, the angular velocity of the body is

$$\omega = \sqrt{\omega_0^2 + \omega'^2} = \sqrt{\omega_0^2 + \beta_0^2 t^2} = 0.6 \text{ rad/s}$$



The angular acceleration,

$$\boldsymbol{\beta} = \frac{d\boldsymbol{\omega}'}{dt} = \frac{d(\boldsymbol{\omega}' + \boldsymbol{\omega}_0)}{dt} = \frac{d\boldsymbol{\omega}'}{dt} + \frac{d\boldsymbol{\omega}_0}{dt}$$

But, $$\frac{d\boldsymbol{\omega}_0}{dt} = \boldsymbol{\omega}' \times \boldsymbol{\omega}_0 \quad \text{and} \quad \frac{d\boldsymbol{\omega}'}{dt} = \boldsymbol{\beta}_0$$

So, $$\boldsymbol{\beta} = (\boldsymbol{\beta}_0 + \boldsymbol{\omega}' \times \boldsymbol{\omega}_0) = \boldsymbol{\beta}_0 + (\boldsymbol{\beta}_0 t \times \boldsymbol{\omega}_0) \quad (\text{because } \boldsymbol{\omega}' = \boldsymbol{\beta}_0 t)$$

As, $\boldsymbol{\beta}_0 \perp \boldsymbol{\omega}_0$ so, $\boldsymbol{\beta} = \sqrt{(\omega_0 \beta_0 t)^2 + \beta_0^2} = \beta_0 \sqrt{1 + (\omega_0 t)^2} = 0.2 \text{ rad/s}^2$

1.2 The Fundamental Equation of Dynamics

1.59 Let R be the constant upward thrust on the aerostat of mass m, coming down with a constant acceleration w. Applying Newton's second law of motion for the aerostat in projection form

$$F_y = m w_y$$

$$mg - R = mw \qquad (1)$$

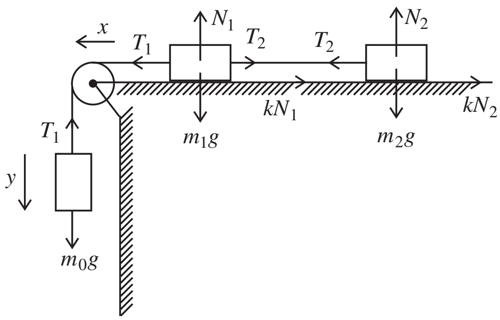
Now, if Δm be the mass to be dumped, then using the equation $F_y = mw_y$

$$R - (m - \Delta m)\, g = (m - \Delta m) w \tag{2}$$

From Eqs. (1) and (2), we get

$$\Delta m = \frac{2mw}{g + w}$$

1.60 Let us write the fundamental equation of dynamics for all the three blocks in terms of projections, having taken the positive direction of x and y axes as shown in the figure and using the fact that kinematical relation between the accelerations is such that the blocks move with same value of acceleration (say w)



$$m_0 g - T_1 = m_0 w \qquad (1)$$

$$T_1 - T_2 - km_1 g = m_1 w \qquad (2)$$

and $$T_2 - km_2 g = m_2 w \qquad (3)$$

The simultaneous solution of Eqs. (1), (2) and (3) yields,

$$w = g\,\frac{[m_0 - k(m_1 + m_2)]}{m_0 + m_1 + m_2}$$

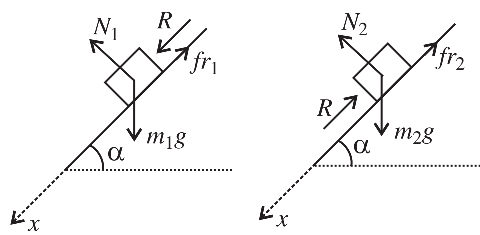
and $$T_2 = \frac{(1 + k)\, m_0}{m_0 + m_1 + m_2}\, m_2 g$$

As the block m_0 moves down with acceleration w, so in vector form

$$\mathbf{w} = \frac{[m_0 - k(m_1 + m_2)]\,\mathbf{g}}{m_0 + m_1 + m_2}$$

1.61 (a) Let us indicate the positive direction of x-axis along the incline (see figure). The figures show the force diagram for the blocks.

Let $\boldsymbol{R}$ be the force of interaction between the bars and they are obviously sliding down with the same constant acceleration w.



Newton's second law of motion in projection form along x-axis for the blocks gives

$$m_1 g \sin\alpha - k_1 m_1\, g \cos\alpha + R = m_1 w \tag{1}$$

$$m_2 g \sin\alpha - R - k_2 m_2\, g \cos\alpha = m_2 w \tag{2}$$

Solving Eqs. (1) and (2) simultaneously, we get

$$w = g \sin \alpha - g \cos \alpha \frac{k_1 m_1 + k_2 m_2}{m_1 + m_2} \tag{3}$$

and
$$R = \frac{m_1 m_2 (k_1 - k_2) g \cos \alpha}{m_1 + m_2}$$

(b) When the blocks just slide down the plane, $w = 0$, then from Eq. (3)

$$g \sin \alpha - g \cos \alpha \frac{k_1 m_1 + k_2 m_2}{m_1 + m_2} = 0$$

or
$$(m_1 + m_2) \sin \alpha = (k_1 m_1 + k_2 m_2) \cos \alpha$$

Hence,
$$\tan \alpha = \frac{(k_1 m_1 + k_2 m_2)}{m_1 + m_2}$$

1.62 **Case 1:** When the body is launched up.

Let k be the coefficient of friction, u the velocity of projection and l the distance traversed along the incline. Retarding force on the block $= mg \sin\alpha + kmg \cos \alpha$ and hence the retardation $= g \sin\alpha + kg \cos\alpha$.
Using the equation of particle kinematics along the incline,

$$0 = u^2 - 2(g \sin\alpha + kg \cos\alpha)\, l$$

or
$$l = \frac{u^2}{2(g \sin \alpha + kg \cos \alpha)} \tag{1}$$

and
$$0 = u - (g \sin\alpha + kg \cos\alpha)\, t$$

or
$$u = (g \sin\alpha + kg \cos\alpha)\, t \tag{2}$$

Using Eq. (2) in Eq. (1) we get
$$l = \frac{1}{2} (g \sin\alpha + kg \cos \alpha)\, t^2 \tag{3}$$

Case 2: When the block comes downward.

The net force on the body $= mg \sin \alpha - kmg \cos \alpha$ and hence its acceleration $= g \sin \alpha - kg \cos \alpha$.

Let, t' be the time required, then

$$l = \frac{1}{2} (g \sin \alpha - kg \cos \alpha) t'^2 \tag{4}$$

From Eqs. (3) and (4)

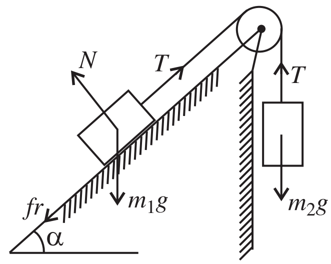
$$\frac{t^2}{t'^2} = \frac{\sin \alpha - k \cos \alpha}{\sin \alpha + k \cos \alpha}$$

But,
$$\frac{t}{t'} = \frac{1}{\eta} \quad \text{(according to the question)}$$

Hence, on solving we get

$$k = \frac{(\eta^2 - 1)}{(\eta^2 + 1)} \tan \alpha = 0.16$$

1.63 At the initial moment, obviously the tension in the thread connecting m_1 and m_2 equals the weight of m_2.



(a) For the block m_2 to come down or the block m_1 to go up, the conditions is

$m_2 g - T \geq 0$ and $T - m_1 g \sin \alpha - fr \geq 0$

where T is tension and fr is friction which in the limiting case equals $km_1 g \cos\alpha$.

Then $$m_2 g - m_1 g \sin \alpha > km_1 g \cos \alpha$$

or $$\frac{m_2}{m_1} > (k \cos\alpha + \sin\alpha)$$

(b) Similarly in the case

$$m_1 g \sin\alpha - m_2 g > fr_{\text{lim}}$$

or $$m_1 g \sin\alpha - m_2 g > km_1 g \cos\alpha$$

or $$\frac{m_2}{m_1} < (\sin\alpha - k \cos\alpha)$$

(c) For this case, neither kind of motion is possible, and fr need not be limiting.

Hence, $$(k \cos\alpha + \sin\alpha) > \frac{m_2}{m_1} > (\sin\alpha - k \cos\alpha)$$

1.64 From the conditions obtained in the previous problem, first we will check whether the mass m_2 goes up or down.

Here, $m_2/m_1 = \eta > \sin\alpha + k \cos\alpha$, (substituting the values). Hence the mass m_2 will come down with an acceleration (say w). From the free body diagram of previous problem,

$$m_2 g - T = m_2 w \tag{1}$$

and $$T - m_1 g \sin \alpha - km_1 g \cos\alpha = m_1 w \tag{2}$$

Adding Eqs. (1) and (2), we get,

$$m_2 g - m_1 g \sin\alpha - km_1 g \cos\alpha = (m_1 + m_2)\, w$$

$$w = \frac{(m_2/m_1 - \sin \alpha - k \cos \alpha)\, g}{(1 + m_2/m_1)} = \frac{(\eta - \sin\alpha - k \cos\alpha) g}{1 + \eta}$$

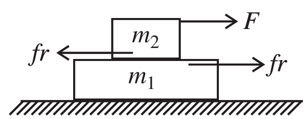
Substituting all the values, $w = 0.048\ g \approx 0.05\ g$

As m_2 moves down with acceleration of magnitude $w = 0.05\ g > 0$, thus in vector form, acceleration of m_2 is given by

$$\mathbf{w}_2 = \frac{(\eta - \sin\alpha - k\cos\alpha)\mathbf{g}}{1 + \eta\ =} = 0.05\ \mathbf{g}$$

1.65 Let us write Newton's second law in projection form along positive x-axis for the plank and the bar

$$fr = m_1 w_1, \quad F - f_r = m_2 w_2 \tag{1}$$



At the initial moment, fr represents the static friction, and as the force F grows so does the friction force fr, but up to its limiting value, i.e., $fr = fr_{s(\max)} = kN = km_2 g$.

Unless this value is reached, both bodies move as a single body with equal acceleration. But as soon as the force fr reaches the limit, the bar starts sliding over the plank, i.e., $w_2 \geq w_1$.

Substituting here the values of w_1 and w_2 taken from Eq. (1) and taking into account that $fr = km_2 g$, we obtain,

$$(at - m_2 g)/m_2 \geq \frac{km_2}{m_1} g$$

(where the sign "=" corresponds to the moment $t = t_0$).

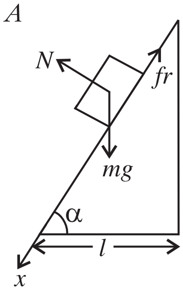
Hence, $$t_0 = \frac{kgm_2(m_1 + m_2)}{am_1}$$

If $t \leq t_0$, then $$w_1 = w_2 = \frac{at}{m_1 + m_2}$$

and if $t > t_0$, then $$w_1 = \frac{km_2 g}{m_1} = \text{constant}, \quad w_2 = \frac{(at - km_2 g)}{m_2}$$

On this basis $w_1(t)$ and $w_2(t)$ plots are as shown in the figure of answer sheet.

1.66 Let us designate the x-axis (see figure) and apply $F_x = mw_x$ for body A



$$mg\sin\alpha - kmg\cos\alpha = mw$$

or $$w = g\sin\alpha - kg\cos\alpha$$

Now, from kinematical equation:

$$l\sec\alpha = 0 + (1/2)\ wt^2$$

or $$t = \sqrt{2l\sec\alpha/(\sin\alpha - k\cos\alpha)\ g}$$

$$= \sqrt{2l/(\sin 2\alpha/2 - k\cos^2\alpha)g} \quad \text{(using Eq. 1)}$$

For $t_{\min}$,
$$\frac{d\left(\dfrac{\sin 2\alpha}{2} - k\cos^2\alpha\right)}{d\alpha} = 0$$

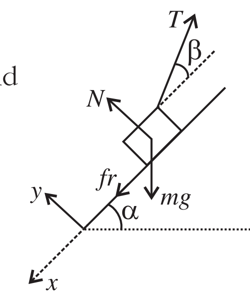
i.e.,
$$\frac{2\cos 2\alpha}{2} + 2k\cos\alpha\sin\alpha = 0$$

or
$$\tan 2\alpha = -\frac{1}{k} \Rightarrow \alpha = 49°$$

and putting the values of α, k and l in Eq. (2) we get $t_{\min} = 1\,\text{s}$.

1.67 Let us fix the $x-y$ co-ordinate system to the wedge, taking the x-axis up along the incline and the y-axis perpendicular to it (see figure).

Now, we draw the free body diagram for the bar.



Let us apply Newton's second law in projection form along x- and y-axes for the bar

$$T\cos\beta - mg\sin\alpha - fr = 0 \qquad (1)$$
$$T\sin\beta + N - mg\cos\alpha = 0$$

or
$$N = mg\cos\alpha - T\sin\beta \qquad (2)$$

But, $fr = kN$ and using Eq. (2) in Eq. (1), we get

$$T = mg\sin\alpha + kmg\cos\alpha/(\cos\beta + k\sin\beta) \qquad (3)$$

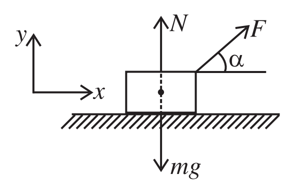
For $T_{\min}$ the value of $(\cos\beta + k\sin\beta)$ should be maximum

So,
$$\frac{d(\cos\beta + k\sin\beta)}{d\beta} = 0 \quad \text{or} \quad \tan\beta = k$$

Putting this value of β in Eq. (3), we get

$$T_{\min} = \frac{mg(\sin\alpha + k\cos\alpha)}{\sqrt{1+k^2}}$$

1.68 First of all let us draw the free body diagram for the small body of mass m and indicate x-axis along the horizontal plane and y-axis, perpendicular to it, as shown in the figure. Let the block break off the plane at $t = t_0$, i.e., $N = 0$.



So,
$$N = mg - at_0\sin\alpha = 0$$

or
$$t_0 = \frac{mg}{a\sin\alpha} \qquad (1)$$

From $F_x = mw_x$, for the body under investigation

$$\frac{mdv_x}{dt} = at\cos\alpha$$

Integrating within the limits for $v(t)$

$$m\int_0^v dv_x = a\cos\alpha\int_0^t t\,dt \quad \text{(using Eq. 1)}$$

So,
$$v = \frac{ds}{dt} = \frac{a\cos\alpha}{2m}t^2 \tag{2}$$

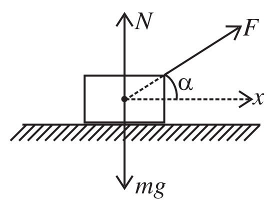
Integrating, Eq. (2) for $s(t)$ we get

$$s = \frac{a\cos\alpha}{2m}\frac{t^3}{3} \tag{3}$$

Using the value of $t = t_0$ from Eq. (1), into Eqs. (2) and (3)

$$v = \frac{mg^2\cos\alpha}{2\,a\sin^2\alpha} \quad \text{and} \quad s = \frac{m^2g^3\cos\alpha}{6\,a^2\sin^3\,\alpha}$$

1.69 Newton's second law of motion in projection form, along horizontal or x-axis, i.e., $F_x = mw_x$ gives



$$F\cos(as) = mv\frac{dv}{ds} \quad (\text{as } \alpha = as)$$

or
$$F\cos(as)\,ds = mv\,dv$$

Integrating, over the limits for $v\,(s)$

$$\frac{F}{m}\int_0^x \cos(ax)\,dx = \frac{v^2}{2}$$

or
$$v = \sqrt{\frac{2F\sin\alpha}{ma}} \quad (\text{as } \alpha = as)$$

$$= \sqrt{2g\sin\alpha/3a} \quad (\text{using } F = mg/3)$$

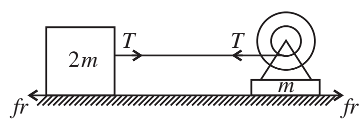
This is the sought relationship.

1.70 From the Newton's second law in projection form

For the bar,
$$T - 2\,kmg = (2m)\,w \tag{1}$$

For the motor,
$$T - kmg = mw' \tag{2}$$

Now, from the equation of kinematics in the frame of bar or motor



$$l = \frac{1}{2}(w + w')\,t^2 \tag{3}$$

From Eqs. (1), (2) and (3) we get on eliminating T and w'

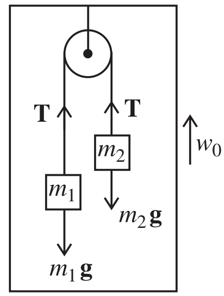
$$t = \sqrt{2l/(kg + 3w)}$$

1.71 (a) Let us write Newton's second law in vector form $\mathbf{F} = m\mathbf{w}$, for both the blocks (in the frame of ground)

$$\mathbf{T} + m_1\mathbf{g} = m_1\mathbf{w}_1 \quad (1)$$

$$\mathbf{T} + m_2\mathbf{g} = m_2\mathbf{w}_2 \quad (2)$$

These two equations contain three unknown quantities $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{T}$. The third equation is provided by the kinematic relationship between the accelerations



$$\mathbf{w}_1 = \mathbf{w}_0 + \mathbf{w}', \mathbf{w}_2 = \mathbf{w}_0 - \mathbf{w}' \quad (3)$$

where $\mathbf{w}$ is the acceleration of the mass m_1 with respect to the pulley or elevator car.

Summing up term wise the left hand and the right-hand sides of these kinematical equations, we get

$$\mathbf{w}_1 + \mathbf{w}_2 = 2\mathbf{w}_0 \quad (4)$$

The simultaneous solution of Eqs. (1), (2) and (4) yields

$$\mathbf{w}_1 = \frac{(m_1 - m_2)\,\mathbf{g} + 2m_2\mathbf{w}_0}{m_1 + m_2}$$

Using this result in Eq. (3), we get,

$$\mathbf{w}_1' = \frac{m_1 - m_2}{m_1 + m_2}(\mathbf{g} - \mathbf{w}_0) \quad \text{and} \quad \mathbf{T} = \frac{2m_1m_2}{m_1 + m_2}(\mathbf{w}_0 - \mathbf{g})$$

Using the results in Eq. (3) we get, $\mathbf{w} = \dfrac{m_1 - m_2}{m_1 + m_2}\,(\mathbf{g} - \mathbf{w}_0)$

(b) Obviously the force exerted by the pulley on the ceiling of the car

$$\mathbf{F} = -2\mathbf{T} = \frac{4m_1m_2}{m_1 + m_2}(\mathbf{g} - \mathbf{w}_0)$$

Note: *One could also solve this problem in the frame of elevator car.*

1.72 Let us write Newton's second law for both, bar 1 and body 2 in terms of projection having taken the positive direction of x_1 and x_2 as shown in the figure and assuming that body 2 starts sliding, say, upward along the incline.

$$T_1 - m_1 g \sin\alpha = m_1 w_1 \quad (1)$$

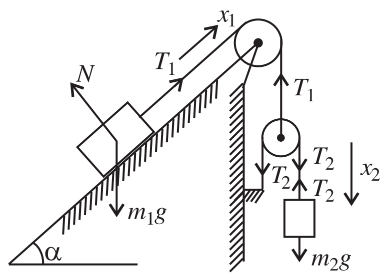
$$m_2 g - T_2 = m_2 w \quad (2)$$

For the pulley, moving in vertical direction from the equation $F_x = mw_x$

$$2T_2 - T_1 = (m_p)\, w_1 = 0$$

(as mass of the pulley $m_p = 0$)

or $$T_1 = 2T_2 \qquad (3)$$



As the length of the threads are constant, the kinematic relationship of accelerations becomes

$$w = 2w_1 \qquad (4)$$

Simultaneous solutions of all these equations yields

$$w = \frac{2g\,(2\,m_2/m_1 - \sin\alpha)}{(4\,m_2/m_1 + 1)} = \frac{2g\,(2\eta - \sin\alpha)}{4\eta + 1}$$

As $\eta > 1$, w is directed vertically downward, and hence in vector form

$$\mathbf{w} = \frac{2\,(2\eta - \sin\alpha)}{4\eta + 1}\,\mathbf{g}$$

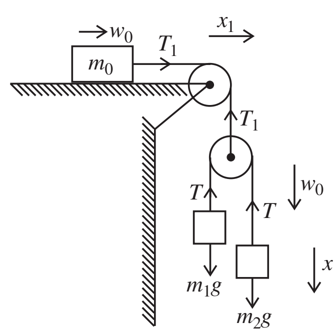
1.73 Let us write Newton's second law for masses m_1 and m_2 and moving pulley in vertical direction along positive x-axis (see figure):

$$m_1 g - T = m_1 w_{1x}$$

$$m_2 g - T = m_2 w_{2x}$$

$$T_1 - 2T = 0 \text{ (as } m = 0)$$

or $$T_1 = 2T$$



Again using Newton's second law in projection form for mass m_0 along positive x_1 direction (see figure), we get

$$T_1 = m_0 w_0$$

The kinematic relationship between the accelerations of masses given in terms of projection on the x-axis

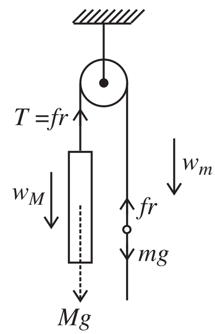
$$w_{1x} + w_{2x} = 2w_0$$

Simultaneous solution of the obtained five equations yields, in vector form

$$\mathbf{w}_1 = \frac{[4m_1 m_2 + m_0\,(m_1 - m_2)]\mathbf{g}}{4m_1 m_2 + m_0(m_1 + m_2)}$$

1.74 As the thread is not tied with m, so if there were no friction between the thread and the ball m, the tension in the thread would be zero and as a result both bodies will have free fall motion. Obviously in the given problem it is the friction force exerted by the ball on the thread, which becomes the tension in the thread. From the condition or language of the problem $w_M > w_m$ and as both are directed downward so, relative acceleration of $M = w_M - w_m$ and is directed downward. Kinematical equation for the ball in the frame of rod in projection form along upward direction gives

$$l = \frac{1}{2}(w_M - w_m)\, t^2 \tag{1}$$



Newton's second law in projection form along vertically down direction for both, rod and ball gives,

$$Mg - fr = Mw_M \tag{2}$$

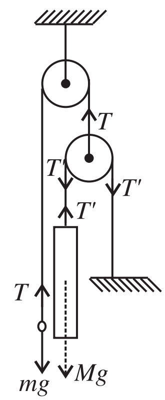
$$mg - fr = mw_m \tag{3}$$

Multiplying Eq. (2) by m and Eq. (3) by M and then subtracting Eq. (3) from Eq. (2) and after using Eq. (1), we get

$$fr = \frac{2l\, Mm}{(M - m)\, t^2}$$

1.75 Suppose, the ball goes up with acceleration w_1 and the rod comes down with the acceleration w_2.

As the length of the thread is constant, $2w_1 = w_2$ (1)



Newton's second law in projection form, vertically upwards for the ball and vertically downwards for the rod respectively gives,

$$T - mg = mw_1 \tag{2}$$

and $$Mg - T' = Mw_2 \tag{3}$$

but, $$T = 2T' \quad \text{(because pulley is massless)} \tag{4}$$

From Eqs. (1), (2), (3) and (4)

$$w_1 = \frac{(2M - m)g}{m + 4M} = \frac{(2 - \eta)g}{\eta + 4} \quad \text{(upwards)}$$

and $$w_2 = \frac{2(2 - \eta)g}{(\eta + 4)} \quad \text{(downwards)}$$

From kinematical equation in projection form, we get

$$l = \frac{1}{2}(w_1 + w_2)\, t^2$$

as w_1 and w_2 are in the opposite direction.

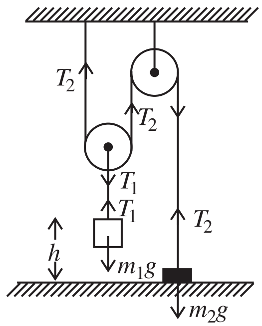
Putting the values of w_1 and w_2, the sought time becomes

$$t = \sqrt{2l(\eta + 4)/3(2 - \eta)g} = 1.4 \text{ s}$$

1.76 Using Newton's second law in projection form along x-axis for the body 1 and along negative x-axis for the body 2, respectively, we get

$$m_1 g - T_1 = m_1 w_1 \qquad (1)$$

$$T_2 - m_2 g = m_2 w_2 \qquad (2)$$



For the pulley lowering in downward direction from along x-axis, Newton's law gives

$$T_1 - 2T_2 = 0 \quad \text{(as pulley is massless)}$$

or

$$T_1 = 2T_2 \qquad (3)$$

As the length of the thread is constant, so

$$w_2 = 2w_1 \qquad (4)$$

The simultaneous solution of above equations yields

$$w_2 = \frac{2(m_1 - 2m_2)\, g}{4m_2 + m_1} = \frac{2(\eta - 2)}{\eta + 4}\, g \left(\text{as } \frac{m_1}{m_2} = \eta\right) \qquad (5)$$

Obviously during the time interval in which the body 1 comes to the horizontal floor covering the distance h, the body 2 moves upward the distance $2h$. At the moment when the body 2 is at the height $2h$ from the floor its velocity is given by the expression

$$v_2^2 = 2w_2(2h) = 2\left[\frac{2(\eta - 2)\, g}{\eta + 4}\right] 2h = \frac{8h\,(\eta - 2) g}{\eta + 4}$$

After the body m_1 touches the floor, the thread becomes slack or the tension in the thread zero, thus as a result body 2 is only under gravity for it's subsequent motion.

Owing to the velocity v_2 at that moment or at the height $2h$ from the floor, the body 2 further goes up under gravity by the distance,

$$h' = \frac{v_2^2}{2g} = \frac{4h\,(\eta - 2)}{\eta + 4}$$

Thus the sought maximum height attained by body 2 is

$$H = 2h + h' = 2h + \frac{4h(\eta - 2)}{(\eta + 4)}$$

$$= \frac{6\eta h}{\eta + 4} = 0.6 \text{ m} \quad \text{(on substituting values)}$$

1.77 Let us draw free body diagram of each body, i.e., of rod A and of wedge B and also draw the kinematical diagram for accelerations, after analysing the directions of motion of A and B. The kinematic relationship of accelerations is

$$\tan\alpha = \frac{w_A}{w_B} \tag{1}$$

Let us write Newton's second law for both bodies in terms of projections having taken positive directions of y- and x-axes as shown in the figure:

$$m_A g - N\cos\alpha = m_A w_A \tag{2}$$

and

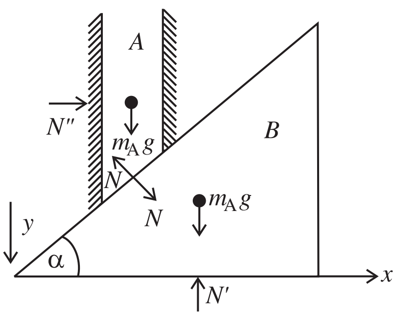
$$N\sin\alpha = m_B w_B \tag{3}$$

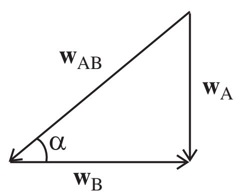
Simultaneous solution of Eqs. (1), (2) and (3) yields

$$w_A = \frac{m_A g \sin\alpha}{m_A \sin\alpha + m_B \cot\alpha \cos\alpha} = \frac{g}{(1 + \eta \cot^2\alpha)}$$

and

$$w_B = \frac{w_A}{\tan\alpha} = \frac{g}{(\tan\alpha + \eta \cot\alpha)}$$

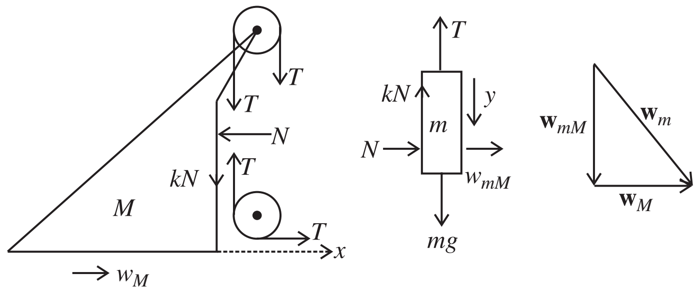




Note: *We may also solve this problem using conservation of mechanical energy instead of Newton's second law.*

1.78 Let us draw free body diagram of each body and fix the coordinate system, as shown in the figure. After analysing the motion of M and m on the basis of force diagrams, let us draw the kinematic diagram for accelerations (see figure).

As the length of thread is constant so,



$ds_{mM} = ds_M$ and as $\mathbf{v}_{mM}$ and $\mathbf{v}_M$ do not change their directions that's why

$$|\mathbf{w}_{mM}| = |\mathbf{w}_M| = w \quad \text{(say)}$$

As
$$\mathbf{w}_m = \mathbf{w}_{mM} + \mathbf{w}_M \tag{1}$$

So, from the triangle law of vector addition

$$w_m = \sqrt{2}\, w$$

From equation $F_x = mw_x$, for the wedge and the block,

$$T - N = Mw \tag{2}$$

and
$$N = mw \tag{3}$$

Now, from equation $F_y = mw_y$, for the block,

$$mg - T - kN = mw \tag{4}$$

Simultaneous solution of Eqs. (2), (3) and (4) yields

$$w = \frac{mg}{(km + 2m + M)} = \frac{g}{(k + 2 + M/m)}$$

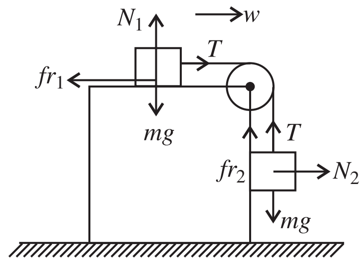
Hence, using Eq. (1)

$$w_m = \frac{g\sqrt{2}}{(2 + k + M/m)}$$

1.79 Bodies 1 and 2 will remain at rest with respect to bar A for $w_{\min} \leq w \leq w_{\max}$ where $w_{\min}$ is the sought minimum acceleration of the bar. Beyond these limits there will be a relative motion between bar and the bodies. For $0 \leq w \leq w_{\min}$, the tendency of body 1 in relation to the bar A is to move towards right and is in the opposite sense for $w \geq w_{\max}$. On the basis of above argument the static friction on 2 by A is directed upward and on 1 by A is directed towards left for the purpose of calculating $w_{\min}$.

Let us write Newton's second law for bodies 1 and 2 in terms of projection along positive x-axis (see figure).

$$T - fr_1 = mw \quad \text{or} \quad fr_1 = T - mw \tag{1}$$



$$N_2 = mw \quad (2)$$

As body 2 has no acceleration in vertical direction, so

$$fr_2 = mg - T \quad (3)$$

From Eqs. (1) and (3)

$$(fr_1 + fr_2) = m(g - w) \quad (4)$$

But, $\quad fr_1 + fr_2 \leq k(N_1 + N_2)$

or $\quad fr_1 + fr_2 \leq k(mg + mw) \quad (5)$

From Eqs. (4) and (5)

$$m(g - w) \leq mk(g + w), \quad \text{or} \quad w \geq \frac{g(1 - k)}{(1 + k)}$$

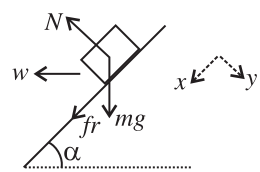
Hence, $$w_{\min} = \frac{g(1 - k)}{(1 + k)}$$

1.80 On the basis of the initial argument of the solution of problem 1.79, the tendency of bar 2 with respect to 1 will be to move up along the plane.

Let us fix x-y coordinate system in the frame of ground as shown in the figure.

From second law of motion in projection form along y- and x-axes

$$mg\cos\alpha - N = mw\sin\alpha$$



or $\quad N = m(g\cos\alpha - w\sin\alpha) \quad (1)$

$$mg\sin\alpha + fr = mw\cos\alpha$$

or $\quad fr = m(w\cos\alpha - g\sin\alpha) \quad (2)$

but $\quad fr \leq kN$

So from Eqs. (1) and (2), $(w\cos\alpha - g\sin\alpha) \leq k(g\cos\alpha + w\sin\alpha)$

or $\quad w(\cos\alpha - k\sin\alpha) \leq g(k\cos\alpha + \sin\alpha)$

or $$w \leq g\frac{(k\cos\alpha + \sin\alpha)}{\cos\alpha - k\sin\alpha}$$

So, the sought maximum acceleration of the wedge is

$$w_{\max} = \frac{(k\cos\alpha + \sin\alpha)\,g}{\cos\alpha - k\sin\alpha} = \frac{(k\cot\alpha + 1)g}{\cot\alpha - k} \quad \text{where } \cot\alpha > k$$

1.81 Let us draw the force diagram of each body, and on this basis we observe that the prism moves towards right (say) with an acceleration w_1 and the bar 2 of mass m_2 moves down the plane with respect to 1, say with acceleration $\mathbf{w}_{21}$, then $\mathbf{w}_2 = \mathbf{w}_{21} + \mathbf{w}_1$ (see figure).

Let us write Newton's second law of both bodies in projection form along positive y_2 and x_1 axes as shown in the figure.

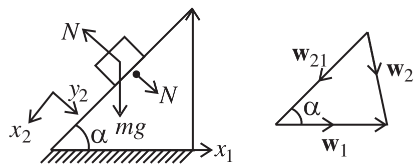
$$m_2 g \cos\alpha - N = m_2 w_{2(y_2)} = m_2 [w_{21(y_2)} + w_{1(y_2)}] = m_2 [0 + w_1 \sin\alpha]$$

or $$m_2 g \cos\alpha - N = m_2 w_1 \sin\alpha \tag{1}$$

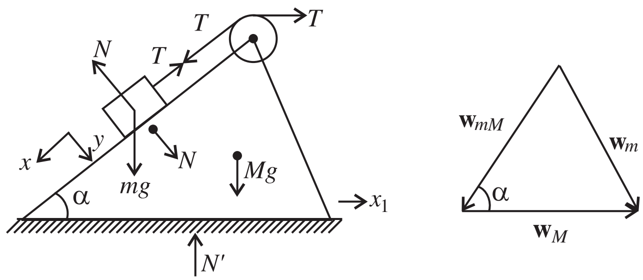
and $$N \sin\alpha = m_1 w_1 \tag{2}$$

Solving Eqs. (1) and (2), we get

$$w_1 = \frac{m_2 g \sin\alpha \cos\alpha}{m_1 + m_2 \sin^2\alpha} = \frac{g \sin\alpha \cos\alpha}{(m_1/m_2) + \sin^2\alpha}$$



1.82 To analyse the kinematical relations between the bodies, sketch the force diagram of each body as shown in the figure.



On the basis of force diagram, it is obvious that the wedge M will move towards right and the block will move down along the wedge. As the length of the thread is constant, the distance travelled by the block on the wedge must be equal to the distance travelled by the wedge on the floor. Hence $ds_{mM} = ds_M$. As $\mathbf{v}_{mM}$ and $\mathbf{v}_M$ do not change their directions and acceleration that's why $\mathbf{w}_{mM} \uparrow\uparrow \mathbf{v}_{mM}$ and $\mathbf{w}_M \uparrow\uparrow \mathbf{v}_M$ and $w_{mM} = w$ (say) and accordingly the diagram of kinematical dependence is shown in figure.

As $\mathbf{w}_m = \mathbf{w}_{mM} + \mathbf{w}_M$, so from triangle law of vector addition

$$w_m = \sqrt{w_M^2 + w_M^2 - 2\, w_{mM}\, w_M \cos\alpha} = w\sqrt{2(1 - \cos\alpha)} \tag{1}$$

From $F_x = mw_x$, (for the wedge),

$$T - T\cos\alpha + N\sin\alpha = Mw \tag{2}$$

For the bar m let us fix x-y coordinate system in the frame of ground. Newton's law in in projection form along x- and y-axes (see figure) gives

$$mg\sin\alpha - T = mw_{m(x)} = m[w_{mM(x)} + w_{M(x)}]$$
$$= m\,[w_{mM} + w_M\cos(\pi - \alpha)] = mw\,(1 - \cos\alpha) \qquad (3)$$
$$mg\cos\alpha - N = mw_{m(y)} = m\,[w_{mM(y)} + w_{m(y)}] = m\,[0 + w\sin\alpha] \qquad (4)$$

Solving the above equations simultaneously, we get

$$w = \frac{mg\sin\alpha}{M + 2m\,(1 - \cos\alpha)}$$

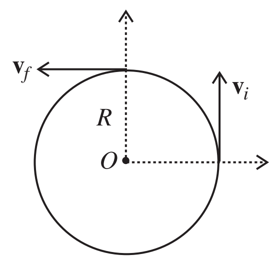
Note: *We can study the motion of the block m in the frame of wedge also, alternately we may solve this problem using conservation of mechanical energy.*

1.83 Let us sketch the diagram for the motion of the particle of mass m along the circle of radius R as shown in the figure.

For a particle, modulus of change in linear momentum $|\Delta\mathbf{p}| = m\,|\Delta\mathbf{v}|$.

(a) In this case $|\Delta\mathbf{v}| = \sqrt{2}v$ (see figure), and the time taken in describing quarter of the circle,

$$\Delta t = \frac{\pi R}{2v}$$



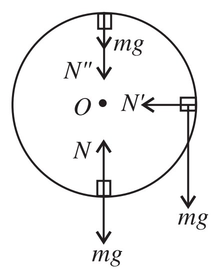
Hence,
$$|\langle\mathbf{F}\rangle| = \frac{|\Delta\,\mathbf{p}|}{\Delta t} = \frac{m|\Delta\mathbf{v}|}{\Delta t} = \frac{\sqrt{2}\,mv}{\pi R/2v} = \frac{2\sqrt{2}\,mv^2}{\pi R}$$

(b) In this case $\mathbf{v}_i = 0$ so, $\mathbf{v}_f = \mathbf{v}(t)$. Thus $|\Delta\mathbf{v}| = |\mathbf{v}(t)| = v(t) = w_t t$.

Hence,
$$|\langle\mathbf{F}\rangle| = \frac{|\Delta\mathbf{p}|}{\Delta t} = \frac{m|\Delta\mathbf{v}|}{\Delta t} = \frac{m|\mathbf{v}(t)|}{t} = mw_t$$

1.84 While moving in a loop, normal reaction exerted by the flyer on the loop at different points and uncompensated weight if any contribute to the weight of flyer at those points.

(a) When the aircraft is at the lowermost point, Newton's second law of motion in projection form, $F_n = mw_n$ gives



$$N - mg = \frac{mv^2}{R}$$

or
$$N = mg + \frac{mv^2}{R} = 2.09\text{ kN}$$

(b) When it is at the upper most point, again from $F_n = mw_n$, we get

$$N'' + mg = \frac{mv^2}{R}$$

$$N'' = \frac{mv^2}{R} - mg = 0.7\text{ kN}$$

(c) When the aircraft is at the middle point of the loop, again from $F_n = mw_n$

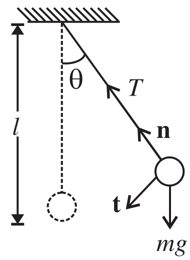
$$N' = \frac{mv^2}{R} = 1.4 \text{ kN}$$

The uncompensated weight is mg. Thus effective weight $= \sqrt{N'^2 + m^2g^2} = 1.56$ kN acts obliquely.

1.85 (a) Let us depict the forces acting on the small sphere m, (at an arbitrary position when the thread makes an angle θ from the vertical) and write equation $\mathbf{F} = m\mathbf{w}$ via projection on the unit vectors $\mathbf{t}$ and $\mathbf{n}$. From $F_t = mw_t$, we have

$$mg \sin\theta = m\frac{dv}{dt}$$

$$= m\frac{vdv}{ds} = m\frac{vdv}{l(-d\theta)}$$



(as vertical is reference line of angular position)

or $$vdv = -gl\sin\theta d\theta$$

Integrating both the sides we get

$$\int_0^v vdv = -gl\int_{\pi/2}^{\theta} \sin\theta \, d\theta$$

or $$\frac{v^2}{2} = gl\cos\theta$$

Hence, $$\frac{v^2}{l} = 2g\cos\theta = w_n \tag{1}$$

Note: *Eq. (1) can be easily obtained by the conservation of mechanical energy.*

From $$F_n = mw_n$$

$$T - mg\cos\theta = \frac{mv^2}{l}$$

Using Eq. (1) we have

$$T = 3mg\cos\theta \tag{2}$$

Again from equation $F_t = mw_t$,

$$mg\sin\theta = mw_t \quad \text{or} \quad w_t = g\sin\theta \tag{3}$$

Hence, $w = \sqrt{w_t^2 + w_n^2} = \sqrt{(g\sin\theta)^2 + (2g\cos\theta)^2}$ (using Eqs. 1 and 3)

$= g\sqrt{1 + 3\cos^2\theta}$.

(b) Vertical component of velocity, $v_y = v\sin\theta$

So, $\quad v_y^2 = v^2\sin^2\theta = 2\,gl\cos\theta\,\sin^2\theta$ (using Eq. 1)

For maximum v_y or v_y^2, $\quad \dfrac{d(\cos\theta\,\sin^2\theta)}{d\theta} = 0$

which yields $\quad \cos\theta = \dfrac{1}{\sqrt{3}}$

Therefore from Eq. (2), $T = 3mg\dfrac{1}{\sqrt{3}} = \sqrt{3}\,mg$

(c) We have $\mathbf{w} = w_t\mathbf{t} + w_n\mathbf{n}$ thus $w_y = w_{t(y)} + w_{n(y)}$

But in accordance with the problem $w_y = 0$

So, $\quad w_{t(y)} + w_{n(y)} = 0$

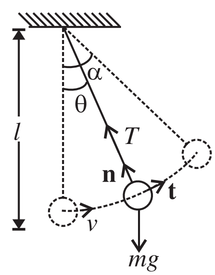
or $\quad -g\sin\theta\sin\theta + 2g\cos^2\theta = 0$

or $\quad \cos\theta = \dfrac{1}{\sqrt{3}}$ or $\theta = 54.7°$

1.86 The ball has only normal acceleration at the lowest position and only tangential acceleration at either of the extreme positions, Let v be the speed of the ball at its lowest position and l be length of the thread, then according to the problem

$$\frac{v^2}{l} = g\sin\alpha \qquad (1)$$

where α is the maximum deflection angle.



From Newton's law in projection form:

$$F_t = mw_t$$

$$-mg\sin\theta = mv\frac{dv}{l\,d\theta}$$

or $\quad -gl\sin\theta\,d\theta = v\,dv$

On integrating both the sides within their limits.

$$-gl\int_0^{\alpha}\sin\theta\,d\theta = \int_v^0 v\,dv$$

or $$v^2 = 2gl(1 - \cos\alpha) \tag{2}$$

Note: *Eq. (2) can easily be obtained by the conservation of mechanical energy of the ball in the uniform field of gravity.*

From Eqs. (1) and (2) with $\theta = \alpha$

$$\sin\alpha = 2(1 - \cos\alpha)$$

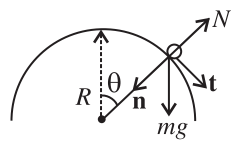
or $$\sin\alpha + 2\cos\alpha = 2$$

On solving we get,

$$\alpha \cong 53°$$

1.87 Let us depict the forces acting on the body A (which are the force of gravity mg and the normal reaction N) and write equation $\mathbf{F} = m\mathbf{w}$ via projection on the unit vectors $\mathbf{t}$ and $\mathbf{n}$ (see figure).

From $F_t = mw_t$



$$mg\sin\theta = m\frac{dv}{dt}$$

$$= m\frac{vdv}{ds} = m\frac{vdv}{Rd\theta}$$

or $$gR\sin\theta d\theta = vdv$$

Integrating both sides for obtaining $v(\theta)$, we get

$$\int_0^\theta gR\sin\theta\, d\theta' = \int_0^v vdv$$

or $$v^2 = 2gR(1 - \cos\theta) \tag{1}$$

From $F_n = mw_n$

$$mg\cos\theta - N = m\frac{v^2}{R} \tag{2}$$

At the moment the body loses contact with the surface, $N = 0$, and therefore Eq. (2) becomes

$$v^2 = gR\cos\theta \tag{3}$$

where v and θ correspond to the moment when the body loses contact with the surface. Solving Eqs. (1) and (3), we obtain

$$\cos\theta = \frac{2}{3} \quad \text{or} \quad \theta = \cos^{-1}(2/3) \cong 48° \text{ and } v = \sqrt{2gR/3}$$

1.88 At first draw the free body diagram of the device as, shown. The forces, acting on the sleeve are its weight, acting vertically downward, spring force, along the length of the spring and normal reaction by the rod, perpendicular to its length.

Let F be the spring force, and Δl be the elongation.

From $F_n = mw_n$,

$$N \sin\theta + F\cos\theta = m\omega^2 r \tag{1}$$

where $r\cos\theta = (l_0 + \Delta l)$.

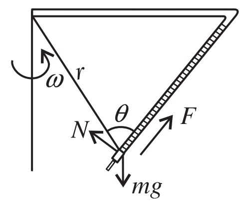
Similarly, from $F_t = mw_t$

$$N\cos\theta - F\sin\theta = 0 \quad \text{or} \quad N = F\sin\theta/\cos\theta \tag{2}$$

From Eqs. (1) and (2)

$$F(\sin\theta/\cos\theta)\cdot\sin\theta + F\cos\theta = m\omega^2 r$$

$$= m\omega^2(l_0 + \Delta l)/\cos\theta$$



On putting $F = \kappa\Delta l$,

$$\kappa\Delta l\sin^2\theta + \kappa\Delta l\cos^2\theta = m\omega^2(l_0 + \Delta l)$$

On solving, we get

$$\Delta l = m\omega^2 \frac{l_0}{\kappa - m\omega^2} = \frac{l_0}{(\kappa/m\,\omega^2 - 1)}$$

It is independent of the direction of rotation.

1.89 According to the question, the cyclist moves along the circular path and the centripetal force is provided by the frictional force. Thus from the equation

$$F_n = mw_n$$

$$fr = \frac{mv^2}{r} \quad \text{or} \quad kmg = \frac{mv^2}{r}$$

or $$k_0\left(1 - \frac{r}{R}\right)g = \frac{v^2}{r} \quad \text{or} \quad v^2 = k_0\,(r - r^2/R)\,g \tag{1}$$

For v_{max}, we should have $$\frac{d\left(r - \dfrac{r^2}{R}\right)}{dr} = 0$$

or $$1 - \frac{2r}{R} = 0, \quad \text{so} \quad r = \frac{R}{2}$$

Hence, $$v_{max} = \frac{1}{2}\sqrt{k_0 g R}$$

1.90 As initial velocity is zero thus,

$$v^2 = 2w_4 s = f(s) \tag{1}$$

As $w_t > 0$ the speed of the car increases with time or distance. Till the moment, sliding starts, the static friction provides the required centripetal accelaration the car.

Thus, $fr = mw$, but $fr \leq kmg$

So, $w^2 \leq k^2g^2$ or $w_t^2 + \left(\dfrac{v^2}{R}\right)^2 \leq k^2g^2$

or $v^2 \leq R\,(k^2g^2 - w_t^2)$

Hence, $v_{\max}^2 = R\sqrt{(k^2g^2 - w_t^2)}$

So, from Eq. (1), the sought distance $s = \dfrac{v_{\max}^2}{2w_t} = \dfrac{R}{2}\sqrt{\left(\dfrac{kg}{w_t}\right) - 1} = 60$ m

1.91 Since the car follows a sinusoidal curve, so the maximum velocity at which it can ride without sliding is the point of minimum radius of curvature obviously in this case the static friction between the car and the road is limiting. Hence, from the equation $F_n = mw_n$

$$\frac{mv^2}{R} \leq kmg$$

or $v \leq \sqrt{kRg}$

So, $$v_{\max} = \sqrt{kR_{\min}g} \tag{1}$$

We know that the radius of curvature for a curve at any point (x, y) is given as

$$R = \left|\frac{[1 + (dy/dx)^2]^{3/2}}{(d^2y)/dx^2}\right| \tag{2}$$

For the given curve,

$$\frac{dy}{dx} = \frac{a}{\alpha}\cos\left(\frac{x}{\alpha}\right) \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{-a}{\alpha^2}\sin\frac{x}{\alpha}$$

Substituting this value in Eq. (2), we get

$$R = \frac{[1 + (a^2/\alpha^2)\cos^2(x/\alpha)]^{3/2}}{(a/\alpha^2)\sin(x/\alpha)}$$

For the minimum R, $$\frac{x}{\alpha} = \frac{\pi}{2}$$

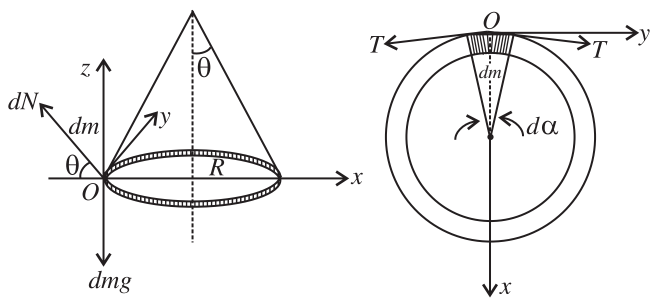
and therefore, corresponding radius of curvature

$$R_{\min} = \frac{\alpha^2}{a} \tag{3}$$

Hence, from Eqs. (1) and (3)

$$v_{\max} = \alpha\sqrt{kg/a}$$

1.92 The sought tensile stress acts on each element of the chain. Hence divide the chain into small, similar elements so that each element may be assumed as a particle. We consider one such element of mass dm, which subtends angle $d\alpha$ at the centre. The chain moves along a circle of known radius R with known angular speed ω and certain forces act on it. We have to find one of these forces.



From Newton's second law in projection form, $F_x = mw_x$, we get

$$2T\sin(d\alpha/2) - dN\cos\theta = dm\,\omega^2 R$$

and from $F_Z = mw_z$, we get

$$dN\sin\theta = (dm)\,g$$

Then putting $dm = md\alpha/2\pi$ and $\sin(d\alpha/2) = d\alpha/2$ and solving, we get

$$T = \frac{m(\omega^2 R + g\cot\theta)}{2\pi}$$

1.93 Let us consider a small element of the thread and draw free body diagram for this element.

(a) Applying Newton's second law of motion in projection form, $F_n = mw_n$ for this element,

$$(T + dT)\sin(d\theta/2) + T\sin(d\theta/2) - dN = dm\omega^2 R = 0$$

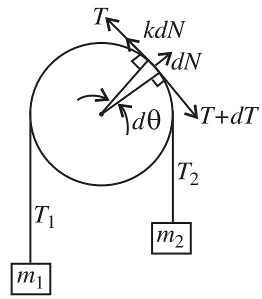
or $$2T\sin(d\theta/2) = dN \quad [\text{neglecting the term}\,(dT\sin d\theta/2)]$$

or $$T\,d\theta = dN \left(\text{as } \sin\frac{d\theta}{2} \approx \frac{d\theta}{2}\right) \tag{1}$$

Also, $$dfr = kdN = (T + dT) - T = dT \tag{2}$$

From Eqs. (1) and (2)

$$kT\,d\theta = dT \text{ or } \frac{dT}{T} = kd\,\theta$$



Integrating θ from $\theta = 0$ to $\theta = \pi$, we get

$$\ln\frac{T_2}{T_1} = k\pi \tag{3}$$

So, $$k = \frac{1}{\pi}\ln\frac{T_2}{T_1} = \frac{1}{\pi}\ln\eta_0$$

$$\left(\text{as } \frac{T_2}{T_1} = \frac{m_g g}{m_1 g} = \frac{m_2}{m_1} = \eta_0\right)$$

(b) When $m_2/m_1 = \eta > \eta_0$, the blocks will move with same value of acceleration (say w) and clearly m_2 moves downward. From Newton's second law in projection form (downward for m_2 and upward for m_1), we get

$$m_2 g - T_2 = m_2 w \tag{4}$$

and $$T_2 - m_1 g = m_1 w \tag{5}$$

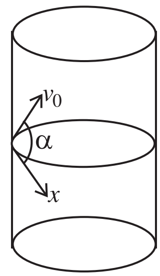
Also $$\frac{T_2}{T_1} = \eta_0 \tag{6}$$

Simultaneous solution of Eqs. (4), (5) and (6) yields

$$w = \frac{(m_2 - \eta_0 m_1)g}{(m_2 + \eta_0 m_1)} = \frac{(\eta - \eta_0)}{(\eta + \eta_0)}g\left(\text{as } \frac{m_2}{m_1} = \eta\right)$$

1.94 The force with which the cylinder wall acts on the particle will provide centripetal force necessary for the motion of the particle, and since there is no acceleration acting in the horizontal direction, horizontal component of the velocity will remain constant throughout the motion.

So, $$v_x = v_0 \cos\alpha$$



Using, $F_n = mw_n$, for the particle of mass m

$$N = \frac{mv_x^2}{R} = \frac{mv_0^2 \cos^2\alpha}{R}$$

which is the required normal force.

1.95 Obviously the radius vector describing the position of the particle relative to the origin of the coordinate is,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} = a \sin\omega\, t\mathbf{i} + b\cos\omega\, t\mathbf{j}$$

Differentiating twice with respect to time

$$\mathbf{w} = \frac{d^2\mathbf{r}}{dt^2} = -\boldsymbol{\omega}^2 (a \sin \omega\, t\, \mathbf{i} + b \cos \omega\, t\, \mathbf{j}) = -\omega^2\mathbf{r} \tag{1}$$

Thus, $\mathbf{F} = m\mathbf{w} = -m\omega^2\mathbf{r}$ (using Eq. 1)

1.96 (a) We have

$$\Delta\mathbf{p} = \int \mathbf{F}dt = \int_0^t m\mathbf{g}\, dt = m\mathbf{g}t \tag{1}$$

(b) Using the solution of problem 1.28 (b), the total time of motion,

$$\tau = -\frac{2(\mathbf{v}_0 \cdot \mathbf{g})}{\mathbf{g}^2}$$

Hence, using $t = \tau$ in Eq. (1)

$$|\Delta\mathbf{p}| = mg\,\tau$$

$$= \frac{-2m\,(\mathbf{v}_0 \cdot \mathbf{g})}{g}$$

1.97 From the equation of the given time dependence force $\mathbf{F} = \mathbf{a}\, t\,(\tau - t)$ at $\mathbf{t} = \tau$, the force vanishes.

(a) Thus,

$$\Delta\mathbf{p} = \mathbf{p} = \int_0^\tau \mathbf{F}dt$$

or

$$\mathbf{p} = \int_0^\tau \mathbf{a}\, t\,(\tau - t)\, dt = \frac{\mathbf{a}\tau^3}{6}$$

But, $\mathbf{p} = m\mathbf{v}$ so, $\mathbf{v} = \dfrac{\mathbf{a}\tau^3}{6m}$

(b) Again from the equation $\mathbf{F} = m\mathbf{w}$

$$\mathbf{a}t(\tau - t) = m\frac{d\mathbf{v}}{dt}$$

or $$\mathbf{a}\,(t\tau - t^2)\,dt = md\mathbf{v}$$

Integrating within the limits for $\mathbf{v}(t)$

$$\int_0^t \mathbf{a}\,(t\,\tau - t^2)\,dt = m\int_0^{\mathbf{v}} d\mathbf{v}$$

or $$\mathbf{v} = \frac{\mathbf{a}}{m}\left(\frac{\tau t^2}{2} - \frac{t^3}{3}\right) = \frac{\mathbf{a}t^2}{m}\left(\frac{\tau}{2} - \frac{t}{3}\right)$$

Thus, $$v = \frac{at^2}{m}\left(\frac{\tau}{2} - \frac{t}{3}\right) \quad \text{for } t \leq \tau$$

Hence, distance covered during the time interval $t = \tau$ is given by

$$s = \int_0^{\tau} vdt = \int_0^{\tau} \frac{at^2}{m}\left(\frac{\tau}{2} - \frac{t}{3}\right)dt = \frac{a}{m}\frac{\tau^4}{12}$$

1.98 We have $$F = F_0 \sin \omega t$$

or $$m\frac{d\mathbf{v}}{dt} = \mathbf{F}_0 \sin\omega t \quad \text{or} \quad md\mathbf{v} = \mathbf{F}_0 \sin\omega t\; dt$$

On integrating,

$$m\mathbf{v} = \frac{-\mathbf{F}_0}{\omega}\cos \omega t + C \quad (\text{where } C \text{ is integration constant})$$

When $t = 0,\ v = 0,$ so $$C = \frac{\mathbf{F}_0}{m\omega}$$

Hence, $$\mathbf{v} = \frac{-\mathbf{F}_0}{m\omega}\cos \omega\, t + \frac{\mathbf{F}_0}{m\omega}$$

As $\cos \omega t \leq 1$ so,

$$\mathbf{v} = \frac{\mathbf{F}_0}{m\omega}(1 - \cos \omega t)$$

Thus, $$s = \int_0^t vdt = \frac{F_0 t}{m\omega} - \frac{F_0 \sin\omega t}{m\omega^2} = \frac{F_0}{m\omega^2}(\omega t - \sin\omega t).$$

(see figure in the answer sheet)

1.99 According to the problem, the force acting on the particle of mass m is, $\mathbf{F} = \mathbf{F}_0 \cos\omega t$.

So, $$m\frac{d\mathbf{v}}{dt} = \mathbf{F}_0 \cos\omega t \quad \text{or} \quad d\mathbf{v} = \frac{\mathbf{F}_0}{m}\cos\omega t\, dt$$

Integrating, within the limits

$$\int_0^{\mathbf{v}} d\,\mathbf{v} = \frac{\mathbf{F}_0}{m}\int_0^t \cos\omega t\, dt \quad \text{or} \quad \mathbf{v} = \frac{\mathbf{F}_0}{m\omega}\sin\omega t \qquad (1)$$

It is clear from Eq. (1), that after starting at $t = 0$, the particle comes to rest for the first time at $t = \pi/\omega$.

From Eq. (1)

$$v = |\mathbf{v}| = \frac{\mathbf{F}_0}{m\omega}\sin\omega t \quad \text{for} \quad t \leq \frac{\pi}{\omega} \qquad (2)$$

Thus during the time interval $t = \pi/\omega$, the sought distance

$$s = \frac{F_0}{mw}\int_0^{\pi/\omega} \sin\omega t\, dt = \frac{2F}{m\omega^2}$$

From Eq. (1)

$$v_{\max} = \frac{F_0}{m\omega} \quad \text{as } |\sin\omega t| \leq 1$$

1.100 (a) From the problem $\mathbf{F} = -r\mathbf{v}$ so, $m\dfrac{d\mathbf{v}}{dt} = -r\mathbf{v}$

Thus, $$m\frac{dv}{dt} = -rv \quad [\text{as } d\mathbf{v} \uparrow\downarrow \mathbf{v}]$$

or $$\frac{dv}{v} = -\frac{r}{m}\,dt$$

On integrating $$\ln v = -\frac{r}{m}t + C$$

But at $t = 0$, $v = v_0$ so, $C = \ln v_0$.

So, $$\ln\frac{v}{v_0} = -\frac{r}{m}t \quad \text{or} \quad v = v_0 e^{-(r/m)t}$$

Thus, for $t \to \infty$, $v = 0$.

(b) We have $m\dfrac{dv}{dt} = -rv$ so, $dv = \dfrac{-r}{m}ds$

Integrating within the given limits to obtain $v(s)$, we get

$$\int_{v_0}^{v} dv = -\frac{r}{m}\int_0^s ds \quad \text{or} \quad v = v_0 - \frac{rs}{m} \tag{1}$$

Thus for $v = 0,\ s = s_{\text{total}} = \dfrac{mv_0}{r}$

(c) We have
$$\frac{mdv}{dt} = -rv \quad \text{or} \quad \frac{dv}{v} = \frac{-r}{m}\,dt$$

or
$$\int_0^{v_0/\eta} \frac{dv}{v} = \frac{-r}{m}\int_0^t dt \quad \text{or} \quad \ln\frac{v_0}{\eta v_0} = -\frac{r}{m}\,t$$

So,
$$t = \frac{-m\ln(1/\eta)}{r} = \frac{m\ln\eta}{r}$$

Now, average velocity over this time interval

$$<v> = \frac{\int v dt}{\int dt} = \frac{\int_0^{\frac{m}{r}\ln\eta} v_0 e^{-\frac{rt}{m}}\,dt}{m/r\ \ln\eta} = \frac{v_0(\eta - 1)}{\eta\ln\eta}$$

1.101 According to the problem

$$m\frac{dv}{dt} = -kv^2 \quad \text{or} \quad m\frac{dv}{v^2} = -kdt$$

Integrating, within the limits

$$\int_{v_0}^{v} \frac{dv}{v^2} = -\frac{k}{m}\int_0^t dt \quad \text{or} \quad t = \frac{m}{k}\frac{(v_0 - v)}{v_0 v} \tag{1}$$

To find the value of k, rewrite

$$mv\frac{dv}{ds} = -kv^2 \quad \text{or} \quad \frac{dv}{v} = -\frac{k}{m}\,ds$$

On integrating

$$\int_{v_0}^{v} \frac{dv}{v} = -\frac{k}{m}\int_0^h ds$$

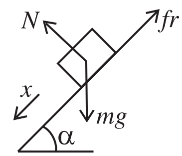
So, $$k = \frac{m}{h} \ln \frac{v_0}{v} \tag{2}$$

Putting the value of k from Eq. (2) in Eq. (1), we get

$$t = \frac{h\,(v_0 - v)}{v_0 v \ln \dfrac{v_0}{v}}$$

1.102 From Newton's second law for the bar in projection form, $F_x = mw_x$, along x direction, we get

$$mg \sin\alpha - kmg \cos\alpha = mw$$



or $$v\frac{dv}{dx} = g \sin\alpha - ax\, g \cos\alpha \quad (\text{as } k = ax)$$

or $$vdv = (g \sin\alpha - ax\, g \cos\alpha)\, dx$$

or $$\int_0^v vdv = g \int_0^x (\sin\alpha - x \cos\alpha)\, dx$$

So, $$\frac{v^2}{2} = g\left(x \sin\alpha - \frac{x^2}{2} a \cos\alpha\right) \tag{1}$$

From Eq. (1),

$$v = 0 \text{ at either } x = 0 \quad \text{or} \quad x = \frac{2}{a} \tan\alpha$$

As the motion of the bar is unidirectional it stops after going through a distance of $2/a \tan\alpha$.

From Eq. (1), for v_{max},

$$\frac{d}{dx}\left(x \sin\alpha - \frac{x^2}{2} a \cos\alpha\right) = 0 \quad \text{or} \quad x = \frac{1}{a} \tan\alpha$$

Hence, the maximum velocity will be at the distance, $x = \tan\alpha/a$.

Putting this value of x in Eq. (1), the maximum velocity,

$$v_{max} = \sqrt{\frac{g \sin\alpha \tan\alpha}{a}}$$

1.103 Since the applied force is proportional to the time and the frictional force also exists, the motion does not start just after applying the force. The body starts its motion when F equals the limiting friction. Let the motion start after time t_0, then

$$F = at_0 = kmg \quad \text{or} \quad t_0 = \frac{kmg}{a}$$

So, for $t \leq t_0$, the body remains at rest and for $t > t_0$ obviously

$$\frac{mdv}{dt} = a(t - t_0) \quad \text{or} \quad mdv = a(t - t_0)\, dt$$

Integrating, and noting $v = 0$ at $t = t_0$, we have for $t > t_0$

$$\int_0^v mdv = a\int_0^t (t - t_0)\, dt \quad \text{or } v = \frac{a}{2m}(t - t_0)^2$$

Thus, $$s = \int vdt = \frac{a}{2m}\int_{t_0}^{t} (t - t_0)^2\, dt = \frac{a}{6m}(t - t_0)^3$$

1.104 While going upward, from Newton's second law, in vertical direction

$$m\frac{vdv}{ds} = -(mg + kv^2) \quad \text{or} \quad \frac{vdv}{(g + kv^2/m)} = -ds$$

At the maximum height h, the speed $v = 0$, so

$$\int_{v_0}^{0} \frac{vdv}{g + (kv^2/m)} = -\int_0^h ds$$

Integrating and solving, we get

$$h = \frac{m}{2k}\ln\left(1 + \quad + \frac{kv_0^2}{mg}\right) \tag{1}$$

When the body falls downward, the net force acting on the body in downward direction equals $(mg - kv^2)$.
Hence net acceleration, in downward direction, according to second law of motion

$$\frac{vdv}{ds} = g - \frac{kv^2}{m} \quad \text{or} \quad \frac{vdv}{g - kv^2/m} = ds$$

Thus, $$\int_0^{v'} \frac{vdv}{g - kv^2/m} = \int_0^h ds$$

Integrating and putting the value of h from Eq. (1), we get

$$v' = \frac{v_0}{\sqrt{1 + kv_0^2/mg}}$$

1.105 Let us fix $x-y$ co-ordinate system to the given plane, taking x-axis in the direction along which the force vector was oriented at the moment $t = 0$, then the fundamental equation of dynamics expressed via the projection on x and y axes gives,

$$F\cos\omega t = m\frac{dv_x}{dt}$$

and
$$F\sin\omega t = m\frac{dv_y}{dt}$$

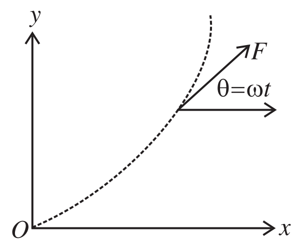
(a) Using the condition $v(0) = 0$, we obtain $v_x = \frac{F}{m\omega}\sin\omega t$

and
$$v_y = \frac{F}{m\omega}(1 - \cos\omega t)$$

Hence,
$$v = \sqrt{v_x^2 + v_y^2} = \left(\frac{2F}{m\omega}\right)\left|\sin\left(\frac{\omega t}{2}\right)\right|$$

(b) It is seen from this that the velocity v turns into zero after the time interval Δt, which can be found from therelation, $\omega\frac{\Delta t}{2} = \pi$. Consequently, the sought distance is

$$s = \int_0^{\Delta t} v\,dt = \frac{8F}{m\omega^2}$$

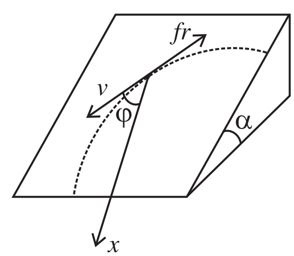


Average velocity,
$$<v> = \frac{\int v\,dt}{\int dt}$$

So,
$$<v> = \int_0^{2\pi/\omega}\frac{2F}{m\omega}\sin\left(\frac{\omega t}{2}\right)dt/(2\pi\omega) = \frac{4F}{\pi m\,\omega}$$

1.106 The acceleration of the disk along the plane is determined by the projection of the force of gravity on this plane $F_x = mg\sin\alpha$ and the friction force $fr = kmg\cos\alpha$. In our case, $k = \tan\alpha$ and therefore

$$fr = F_x = mg\sin\alpha$$



Let us find the projection of acceleration on the direction of the tangent to the trajectory and on the x-axis

$$mw_t = F_x\cos\varphi - fr = mg\sin\alpha(\cos\varphi - 1)$$

$$mw_x = F_x - fr\cos\varphi = mg\sin\alpha(1 - \cos\varphi)$$

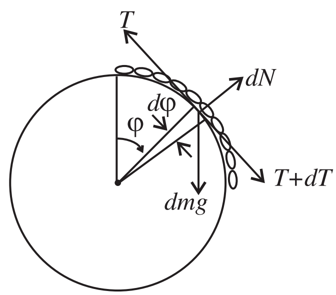
It is seen from this that $w_t = -w_x$, which means that the velocity v and its projection v_x differ only by a constant value C which does not change with time, i.e., $v = -v_x + C$, where $v_x = v\cos\varphi$. The constant C is found from the initial condition $v = v_0$, $U_x = 0$ since $\varphi = \pi/2$ initially. Finally, we obtain

$$C = v_0$$

$$v = \frac{v_0}{(1 + \cos\varphi)}$$

In the course of time $\varphi \to 0$ and $v \to v_0/2$. (Motion then is unaccelerated.)

1.107 Let us consider an element of length ds at an angle φ from the vertical diameter. As the speed of this element is zero at initial instant of time, its centripetal acceleration is zero, and hence, $dN - \lambda ds\, g\cos\varphi = 0$, where λ is the linear mass density of the chain. Let T and $T + dT$ be the tension at the upper and the lower ends of ds, then we have from $F_t = mw_t$,



$$(T + dT) + \lambda\, ds\, g\sin\varphi - T = \lambda ds w_t$$

or $$dT + \lambda\, Rd\varphi g\sin\varphi = \lambda ds w_t$$

If we sum the above equation for all elements, the term $\int dT = 0$, because there is no tension at the free ends, so

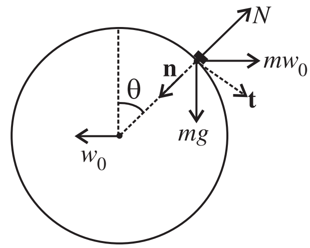
$$\lambda g R \int_0^{l/R} \sin\varphi\, d\varphi = \lambda w_t \int ds = \lambda l w_t$$

Hence, $$w_t = \frac{gR}{l}\left(1 - \cos\frac{l}{R}\right)$$

As $w_n = a$ at initial moment, so

$$w = |w_t| = \frac{gR}{l}\left(1 - \cos\frac{l}{R}\right)$$

1.108 In the problem, we require the velocity of the body, relative to the sphere, which itself moves with an acceleration w_0 in horizontal direction (say towards left). Hence it is advisable to solve the problem in the frame of sphere (*a translating frame here*).



At an arbitrary moment, when the body is at an angle θ with the vertical, we sketch the force diagram for the body and write the second law of motion in projection form $F_n = mw_n$ as,

$$mg\cos\theta - N - mw_0\sin\theta = \frac{mv^2}{R} \qquad (1)$$

At the break off point, $N = 0$, $\theta = \theta_0$ and let $v = v_0$, so Eq. (1) becomes,

$$\frac{v_0^2}{R} = g\cos\theta_0 - w_0\sin\theta_0 \qquad (2)$$

From $F_t = mw_t$

$$mg\sin\theta - mw_0\cos\theta = m\frac{vdv}{ds} = m\frac{v\,dv}{Rd\theta}$$

or $$vdv = R(g\sin\theta + w_0\cos\theta)\,d\theta$$

Integrating $$\int_0^{v_0} v\,dv = \int_0^{\theta_0} R\,(g\sin\theta + w_0\cos\theta)\,d\theta$$

$$\frac{v_0^2}{2R} = g(1 - \cos\theta_0) + w_0\sin\theta_0 \qquad (3)$$

Note that Eq. (3) can also be obtained by the work-energy theorem $A = \Delta T$ (in the frame of sphere). Therefore,

$$mgR\,(1 - \cos\theta_0) + mw_0\,R\sin\theta_0 = \frac{1}{2}mv_0^2$$

[here $mw_0\,R\sin\theta_0$ is the work done by the pseudoforce $(-m\mathbf{w}_0)$]

or $$\frac{v_0^2}{2R} = g(1 - \cos\theta_0) + w_0\sin\theta_0$$

Solving Eqs. (2) and (3), we get

$$v_0 = \sqrt{\frac{2gR}{3}} \quad \text{and } \theta_0 = \cos^{-1}\left[\frac{2 + k\sqrt{5 + 9k^2}}{3\,(1 + k^2)}\right] \left(\text{where } k = \frac{w_0}{g}\right)$$

Hence, $$\theta_0\Big|_{w_0 = g} = 17^\circ$$

1.109 This is not central force problem unless the path is a circle about the said point. Rather here F_t (tangential force) vanishes. Thus, the equation of motion becomes

$$v_t = v_0 = \text{constant}$$

and
$$\frac{m\,v_0^2}{r} = \frac{A}{r^n} \quad (\text{for} \quad r = r_0)$$

We can consider the latter equation as the equilibrium under two forces. When the motion is perturbed, we write $r = r_0 + x$ and the net force acting on the particle is

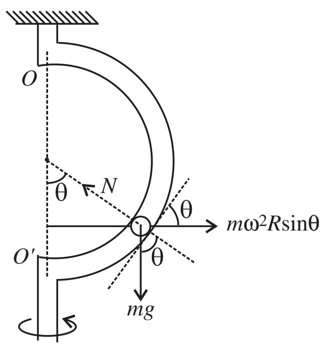
$$-\frac{A}{(r_0 + x)^n} + \frac{m\,v_0^2}{r_0 + x} = \frac{-A}{r_0^n}\left(1 - \frac{nx}{r_0}\right) + \frac{m\,v_0^2}{r_0}\left(1 - \frac{x}{r_0}\right)$$
$$= -\frac{m\,v_0^2}{r_0^2}(1 - n)\,x$$

This opposes the displacement x, if $n < 1$. ($m\,v_0^2/r$ is an outward directed centrifugal force while $-A/r^n$ is the inward directed external force.)

1.110 Let us observe the behaviour of the sleeve in the frame fixed to the rotating rod bent into the shape of half circle. We resolve all the forces into tangential and normal components, then the net downward tangential force on the sleeve is

$$mg\sin\theta\left(1 - \frac{\omega^2 R}{g}\cos\theta\right)$$

This vanishes for $\theta = 0$ and for $\theta = \theta_0 = \cos^{-1}(g/\omega^2 R)$, which is real if $\omega^2 R > g$.



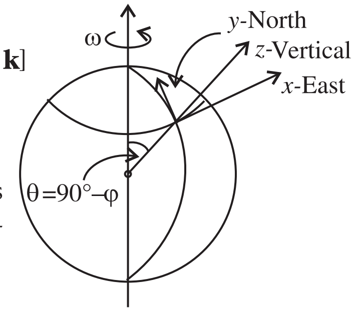
If $\omega^2 R < g$, then $[1-(\omega^2 R/g)\cos\theta]$ is always positive for small values of θ and hence the net tangential force near $\theta = 0$ opposes any displacement away from it, and $\theta = 0$ is then stable.

If $\omega^2 R > g$, then $[1-(\omega^2 R/g)\cos\theta]$ is negative for small θ near $\theta = 0$, and $\theta = 0$ is then unstable.

However $\theta = \theta_0$ is stable because the force tends to bring the sleeve near the equilibrium position $\theta = \theta_0$.

If $\omega^2 R = g$, the two positions coincide and becomes a stable equilibrium point.

1.111 Define the axes as shown with **z** along the local vertical, x due east and y due north. (We assume we are in the northern hemisphere.) Then the Coriolis force has the components as



$$F_{cor} = -2m\,(\omega \times \mathbf{v})$$

$$= 2m\omega\,[v_y \cos\theta - v_z \sin\theta)\mathbf{i} - v_x \cos\theta\,\mathbf{j} + v_x \cos\theta\,\mathbf{k}]$$

$$= 2m\omega\,(v_y \cos\theta - v_z \sin\theta)\mathbf{i}$$

Since v_x is small when the direction in which the gun is fired is due north. Thus the equations of motion (neglecting centrifugal forces) are

$$x = 2m\omega(v_y \sin\varphi - v_z \cos\varphi),\ \ddot{y} = 0 \text{ and } \ddot{z} = -g$$

Integrating we get $y = v$ (constant)

$$\mathbf{z} = -gt \text{ and } x = 2\omega v \sin\varphi t + \omega g t^2 \cos\varphi$$

Finally,

$$x = \omega v t^2 \sin\varphi + \frac{1}{3}\omega g t^3 \cos\varphi$$

Now, $v >> gt$ in the present case. So,

$$x = \omega v \sin\varphi \left(\frac{s}{v}\right)^2 = \omega \sin\theta\,\frac{s^2}{v}$$

$$= 7 \text{ cm (to the east)}$$

1.112 For the observer attached in the frame of disks

$$\mathbf{N} + \mathbf{F}_{cor} + m\mathbf{g} + \mathbf{F}_{cf} = 0$$

So, $$\mathbf{N} = -(\mathbf{F}_{cor} + m\mathbf{g} + \mathbf{F}_{cf})$$

So, $$|\mathbf{N}| = \sqrt{(2m\,v'\omega)^2 + (mg)^2 + (m\omega^2 r)^2}$$

1.113 The sleeve is free to slide along the rod AB. Thus the centrifugal force acts on it in the outward direction along the river.

So, we have $$m\omega' = m\omega^2 r$$

where $$v' = \frac{dr}{dt}$$

But, $$w' = \frac{vdv}{dr} = \frac{d}{dr}\left(\frac{1}{2}v^2\right)$$

So, $$\frac{1}{2} v^2 = \frac{1}{2} \omega^2 r^2 + \text{constant}$$

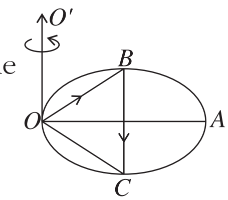
or $$v^2 = v_0^2 + \omega^2 r^2$$

v_0 being the initial velocity when $r = 0$. The Coriolis force is then

$$2m\omega\sqrt{v_0^2 + \omega^2 r^2} = 2m\omega^2 r\sqrt{1 + v_0^2/\omega^2 r^2)} = 2.83 \text{ N} \quad \text{(on substituting values)}$$

1.114 The disk $OBAC$ is rotating with angular velocity ω about the axis OO' passing through the edge point O. The equation of motion in rotating frame is

$$m\mathbf{w}' = \mathbf{F} + m\omega^2 \mathbf{R} + 2m(\mathbf{v}' \times \boldsymbol{\omega}) = \mathbf{F} + \mathbf{F}_{in}$$



where $\mathbf{F}_{in}$ is the resultant inertial force (pseudo force) which is the vector sum of centrifugal and Coriolis forces.

(a) At A, F_{in} vanishes.

Thus, $$0 = -2m\,\omega^2 R\,\mathbf{n} + 2mv'\omega\mathbf{n}$$

where $\mathbf{n}$ is the inward drawn unit vector to the centre, from the point in question, here A.

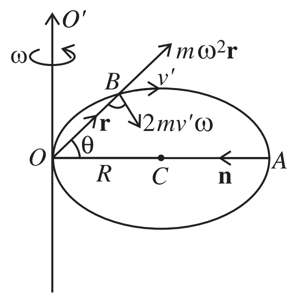
Thus, $$v' = \omega R$$

So, $$w = \frac{v'^2}{\rho} = \frac{v'^2}{R} = \omega^2 R$$

(b) At B, $$\mathbf{F}_{in} = m\omega^2\mathbf{r} + 2mv'\omega^2(-\mathbf{R})$$

$$v' = \omega r = \text{constant} \qquad \text{(from question)}$$

$$\mathbf{F}_{in} = m\omega^2\mathbf{r} + 2m\omega^2\mathbf{R}$$



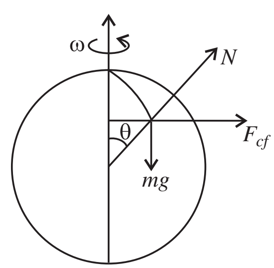
From geometry,

$$r = 2R\cos\theta \text{ (see figure)}$$

So, $$|\mathbf{F}_{in}| = m\omega^2\sqrt{4R^2 - r^2}$$

1.115 The equation of motion in the rotating coordinate system is

$$m\mathbf{w}' = \mathbf{F} + m\omega^2\boldsymbol{\rho} + 2m(\mathbf{v}' \times \boldsymbol{\omega})$$



Now, $$\mathbf{v}' = \mathbf{R}\dot{\theta}\,\mathbf{e}_\theta + R\sin\theta\,\dot{\varphi}\,\mathbf{e}_\varphi$$

and $$\boldsymbol{\omega} = \omega\cos\theta\,\mathbf{e}_r - \omega\sin\theta\,\mathbf{e}_\theta$$

$$\frac{1}{2m}\mathbf{F}_{cor} = \begin{vmatrix} \mathbf{e}_r & \mathbf{e}_\theta & \mathbf{e}_\varphi \\ 0 & R\dot{\theta} & (R\sin\theta)\dot{\varphi} \\ \omega\cos\theta & -\omega\sin\theta & 0 \end{vmatrix}$$

$$= \mathbf{e}_r(\omega R\sin^2\theta\,\dot{\varphi}) + [\omega R\sin\theta\cos\theta\,\dot{\varphi}]\mathbf{e}_\theta - (\omega R\dot{\theta}\cos\theta)\mathbf{e}_\varphi$$

Now, on the sphere

$$\mathbf{w}' = (-R\dot{\theta}^2 - R\sin\theta^2\,\dot{\varphi}^2)\mathbf{e}_r + (R\ddot{\theta} - R\sin\dot{\theta}\cos\theta\,\dot{\varphi}^2)\,\mathbf{e}_\theta$$
$$+ (R\sin\theta\ddot{\varphi} + 2R\cos\theta\,\dot{\theta}\,\dot{\varphi})\,\mathbf{e}_\varphi$$

Thus the equations of motion are

$$m(-R\dot{\theta}^2 - R\sin^2\theta\ddot{\varphi}^2) = N - mg\cos\theta + m\omega^2 R\sin^2\theta + 2m\omega R\sin^2\theta\dot{\varphi}$$
$$m(R\ddot{\theta} - R\sin\theta\cos\theta\,\dot{\varphi}^2) = mg\sin\theta + m\omega^2 R\sin\theta\cos\theta + 2m\omega R\sin\theta\cos\theta\,\dot{\varphi}$$
$$m(R\sin\theta\,\ddot{\varphi} + 2R\cos\theta\,\dot{\theta}\,\,\dot{\varphi}) = -2\,m\omega R\ddot{\theta}\,\,\cos\theta$$

From the third equation, we get, $\dot{\varphi} = -\omega$.
A result that is easy to understand by considering the motion in non-rotating frame.

Eliminating $\dot{\varphi}$ we get,
$$mR\dot{\theta}^2 = mg\cos - N$$
$$mR\ddot{\theta} = mg\sin\theta$$

Integrating the last equation

$$\frac{1}{2}mR\dot{\theta}^2 = mg(1 - \cos\theta) \tag{1}$$

Hence,
$$N = (3\cos\theta - 2)mg$$

So the body must fly off for $\theta = \theta_0 = \cos^{-1}\frac{2}{3}$, exactly as if the sphere were non-rotating.

Now, at this point F_{cf} = centrifugal force = $m\,\omega^2 R\sin\theta_0 = \sqrt{\frac{5}{9}}\,m\omega^2 R$

$$F_{cor} = 2m\sqrt{(\omega^2 R)^2\sin^2\theta\cos^2\theta + (\omega^2 R^2)^2\sin^4\theta + (\omega R)^2\cos^2\theta\dot{\theta}^2}$$

Putting the value of θ from Eq. (1)

$$\theta = \theta_0$$

we get,
$$F_{cor} = \frac{2m\omega^2 R}{3}\sqrt{5 + \frac{8g}{3\omega^2 R}} = 17\text{ N} \quad \text{(on substituting values)}$$

1.116 (a) When the train is moving along a meridian only the Coriolis force has a lateral component and its magnitude (see the previous problem) is,

$$2m\omega v\cos\theta = 2m\omega\sin\lambda \quad \text{(here we have put } R\dot{\theta} \to v\text{)}$$

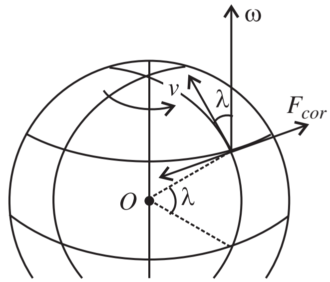
So, $$F_{\text{lateral}} = 2 \times 2000 \times 10^3 \times \frac{2\pi}{86400} \times \frac{54000}{3600} \times \frac{\sqrt{3}}{2}$$

$$= 3.77 \text{ kN} \quad \text{(we write } \lambda \text{ for the latitude)}$$

(b) The resultant of the inertial forces acting on the train is,

$$\mathbf{F}_{in} = -2m\omega R\dot{\theta}\cos\theta\,\mathbf{e}_\varphi + (m\omega^2 R\sin\theta\cos\theta + 2m\,\omega R\sin\theta\cos\theta\,\dot{\varphi})\,\mathbf{e}_\theta$$

$$+ (m\omega^2 R\sin^2\theta + 2m\,\omega R\sin^2\theta\,\dot{\varphi})\,\mathbf{e}_r$$



This vanishes if $\dot{\theta} = 0$, $\dot{\varphi} = -\frac{1}{2}\omega$

Thus, $$\mathbf{v} = v_\varphi \mathbf{e}_\varphi, v_\varphi = -\frac{1}{2}\omega R\sin\theta = -\frac{1}{2}\omega R\cos\lambda$$

(we write λ for the latitude here)

Thus the train must move from the east to west along the 60th parallel with a speed,

$$\frac{1}{2}\omega R\cos\lambda = \frac{1}{4} \times \frac{2\pi}{8.64} 10^{-4} \times 6.37 \times 10^6 = 115.8 \text{ m/s} \approx 420 \text{ km/h}$$

1.117 We go to the equation given in problem 1.111. Here $v_y = 0$ so we can take $y = 0$; thus, we get for the motion in the $x\,z$ plane.

$$\ddot{x} = -2\omega v_z \cos\varphi$$

and $$\dot{z} = -g$$

Integrating, $$z = -\frac{1}{2}gt^2$$

$$\dot{x} = \omega g\cos\varphi t^2$$

So $$x = \frac{1}{3}\omega g\cos\varphi t^3 = \frac{1}{3}\omega g\cos\varphi\left(\frac{2h}{g}\right)^{3/2} = \frac{2\omega h}{3}\cos\varphi\sqrt{\frac{2h}{g}}$$

There is thus a displacement to the east of

$$\frac{2}{3} \times 500 \times \frac{2\pi}{8.64} \times 10^{-4} \times 1 \times \sqrt{\frac{2 \times 500}{9.8}} = 26 \text{ cm}$$

1.3 Law of Conservation of Energy, Momentum and Angular Momentum

1.118 As **F** is constant so the sought work done

$$A = \mathbf{F} \cdot \Delta \mathbf{r} = \mathbf{F} \cdot (\mathbf{r}_2 - \mathbf{r}_1)$$

or $$A = (3\mathbf{i} + 4\mathbf{j}) \cdot [(2\mathbf{i} - 3\mathbf{j}) - (\mathbf{i} + 2\mathbf{j})] = (3\mathbf{i} + 4\mathbf{j}) \cdot (\mathbf{i} - 5\mathbf{j}) = 17 \text{ J}$$

1.119 As locomotive is in unidirectional motion, its acceleration

$$w = \frac{dv}{dt} = \frac{1}{2} \frac{dv^2}{ds} = \frac{a^2}{2}$$

Hence, force acting on the locomotive $F = mw = \dfrac{ma^2}{2}$

Let, $v = 0$ at $t = 0$, then the distance covered during the first t seconds

$$s = \frac{1}{2} wt^2 = \frac{1}{2}\frac{a^2}{2} t^2 = \frac{a^2}{4} t^2$$

Hence the sought work,

$$A = Fs = \frac{ma^2}{2} \frac{(a^2 t^2)}{4} = \frac{ma^4 t^2}{8}$$

1.120 We have $$T = \frac{1}{2} mv^2 = as^2 \quad \text{or} \quad v^2 = \frac{2as^2}{m} \tag{1}$$

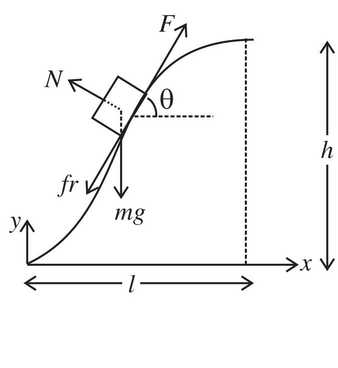
Differentiating Eq. (1) with respect to time

$$2vw_t = \frac{4as}{m} v \quad \text{or} \quad w_t = \frac{2as}{m} \tag{2}$$

Hence net acceleration of the particle

$$w = \sqrt{w_t^2 + w_n^2} = \sqrt{\left(\frac{2as}{m}\right)^2 + \left(\frac{2as^2}{mR}\right)^2} = \frac{2as}{m}\sqrt{1 + (s/R)^2}$$

Hence the sought, force $F = mw = 2as\sqrt{1 + (s/R)^2}$



1.121 Let **F**, make an angle θ with the horizontal at an arbitrary instant of time (see figure). Newton's second law in projection form along the direction of the force gives
$F = kmg \cos\theta + mg \sin\theta$ (because there is no acceleration of the body)

As $\mathbf{F}\uparrow\uparrow d\mathbf{r}$, the differential work done by the force $\mathbf{F}$

$$\begin{aligned} dA = \mathbf{F}\cdot d\mathbf{r} &= Fds \quad (\text{where } ds = |d\mathbf{r}|) \\ &= kmg\, ds\,(\cos\theta) + mg\, ds \sin\theta \\ &= kmg\, dx + mg\, dy \end{aligned}$$

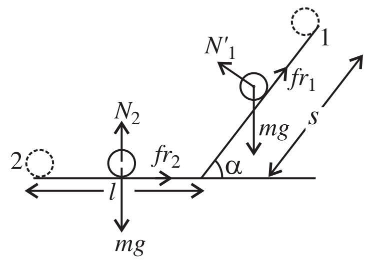
Hence,

$$\begin{aligned} A &= kmg\int_0^l dx + mg\int_0^h dy \\ &= kmgl + mgh = mg(kl + h) \end{aligned}$$

1.122 Let s be the distance covered by the disk along the incline, from the equation of increment of mechanical energy of the disk in the field of gravity: $\Delta T + \Delta U - A_{fr}$

$$0 + (-mgs\sin\alpha) = -kmg\cos\alpha\; s - kmgl$$

or $$s = \frac{kl}{\sin\alpha - k\cos\alpha} \tag{1}$$



Hence the sought work

$$A_{fr} = -kmg\,[s\cos\alpha + l]$$

$$A_{fr} = -\frac{klmg}{1 - k\cot\alpha} \qquad (\text{using Eq. 1})$$

On putting the values

$$A_{fr} = -0.05 \text{ J}$$

1.123 Let x be the compression in the spring when the bar m_2 is about to shift. Therefore at this moment spring force on m_2 is equal to the limiting friction between the bar m_2 and horizontal floor. Hence

$$\kappa x = km_2 g \quad [\text{where } \kappa \text{ is the spring constant (say)}] \tag{1}$$

For the block m_1 from work-energy theorem:

$A = \Delta T = 0$ for minimum force. (A here includes the work done in stretching the spring.)

So, $$Fx - \frac{1}{2}\kappa x^2 - kmgx = 0 \quad \text{or} \quad \kappa\frac{x}{2} = F - km_1 g \tag{2}$$

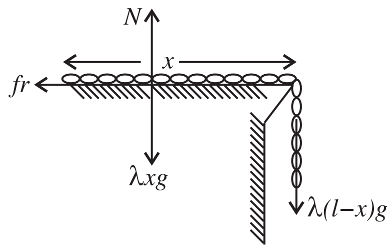
From Eqs. (1) and (2), $$F = kg\left(m_1 + \frac{m_2}{2}\right)$$

1.124 From the initial condition of the problem, the limiting friction between the chain lying on the horizontal table equals the weight of the over hanging part of the chain, i.e., $\lambda \eta l g = k\lambda(1-\eta) l g$ (where λ is the linear mass density of the chain).

So, $$k = \frac{\eta}{1-\eta} \qquad (1)$$

Let (at an arbitrary moment of time) the length of the chain on the table be x. So the net friction force between the chain and the table, at this moment

$$fr = kN = k\lambda x g \qquad (2)$$



The differential work done by the friction forces

$$dA = \mathbf{f}r \cdot d\mathbf{r} = -\mathbf{f}r\, ds$$

$$= -k\lambda x g\,(-dx) = \lambda g\left(\frac{\eta}{1-\eta}\right) x\,dx \qquad (3)$$

(Note that here we have written $ds = -dx$, because ds is essentially a positive term and as the length of the chain decreases with time, dx is negative.)

Hence, the sought work done

$$A = \int_{(1-\eta)l}^{0} \lambda g \frac{\eta}{1-\eta} x\,dx = -(1-\eta)\eta\frac{mgl}{2} = -1.3 \text{ J}$$

1.125 The velocity of the body, t seconds after the beginning of the motion becomes $\mathbf{v} = \mathbf{v}_0 + \mathbf{g}t$. The power developed by the gravity ($m\mathbf{g}$) at that moment, is

$$P = m\mathbf{g}\cdot\mathbf{v} = m(\mathbf{g}\cdot\mathbf{v}_0 + g^2 t) = mg(gt - v_0 \sin\alpha)$$

As $m\mathbf{g}$ is a constant force, so the average power

$$\langle P \rangle = \frac{A}{\tau} = \frac{m\mathbf{g}\cdot\Delta\mathbf{r}}{\tau}$$

where $\Delta\mathbf{r}$ is the net displacement of the body during time of flight.

As, $$m\mathbf{g} \perp \Delta\mathbf{r}, \quad \text{so} \quad \langle P \rangle = 0$$

1.126 We have $$w_n = v^2/R = at^2 \quad \text{or} \quad v = \sqrt{aR}\; t$$

where t is defined to start from the beginning of motion from rest.

So, $$w_t = \frac{dv}{dt} = \sqrt{aR}$$

Instantaneous power, $P = \mathbf{F} \cdot \mathbf{v} = m\,(w_t\,\mathbf{t} + w_n \mathbf{n}) \cdot (\sqrt{aR}\,t\,\mathbf{t})$, where $\mathbf{t}$ and $\mathbf{n}$ are unit vectors along the direction of tangent (velocity) and normal, respectively.

So, $$P = mw_t \sqrt{aR}\,t = maRt$$

Hence the sought average power

$$<P> = \frac{\int_0^t P\,dt}{\int_0^t dt} = \frac{\int_0^t ma\,Rt\,dt}{t}$$

Hence, $$<P> = \frac{maRt^2}{2t} = \frac{maRt}{2}$$

1.127 Let the body m acquire the horizontal velocity v_0 along positive x-axis at the point O.

(a) Velocity of the body t seconds after the beginning of the motion,

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{w}t = (v_0 - kg\,t)\,\mathbf{i} \tag{1}$$

Instantaneous power $P = \mathbf{F} \cdot \mathbf{v} = (-kmg\,\mathbf{i}) \cdot (v_0 - kgt)\,\mathbf{i} = -kmg\,(v_0 - kgt)$

From Eq. (1), the time of motion $\tau = v_0/kg$.

Hence sought average power during the time of motion

$$<P> = \frac{\int_0^\tau -kmg\,(v_0 - kgt)dt}{\tau} = -\frac{kmg\,v_0}{2} = -2\text{ W} \quad \text{(on substituting values)}$$

(b) From $$F_x = mw_x$$

$$-kmg = mw_x = mv_x \frac{dv_x}{dx}$$

or $$v_x dv_x = -kgdx = -\alpha gxdx$$

To find $v\,(x)$, let us integrate the above equation

$$\int_{v_0}^{v} v_x dv_x = -\alpha g \int_0^x xdx \quad \text{or} \quad v^2 = v_0^2 - \alpha g x^2$$

Now, $$\mathbf{P} = \mathbf{F} \cdot \mathbf{v} = -m\alpha\, xg\sqrt{v_0^2 - \alpha\, gx^2} \tag{2}$$

For maximum power,

$$\frac{d}{dt}\left(\sqrt{v_0^2 x^2 - \lambda g x^4}\right) = 0 \text{ which yields } x = \frac{v_0}{\sqrt{2\alpha g}}$$

Putting this value of x in Eq. (2), we get

$$P_{\max} = -\frac{1}{2} m v_0^2 \sqrt{\alpha g}$$

1.128 Centrifugal force of inertia is directed outward along radial line, thus the sought work

$$A = \int_{r_1}^{r_2} m\omega^2\, r dr = \frac{1}{2}\, m\omega^2\, (r_2^2 - r_1^2) = 0.20 \text{ J} \quad \text{(on substituting values)}$$

1.129 Since the springs are connected in series, the combination may be treated as a single spring of spring constant.

So, $$\kappa = \frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}$$

From the equation of increment of mechanical energy,

$$\Delta T + \Delta U_{\text{spring}} = A_{\text{ext}}$$

$$0 + \frac{1}{2}\, \kappa \Delta l^2 = A_{\text{ext}} \quad \text{or} \quad A_{\text{ext}} = \frac{1}{2}\left(\frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}\right) \Delta l^2$$

1.130 First, let us find the total height of ascent. At the beginning and at the end of the path, velocity of the body is equal to zero, and therefore the increment of the kinetic energy of the body is also equal to zero. On the other hand, according to work-energy theorem, ΔT is equal to the algebraic sum of the work A performed by all the forces, i.e. by the force F and gravity, over this path. However, since $\Delta T = 0$ then $A = 0$. Taking into account that the upward direction is assumed to coincide with the positive direction of the y-axis, we can write

$$A = \int_0^h (\mathbf{F} + m\mathbf{g}) \cdot d\mathbf{r} = \int_0^h (F_y - mg)\, dy$$

$$= mg \int_0^h (1 - 2\, ay)\, dy = mgh\, (1 - ah) = 0 \text{ (when } h = 1/a)$$

The work performed by the force F over the first half of the ascent is

$$A_F = \int_0^{h/2} F_y dy = 2mg \int_0^{h/2} (1 - ay)\, dy = \frac{3\,mg}{4a}$$

The corresponding increment of the potential energy is

$$\Delta U = \frac{mgh}{2} = \frac{mg}{2a}$$

1.131 From the equation $F_r = -\dfrac{dU}{dr}$, we get $F_r = \left[-\dfrac{2a}{r^3} + \dfrac{b}{r^2}\right]$

(a) We have at $r = r_0$, the particle is in equilibrium position, i.e, $F_r = 0$.

So, $$r_0 = \frac{2a}{b}$$

To check whether the position is steady (the position of stable equilibrium), we have to satisfy

$$\frac{d^2U}{dr^2} > 0$$

We have $$\frac{d^2U}{dr^2} = \left[\frac{6a}{r^4} - \frac{2b}{r^3}\right]$$

Putting the value of $r = r_0 = 2a/b$, we get

$$\frac{d^2U}{dr^2} = \frac{b^4}{8a^3} \quad \text{(as } a \text{ and } b \text{ are positive constants)}$$

So, $$\frac{d^2U}{dr^2} = \frac{b^2}{8a^3} > 0$$

which indicates that the potential energy of the system is minimum, hence this position is steady.

(b) We have $$F_r = -\frac{dU}{dr} = \left[-\frac{2a}{r^3} + \frac{b}{r^2}\right]$$

For F to be maximum, $\dfrac{dF_r}{dr} = 0$

So, $$r = \frac{3a}{b} \quad \text{and} \quad F_{r(\max)} = \frac{-b^3}{27a^2}$$

As F_r is negative, the force is attractive.

1.132 (a) We have

$$F_x = -\frac{\partial U}{\partial x} = -2\alpha x \quad \text{and} \quad F_y = \frac{-\partial U}{\partial y} = -2\beta y$$

So, $$\mathbf{F} = 2\alpha x\mathbf{i} - 2\beta y\ \mathbf{i} \quad \text{and} \quad F = 2\sqrt{\alpha^2 x^2 + \beta^2 y^2}$$

For a central force, $\mathbf{r} \times \mathbf{F} = 0$.

Here, $$\mathbf{r} \times \mathbf{F} = (x\mathbf{i} + y\mathbf{j}) \times (-2\alpha\ x\mathbf{i} - 2\beta y\mathbf{j})$$
$$= -2\beta xy\,\mathbf{k} - 2\alpha xy\,(\mathbf{k}) \neq 0$$

Hence the force is not a central force.

(b) As $U = \alpha x^2 + \beta y^2$

So, $$F_x = \frac{\partial U}{\partial x} = -2\ \alpha x \quad \text{and} \quad F_y = \frac{-\partial U}{\partial y} = -2\beta y$$

So, $$F = \sqrt{F_x^2 + F_y^2} = \sqrt{4\alpha^2 x^2 + 4\beta^2 y^2}$$

According to the problem

$$F = 2\sqrt{\alpha^2 x^2 + \beta^2 y^2} = C\,(\text{constant})$$

or $$\alpha^2 x^2 + \beta^2 y^2 = \frac{C^2}{2}$$

or $$\frac{x^2}{\beta^2} + \frac{y^2}{\alpha^2} = \frac{C^2}{2\alpha^2\beta^2} = k\,(\text{say})$$

Therefore, the surface for which F is constant is an ellipse.

For an equipotential surface U is constant.

So, $$\alpha x^2 + \beta y^2 = C_0\ (\text{constant})$$

or $$\frac{x^2}{\sqrt{\beta^2}} + \frac{y^2}{\sqrt{\alpha^2}} = \frac{C_0}{\alpha\beta} = K_0\ (\text{constant})$$

Hence, the equipotential surface is also an ellipse.

1.133 Let us calculate the work performed by the forces of each field over the path from a certain point 1 (x_1, y_1) to another point 2 (x_2, y_2)

(i) $$dA = \mathbf{F} \cdot d\mathbf{r} = ay\mathbf{i} \cdot d\mathbf{r} = ay\,dx \quad \text{or} \quad A = a\int_{x_1}^{x_2} y\,dx$$

(ii) $$dA = \mathbf{F} \cdot d\mathbf{r} = (ax\,\mathbf{i} + by\,\mathbf{j}) \cdot d\mathbf{r} = ax\,dx + by\,dy$$

Hence, $$A = \int_{x_1}^{x_2} axdx + \int_{y_1}^{y_2} bydy$$

In the first case, the integral depends on the function of type $y(x)$, i.e., on the shape of the path. Consequently, the first field of force is not potential. In the second case, both the integrals do not depend on the shape of the path. They are defined only by the coordinate, of the initial and final points of the path, therefore the second field of force is potential.

1.134 Let s be the sought distance, then from the equation of increment of mechanical energy.

$$\Delta T + \Delta U_{\text{gr}} = A_{fr}$$

$$\left(0 - \frac{1}{2} mv_0^2\right) + (mg\, s \sin\alpha) = -kmg \cos\alpha s$$

or $$s = \frac{v_0^2/2g}{(\sin\alpha + k\cos\alpha)}$$

Hence, $$A_{fr} = -kmg\cos\alpha s = \frac{-km\, v_0^2}{2(k + \tan\alpha)}$$

1.135 Velocity of the body at height h, $v_h = \sqrt{2g(H - h)}$, horizontally (from the figure given in the problem book). Time taken in falling through the distance h is given by

$$t = \sqrt{\frac{2h}{g}} \text{ (as initial vertical component of the velocity is zero)}$$

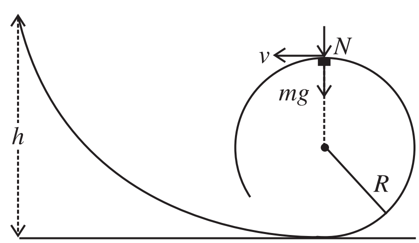
Now, $$s = v_h t = \sqrt{2g(H + h)} \times \sqrt{\frac{2h}{g}} = \sqrt{4(Hh - h^2)}$$

For $s_{\max}$, $$\frac{d}{ds}(Hh - h^2) = 0, \text{ which yields } h = \frac{H}{2}$$

Putting this value of h in the expression obtained for s, we get

$$s_{\max} = H$$

1.136 To complete a smooth vertical track of radius R, the minimum height at which a particle starts, must be equal to $\frac{5}{2} R$. Let us first prove this.



In the figure there is a smooth track and a small body is released at height h. We have to find minimum value for h so that the body can complete smooth vertical track of radius R. As the track is smooth and the body is under the action of only two forces, normal reaction and gravitational pull. The mechanical energy of the block in the gravitational field is conversed, because normal reaction does zero work where as the gravitational force is conservative. From the conservation of mechanical energy between the point of release and the upper most point of the circular track

$$\Delta T + \Delta U_{gr} = 0$$

$$\left(\frac{1}{2} mv^2 - 0\right) - mg(h - 2R) = 0$$

which yields $$v^2 = 2g(h - 2R) \tag{1}$$

To complete the vertical circular track the condition is that the normal reaction at the upper most point of the circular track $N \geq 0$.

From Newtons second law in projection form towards the centre of the track

$$F_n = mw_n \text{ (where } w_n \text{ is the normal acceleration)}$$

$$N + mg = m\frac{v^2}{R}$$

$$N = m\left(\frac{v^2}{R} - g\right)$$

As $N \geq 0$ at upper most point of the circular track

So, $$m\left(\frac{v^2}{R} - g\right) \geq 0$$

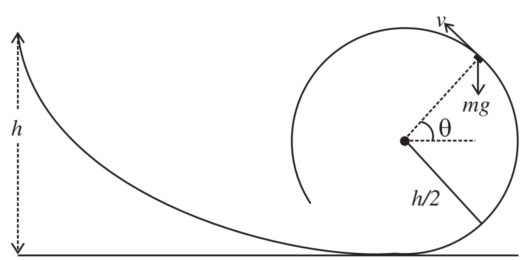
or $$v^2 \geq gR \tag{2}$$

From Eqs. (1) and (2)

$$2g(h - 2R) \geq gR$$

Hence, $$h \geq \frac{5}{2}R$$

Thus in our problem, the body could not reach the upper most point of the vertical track of radius $R/2$. Let the particle A leave the track at some point with speed v (see figure). Now from energy conservation for the body A in the field of gravity



$$mg\left[h - \frac{h}{2}(1 + \sin\theta)\right] = \frac{1}{2}mv^2$$

or $$v^2 = gh(1 - \sin\theta) \tag{3}$$

At the break off point the normal reaction $N = 0$, so from Newton's second law at this point

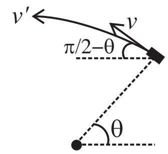
$$F_n = mw_n$$

or
$$mg\sin\theta = \frac{mv^2}{(h/2)}$$

So,
$$v^2 = \frac{gh}{2}\sin\theta \tag{4}$$

From Eqs. (3) and (4), $\sin\theta = \dfrac{2}{3}$ and $v = \sqrt{\dfrac{gh}{3}}$

After leaving the track, the body A comes in air and further goes up and at maximum height of it's trajectory in air and it's velocity (say v') becomes horizontal (Fig.). Hence, the sought velocity of A at this point.



$$v' = v\cos\left(\frac{\pi}{2} - \theta\right) = v\sin\theta = \frac{2}{3}\sqrt{\frac{gh}{3}}$$

1.137 Let the point of suspension be shifted with velocity v_A in the horizontal direction towards left, then in the rest frame of point of suspension the ball starts with same velocity horizontally towards right. Let us work in this frame. From Newton's second law in projection form towards the point of suspension at the upper most point (say B):

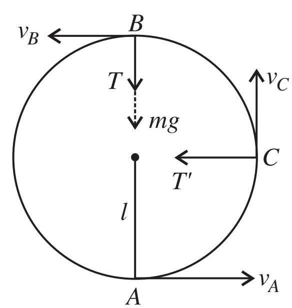
$$mg + T = \frac{mv_B^2}{l} \quad \text{or} \quad T = \frac{mv_B^2}{l} - mg \tag{1}$$

Condition required to complete the vertical circle is that $T \geq 0$. (2)

But,
$$\frac{1}{2}mv_A^2 = mg(2l) + \frac{1}{2}mv_B^2$$

So,
$$v_B^2 = v_A^2 - 4gl \tag{3}$$

From Eqs. (1), (2) and (3)



$$T = \frac{m(v_A^2 - 4gl)}{l} - mg \geq 0 \quad \text{or} \quad v_A \geq \sqrt{5gl}$$

Thus
$$v_{A\,(\min)} = \sqrt{5gl}$$

From the equation $F_n = mw_n$ at point C

$$T' = \frac{mv_C^2}{l} \tag{4}$$

Again from energy conservation

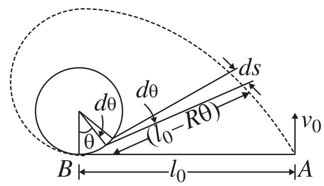
$$\frac{1}{2} mv_A^2 = \frac{1}{2} mv_C^2 + mgl \tag{5}$$

From Eqs. (4) and (5)

$$T = 3mg$$

1.138 Since the tension is always perpendicular to the velocity vector, the work done by the tension force will be zero. Hence, according to the work energy theorem, the kinetic energy or velocity of the disk will remain constant during it's motion. Hence, the sought time $t = s/v_0$, where s is the total distance traversed by the small disk during it's motion.

Now, at an arbitrary position (see figure)



$$ds = (l_0 - R\,\theta)\; d\theta$$

So, $$s = \int_0^{l/R} (l_0 - R\,\theta)\; d\theta$$

or $$s = \frac{l_0^2}{R} - \frac{R l_0^2}{2R^2} = \frac{l_0^2}{2R}$$

Hence, the required time $$t = \frac{l_0^2}{2Rv_0}$$

Note: *It should be clearly understood that the only uncompensated force acting on the disk A in this case is the tension T, of the thread. It is easy to see that there is no point here, relative to which the moment of force T is invariable in the process of motion. Hence conservation of angular momentum is not applicable here.*

1.139 Suppose that Δl is the elongation of the rubber cord. Then, from energy conservation

$$\Delta U_{gr} + \Delta U_{el} = 0 \text{ (as } \Delta T = 0)$$

or $$-mg\,(l + \Delta l) + \frac{1}{2}\kappa\, \Delta l^2 = 0$$

or $$\frac{1}{2}\kappa \Delta l^2 - mg\, \Delta l - mgl = 0$$

or $$\Delta l = \frac{mg \pm \sqrt{(mg)^2 + 4 \times \frac{\kappa}{2}\, mgl}}{2 \times \frac{\kappa}{2}} = \frac{mg}{\kappa}\left[1 + \sqrt{1 \pm \frac{2\kappa l}{mg}}\right]$$

Since the value of $\sqrt{1+\frac{2\kappa l}{mg}}$ is certainly greater than 1, hence negative sign is avoided.

So, $$\Delta l = \frac{mg}{\kappa}\left(1+\sqrt{1+\frac{2\kappa l}{mg}}\right)$$

1.140 When the thread PA is burnt, obviously the speed of the bars will be equal at any instant of time until it breaks off. Let v be the speed of each block and θ be the angle, which the elongated spring makes with the vertical at the moment when the bar A breaks off the plane. At this stage the elongation in the spring is

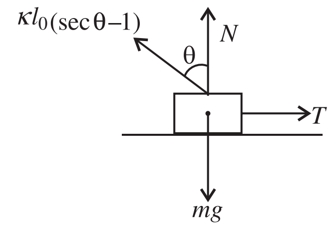
$$\Delta l = l_0 \sec\theta - l_0 = l_0(\sec\theta - 1) \quad (1)$$

Since the problem is concerned with position and there are no forces other than conservative forces, the mechanical energy of the system (both bars + spring) in the field of gravity is conserved, i.e., $\Delta T + \Delta U = 0$.

So, $$2\left(\frac{1}{2}mv^2\right) + \frac{1}{2}\kappa l_0^2(\sec\theta - 1)^2 - mgl_0\tan\theta = 0 \quad (2)$$

From Newton's second law in projection form along vertical direction:

$$mg = N + \kappa l_0(\sec\theta - 1)\cos\theta$$



But, at the moment of break off, $N = 0$.

Hence, $$\kappa l_0(\sec\theta - 1)\cos\theta = mg$$

or $$\cos\theta = \frac{\kappa l_0 - mg}{\kappa l_0} \quad (3)$$

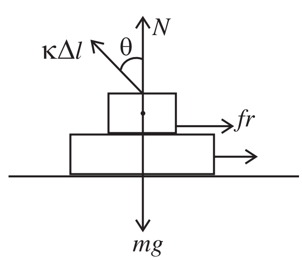
Taking $\kappa = \frac{5mg}{l_0}$, simultaneous solution of Eqs. (2) and (3) yields

$$v = \sqrt{\frac{19gl_0}{32}} = 1.7 \text{ m/s}$$

1.141 Obviously the elongation in the cord, $\Delta l = l_0(\sec\theta - 1)$, at the moment the sliding first starts and at the moment horizontal projection of spring force equals the limiting friction.

So, $$\kappa\Delta l\sin\theta = kN \quad (1)$$

(where κ is the elastic constant)



From Newton's law in projection form along vertical direction:

$$\kappa\Delta l\cos\theta + N = mg$$

or $$N = mg - \kappa\Delta l\cos\theta \quad (2)$$

From Eqs. (1) and (2),

$$\kappa \Delta l \sin\theta = k(mg - \kappa \Delta l \cos\theta)$$

or
$$\kappa = \frac{kmg}{\Delta l \sin\theta + k\Delta l \cos\theta}$$

From the equation of the increment of mechanical energy: $\Delta U + \Delta T = A_{fr}$

or
$$\left(\frac{1}{2}\kappa \Delta l^2\right) = A_{fr}$$

or
$$\frac{kmg\,\Delta l^2}{2\Delta l(\sin\theta + k\cos\theta)} = A_{fr}$$

Thus,
$$A_{fr} = \frac{(kmgl_0(\sec\theta - 1)}{2(\sin\theta - k\cos\theta)} = 0.09\text{ J} \quad \text{(on substituting values)}$$

1.142 Let the deformation in the spring be Δl, when the rod AB has attained the angular velocity ω. From the second law of motion in projection form $F_n = mw_n$.

$$\kappa \Delta l = m\omega^2(l_0 + \Delta l) \quad \text{or} \quad \Delta l = \frac{m\omega^2 l_0}{\kappa - m\omega^2}$$

From the energy equation,
$$A_{\text{ext}} = \frac{1}{2}mv^2 + \frac{1}{2}\kappa\Delta l^2$$

$$= \frac{1}{2}m\omega^2(l_0 + \Delta l)^2 + \frac{1}{2}\kappa\Delta l^2$$

$$= \frac{1}{2}m\omega^2\left(l_0 + \frac{m\omega^2 l_0}{\kappa - m\omega^2}\right)^2 + \frac{1}{2}\kappa\left(\frac{m\omega^2 l_0^2}{\kappa - m\omega^2}\right)^2$$

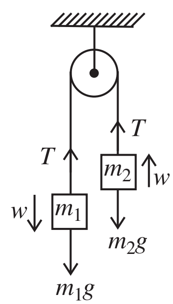
On solving
$$A_{\text{ext}} = \frac{\kappa}{2}\frac{l_0^2\,\eta(1+\eta)}{(1-\eta)^2}, \text{ where } \eta = \frac{m\omega^2}{\kappa}$$

1.143 We know that acceleration of centre of mass of the system is given by the expression.

$$\mathbf{w}_C = \frac{m_1\mathbf{w}_1 + m_2\mathbf{w}_2}{m_1 + m_2}$$

Since
$$\mathbf{w}_1 = -\mathbf{w}_2$$

$$\mathbf{w}_C = \frac{(m_1 - m_2)\,\mathbf{w}_1}{m_1 + m_2} \qquad (1)$$



Now from Newton's second law $\mathbf{F} = m\mathbf{w}$, for bodies m_1 and m_2, respectively,

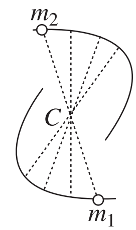
$$\mathbf{T} + m_1\mathbf{g} = m_1\mathbf{w}_1 \qquad (2)$$

and $$\mathbf{T} + m_2\mathbf{g} = m_2\mathbf{w}_2 = -m_2\mathbf{w}_1 \tag{3}$$

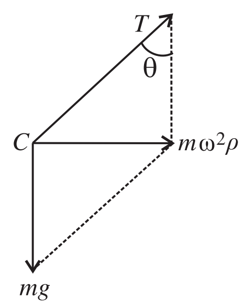
Solving Eqs. (2) and (3) $$\mathbf{w}_1 = \frac{(m_1 - m_2)\,\mathbf{g}}{m_1 + m_2} \tag{4}$$

Thus, from Eqs. (1) and (4), $$\mathbf{w}_C = \frac{(m_1 - m_2)^2\mathbf{g}}{(m_1 + m_2)^2}$$

1.144 As the closed system consisting of two particles m_1 and m_2 is initially at rest, the centre of mass of the system will remain at rest. Further as $m_2 = m_1/2$, the centre of mass of the system divides the line joining m_1 and m_2 at all the moments of time in the ratio 1:2. In addition to it, the total linear momentum of the system at all the times is zero. So, $\mathbf{p}_1 = -\mathbf{p}_2$ and therefore the velocities of m_1 and m_2 are also directed in opposite sense. Bearing in mind all these things, the sought trajectory is as shown in the figure.



1.145 First of all, it is clear that the chain does not move in the vertical direction during the uniform rotation. This means that the vertical component of the tension T balances gravity. As for the horizontal component of the tension T, it is constant in magnitude and permanently directed toward the rotation axis. It follows from this that the centre of mass of the chain, the point C, travels along horizontal circle of radius ρ (say). Therefore we have

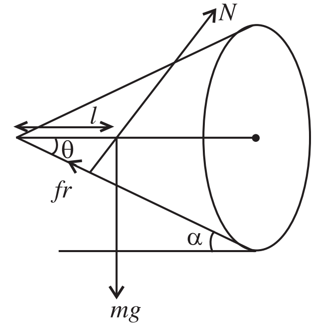


$$T\cos\theta = mg \quad \text{and} \quad T\sin\theta = m\omega^2\rho$$

Thus $$\rho = \frac{g\tan\theta}{\omega^2} = 0.8 \text{ cm}$$

and $$T = \frac{mg}{\cos\theta} = 5 \text{ N} \quad \text{(on substituting values)}$$

1.146 (a) Let us draw free body diagram and write Newton's second law in terms of projection along vertical and horizontal directions, respectively,



$$N\cos\alpha - mg + fr\sin\alpha = 0 \tag{1}$$

$$fr\cos\alpha - N\sin\alpha = m\omega^2 l \tag{2}$$

From Eqs. (1) and (2)

$$fr\cos\alpha - \frac{\sin\alpha}{\cos\alpha}(-fr\sin\alpha + mg) = m\omega^2 l$$

So, $$fr = mg\left(\sin\alpha + \frac{\omega^2 l}{g}\cos\alpha\right) \tag{3}$$

$$= 6 \text{ N} \quad \text{(on substituting values)}$$

(b) For rolling, without sliding,

$$fr \leq k\,N$$

but, $N = mg\cos\alpha - m\omega^2 l\sin\alpha$

$$mg\left(\sin\alpha + \frac{\omega^2 l}{g}\cos\alpha\right) \leq k\,(mg\cos\alpha - m\,\omega^2\,l\sin\alpha) \qquad \text{(using Eq. 3)}$$

Rearranging, we get

$$m\omega^2\,l\,(\cos\alpha + k\sin\alpha) \leq (k\,mg\cos\alpha - mg\sin\alpha)$$

Thus, $$\omega \leq \sqrt{g\,(k - \tan\alpha)/(1 + k\tan\alpha)l} = 2 \text{ rad/s}$$

1.147 (a) Total kinetic energy in frame K' is

$$T = \frac{1}{2}\,m_1\,(\mathbf{v}_1 - \mathbf{V})^2 + \frac{1}{2}\,m_2\,(\mathbf{v}_2 - \mathbf{V})^2$$

This is minimum with respect to variation in $\mathbf{V}$, when

$$\frac{\delta T}{\delta \mathbf{V}} = 0, \quad \text{i.e.,} \quad m_1(\mathbf{v}_1 - \mathbf{V}) + m_2(\mathbf{v}_2 - \mathbf{V}) = 0$$

or $$\mathbf{V} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} = \mathbf{v}_C$$

Hence, it is the frame of centre of mass in which kinetic energy of a system is minimum.

(b) Linear momentum of the particle 1 in the K' or C frame

$$\tilde{\mathbf{p}}_1 = m_1(\mathbf{v}_1 - \mathbf{v}_C) = \frac{m_1 m_2}{m_1 + m_2}(\mathbf{v}_1 - \mathbf{v}_2)$$

or $$\tilde{\mathbf{p}}_1 = \mu(\mathbf{v}_1 - \mathbf{v}_2), \text{ where } \mu = \frac{m_1 m_2}{m_1 + m_2} = \text{ reduced mass}$$

Similarly, $$\tilde{\mathbf{p}}_2 = \mu(\mathbf{v}_2 - \mathbf{v}_1)$$

So, $$|\tilde{\mathbf{p}}_1| = |\tilde{\mathbf{p}}_2| = \tilde{p} = \mu\,v_{\text{rel}}, \text{ where } v_{\text{rel}} = |\mathbf{v}_1 - \mathbf{v}_2|$$

Now the total kinetic energy of the system in the C frame is

$$\tilde{T} = \tilde{T}_1 + \tilde{T}_2 = \frac{\tilde{p}^2}{2m_1} + \frac{\tilde{p}^2}{2m_2} = \frac{\tilde{p}^2}{2\mu}$$

Hence, $$\tilde{T} = \frac{1}{2}\mu v_{\text{rel}}^2 = \frac{1}{2}\mu |\mathbf{v}_1 - \mathbf{v}_2|^2$$

1.148 To find the relationship between the values of the mechanical energy of a system in the K and C reference frames, let us begin with the kinetic energy T of the system. The velocity of the i-th particle in the K frame may be represented as $\mathbf{v}_i = \tilde{\mathbf{v}}_i + \mathbf{v}_C$. Now we can write

$$T = \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i (\tilde{\mathbf{v}}_i + \mathbf{v}_C) \cdot (\tilde{\mathbf{v}}_i + \mathbf{v}_C)$$
$$= \sum \frac{1}{2} m_i \tilde{v}_1^2 + \mathbf{v}_C \sum m_i \tilde{\mathbf{v}}_i + \sum \frac{1}{2} m_i v_C^2$$

Since in the C frame $\sum m_i \tilde{\mathbf{v}}_i = 0$, the previous expression takes the form

$$T = \tilde{T} + \frac{1}{2} m \, \tilde{v}_C^2 = \tilde{T} + \frac{1}{2} mV^2 \text{ (since according to the problem } v_C = V) \qquad (1)$$

Since the internal potential energy U of a system depends only on its configuration, the magnitude U is the same in all reference frames. Adding U to the left and right hand sides of Eq. (1), we obtain the sought relationship,

$$E = \tilde{E} + \frac{1}{2} mV^2$$

1.149 As initially $U = \tilde{U} = 0$, so, $\tilde{E} = \tilde{T}$. From the solution of problem 1.147(b)

$$\tilde{T} = \frac{1}{2}\mu |\mathbf{v}_1 - \mathbf{v}_2|$$

As $$\mathbf{v}_1 \perp \mathbf{v}_2$$

Thus, $$\tilde{T} = \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (v_1^2 + v_2^2)$$

1.150 Velocities of masses m_1 and m_2 after t seconds are, respectively,

$$\mathbf{v}'_1 = \mathbf{v}_1 + \mathbf{g}t \quad \text{and} \quad \mathbf{v}'_2 = \mathbf{v}_2 + \mathbf{g}t$$

Hence the final momentum of the system

$$\mathbf{p} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{v}_1 + m_1\mathbf{v}_2 + (m_1 + m_2)\,\mathbf{g}t$$

$$= \mathbf{p}_0 + m\mathbf{g}t \ \ (\text{where, } \mathbf{p}_0 = m_2\mathbf{v}_1 + m_2\mathbf{v}_2 \text{ and } m = m_1 + m_2)$$

Radius vector, is given by

$$\mathbf{r}_C = \mathbf{v}_C\, t + \frac{1}{2}\,\mathbf{w}_C t^2$$

$$= \frac{(m_1\mathbf{v}_1 + m_2\mathbf{v}_2)t}{(m_1 + m_2)} + \frac{1}{2}\,\mathbf{g}t^2 = \mathbf{v}_0 t + \frac{1}{2}\,\mathbf{g}t^2, \quad \text{where } \mathbf{v}_0 = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}$$

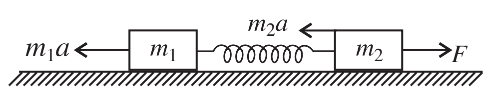
1.151 After releasing, bar 2 acquires velocity v_2, obtained by the energy conservation:

$$\frac{1}{2}\,m_2 v_2^2 = \frac{1}{2}\kappa x^2 \quad \text{or} \quad v_2 = x\sqrt{\kappa/m_2}$$

Thus, the sought velocity of C.M.

$$v_C = \frac{0 + m_2 x\sqrt{\kappa/m_2}}{m_1 + m_2} = \frac{x\sqrt{m_2\kappa}}{m_1 + m_2}$$

1.152 Let us consider both blocks and spring as the physical system. The centre of mass of the system moves with acceleration $a = F/(m_1 + m_2)$ towards right. Let us work in the frame of centre of mass. As this frame is a non-inertial frame (accelerated with respect to the ground) we have to apply a pseudo force $m_1 a$ towards left on the block m_1 and $m_2 a$ towards left on the block m_2.



As the centre of mass is at rest in this frame, the blocks move in opposite directions and come to instantaneous rest at some instant. The elongation of the spring will be maximum or minimum at this instant. Assume that the block m_1 is displaced by the distance x_1 and the block m_2 through a distance x_2 from the initial positions.

From the equation of increment of mechanical energy in C.M. frame

$$\Delta\tilde{T} + \Delta U = A_{\text{ext}}$$

where A_{ext} also includes the work done by the pseudo forces.

Here, $\Delta\tilde{T} = 0$, $\Delta U = \dfrac{1}{2}k\,(x_1 + x_2)^2$ and

$$W_{\text{ext}} = \left(F - \frac{m_2 F}{m_1 + m_2}\right)x_2 + \frac{m_1 F}{m_1 + m_2}\,x_1 = \frac{m_1\,F(x_1 + x_2)}{m_1 + m_2}$$

or $$\frac{1}{2}k(x_1 + x_2)^2 = \frac{m_1(x_1 + x_2)F}{m_1 + m_2}$$

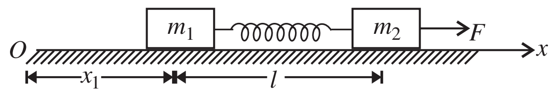
So, $$x_1 + x_2 = 0 \quad \text{or} \quad x_1 + x_2 = \frac{2m_1F}{k(m_1 + m_2)}$$

Hence, the maximum and minimum separations between the blocks are

$$l_0 + \frac{2m_1F}{k(m_1 + m_2)} \text{ and } l_0, \text{ respectively.}$$

Alternate:

Let us link the inertial frame with horizontal floor and point the coordinate axis as shown in figure.



At an arbitrary instant of time the separation between the blocks is l and the x coordinate of the block m_1 is x_1. From Newton's second law for the block m_1,

$$m_1\frac{d^2x_1}{dt^2} = k(l - l_0) \tag{1}$$

Similarly for block m_2,

$$m_2\frac{d^2(x_1 + l)}{dt^2} = F - k(l - l_0) \tag{2}$$

Multiplying Eq. (2) by m_1 and Eq. (1) by m_2 and further subtracting Eq. (1) from Eq. (2), we get

$$m_1m_2\frac{d^2(x + l)}{dt^2} - m_1m_2\frac{d^2x_1}{dt^2} = m_1F = (m_1 + m_2)\,k(l - l_0)$$

or $$m_1m_2\frac{d^2l}{dt^2} = m_1F - (m_1 + m_2)\,k(l - l_0) \tag{3}$$

or $$m_1m_2\frac{d^2(l - l_0)}{dt^2} = m_1F - (m_1 + m_2)\,k(l - l_0)$$

or $$\frac{d^2(l - l_0)}{dt^2} + \frac{k(m_1 + m_2)(l - l_0)}{m_1m_2} = \frac{F}{m_2} \tag{4}$$

Putting $$\frac{k(m_1 + m_2)}{m_1m_2} = \omega^2 \quad \text{and} \quad \frac{F}{m_2} = A$$

and comparing with differential equation $\dfrac{d^2x}{dt^2} + \omega^2x = A$, we get

$$l - l_0 = \frac{A}{\omega^2} + B\sin(\omega t + \delta) \tag{5}$$

But at time $t = 0$, $\quad l - l_0 = 0 \quad$ and $\quad \dfrac{d(l - l_0)}{dt} = 0$

So, $B\omega \cos(\omega t + \delta)|_{t=0} = 0$ and hence $\delta = \pi/2$.

Therefore, Eq. (5) becomes

$$l - l_0 = \frac{A}{\omega\theta^2} + B\cos\omega t \tag{6}$$

But at $t = 0$, $\quad l - l_0 = 0$, so, $B = \dfrac{-A}{\omega^2}$

Hence, Eq. (6) becomes

$$l - l_0 = \frac{A}{\omega^2}(1 - \cos\omega t) \tag{7}$$

But $(l - l_0)$ will be maximum, when $\cos\omega t = -1$

Therefore, from Eq. (7)

$$l - l_0 = \frac{2A}{\omega^2} = \frac{2Fm_1 m_2}{km_2(m_1 + m_2)}$$

or

$$l = l_0 + \frac{2Fm_1}{k(m_1 + m_2)}$$

Similarly $(l - l_0)$ will be minimum when $\cos\omega t = 1$

Therefore, from Eq. (7)

$$l - l_0 = 0 \quad \text{or} \quad l = l_0$$

1.153 (a) The initial compression in the spring Δl must be such that after burning of the thread, the upper cube rises to a height that produces a tension in the spring that is atleast equal to the weight of the lower cube. Actually, the spring will first go from its compressed state to its natural length and then get elongated beyond this natural length. Let l be the maximum elongation produced under these circumstances.

Then
$$\kappa l = mg \tag{1}$$

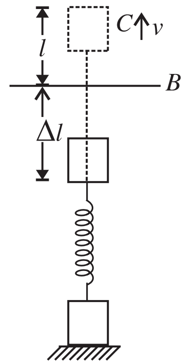
Now, from energy conservation,

$$\frac{1}{2}\kappa\Delta l^2 = mg(\Delta l + l) + \frac{1}{2}\kappa l^2 \tag{2}$$

(because at maximum elongation of the spring, the speed of upper cube becomes zero).

From Eqs. (1) and (2),

$$\Delta l^2 - \frac{2mg\Delta l}{\kappa} - \frac{3m^2g^2}{\kappa^2} = 0 \quad \text{or} \quad \Delta l = \frac{3mg}{\kappa}, \frac{-\ mg}{\kappa}$$



Therefore, acceptable solution of Δl equals $3mg/\kappa$.

(b) Let v be the velocity of upper cube at the position (say, at 2) when the lower block breaks off the floor, then from energy conservation,

$$\frac{1}{2}\, mv^2 = \frac{1}{2}\kappa\, (\Delta l^2 - l^2) - mg\,(l + \Delta l)$$

$$\left(\text{where } l = \frac{mg}{\kappa} \quad \text{and} \quad \Delta l = 7\frac{mg}{\kappa}\right)$$

or
$$v^2 = 32\frac{mg^2}{\kappa} \tag{3}$$

The displacement of C.M. (of two blocks systems) in the interval in which upper block reaches position 2 is

$$\Delta y_{C_1} = \frac{m\cdot 0 + m\,(7mg/\kappa + mg/\kappa)}{2m} = \frac{4\ mg}{\kappa}$$

When the upper block reaches position 2, the lower block just leaves the floor with zero initial velocity. Now treat two blocks system like a single particle of mass $2m$ projected vertically upward with the velocity of C.M.

$$v_C = \frac{mv + 0}{2m} = \frac{v}{2}$$

So, the further vertical displacement of C.M. of two block system

$$\Delta y_{C_2} = \frac{v_C^2}{2g} = \frac{(v/2)^2}{2g} = \frac{4mg}{\kappa} \qquad \text{(using Eq. 3)}$$

Hence, the net displacement of the C.M. of the system, in upward direction

$$\Delta y_C = \Delta y_{C_1} + \Delta y_{C_2} = \frac{8mg}{\kappa}$$

1.154 Due to ejection of mass from a moving system (which moves due to inertia) in a direction perpendicular to it, the velocity of moving system does not change. The momentum change being adjusted by the forces on the rails. Hence in our problem velocities of buggies change only due to the entrance of the man coming from the other buggy. From conservation of linear momentum for the system (buggy 1 + man coming from buggy 2) in the direction of motion of buggy 1,

$$Mv_1 - mv_2 = 0 \tag{1}$$

Similarly from conservation of linear momentum for the system (buggy 2 + man coming from buggy 1) in the direction of motion of buggy 2,

$$Mv_2 - mv_2 = (M + m)v \tag{2}$$

Solving Eqs. (1) and (2), we get

$$v_1 = \frac{mv}{M - m} \quad \text{and} \quad v_2 = \frac{Mv}{M - m}$$

As $\quad \mathbf{v}_1 \uparrow\downarrow \mathbf{v} \quad \text{and} \quad \mathbf{v}_2 \uparrow\uparrow \mathbf{v}$

So, $\quad \mathbf{v}_1 = \dfrac{-m\mathbf{v}}{(M - m)} \quad \text{and} \quad \mathbf{v}_2 = \dfrac{M\mathbf{v}}{(M - m)}$

1.155 From momentum conservation, for the system (rear buggy + man)

$$(M + m)\mathbf{v}_0 = m(\mathbf{u} + \mathbf{v}_R) + M\mathbf{v}_R \tag{1}$$

From momentum conservation, for the system (front buggy + man coming from rear buggy)

$$M\mathbf{v}_0 + m(\mathbf{u} + \mathbf{v}_R) = (M + m)\,\mathbf{v}_F$$

So, $\quad \mathbf{v}_F = \dfrac{M\mathbf{v}_0}{M + m} + \dfrac{m}{M + m}(\mathbf{u} + \mathbf{v}_R)$

Putting the value of $\mathbf{v}_R$ from Eq. (1), we get

$$\mathbf{v}_F = \mathbf{v}_0 + \frac{mM}{(M + m)^2}\,\mathbf{u}$$

1.156 (i) Let $\mathbf{v}_1$ be the velocity of the buggy after both men jump off simultaneously. For the closed system (two men + buggy), from the conservation of linear momentum,

$$M\mathbf{v}_1 + 2m(\mathbf{u} + \mathbf{v}_1) = 0$$

or $\quad \mathbf{v}_1 = \dfrac{-2m\mathbf{u}}{M + 2m} \tag{1}$

(ii) Let $\mathbf{v}$ be the velocity of buggy with man, when one man jumps off the buggy. For the closed system (buggy with one man + other man) from the conservation of linear momentum:

$$0 = (M + m)\,\mathbf{v} + m(\mathbf{u} + \mathbf{v}) \tag{2}$$

Let $\mathbf{v}_2$ be the sought velocity of the buggy when the second man jumps off the buggy; then from conservation of linear momentum of the system (buggy + one man):

$$(M + m)\mathbf{v} = M\mathbf{v}_2 + m(\mathbf{u} + \mathbf{v}_2) \tag{3}$$

Solving Eqs. (2) and (3) we get

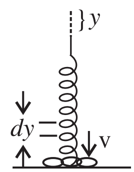
$$\mathbf{v}_2 = \frac{m(2M + 3m)\mathbf{u}}{(M + m)(M + 2m)} \tag{4}$$

From Eqs. (1) and (4)

$$\frac{v_2}{v_1} = 1 + \frac{m}{2(M + m)} > 1$$

Hence, $$v_2 > v_1$$

1.157 The descending part of the chain is in free fall, it has speed $v = \sqrt{2gy}$ at the instant when the chain descended a distance y. The length of the chain which lands on the floor, during the differential time interval dt following this instant is vdt.



For the incoming chain element on the floor, from $dp_y = F_y\, dt$ (where y-axis is directed down)

$$0 - (\lambda\ vdt)\ v = F_y dt$$

or $$F_y = -\lambda v^2 = -2\lambda gy$$

Hence, the force exerted on the falling chain equals λv^2 and is directed upward. Therefore from third law the force exerted by the falling chain on the table at the same instant of time becomes λv^2 and is directed downward.

Since a length of chain of weight (λyg) already lies on the table and the table is at rest, the total force on the floor is $(2\lambda yg) + (\lambda yg) = (3\lambda yg)$ or the weight of a length $3y$ of chain.

1.158 Velocity of the ball, with which it hits the slab, $v = \sqrt{2gh}$.
After first impact $v' = ev$ (upward) but according to the problem

$$v' = \frac{v}{\eta}, \quad \text{so} \quad e = \frac{1}{\eta} \tag{1}$$

and momentum imparted to this slab equals

$$mv - (-mv') = mv\,(1 + e)$$

Similarly, velocity of the ball after second impact is

$$v'' = ev' = e^2 v$$

and momentum imparted is

$$m(v' + v'') = m(1 + e)ev$$

Again, momentum imparted during third impact is

$$m\,(1 + e)e^2 v, \quad \text{and so on}$$

$$\text{Hence, net momentum imparted} = mv(1+e) + mve'(1+e) + \dots$$
$$= mv(1+e)(1+e+e^2+\dots)$$
$$= mv\frac{(1+e)}{1-e} \text{ (from summation of G.P.)}$$
$$= \sqrt{2gh}\ \frac{(1+1/\eta)}{(1-1/\eta)} = m\sqrt{2gh}\Big/(\eta+1)/(\eta-1) \qquad \text{(using Eq. 1)}$$
$$= 0.2 \text{ kg m/s (on substituting values)}$$

1.159 (a) Since the resistance of water is negligibly small the resultant of all external forces acting on the system "a man and a raft" is equal to zero. This means that the position of the C.M. of the given system does not change in the process of motion.

i.e., $\mathbf{r}_C = \text{constant}$ or $\Delta\mathbf{r}_C = 0$, $\sum m_i\Delta\mathbf{r}_i = 0$

or $m(\Delta\mathbf{r}_{mM} + \Delta\mathbf{r}_M) + M\Delta\mathbf{r}_M = 0$

Thus, $m(\mathbf{l}' + \mathbf{l}) + M\mathbf{l} = 0$ or $\mathbf{l} = -\dfrac{m\mathbf{l}'}{m+M}$

(b) As net external force on "man-raft" system is equal to zero, therefore the momentum of this system does not change.

So, $$0 = m[\mathbf{v}'(t) + \mathbf{v}_2(t)] + M\mathbf{v}_2(t)$$

or $$\mathbf{v}_2(t) = -\frac{m\mathbf{v}'(t)}{m+M} \tag{1}$$

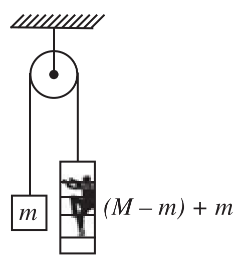
Thus, the sought force on the raft

$$M\frac{d\mathbf{v}_2(t)}{dt} = -\frac{Mm}{m+M}\frac{d\mathbf{v}'(t)}{dt}$$

Note: *We may get the result of part (a), if we integrate Eq. (1) over the time of motion of man or raft.*

1.160 The displacement of the C.M. of the system, man (m), ladder ($M-m$) and the counterweight (M), is described by radius vector

$$\Delta\mathbf{r}_C = \frac{\Sigma m_i\,\Delta\mathbf{r}_i}{\Sigma m_i} = \frac{M\Delta\mathbf{r}_M + (M-m)\,\Delta\mathbf{r}_{(M-m)} + m\,\Delta\mathbf{r}_m}{2M} \tag{1}$$



But, $\Delta\mathbf{r}_m = -\Delta\mathbf{r}_{(M-m)}$

and $$\Delta\mathbf{r}_m = \Delta\mathbf{r}_{m(M-m)} + \Delta\mathbf{r}_{(M-m)} \tag{2}$$

Using Eq. (2) in Eq. (1) we get

$$\Delta \mathbf{r}_C = \frac{ml}{2M}$$

Alternate:

Initially all the bodies of the system are at rest, and therefore the increments of linear momentum of the bodies in their motion are equal to the momentum themselves. The rope tension is the same both on the left and on the right-hand side, and consequently the momentum of the counter-balancing mass ($\mathbf{p}_1$) and the ladder with the man ($\mathbf{p}_2$) are equal at any instant of time, i.e., $\mathbf{p}_1 = \mathbf{p}_2$,

or
$$M\mathbf{v}_1 = m\mathbf{v} + (M - m)\mathbf{v}_2$$

where $\mathbf{v}_1$, $\mathbf{v}$, and $\mathbf{v}_2$ are the velocities of the mass, the man, and the ladder, respectively. Taking into account that $\mathbf{v}_2 = -\mathbf{v}_1$ and $\mathbf{v} = \mathbf{v}_2 + \mathbf{v}'$, where $\mathbf{v}'$ is the man's velocity relative to the ladder, we obtain

$$\mathbf{v}_1 = (m/2M)\mathbf{v}'$$

On the other hand, the momentum of the whole system

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 = 2\mathbf{p}_1 \quad \text{or} \quad 2M\,\mathbf{V}_C = 2M\,\mathbf{v}_1 \tag{1}$$

where $\mathbf{V}_C$ is the velocity of the centre of inertia of the system. With allowance made for Eq. (1) we get

$$\mathbf{V}_C = \mathbf{v}_1 = (m/2M)\mathbf{v}'$$

Finally, the sought displacement is

$$\Delta \mathbf{r}_C = \int \mathbf{V}_C\,dt = (m/2M)\int \mathbf{v}'\,dt = (m/2M)\,\Delta \mathbf{r}'$$

1.161 Velocity of cannon as well as that of shell equals $\sqrt{2gl\sin\alpha}$ down the inclined plane taken as the positive x-axis. From the linear impulse momentum theorem in projection form along x-axis for the system (cannon + shell),

$$\Delta p_x = F_x \Delta t$$

i.e.,
$$p\cos\alpha - M\sqrt{2gl\sin\alpha} = Mg\sin\alpha\,\Delta t \text{ (as mass of the shell is negligible)}$$

or
$$\Delta t = \frac{p\cos\alpha - M\sqrt{2gl\sin\alpha}}{Mg\sin\alpha}$$

1.162 From conservation of momentum, for the system "bullet +body" along the initial direction of bullet

$$mv_0 = (m + M)\,v \quad \text{or} \quad v = \frac{mv_0}{m + M} \tag{1}$$

where v_0 is the initial velocity of the bullet and v is combined velocity, after the collision.

From conservation of mechanical energy of the system (bullet + body) in the field of gravity:

$$\frac{1}{2}(m+M)v^2 = (M+m)gl(1-\cos\theta) \quad \text{or} \quad v^2 = 2gl(1-\cos\theta) \tag{2}$$

(a) From Eqs. (1) and (2),

$$v_0 = \frac{(m+M)}{m}\sqrt{2gl(1-\cos\theta)} = \frac{2(M+m)}{m}\sqrt{gl}\sin(\theta/2)$$

As $m \ll M$, $$v_0 \cong \frac{2M}{m}\sqrt{gl}\sin(\theta/2)$$

(b) Fraction of initial kinetic energy turned into heat

$$= \frac{T_i - T_f}{T_i} = 1 - \frac{T_f}{T_i} = 1 - \frac{\frac{1}{2}(m+M)v^2}{\frac{1}{2}mv_0^2}$$

$$= 1 - \frac{m}{(m+M)} \cong \left(1 - \frac{m}{M}\right) \quad \text{(using Eq. 1)}$$

1.163 When the disk breaks off the body M, its velocity towards right (along x-axis) equals the velocity of the body M, and let the disk's velocity in upward direction (along y-axis) at that moment be v_y'.

From conservation of momentum, along x-axis for the system (disk + body)

$$mv = (m+M)\,v_x' \quad \text{or} \quad v_x' = \frac{mv}{m+M} \tag{1}$$

And from energy conservation, for the same system in the field of gravity:

$$\frac{1}{2}mv^2 = \frac{1}{2}(m+M)v_x'^2 + \frac{1}{2}mv_y'^2 + mgh'$$

where h' is the height of break-off point from initial level.

So, $$\frac{1}{2}mv^2 = \frac{1}{2}(m+M)\frac{m^2v^2}{(M+m)} + \frac{1}{2}mv_y'^2 + mgh' \quad \text{(using Eq. 1)}$$

or $$v_y'^2 = v^2 - \frac{mv^2}{(m+m)} - 2gh'$$

Also, if h'' is the height of the disk from the break-off point, then,

$$v_y'^2 = 2gh''$$

So, $$2g(h''+h') = v^2 - \frac{mv^2}{(M+m)}$$

Hence, the total height, raised from the initial level is

$$h' + h'' = \frac{Mv^2}{2g(M+m)}$$

1.164 (a) When the disk slides and comes to the plank, it has a velocity equal to $v = \sqrt{2gh}$. Due to friction between the disk and the plank the disk slows down and after some time the disk moves with the plank with velocity v' (say).

From the momentum conservation for the system (disk + plank) along horizontal towards right:

$$mv = (m + M)\,v' \quad \text{or} \quad v' = \frac{mv}{m+M}$$

Now from the equation of the increment of total mechanical energy of a system:

$$\frac{1}{2}(M+m)v'^2 - \frac{1}{2}mv^2 = A_{fr}$$

or $$\frac{1}{2}(M+m)\frac{m^2v^2}{(m+M)^2} - \frac{1}{2}mv^2 = A_{fr}$$

so, $$\frac{1}{2}v^2\left[\frac{m^2}{M+m} - m\right] = A_{fr}$$

Hence, $$A_{fr} = -\left(\frac{mM}{m+M}\right)gh = -\mu gh$$

$\left(\text{where } \mu = \dfrac{mM}{m+M} = \text{reduced mass}\right)$.

(b) We look at the problem from a frame in which the hill is moving (together with the disk on it) to the right with speed u. Then in this frame the speed of the disk when it just gets onto the plank is, by the law of addition of velocities, $v = u + \sqrt{2gh}$. Similarly the common speed of the plank and the disk when they move together is

$$v = u + \frac{m}{m+M}\sqrt{2gh}$$

Then as above $$A_{fr} = \frac{1}{2}(m+M)\,v^2 - \frac{1}{2}mv^2 - \frac{1}{2}Mu^2$$

$$= \frac{1}{2}(m+M)\left\{u^2 + \frac{2m}{m+M}u\sqrt{2gh} + \frac{m^2}{(m+M)^2}2gh\right\}$$

$$- \frac{1}{2}(m+M)u^2 - \frac{1}{2}m2u\sqrt{2gh} - mgh$$

We see that A_{fr} is independent of u and is in fact just $-\mu gh$ as in (a). Thus the result obtained does not depend on the choice of reference frame.

Do note however that it will be incorrect to apply "conservation of energy" formula in the frame in which the hill is moving. The energy carried by the hill is not negligible in this frame. See also the next problem.

1.165 In a frame moving relative to the Earth, one has to include the kinetic energy of the Earth as well as Earth's acceleration to be able to apply conservation of energy to the problem. In a reference frame falling to the Earth with velocity v_0, the stone is initially going up with velocity v_0 and so is the Earth. The final velocity of the stone is $0 = v_0 - gt$ and that of the Earth is $v_0 + (m/M)gt$ (M is the mass of the Earth), from Newton's third law, where t = time of fall. From conservation of energy

$$\frac{1}{2}mv_0^2 + \frac{1}{2}Mv_0^2 + mgh = \frac{1}{2}M\left(v_0 + \frac{m}{M}v_0\right)^2$$

Hence,
$$\frac{1}{2}v_0^2\left(m + \frac{m^2}{M}\right) = mgh$$

Neglecting m/M in comparison with 1, we get

$$v_0^2 = 2gh \text{ or } v_0 = \sqrt{2gh}$$

The point in this is that in the Earth's rest frame the effect of Earth's acceleration is of order m/M and can be neglected but in a frame moving with respect to the Earth the effect of Earth's acceleration must be kept because it is of order one (i.e., large).

1.166 From conservation of momentum for the closed system "both colliding particles"

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\,\mathbf{v}$$

or
$$\mathbf{v} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2} = \frac{1(3\mathbf{i} - 2\mathbf{j}) + 2(4\mathbf{j} - 6\mathbf{k})}{3} = \mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$$

Hence,
$$|\mathbf{v}| = \sqrt{1 + 4 + 16}\text{ m/s} = 4.6\text{ m/s}$$

1.167 For perfectly inelastic collision, in the C.M. frame, final kinetic energy of the colliding system (both spheres) becomes zero. Hence initial kinetic energy of the system in C.M. frame completely turns into the internal energy (Q) of the formed body.

Hence,
$$Q = T_i = \frac{1}{2}\mu|\mathbf{v}_1 - \mathbf{v}_2|^2$$

Now from energy conservation
$$\Delta T = -Q = -\frac{1}{2}\mu|\mathbf{v}_1 - \mathbf{v}_2|^2$$

In laboratory frame the same result is obtained as

$$\Delta \tilde{T} = \frac{1}{2}\frac{(m_1\mathbf{v}_1 + m_2\mathbf{v}_2)^2}{m_1 + m_2} - \frac{1}{2}m_1|\mathbf{v}_1|^2 + \frac{1}{2}m_2|\mathbf{v}_2|^2$$

$$= -\frac{1}{2}\mu|\mathbf{v}_1 - \mathbf{v}_2|^2$$

1.168 (a) From conservation of linear momentum

$$\mathbf{p}_1 = \mathbf{p}_1' + \mathbf{p}_2' \quad \text{or} \quad \mathbf{p}_1 - \mathbf{p}_1' = \mathbf{p}_2'$$

As $$\mathbf{p}_1' \perp \mathbf{p}_1 \quad \text{so,} \quad {p_1}^2 + {p_1'}^2 = {p_2'}^2 \tag{1}$$

From conservation of kinetic energy $$\frac{p_1^2}{2m_1} = \frac{{p_1'}^2}{2m_1} + \frac{{p_2'}^2}{2m_2} \tag{2}$$

Using Eq. (1) in Eq. (2) we get $$\frac{p_1^2}{2m_1} = \frac{{p_1'}^2}{2m_1} + \frac{(p_1^2 + {p_1'}^2)}{2m_2}$$

which yields $$\frac{{p_1'}^2}{{p_1}^2} = \left(\frac{m_2 - m_1}{m_2 + m_1}\right) \tag{3}$$

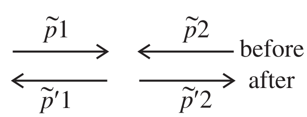
So, sought fraction of kinetic energy $$\eta = 1 - \frac{T_1'}{T_1} = 1 - \frac{{p'_1}^2/2m_1}{{p_1}^2/2m_1} = 1 - \frac{{p'_1}^2}{{p_1}^2}$$

Hence, $$\eta = 1 - \left(\frac{m_2 - m_1}{m_2 + m_1}\right) = \frac{2m_1}{m_1 + m_2}$$

(b) For two particles closed system momentum of each particle in their C.M. frame are always equal and opposite. In C.M. frame the kinetic energy of the two particle system $\tilde{T} = \tilde{p}^2/2\mu$ where μ is the reduced mass. In perfectly elastic collision, the kinetic energy of the system is conserved.

So, $$\frac{\tilde{p}'^2}{2\mu} = \frac{\tilde{p}^2}{2\mu}, \quad \text{which gives} \quad \tilde{p}' = \tilde{p}$$

Being head on collision, both the particles have to keep their motion along the same straight line before and after the collision. On collision, momentum vector of each particle has to change due to the reaction force by the other particle. So only choice left to each of the particle is to reverse the direction of its momentum after the collision keeping the magnitude constant, i.e., $\tilde{\mathbf{p}}_i' = -\tilde{\mathbf{p}}_i$, where $i = 1, 2$.



The same can be said about the velocity of each particle in the C.M. frame

$$\tilde{\mathbf{v}}'_i = -\tilde{\mathbf{v}}_i$$

but $\quad \mathbf{v}'_i = \mathbf{V}_C + \tilde{\mathbf{v}}'_i = \mathbf{V}_C - \tilde{\mathbf{v}}_i = \mathbf{V}_C - (\mathbf{v}_i - \mathbf{V}_C) = 2\mathbf{V}_C - \mathbf{v}_i$

Hence the velocity of i-th particle after collision.

$$\mathbf{v}'_i = 2\mathbf{V}_C - \mathbf{v}_i \quad (\text{where } i = 1, 2)$$

So velocity of particle 1 just after the collision from above relation is

$$\mathbf{v}'_1 = 2\mathbf{V}_C - \mathbf{v}_1 = 2\left(\frac{m_1\mathbf{v}_1}{m_1 + m_2}\right) - \mathbf{v}_1 = \frac{(m_1 - m_2)\,\mathbf{v}_1}{m_1 + m_2}$$

Therefore, $\quad \dfrac{v'^2_1}{v_1^2} = \dfrac{(m_1 - m_2)^2}{(m_1 + m_2)^2}$

Thus sought fraction $= 1 - \dfrac{T'_1}{T_1} = 1 - \dfrac{v'^2_1}{v_1^2} = 1 - \dfrac{(m_1 - m_2)^2}{(m_1 + m_2)^2} = \dfrac{4m_1m_2}{(m_1 + m_2)^2}$

1.169 (a) Being perfectly elastic head on collision, the velocity of i-th particle after collision

$$\mathbf{v}'_i = 2\mathbf{V}_C - \mathbf{v}_i \quad (\text{where } i = 1, 2) \ (\text{see previous problem})$$

So, $\quad \mathbf{v}'_1 = 2\left(\dfrac{m_1\mathbf{v}_1}{m_1 + m_2}\right) - \mathbf{v}_1 = \dfrac{(m_1 - m_2)\,\mathbf{v}_1}{m_1 + m_2}$ and (1)

$$\mathbf{v}'_2 = 2\left(\frac{m_1\mathbf{v}_1}{m_1 + m_2}\right) \tag{2}$$

According to the problem

$$\mathbf{v}'_2 = -\mathbf{v}'_1$$

so, $\quad 2\left(\dfrac{m_1\mathbf{v}_1}{m_1 + m_2}\right) = \dfrac{(m_2 - m_1)\,\mathbf{v}_1}{m_1 + m_2}$ which gives $2m_1 = (m_2 - m_1)$

Hence, $\quad \dfrac{m_1}{m_2} = \dfrac{1}{3}$

(b) From conservation of linear momentum in the direction perpendicular to initial motion direction of striking particle 1 gives

$$p'_1 \sin 30° = p'_2 \sin 30°$$

So, $\quad p'_1 = p'_2 \qquad (1)$

From conservation of linear momentum

$$\mathbf{p}_1 = \mathbf{p}'_1 + \mathbf{p}'_2 \quad \text{or,} \quad \mathbf{p}_1 - \mathbf{p}'_1 = \mathbf{p}'_2$$

so, $\quad p_1^2 + p'^2_1 - 2p_1\,p'_1 \cos 30° = p'^2_2 \qquad (2)$

On using Eq. (1) in Eq. (2) we get on manipulation

$$p_1' = \frac{p_1}{2\cos 30°} = \frac{p_1}{\sqrt{3}} \tag{3}$$

From conservation of kinetic energy

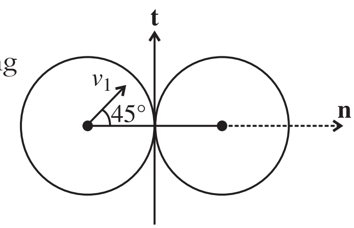
$$\frac{p_1^2}{2m_1} = \frac{p_1'^2}{2m_1} + \frac{p_2'^2}{2m_2}$$

On using Eq. (1) and Eq. (3), we get on solving

$$\frac{m_1}{m_2} = 2$$

1.170 At the moment of maximum deformation the velocity of colliding bodies along their common normal **n** (here the line joining the mass centres of the balls) must be equal so

$$v_{1n}' = v_{2n}' = v_n' \text{ (say)} \tag{1}$$



As the balls are smooth so, velocity of each ball along their common tangent **t** remains constant.

So, $v'_{1t} = v_{1t} = v_1 \sin 45°$ and $v'_{2t} = v_{2t} = 0$ (2)

Now, from conservation of linear momentum

$$mv_1 \cos 45° = 2m\, v_n' \quad \text{so,} \quad v_n' = \frac{v_1}{2\sqrt{2}} \tag{3}$$

From energy conservation, gain in internal potential energy is due to loss of kinetic energy of the system. Initial kinetic energy of the system is the kinetic energy T_1 of striking ball only so, the sought fraction $\eta = 1 - T'_{\text{system}}/\text{T}_1$.

But, $$T'_{\text{system}} = \frac{1}{2} m\,[v_{1n}'^{\,2} + v_{1t}'^{\,2}] + \frac{1}{2} m\,[v_{2n}'^{\,2} + v_{2t}'^{2}]$$

On using Eqs. (1), (2) and (3), we get

$$T'_{\text{system}} = \frac{1}{2} m \frac{3v_1^2}{4}$$

So, $$\frac{T'_{\text{system}}}{T_1} = \frac{1}{2} m \frac{3v_1^2}{4} \Big/ \frac{1}{2} mv_1^2 = \frac{3}{4}$$

Hence, sought fraction $\eta = 1 - \frac{3}{4} = \frac{1}{4}$

1.171 From the conservation of linear momentum of the shell just before and after its fragmentation

$$3\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 \tag{1}$$

where $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are the velocities of its fragments.

From the energy conservation

$$3\eta v^2 = v_1^2 + v_2^2 + v_3^2 \tag{2}$$

Now $$\tilde{\mathbf{v}} \quad \text{or} \quad \mathbf{v}_{iC} = \mathbf{v}_i - \mathbf{v}_C = \mathbf{v}_i - \mathbf{v} \tag{3}$$

where $\mathbf{v}_C = \mathbf{v} =$ velocity of the C.M. of the fragments the velocity of the shell. Obviously in the C.M. frame the linear momentum of a system is equal to zero so

$$\tilde{\mathbf{v}}_1 + \tilde{\mathbf{v}}_2 + \tilde{\mathbf{v}}_3 = 0 \tag{4}$$

Using Eqs. (3) and (4) in Eq. (2), we get

$$3\eta v^2 = (\mathbf{v} + \tilde{\mathbf{v}}_1)^2 + (\mathbf{v} + \tilde{\mathbf{v}}_2)^2 + (\mathbf{v} - \tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2)^2 = 3v^2 + 2\tilde{\mathbf{v}}_1^2 + 2\tilde{\mathbf{v}}_1^2 + 2\tilde{\mathbf{v}}_1 \cdot \tilde{\mathbf{v}}_2$$

or $$2\tilde{v}_1^2 + 2\tilde{v}_1\tilde{v}_2 \cos\theta + 2\tilde{v}_2^2 + 3(1-\eta)v^2 = 0 \tag{5}$$

If we have had used $\tilde{\mathbf{v}}_2 = -\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_3$, then Eq. 5 would contain $\tilde{v}_3$ instead of $\tilde{v}_2$ and so on. The problem being symmetrical we can look for the maximum of any one. Obviously it will be the same for each.

For $\quad \tilde{v}_2^2 \cos^2\theta \geq 8(2\tilde{v}_2^2 + 3(1-\eta)v^2)$ or $6(\eta - 1)v^2 \geq (4 - \cos^2\theta)\tilde{v}_2^2$

So, $$\tilde{v}_2 \leq v\sqrt{\frac{6(\eta-1)}{4-\cos^2\theta}} \quad \text{or} \quad \tilde{v}_{2(\max)} = \sqrt{2(\eta-1)}\,v$$

Hence, $v_{2(\max)} = |\mathbf{v} + \tilde{\mathbf{v}}_2|_{\max} = v + \sqrt{2(\eta-1)}v = v\left(1 + \sqrt{2(\eta-1)}\right) = 1$ km/s

Thus, owing to the symmetry

$$v_{1(\max)} = v_{2(\max)} = v_{3(\max)} = v\left(1 + \sqrt{2(\eta-1)}\right) = 1 \text{ km/s}$$

Alternate:

The maximum velocity can be attained by that fragment (say 1) which has experienced the reaction force by the rest of two fragment (2 and 3) in the initial motion direction of the shell. Symmetry of the problem tells us that the velocities of fragments 2 and 3 are same and so is the reaction force exerted by the fragment 2 and 3 on fragment 1.

Let v' is the velocity of each of the fragments 2 and 3 and v_1' of fragment 1 after the collision.

From conservation of linear momentum

$$3mv = mv_1' - 2\,(mv')$$

$$3v = v_1' - 2v' \tag{1}$$

From conservation of energy

$$\frac{\eta}{2}(3m)\,v^2 = \frac{1}{2}\,mv_1'^{\,2} + \left(\frac{1}{2}\,mv'^2\right)$$

So,
$$\eta\,3v^2 = v_1'^{\,2} + v^2 \tag{2}$$

Solving Eqs. (1) and (2), we get

$$v_1' = v\left(1 + \sqrt{2(\eta - 1)}\right) = 1 \text{ km/s} \quad \text{(on substituting values)}$$

1.172 From the conservation of momentum

$$mv_1 = mv_1' + mv_2' \quad \text{or} \quad v_1 = v_1' + v_2' \tag{1}$$

From the given condition,

$$\eta = \frac{T - T'}{T} = 1 - \frac{T'}{T} = 1 - \frac{v'^2_1 + v'^2_2}{v_1^{\,2}}$$

or
$$v'^2_1 + v'^2_2 = (1 - \eta)\,v_1^2 \tag{2}$$

From Eqs. (1) and (2)

$$v'^2_1 + (v_1 - v_1')^2 = (1 - \eta)\,v_1^2$$

or
$$2v'^2_1 - 2v_1'\,v_1 + \eta v_1^2 = 0$$

So,
$$v_1' = \frac{2v_1 \pm \sqrt{4v_1^2 - 8\eta\,v_1^2}}{4}$$

$$= \frac{1}{2}\left[v_1 \pm \sqrt{v_1^{\,2} - 2\eta\,v_1^2}\right] = \frac{1}{2}\,v_1\left(1 \pm \sqrt{1 - 2\eta}\right)$$

Positive sign gives the velocity of the second particle which lies ahead. The negative sign is correct for v_1. So, $v_1' = 1/2\,v_1\,(1 - \sqrt{1 - 2\eta}) = 5$ m/s will continue moving in the same direction.

Note that $v_1' = 0$ if $\eta = 0$.

Alternate:

As internal energy is independent of reference frame, so loss in kinetic energy of the system in the C.M. frame is the same.

So,
$$\tilde{T} - \tilde{T}' = T - T'$$

or
$$\left(1 - \frac{\tilde{T}'}{\tilde{T}}\right) = \frac{T}{\tilde{T}}\left(1 - \frac{T'}{T}\right)$$

Using $\left(1 - \frac{T'}{\tilde{T}}\right) = \eta$, $T = \frac{1}{2}\, mv^2$, and $\tilde{T} = \frac{1}{2}\mu v_{rel}^2 = \frac{1}{2}\left(\frac{m}{2}\right) v^2$, where m is mass of each particle, we get

$$\left(1 - \frac{T'}{\tilde{T}}\right) = 2\eta$$

Now, according to the problem

$$\tilde{T} - \tilde{T}' = 2\eta \tilde{T}$$

$$\frac{\tilde{p}^2}{2\mu} - \frac{\tilde{p}'^2}{2\mu} = 2\eta \frac{\tilde{p}^2}{2\mu}$$

which yields $\quad \tilde{p}' = \tilde{p}\sqrt{(1 - 2\eta)}$

Being head-on collision, final momentum in C.M. frame of any particle should be reversed, so

$$\tilde{\mathbf{p}}_i' = \tilde{\mathbf{p}}_i \sqrt{(1 - 2\eta)} \quad (\text{where } i = 1, 2)$$

or $\quad \tilde{\mathbf{v}}_i' = -\tilde{\mathbf{v}}_i \sqrt{(1 - 2\eta)},$

But, $\quad \mathbf{v}_i' = \tilde{\mathbf{v}}_i' + \mathbf{v}_C = -\tilde{\mathbf{v}}_i\sqrt{(1 - 2\eta)} + \mathbf{v}_C = -(\mathbf{v}_i - \mathbf{v}_C)\sqrt{(1 - 2\eta)} + \mathbf{v}_C$

Thus, $\quad \mathbf{v}_i' = \mathbf{v}_c[1 + \sqrt{1 - 2\eta}] - \mathbf{v}_i\sqrt{1 - 2\eta}$

Using $\quad \mathbf{v}_C = \frac{m\mathbf{v}}{2m} = \frac{\mathbf{v}}{2}$ (where $\mathbf{v}$ is the velocity of striking particle)

$$\mathbf{v}_i = \frac{\mathbf{v}}{2}(1 - \sqrt{1 - 2\eta})$$

1.173 From conservation of linear momentum in the initial motion direction of striking particle 1 and perpendicular to it gives

$$mv_1 = Mv_2' \cos\theta \quad \text{and } mv_1' = Mv_2' \sin\theta, \text{ respectively}$$

Thus, $\quad v_1' = v_1 \tan\theta$

Percentage change in the kinetic energy of the system $= \dfrac{T'_{\text{system}} - T_{\text{system}}}{T_{\text{system}}} \times 100$

$$= \left(\frac{T'_{\text{system}}}{T_{\text{system}}} - 1\right) \times 100$$

$$= \left[\frac{\frac{1}{2}\, mv_1^2 \tan^2\theta + \frac{1}{2}\, M\left(\frac{mv_1}{M\cos\theta}\right)^2}{\frac{1}{2}\, mv_1^2} - 1\right] \times 100 = \left(\tan^2\theta + \frac{m}{M}\sec^2\theta - 1\right) \times 100$$

Putting the values of θ and m/M, we get % of change in kinetic energy $= -40\%$.

1.174 (a) Let the particles m_1 and m_2 move with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. On the basis of solution of problem 1.147(b):

$$\tilde{p} = \mu v_{\text{rel}} = \mu|\mathbf{v}_1 - \mathbf{v}_2|$$

As $$\mathbf{v}_1 \perp \mathbf{v}_2$$

so, $$\tilde{p} = \mu\sqrt{v_1^2 + v_2^2} \quad \text{where } \mu = \frac{m_1 m_2}{m_1 + m_2}$$

(b) Again from solution of problem 1.147(b):

$$\tilde{T} = \frac{1}{2}\mu v_{\text{rel}}^2 = \frac{1}{2}\mu|\mathbf{v}_1 - \mathbf{v}_2|^2$$

So, $$\tilde{T} = \frac{1}{2}\mu\,(v_1^2 + v_2^2)$$

1.175 From conservation of momentum

$$\mathbf{p}_1 = \mathbf{p}_1' + \mathbf{p}_2 \quad \text{or} \quad \mathbf{p}_1 - \mathbf{p}_1' = \mathbf{p}_2$$

So, $p_1^2 - 2\,p_1 p_1' \cos\theta_1 + p_1'^2 = p_2'^2$, where θ is the angle between $\mathbf{p}_1$ and $\mathbf{p}_1'$.

From conservation of energy

$$\frac{p_1^2}{2m_1} = \frac{p_1'^2}{2m_1} + \frac{p_2'^2}{2m_2}$$

Eliminating p_2' we get

$$0 = p_1'^2\left(1 + \frac{m_2}{m_1}\right) - 2p_1' p_1 \cos\theta_1 + p_1^2\left(1 - \frac{m_2}{m_1}\right)$$

This quadratic equation for p_1' has a real solution in terms of p_1 and $\cos\theta_1$ only if

$$4\cos^2\theta_1 \geq 4\left(1 - \frac{m_2^2}{m_1^2}\right)$$

or $$\sin^2\theta_1 \leq \frac{m_2^2}{m_1^2} \quad \text{or} \quad \sin\theta_1 \geq -\frac{m_2}{m_1}$$

This clearly implies (since only + sign makes sense) that

$$\sin\theta_{1\max} = \frac{m_2}{m_1}$$

Alternate:

The solution of the problem becomes standard in the frame of C.M., which is moving with velocity $\mathbf{v}_C = m_1\mathbf{v}_1/(m_1 + m_2)$ in the frame of laboratory. In the frame of C.M., the momenta of the particles must always be equal and opposite.

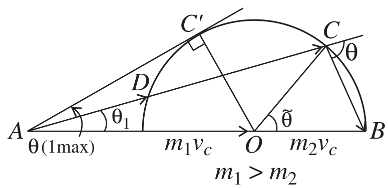
So, $$\tilde{\mathbf{p}}_1 = \tilde{\mathbf{p}}_2,$$

but $|\tilde{\mathbf{p}}_1| = |\tilde{\mathbf{p}}_2| = \tilde{p} = \mu\ v_{\text{rel}} = \mu\ v_1$ (where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass of the system).

The initial kinetic energy of the system in the frame of C.M. $\tilde{T}_i = \tilde{p}^2/2\mu$.

As the collision is perfectly elastic and the reduced mass the system is constant.

$$\tilde{T}_i = \tilde{T}_f \quad \text{or} \quad \frac{\tilde{p}^2}{2\mu} = \frac{\tilde{p}^2}{2\mu}$$



Hence, $$\tilde{p} = \tilde{p}'$$

In the frame of laboratyry from the conservation of linear momentum

$$\mathbf{p}_1' + \mathbf{p}_2' = \mathbf{p}_1 = (m_1 + m_2)\mathbf{v}_C$$

Now let us draw the so-called vector diagram (see figure) of momenta. First we depict the vector $\mathbf{p}_1$ as the section AB and the $\mathbf{p}_1$ and $\mathbf{p}_2$ each of which represents according to

$$\mathbf{p}_1' = \tilde{\mathbf{p}}_1' + m_1\mathbf{v}_C$$

and $$\mathbf{p}_2' = \tilde{\mathbf{p}}_2' + m_2\mathbf{v}_C$$

Note that this diagram is valid regardless of the angle $\tilde{\theta}$. The point C (centre of mass) therefore can be located only on the circle of radius $\tilde{p}$ having it's centre at the point O, which divides the section AB into two parts in the ratio $AO : OB = m_1 : m_2$. In our case, this circle passes through the point B, the end point of vector $\tilde{\mathbf{p}}_1$ since the section $OB = m_2 v_C = \mu v_1 = \tilde{p}$. This circle is the locus of all possible locations of the apex C of the momenta triangle ABC, whose side AC and CB represent the possible momentum of particles $\mathbf{p}_1'$ and $\mathbf{p}_2'$ after the collision in the frame of laboratory.

Depending on the particle mass ratio ($m_1 = m_2$) the point A, the beginning of the vector $\mathbf{p}_1$ can be located inside the given circle, on it, or outside it. In our case point A will lie outside the circle and it is also another intersting fact that the particle m_1 can be scattered by the same angle θ_1, where it possesses the momentum AC or AD (figure) The same is true for particle m_2.The maximum scattering in the laboratory $\theta_{1\,\max}$ corresponds to the case when the velocity vector in the frame of laboratory become a tangent (AC') to the circle (see figure).

It then follows that

$$\sin\theta_{1\,\max} = \frac{OC'}{AO} = \frac{\tilde{p}}{m_1 v_C} = \frac{(m_1 m_2/m_1 + m_2)\ v_1}{(m_1^2/m_1 + m_2)\ v_1} = \frac{m_2}{m_1}$$

Note: *It is clear that θ_m is limited to $\pi/2$ for $m_1 = m_2$, for the case $m_2 > m_1$, point A will be inside the sphere, and all angles of scattering from 0 to $\pi/2$ are permitted.*

1.176 From the symmetry of the problem, the velocity of the disk A will be directed either in the initial direction or opposite to it just after the impact. Let the velocity of the disk A after the collision be v' and be directed towards right after the collision. It is also clear from the symmetry of the problem that the disks B and C have equal speed (say v'') in the directions shown. From the condition of the problem,

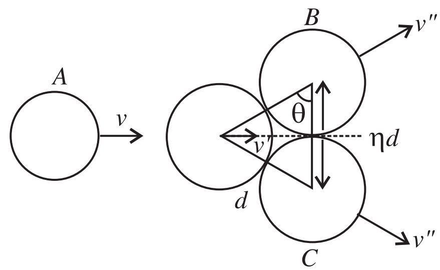
$$\cos\theta = \frac{\eta\frac{d}{2}}{d} = \frac{\eta}{2} \quad \text{so,} \quad \sin\theta = \sqrt{4-\eta^2}/2 \tag{1}$$

For the three disk system, from the conservation of linear momentum in the symmetry direction (towards right)

$$mv = 2mv''\sin\theta + mv' \quad \text{or} \quad v = 2v''\sin\theta + v' \tag{2}$$

From the definition of the coefficient of restitution, we have for disks A, B and C

$$e = \frac{v'' - v'\sin\theta}{v\sin\theta - 0}$$



But $e = 1$, for perfectly elastic collision.

So, $$v\sin\theta = v'' - v'\sin\theta \tag{3}$$

From Eqs. (2) and (3)

$$v' = \frac{v(1-2\sin^2\theta)}{(1+2\sin^2\theta)}$$

$$= \frac{v(\eta^2-2)}{6-\eta^2} \quad \text{(using Eq. 1)}$$

Hence, we have,

$$v' = \frac{v(\eta^2-2)}{6-\eta^2}$$

Therefore, the disk A will recoil if $\eta < \sqrt{2}$ and stop if $\eta = \sqrt{2}$.

Note: *One can write the equations of momentum conservation along the direction perpendicular to the initial direction of disk A and the conservation of kinetic energy instead of the equation of restitution.*

1.177 (a) Let a molecule come with velocity $\mathbf{v}_1$ to strike another stationary molecule an just after collision their velocities become $\mathbf{v}_1'$ and $\mathbf{v}_2'$, respectively. As the mass of each molecule is same, conservation of linear momentum and conservation of kinetic energy for the system (both molecules), respectively, gives

$$\mathbf{v}_1 = \mathbf{v}_1' + \mathbf{v}_2'$$

and $$v_1^2 = v_1'^2 + v_2'^2$$

From the property of vector addition it is obvious from the obtained equations that

$$\mathbf{v}_1 \perp \mathbf{v}_2 \quad \text{or} \quad \mathbf{v}_1 \cdot \mathbf{v}_2 = 0$$

(b) Due to the loss of kinetic energy in inelastic collision $v_1^2 > v_1'^2 + v_2'^2$ so, $\mathbf{v}_1 \cdot \mathbf{v}_2 > 0$ and therefore angle of divergence $< 90°$.

1.178 Suppose that at time t, the rocket has the mass m and the velocity $\mathbf{v}$, relative to the reference frame, employed. Now consider the inertial frame moving with the velocity that the rocket has at the given moment. In this reference frame, the momentum increment that the rocket and ejected gas system acquires during time dt is,

$$d\mathbf{p} = m d\mathbf{v} + \mu dt\, \mathbf{u} = \mathbf{F}\, dt$$

or
$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} - \mu\mathbf{u}$$

or
$$m\mathbf{w} = \mathbf{F} - \mu\mathbf{u}$$

1.179 According to the question, $\mathbf{F} = 0$ and $\mu = -dm/dt$ so the equation for this system becomes,

$$m\frac{d\mathbf{v}}{dt} = \frac{dm}{dt}\mathbf{u}$$

As $d\mathbf{v} \uparrow\downarrow \mathbf{u}$, so, $m\, dv = -udm$.

Integrating within the limits

$$\frac{1}{u}\int_0^v dv = -\int_{m_0}^0 \frac{dm}{m} \quad \text{or} \quad \frac{v}{u} = \ln\frac{m_0}{m}$$

Thus,
$$v = u \ln\frac{m_0}{m}$$

As $d\mathbf{v} \uparrow\downarrow \mathbf{u}$, so in vector form $\mathbf{v} = -\mathbf{u} \ln m_0/m$.

1.180 According to the question, $\mathbf{F}$ (external force) $= 0$

So,
$$m\frac{d\mathbf{v}}{dt} = \frac{dm}{dt}\mathbf{u}$$

As,
$$d\mathbf{v} \uparrow\downarrow \mathbf{u}$$

So, in scalar form
$$mdv = -udm$$

or
$$\frac{wdt}{u} = -\frac{dm}{m}$$

Integrating within the limit for $m(t)$

$$\frac{wt}{u} = -\int_{m_0}^{m} \frac{dm}{m} \quad \text{or} \quad \frac{v}{u} = -\ln \frac{m}{m_0}$$

Hence, $$m = m_0 e^{-(wt/u)}$$

1.181 As $\mathbf{F} = 0$, from the equation of dynamics of a body with variable mass,

$$m\frac{d\mathbf{v}}{dt} = \mathbf{u}\frac{dm}{dt} \quad \text{or} \quad d\mathbf{v} = \mathbf{u}\frac{dm}{m} \tag{1}$$

Now $dv \uparrow\downarrow \mathbf{u}$ and since $\mathbf{u} \perp \mathbf{v}$, we must have $|d\mathbf{v}| = v_0 d\alpha$ (because v_0 is constant), where $d\alpha$ is the angle by which the spaceship turns in time dt.

So, $$-u\frac{dm}{m} = v_0 d\alpha \quad \text{or} \quad d\alpha = -\frac{u}{v_0}\frac{dm}{m}$$

or $$\alpha = -\frac{u}{v_0}\int_{m_0}^{m} \frac{dm}{m} = \frac{u}{v_0}\ln\left(\frac{m_0}{m}\right)$$

1.182 We have $$\frac{dm}{dt} = -\mu \quad \text{or} \quad dm = -\mu\, dt$$

Integrating $$\int_{m_0}^{m} dm = -\mu\int_0^t dt \quad \text{or} \quad m = m_0 - \mu t$$

As $\mathbf{u} = 0$ so, from the equation of variable mass system

$$(m_0 - \mu t)\frac{d\mathbf{v}}{dt} = \mathbf{F} \quad \text{or} \quad \frac{d\mathbf{v}}{dt} = \mathbf{w} = \frac{\mathbf{F}}{(m_0 - \mu t)}$$

or $$\int_0^{\mathbf{v}} d\mathbf{v} = \mathbf{F}\int_0^t \frac{dt}{(m_0 - \mu t)}$$

Hence, $$\mathbf{v} = \frac{\mathbf{F}}{\mu}\ln\left(\frac{m_0}{m_0 - \mu t}\right)$$

1.183 Let the car be moving in a reference frame to which the hopper is fixed and at any instant of time, let its mass be m and velocity $\mathbf{v}$.

Then from the general equation, for variable mass system

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} + \mathbf{u}\frac{dm}{dt}$$

We write the equation, for our system as,

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} - \mathbf{v}\frac{dm}{dt} \quad \text{as} \quad \mathbf{u} = -\mathbf{v}$$

So,
$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F}$$

and
$$\mathbf{v} = \frac{\mathbf{F}t}{m} \quad \text{on integration}$$

But
$$m = m_0 + \mu t$$

so,
$$\mathbf{v} = \frac{\mathbf{F}t}{m_0\left(1 + \frac{\mu t}{m_0}\right)}$$

Thus, the sought acceleration is

$$\mathbf{w} = \frac{d\mathbf{v}}{dt} = \frac{\mathbf{F}}{m_0\left(1 + \frac{\mu t}{m_0}\right)^2}$$

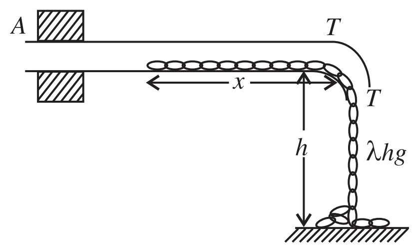
1.184 Let the length of the chain inside the smooth horizontal tube at an arbitrary instant be x. From the equation,

$$m\mathbf{w} = \mathbf{F} + \mathbf{u}\frac{dm}{dt}$$

as $\mathbf{u} = 0$, $\mathbf{F}\uparrow\uparrow\mathbf{w}$, for the chain inside the tube

$$\lambda x w = T, \quad \text{where} \quad \lambda = \frac{m}{l} \tag{1}$$

Similarly for the overhanging part,



$$\mathbf{u} = 0$$

Thus,
$$mw = F$$

or
$$\lambda\, hw = \lambda hg - T \tag{2}$$

From Eqs. (1) and (2)

$$\lambda(x + h)\, w = \lambda\, hg \quad \text{or} \quad (x + h)\, v\frac{dv}{ds} = hg$$

or
$$(x + h)\, v\frac{dv}{(-dx)} = gh$$

(As the length of the chain inside the tube decreases with time, $ds = -dx$.)

or $$v dv = -gh \frac{dx}{x+h}$$

Integrating, $$\int_0^v v dv = -gh \int_{(l-h)}^0 \frac{dx}{x+h}$$

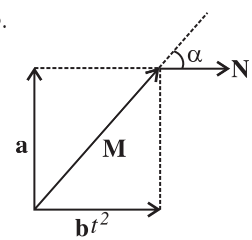
or $$\frac{v^2}{2} = gh \ln\left(\frac{l}{h}\right) \quad \text{or} \quad v = \sqrt{2gh \ln\left(\frac{l}{h}\right)}$$

1.185 The angular momentum of the particle relative to point O is given as function of a time as $\mathbf{M} = \mathbf{a} + \mathbf{b}t^2$.

So, force moment relative to point O is given by

$$\mathbf{N} = \frac{d\mathbf{M}}{dt} = 2\mathbf{b}t$$

As force moment $\mathbf{N} = 2\mathbf{b}t$, so vector $\mathbf{N}$ coincides with vector $\mathbf{b}$.



Let us depict the vectors $\mathbf{N}$ and $\mathbf{M}$ at an arbitrary instant of time t, when angle between $\mathbf{M}$ and $\mathbf{N}$ is α (see figure). It is obvious from the figure that $\tan\alpha = a/bt^2$, therefore $\alpha = 45°$, the time $t = t_0 = \sqrt{a/b}$.

Thus, $\mathbf{N} = 2\mathbf{b}\sqrt{a/b}$, when vector $\mathbf{N}$ forms angle $\alpha = 45°$ with vector $\mathbf{M}$.

Alternate:

Let the angle between $\mathbf{M}$ and $\mathbf{N}$, $\alpha = 45°$ at $t = t_0$.

Then $$\frac{1}{\sqrt{2}} = \frac{\mathbf{M}\cdot\mathbf{N}}{|\mathbf{M}|\,|\mathbf{N}|} = \frac{(\mathbf{a} + \mathbf{b}t_0^2)\cdot(2\mathbf{b}t_0)}{\sqrt{a^2 + b^2t_0^4}\; 2bt_0}$$

$$= \frac{2b^2t_0^3}{\sqrt{a^2 + b^2t_0^4}\; 2bt_0} = \frac{bt_0^2}{\sqrt{a^2 + b^2t_0^4}}$$

So, $$2b^2t_0^4 = a^2 + b^2t_0^4 \quad \text{or} \quad t_0 = \sqrt{\frac{a}{b}} \quad \text{(as } t_0 \text{ cannot be negative)}$$

Hence, $$\mathbf{N} = 2\mathbf{b}\sqrt{\frac{a}{b}}$$

1.186 $$\mathbf{M}(t) = \mathbf{r} \times \mathbf{p} = \left(\mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2\right) \times m\,(\mathbf{v}_0 + \mathbf{g}t)$$

$$= mv_0\, gt^2 \sin\left(\frac{\pi}{2} + \alpha\right)(-\,\mathbf{k}) + \frac{1}{2}\, mv_0 gt^2 \sin\left(\frac{\pi}{2} + \alpha\right)(\mathbf{k})$$

$$= \frac{1}{2}\, mv_0 gt^2 \cos\alpha(-k)$$

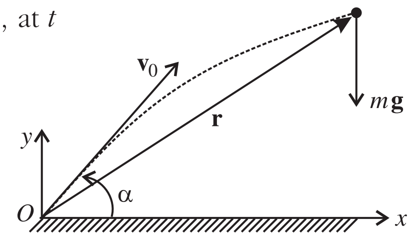
Thus $$M(t) = \frac{mv_0 gt^2 \cos\alpha}{2}$$

Thus, angular momentum at maximum height, i.e., at t

$$\frac{\tau}{2} = \frac{v_0 \sin\alpha}{g}$$

$$M\left(\frac{\tau}{2}\right) = \left(\frac{mv_0^3}{2g}\right)\sin^2\alpha \cos\alpha$$

$$= 37 \text{ kg m}^2/\text{s}$$

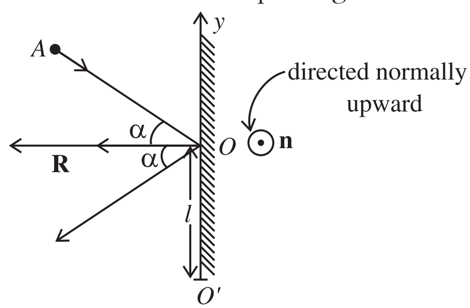


Alternate:

$$\mathbf{M}(0) = 0 \text{ so } \quad \mathbf{M}(t) = \int_0^t \mathbf{N}\, dt = \int_0^t (\mathbf{r} \times m\mathbf{g})$$

$$= \int_0^t \left[\left(\mathbf{v}_0 t + \frac{1}{2}\mathbf{g}t^2\right) \times m\mathbf{g}\right] dt = (\mathbf{v}_0 \times m\mathbf{g})\,\frac{t^2}{2}$$

1.187 (a) The disk experiences gravity, the force of reaction of the horizontal surface, and the force R of reaction of the wall at the moment of the impact against it. The first two forces counter-balance each other, leaving only the force **R.** It's moment relative to any point of the line along which the vector **R** acts or along normal to the wall is equal to zero and therefore the angular momentum of the disk relative to any of these points does not change in the given process.



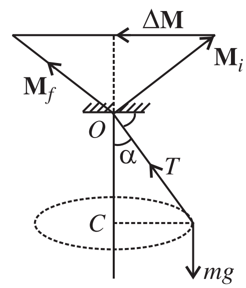
(b) During the course of collision with wall the position of disk is same and is equal to $\mathbf{r}_{00'}$. Obviously the increment in momentum

$$\Delta\mathbf{M} = \mathbf{r}_{00'} \times \Delta\mathbf{p} = 2mvl\cos\alpha\ \mathbf{n}$$

Here, $$|\Delta\mathbf{M}| = 2mvl\ \cos\alpha$$

1.188 (a) The ball is under the influence of forces **T** and $m\mathbf{g}$ at all the moments of time, while moving along a horizontal circle. Obviously the vertical component of **T** balances $m\mathbf{g}$ and so the net moment of these two about any point becomes zero. The horizontal component of **T**, which provides the centripetal acceleration to ball is already directed toward the centre (C) of the horizontal circle, thus its moment about the point C equals zero at all the moments of time. Hence the net moment of the force acting on the ball about point C equals zero and that's why the angular momentum of the ball is conserved about the horizontal circle.

(b) Let α be the angle which the thread forms with the vertical. Now from equation of particle dynamics



$$T\cos\alpha = mg \quad \text{and} \quad T\sin\alpha = m\omega^2 l\sin\alpha$$

Hence on solving,

$$\cos\alpha = \frac{g}{\omega^2 l} \tag{1}$$

As $|\mathbf{M}|$ is constant in magnitude, so from the figure

$$|\Delta\mathbf{M}| = 2M\cos\alpha$$

where $M = |\mathbf{M}_i| = |\mathbf{M}_f|$

$$= |\mathbf{r}_{bo} \times m\mathbf{v}| = mvl \;\; (\text{as } \mathbf{r}_{bo} \perp \mathbf{v})$$

Thus, $|\Delta M| = 2mvl\cos\alpha = 2\,m\omega l^2 \sin\alpha\cos\alpha$

$$= \frac{2mgl}{\omega}\sqrt{1 - \left(\frac{g}{\omega^2 t}\right)^2} \quad (\text{using Eq. 1})$$

1.189 During the free fall, time $\tau = \sqrt{2h/g}$, the reference point O moves in horizontal direction by the distance $V\tau$. In the translating frame as $\mathbf{M}_i(O) = 0$, so

$$\Delta\mathbf{M} = \mathbf{M}_f = \mathbf{r}_f \times m\mathbf{v}_f$$

$$= \left(\frac{1}{2}\mathbf{g}\tau^2 - \mathbf{V}\tau\right) \times m(\mathbf{g}\tau - \mathbf{V})$$

$$= \frac{1}{2}m(\mathbf{g}\times\mathbf{V})\tau^2$$

So, $|\mathbf{M}_f| = \frac{1}{2}mgV\left(\frac{2h}{g}\right)$ (using the value of τ and taking into account $\mathbf{V}\perp\mathbf{g}$)

Hence, $|\Delta\mathbf{M}| = mVh$

1.190 The Coriolis force is $2m(\mathbf{v}' \times \boldsymbol{\omega})$. Here $\boldsymbol{\omega}$ is along the z-axis (vertical). The moving disk is moving with velocity v_0 which is constant, the motion is along the x-axis, then the Coriolis force is along y-axis and has the magnitude $2mv_0\omega$. At time t, the distance of the centre of moving disk from O is $v_0 t$ (along x-axis). Thus the torque N due to the Coriolis force is $N = 2m\, v_0\omega \cdot v_0 t$ along the z-axis. Hence, equating this to dM/dt, we get

$$\frac{dM}{dt} = 2mv_0^2\, \omega t$$

or

$$M = mv_0^2\, \omega t^2 + \text{constant}$$

The constant is irrelevant and may be put equal to zero if the disk is originally set in motion from the point O.

This discussion is approximate. The Coriolis force which causes the disk to swerve from straight line motion and thus causes deviation from the above formula will be substantial for large t.

1.191 If $\dot{r}$ = radial velocity of the particle then the total energy of the particle at any instant is

$$\frac{1}{2}m\dot{r}^2 + \frac{M^2}{2mr^2} + kr^2 = E \tag{1}$$

where the second term is the kinetic energy of angular motion about the centre O.

Then the extreme values of r are determined by $\dot{r} = 0$ and solving the resulting quadratic equation

$$k(r^2)^2 - Er^2 + \frac{M^2}{2m} = 0$$

we get
$$r^2 = \frac{E \pm \sqrt{E^2 - 2M^2k/m}}{2k}$$

From this we see that

$$E = k(r_1^2 + r_1^2)$$

So,
$$\frac{1}{2}mv_2^2 + kr_2^2 = k(r_1^2 + r_2^2)$$

Hence,
$$m = \frac{2kr_1{}^2}{v_2^2}$$

Note: *Eq. (1) can be derived from the standard expression for kinetic energy and angular momentum in plane polar coordinates:*

$$T = \frac{1}{2} m\dot{r}^2 + \frac{1}{2} mr^2 \dot{\theta}^2 \quad \text{and} \quad M = mr^2\dot{\theta}, \text{ respectively}$$

1.192 The swinging sphere experiences two forces: the gravitational force and the tension of the thread. Now, it is clear from the condition, given in the problem, that the moment of these forces about the vertical axis, passing through the point of suspension $N_Z = 0$. Consequently, the angular momentum M_Z of the sphere relative to the given axis (z) is constant.

Thus, $$mv_0 \,(l \sin\theta) = mvl \tag{1}$$

where m is the mass of sphere and v is its velocity in the position, when the thread forms an angle $\pi/2$ with the vertical. Mechanical energy is also conserved, as the sphere is under the influence of only one other force, i.e. tension, which does not perform any work, as it is always perpendicular to the velocity.

So, $$\frac{1}{2}\, mv_0^2 + mgl\cos\theta = \frac{1}{2}\, mv^2 \tag{2}$$

From Eqs. (1) and (2), we get

$$v_0 = \sqrt{2gl/\cos\theta}$$

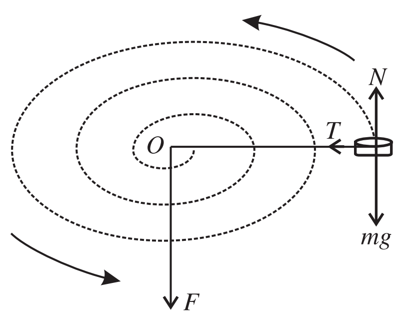
1.193 Forces, acting on the mass m are shown in the figure. As $\mathbf{N} = m\mathbf{g}$, the net torque of these two forces about any fixed point must be equal to zero. Tension T, acting on the mass m is a central force, which is always directed towards the centre O. Hence the moment of force T is also zero about the point O and therefore the angular momentum of the particle m is conserved about O.

So, $$mr^2\dot{\theta} = mr_0^2\omega_0$$

or $$\omega = \dot{\theta} = \frac{\omega_0 r_0^2}{r^2}$$

From $$F_r = ma_r$$

$$-T = m(\ddot{r} - r\dot{\theta}^2)$$



As the thread is pulled with constant speed so, $\dot{r}$ = constant and $\ddot{r} = 0$ (where dot stands for time derivative).

Thus $$T = F = mr\dot{\theta}^2$$

Hence, the sought tension is

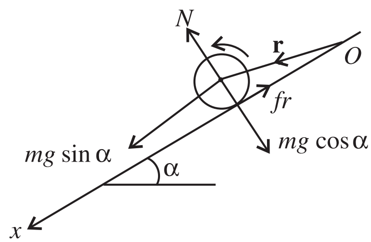
$$F = mr\,\dot{\theta}^2 = mr\left(\frac{\omega_0 r_0^2}{r^2}\right)^2 = \frac{m\omega_0^2 r_0^4}{r^3}$$

1.194 On the given system the weight of the body m is the only force whose moment is effective about the axis of pulley. Let us take the sense of $\boldsymbol{\omega}$ of the pulley at an arbitrary instant as the positive sense of axis of rotation (z-axis).

As $$M_z(0) = 0 \quad \text{so} \quad \Delta M_z = M_z(t) = \int N_z\,dt$$

So, $$M_z(t) = \int_0^t mgR\,dt = mgRt$$

1.195 Let the point of contact of sphere at initial moment ($t = 0$) be at O. At an arbitrary moment, the forces acting on the sphere are shown in the figure.



As $N = mg\cos\alpha$ and both (N and $mg\cos\alpha$) pass through the same line, their net torque about any point is zero. The force of static friction passes through the point O, thus its moment about point O becomes zero. Hence $mg\sin\alpha$ is the only force which has effective torque about point O, and is given by $|\mathbf{N}| = mgR\sin\alpha$ normally emerging from the plane of figure.

As $$\mathbf{M}(t = 0) = 0 \quad \text{so} \quad \Delta\mathbf{M} = \mathbf{M}(t) = \int \mathbf{N}\,dt$$

Hence, $$M(t) = Nt = mg\,R\sin\alpha t$$

1.196 Let position vectors of the particles of the system be $\mathbf{r}_i$ and $\mathbf{r}'_i$ with respect to the points O and O' respectively, then we have

$$\mathbf{r}_i = \mathbf{r}'_i + \mathbf{r}_0 \tag{1}$$

(where $\mathbf{r}_0$ is the radius vector of O' with respect to O).

Now, the angular momentum of the system relative to the point O can be written as follows:

$$\mathbf{M} = \Sigma\mathbf{r}_i \times \mathbf{p}_i = \Sigma(\mathbf{r}'_i \times \mathbf{p}_i) + \Sigma\int(\mathbf{r}_0 \times \mathbf{p}_i) \quad \text{(using Eq. 1)}$$

or $$\mathbf{M} = \mathbf{M}' + (\mathbf{r}_0 \times \mathbf{p}) \quad (\text{where } \mathbf{p} = \Sigma\mathbf{p}_i) \tag{2}$$

From Eq. (2), if the total linear momentum of the system, $\mathbf{p} = 0$, then its angular momentum does not depend on the choice of the point O.

Note In the C.M. frame, the system of particles, as whole is at rest.

1.197 On the basis of solution of problem 1.196. we have concluded that; "in the C.M. frame, the angular momentum of system of particles is independent of the choice of the point, relative to which it is determined" and in accordance with the problem this is denoted by **M**.

We denote the angular momentum of the system of particles, relative to the point O, by **M**. Since the internal and proper angular momentum $\tilde{\mathbf{M}}$, in the C.M. frame, does not depend on the choice of the point O', this point may be taken coincident with the point O of the K-frame, at a given moment of time. Then at that moment, the radius vectors of all the particles, in both reference frames, are equal ($\mathbf{r}'_i = \mathbf{r}_i$) and the velocities are related by the equation

$$\mathbf{v}_i = \tilde{\mathbf{v}}_i + \mathbf{v}_C$$

where $\mathbf{v}_C$ is the velocity of C.M. frame, relative to the K-frame. Consequently, we may write

$$\mathbf{M} = \Sigma m_i(\mathbf{r}_i \times \mathbf{v}_\mathrm{i}) = \Sigma m_i\,(\mathbf{r}_i \times \tilde{\mathbf{v}}_i) + \Sigma m_i(\mathbf{r}_i \times \mathbf{v}_C)$$

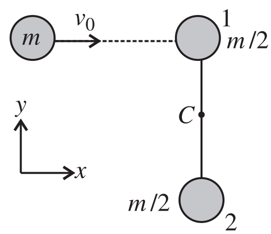
or $$\mathbf{M} = \tilde{\mathbf{M}} + m(\mathbf{r}_C \times \mathbf{v}_C), \quad \text{as} \quad \Sigma m_i\mathbf{r}_i = m\mathbf{r}_C \ \ (\text{where } m = \Sigma m_i)$$

or $$\mathbf{M} = \tilde{\mathbf{M}} + (\mathbf{r}_C \times m\mathbf{v}_C) = \tilde{\mathbf{M}} + (\mathbf{r}_C \times \mathbf{p})$$

1.198 Taking striking ball as particle 1 and upper ball or the dumbbell as particle 2 the velovity acquired by particle 2 on collsion, from the formula

$$\mathbf{v}'_i = 2\mathbf{v}_C - \mathbf{v}_i \quad \text{(see solution of problem 1.168)}$$

$$\mathbf{v}'_2 = 2\left(\frac{m\mathbf{v}_0}{m + m/2}\right) - 0 = \frac{4}{3}\mathbf{v}_0$$



To find the proper angular momentum of the dumbbell, let us denote upper ball of the dumbbell as particle 1 and lower ball as particle 2.

Then proper angular momentum vector is

$$\tilde{\mathbf{M}} = \mathbf{r}_{1C} \times m_1\mathbf{v}_{1C} + \mathbf{r}_{2C} \times m_2\mathbf{v}_{2C} = \mathbf{r}_{1C} \times \tilde{\mathbf{p}}_1 + \mathbf{r}_{2C} \times \tilde{\mathbf{p}}_2$$

In C.M. frame net linear momentum of any system is zero, so, $\tilde{\mathbf{p}}_1 = -\tilde{\mathbf{p}}_2$.

So, $$\tilde{\mathbf{M}} = (\mathbf{r}_{1C} - \mathbf{r}_{2C}) \times \tilde{\mathbf{p}}_1 = \mathbf{r}_{12} \times \tilde{\mathbf{p}}_1$$

We have $$\tilde{\mathbf{p}}_1 = \mu\,(\mathbf{v}_1 - \mathbf{v}_2) \text{ (see solution of problem 1.147)}$$
$$= \frac{m}{4}\left(\frac{4v_0}{3}\mathbf{i} - 0\right) = \frac{mv_0}{3}\mathbf{i}$$

Hence, $$\tilde{\mathbf{M}} = [\mathbf{r}_{12} \times \tilde{\mathbf{p}}_1] = \left[l\,\mathbf{j} \times \frac{mv_0}{3}\mathbf{i}\right] = \frac{mv_0\,l}{3}(-\mathbf{k})$$

Hence, $$\tilde{M} = \frac{mv_0\,l}{3}$$

1.199 In the C.M. frame of the system (both the disks + spring), the linear momentum of the disks are related by the relation, $\tilde{\mathbf{p}}_1 = -\tilde{\mathbf{p}}_2$, at all the moments of time.

Where, $$\tilde{p}_1 = \tilde{p}_2 = \tilde{p} = \mu v_{\text{rel}}$$

And the total kinetic energy of the system,

$$T = \frac{1}{2}\mu v_{\text{rel}}^2 \quad \text{[see solution of problem 1.147(b)]}$$

Bearing in mind that at the moment of maximum deformation of the spring, the projection of $\mathbf{v}_{\text{rel}}$ along the length of the spring becomes zero, i.e., $v_{\text{rel}\,(x)} = 0$.

The conservation of mechanical energy of the considered system in the C.M. frame gives

$$\frac{1}{2}\left(\frac{m}{2}\right)v_0^2 = \frac{1}{2}\kappa x^2 + \frac{1}{2}\left(\frac{m}{2}\right)v_{\text{rel}\,(y)}^2 \tag{1}$$

Now from the conservation of angular momentum of the system about the C.M.,

$$\frac{1}{2}\left(\frac{l_0}{2}\right)\left(\frac{m}{2}v_0\right) = 2\left(\frac{l_0 + x}{2}\right)\frac{m}{2}v_{\text{rel}\,(y)}$$

or $$v_{\text{rel}\,(y)} = \frac{v_0 l_0}{(l_0 + x)} = v_0\left(1 + \frac{x}{l_0}\right)^{-1} = v_0\left(1 - \frac{x}{l_0}\right) \quad (\text{as } x \ll l_0) \tag{2}$$

Using Eq. (2) in Eq. (1), $$\frac{1}{2}mv_0^2\left[1 - \left(1 - \frac{x}{l_0}\right)^2\right] = \kappa x^2$$

or $$\frac{1}{2}mv_0^2\left[1 - \left(1 - \frac{2x}{l_0} + \frac{x^2}{l_0^2}\right)\right] = \kappa x^2$$

or $$\frac{mv_0^2 x}{l_0} = \kappa x^2 \quad [\text{neglecting } x^2/l_0^{\,2}]$$

As $$x \neq 0, \text{ thus } x = \frac{mv_0^2}{\kappa l_0}$$

1.4 Universal Gravitation

1.200 We have $\frac{Mv^2}{r} = \frac{\gamma M M_S}{r^2}$ or $r = \frac{\gamma M_S}{v^2}$ (here M_S is the mass of the Sun)

Thus $$\omega = \frac{v}{r} = \frac{v}{\gamma M_S/v^2} = \frac{v^3}{\gamma M_S}$$

So, $$T = \frac{2\pi\gamma M_S}{v^3} = \frac{2\pi \times 6.67 \times 10^{-11} \times 1.97 \times 10^{30}}{(34.9 \times 10^3)^3} = 1.94 \times 10^7 \text{ s} = 225 \text{ days}$$

(The answer is incorrectly written in terms of the planetary mass M in answer sheet.)

1.201 For any planet

$$MR\omega^2 = \frac{\gamma M M_S}{R^2} \quad \text{or} \quad \omega = \sqrt{\frac{\gamma M_S}{R^3}}$$

So, $$T = \frac{2\pi}{\omega} = 2\pi R^{3/2}/\sqrt{\gamma M_S}$$

(a) Thus, $$\frac{T_J}{T_E} = \left(\frac{R_J}{R_E}\right)^{3/2}$$

So, $$\frac{R_J}{R_E} = \left(\frac{T_J}{T_E}\right)^{2/3} = (12)^{2/3} = 5.24$$

(b) $$V_J^2 = \frac{\gamma M_S}{R_J} \quad \text{and} \quad R_J = \left(T\frac{\sqrt{\gamma M_S}}{2\pi}\right)^{2/3}$$

So, $$V_J^2 = \frac{(\gamma M_S)^{2/3}\,(2\pi)^{2/3}}{T^{2/3}} \quad \text{or} \quad V_J = \left(\frac{2\pi\gamma M_S}{T}\right)^{2/3}$$

where $T = 12$ years, M_S = mass of the Sun.

Putting the values, we get $V_J = 12.97$ km/s

$$\text{Acceleration} = \frac{V_J^2}{R_J} = \left(\frac{2\pi\gamma M_S}{T}\right)^{2/3} \times \left(\frac{2\pi}{T\sqrt{\gamma M_S}}\right)^{2/3}$$

$$= \left(\frac{2\pi}{T}\right)^{4/3} (\gamma M_s)^{1/3}$$

$$= 2.15 \times 10^{-4} \text{ km/s}^2$$

1.202 Semi-major axis = $(r+R)/2$. It is sufficient to consider the motion be along a circle of semi-major axis $(r + R)/2$ for T does not depend on eccentricity.

Hence,
$$T = \frac{[2\pi(r + R)/2]^{3/2}}{\sqrt{\gamma M_S}}$$
$$= \pi\sqrt{(r + R)^3/2\gamma M_S} \text{ (again } M_S \text{ is the mass of the Sun)}$$

1.203 We can think of the body as moving in a very elongated orbit of maximum distance R and minimum distance 0 so semi major axis = $R/2$. Hence if τ is the time of fall then

$$\left(\frac{2\tau}{T}\right)^2 = \left(\frac{R/2}{R}\right)^3 \quad \text{or} \quad \tau^2 = \frac{T^2}{32}$$

or
$$\tau = T/4\sqrt{2} = 365/4\sqrt{2} = 64.5 \text{ days}$$

1.204 $T = 2\pi R^{3/2}/\sqrt{rM_s}$. If the distances are scaled down, $R^{3/2}$ decreases by a factor $\eta^{3/2}$ and so does M_s. Hence T does not change.

1.205 The double star can be replaced by a single star of mass $m_1m_2/(m_1 + m_2)$ moving about the centre of mass subjected to the force $\gamma m_1m_2/r^2$. Then

$$T = \frac{2\pi r^{3/2}}{\sqrt{\gamma m_1 m_2 \Big/ \dfrac{m_1 m_2}{m_1 + m_2}}}$$
$$= \frac{2\pi r^{3/2}}{\sqrt{\gamma M}}$$

So,
$$r^{3/2} = \frac{T}{2\pi}\sqrt{\gamma M}$$

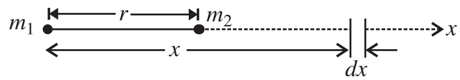
or
$$r = \left(\frac{T}{2\pi}\right)^{2/3}(\gamma M)^{1/3} = 3\sqrt{\gamma M\,(T/2\pi)^2}$$

1.206 (a) The gravitational potential due to m_1 at the point of location of m_2

$$V_2 = \int_r^\infty \mathbf{G}\cdot d\mathbf{r} = \int_r^\infty -\frac{\gamma m_1}{x^2}\,dx = -\frac{\gamma m_1}{r}$$

So, $$U_{21} = m_2 V_2 = -\frac{\gamma m_1 m_2}{r}$$

Similarly, $$U_{12} = -\frac{\gamma m_1 m_2}{r}$$

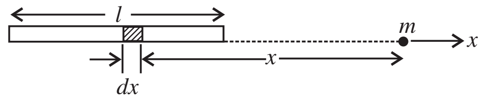


Hence, $$U_{12} = U_{21} = U = -\frac{\gamma m_1 m_2}{r}$$

(b) Choose the location of the point mass as the origin. Then the potential energy dU of an element of mass $dM = M/l\ dx$ of the rod in the field of the point mass is

$$dU = -\gamma m \frac{M}{l}\, dx \frac{1}{x}$$

where x is the distance between the element and the point. (Note that the rod and the point mass are on a straight line.) Then if a is the distance of the nearer end of the rod from the point mass,



$$U = -\gamma \frac{mM}{l} \int_a^{a+l} \frac{dx}{x} = -\gamma m \frac{M}{l} \ln\left(1 + \frac{l}{a}\right)$$

The force of interaction is

$$F = -\frac{\partial U}{\partial a} = \gamma \frac{mM}{l} \times \frac{1}{1 + \frac{l}{a}} \left(-\frac{l}{a^2}\right) = -\frac{\gamma m M}{a\,(a + l)}$$

Minus sign means attraction.

1.207 As the planet is under central force (gravitational interaction), its angular momentum is conserved about the Sun (which is situated at one of the focii of the ellipse).

So, $$m v_1 r_1 = m v_2 r_2 \quad \text{or} \quad v_1^2 = \frac{v_2^2 r_2^2}{r_1^2} \tag{1}$$

From the conservation of mechanical energy of the system (Sun + planet),

$$-\frac{\gamma M_S m}{r_1} + \frac{1}{2} m v_1^2 = -\frac{\gamma M_S m}{r_2} + \frac{1}{2} m v_2^2$$

or $$-\frac{\gamma M_S}{r_1} + \frac{1}{2} v_2^2 \frac{r_2^2}{r_2^2} = -\left(\frac{\gamma M_S}{r_2}\right) + \frac{1}{2} v_2^2 \qquad \text{(using Eq. 1)}$$

Thus, $$v_2 = \sqrt{2\gamma\, M_S\, r_1/r_2(r_1 + r_2)} \tag{2}$$

Hence, $$M = m v_2 r_2 = m\sqrt{2\gamma M_S r_1 r_2/(r_1 + r_2)}$$

1.208 From the previous problem, if r_1, r_2 are the maximum and minimum distances from the Sun to the planet and v_1, v_2 are the corresponding velocities, then, say,

$$E = \frac{1}{2}mv_2^2 - \frac{\gamma m M_S}{r_2}$$

$$= \frac{\gamma m M_S}{r_1 + r_2} \cdot \frac{r_1}{r_2} - \frac{\gamma m M_S}{r_2} = \frac{-\gamma m M_S}{r_1 + r_2} = \frac{\gamma m M_S}{2a} \text{ (using Eq. 2 of problem 1.207)}$$

where $2a$ = major axis $= r_1 + r_2$.

The same result can also be obtained directly by writing an equation analogous to Eq. (1) of problem 1.191

$$E = \frac{1}{2}mr^2 - \frac{M^2}{2mr^2} - \frac{\gamma m M_S}{r}$$

(Here M is angular momentum of the planet and m is its mass.) For extreme position $\dot{r} = 0$ and we get the quadratic equation

$$Er^2 + \gamma m M_S r - \frac{M^2}{2m} = 0$$

The sum of the two roots of this equation is

$$r_1 + r_2 = -\frac{\gamma m M_S}{E} = 2$$

Thus,
$$E = -\frac{\gamma m M_S}{2a} = \text{constant}$$

1.209 From the conservation of angular momentum about the Sun.

$$mv_0 r_0 \sin\alpha = mv_1 r_1 = mv_2 \text{ or } v_1 r_1 = v_2 r_2 = v_0 r_2 \sin\alpha \tag{1}$$

From conservation of mechanical energy,

$$\frac{1}{2}mv_0^2 - \frac{\gamma M_S m}{r_0} = \frac{1}{2}mv_1^2 - \frac{\gamma M_S m}{r_1}$$

or
$$\frac{v_0^2}{2} - \frac{\gamma M_S}{r_0} = \frac{v_0^2 r_0^2 \sin^2\alpha}{2r_1^2} - \frac{\gamma M_S}{r_1} \text{ (using Eq. 1)}$$

or
$$\left(v_0^2 - \frac{2\gamma M_S}{r_0}\right) r_1^2 + 2\gamma M_S r_1 - v_0^2 r_0^2 \sin\alpha = 0$$

So,
$$r_1 = \frac{-2\gamma M_S \pm \sqrt{4\gamma^2 M_S^2 + 4\left(v_0^2 r_0^2 \sin^2\alpha\right)\left(v_0^2 - \frac{2\gamma M_S}{r_0}\right)}}{2\left(v_0^2 - \frac{2\gamma M_S}{r_0}\right)}$$

$$= \frac{1 \pm \sqrt{1 - \dfrac{v_0^2 r_0^2 \sin^2 \alpha}{\gamma M_S}\left(\dfrac{2}{r_0} - \dfrac{v_0^2}{rM_S}\right)}}{\left(\dfrac{2}{r_0} - \dfrac{v_0}{\gamma M_S}\right)} = \frac{r_0\left[1 \pm \sqrt{1 - (2 - \eta)\sin^2 \alpha}\right]}{(2 - \eta)}$$

where $\eta = v_0^2\, r_0/\gamma M_S$.

1.210 At the minimum separation with the Sun, the cosmic body's velocity is perpendicular to its position vector relative to the Sun. If $r_{\min}$ be the sought minimum distance, from conservation of angular momentum about the Sun (C),

$$mv_0 l = mvr_{\min} \quad \text{or} \quad v = \frac{v_0 l}{r_{\min}} \tag{1}$$

From conservation of mechanical energy of the system (Sun 1 cosmic body),

$$\frac{1}{2}mv_0^2 = -\frac{\gamma M_S m}{r_{\min}} + \frac{1}{2}mv^2$$

So,
$$\frac{v_0^2}{2} = -\frac{\gamma M_S}{r_{\min}} + \frac{v_0^2}{2r_{\min}^2} \quad \text{(using Eq. 1)}$$

or
$$v_0^2\, r_{\min}^2 + 2\gamma M_S r_{\min} - v_0^2\, l^2 = 0$$

So,
$$r_{\min} = \frac{-2\gamma M_S \pm \sqrt{4\gamma^2 m^2 + 4v_0^2 v_0^2\, l^2}}{2v_0^2} = \frac{-\gamma M_S \pm \sqrt{\gamma^2 M_S^2 + v_0^4\, l^2}}{v_0^2}$$

Hence, taking positive root

$$r_{\min} = (\gamma M_S/v_0^2)\left[\sqrt{1 + (lv_0^2/\gamma M_S)^2 - 1}\right]$$

1.211 (a) Suppose that the sphere has a radius equal to a. We may imagine that the sphere is made up of concentric thin spherical shells (layers) with radii ranging from 0 to a, and each spherical layer is made up of elementary bands (rings). Let us first calculate potential due to an elementary band of a spherical layer at the point of location of the point mass m (say point P) (see figure). As all the points of the band are located at the distance l from the point P, so,

$$\partial\varphi = -\frac{\gamma \partial M}{l} \quad \text{(where } \partial M \text{ is mass of the band)} \tag{1}$$

$$\partial M = \left(\frac{dM}{4\pi a^2}\right)(2\pi a \sin\theta)\,(a d\theta)$$

$$= \left(\frac{dM}{2}\right)\sin\theta\, d\theta \tag{2}$$

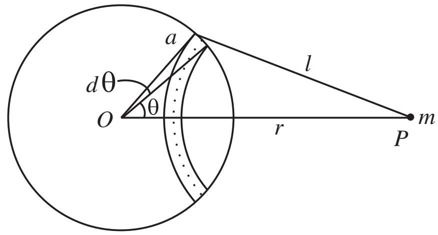
and $\qquad l^2 = a^2 + r^2 - 2\,ar\cos\theta \qquad (3)$

Differentiating Eq. (3), we get

$$l\,dl = ar\sin\theta\, d\theta \qquad (4)$$

Hence using above equations

$$\partial\varphi = -\left(\frac{\gamma dM}{2ar}\right) dl \qquad (5)$$



Now integrating Eq. (5) over the whole spherical layer

$$d\varphi = \int \partial\varphi = -\frac{\gamma dM}{2ar}\int_{r-a}^{r+a} dl$$

So, $\quad d\varphi = -\dfrac{\gamma dM}{r} \qquad (6)$

Eq. (6) demonstrates that the potential produced by a thin uniform spherical layer outside the layer is such as if the whole mass of the layer were concentrated at its centre. Hence the potential due to the sphere at point P is given by

$$\varphi = \int d\varphi - \frac{\gamma}{r}\int dM = -\frac{\gamma M}{r} \qquad (7)$$

This expression is similar to that of Eq. (6).

Hence the sought potential energy of gravitational interaction of the particle m and the sphere,

$$U = m\varphi = -\frac{\gamma Mm}{r}$$

(b) Using the equation $\quad G_r = -\dfrac{\partial\varphi}{\partial r}$

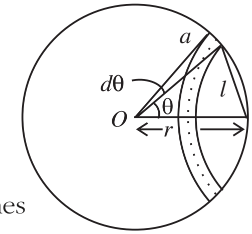
$$G_r = -\frac{\gamma M}{r^2} \quad \text{(using Eq. 7)}$$

So, $\quad \mathbf{G} = -\dfrac{\gamma M}{r^3}\,\mathbf{r} \quad$ and $\quad \mathbf{F} = m\mathbf{G} = -\dfrac{\gamma mM}{r^3}\,\mathbf{r} \qquad (8)$

1.212 (The problem has already a clear hint in the answer sheet of the problem book. Here we adopt a different method.)

Let m be the mass of the spherical layer, which is imagined to be made up of rings. At a point inside the spherical layer at distance r from the centre, the gravitational potential due to a ring element of radius a equals

$$d\varphi = -\frac{\gamma m}{2ar}\,dl \quad \text{(see Eq. (5) of solution of problem 1.211)}$$



So, $$\varphi = \int d\varphi = -\frac{\gamma m}{2ar}\int_{a-r}^{a+r} dl = -\frac{\gamma m}{a}$$

Hence, $G_r = -\dfrac{\partial \varphi}{\partial r} = 0$

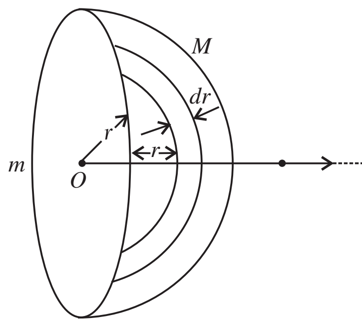
Hence, gravitational field strength as well as field force becomes zero inside a thin spherical layer.

1.213 One can imagine that the uniform hemisphere is made up of thin hemispherical layers of radii ranging from 0 to R. Let us consider such a layer (see figure). Potential at point O due to this layer is

$$d\varphi = -\frac{\gamma dm}{r} = -\frac{3\gamma M}{r^3}\, r dr, \quad \text{where } dm = \frac{M}{(2/3)\,\pi R^3}\left(\frac{4\pi r^2}{2}\right) dr$$

(This is because all points of each hemispherical shell are equidistance from O.)

Hence, $$\varphi = \int d\varphi = -\frac{3\gamma M}{R^3}\int_0^R r dr = -\frac{3\gamma M}{2R}$$



Hence, the work done by the gravitational field force on the particle of mass m, to remove it to infinity is given by the formula

$$A = m\varphi, \text{ since } \varphi = 0 \text{ at infinity}$$

Hence the sought work,

$$A_{0\to\infty} = -\frac{3\gamma m M}{2R}$$

(The work done by the external agent is – A.)

1.214 In the solution of problem 1.211, we have obtained φ and $\mathbf{G}$ due to a uniform sphere, at a distance r from it's centre outside it. We have from Eqs. (7) and (8) of problem 1.211,

$$\varphi = -\frac{\gamma M}{r} \quad \text{and} \quad \mathbf{G} = -\frac{\gamma M}{r^3}\mathbf{r} \tag{A}$$

In accordance with the Eq. (1) of the solution of 1.212, potential due to a spherical shell of radius a, at any point, inside it becomes

$$\varphi = \frac{\gamma M}{a} = \text{constant} \quad \text{and} \quad G_r = -\frac{\partial\varphi}{\partial r} = 0 \tag{B}$$

For a point (say P) which lies inside the uniform solid sphere, the potential φ at that point may be represented as a sum

$$\varphi_{\text{inside}} = \varphi_1 + \varphi_2$$

where φ_1 is the potential of a solid sphere having radius r and φ_2 is the potential of the layer of radii r and R. In accordance with Eq. (A)

$$\varphi_1 = -\frac{\gamma}{r}\left(\frac{M}{(4/3)\,\pi R^3}\,\frac{4}{3}\,\pi r^3\right) = \frac{\gamma M}{R^3}\,r^2$$

The potential φ_2 produced by the layer (thick shell) is the same at all points inside it. The potential φ_2 is easiest to calculate, for the point positioned at the layer's centre. Using Eq. (B)

$$\varphi_2 = -\gamma\int_r^R \frac{dM}{r} = -\frac{3}{2}\,\frac{\gamma M}{R^3}\,(R^2 - r^2)$$

where $$dM = \frac{M}{(4/3)\,\pi R^3}\,4\pi r^2 dr = \left(\frac{3M}{R^3}\right) r^2 dr$$

is the mass of a thin layer between the radii r and $r + dr$.

Thus, $$\varphi_{\text{inside}} = \varphi_1 + \varphi_2 = \left(\frac{\gamma M}{2R}\right)\left(3 - \frac{r^2}{R^2}\right) \qquad \text{(C)}$$

From Eq. (B) $$G_r = \frac{-\,\partial\varphi}{\partial r}$$

$$G_r = \frac{\gamma M r}{R^3}$$

or $$\mathbf{G} = -\frac{\gamma M}{R^3}\,\mathbf{r} = -\gamma\,\frac{4}{3}\,\pi\rho\mathbf{r}$$

where $\rho = \dfrac{M}{4/3\,\pi R^3}$ is the density of the sphere.

The plots $\varphi\,(r)$ and $G\,(r)$ for a uniform sphere of radius R are shown in the figure of answer sheet.

Alternate:

Like Gauss's theorem of electrostatics, one can derive Gauss's theorem for gravitation in the form $\mathbf{G}.d\mathbf{S} = -4\pi\gamma m_{\text{enclosed}}$. For calculation of $\mathbf{G}$ at a point inside the sphere at a distance r from its centre, let us consider a Gaussian surface of radius r. Then

$$G_r 4\pi r^2 = -4\pi\gamma\left(\frac{M}{r^3}\right) r^3 \qquad \text{or} \qquad G_r = -\frac{\gamma M}{R^3}\,r$$

Hence, $$\mathbf{G} = -\frac{\gamma M}{R^3}\,\mathbf{r} = -\gamma\frac{4}{3}\,\pi\rho\,\mathbf{r}\left(\text{as } \rho = \frac{M}{(4/3)\pi R^3}\right)$$

So, $$\varphi = \int_r^\infty G_r dr = \int_r^R -\frac{\gamma M}{R^3} r dr + \int_R^\infty -\frac{\gamma M}{r^2} dr$$

Integrating and summing up, we get

$$\varphi = -\frac{\gamma M}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

From Gauss' theorem for a point outside it,

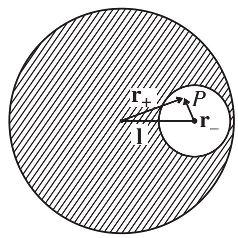
$$G_r 4\pi r^2 = -4\pi\gamma M \quad \text{or} \quad G_r = -\frac{\gamma M}{r^2}$$

Thus, $$\varphi(r) = \int_r^\infty G_r dr = -\frac{\gamma M}{r}$$

1.215 Treating the cavity as negative mass of density $-\rho$ in a uniform sphere density $+\rho$ and using the superposition principle, the sought field strength is

$$\mathbf{G} = \mathbf{G}_1 + \mathbf{G}_2$$

or $$\mathbf{G} = -\frac{4}{3}\pi\gamma\rho\mathbf{r}_+ + -\frac{4}{3}\gamma\pi(-\rho)\,\mathbf{r}_-$$



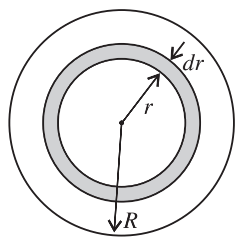
where $\mathbf{r}_+$ and $\mathbf{r}_-$ are the position vectors of an arbitrary point P inside the cavity with respect to centre of sphere and cavity, respectively.

Thus, $$\mathbf{G} = -\frac{4}{3}\pi\gamma\rho(\mathbf{r}_+ - \mathbf{r}_-) = -\frac{4}{3}\pi\gamma\rho l$$

1.216 We partition the solid sphere into thin spherical layers and consider a layer of thickness dr lying at a distance r from the centre of the ball. Each spherical layer presses on the layers within it. The considered layer is attracted to the part of the sphere lying within it (the outer part does not act on the layer). Hence for the considered layer

$$dp\, 4\pi r^2 = dF$$

or $$dp\, 4\pi r^2 = \frac{\gamma\,(4/3\ \pi\ r^3\rho)\,(4\pi r^2 dr\rho)}{r^2}$$



(where ρ is the mean density of sphere)

or $$dp = \frac{4}{3}\pi\gamma\rho^2\ r dr$$

Thus, $$p = \int_r^R dp = \frac{2\pi}{3}\gamma\rho^2 (R^2 - r^2)$$

(the pressure must vanish at $r = R$)

or $$p = \frac{3}{8}(1 - (r^2/R^2))\ \gamma M^2/\pi R^4, \text{ putting } \rho = M/(4/3)\ \pi R^3$$

Putting $r = 0$, we have the pressure at sphere's centre, and treating it as the Earth where mean density is equal to $\rho = 5.5 \times 10^3$ kg/m^3 and $R = 64 \times 10^2$ km we have, $p = 1.73 \times 10^{11}$ Pa or 1.72×10^6 atm.

1.217 (a) Since the potential at each point of a spherical surface (shell) is constant and is equal to $\varphi = -\gamma m/R$ [as we have in Eq. (1) of solution of problem 1.212].

We obtain in accordance with the equation

$$U = \frac{1}{2}\int dm\varphi = \frac{1}{2}\varphi\int dm$$

$$= \frac{1}{2}\left(-\frac{\gamma m}{R}\right) m = -\frac{\gamma m^2}{2R}$$

(The factor 1/2 is needed otherwise contribution of different mass elements is counted twice.)

(b) In this case the potential inside the sphere depends only on r (see Eq. (C) of the solution of problem 1.214)

$$\varphi = -\frac{3\gamma M}{2R}\left(1 - \frac{r^2}{3R^2}\right)$$

Here dm is the mass of an elementary spherical layer confined between the radii r and $r + dr$ given by

$$dm = (4\pi r^2\ dr\rho) = \left(\frac{3m}{R^3}\right) r^2 dr$$

$$U = \frac{1}{2}\int dm\ \varphi$$

$$= \frac{1}{2}\int_0^R \left(\frac{3m}{R^3}\right) r^2\ dr - \frac{3\gamma m}{2R}\left(1 - \frac{r^2}{3R^2}\right)$$

After integrating, we get

$$U = -\frac{3}{5}\frac{\gamma m^2}{R}$$

1.218 Let $\omega = \sqrt{\frac{\gamma M_E}{r^3}}$ = circular frequency of the satellite in the outer orbit,

$\omega_0 = \sqrt{\frac{\gamma M_E}{(r - \Delta r)^3}}$ = circular frequency of the satellite in the inner orbit.

So, relative angular velocity $= \omega_0 \pm \omega$, where $-ve$ sign is to be taken when the satellites are moving in the same sense and $+ve$ sign if they are moving in opposite sense.

Hence, time between closest approaches

$$\frac{2\pi}{\omega_0 \pm \omega} = \frac{2\pi}{\sqrt{\gamma M_E}/r^{3/2}}\frac{1}{3\Delta r/2r + \delta} = \begin{cases} 4.5 \text{ days } (\delta = 0) \\ 0.80 \text{ hour } (\delta = 2) \end{cases}$$

where δ is 0 in the first case and 2 in the second case.

1.219 $\omega_1 = \frac{\gamma M}{R^2} = \frac{6.67 \times 10^{-11} \times 5.96 \times 10^{24}}{(6.37 \times 10^6)^2} = 9.8 \text{ m/s}^2$

$$\omega_2 = \omega^2 R = \left(\frac{2\pi}{T}\right)^2 R = \left(\frac{2 \times 22}{24 \times 3600 \times 7}\right)^2 6.37 \times 10^6 = 0.034 \text{ m/s}^2$$

and $\omega_3 = \frac{\gamma M_S}{R^2_{\text{mean}}} = \frac{6.67 \times 10^{-11} \times 1.97 \times 10^{30}}{(149.50 \times 10^6 \times 10^3)^2} = 5.9 \times 10^{-3} \text{ m/s}^2$

Then, $\omega_1 : \omega_2 : \omega_3 = 1 : 0.0034 : 0.0006$

1.220 Let h be the sought height in the first case, so

$$\frac{99}{100} g = \frac{\gamma M}{(R + h)^2}$$

$$= \frac{\gamma M}{R^2 (1 + h/R)^2} = \frac{g}{(1 + h/R)^2}$$

or $$\frac{99}{100} = \left(1 + \frac{h}{R}\right)^{-2}$$

From the statement of the problem, it is obvious that in this case $h \ll R$

Thus, $$\frac{99}{100} = \left(1 - \frac{2h}{R}\right) \quad \text{or} \quad h = \frac{R}{200} = \left(\frac{6400}{200}\right) \text{km} = 32 \text{ km}$$

In the other case, if h' be the sought height, then

$$\frac{g}{2} = g\left(1 + \frac{h'}{R}\right)^{-2} \quad \text{or} \quad \frac{1}{2} = \left(1 + \frac{h'}{R}\right)^{-2}$$

From the language of the problem, in this case h' is not very small in comparison with R. Therefore in this case we cannot use the approximation adopted in the previous case.

Here, $$\left(1 + \frac{h'}{R}\right)^{2} = 2$$

So, $$\frac{h'}{R} = \pm\sqrt{2} - 1$$

As $-ve$ sign is not acceptable

$$h' = (\sqrt{2} - 1)\, R = (\sqrt{2} - 1) 6400 \text{ km} = 2650 \text{ km}$$

1.221 Let the mass of the body be m and let it go upto a height h.

From conservation of mechanical energy of the system

$$-\frac{\gamma Mm}{R} + \frac{1}{2} mv_0^2 = \frac{-\gamma Mm}{(R + h)} + 0$$

Using $\gamma M/R^2 = g$, in above equation and on solving we get

$$h = \frac{Rv_0^2}{2gR - v_0^2}$$

1.222 Gravitational pull provides the required centripetal acceleration to the satelite. Thus if h be the sought distance, we have

$$\frac{mv^2}{(R + h)} = \frac{\gamma mM}{(R + h)^2} \quad \text{or} \quad (R + h)\, v^2 = \gamma M$$

or $$Rv^2 + hv^2 = gR^2 \quad \text{as} \quad g = \frac{\gamma M}{R^2}$$

Hence, $$h = \frac{gR^2 - Rv^2}{v^2} = R\left[\frac{gR}{v^2} - 1\right]$$

1.223 A satellite that hovers above the Earth's equator and co-rotates with it moving from west to east with the diurnal angular velocity of the Earth appears stationary to an observer on the Earth. It is called geostationary. For this calculation we may neglect the annual motion of the Earth as well as all other influences. Then, by Newton's law,

$$\frac{\gamma Mm}{r^2} = m\left(\frac{2\pi}{T}\right)^2 r$$

where M = mass of the Earth, $T = 86400$ seconds = period of daily rotation of the Earth and r = distance of the satellite from the centre of the Earth.

Then,
$$r = \sqrt[3]{\gamma M\left(\frac{T}{2\pi}\right)^2}$$

Substitution of $M = 5.96 \times 10^{24}$ kg, gives $r = 4.220 \times 10^4$ km.

Thus instantaneous velocity with respect to an inertial frame fixed to the centre of the Earth at that moment will be

$$\left(\frac{2\pi}{T}\right) r = 3.07 \text{ km/s}$$

and the acceleration will be the centripetal acceleration

$$\left(\frac{2\pi}{T}\right)^2 r = 0.223 \text{ m/s}^2$$

1.224 We know from the previous problem that a satellite moving west to east at distance $R = 2.00 \times 10^4$ km from the centre of the Earth will be revolving round the Earth with an angular velocity faster than the Earth's diurnal angular velocity. Let

$$\omega = \text{angular velocity of the satellite,}$$

$$\omega_0 = 2\pi/T = \text{angular velocity of the Earth.}$$

Then
$$\omega - \omega_0 = 2\pi/\tau$$

is the relative angular velocity with respect to Earth. Now by Newton's law

$$\frac{\gamma M}{R^2} = \omega^2 R$$

So,
$$M = \frac{R^3}{\gamma}\left(\frac{2\pi}{\tau} + \frac{2\pi}{T}\right)^2$$

$$= \frac{4\,\pi^2 R^3}{\gamma T^2}\left(1 + \frac{T}{\tau}\right)^2$$

Substitution gives

$$m = 6.27 \times 10^{24} \text{ kg}$$

1.225 The velocity of the satellite in the inertial space fixed frame is $\sqrt{\gamma M/R}$ east to west. With respect to the Earth fixed frame, from the $\mathbf{v}' = \mathbf{v} - (\boldsymbol{\omega} \times \mathbf{r})$, the velocity is

$$v' = \frac{2\pi R}{T} + \sqrt{\frac{\gamma M}{R}} = 7.03 \text{ km/s}$$

Here M is the mass of the Earth and T is its period of rotation about its own axis. If the satellite were moving from west to east, velocity would be

$$-\frac{2\pi R}{T} + \sqrt{\frac{\gamma m}{R}}$$

To find the acceleration we note the formula

$$m\mathbf{w}' = \mathbf{F} + 2\, m\,(\mathbf{v}' \times \boldsymbol{\omega}) + m\omega^2\mathbf{R}$$

Here $\mathbf{F} = \gamma Mm/R^3\, \mathbf{R}$ and $\mathbf{v}\ \perp\ \boldsymbol{\omega}$ and $\mathbf{v}' \times \boldsymbol{\omega}$ is directed towards the centre of the Earth.

$$\text{Thus, } \mathbf{w}' = \frac{\gamma M}{R^2} + 2\left(\frac{2\pi R}{T} + \sqrt{\frac{\gamma M}{R}}\right)\frac{2\pi}{T} - \left(\frac{2\pi}{T}\right)^2 R \text{ (toward the Earth's rotation axis)}$$

$$= \frac{\gamma M}{R^2} + \frac{2\pi}{T}\left[\frac{2\pi R}{T} + 2\sqrt{\frac{\gamma M}{R}}\right] = 4.94 \text{ m/s}^2 \text{ on substitution}$$

1.226 From the well known relationship between the velocities of a particle with respect to a space fixed frame (K) and rotating frame (K'),

$$\mathbf{v} = \mathbf{v}' + (\boldsymbol{\omega} \times \mathbf{r})$$

we get,
$$\mathbf{v}_1' = v - \left(\frac{2\pi}{T}\right) R$$

Thus kinetic energy of the satellite in the Earth's frame

$$T_1' = \frac{1}{2}\, m v_1'^{\,2} = \frac{1}{2}\, m\left(v - \frac{2\pi R}{T}\right)^2 \tag{1}$$

Obviously when the satellite moves in opposite sense compared to the rotation of the Earth, its velocity relative to the same frame would be

$$v_2' = v + \left(\frac{2\pi}{T}\right) R$$

and kinetic energy

$$T_2' = \frac{1}{2}\, m v_2'^2 = \frac{1}{2}\, m\left(v + \frac{2\pi R}{T}\right)^2 \tag{2}$$

From Eqs. (1) and (2)

$$T' = \frac{(v + 2\pi R/T)^2}{(v - 2\pi R/T)^2} \tag{3}$$

Now from Newton's second law

$$\frac{\gamma Mm}{R^2} = \frac{mv^2}{R} \quad \text{or} \quad v = \sqrt{\frac{\gamma M}{R}} = \sqrt{gR} \tag{4}$$

Using Eqs. (3) and (4)

$$\frac{T_2'}{T_1'} = \frac{(\sqrt{gR} + 2\pi R/T)^2}{(\sqrt{gR} - 2\pi R/T)^2} = 1.27 \quad \text{nearly} \quad \text{(using Appendices)}$$

1.227 For a satellite in a circular orbit about any massive body, the following relation holds between kinetic, potential and total energy

$$T = -E, \; U = 2E \tag{1}$$

Thus, since total mechanical energy must decrease due to resistance of the cosmic dust, the kinetic energy will increase and the satellite will "fall". We see then, by work energy theorem

$$dT = -dE = -dA_{fr}$$

So,
$$mvdv = \alpha v^2 v dt \quad \text{or} \quad \frac{\alpha dt}{m} = \frac{dv}{v^2}$$

Now from Newton's law at an arbitrary radius r from the Moon's centre,

$$\frac{v^2}{r} = \frac{\gamma M}{r^2} \quad \text{or} \quad v = \sqrt{\frac{\gamma M}{r}} \quad \text{(where } M \text{ is the mass of the moon)}$$

Then,
$$v_i = \sqrt{\frac{\gamma M}{\eta R}} \quad \text{and} \quad v_f = \sqrt{\frac{\gamma M}{R}} \quad \text{(where } R = \text{Moon's radius)}$$

So,
$$\int_{v_i}^{v_f} \frac{dv}{v^2} = \frac{\alpha m}{m} \int_0^{\tau} dt = \frac{\alpha \tau}{m}$$

or
$$\tau = \frac{m}{\alpha}\left(\frac{1}{v_i} - \frac{1}{v_f}\right) = \frac{m}{\alpha\sqrt{\gamma M/R}}(\sqrt{\eta} - 1)$$

$$= \frac{m}{\alpha\sqrt{gR}}(\sqrt{\eta} - 1) \quad \text{(where } g \text{ is Moon's gravity)}$$

The averaging implied by Eq. (1) (for non-circular orbits) makes the result approximate.

1.228 From Newton's second law

$$\frac{\gamma Mm}{R^2} = \frac{mv_0^2}{R} \quad \text{or} \quad v_0 = \sqrt{\frac{\gamma M}{R}} = 1.67 \text{ km/s} \tag{1}$$

From conservation of mechanical energy

$$\frac{1}{2}\, m v_e^2 - \frac{\gamma M m}{R} = 0 \quad \text{or} \quad v_e = \sqrt{\frac{2\gamma M}{R}} = 2.37 \text{ km/s} \tag{2}$$

In Eqs. (1) and (2), M and R are the mass of the Moon and its radius. In Eq. (1) if M and R represent the mass of the Earth and its radius, then, using appendices, we can easily get orbital and escape velocities for Earth as

$$v_0 = 7.9 \text{ km/s} \quad \text{and} \quad v_e = 11.2 \text{ km/s}$$

1.229 In a parabolic orbit, $E = 0$

So, $$\frac{1}{2}\, m v_i^2 - \frac{\gamma M m}{R} = 0 \quad \text{or} \quad v_i = \sqrt{2}\sqrt{\frac{\gamma M}{R}}$$

where M = mass of the Moon, R = its radius and this is just the escape velocity. On the other hand in orbit,

$$m v_f^2\, R = \frac{\gamma M m}{R^2} \quad \text{or} \quad v_f = \sqrt{\frac{\gamma M}{R}}$$

Thus, $$\Delta v = (1 - \sqrt{2})\sqrt{\frac{\gamma M}{R}} = -0.70 \text{ km/s}$$

1.230 From solution of problem 1.228 for the Earth surface

$$v_0 = \sqrt{\frac{\gamma M}{R}} \quad \text{and} \quad v_e = \sqrt{\frac{2\gamma M}{R}}$$

Thus the sought additional velocity

$$\Delta v = v_e - v_0 = \sqrt{\frac{\gamma M}{R}}\,(\sqrt{2} - 1) = \sqrt{gR}\,(\sqrt{2} - 1)$$

This "kick" in velocity must be given along the direction of motion of the satellite in its orbit.

1.231 Let r be the sought distance then

$$\frac{\gamma \eta M}{(nR - r)^2} = \frac{\gamma M}{r^2} \quad \text{or} \quad \eta r^2 = (nR - r)^2$$

or $$\sqrt{\eta}\, r = (nR - r) \quad \text{or} \quad r = \frac{nR}{\sqrt{\eta} + 1} = 3.8 \times 10^4 \text{ km}$$

1.232 Between the Earth and the Moon, the potential energy (P.E.) of the spaceship will have a maximum at the point where the attractions of the Earth and the moon balance each other. This maximum P.E. is approximately zero. We can also neglect the contribution of either body to the P.E. of the spaceship sufficiently near the other body. Then the minimum energy that must be imparted to the spaceship to cross the maximum of the P.E. is clearly (using E to denote the Earth)

$$\frac{\gamma M_E m}{R_E} \tag{1}$$

With this energy the spaceship will cross over the hump in the P.E. and coast down the hill of P.E. towards the moon and crash land on it. What the problem seeks is the minimum energy required for soft-landing. That requires the use of rockets to lowering and braking of the spaceship, and since the kinetic energy of the gases ejected from the rocket will always be positive, the total energy required for soft-landing is greater than that required for crash-landing. To calculate this energy we assume the rockets are used fairly close to the moon when the spaceship has nearly attained its terminal velocity on the moon $\sqrt{2\gamma M_0/R_0}$ where M_0 is the mass of the moon and R_0 is its radius. In general $dE = vdp$ and since the speed of the ejected gases is not less than the speed of the rocket, and momentum transferred to the gases must equal the momentum of the spaceship the energy E of the gas ejected is not less than the kinetic energy of spaceship

$$\frac{\gamma M_0 m}{R_0} \tag{2}$$

Adding Eqs. (1) and (2) we get the minimum work done on the ejected gases to bring about the soft landing.

$$A_{\min} = \gamma m \left(\frac{M_E}{R_E} + \frac{M_0}{R_0} \right)$$

On substitution we get 1.3×10^8 kJ.

1.233 Assume first that the attraction of the Earth can be neglected. Then the minimum velocity, that must be imparted the body to escape from the Sun's pull, is, as in solutio of problem 1.230, equal to

$$(\sqrt{2} - 1)\, v_1$$

where $v_1^2 = \gamma M_S/r$, r = radius of the Earth's orbit, M_S = mass of the Sun.

In the actual case near the Earth, the pull of the Sun is small and does not change much over distances, which are several times the radius of the Earth. The velocity v_3 in question is that which overcomes the Earth's pull with sufficient velocity to escape the Sun's pull.

Thus, $$\frac{1}{2} m v_3^2 - \frac{\gamma M_E}{R} = \frac{1}{2} m (\sqrt{2} - 1)^2 v_1^2$$

where R = radius of the Earth and M_E = mass of the Earth.

Writing $v_1{}^2 = \gamma M_E/R$, we get

$$v_3 = \sqrt{2\, v_2^2 + (\sqrt{2} - 1)^2 v_1^2} \doteq 16.6 \text{ km/s}$$

1.5 Dynamics of a Solid Body

1.234 Since, motion of the rod is purely translational, net torque about the C.M. of the rod should be equal to zero.

Thus, $$F_1 \frac{l}{2} = F_2\left(\frac{l}{2} - a\right) \quad \text{or} \quad \frac{F_1}{F_2} = 1 - \frac{a}{l/2} \tag{1}$$

For the translational motion of rod,

$$F_2 - F_1 = mw_C \quad \text{or} \quad 1 - \frac{F_1}{F_2} = \frac{mw_C}{F_2} \tag{2}$$

From Eqs. (1) and (2)

$$\frac{a}{l/2} = \frac{mw_C}{F_2} \quad \text{or} \quad l = \frac{2\,aF_2}{mw_C} = 1 \text{ m}$$

1.235 Sought moment $\mathbf{N} = \mathbf{r} \times \mathbf{F} = (a\mathbf{i} + b\mathbf{j}) \times (A\mathbf{i} + B\,\mathbf{j})$

$$= aB\mathbf{k} + Ab(-\mathbf{k}) = (aB - Ab)\,\mathbf{k}$$

and arm of the force $$l = \frac{N}{F} = \frac{aB - Ab}{\sqrt{A^2 + B^2}}$$

1.236 Relative to point O, the net moment of force is given by

$$N = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 = (a\mathbf{i} \times A\mathbf{j}) + (b\mathbf{j} \times B\mathbf{i})$$
$$= ab\mathbf{k} + AB\,(-\mathbf{k}) = (ab - AB)\mathbf{k} \tag{1}$$

Resultant of the external force

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = A\mathbf{j} + B\mathbf{i} \tag{2}$$

$\mathbf{N} \cdot \mathbf{F} = 0$ (as $\mathbf{N} \perp \mathbf{F}$), so the sought arm l of the force $\mathbf{F}$ is given by

$$l = \frac{N}{F} = \frac{ab - AB}{\sqrt{A^2 + B^2}}$$

1.237 For coplanar forces, about any point in the same plane,

$$\Sigma \mathbf{r}_i \times \mathbf{F}_i = \mathbf{r} \times \mathbf{F}_{\text{net}} \text{ (where } \mathbf{F}_{\text{net}} = \Sigma \mathbf{F}_i = \text{resultant force)}$$

or $$\mathbf{N}_{\text{net}} = \mathbf{r} \times \mathbf{F}_{\text{net}}$$

Thus length of the arm, $$l = \frac{N_{\text{net}}}{F_{\text{net}}}$$

Here obviously $|\mathbf{F}_{net}| = 2F$ and it is directed toward right along AC. Take the origin at C. Then about C,

$$\mathbf{N} = \left(\sqrt{2}\,aF + \frac{aF}{\sqrt{2}}F - \sqrt{2}\,aF\right), \text{ directed normally into the plane of figure}$$

(Here a = side of the square.)

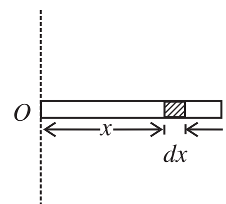
Thus, $\mathbf{N} = F\dfrac{a}{\sqrt{2}}$, directed into the plane of the figure.

Hence, $$l = \frac{F(a/\sqrt{2})}{2F} = \frac{a}{2\sqrt{2}} = \frac{a}{2}\sin 45°$$

Thus the point of application of force is at the mid point of the side BC.

1.238 (a) Consider a strip of length dx at a perpendicular distance x from the axis about which we have to find the moment of inertia of the rod. The elemental mass of the rod equals

$$dm = \frac{m}{l}dx$$



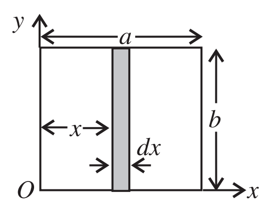
Moment of inertia of this element about the axis

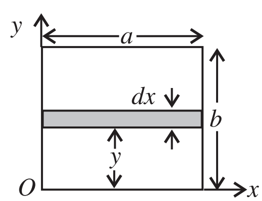
$$dI = dmx^2 = \frac{m}{l}dx \cdot x^2$$

Thus, moment of inertia of the rod, as a whole about the given axis

$$I = \int_0^l \frac{m}{l}x^2\,dx = \frac{ml^2}{3}$$

(b) Let us imagine the plane of plate as x–y plane taking the origin at the intersection point of the sides of the plate (see figure).





Obviously, $$I_x = \int dm\, y^2$$

$$= \int_0^a \left(\frac{m}{ab}\,b\,dy\right)y^2$$

$$= \frac{ma^2}{3}$$

Similarly, $$I_y = \frac{mb^2}{3}$$

Hence from perpendicular axis theorem

$$I_z = I_x + I_y = \frac{m}{3}(a^2 + b^2)$$

which is the sought moment of inertia.

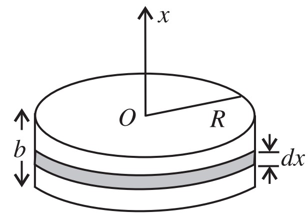
1.239 (a) Consider an elementary disk of thickness dx. Moment of inertia of this element about the z-axis, passing through its C.M. is

$$dI_z = \frac{(dm)R^2}{2} = \rho\ Sdx\frac{R^2}{2}$$

where ρ = density of the material of the plate and S = area of cross section of the plate. Thus, the sought moment of inertia

$$I_z = \frac{\rho S R^2}{2}\int_0^b dx = \frac{R^2}{2}\rho S b$$

$$= \frac{\pi}{2}\rho b R^4 \text{ (as } S = \pi R^2)$$



Putting all the values we get $I_z = 2.8\ \text{g/m}^2$.

(b) Consider an element disk of radius r and thickness dx at a distance x from the point O. Then $r = x\tan\alpha$ and volume of the disk

$$= \pi x^2 \tan^2\alpha\ dx$$

Hence, its mass $\quad dm = \pi x^2 \tan\alpha\ dx \cdot \rho$

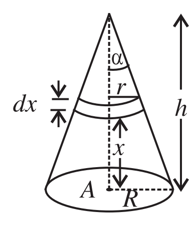
where ρ = density of the cone $= m\Big/\frac{1}{3}\pi R^2 h$.

Moment of inertia of this element, about the axis OA,

$$dI = dm\frac{r^2}{2}$$

$$= (\pi x^2 \tan^2\alpha\ dx)\frac{x^2\tan^2\alpha}{2}$$

$$= \frac{\pi\rho}{2}x^4\tan^4\alpha\ dx$$



Thus the sought moment of inertia

$$I = \frac{\pi\rho}{2}\tan^4\alpha\int_0^h x^4\,\mathbf{dx}$$

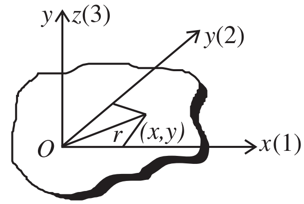
$$= \frac{\pi\rho R^4\cdot h^5}{10h^4}\quad\left(\text{as } \tan\alpha = \frac{R}{h}\right)$$

Hence, $$I = \frac{3mR^2}{10} \quad \left(\text{putting } \rho = \frac{3m}{\pi R^2 h}\right)$$

1.240 (a) Let us consider a lamina of an arbitrary shape and indicate by 1,2 and 3, three axes coinciding with x, y and z-axes and the plane of lamina as $x-y$ plane.

Now, moment of inertia of a point mass about x-axis,

$$dI_x = dm\, y^2$$



Thus moment of inertia of the lamina about this axis,

$$I_x = \int dmy^2$$

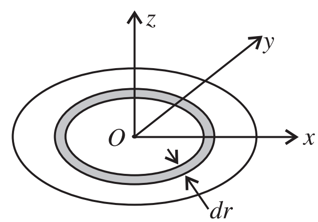
Similarly, $$I_y = \int dmx^2$$

and $$I_z = \int dmr^2$$

$$= \int dm(x^2 + y^2) \text{ as } r = \sqrt{x^2 + y^2}$$

Thus, $$I_z = I_x + I_y \quad \text{or} \quad I_3 = I_1 + I_2$$

(b) Let us take the plane of the disk as $x-y$ plane and origin to the centre of the disk (see figure). From the symmetry $I_x = I_y$. Let us consider a ring element of radius r and thickness dr, then the moment of inertia or the ring element about the y-axis is given by



$$dI_z = dmr^2 = \frac{m}{\pi R^2}(2\pi r dr)r^2$$

Thus the moment of inertia of the disk about z-axis

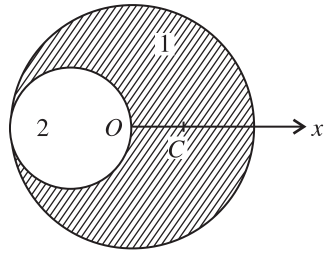
$$I_z = \frac{2m}{R^2}\int_0^R r^3 dr = \frac{mR^2}{2}$$

But we have $$I_x = I_x + I_y = 2I_x$$

Thus, $$I_z = \frac{I_z}{2} = \frac{mR^2}{4}$$

1.241 For simplicity let us use a mathematical trick. We consider the portion of the given disk as the superposition of two complete disks (without holes), one of positive density and radius R and other of negative density but of same magnitude and radius $R/2$.

As (area) $\propto$ (mass), the respective masses of the considered disks are $(4m/3)$ and $(-m/3)$, respectively, and these masses can be imagined to be situated at their respective centres (C.M.). Let us take point O as origin and point x-axis towards right. Obviously the C.M. of the shaded position of given shape lies on the x-axis. Hence the C.M. (**C**) of the shaded portion is given by



$$x_C = \frac{(-m/3)(-R/2) + (4m/3)0}{\left(-\frac{m}{3}\right) + \frac{4m}{3}} = \frac{R}{6}$$

Thus C.M. of the shape is at a distance $R/6$ from point O toward x-axis.

Using parallel axis theorem and bearing in mind that the moment of inertia of a complete homogeneous disk of radius m_0 and radius r_0 equals $1/2m_0r_0^2$; the moment of inertia of the small disk of mass $(-m/3)$ and radius $R/2$ about the axis passing through point C and perpendicular to the plane of the disk is given by

$$I_{2C} = \frac{1}{2}\left(-\frac{m}{3}\right)\left(\frac{R}{2}\right)^2 + \left(-\frac{m}{3}\right)\left(\frac{R}{2} + \frac{R}{6}\right)^2$$

$$= -\frac{mR^2}{24} - \frac{4}{27}mR^2$$

Similarly,
$$I_{1C} = \frac{1}{2}\left(\frac{4m}{3}\right)R^2 + \left(\frac{4m}{3}\right)\left(\frac{R}{6}\right)^2$$

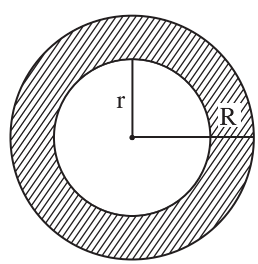
$$= \frac{2}{3}mR^2 + \frac{mR^2}{27}$$

Thus, the sought moment of inertia,

$$I_C = I_{1C} + I_{2C} = \frac{37}{72}mR^2$$

$$= 0.15\ \text{kg}\cdot\text{m}^2$$

1.242 Moment of inertia of the shaded portion, about the axis passing through its centre,



$$I = \frac{2}{5}\left(\frac{4}{3}\pi\ R^3\rho\right)R^2 - \frac{2}{5}\left(\frac{4}{3}\pi r^3\rho\right)r^2$$

$$= \frac{2}{5}\cdot\frac{4}{3}\pi\rho(R^5 - r^5)$$

Now, if $R = r + dr$, the shaded portion becomes a shell, which is the required shape to calculate the moment of inertia.

Now, $$I = \frac{2}{5}\cdot\frac{4}{3}\pi\rho\,\{(r + dr)^5 - r^5\}$$

$$= \frac{2}{5}\cdot\frac{4}{3}\pi\rho\,(r^5 + 5r^4\,dr + \cdots - r^5)$$

Neglecting higher terms, we get

$$= \frac{2}{3}(4\pi r^2\,dr\rho)\,r^2 = \frac{2}{3}mr^2$$

1.243 (a) Net force which is effective on the system (cylinder M + body m) is the weight of the body m in a uniform gravitational field, which is a constant. Thus the initial acceleration of the body m is also constant.

From the conservation of mechanical energy of the said system in the uniform field of gravity at time $t = \Delta t$: $\Delta T + \Delta U_{gr} = 0$

or $$\frac{1}{2}mv^2 + \frac{1}{2}\frac{MR^2}{2}\omega^2 - mg\,\Delta h = 0$$

or $$\frac{1}{4}(2m + M)v^2 - mg\,\Delta h = 0 \quad (\text{as } v = \omega R \text{ at all times}) \qquad (1)$$

But $$v^2 = 2w\Delta h$$

Hence using it in Eq. (1), we get

$$\frac{1}{4}(2m + M)\,2w\,\Delta h - mg\,\Delta h = 0 \quad \text{or} \quad w = \frac{2mg}{(2m + M)}$$

From the kinematic relationship, $\beta = \dfrac{w}{R} = \dfrac{2mg}{(2m + M)R}$

Thus the sought angular velocity of the cylinder

$$\omega(t) = \beta t = \frac{2mg}{(2m + M)R}t = \frac{gt}{(1 + M/2m)R}$$

(b) Sought kinetic energy

$$T(t) = \frac{1}{2}mv^2 + \frac{1}{2}\frac{ml^2}{2}\omega^2$$

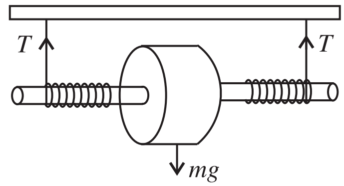
$$= \frac{1}{4}(2m + M)R^2\omega^2$$

$$= \frac{1}{2}mg^2\left(1 + \frac{M}{2m}\right) \quad (\text{on substituting value of } \omega\,)$$

1.244 For equilibrium of the disk and axle

$$2T = mg \quad \text{or} \quad T = mg/2$$

As the disk unwinds, it has an angular acceleration β given by



$$2Tr = I\beta \quad \text{or} \quad \beta = \frac{2Tr}{I} = \frac{mgr}{I}$$

The corresponding linear acceleration is

$$r\beta = w = \frac{mgr^2}{I}$$

Since the disk remains stationary under the combined action of this acceleration and the acceleration $(-w)$ of the bar which is transmitted to the axle, we must have $w = mgr^2/I$.

1.245 Let the rod be deviated through an angle φ from its initial position at an arbitrary instant of time, measured relative to the initial position in the positive direction. From the equation of the increment of the mechanical energy of the system

$$\Delta T = A_{\text{ext}}$$

or $$\frac{1}{2}I\omega^2 = \int N_z d\varphi$$

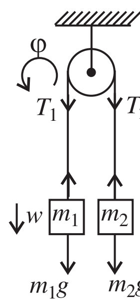
or $$\frac{1}{2}\frac{Ml^2}{3}\omega^2 = \int_0^{\varphi} Fl\cos\varphi \, d\varphi = Fl\sin\varphi$$

Thus, $$\omega = \sqrt{\frac{6F\sin\varphi}{Ml}}$$

1.246 First of all, let us sketch a free diagram of each body. Since the cylinder is rotating and massive, the tension will be different in both the sections of threads. From Newton's law in projection form for the bodies m_1 and m_2 and noting that $w_1 = w_2 = w = \beta R$ (as the thread is not slipping), we have $(m_1 > m_2)$:

$$m_1 g - T_1 = m_1 w = m_1 \beta R$$

and $$T_2 - m_2 g = m_2 w \quad (1)$$



Now from the equation of rotational dynamics of a solid about stationary axis of rotation, i.e.,

$$N_z = I\beta_z \text{ (for the cylinder)}$$

or $$(T_1 - T_2)R = I\beta = \frac{mR^2\beta}{2} \quad (2)$$

Simultaneous solution of the above equations yields:

$$\beta = \frac{(m_1 - m_2)g}{R\left(m_1 + m_2 + m/2\right)} \quad \text{and} \quad \frac{T_1}{T_2} = \frac{m_1(m + 4m_2)}{m_2(m + 4m_1)}$$

1.247 As the system $(m + m_1 + m_2)$ is under constant forces, the acceleration of body m_1 and m_2 is constant. In addition to it the velocities and acceleration of bodies m_1 and m_2 are equal in magnitude (say v and w) because the length of the thread is constant.

From the equation of increment of mechanical energy, i.e., $\Delta T + \Delta U = A_{fr}$, at time t when block m_1 is distance h below from initial position corresponding to $t = 0$,

$$\frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}\left(\frac{mR^2}{2}\right)\frac{v^2}{R^2} - m_2gh = -km_1gh \tag{1}$$

(as angular velocity $\omega = V/R$ for no slipping of thread).

But $\qquad v^2 = 2wh$

So using it in Eq. (1), we get

$$w = \frac{2(m_2 - km_1)g}{m + 2(m_1 + m_2)} \tag{2}$$

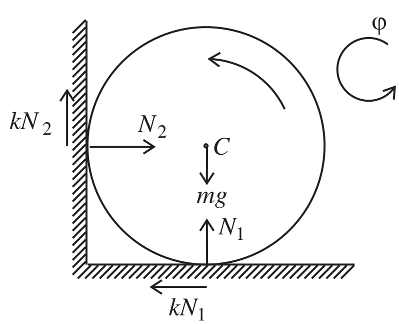
Thus the work done by the friction force on m_1

$$A_{fr} = -km_1gh = -km_1g\left(\frac{1}{2}wt^2\right)$$

$$= -\frac{km_1(m_2 - km_1)g^2t^2}{m + 2(m_1 + m_2)} \quad \text{(using Eq. 2)}$$

1.248 In the problem, the rigid body is in translation equilibrium but there is an angular retardation. We first sketch the free body diagram of the cylinder. Obviously the friction forces, acting on the cylinder, are kinetic. From the condition of translational equilibrium for the cylinder,

$$mg = N_1 + kN_2;\ N_2 = kN_1$$



Hence, $N_1 = \dfrac{mg}{1 + k^2};\ N_2 = k\dfrac{mg}{1 + k^2}$

For pure rotation of the cylinder about its rotation axis, $N_z = I\beta_z$

or $\qquad -kN_1R - kN_2R = \dfrac{mR^2}{2}\beta_z$

or $$-\frac{kmgR(1+k)}{1+k^2} = \frac{mR^2}{2}\beta_z$$

or $$\beta_z = -\frac{2k(1+k)g}{(1+k^2)R}$$

Now, from the kinematical equation,

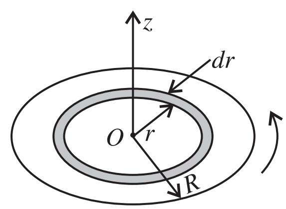
$$\omega^2 = \omega_0^2 + 2\beta_z \Delta\varphi$$

$$\text{we have, } \Delta\varphi = \frac{\omega_0^2(1+k^2)R}{4k(1+k)g} \text{ (because } \omega = 0)$$

Hence, the sought number of turns,

$$n = \frac{\Delta\varphi}{2\pi} = \frac{\omega_0^2(1+k^2)R}{8\pi k(1+k)g}$$

1.249 It is the moment of friction force which brings the disk to rest. The force of friction is applied to each section of the disk, and since these sections lie at different distances from the axis, the moments of the forces of friction differ from section to section.



To find N_z, where z is the axis of rotation of the disk let us partition the disk into thin rings (see figure). The force of friction acting on the considered element $dfr = k(2\pi r dr \sigma)g$, (where σ is the density of the disk).

The moment of this force of friction is

$$dN_z = -r dfr = -2\pi k \sigma g r^2 dr$$

Integrating with respect to r from zero to R, we get

$$N_z = -2\pi k\sigma g \int_0^R r^2 dr = -\frac{2}{3}\pi k\sigma g R^3$$

For the rotation of the disk about the stationary axis z, from the equation $N_z = I\beta_z$

$$-\frac{2}{3}\pi k\sigma g R^3 = \frac{(\pi R^2\sigma)R^2}{2}\beta_z \quad \text{or} \quad \beta_z = -\frac{4kg}{3R}$$

Thus, from the angular kinematical equation $\omega_z = \omega_{0z} + \beta_z t$

$$0 = \omega_0 + \left(-\frac{4kg}{3R}\right)t \quad \text{or} \quad t = \frac{3R\omega_0}{4kg}$$

1.250 According to the question,

$$I\frac{d\omega}{dt} = -k\sqrt{\omega} \quad \text{or} \quad I\frac{d\omega}{\sqrt{\omega}} = -kdt$$

Integrating, $\sqrt{\omega} = -\dfrac{kt}{2I} + \sqrt{\omega_0}$

or $$\omega = \frac{k^2t^2}{4I^2} - \frac{\sqrt{\omega_0}\, kt}{I} + \omega_0 \quad \text{(noting that at } t = 0,\ \omega = \omega_0\text{)}$$

Let the flywheel stop at $t = t_0$, then from Eq. (1), $t_0 = 2I\sqrt{\omega_0}/k$

Hence sought average angular velocity

$$\langle\omega\rangle = \frac{\displaystyle\int_0^{2I\sqrt{\omega_0}/k} \left(k^2t^2/4I^2 - \sqrt{\omega_0}\, kt/I + \omega_0\right) dt}{\displaystyle\int_0^{2I\sqrt{\omega_0}/k} dt} = \frac{\omega_0}{3}$$

1.251 Let us use the equation $dM_z/dt = N_z$, relative to the axis through O. (1)

For this purpose, let us find the angular momentum of the system M_z about the given rotation axis and the corresponding torque N_z. The angular momentum is

$$M_z = I\omega + mvR = \left(\frac{m_0}{2} + m\right)R^2\omega$$

where $I = \dfrac{m_0}{2}R^2$ and $v = \omega R$ (no cord slipping).

So, $$\frac{dM_z}{dt} = \left(\frac{MR^2}{2} + mR^2\right)\beta_z \tag{2}$$

The downward pull of gravity on the overhanging part is the only external force, which exerts a torque about the z-axis, passing through O and is given by,

$$N_z = \left(\frac{m}{l}\right)xgR$$

Hence, from the equation $\dfrac{dM_z}{dt} = N_z$

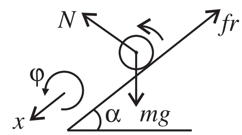
$$\left(\frac{MR^2}{2} + mR^2\right)\beta_z = \frac{m}{l}xgR$$

Thus, $$\beta_z = \frac{2mgx}{lR\,(M + 2m)} > 0$$

Note: *We may solve this problem using conservation of mechanical energy of the system (cylinder + thread) in the uniform field of gravity.*

1.252 (a) Let us indicate the forces acting on the sphere and their points of application. Choose positive direction of x and φ (rotation angle) along the incline in downward direction and in the sense of ω (for unidirectional rotation) respectively. Now from equations of dynamics of rigid body, i.e., $F_x = mw_{Cx}$ and $N_{Cz} = I_C \beta_z$ we get

$$mg \sin\alpha - fr = mw \qquad (1)$$



and $$frR = \frac{2}{5} mR^2 \beta \qquad (2)$$

But, $$fr \le kmg \cos\alpha \qquad (3)$$

In addition, the absence of slipping provides the kinematical relationship between the accelerations as.

$$w = \beta R \qquad (4)$$

The simultaneous solution of all the four equations yields

$$k \cos\alpha \ge \frac{2}{7} \sin\alpha \quad \text{or} \quad k \ge \frac{2}{7} \tan\alpha$$

(b) Solving Eqs. (1) and (2) [of part (a)], we get

$$w_C = \frac{5}{7} g \sin\alpha$$

As the sphere starts at $t = 0$ along positive x-axis, for pure rolling

$$v_C(t) = w_C t = \frac{5}{7} g \sin\alpha t \qquad (5)$$

Hence, the sought kinetic energy is

$$T = \frac{1}{2} mv_C^2 + \frac{1}{2}\frac{2}{5} mR^2\omega^2 = \frac{7}{10} mv_C^2 \quad (\text{as } \omega = v_C/R)$$

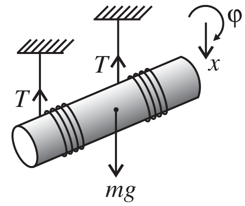
$$= \frac{7}{10} m\left(\frac{5}{7} g \sin\alpha t\right)^2 = \frac{5}{14} mg^2 \sin^2\alpha t^2$$

1.253 (a) Let us indicate the forces and their points of application for the cylinder. Choosing the positive direction for x and φ as shown in the figure, we write the equation of motion of the cylinder axis and the equation of moments in the C.M. frame relative to that axis, i.e., from equation $F_x = mw_C$ and $N_z = I_C\beta_z$ we get

$$mg - 2T = mw_C \quad \text{and} \quad 2RT = \frac{mR^2}{2}\beta$$

As there is no slipping of thread on the cylinder

$$w_C = \beta R$$



From these three equations

$$T = \frac{mg}{6} = 13 \text{ N}, \ \beta = \frac{2}{3}\frac{g}{R}$$

$$= 5 \times 10^2 \text{ rad/s}^2$$

(b) We have $\qquad \beta = \dfrac{2}{3}\dfrac{g}{R}$

So, $\qquad w_C = \dfrac{2}{3}\text{g} > 0 \quad$ or $\quad$ in vector from $\mathbf{w}_C = \dfrac{2}{3}\mathbf{g}$

$$P = \mathbf{F}\cdot\mathbf{v} = \mathbf{F}\cdot(\mathbf{w}_C t)$$

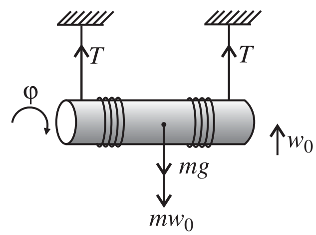
$$= m\mathbf{g}\cdot\left(\frac{2}{3}\mathbf{g}t\right) = \frac{2}{3}mg^2 t$$

1.254 Let us depict the forces and their points of application corresponding to the cylinder attached with the elevator. Newton's second law for solid in vector form in the frame of elevator, gives

$$2\mathbf{T} + m\mathbf{g} + m(-\mathbf{w}_0) = m\mathbf{w}' \qquad (1)$$

The equation of moment in the C.M. frame relative to the cylinder axis, i.e., from $N_z = I_C \beta_z$, is

$$2TR = \frac{mR^2}{2}\beta = \frac{mR^2}{2}\frac{w'}{R}$$



(as thread does not slip on the cylinder, $w' = \beta R$)

or $\qquad T = \dfrac{mw'}{4}$

From Eq. (1), $\mathbf{T} \uparrow\downarrow \mathbf{w}$, and so in vector form,

$$\mathbf{T} = -\frac{m\mathbf{w}}{4} \qquad (2)$$

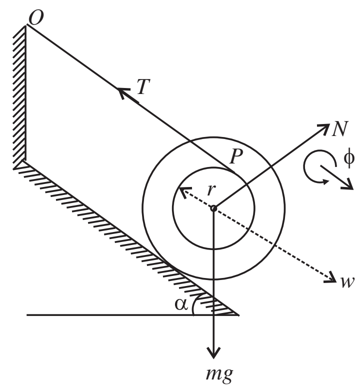
Solving Eqs. (1) and (2), we get

$$\mathbf{w} = \frac{2}{3}(\mathbf{g} - \mathbf{w}_0)$$

and sought force $\qquad \mathbf{F} = 2\mathbf{T} = \dfrac{1}{3}m(\mathbf{g} - \mathbf{w}_0)$

1.255 Let us depict the forces and their points of application for the spool. Choosing the positive direction for x and φ as shown in the figure, we apply $F_x = mw_x$ and $N_{Cz} = I_C\beta_z$ and get $mg\sin\alpha - \text{T} = mw$, $Tr = I\beta$.

"Notice that if a point of a solid in plane motion is connected with a thread, the projection of velocity vector of the solid's point of contact along the length of the thread equals the velocity of the other end of the thread (if it is not slacked)".



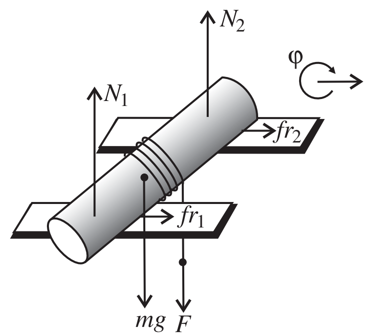
Thus in our problem, $v_P = v_0$ but $v_0 = 0$, hence point P is the instantaneous centre of rotation of zero velocity for the spool. Therefore $v_C = \omega r$ and subsequently $w_C = \beta r$.

Solving the equations simultaneously, we get

$$w = \frac{g \sin\alpha}{1 + I/mr^2} = 1.6 \text{ m/s}^2$$

1.256 Let us sketch the force diagram for solid cylinder and apply Newton's second law in projection form along x- and y-axes (see figure). Then,

$$fr_1 + fr_2 = mw_C \quad (1)$$



and $$N_1 + N_2 - mg - F = 0$$

or $$N_1 + N_2 = mg + F \quad (2)$$

Now choosing positive direction of φ as shown in the figure and using $N_{Cz} = I_C\beta_z$, we get

$$FR - (fr_1 + fr_2)\,R = \frac{mR^2}{2}\beta = \frac{mR^2}{2}\frac{w_C}{R} \quad (3)$$

(as for pure rolling $w_C = \beta R$).

In addition,

$$fr_1 + fr_2 \le k\,(N_1 + N_2) \quad (4)$$

Solving the equations, we get

$$F \le \frac{3k\,mg}{2 - 3k}$$

or $$F_{\max} = \frac{3k\,mg}{2 - 3k}$$

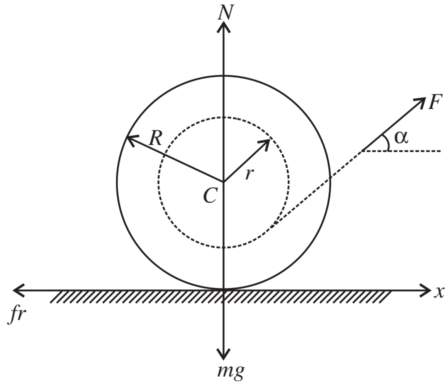
and $$w_{C(\max)} = \frac{k\,(N_1 + N_2)}{m}$$

$$= \frac{k}{m}[mg + F_{\max}] = \frac{k}{m}\left[mg + \frac{3kmg}{2 - 3k}\right] = \frac{2kg}{2 - 3k}$$

1.257 (a) Let us choose the positive direction of the rotation angle φ, such that w_{Cx} and β_z have identical signs (see figure). Equation of motion, $F_x = mw_{Cx}$ and $N_{Cz} = I_C\beta_z$ gives

$$F\cos\alpha - fr = mw_{Cx} \quad \text{and} \quad frR - Fr = I_C\beta_z = \gamma mR^2\beta_z$$

In the absence of the slipping of the spool, $w_{Cx} = \beta_z R$.



From the three equations we get

$$w_{Cx} = w_C = \frac{F\,[\cos\alpha - (r/R)]}{m(1+\gamma)}$$

(where $\cos\alpha > r/R$).

(b) As static friction (fr) does not work on the spool, from the equation of the increment of mechanical energy $A_{\text{ext}} = \Delta T$.

$$A_{\text{ext}} = \frac{1}{2}mv_C^2 + \frac{1}{2}\gamma\, mR^2\frac{v_C^2}{R^2} = \frac{1}{2}m\,(1+\gamma)v_C^2$$

$$= \frac{1}{2}m(1+\gamma)2w_C x = \frac{1}{2}m(1+\gamma)\,2w_C\left(\frac{1}{2}\,w_C t^2\right)$$

$$= \frac{F^2\left(\cos\alpha - r/R\right)^2 t^2}{2m(1+\gamma)}$$

Note: *At* $\cos\alpha = r/R$, *there is no rolling and for* $\cos\alpha < r/R$, $w_{cx} < 0$, *i.e., the spool will move towards negative x-axis and rotate in anticlockwise sense.*

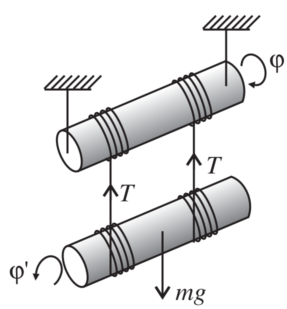
1.258 For the cylinder, from equation $Nz = Iz$ about its stationary axis of rotation,

$$2Tr = \frac{mr^2}{2}\beta \quad \text{or} \quad \beta = \frac{4T}{mr} \tag{1}$$

For the rotation of the lower cylinder, from equation $N_{Cz} = I_C\beta_z$,

$$2Tr = \frac{mr^2}{2}\beta'$$

or
$$\beta' = \frac{4T}{mr} = \beta$$



Now for the translational motion of lower cylinder, from equation $F_x = mw_{Cx}$,

$$mg - 2T = mw_C \tag{2}$$

As there is no slipping of threads on the cylinders,

$$w_C = \beta' r + \beta r = 2\beta r \tag{3}$$

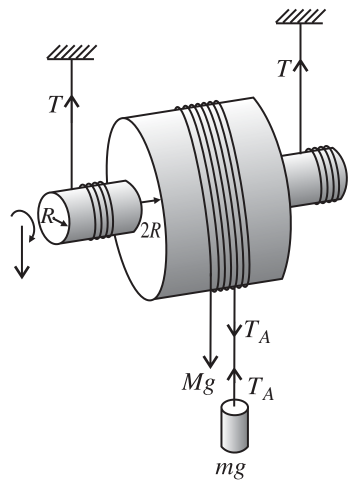
Simultaneous solution of Eqs. (1), (2) and (3) yields

$$T = \frac{mg}{10}$$

1.259 Let us depict the forces acting on the pulley and weight A, and indicate positive direction for x and φ as shown in the figure. For the cylinder from the equation, $F_x = m w_x$ and $N_{Cz} = I_C \beta_z$, we get

$$Mg + T_A - 2T = Mw_C \tag{1}$$

and $$2TR + T_A(2R) = I\beta = \frac{Iw_C}{R} \tag{2}$$



For the weight A from the equation

$$F_x = mw_x$$

We get, $$mg - T_A = mw_A \tag{3}$$

As there is no slipping of the threads on the pulleys,

$$w_A = w_C + 2\beta R = w_C + 2w_C = 3w_C \tag{4}$$

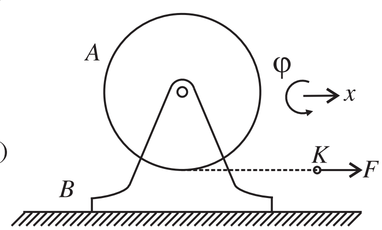
Simultaneous solution of above four equations gives

$$w_A = \frac{3\,(M + 3m)\,g}{M + 9m + 1/R^2}$$

1.260 (a) For the translational motion of the system $(m_1 + m_2)$, from the equation: $F_x = mw_{Cx}$, we get

$$F = (m_1 + m_2)\,w_C \quad \text{or} \quad w_C = F/(m_1 + m_2) \tag{1}$$

Now for the rotational motion of cylinder from the equation: $N_{Cx} = I_C\beta_z$, we get



$$Fr = \frac{m_1 r^2}{2}\beta \quad \text{or} \quad \beta r = \frac{2F}{m_1} \tag{2}$$

But, $$w_K = w_C + \beta r$$

So, $$w_K = \frac{F}{m_1 + m_2} + \frac{2F}{m_1} = \frac{F(3m_1 + 2m_2)}{m_1(m_1 + m_2)} \tag{3}$$

(b) From the equation of increment of mechanical energy: $\Delta T = A_{ext}$

Here, $$\Delta T = T(t) \quad \text{so,} \quad T(t) = A_{ext}$$

As force F is constant and is directed along x-axis the sought work done is given by

$$A_{\text{ext}} = Fx$$

(where x is the displacement of the point of application of the force F during time interval t) or

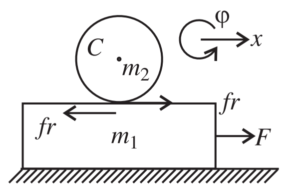
$$F\left(\frac{1}{2}w_K t^2\right) = \frac{F^2 t^2(3m_1 + 2m_2)}{2m_1(m_1 + m_2)} = T(t) \quad \text{(using Eq. 3)}$$

Alternate:

$$T(t) = T_{\text{translation}}(t) + T_{\text{rotation}}(t)$$

$$= \frac{1}{2}(m_1 + m_2)\left(\frac{Ft}{(m_1 + m_2)}\right)^2 + \frac{1}{2}\frac{m_1 r^2}{2}\left(\frac{2Ft}{m_1 r}\right)^2 = \frac{F^2 t^2(3m_1 + 2m_2)}{2m_1(m_1 + m_2)}$$

1.261 Choosing the positive direction for x and φ as shown in figure, let us we write the equation of motion for the sphere $F_x = mw_{Cx}$ and $N_{Cz} = I_C\beta_z$



$$fr = m_2 w_2;$$

$$(fr)r = \frac{2}{5}m_2 r^2 \beta$$

(where w_2 is the acceleration of the C.M. of sphere).

For the plank, from equation $F_x = mw_x$, we have

$$F - fr = m_1 w_1$$

In addition, the condition for the absence of slipping of the sphere yields the kinematic relation between the accelerations

$$w_1 = w_2 + \beta r$$

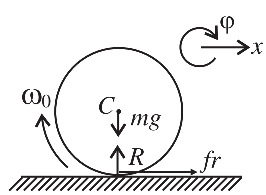
Simultaneous solution of the four equations yields

$$w_1 = \frac{F}{m_1 + 2/7m_2} \quad \text{and} \quad w_2 = \frac{2}{7}w_1$$

1.262 (a) Let us depict the forces acting on the cylinder and their point of applications for the cylinder and indicate positive direction of x and φ as shown in the figure. From the equations for the plane motion of a solid $F_x = mw_{Cx}$ and $N_{Cz} = I_C\beta_z$,

$$kmg = mw_{Cx} \quad \text{or} \quad w_{Cx} = kg \tag{1}$$

$$-kmg\,R = \frac{mR^2}{2}\beta_z \quad \text{or} \quad \beta_z = -2\frac{kg}{R} \tag{2}$$



Let the cylinder start pure rolling at $t = t_0$ after releasing on the horizontal floor at $t = 0$.

From the angular kinematical equation,

$$\omega_z = \omega_{oz} + \beta_z t,$$

or $$\omega = \omega_0 - 2\frac{kg}{R}t \tag{3}$$

From the equation of the linear kinematics,

$$v_{Cx} = v_{oCx} + w_{Cx}t$$

or $$v_C = 0 + kgt_0 \tag{4}$$

But at the moment $t = t_0$, when pure rolling starts $v_C = \omega R$.

So, $$kgt_0 = \left(\omega_0 - 2\frac{kg}{R}t_0\right)R$$

Thus, $$t_0 = \frac{\omega_0 R}{3\,kg}$$

(b) As the cylinder picks up speed till it starts rolling, the point of contact has purely translatory movement equal to $1/2\ w_C t_0^2$ in the forward direction but there is also a backward movement of the point of contact of magnitude $(\omega_0\tau_0 - 1/2\ \beta t_0^2)R$. Because of slipping the net displacement is backwards. The total work done is then,

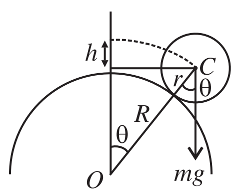
$$\begin{aligned}A_{fr} &= kmg\left[\frac{1}{2}w_C t_0^2 - (\omega_0 t_0 + \frac{1}{2}\beta t_0^2)R\right]\\ &= kmg\left[\frac{1}{2}kg\ t_0^2 - \frac{1}{2}\left(-\frac{2kg}{R}\right)t_0^2\ R - \omega_0 t_0 R\right]\\ &= kmg\frac{\omega_0 R}{3kg}\left[\frac{\omega_0 R}{6} + \frac{\omega_0 R}{3} - \omega_0 R\right] = -\frac{m\omega_0^2 R^2}{6}\end{aligned}$$

The same result can also be obtained by the work-energy theorem, $A_{fr} = \Delta T$.

1.263 Let us write the equation of motion for the centre of the sphere at the moment of breaking-off

$$\frac{mv^2}{R + r} = mg\cos\theta$$

where v is the velocity of the centre of the sphere at that moment, and θ is the corresponding angle (see figure). The velocity v can be found from the energy conservation law



$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

where I is the moment of inertia of the sphere relative to the axis passing through the sphere's centre. i.e. $I = 2/5\ mr^2$. In addition,

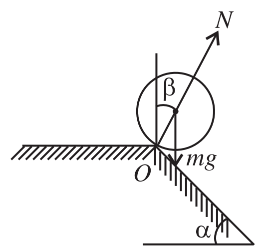
$$v = \omega r \quad \text{and} \quad h = (R + r)(1 + \cos\theta)$$

From these four equations we obtain

$$\omega = \sqrt{10g(R + r)/17r^2}$$

1.264 Since the cylinder moves without sliding, the centre of the cylinder rotates about the point O, while passing through the common edge of the planes. In other words, the point O becomes the foot of the instantaneous axis of rotation of the cylinder.

At any instant during this motion the velocity of the C.M. is v_1.
When the angle (as shown in the figure) is β, we have



$$\frac{mv_1^2}{R} = mg\cos\beta - N$$

(where N is the normal reaction of the edge)

or $$v_1^2 = gR\cos\beta - \frac{NR}{m} \tag{1}$$

From the energy conservation law,

$$\frac{1}{2}I_0\frac{v_1^2}{R^2} - \frac{1}{2}I_0\frac{v_0^2}{R^2} = mgR(1 - \cos\beta)$$

But, $$I_0 = \frac{mR^2}{2} + mR^2 = \frac{3}{2}mR^2$$

(from the parallel axis theorem).

Thus, $$v_1^2 = v_0^2 + \frac{4}{3}g\,R(1 - \cos\beta) \tag{2}$$

From Eqs. (1) and (2)

$$v_0^2 = \frac{gR}{3}(7\cos\beta - 4) - \frac{NR}{m}$$

The angle β in this equation is clearly smaller than or equal to α, so putting $\beta = \alpha$, we get

$$v_0^2 = \frac{gR}{3}(7\cos\alpha - 4) - \frac{N_0R}{M} \quad \text{(where } N_0 \text{ is the corresponding reaction.)}$$

Note that $N \geq N_0$. No jumping occurs during this turning if $N_0 > 0$. Hence, v_0 must be less than

$$v_{\max} = \sqrt{\frac{gR}{3}(7\cos\alpha - 4)}$$

$$= 1.0 \text{ m/s} \quad \text{(on substituting values)}$$

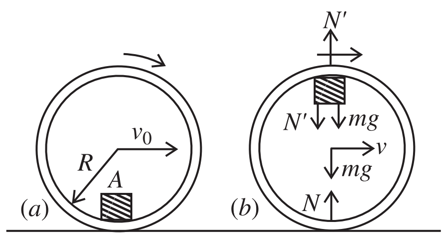
1.265 Clearly the tendency of bouncing of the hoop will be maximum when the small body A will be at the highest point of the hoop during its rolling motion. Let the velocity of C.M. of the hoop equal v at this position. The static friction does no work on the hoop, so from conservation of mechanical energy: $E_1 = E_2$

or $$0 + \frac{1}{2}mv_0^2 + \frac{1}{2}mR^2\left(\frac{v_0}{R}\right)^2 - mgR = \frac{1}{2}\, m\,(2v)^2 + \frac{1}{2}mv^2 + \frac{1}{2}mR^2\left(\frac{v}{R}\right)^2 + mgR$$

or $$3v^2 = v_0^2 - 2gR \tag{1}$$

From the equation $F_n = mw_n$ for body A at final position, as shown in Fig. (b):

$$mg + N' = m\omega^2 R = m\left(\frac{v}{R}\right)^2 R \tag{2}$$



As the hoop has no acceleration in vertical direction, so for the hoop,

$$N + N' = mg \tag{3}$$

From Eqs. (2) and (3),

$$N = 2mg - \frac{mv^2}{R} \tag{4}$$

As the hoop does not bounce,

$$N \geq 0 \tag{5}$$

So from Eqs. (1), (4) and (5),

$$\frac{8\,gR - v_0^2}{3R} \geq 0 \quad \text{or} \quad 8gR \geq v_0^2$$

Hence, $$v_0 \leq \sqrt{8\,gR}$$

1.266 Since the lower part of the belt is in contact with the rigid floor, velocity of this part becomes zero. The crawler moves with velocity v, hence the velocity of upper part of the belt becomes $2v$ by the rolling condition and kinetic energy of upper part is given by

$$\frac{1}{2}\left(\frac{m}{2}\right)(2v)^2 = mv^2$$

which is also the sought kinetic energy, assuming that the length of the belt is much larger than the radius of the wheels.

1.267 The sphere has two types of motion, one is the rotation about its own axis and the other is motion in a circle of radius R. Hence the sought kinetic energy is,

$$T = \frac{1}{2}I_1\,\omega_1^2 + \frac{1}{2}I_2\,\omega_2^2 \tag{1}$$

where I_1 is the moment of inertia about its own axis, and I_2 is the moment of inertia about the vertical axis, passing through O.

But, $I_1 = \frac{2}{5}mr^2$ and $I_2 = \frac{2}{5}mr^2 + mR^2$ (using parallel axis theorem) (2)

In addition

$$\omega_1 = \frac{v}{r} \quad \text{and} \quad \omega_2 = \frac{v}{R} \tag{3}$$

Using Eqs. (2) and (3) in Eq. (1), we get

$$T' = \frac{7}{10}mv^2\left(1 + \frac{2r^2}{7R^2}\right)$$

1.268 For a point mass of mass dm, looked at from C rotating frame, the equation is

$$dm\mathbf{w}' = \mathbf{f} + dm\,\omega^2\mathbf{r}' + 2dm(\mathbf{v}' \times \boldsymbol{\omega})$$

where $\mathbf{r}'$ = radius vector in the rotating frame with respect to rotation axis and $\mathbf{v}'$ = velocity in the same frame. The total centrifugal force is clearly

$$\mathbf{F}_{\text{cf}} = \sum dm\,\omega^2\mathbf{r}' = m\omega^2\mathbf{R}_C$$

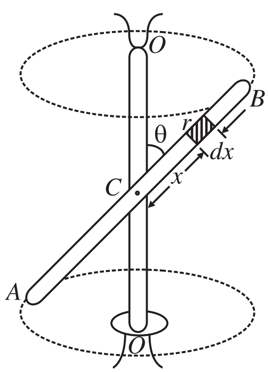
where $\mathbf{R}_C$ is the radius vector of the C.M. of the body with respect to rotation axis. Also

$$\mathbf{F}_{\text{cor}} = 2m\mathbf{v}'_C \times \boldsymbol{\omega}$$

where we have used the definitions,

$$m\mathbf{R}_C = \sum dm\mathbf{r}' \quad \text{and} \quad m\mathbf{v}'_C = \sum dm\mathbf{v}'.$$

1.269 Consider a small element of length dx at a distance x from the point C, which is rotating in a circle of radius $r = x\sin\theta$.



Now, mass of the element $= \left(\frac{m}{l}\right)dx$

So, centrifugal force acting on this element

$$= \left(\frac{m}{l}\right)dx\,\omega^2 x\sin\theta$$

and moment of this force about C is

$$|dN| = \left(\frac{m}{l}\right)dx\,\omega^2 x\sin\theta \cdot x\cos\theta$$

$$= \frac{m\omega^2}{2l}\sin 2\theta x^2 dx$$

and hence, total moment

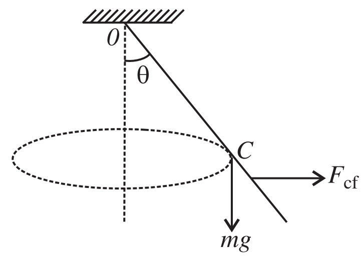
$$N = 2\int_0^{l/2}\frac{m\omega^2}{2l}\sin 2\theta x^2 dx = \frac{1}{24}m\omega^2 l^2 \sin 2\theta$$

1.270 Let us consider the system in a frame rotating with the rod. In this frame, the rod is at rest and experiences not only the gravitational force $m\mathbf{g}$ and the reaction force $\mathbf{R}$, but also the centrifugal $\mathbf{F}_{cf}$.

In the considered frame, from the condition of equilibrium, i.e., $N_{0z} = 0$

or $$N_{cf} = mg\frac{l}{2}\sin\theta \tag{1}$$

where N_{cf} is the moment of centrifugal force about O. To calculate N_{cf}, let us consider an element of length dx, situated at a distance x from the point O. This element is subjected to a horizontal pseudo force $(m/l)dx\omega^2 x\sin\theta$. The moment of this pseudo force about the axis of rotation through the point O is



$$dN_{cf} = \left(\frac{m}{l}\right)dx\omega^2\, x\sin\theta x\cos\theta$$

$$= \frac{m\omega^2}{l}\sin\theta\,\cos\theta x^2 dx$$

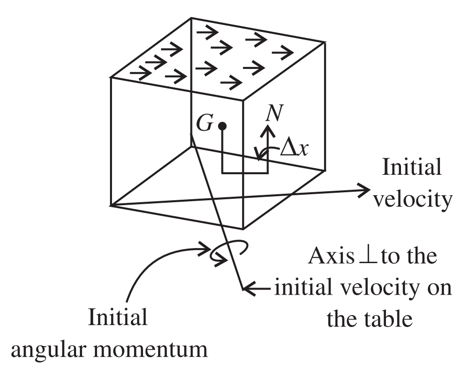
So, $$N_{cf} = \int_0^l \frac{m\omega^2}{l}\sin\theta\,\cos\theta x^2 dx$$

$$= \frac{m\omega^2 l^2}{3}\sin\theta\,\cos\theta \tag{2}$$

It follows from Eqs. (1) and (2) that,

$$\cos\theta = \left(\frac{3g}{2\omega^2 l}\right) \quad \text{or} \quad \theta = \cos^{-1}\left(\frac{3g}{2\omega^2 l}\right) \tag{3}$$

1.271 When the cube is given an initial velocity on the table in some direction (as shown in figure) it acquires an angular momentum about an axis on the table perpendicular to the initial velocity and (say) just below the centre of gravity (C.G.) This angular momentum will disappear when the cube stops and this can only be due to a torque. Frictional forces cannot do this by themselves because they act in the plane containing the axis.



But if the forces of normal reaction act eccentrically (as shown in figure), their torque can bring about the vanishing of the angular momentum. We can calculate the distance Δx between the point of application of the normal reaction and the C.G. of the cube as follows. Take the moment about C.G. of all the forces. This must vanish because the cube does not turn on the table.

Then if the force of friction is fr

$$fr\frac{a}{2} = N\Delta x$$

But $N = mg$ and $fr = kmg$, so

$$\Delta x = \frac{ka}{2}$$

1.272 As the rod is smooth, in the process of motion of the given system, the kinetic energy and the angular momentum relative to rotation axis do not vary. Hence, it follows that

$$\frac{1}{2}\frac{Ml^2}{3}\omega_0^2 = \frac{1}{2}\,m\,(\omega^2 l^2 + v'^2) + \frac{1}{2}\frac{Ml^2}{3}\omega^2$$

(where ω is the final angular velocity of the rod)

and $$\frac{Ml^2}{3}\omega_0 = \frac{Ml^2}{3}\omega + ml^2\omega$$

From these equations we obtain

$$\omega = \frac{\omega_0}{(1 + 3m/M)}$$

and $$v' = \frac{\omega_0 l}{\sqrt{1 + 3m/M}}$$

1.273 As surface is smooth, for further motion the frame attached with the C.M. is inertial. Due to hitting of the ball, the angular impulse received by the rod about the C.M. is equal to $pl/2$. If ω is the angular velocity acquired by the rod, we have

$$\frac{ml^2}{12}\omega = \frac{pl}{2} \quad \text{or} \quad \omega = \frac{6p}{ml} \tag{1}$$

In the frame of C.M., the rod is rotating about an axis passing through its mid point with the angular velocity ω. Hence the force exerted by one half on the other = mass of one half $\times$ acceleration of C.M. of that part, in the frame of C.M.

$$= \frac{m}{2}\left(\omega^2\frac{l}{4}\right) = m\frac{\omega^2 l}{8} = \frac{9p^2}{2ml} = 9\text{ N}$$

1.274 (a) In the process of motion of the given system the kinetic energy and the angular momentum relative to rotation axis do not vary. Hence it follows that

$$\frac{1}{2}mv^2 = \frac{1}{2}mv'^2 + \frac{1}{2}\left(\frac{Ml^2}{3}\right)\omega^3$$

and
$$mv\frac{l}{2} = mv'\frac{l}{2} + \frac{Ml^2}{3}\omega$$

From these equations we obtain

$$v' = \left(\frac{3m - 4m}{3m + 4M}\right)v \quad \text{and} \quad \omega = \frac{4v}{l\,(1 + 4m/3M)}$$

As $\mathbf{v}' \uparrow\uparrow \mathbf{v}$, so in vector form

$$\mathbf{v}' = \left(\frac{3m - 4M}{3m + 4M}\right)\mathbf{v}$$

(b) Obviously the sought force provides the centripetal acceleration to the C.M. of the rod and is

$$F_n = mw_{cn}$$
$$= M\omega^2\frac{l}{2} = \frac{8Mv^2}{l(1 + 4M/3m)^2}$$

1.275 (a) About the axis of rotation of the rod, the angular momentum of the system is conserved. Thus if the velocity of the flying bullet is v,

$$mvl = \left(ml^2 + \frac{Ml^2}{3}\right)\omega$$

$$\omega = \frac{mv}{m + (M/3)l} = \frac{3mv}{Ml} \quad (\text{as } m \ll M) \tag{1}$$

Now from the conservation of mechanical energy of the system (rod with bullet) in the uniform field of gravity,

$$\frac{1}{2}\left(ml^2 + \frac{Ml^2}{3}\right)\omega^2 = (M + m)\,g\frac{l}{2}(1 - \cos\alpha) \tag{2}$$

[because C.M. of rod raises by the height $\frac{l}{2}(1 - \cos\alpha)$].

Solving Eqs. (1) and (2), we get

$$v = \left(\frac{M}{m}\right)\sqrt{\frac{2}{3}gl}\,\sin\frac{\alpha}{2} \text{ and } \omega = \sqrt{\frac{6g}{l}}\,\sin\frac{\alpha}{2}$$

(b) Sought $$\Delta p = \left[m(\omega l) + M\left(\omega \frac{l}{2}\right)\right] - mv$$

where ωl is the velocity of the bullet and $\omega(l/2)$ equals the velocity of C.M. of the rod after the impact. Putting the value of v and ω, we get

$$\Delta p = \frac{1}{2} mv = M\sqrt{\frac{gl}{6}} \sin \frac{\alpha}{2}$$

This is caused by the reaction at the hinge on the upper end.

(c) Let the rod start swinging with angular velocity ω' in this case. Then, like in part (a),

$$mvx = \left(\frac{Ml^2}{3} + mx^2\right)\omega' \quad \text{or} \quad \omega' \approx \frac{3mvx}{Ml^2}$$

Final momentum is

$$p_f = mx\,\omega' + \int_0^t y\,\omega' \frac{M}{l}\, dy \approx \frac{M}{2}\,\omega' l \approx \frac{3}{2}\, mv \frac{x}{l}$$

So, $$\Delta p = p_f - p_i \approx mv\left(\frac{3x}{2l} - 1\right)$$

This vanishes for $x \approx 2/3l$.

1.276 (a) As force F on the body is radial so its angular momentum about the axis becomes zero and the angular momentum of system about the given axis is conserved. Thus

$$\frac{MR^2}{2}\omega_0 + m\omega_0 R^2 = \frac{MR^2}{2}\omega \quad \text{or} \quad \omega = \omega_0\left(1 + \frac{2m}{M}\right)$$

(b) From the equation of the increment of the mechanical energy of the system

$$\Delta T = A_{\text{ext}}$$

$$\frac{1}{2}\frac{MR^2}{2}\omega^2 - \frac{1}{2}\left(\frac{MR^2}{2} + mR^2\right)\omega_0^2 = A_{\text{ext}}$$

Putting the value of ω from part (a) and solving we get

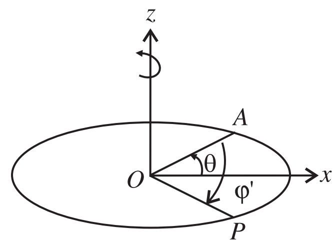
$$A_{\text{ext}} = \frac{m\omega_0^2 R^2}{2}\left(1 + \frac{2m}{M}\right)$$

1.277 (a) Among the external forces are, the reaction force by the support at the centre of the disk, weight of disk, and weight of the man, acting on the system (disk+man).

First two forces pass through the centre O of the disk, so no torque about O, that's why about the vertical axis of rotations OZ is also zero. The weight of the man is collinear to axis OZ, so no torque about this axis.

Let the man m_1 stand at point A (say) at the edge of the disk. Let Ox be a fixed line and such that at $t = 0$ the line joining O to A coincides with it. At time t, let $\angle AOx = \theta$ and $\angle AOP = \varphi'$ where P denotes the position of the man on the disk (relative to disk). The angle xOP increases at the rate $\dot{\varphi}' - \dot{\theta}$, therefore the velocity of the man at right angles to OP is $R(\dot{\varphi} - \dot{\theta})$ and its angular momentum about the rotation axis Oz of the disk is $mR^2(\dot{\varphi}' - \dot{\theta})$.

The disk has angular momentum about the Oz axis which becomes $(m_2R^2/2)\dot{\theta}$ in the opposite sense to conserve the angular momentum of the (disk + man) system, which is zero.



Therefore, $$\frac{m_2R^2}{2}\dot{\theta} - m_1R^2(\dot{\varphi}' - \dot{\theta}) = 0$$

$$m_1R^2\dot{\varphi}' = \left(\frac{m_2R^2}{2} + m_1R^2\right)\theta$$

or $$d\theta = \left[\frac{m_1}{m_1 + \left(\frac{m_2}{2}\right)}\right]d\varphi'$$

On integrating $$\int_0^\theta d\theta = -\int_0^{\varphi'}\left(\frac{m_1}{m_1 + \frac{m_2}{2}}\right)d\varphi'$$

or $$\theta = \left(\frac{m_1}{m_1 + m_2/2}\right)^{\varphi'} \tag{1}$$

This gives the total angle of rotation of the disk.

(b) From Eq. (1)

$$\frac{d\theta}{dt} = -\left(\frac{m_1}{m_1 + m_2/2}\right)\frac{d\varphi'}{dt} = -\left(\frac{m_1}{m_1 + m_2/2}\right)\frac{v'(t)}{R}$$

Differentiating with respect to time

$$\frac{d^2\theta}{dt^2} = -\left(\frac{m_1}{m_1 + m_2/2}\right)\frac{1}{R}\frac{dv'(t)}{dt}$$

Thus, the sought force moment from the equation $N_z = I\beta_z$ is

$$N_z = \frac{m_2R^2}{2}\frac{d^2\theta}{dt^2} = -\frac{m_2R^2}{2}\left(\frac{m_1}{m_1 + m_2/2}\right)\frac{1}{R}\frac{dv'(t)}{dt}$$

Hence, $$N_z = -\frac{m_1m_2R}{2m_1 + m_2}\frac{dv'(t)}{dt}$$

1.278 (a) From the law of conservation of angular momentum of the system relative to vertical axis z, it follows that

$$I_1\,\omega_{1z} + I_2\omega_{2z} = (I_1 + I_2)\omega_z$$

Hence, $$\omega_z = (I_1\omega_{1z} + I_2\omega_{2z})/(I_1 + I_2) \tag{1}$$

Note that for $\omega_z > 0$, the corresponding vector $\boldsymbol{\omega}$ coincides with the positive direction to the z-axis, and vice versa. As both disks rotate about the same vertical axis z, thus in vector form

$$\boldsymbol{\omega} = (I_1\boldsymbol{\omega}_1 + I_2\boldsymbol{\omega}_2)/(I_1 + I_2)$$

However, the problem makes sense only if $\boldsymbol{\omega}_1 \uparrow\uparrow \boldsymbol{\omega}_2$ or $\boldsymbol{\omega}_1 \uparrow\downarrow \boldsymbol{\omega}_2$.

(b) From the equation of increment of mechanical energy of a system: $A_{fr} = \Delta T$

$$A_{fr} = \frac{1}{2}(I_1 + I_2)\,\omega_z^2 - \frac{1}{2}I_1\,\omega_1^2 + \frac{1}{2}I_2\,\omega_{2z}^2$$

Using Eq. (1)

$$A_{fr} = \frac{I_1I_2}{2(I_1 + I_2)}(\omega_{1z} - \omega_{2z})^2$$

1.279 For the closed system (disk + rod), the angular momentum is conserved about any axis. Thus from the conservation of angular momentum of the system about the rotation axis of rod passing through its C.M. gives:

$$mv\frac{l}{2} = mv'\frac{l}{2} + \frac{\eta ml^2}{12}\omega \tag{1}$$

(where v' is the final velocity of the disk and ω angular velocity of the rod).

For the closed system linear momentum is also conserved. Hence,

$$mv = mv' + \eta\, mv_C \tag{2}$$

(where v_C is the velocity of C.M. of the rod).

From Eqs. (1) and (2) we get

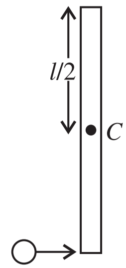
$$v_C = \frac{l\omega}{3} \quad \text{and} \quad v - v' = \eta v_C$$

Applying conservation of kinetic energy, as the collision is elastic

$$\frac{1}{2}mv^2 = \frac{1}{2}mv'^2 + \frac{1}{2}\eta m v_C^2 + \frac{1}{2}\frac{\eta m l^2}{12}\omega^2 \tag{3}$$

or $\quad v^2 - v'^2 = 4\eta v_C^2$ and hence $v + v' = 4v_C$

Then $\quad v' = \dfrac{4-\eta}{4+\eta}v$ and $\omega = \dfrac{12v}{(4+\eta)l}$



Vectorially, noting that we have taken $\mathbf{v}'$ parallel to $\mathbf{v}$

$$\mathbf{v}' = \left(\frac{4-\eta}{4+\eta}\right)\mathbf{v}$$

So, $\quad \mathbf{v}' = 0$ for $\eta = 4$ and $\mathbf{v}' \downarrow\uparrow \mathbf{v}$ for $\eta > 4$

Note: *Instead of Eq. 3 one can also write the equation of coefficient of restitution.*

1.280 See the diagram in the book (Fig. 1.72).

(a) When the shaft BB' is turned through 90°, the platform must start turning with angular velocity $\boldsymbol{\Omega}$ so that the angular momentum remains constant. Here,

$$(I + I_0)\boldsymbol{\Omega} = I_0\omega_0 \quad \text{or} \quad \boldsymbol{\Omega} = \frac{I_0\omega_0}{I + I_0}$$

The work performed by the motor is therefore,

$$\frac{1}{2}(I + I_0)\Omega^2 = \frac{1}{2}\frac{I_0^2\omega_0^2}{I + I_0}$$

If the shaft is turned through 180°, angular velocity of the sphere changes sign. Thus from conservation of angular momentum,

$$I\boldsymbol{\Omega} - I_0\omega_0 = I_0\omega_0$$

Here $-I_0\omega_0$ is the complete angular momentum of the sphere, i.e., we assume that the angular velocity of the sphere is just $-\omega_0$. Then

$$\boldsymbol{\Omega} = 2I_0\frac{\omega_0}{I}$$

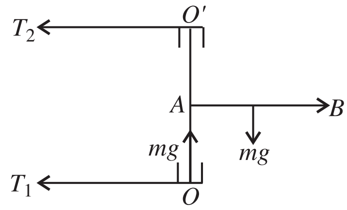
and the work done must be

$$\frac{1}{2}I\Omega^2 + \frac{1}{2}I_0\omega_0^2 - \frac{1}{2}I_0\omega_0^2 = \frac{2I_0^2\omega_0^2}{I}$$

(b) In the first part of (a), the angular momentum vector of the sphere is processing with angular velocity $\boldsymbol{\Omega}$. Thus torque needed is

$$I_0\omega_0\boldsymbol{\Omega} = \frac{I_0^2\omega_0^2}{I + I_0}$$

1.281 The total centrifugal force can be calculated by,



$$\int_0^{l_0} \frac{m}{l_0}\omega^2 x dx = \frac{1}{2} m l_0 \omega^2$$

Then for equilibrium,

$$(T_2 - T_1)\frac{l}{2} = mg\frac{l_0}{2}$$

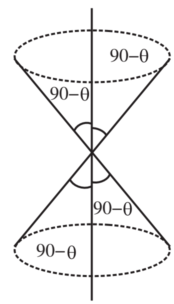
and $$T_2 + T_1 = \frac{1}{2} m l_0 \omega^2$$

Thus T_1 vanishes, when

$$\omega^2 = \frac{2g}{l}, \ \omega = \sqrt{\frac{2g}{t}} = 6 \text{ rad/s}$$

Then $$T_2 = mg\frac{l_0}{l} = 25 \text{ N}$$

1.282 See the diagram in the book (Fig. 1.72).



(a) The angular velocity $\boldsymbol{\omega}$ about OO' can be resolved into a component parallel to the rod and a component $\omega \sin\theta$ perpendicular to the rod through C. Then component parallel to the rod does not contribute to the angular momentum.

$$M = I\omega \sin\theta = \frac{1}{12} ml^2 \omega \sin\theta$$

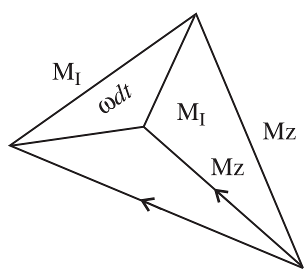
Also, $$M_z = M\sin\theta = \frac{1}{12} ml^2 \omega \sin^2\theta$$

This can be obtained directly also.

(b) The modulus of **M** does not change but the modulus of the change of **M** is

$$|\Delta\mathbf{M}| = 2M\sin(90° - \theta) = \frac{1}{12} ml^2 \omega \sin 2\theta$$

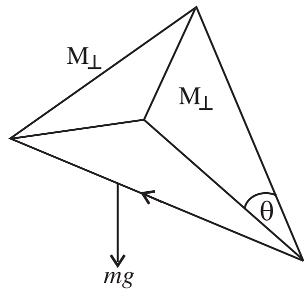
(c) Here, $$M_\perp = M\cos\theta = I\omega \sin\theta \cos\theta$$



Now, $$\left|\frac{d\mathbf{M}}{dt}\right| = I\omega\sin\theta\cos\theta \frac{\omega dt}{dt} = \frac{1}{24} ml^2\omega^2 \sin 2\theta$$

as **M** processes with angular velocity $\boldsymbol{\omega}$.

1.283 (a) Here $M = I\omega$ is along the symmetry axis. It has two components, the part $I\omega\cos\theta$ is constant and the part $M_\perp = I\omega\sin\theta$ precesses. Then,



$$\left|\frac{d\mathbf{M}}{dt}\right| = I\,\omega \sin\theta\ \omega' = mgl \sin\theta$$

or

$$\omega' = \text{precession frequency} = \frac{mgl}{I\omega} = 0.7 \text{ rad/s}$$

(b) This force is the centripetal force due to precession. It acts inward and has the magnitude

$$|\mathbf{F}| = \left|\sum m_i \omega'^2 \rho_i\right| = m\,\omega'^2\, l \sin\theta = 12 \text{ mN}$$

ρ_i is the distance of the ith element from the axis. This is the force that the table will exert on the top. See the diagram in the answer sheet.

1.284 See the diagram in the book (Fig. 1.73).

The moment of inertia of the disk about its symmetry axis is $1/2\ mR^2$. If the angular velocity of the disk is ω, then the angular moment is $1/2$ m $R^2\omega$. The precession frequency being $2\pi n$, we have

$$\left|\frac{d\mathbf{M}}{dt}\right| = \frac{1}{2}\, mR^2\,\omega \times 2\pi n$$

This must equal $m\,(g + w)\,l$, the effective gravitational torques (g being replaced by $g + w$ in the elevator). Thus,

$$\omega = \frac{(g + w)l}{\pi R^2 n} = 300 \text{ rad/s}$$

1.285 The effective g is $\sqrt{g^2 + w^2}$ inclined at angle $\tan^{-1}\ w/g$ with the vertical. Then with reference to the new "vertical" we proceed as in solution of problem 1.283. Thus,

$$\omega' = \frac{ml\sqrt{g^2 + w^2}}{I\omega} = 0.8 \text{ rad/s}$$

The vector $\boldsymbol{\omega}'$ forms an angle $\theta = \tan^{-1}\dfrac{\omega}{g} = 6°$ with the normal vertical.

1.286 The moment of inertia of the sphere is $2/5\ mR^2$ and hence the value of angular momentum is $2/5\ mR^2\,\omega$. Since it processes at speed ω' the torque required is

$$\frac{2}{5}\ mR^2\omega\omega' = F'l$$

So, $$F' = \frac{2}{5}\ mR^2\omega\omega'/l = 300\text{ N} \quad \text{(the force } F' \text{ must be vertical)}$$

1.287 The moment of inertia is $1/2\ mr^2$ and angular momentum is $1/2\ mr^2\omega$. The axle oscillates about a horizontal axis making an instantaneous angle

$$\varphi = \varphi_m \sin\frac{2\pi t}{T}$$

This means that there is a variable precession with a rate of precession $d\varphi/dt$. The maximum value of this is $2\pi\varphi_m/T$. When the angle between the axle and the axis is at its maximum value, a torque $I\omega\Omega$ acts on it which is equal to

$$\frac{1}{2}\ mr^2\omega\ \frac{2\pi\varphi}{T} = \frac{\pi\, mr^2\omega\,\varphi}{T}$$

The corresponding gyroscopic force will be

$$\frac{\pi\, mr^2\omega\,\varphi}{lT} = 90\text{ N}$$

1.288 The revolutions per minute of the flywheel being n, the angular momentum of the flywheel is $l \times 2\pi n$. The rate of precession is v/R.

Thus, $$N = 2\pi INv/R = 5.97\text{ kNm}$$

1.289 As in the previous problem, a couple $2\pi Inv/R$ must come in play. This can be done if a force, $2\pi Inv/Rl$ acts on the rails in opposite directions in addition to the centrifugal and other forces. The force on the outer rail is increased and that on the inner rail decreased. The additional force in this case has the magnitude 1.4 kN.

1.6 Elastic Deformations of a Solid Body

1.290 Variation of length with temperature is give by

$$l_t = l_0\,(1 + \alpha\Delta t) \quad \text{or} \quad \frac{\Delta l}{l_0} = \alpha\Delta t = \varepsilon \tag{1}$$

But, $\quad \varepsilon = \dfrac{\sigma}{E}$

Thus, $\sigma = \alpha\,\Delta tE$, which is the sought stress of pressure.

Putting the value of α and E from Appendix and taking $\Delta t = 100°\text{C}$, we get

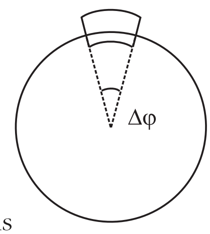
$$\sigma = 2.2 \times 10^3 \text{ atm}$$

1.291 (a) Consider a transverse section of the tube and concentrate on an element which subtends angle $\Delta\varphi$ at the centre. The forces acting on a portion of length Δl on the element are:

(1) Tensile forces side ways of magnitude $\sigma\Delta r\Delta l$.

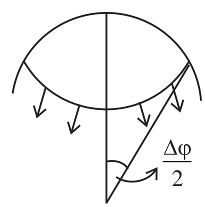
The resultant of these is $2\sigma\Delta r\,\Delta l \sin\Delta\varphi/2 \approx \sigma\Delta rl\,\Delta\varphi$ radically towards the centre.

(2) The force due to fluid pressure $= pr\,\Delta\varphi\Delta l$.
Since these balance, we get $p_{\max} \approx \sigma_m\,\Delta r/r$
where σ_m is the maximum tensile force.
Putting the values, we get $p_{\max} = 19.7$ atm.



(b) Consider an element of area $dS = \pi\,(r\,\Delta\theta/r)^2$ about z-axis chosen arbitrarily. There are tangential tensile forces all around the ring of the cap. Their resultant is

$$\sigma\left[2\pi\left(r\frac{\Delta\theta}{2}\right)\Delta r\right]\sin\frac{\Delta\theta}{2}$$



Hence, in the limit,

$$p_m\pi\left(\frac{r\Delta\theta}{2}\right)^2 = \sigma_m\pi\left(\frac{r\Delta\theta}{2}\right)\Delta r\,\Delta\theta$$

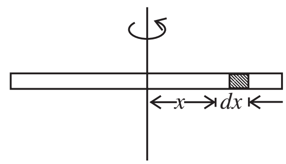
or $$p_m = \frac{2\sigma_m\Delta r}{r} = 39.5 \text{ atm}$$

1.292 Let us consider an element of rod at a distance x from its rotation axis (see figure). From Newton's second law in projection form directed towards the rotation axis,

$$-dT = (dm)\,\omega^2 x = \frac{m}{l}\,\omega^2 x\,dx$$

On integrating $$-T = \frac{m\omega^2}{l}\frac{x^2}{2} + C\,(\text{constant})$$

But at $$x = \pm\frac{l}{2} \quad \text{or} \quad \text{free end, } T = 0$$



Thus, $$0 = \frac{m\omega^2}{2}\frac{l^2}{4} + C \quad \text{or} \quad C = -\frac{m\omega^2 l}{8}$$

Hence $$T = \frac{m\omega^2}{2}\left(\frac{l}{4} - \frac{x^2}{l}\right)$$

Thus $$T_{\max} = \frac{m\omega^2 l}{8}\ \text{(at mid point)}$$

Condition required for the problem is

$$T_{\max} = S\sigma_m$$

So, $$\frac{m\omega^2 l}{8} = S\sigma_m \quad \text{or} \quad \omega = \frac{2}{l}\sqrt{\frac{2\sigma_m}{\rho}}$$

Hence the sought number of rps

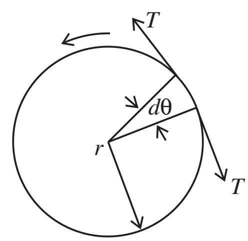
$$n = \frac{\omega}{2\pi} = \frac{1}{\pi l}\sqrt{\frac{2\sigma_m}{\rho}} \quad (\text{using the table } n = 0.8\times 10^3 \text{ rps})$$

1.293 Let us consider an element of the ring (see figure). From Newton's law $F_n = mw_n$ for this element, we get

$$Td\theta = \left(\frac{m}{2\pi}\,d\theta\right)\omega^2 r \quad (\text{see solution of problem 1.93 or 1.92})$$

So, $$T = \frac{m}{2\pi}\omega^2 r$$

Condition for the problem is



$$\frac{T}{\pi r^2} \le \sigma_m \quad \text{or} \quad \frac{m\omega^2 r}{2\pi^2 r^2} \le \sigma_m$$

or $$\omega^2_{\max} = \frac{2\pi^2\sigma_m r}{\pi r^2(2\pi r\rho)} = \frac{\sigma_m}{\rho r^2}$$

Thus, sought number of rps is

$$n = \frac{\omega_{\max}}{2\pi} = \frac{1}{2\pi r}\sqrt{\frac{\sigma_m}{\rho}}$$

Using the table of appendices $n = 23$ rps.

1.294 Let the point O descend by the distance x (see figure). From the condition of equilibrium of point O,

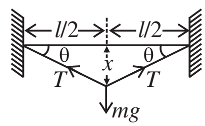
$$2T\sin\theta = mg \quad \text{or} \quad T = \frac{mg}{2\sin\theta} = \frac{mg}{2x}\sqrt{(l/2)^2 + x^2} \tag{1}$$

Now, $$\frac{T}{\pi(d/2)^2} = \sigma = \varepsilon E \quad \text{or} \quad T = \varepsilon E\pi\frac{d^2}{4} \tag{2}$$

(Here σ is stress and ε is strain.)

In addition to this,

$$\varepsilon = \frac{\sqrt{(l/2)^2 + x^2} - l/2}{l/2} = \sqrt{1 + \left(\frac{2x}{l}\right)^2} - 1 \tag{3}$$



From Eqs. (1), (2) and (3)

$$x - \frac{x}{\sqrt{1 + (2x/l)^2}} = \frac{mgl}{\pi E d^2}$$

or $$x = l\left(\frac{mg}{2\pi E d^2}\right)^{1/3} = 2.5 \text{ cm}$$

1.295 Let us consider an element of the rod at a distance x from the free end (see figure). For the considered element $T - T'$ are internal restoring forces which produce elongation and dT provide the acceleration to the element. For the element, from Newton's law

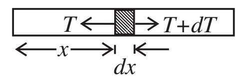
$$dT = (dm)\, w = \left(\frac{m}{l}dx\right)\frac{F_0}{m} = \frac{F_0}{l}\,dx$$

As free end has zero tension, on integrating the above expression,

$$\int_0^T dT = \frac{F_0}{l}\int_0^x dx \quad \text{or} \quad T = \frac{F_0}{l}\,x$$

Elongation in the considered element of length dx is given by

$$\partial\xi = \frac{\sigma}{E}(x)\, dx = \frac{T}{SE}\, dx = \frac{F_{xo}\, xdx}{SEl}$$



Thus, total elongation $\xi = \dfrac{F_0}{SEl}\displaystyle\int_x^l xdx = \dfrac{F_0 l}{2SE}$

Hence, the sought strain is

$$\sigma = \frac{\xi}{l} = \frac{F_0}{2SE}$$

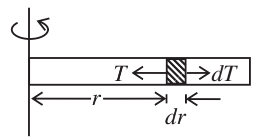
1.296 Let us consider an element of the rod at a distance r from its rotation axis. As the element rotates in a horizontal circle of radius r, we have from Newton's second law in projection form directed toward the axis of rotation

$$T - (T + dT) = (dm)\omega^2 r$$

or
$$-dT = \left(\frac{m}{l}\, dr\right)\omega^2 r = \frac{m}{l}\omega^2 r dr$$

At the free end tension becomes zero. Integrating the above expression we get

$$-\int_T^0 dT = \frac{m}{l}\omega^2 \int_r^l r dr$$



Thus,
$$T = \frac{m\omega^2}{l}\left(\frac{l^2 - r^2}{2}\right) = \frac{m\omega^2 l}{2}\left(1 - \frac{r^2}{l^2}\right)$$

Elongation in elemental length dr is given by

$$\partial\xi = \frac{\sigma(r)}{E}\, dr = \frac{T}{SE}\, dr$$

(where S is the cross sectional area of the rod and T is the tension in the rod at the considered element)

or
$$\partial\xi = \frac{m\omega^2 l}{2SE}\left(1 - \frac{r^2}{l^2}\right) dr$$

Thus, the sought elongation

$$\xi = \int d\xi = \frac{m\omega^2 l}{2SE}\int_0^l \left(1 - \frac{r^2}{l^2}\right) dr$$

or $$\xi = \frac{m\omega^2 l}{2SE}\frac{2l}{3} = \frac{(Sl\,\rho)}{3SE}\,\omega^2 l^3$$

$$= \frac{1}{3}\frac{\rho\omega^2 l^3}{E} \quad \text{(where } \rho \text{ is the density of the copper)}$$

1.297 Volume of a solid cylinder

$$V = \pi r^2 l$$

So, $$\frac{\Delta V}{V} = \frac{\pi 2r\Delta rl}{\pi r^2 l} + \frac{\pi r^2 \Delta l}{\pi r^2 l} = \frac{2\Delta r}{r} + \frac{\Delta l}{l} \tag{1}$$

But longitudinal strain $\Delta l/l$ and accompanying lateral strain $\Delta r/r$ are related as

$$\frac{\Delta r}{r} = -\mu\,\frac{\Delta l}{l} \tag{2}$$

Using Eq. (2) in Eq. (1), we get

$$\frac{\Delta V}{V} = \frac{\Delta l}{l}(1 - 2\mu) \tag{3}$$

But $$\frac{\Delta l}{l} = \frac{-F/\pi r^2}{E}$$

(Because the increment in the length of cylinder Δl is negative.)

So, $$\frac{\Delta V}{V} = \frac{-F}{\pi r^2 E}(1 - 2\mu) \quad \text{(where } \mu \text{ is the Poisson's ratio for copper)}$$

Thus, $$\Delta V = \frac{-Fl}{E}(1 - 2\mu)$$

$$= -1.6 \text{ mm}^3$$

(Negative sign means that the volume of the cylinder has decreased.)

1.298 (a) As free end has zero tension, thus the tension in the rod at a vertical distance y from its lower end

$$T = \frac{m}{l}\,gy \tag{1}$$

Let ∂l be the elongation of the element of length dy, then

$$\partial l = \frac{\sigma(y)}{E}\,dy$$

$$= \frac{T}{SE}\,dy = \frac{mgydy}{SlE} = \frac{\rho gydy}{E} \quad \text{(where } \rho \text{ is the density of copper)}$$

Thus, the sought elongation is

$$\Delta l = \int \partial l = \rho g \int_0^l \frac{y dy}{E} = \frac{1}{2}\rho g l^2 / E \tag{2}$$

(b) If the longitudinal (tensile) strain is $\varepsilon = \Delta l / l$, the accompanying lateral (compressive) strain is given by

$$\varepsilon' = \frac{\Delta r}{r} = -\mu \varepsilon \tag{3}$$

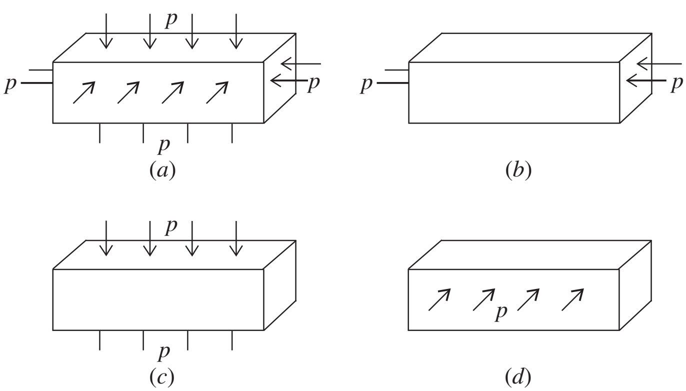
Then, since $V = \pi r^2 l$, we have

$$\frac{\Delta V}{V} = \frac{2\Delta r}{r} + \frac{\Delta l}{l}$$

$$= (1 - 2\mu)\frac{\Delta l}{l} \quad \text{(using Eq. 3)}$$

(where $\Delta l/l$ is given in part (a) and μ is the Poisson ratio for copper).

1.299 (a) Since the hydrostatic pressure is uniform, the stress on each face of the bar is the same (see figure (a)). The change in length of the bar can be thought of as the sum of changes in length that would occur in the three independent cases shown in the figures (b), (c) and (d).



If we push on the ends of the block with a pressure p, the compressional strain is p/E, and it is negative,

$$\frac{\Delta l_1}{l} = -\frac{p}{E}$$

If we push on the two sides of the block with pressure p, the compressional st-rain is again p/Y, but now we want the lengthwise strain. We can get that from the sideways strain multiplied by $-\mu$. The sideways strain is

$$\frac{\Delta w}{w} = -\frac{p}{E}$$

So,
$$\frac{\Delta l_2}{l} = +\mu\frac{p}{E}$$

If the we push on the top of the block, the compressional strain in once more p/E, and the corresponding strain in the sideways direction is again $-\mu p/E$. We get

$$\frac{\Delta l_3}{l} = +\mu\frac{p}{E}$$

Combining the results of the three problems i.e., taking $\Delta l = \Delta l_1 + \Delta l_2 + \Delta l_3$, we get

$$\frac{\Delta l}{l} = -\frac{p}{E}(1-2\mu) \qquad (1)$$

The problem is, of course, symmetrical in all three directions; it follows that

$$\frac{\Delta w}{w} = \frac{\Delta h}{h} = -\frac{p}{E}(1-2\mu) \qquad (2)$$

The change in the volume under hydrostatic pressure is also of some interest. Since $V = lwh$, we can write, for small displacements.

$$\frac{\Delta V}{V} = \frac{\Delta l}{l} + \frac{\Delta w}{w} + \frac{\Delta h}{h}$$

Using Eqs. (1) and (2), we have

$$\frac{\Delta V}{V} = -3\,\frac{p}{E}(1-2\mu) \qquad (3)$$

(b) But volume strain

$$\frac{\Delta V}{V} = -\frac{p}{K} \quad \text{(where } K \text{ is called the bulk modulus)} \qquad (4)$$

From Eqs. (3) and (4), we have

$$K = \frac{E}{3\,(1-2\mu)}$$

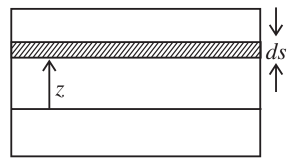
Hence the compressibility

$$\beta = \frac{1}{K} = \frac{3\,(1-2\mu)}{E}$$

1.300 A beam clamped at one end and supporting an applied load at the free end is called a cantilever. The theory of cantilevers is discussed in the advanced text book on mechanics. The key result is that elastic forces in the beam generate a couple, whose moment, called the moment of resistances, balances the external bending moment due to weight of the beam, load etc. The moment of resistance, also called internal bending moment (I.B.M) is given by

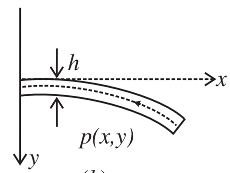
$$\text{I.B.M.} = EI/R$$

Here R is the radius of curvature of the beam at the representative point (x, y). I is called the geometrical moment of inertia of the cross section relative to the axis passing through the natural layer which remains unscratched Fig. (a).



(*a*)

$$I = \int z^2 \, dS \qquad (1)$$



(*b*)

The section of the beam beyond P exerts the bending moment $N(x)$ and we have,

$$\frac{EI}{R} = N(x) \qquad (2)$$

If there is no load other than that due to the weight of the beam, then

$$N(x) = \frac{1}{2}\rho g(l - x)^2 \, bh$$

(where ρ = density of steel).

Hence, at $x = 0$

$$\left(\frac{I}{R}\right)_0 = \frac{\rho g l^2 bh}{2EI}$$

Here, b = width of the beam perpendicular to paper.

Also, $$I = \int_{-h/2}^{h/2} z^2 \, b dz = \frac{bh^3}{12}$$

Hence, $$\left(\frac{I}{R}\right)_0 = \frac{6\rho g l^2}{Eh^2}$$

$$R = \frac{1}{6}\frac{Eh^2}{\rho g l^2} = 0.121 \text{ km}$$

1.301 We use the equation given above and use the result that when y is small

$$\frac{1}{R} = \frac{d^2y}{dx^2}; \quad \text{thus,} \quad \frac{d^2y}{dx^2} = \frac{N(x)}{EI}$$

(a) Here $N(x) = N_0$ is a constant. Then integration gives

$$\frac{dy}{dx} = \frac{N_0 x}{EI} + C_1$$

But, $\left(\frac{dy}{dx}\right) = 0, \quad \text{for} \quad x = 0, \quad \text{so} \quad C_1 = 0.$

Integrating again, $y = \frac{N_0 x^2}{2EI}$

where we have used $y = 0$ for $x = 0$ to set the constant of integration at zero. This is the equation of a parabola. The sag of the free end is

$$\lambda = y\,(x = l) = \frac{N_0 l^2}{2EI}$$

(b) In this case $N(x) = F(l - x)$ because the load F at the extremity is balanced by a similar force at F directed upward and they constitute a couple. Then

$$\frac{d^2y}{dx^2} = \frac{F(l - x)}{EI}$$

Integrating, $\frac{dy}{dx} = \frac{F(lx - x^2/2)}{El} + C_1$

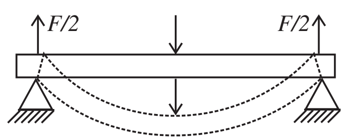
As before $C_1 = 0$. Integrating again, using $y = 0$ for $x = 0$,

$$y = \frac{F\,(lx^2/2 - x^3/6)}{EI} \quad \left(\text{here } \lambda = \frac{Fl^3}{3EI}\right)$$

Here for a square cross section

$$I = \int_{-a/2}^{a/2} z^2\, a dz = \frac{a^4}{12}$$

1.302 One can think of it as analogous to the previous case but with a beam of length $l/2$ loaded upward by a force $F/2$.



Thus, $\lambda = \frac{Fl^3}{48\ EI}$

on using the last result of the previous problem.

1.303 (a) In this case $N(x) = \frac{1}{2}\rho g b h\,(l - x)^2$ (where b = width of the girder).

Also $$I = \frac{bh^3}{12}$$

Then, $$\frac{Ebh^2}{12}\frac{d^2y}{dx^2} = \frac{\rho g b h}{2}(l^2 - 2lx + x^2)$$

Integrating, $$\frac{dy}{dx} = \frac{6\rho g}{Eh^2}\left(l^2x - lx^2 + \frac{x^3}{3}\right)$$

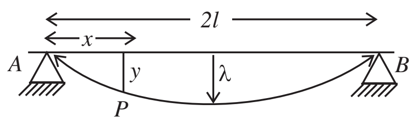
Using $$\frac{dy}{dx} = 0 \text{ for } x = 0$$

Integrating again, $$y = \frac{6\rho g}{Eh^2}\left(\frac{l^2x^2}{2} - \frac{lx^3}{3} + \frac{x^4}{12}\right)$$

Thus, $$\lambda = \frac{6\rho g l^4}{Eh^2}\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{12}\right) = \frac{6\rho g l^4}{Eh^2}\frac{3}{12} = \frac{3\rho g l^4}{2Eh^2}$$

(b) As before, $EI\dfrac{d^2y}{dx^2} = N(x)$ where $N(x)$ is the bending moment due to section PB.

Thus, bending moment is clearly



$$N = \int_x^{2l} w\,d\xi(\xi - x) - wl(2l - x)$$

$$= w\left(2l^2 - 2xl + \frac{x^2}{2}\right) - wl\,(2l - x)$$

$$= w\left(\frac{x^2}{2} - xl\right)$$ (here $w = \rho\, gb\, h$ is weight of the beam per unit length)

Now integrating, $$El\frac{dy}{dx} = w\left(\frac{x^3}{6} - \frac{x^2 l}{2}\right) + c_0$$

or since $$\frac{dy}{dx} = 0 \text{ for } x = l,\ c_0 = \frac{wl^3}{3}$$

Integrating again, $$EI_y = w\left(\frac{x^4}{24} - \frac{x^3 l}{6}\right) + \frac{wl^3 x}{3} + c_1$$

As $y = 0$ for $x = 0$, $c_1 = 0$. From this we find

$$\lambda = y\,(x = l) = \frac{5wl^4/24}{EI} = \frac{5\rho g l^4}{2Eh^2}$$

1.304 The deflection of the plate can be noticed by going to a co-rotating frame. In this frame each element of the plate experiences a pseudo force proportional to its mass. These forces have a moment which constitutes the bending moment of the problem. To calculate this moment we note that the acceleration of an element at a distance ξ from the axis is $a = \xi\beta$ and the moment of the forces exerted by the section between x and l is

$$N = \rho l h \beta \int_l^x \xi^2 d\xi = \frac{1}{3} \rho l h \, \beta \, (l^3 - x^3)$$

From the fundamental equation

$$EI \frac{d^2 y}{dx^2} = \frac{1}{3} \rho l h \, \beta (l^3 - x^3)$$

The moment of inertia
$$I = \int_{-h/2}^{+h/2} z^2 I dz = \frac{lh^3}{12}$$

Note that the neutral surface (i.e. the surface which contains lines which are neither stretched nor compressed) is a vertical plane here and **z** is perpendicular to it.

$$\frac{d^2 y}{dx^2} = \frac{4\rho\beta}{Eh^2} (l^3 - x^3)$$

Integrating
$$\frac{dy}{dx} = \frac{4\rho\beta}{Eh^2} \left(l^3 x - \frac{x^4}{4} \right) + c_1$$

Since $\frac{dy}{dx} = 0$, for $x = 0$, $c_1 = 0$. Integrating again,

$$y = \frac{4\rho\beta}{Eh^2} \left(\frac{l^3 x^2}{2} - \frac{x^5}{20} \right) + c_2$$

$$c_2 = 0 \quad (\text{because } y = 0 \text{ for } x = 0)$$

Thus,
$$\lambda = y\,(x = l) = \frac{9\rho\beta l^5}{5Eh^2}$$

1.305 (a) Consider a hollow cylinder of length l, outer radius $r + \Delta r$, inner radius r, fixed at one end and twisted at the other by means of a couple of moment N. The angular displacement φ, at a distance l from the fixed end, is proportional to both l and N. Consider an element of length dx at the twisted end. It is moved by an angle φ as shown. A vertical section is also shown and the twisting of the parallelopipe of length l and area $\Delta r dx$ under the action of the twisting couple can be discussed by elementary means. If f is the tangential force generated, then shearing stress is $f/\Delta r\, dx$ and this must equal

$$G\theta = G\frac{r\varphi}{l}, \quad \text{since } \theta = \frac{r\varphi}{l}$$

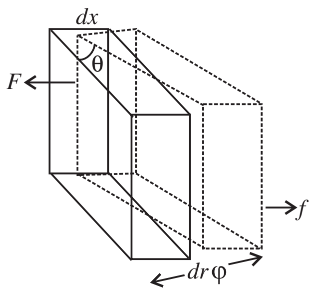
Hence,
$$f = G\Delta r\, dx\frac{r\varphi}{l}$$

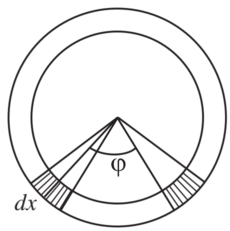
The force f has moment fr about the axis and so the total moment is

$$N = G\Delta r\frac{\varphi}{l}\, r^2 \int dx = \frac{2\pi r^3 \Delta r\varphi}{l}\, G$$

(b) For a solid cylinder we must integrate over r. Thus

$$N = \int_0^r \frac{2\pi r^3\, dr\varphi\, G}{l} = \frac{\pi r^4\, G\varphi}{2l}$$





1.306 Clearly $N = \displaystyle\int_{d_1/2}^{d_2/2} \frac{2\pi r^3\, dr\varphi G}{l} = \frac{\pi}{32l}\, G\varphi(d_2^4 - d_1^4)$

Using $G = 81 \text{ GPa} = 8.1 \times 10^{10} \text{ Nm}^{-2}$

$$d_2 = 5 \times 10^{-2} \text{ m}, \; d_1 = 3 \times 10^{-2} \text{ m}$$

$$\varphi = 2.0° = \frac{\pi}{90} \text{ radians}, \; l = 3\,\text{m}$$

$$N = \frac{\pi \times 8.1 \times \pi}{32 \times 3 \times 90}(625 - 81) \times 10^2 \text{ Nm}$$

$$= 0.5033 \times 10^3 \text{ N}\cdot\text{m} \approx 0.5 \text{ kNm}$$

1.307 The maximum power that can be transmitted by means of a shaft rotating about its axis is clearly $N\omega$ where N is moment of the couple producing the maximum permissible torsion, φ. Thus,

$$P = \frac{\pi r^4 G\varphi}{2l} \cdot \omega = 16.9 \text{ kW}$$

1.308 Consider an elementary ring of width dr at a distant r from the axis. The part outside exerts couple $N + \frac{dN}{dr}\, dr$ on this ring while the part inside exerts a couple N in the opposite direction. We have for equilibrium

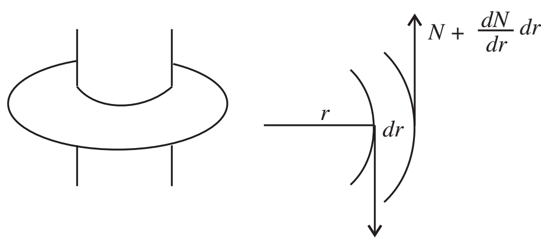
$$\frac{dN}{dr}\, dr = -dI\beta$$

where dI is the moment of inertia of the elementary ring, β is the angular acceleration and minus sign is needed because the couple $N(r)$ decreases, with distance, vanishing at the outer radius, $N(r_2) = 0$. Now

$$dI = \frac{m}{\pi\ (r_2^2 - r_1^2)}\, 2\pi r\, dr\, r^2$$

Thus, $$dN = \frac{2m\beta}{r_2^2 - r_1^2}\, r^3\, dr$$

On integration, $$N = \frac{1}{2}\frac{m\beta}{(r_2^2 - r_1^2)}\,(r_2^4 - r_1^4)$$



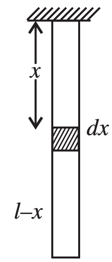
1.309 We assume that the deformation is wholly due to external load, neglecting the effect of the weight of the rod (see next problem). Then a well known formula says, elastic energy per unit volume

$$= \frac{1}{2} \text{ stress} \times \text{strain} = \frac{1}{2}\sigma\varepsilon$$

This gives $\frac{1}{2}\frac{m}{\rho} E\varepsilon^2 \approx 0.04$ kJ for the total deformation energy.

1.310 When a rod is deformed by its own weight the stress increases as one moves up, the stretching force being the weight of the portion below the element considered.

The stress on the element dx is



$$\rho\pi r^2\,(l - x)\,g/\pi r^2 = \rho g(l - x)$$

The extension of the element is

$$\Delta dx = d\Delta x = \rho g(l - x)\,dx/E$$

Integrating we get the extension of the whole rod as

$$\Delta l = \frac{1}{2}\frac{\rho g l^2}{E}$$

The elastic energy of the element is

$$\frac{1}{2}\rho g(l - x)\,\frac{\rho g(\,l - x)}{E}\,\pi r^2 dx$$

Integrating

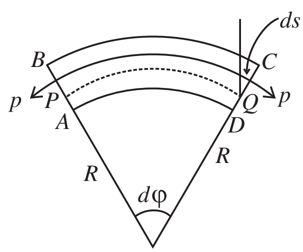
$$\Delta U = \frac{1}{6}\frac{\pi r^2 \rho^2 g^2 l^3}{E} = \frac{2}{3}\pi r^2 l E\left(\frac{\Delta l}{l}\right)^2$$

1.311 The work done to make a loop out of a steel band appears as the elastic energy of the loop and may be calculated from the same.

If the length of the band is l, the radius of the loop $R = l/2\pi$. Now consider an element $ABCD$ of the loop. The elastic energy of this element can be calculated by the same sort of arguments as used to derive the formula for internal bending moment. Consider a fiber at a distance z from the neutral surface PQ. This fiber experiences a force p and undergoes an extension ds where $d = Zd\varphi$, while $PQ = s = Rd\varphi$. Thus strain $ds/s = Z/R\cdot$

If α is the cross sectional area of the fiber, the elastic energy associated with it is

$$\frac{1}{2}\,E\left(\frac{Z}{R}\right)^2 Rd\varphi\alpha$$



Summing over all the fibres we get

$$\frac{EI\varphi}{2R}\sum \alpha Z^2 = \frac{E\,ld\varphi}{2R}$$

For the whole loop this gives, using $\int d\varphi = 2\pi$

$$\frac{El\,\pi}{R} = \frac{2EI\,\pi^2}{l}$$

Now, $$I = \int_{-\delta/2}^{\delta/2} z^2 h \, dZ = \frac{h\delta^3}{12}$$

So the energy is

$$\frac{1}{6}\frac{\pi^2 E h \delta^3}{l} = 0.08 \text{ kJ}$$

Alternate:

Suppose that the steel band was made into a hoop of radius R, then length of the hoop $l = 2\pi R$.

Consider an infinitesimally thin section of radius ρ and thickness $d\rho$ in the hoop. The length of this section of the hoop is $2\pi\rho$. Hence the longitudinal strain corresponding to this section is

$$\varepsilon = \frac{2\pi\rho - 2\rho R}{2\pi R} = \frac{\rho}{R} - 1$$

So elastic energy density is

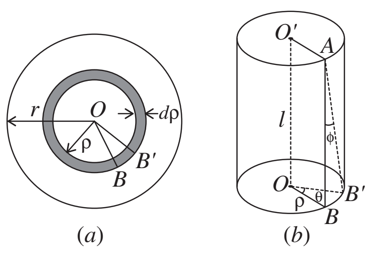
$$u = \frac{1}{2}E\varepsilon^2 = \frac{1}{2}E\left(\frac{\rho}{R} - 1\right)^2$$

Thus the sought work done is nothing but elastic energy stored.

$$U = \int u dV = \int_{R-\frac{\delta}{2}}^{R+\frac{\delta}{2}} \frac{1}{2}E\left(\frac{x}{R} - 1\right)^2 2\pi\rho d\rho \; h = \frac{1}{l}\frac{\pi^2 E h\delta^3}{} \quad \text{(on integrating)}$$

1.312 A rod should be treated as a solid cylinder. Let the upper end of the cylindrical rod be fixed and let a couple be applied to the lower end in a plane perpendicular to its length (with its axis coinciding with that of the cylinder) twisting it through an angle θ (radians). This is an example of "pure shear". In equilibrium position the twisting couple is equal and opposite to the restoring couple. Let us calculate the value of this couple.

Imagine the cylinder to consist of a large number of co-axial, hollow cylinders, and consider one such hollow cylinder of radius v, and radial thickness $d\rho$ Fig. (a). Each radius of the lower end is turned through the same angle θ, but the displacement is the greatest at the rim, decreasing as the centre is approached, where it is reduced to zero.



(a) (b)

Let AB Fig. (b) be a line, parallel to the axis, before the cylinder is twisted. On twisting, since the point B shifts to B', the line AB takes up the position AB'. The angle through which this hollow cylinder is sheared is, therefore, $BAB' = \phi$, say. Then clearly,

$$BB' = l\phi \quad \text{(see Figs. a and b)}$$

Also $BB' = \rho\theta$, so, angle of shear or shear strain

$$\phi = \frac{\rho\theta}{l}$$

Obviously, ϕ will have the maximum value where ρ is the greatest, i.e., the maximum strain is on the outermost part of the cylinder, and the least, on the innermost. In other words, the shearing stress is not uniform all through.

Thus, although the angle of shear is the same for any one hollow cylinder, it is different for different cylinders, being the greatest for the outermost and the least for the innermost one.

Since $$G = \frac{\text{shearing stress}}{\text{strain or angle of shear}} = \frac{F}{\phi}$$

We have $$F = G\phi = \frac{G\rho\theta}{l}$$

Now, face area of this hollow cylinder $= 2\pi\rho\, d\rho$.

Therefore total shearing force this on area $= 2\pi\rho\, d\rho \times \dfrac{G\rho\theta}{t} = 2\pi G\dfrac{\theta}{l}\rho^2 d\rho$

Therefore, moment of this force about the axis OO' Fig. (b) of the cylinder is equal to

$$\frac{2\pi G\,\theta\rho^2 d\rho\,\rho}{l} = \frac{2\pi G\,\theta\rho^3 d\rho}{l}$$

Integrating this expression between the limits, $\rho = 0$ and $\rho = r$, we have total twisting couple on the cylinder,

$$N(\theta) = \int_0^r 2\pi G\, 2\pi\, G\frac{\theta}{l}\rho^3 d\rho$$

$$= \frac{2\pi G\theta}{l}\int_0^r \rho^3 d\rho = \frac{2\pi G\theta}{l}\left[\frac{\rho^4}{4}\right]_0^r = \frac{2\pi G\theta r^4}{2l} = \frac{\pi G\theta r^4}{2l} = C\theta$$

where $C = \pi G r^4/2l$ is twisting couple per unit twist of the cylinder or torsional rigidity of the cylinder.

Thus, when the rod is twisted through an angle θ, a couple

$$N(\theta) = \frac{\pi r^4 G}{2l}\theta$$

appears to resist this. Work done in twisting the rod by an angle φ or elastic energy stored is then

$$U = \int_0^{\varphi} N(\theta)\, d(\theta) = \frac{\pi r^4 G}{4l}\varphi^2 = 7\text{ J (on substituting values)}$$

1.313 The energy between radii r and $r + dr$ is given by differentiation of

$$U = \frac{\pi r^4 G}{4l}\varphi^2 \text{ (see solution of problem 1.312)}$$

So, $$dU = \frac{\pi r^3 dr}{l} G\varphi^2$$

Its density is $$u = \frac{dU}{dV} = \frac{\pi r^3 dr}{2\pi r dr\, l}\frac{G\varphi^2}{l} = \frac{1}{2}\frac{G\varphi^2 r^2}{l^2}$$

Alternate:

The sought energy density

$$u = \frac{1}{2}G\phi^2 \text{ (where } \phi \text{ is angle of shear or shear strain)}$$

But, $$\phi = \left(r\frac{\varphi}{l}\right) \text{ (see solution of problem 1.312)}$$

So, $$u = \frac{1}{2}\frac{G\varphi^2 r^2}{l^2}$$

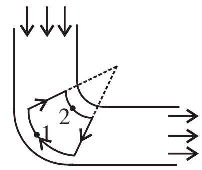
1.314 The energy density is as usual 1/2 stress $\times$ stress. Stress is the pressure $\rho g h$. Strain is $\beta \times \rho g h$ by definition of β. Thus

$$u = \frac{1}{2}\beta(\rho g h)^2 = 23.5\text{ kJ/m}^3 \text{ (on substituting values)}$$

1.7 Hydrodynamics

1.315 Between points 1 and 2, fluid particles are in nearly circular motion and therefore have centripetal acceleration. The force for this acceleration, like for any other situation in an ideal fluid, can only come from the pressure variation along the line joining 1 and 2. This requires that pressure at 1 should be greater than the pressure at 2 so that the fluid particles can have required acceleration. If there is no turbulence, the motion can be taken as irrotational. Then by considering

$$\oint \mathbf{v} \cdot d\mathbf{l} = 0$$



along the circuit shown, we infer that $v_2 > v_1$.

The portion of the circuit near 1 and 2 are streamlines while the other two arms are at right angle to streamlines.

(By electrostatic analogy, the density of streamlines is proportional to the velocity.)

1.316 From the conservation of mass

$$v_1 S_1 = v_2 S_2 \tag{1}$$

But $S_1 < S_2$ as shown in the figure of the problem book, therefore

$$v_1 > v_2$$

As every streamline is horizontal between 1 and 2, Bernoulli's theorem becomes

$$p + \frac{1}{2}\rho v^2 = \text{constant}$$

which gives $$p_1 < p_2 \quad \text{as} \quad v_1 > v_2$$

As the difference in height of the water column is Δh, therefore

$$p_2 - p_1 = \rho g \Delta h \tag{2}$$

From Bernoulli's theorem between points 1 and 2 of a streamline

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

or $$p_2 - p_1 = \frac{1}{2}\rho\,(v_1^2 - v_2^2)$$

or $$\rho g\Delta h = \frac{1}{2}\rho\,(v_1^2 - v_2^2) \quad \text{(using Eq. 2)} \tag{3}$$

Using Eqs. (1) in (3), we get

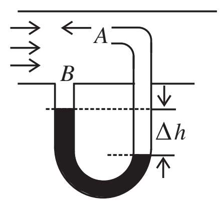
$$v_1 = S_2\sqrt{\frac{2g\Delta h}{S_2^2 - S_1^2}}$$

Hence the sought volume of water flowing per second

$$Q = v_1 S_1 = S_1 S_2\sqrt{\frac{2g\Delta h}{S_2^2 - S_1^2}}$$

1.317 Applying Bernoulli's theorem for points A and B,

$$p_A = p_B + \frac{1}{2}\rho v^2 \quad \text{as} \quad v_A = 0$$



or $$\frac{1}{2}\rho v^2 = p_A - p_B = \Delta h \rho_0 g$$

So, $$v = \sqrt{\frac{2\Delta h\rho_0 g}{\rho}}$$

Thus, rate of flow of gas,

$$Q = Sv = S\sqrt{\frac{2\Delta h\rho_0 g}{\rho}}$$

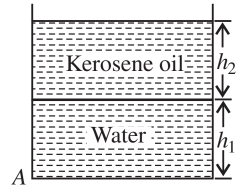
The gas flows over the tube past it at B. But at A the gas becomes stationary as the gas will move into the tube which already contains gas.

In applying Bernoulli's theorem we should remember that

$$\frac{p}{\rho} + \frac{1}{2}v^2 + gz$$

is constant along a streamline. In the present case, we are really applying Bernoulli's theorem somewhat indirectly. The streamline at A is not the streamline at B. Nevertheless the result is correct. To be convinced of this, we need only apply Bernoulli's theorem to the streamline that goes through A by comparing the situation at A with that above B on the same level. In steady conditions, this agrees with the result derived because there cannot be a transverse pressure differential.

1.318 Since, the density of water is greater than that of kerosene oil, it will collect at the bottom. Now, pressure due to water level equals $h_1\rho_1 g$ and pressure due to kerosene oil level equals $h_2\rho_2 g$. So, net pressure becomes $h_1\rho_1 g + h_2\rho_2 g$.

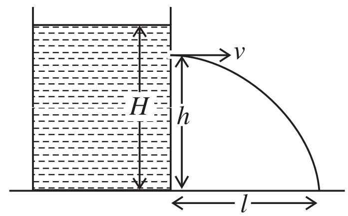


From Bernoulli's theorem, this pressure energy will be converted into kinetic energy while flowing through the hole A.

i.e., $$h_1\rho_1 g + h_2\rho_2 g = \frac{1}{2}\rho_1 v^2$$

Hence, $$v = \sqrt{2\left(h_1 + h_2\frac{\rho_2}{\rho_1}\right)g} = 3 \text{ m/s}$$

1.319 Let H be the total height of the water column and the hole is made at a height h from the bottom.



Then from Bernoulli's theorem

$$\frac{1}{2}\rho v^2 = (H - h)\rho g$$

or $$v = \sqrt{(H - h)\,2g}$$

which is directed horizontally.

For the horizontal range, $l = vt$

$$= \sqrt{2g(H - h)}\cdot\sqrt{\frac{2h}{g}}$$

$$= 2\sqrt{(Hh - h^2)}$$

Now, for maximum l

$$\frac{d\,(Hh - h^2)}{dh} = 0$$

which yields $$h = \frac{H}{2} = 25 \text{ cm}$$

1.320 Let the velocity of the water jet near the orifice be v'. Then applying Bernoulli's theorem,

$$\frac{1}{2}\rho v^2 = h_0\rho g + \frac{1}{2}\rho v^2$$

or $$v' = \sqrt{v^2 - 2gh_0} \tag{1}$$

Here the pressure term on both sides is the same and is equal to the atmospheric pressure.

Now, if water rises up to a height h, then at this height, whole of its kinetic energy will be converted into potential energy. So,

$$\frac{1}{2}\rho v'^2 = \rho g h \quad \text{or} \quad h = \frac{v'^2}{2g}$$

$$= \frac{v^2}{2g} - h_0 = 20 \text{ cm} \quad \text{(using Eq. 1)}$$

1.321 Water flows through the small clearance into the orifice. Let d be the clearance, then from the equation of continuity

$$(2\pi R_1 d)v_1 = (2\pi rd)\,v = (2\pi R_2 d)\,v_2$$

or
$$v_1 R_1 = vr = v_2 R_2 \tag{1}$$

where v_1, v_2 and v are respectively the inward radial velocities of the fluid at 1, 2 and 3. Now by Bernoulli's theorem just before 2 and just after it in the clearance

$$p_0 + h\rho g = p_2 + \frac{1}{2}\rho v_2^2 \tag{2}$$

Applying the same theorem at 3 and 1, we find that this also equals

$$p + \frac{1}{2}\rho v^2 = p_0 + \frac{1}{2}\rho v_1^2 \tag{3}$$

(since the pressure in the orifice is p_0).

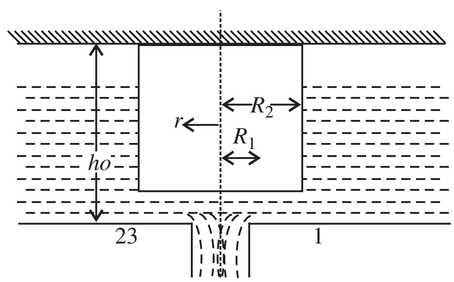
From Eqs. (2) and (3) we also have

$$v_1 = \sqrt{2gh} \tag{4}$$

and
$$p = p_0 + \frac{1}{2}\rho v_1^2\left(1 - \left(\frac{v}{v_1}\right)^2\right)$$

$$= p_0 + h\rho g\left(1 - \left(\frac{R_1}{r}\right)^2\right)$$

(using Eqs. 1 and 4)



1.322 For diagram of piston confirm that

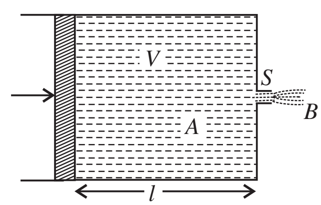
$$f = (p - p_0)\,S \tag{1}$$

as f is constant and p_0 is constant, p is also constant.

From Bernouli's equation from just inside and outside points of the orifice we get

$$p = p_0 + \frac{1}{2}\rho v_{\text{efflux}}^2$$

or
$$(p - p_0) = \frac{1}{2}\rho v_{\text{efflux}}^2 \tag{2}$$



If l is the length of liquid column at an arbitrary instant of time. Then from mass conservation

$$\left(-\frac{dl}{dt}\right) S = v_{\text{efflux}} \tag{3}$$

(where S is the piston area).

From Eq. (2) for constant p, we have $v_{\text{efflux}} =$ constant also.

So, $$V = v_{\text{efflux}} St$$

(where V and $+ve$ have meaning in accordance with the piston).

So, $$v_{\text{efflux}} = \frac{V}{St} \tag{4}$$

Now the differential work done by the constant for $F = (p - p_0)\, S$ is given by

$$dA = F\left(-\frac{dl}{dt}\right) dt = (p - p_0)\, S\left(\frac{-dl}{dt}\right) dt$$

As both p and $(-dl/dt)$ are constants, so, total work

$$W = \int dA = (p - p_0)\, S\left(-\frac{dl}{dt}\right) t$$

$$= \left(\frac{1}{2}\rho v_{\text{efflux}}^2\right) S\left(\frac{v_{\text{eff}}\, s}{S}\right) t \qquad \text{(using Eqs. 2 and 3)}$$

$$= \frac{1}{2}\rho v_{\text{efflux}}^3\, St$$

$$= \frac{1}{2}\rho\left(\frac{V}{St}\right)^3 St \qquad \text{(using Eq. 4)}$$

$$= \frac{1}{2}\rho\, \frac{V^3}{S^2 t^2}$$

1.323 Water jet coming out from the orifice of area s at an arbitrary instant of time when the height of water column in the cylindrical vessel is H.

$$v_{\text{efflux}} = \sqrt{2gH}$$

As the rate of out-flux should be equal to the rate with which volume of the water in the vessel decreases

$$-S\frac{dH}{dt} = s\sqrt{2gH}$$

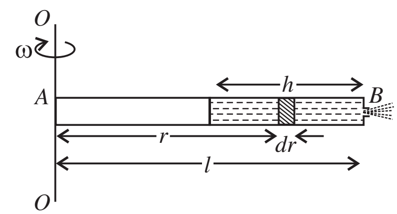
Hence, $$t = -\frac{S}{s\sqrt{2h}}\int_h^0 \frac{dh}{\sqrt{H}} = \frac{S}{s}\sqrt{\frac{2h}{g}}$$

1.324 In a rotating frame (with constant angular velocity), the Eulerian equation is

$$-\nabla p + \rho \mathbf{g} + 2\rho(\mathbf{v}' \times \boldsymbol{\omega}) + \rho\omega^2 \mathbf{r} = \rho \frac{d\mathbf{v}'}{dt}$$

In the frame of rotating tube the liquid in the "column" is practically static because the orifice is sufficiently small. Thus the Eulerian equation in projection form along **r** (which is the position vector of an arbitrary liquid element of length dr relative to the rotation axis) reduces to

$$\frac{-dp}{dr} + \rho\omega^2 r = 0$$



or $$dp = \rho\omega^2 r\,dr$$

So, $$\int_{p_0}^{p} dp = \rho\omega^2 \int_{(l-h)}^{r} r\,dr$$

Thus, $$p(r) = p_0 + \frac{\rho\omega^2}{2}[r^2 - (l - h)^2] \tag{1}$$

Hence the pressure at the end B just before the orifice is

$$p(l) = p_0 + \frac{\rho\omega^2}{2}(2lh - h^2) \tag{2}$$

Then applying Bernoulli's theorem at the orifice for the points just inside and outside of the end B

$$p_0 + \frac{1}{2}\rho\omega^2(2lh - h^2) = p_0 + \frac{1}{2}\rho v^2 \text{ (where } v \text{ is the sought velocity)}$$

So, $$v = \omega h\sqrt{\frac{2l}{h} - 1}$$

1.325 The Euler's equation is

$$\rho\frac{d\mathbf{v}}{dt} = \mathbf{f} - \nabla p = -\nabla(p + \rho g z) \text{ (where } z \text{ is vertically upwards)}$$

Now, $$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} \tag{1}$$

But $$(\mathbf{v}\cdot\nabla)\mathbf{v} = \nabla\left(\frac{1}{2}v^2\right) - \mathbf{v}\times \text{Curl } \mathbf{v} \tag{2}$$

If we consider the steady (i.e. $\partial \mathbf{v}/\partial t = 0$) flow of an incompressible fluid then ρ = constant and as the motion in irrotational Curl $\mathbf{v} = 0$

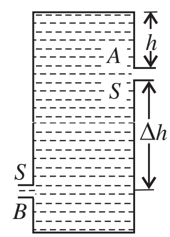
So from Eqs. (1) and (2)

$$\rho \nabla \left(\frac{1}{2} v^2 \right) = -\nabla (p + \rho g z)$$

$$\nabla \left(p + \frac{1}{2} \rho v^2 + \rho g z \right) = 0$$

or
$$p + \frac{1}{2} \rho v^2 + \rho g z = \text{constant}$$

1.326 Let the velocity of water flowing through A be v_A and that through B be v_B, then discharging rate through $A = Q_A = S v_A$ and similarly through $B = S v_B$.



Now, force of reaction at A is

$$F_A = \rho Q_A v_A = \rho S v_B^2$$

Hence, the net force is

$$F = \rho S \left(v_A^2 - v_B^2 \right) \text{ as } \mathbf{F}_A \uparrow\downarrow \mathbf{F}_B \qquad (1)$$

Applying Bernoulli's theorem to the liquid flowing out of A we get

$$\rho_0 + \rho g h = \rho_0 + \frac{1}{2} \rho v_A^2$$

And similarly at B

$$\rho_0 + \rho g (h + \Delta h) = \rho_0 + \frac{1}{2} \rho v_B^2$$

Hence,
$$\left(v_B^2 - v_A^2 \right) \frac{\rho}{2} = \Delta h \rho g$$

Thus,
$$F = 2 \rho g S \, \Delta h = 0.50 \text{ N}$$

1.327 Consider an element of height dy at a distance y from the top. The velocity of the fluid coming out of the element is

$$v = \sqrt{2gy}$$

The force of reaction dF due to this is $dF = \rho dA v^2 = \rho (b dy) 2gy$, as in the previous problem.

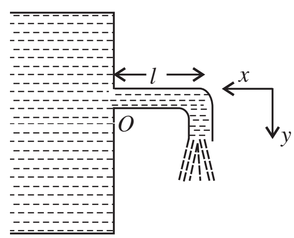
Integrating $$F = \rho\, gb \int_{h-1}^{h} 2y dy$$

$$= \rho gb\,[h^2 - (h - l)^2] = \rho\, gbl\,(2h - l)$$

$$= 5\text{ N} \quad \text{(on substituting values)}$$

(The slit runs from a depth $h - l$ to a depth h from the top.)

1.328 Let the velocity of water flowing through the tube at a certain instant of time be u, then $u = Q/\pi r^2$, where Q is the rate of flow of water and πr^2 is the cross section area of the tube.



From impulse momentum theorem, for the stream of water striking the tube corner, in x-direction in the time interval dt,

$$F_x dt = -\rho Q\, udt \quad \text{or} \quad F_x = -\rho Q u$$

Similarly, $\quad F_y = \rho Q u$

Therefore, the force exerted on the water stream by the tube,

$$\mathbf{F} = -\rho Q u\mathbf{i} + \rho Q u\,\mathbf{j}$$

According to third law, the reaction force on the tube's wall by the stream equals $(-\mathbf{F})$

$$\rho Q u\mathbf{i} - \rho Q u\,\mathbf{j}$$

Hence, the sought moment of force about O becomes

$$\mathbf{N} = -l_i \times (\rho Q u\mathbf{i} - Q u\mathbf{j}) = \rho Q u l\mathbf{k} = \frac{\rho Q^2}{\pi r^2} l\mathbf{k}$$

and $$|\mathbf{N}| = \frac{\rho Q^2 l}{\pi r^2} = 0.70\text{ Nm}$$

1.329 Let us take an arbitrary cross section of radius r of the narrowing tube. If p is the inside pressure at the location of taken ring element of radius r and width dl, and p_0 is outside atmosphere pressure, from symmetry of the problem, net force exerted on the taken ring element due to the inside and outside pressure difference.

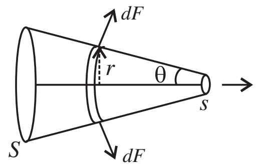
$$= dF\sin\theta = (p - p_0)\, 2\pi r dl \sin\theta$$

$$= (p - p_0)\, 2\pi\ rdr \text{ (because } dl\sin\theta = dr)$$

Hence the sought net force on the tube

$$F = F_x = \int dF\sin\theta = \int (p - p_0)\, 2\pi r dr \tag{1}$$

From Bernoulli's equation



$$p - p_0 = \frac{1}{2}\rho\,(v^2_{\text{efflux}} - v^2),$$

where $$v_{\text{efflux}} = \sqrt{2gh} \tag{2}$$

and v is the speed of water at the location of taken ring element.

From conservation of mass

$$\rho v \pi r^2 = \rho v_{\text{efflux}}\, \pi r_1^{\,2}$$

or $$v = v_{\text{efflux}} \frac{r_1^{\,2}}{r^2} \tag{3}$$

(where r_1 is the radius at open end).

Using Eq. (3) in Eq. (2), we get

$$p - p_0 = \frac{1}{2}\rho\left(v^2_{\text{efflux}} - v^2_{\text{efflux}}\frac{r_1^4}{r^4}\right)$$

$$= \frac{1}{2}\rho v^2_{\text{efflux}}\left(1 - \frac{r_1^{\,2}}{r^4}\right) = \rho g h\left(1 - \frac{r_1^{\,2}}{r^4}\right) \tag{4}$$

Using Eq. (4) in Eq. (1), we get

$$F = \int_{r_1}^{r_2} \rho g h\left(1 - \frac{r_1^{\,2}}{r^4}\right) 2\pi r dr \tag{5}$$

After integration and using $\pi r_2^{\,2} = S$ and $\pi r_1^{\,2} = s$, we get

$$F = \frac{\rho g h\,(S - s)^2}{S} = 6\text{ N} \quad \text{(on substituting the values)}$$

Note: *If we try to calculate F from the momentum change of the liquid flowing out we will be wrong even as regards the sign of the force. There is of course the effect of pressure at S and s but quantitative derivation of F from Newton's law is difficult.*

1.330 The Euler's equation is

$$\rho\frac{d\mathbf{v}}{dt} = \mathbf{f} - \nabla p$$

in the space fixed frame where $\mathbf{f} = -\rho g\mathbf{k}$ downward. We assume incompressible fluid so ρ is constant.

Then $\mathbf{f} = -\nabla(\rho g z)$ where z is the height vertically upwards from some fixed origin. We go to the rotating frame where the equation becomes

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla(p + \rho g z) + \rho\omega^2\mathbf{r} + 2\rho\,(\mathbf{v}\times\boldsymbol{\omega})$$

The additional terms on the right are the well known coriolis and centrifugal forces. In the frame rotating with the liquid $\mathbf{v} = 0$, so

$$\nabla\left(p + \rho g z - \frac{1}{2}\rho\omega^2 r^2\right) = 0$$

or
$$p + \rho g z - \frac{1}{2}\rho\omega^2 r^2 = \text{constant}$$

On the free surface $p =$ constant; thus

$$z = \frac{\omega^2}{2g} r^2 + \text{constant}$$

If we choose the origin at point $r = 0$ (i.e., the axis) of the free surface then "constant" = 0 and

$$z = \frac{\omega^2}{2g} r^2 \quad \text{(the parabolic of revolution)}$$

At the bottom $z =$ constant.

So,
$$p = \frac{1}{2}\rho\omega^2 r^2 + \text{constant}$$

If $p = p_0$ on the axis at the bottom, then

$$p = p_0 + \frac{1}{2}\rho\omega^2 r^2$$

1.331 When the disk rotates the fluid in contact with it co-rotates, but the fluid in contact with the walls of the cavity does not rotate. A velocity gradient is then set up leading to viscous forces.

At a distance r from the axis, the linear velocity is ωr so there is a velocity gradient $\omega r/h$ both in the upper and lower clearance. The corresponding force on the element whose radial width is dr is

$$\eta 2\pi r dr \frac{\omega r}{h} \left(\text{from the formula } F = \eta A \frac{dv}{dx}\right)$$

The torque due to this force is

$$\eta 2\pi r dr \frac{\omega r}{h} r$$

and the net torque considering both the upper and lower clearance is

$$2\int_0^R \eta 2\pi r^3 \, dr \frac{\omega}{h} = \pi R^4 \eta \frac{\omega}{h}$$

So power developed is

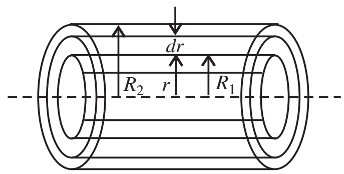
$$P = \frac{\pi R^4 \omega^2 \eta}{h} = 9.05 \text{ W} \quad \text{(on substituting values)}$$

(As instructed, end effects, i.e., rotation of fluid in the clearance $r \geq R$, has been neglected.)

1.332 Let us consider a coaxial cylinder of radius r and thickness dr, then force of friction or viscous force on this elemental layer is given by

$$F = 2\pi r l \eta \frac{dv}{dr}$$

This force must be constant from layer so that steady motion may be possible.



So, $$\frac{F dr}{r} = 2\pi l \eta \int_0^v dv \qquad (1)$$

Integrating, $$F \int_{R_2}^{r} \frac{dr}{r} = 2\pi l \eta \int_0^v dv$$

or $$F \ln\left(\frac{r}{R_2}\right) = 2\pi l \eta v \qquad (2)$$

Putting $r = R_1$, we get

$$F \ln \frac{R_1}{R_2} = 2\pi l \eta v_0 \qquad (3)$$

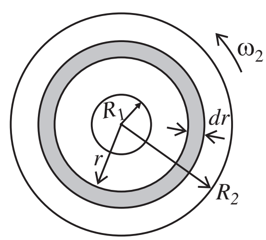
Dividing Eqs. (2) by (3), we get

$$v = v_0 \frac{\ln r/R_2}{\ln R_1/R_2}$$

Note: *The force F is supplied by the agency which tries to carry the inner cylinder with velocity v_0.*

1.333 (a) Let us consider an elemental cylinder of radius r and thickness dr then from Newton's formula

$$F = 2\pi r l \eta r \frac{d\omega}{dr} = 2\pi l \eta r^2 \frac{d\omega}{dr} \qquad (1)$$



and moment of this force acting on the element,

$$N = 2\pi r^2 l \eta \frac{d\omega}{dr} r = 2\pi r^3 l \eta \frac{d\omega}{dr}$$

or $$2\pi l \eta d\omega = N \frac{dr}{r^3} \qquad (2)$$

As in the previous problem N is constant when conditions are steady

Integrating, $$2\pi l\eta \int_0^{\omega} d\omega = N\int_{R_1}^{r} \frac{dr}{r^3}$$

or $$2\pi l\eta\omega = \frac{N}{2}\left[\frac{1}{R_1^2} - \frac{1}{r^2}\right] \tag{3}$$

Putting $r = R_2$, $\omega = \omega_2$, we get

$$2\pi l\eta\omega_2 = \frac{N}{2}\left[\frac{1}{R_1^2} - \frac{1}{R_2^2}\right] \tag{4}$$

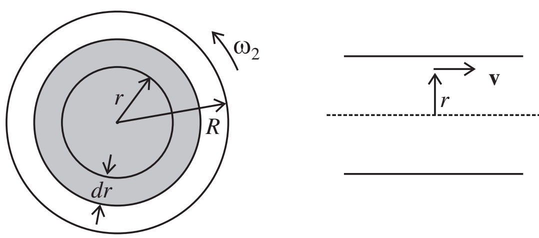
From Eqs. (3) and (4)

$$\omega = \omega_2 \frac{R_1^2\, R_2^2}{R_2^2 - R_1^2}\left[\frac{1}{R_1^2} - \frac{1}{r^2}\right]$$

(b) From Eq. (4)

$$N_1 = \frac{N}{l} = 4\,\pi\eta\omega_2 \frac{R_1^2\, R_2^2}{R_2^2 - R_1^2}$$

1.334 (a) Let dV be the volume flowing per second through the cylindrical shell of thickness dr then,



$$dV = -(2\pi r dr)\, v_0\left(1 - \frac{r^2}{R^2}\right)dr = 2\pi v_0\left(r - \frac{r^3}{R^2}\right)dr$$

The total volume is

$$V = 2\pi v_0 \int_0^R \left(r - \frac{r^3}{R^2}\right) dr = 2\pi v_0 \frac{R^2}{4} = \frac{\pi}{2} R^2 v_0$$

(b) Let dE be the kinetic energy within the above cylindrical shell. Then,

$$dT = \frac{1}{2}(dm)\, v^2 = \frac{1}{2}(2\pi r l d r\rho)\, v^2$$

$$= \frac{1}{2}(2\pi l\rho) r\, dr v_0^2\left(1 - \frac{r^2}{R^2}\right)^2 = \pi l\rho v_0^2\left[r - \frac{2r^3}{R^2} + \frac{r^5}{R^4}\right] dr$$

Hence, total energy of the fluid,

$$T = \pi l\rho v_0^2 \int_0^R \left(r - \frac{2r^3}{R^2} + \frac{r^5}{R^4} \right) dr = \frac{\pi R^2 \rho l\, v_0^2}{6}$$

(c) Here frictional force is the shearing force on the tube, exerted by the fluid, which equals $-\eta S\, dv/dt$.

Given, $$v = v_0 \left(1 - \frac{r^2}{R^2}\right)$$

So, $$\frac{dv}{dr} = -2v_0 \frac{r}{R^2}$$

At $r = R$, $$\frac{dv}{dr} = -\frac{2v_0}{R}$$

Then, viscous force is given by $F = -\eta\,(2\pi Rl) \left(\frac{dv}{dr}\right)_{r=R}$

$$= -2\pi R\eta l \left(-\frac{2v_0}{R}\right) = 4\pi\eta v_0 l$$

(d) Taking a cylindrical shell of thickness dr and radius r, viscous force,

$$F = -\eta(2\pi rl)\frac{dv}{dr}$$

Let Δp be the pressure difference, then net force on the element is

$$\Delta p \pi r^2 + 2\pi\eta lr \frac{dv}{dr}$$

But, since the flow is steady, $F_{\text{net}} = 0$

or $$\Delta p = \frac{-2\pi l\eta r\, dv/dr}{\pi r^2} = \frac{-2\pi l\eta r\,(-2v_0\, r/R^2)}{\pi r^2} = \frac{4\eta v_0 l}{R^2}$$

1.335 The loss of pressure head in travelling a distance l is seen from the middle section to be $h_2 - h_1 = 10$ cm. Since $h_2 - h_1 = h_1$ in our problem and $h_3 - h_2 = 15$ cm $= 5 + h_2 - h_1$, we see that pressure head of 5 cm remains in compensated and must be converted into kinetic energy of the liquid flowing out. Thus,

$$\frac{\rho v^2}{2} = \rho g \Delta h \quad (\text{where } \Delta h = {}_3h - {}_2h\,)$$

Thus, $$v = \sqrt{2g\Delta h} \cong 1 \text{ m/s}$$

1.336 We know that, Reynold's number (R_e) is defined as, $R_e = \rho v l/\eta$, where v is the velocity, l is the characteristic length, and η the coefficient of viscosity. In the case of circular cross-section the characteristic length is the diameter of cross-section d, and v is taken as average velocity of flow of liquid.

Now, R_{e_1} (Reynold's number at x_1 from the pipe end) $= \dfrac{\rho d_1 v_1}{m\eta}$, where v_1 is the velocity at distance x_1.

Similarly, $$R_{e_2} = \frac{\rho d_2 v_2}{\eta}$$

So, $$\frac{R_{e_1}}{R_{e_2}} = \frac{d_1 v_1}{d_2 v_2}$$

From equation of continuity,

$$A_1 v_1 = A_2 v_2$$

or $$\pi r_1^2 v_1 = \pi r_2^2 v_2 \quad \text{or} \quad d_1 v_1 r_1 = d_2 v_2 r_2$$

$$\frac{d_1 v_1}{d_2 v_2} = \frac{r_2}{r_1} = \frac{r_0 e^{-\alpha x_2}}{r_0 e^{-\alpha x_1}} = e^{-\alpha \Delta x} \quad (\text{as } x_2 - x_1 = \Delta x)$$

Thus, $$\frac{R_{e_2}}{R_{e_1}} = e^{\alpha \Delta x} = 5$$

1.337 We know that Reynold's number for turbulent flow is greater than that for laminar flow.

Now, $$(R_e)_l = \frac{\rho v d}{\eta} = \frac{2\rho_1 v_1 r_1}{\eta_1} \quad \text{and} \quad (R_e)_t = \frac{2\rho_2 v_2 r_2}{\eta}$$

But, $$(R_e)_t \geq (R_e)_l$$

So $$v_{2_{\min}} = \frac{\rho_1 v_1 r_1 \eta_2}{\rho_2 r_2 \eta_1} = 5 \ \mu\text{m/s} \quad (\text{on substituting values})$$

1.338 We have $$R = \frac{v\rho_0 d}{\eta}$$

and v is given by $$6\pi\eta r v = \frac{4\pi}{3} r^2 (\rho - \rho_0)\, g$$

(where ρ = density of lead, ρ_0 = density of glycerine)

$$v = \frac{2}{9\eta}(\rho - \rho_0) g r^2 = \frac{1}{18\eta}(\rho - \rho_0)\, g d^2$$

Thus, $$\frac{1}{2} = \frac{1}{18\eta^2}(\rho - \rho_0)\, g\rho_0 d^3$$

and $$d = [9\eta^2/\rho_0(\rho - \rho_0)g]^{1/3} = 5.2 \text{ mm} \quad \text{(on substituting values)}$$

1.339 $$m\frac{dv}{dt} = mg - 6\pi\eta r v$$

or $$\frac{dv}{dt} + \frac{6\pi\eta r}{m}v = g$$

or $$\frac{dv}{dt} + kv = g \quad \left(\text{where } k = \frac{6\pi\eta r}{m}\right)$$

or $$e^{kt}\frac{dv}{dt} + ke^{kt}v = ge^{kt} \quad \text{or} \quad \frac{d}{dt}e^{kt}\,v = ge^{kt}$$

or $$ve^{kt} = \frac{g}{k}e^{kt} + C \quad \text{or} \quad v = \frac{g}{k} + Ce^{-kt} \quad \text{(where } C \text{ is constant)}$$

Since $$v = 0 \text{ for } t = 0, \quad \text{so} \quad 0 = \frac{g}{k} + C$$

or, $$C = -\frac{g}{k}$$

Thus, $$v = \frac{g}{k}(1 - e^{-kt})$$

The steady state velocity is g/k.

v differs from g/k by n, where

$$e^{-kt} = n$$

or $$t = \frac{1}{k}\ln n$$

Thus, $$\frac{1}{k} = -\frac{(4\pi/3)r^3P}{6\pi\eta r} = -\frac{4r^2\rho}{18\eta} = -\frac{d^2\rho}{18\eta}$$

(We have neglected buoyancy in olive oil.)

$$t = \frac{-\rho d^2}{18\,\eta}\ln n$$

$$= 0.20 \text{ s (on substituting values)}$$

1.8 Relativistic Mechanics

1.340 From the formula for length contraction

$$\left(l_0 - l_0\sqrt{1 - \frac{v^2}{c^2}}\right) = \eta l_0$$

So, $$1 - \frac{v^2}{c^2} = (1 - \eta)^2 \quad \text{or} \quad v = c\sqrt{\eta(2 - \eta)}$$

1.341 (a) In the frame in which the triangle is at rest, the space coordinates of the vertices are (000), $\left(a\frac{\sqrt{3}}{2}, +\frac{a}{2}, 0\right)\left(a\frac{\sqrt{3}}{2}, -\frac{a}{2}, 0\right)$, all measured at the same time t. In the moving frame the corresponding coordinates at time t' are

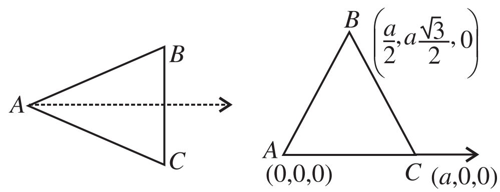
$$A : (vt', 0, 0),\ B : \left(\frac{a}{2}\sqrt{3}\sqrt{1 - \beta^2} + vt', \frac{a}{2}, 0\right) \text{ and}$$

$$C : \left(\frac{a}{2}\sqrt{3}\sqrt{1 - \beta^2} + vt', -\frac{a}{2}, 0\right)$$

The perimeter P is then

$$P = a + 2a\left(\frac{3}{4}(1 - \beta^2) + \frac{1}{4}\right)^{1/2} = a\left[1 + \sqrt{4 - 3\beta^2}\right]$$

(b) The coordinates in the first frame are shown at time t. The coordinates in the moving frame are



$$A : (vt', 0, 0),\ B : \left(\frac{a}{2}\sqrt{1 - \beta^2} + vt', a\frac{\sqrt{3}}{2}, 0\right),\ C : (a\sqrt{1 - \beta^2} + vt', 0, 0)$$

The perimeter P is then

$$P = a\sqrt{1 - \beta^2} + \frac{a}{2}[1 - \beta^2 + 3]^{1/2} \times 2$$

$$= a(\sqrt{1 - \beta^2} + \sqrt{4 - \beta^2}) \left(\text{where } \beta = \frac{v}{c}\right)$$

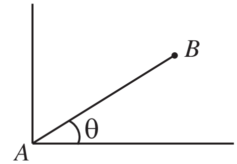
1.342 In the rest frame, the coordinates of the ends of the rod in terms of proper length l_0

$$A:(0,0,0),\ B:(l_0\cos\theta_0,\ l_0\sin\theta_0,\ 0)$$

at time t. In the laboratory frame the coordinates at time t' are

$$A:(vt',0,0),\ B:(l_0\cos\theta_0\sqrt{1-\beta^2}+vt',\ l_0\sin\theta_0,\ 0)$$

Therefore, we can write



$l\cos\theta_0 = l_0\cos\theta_0\sqrt{1-\beta^2}$ and $l\sin\theta = l_0\sin\theta_0$

Hence, $$l_0^2 = (l^2)\left(\frac{\cos^2\theta + (1-\beta^2)\sin^2\theta}{1-\beta^2}\right)$$

or $$l_0 = \sqrt{\frac{1-\beta^2\sin^2\theta}{1-\beta^2}}\ \left(\text{where } \beta = \frac{v}{c}\right)$$

$$= 1.08 \text{ m} \quad \text{(on substituting values)}$$

1.343 In the frame K in which the cone is at rest, the coordinates of A are (0, 0, 0) and of B are $(h,\ h\tan\theta,\ 0)$. In the frame K', which is moving with velocity v along the axis of the cone, the coordinates of A and B at time t' are

$$A:(-vt',0,0),\quad B:(h-\sqrt{1-\beta^2}-vt',\ h\tan\theta,\ 0)$$

Thus the taper angle in the frame K' is

$$\tan\theta' = \frac{\tan\theta}{\sqrt{1-\beta^2}} = \left(\frac{y'_B - y'_A}{x'_B - x_A}\right) \text{ or } \theta' = 59° \quad \text{(on substituting values)}$$

The lateral surface area is

$$S = \pi h'^2 \sec\theta' \tan\theta'$$

$$= \pi h^2(1-\beta^2)\frac{\tan\theta}{\sqrt{1-\beta^2}}\sqrt{1+\frac{\tan^2\theta}{1-\beta^2}}$$

$$= S_0\sqrt{1-\beta^2\cos^2\theta}$$

$$= 3.3 \text{ m}^2 \quad \text{(on substituting values)}$$

Here, $S_0 = \pi h^2\sec\theta\tan\theta$ is the lateral surface area in the rest frame and

$$h' = h\sqrt{1-\beta^2}\ \left(\text{where } \beta = \frac{v}{c}\right)$$

1.344 Because of time dilation, a moving clock reads less time. We write,

$$t - \Delta t = t\sqrt{1-\beta^2}\ \left(\text{where } \beta = \frac{v}{c}\right)$$

Thus, $$1 - \frac{2\Delta t}{t} + \left(\frac{\Delta t}{t}\right)^2 = 1 - \beta^2$$

or $$v = c\sqrt{\frac{\Delta t}{t}\left(2 - \frac{\Delta t}{t}\right)}$$

$$= 0.6 \times 10^8 \text{ m/s} \quad \text{(on substituting values)}$$

1.345 In the frame *K*, the length l of the rod is related to the time of flight Δt by

$$l = v\Delta t$$

In the reference frame fixed to the rod (frame k') the proper length l_0 of the rod is given by

$$l_0 = v\Delta t'$$

But, $$l_0 = \frac{l}{\sqrt{1-\beta^2}} = \frac{v\Delta t}{\sqrt{1-\beta^2}} \quad \left(\text{where } \beta = \frac{v}{c}\right)$$

Thus, $$v\Delta t' = \frac{v\Delta t}{\sqrt{1-\beta^2}}$$

So, $$1-\beta^2 = \left(\frac{\Delta t}{\Delta t'}\right)^2 \quad \text{or} \quad v = c\sqrt{1-\left(\frac{\Delta t}{\Delta t'}\right)^2}$$

and $$l_0 = c\sqrt{(\Delta t')^2 - (\Delta t)^2} = c\Delta t'\sqrt{1-\left(\frac{\Delta t}{\Delta t'}\right)} = 4.5 \text{ m} \quad \text{(on substituting values)}$$

1.346 The distance traveled in the laboratory frame of reference is $v\Delta t$ where v is the velocity of the particle. But by time dilation

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}}$$

So, $$v = c\sqrt{1 - (\Delta t_0/\Delta t)^2}$$

Thus the distance traversed is

$$s = c\Delta t\sqrt{1 - (\Delta t_0/\Delta t)^2}$$
$$= 5 \text{ m} \quad \text{(on substituting values)}$$

1.347 (a) If τ_0 is the proper life time of the mount, the life time in the moving frame is

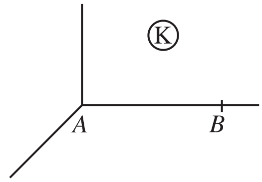
$$\frac{\tau_0}{\sqrt{1 - v^2/c^2}} \quad \text{and hence} \quad l = \frac{v\tau_0}{\sqrt{1 - v^2/c^2}}$$

Thus, $$\tau_0 = \frac{l}{v}\sqrt{1 - v^2/c^2}$$
$$= 1.4\ \mu\text{s} \quad \text{(on substituting values)}$$

(b) The words "from the muon's stand point" are not part of any standard terminology.

1.348 In the frame K in which the particles are at rest, their positions are A and B whose coordinates may be taken as,

$$A : (0, 0, 0),\ B = (l_0, 0, 0)$$



In the frame K' with respect to which K is moving with a velocity v, the coordinates of A and B at time t' in the moving frame are

$$A = (vt', 0, 0),\ B = (l_0\sqrt{1-\beta^2} + vt', 0, 0) \left(\text{where } \beta = \frac{v}{c}\right)$$

Suppose B hits a stationary target in K' after time t'_B while A hits it after time $t_B + \Delta t$. Then,

$$l_0\sqrt{1-\beta^2} + vt'_B = v(t'_B + \Delta t)$$

So,
$$l_0 = \frac{v\Delta t}{\sqrt{1 - v^2/c^2}}$$
$$= 17 \text{ m} \quad (\text{on substituting values})$$

1.349 In the reference frame fixed to the ruler the rod is moving with a velocity v and suffers Lorentz contraction. If l_0 is the proper length of the rod, its measured length will be

$$\Delta x_1 = l_0\sqrt{1-\beta^2} \left(\text{where } \beta = \frac{v}{c}\right)$$

In the reference frame fixed to the rod, the ruler suffers Lorentz contraction and we must have

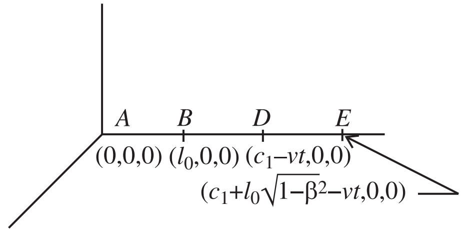
$$\Delta x_2\sqrt{1-\beta^2} = l_0$$

Thus,
$$l_0 = \sqrt{\Delta x_1 \Delta x_2} = 6.0 \text{ m} \quad (\text{on substituting values})$$

and
$$1 - \beta^2 = \frac{\Delta x_1}{\Delta x_2}$$

or
$$v = c\sqrt{1 - \frac{\Delta x_1}{\Delta x_2}} = 2.2 \times 10^8 \text{ m/s} \quad (\text{on substituting values})$$

1.350 The coordinates of the ends of the rods in the frame fixed to the left rod are shown in the figure. The points B and D coincide when



$$l_0 = c_1 - vt_0 \quad \text{or} \quad t_0 = \frac{c_1 - l_0}{v}$$

The points A and E coincide when

$$0 = c_1 + l_0\sqrt{1-\beta^2} - vt_1$$

$$t_1 = \frac{c_1 + l_o\sqrt{1-\beta^2}}{v}$$

Thus, $$\Delta t = t_1 - t_0 = \frac{l_0}{v}(1 + \sqrt{1-\beta^2})$$

or $$\left(\frac{v\Delta t}{l_0} - 1\right)^2 = 1 - \beta^2 = 1 - \frac{v^2}{c^2}$$

From this $$v = \frac{2c^2\Delta t/l_0}{1 + c^2\Delta t^2/l_0^2} = \frac{2l_0/\Delta t}{1 + (l_0/c\Delta t)^2}$$

1.351 In K_0, the rest frame of the particles, the events corresponding to the decay of the particles are

$$A : (0, 0, 0, 0) \quad \text{and} \quad B : (0, l_0, 0, 0)$$

In the reference frame K, the corresponding coordinates are by Lorentz transformation

$$A : (0, 0, 0, 0),\ B : \left(\frac{vl_0}{c^2\sqrt{1-\beta^2}}, \frac{l_0}{\sqrt{1-\beta^2}}, 0, 0\right)$$

Now, $l_0\sqrt{1-\beta^2} = l$, by Lorentz Fitzgerald contraction formula.

Thus the time lag of the decay time of B is

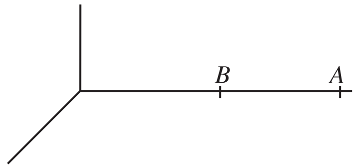
$$\Delta t = \frac{vl_0}{c^2\sqrt{1-\beta^2}} = \frac{vl}{c^2(1-\beta^2)} = \frac{vl}{c^2 - v^2}$$

$$= 20\ \mu\text{s (on substituting values)}$$

B decays later. (B is the forward particle in the direction of motion.)

1.352 (a) In the reference frame K with respect to which the rod is moving with velocity v, the coordinates of A and B are

$$A: t,\ x_A + v(t - t_A),\ 0,\ 0$$
$$B: t,\ x_B + v(t - t_B),\ 0,\ 0$$



Thus, $\quad l = x_A - x_B - v(t_A - t_B) = l_0\sqrt{1-\beta^2}$

So, $$l_0 = \frac{x_A - x_B - v(t_A - t_B)}{\sqrt{1 - v^2/c^2}}$$

(b) $\pm l_0 - v(t_A - t_B) = l = l_0\sqrt{1 - v^2/c^2}$

(Since $x_A - t_B$ can be either $+\,l_0$ or $-l_0$.)

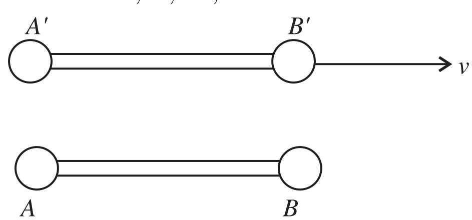
Thus, $\quad v(t_A - t_B) = (\pm 1 - \sqrt{1 - v^2/c^2})l_0$

i.e., $$t_A - t_B = \frac{l_0}{v}\left(1 - \sqrt{1 - v^2/c^2}\right)$$

or $$t_B - t_A = \frac{l_0}{v}\,(1 + \sqrt{1 - v^2/c^2})$$

1.353 At the instant the picture is taken the coordinates of A, B, A', B' in the rest frame of AB are

$$A : (0,\ 0,\ 0,\ 0)$$
$$B : (0,\ l_0,\ 0,\ 0)$$
$$B' : (0,\ 0,\ 0,\ 0)$$
$$A' : (0,\ \text{or}\ -l_0\sqrt{1 - v^2/c^2},\ 0,\ 0)$$



(a) In this frame the coordinates of B' at other times are $B' : (t,\ vt,\ 0,\ 0)$. So B' is opposite to B at time $t(B) = l_0/v$. In the frame in which B', A' is at rest, the time corresponding to this is given by Lorentz transformation.

$$t_0(B') = \frac{1}{\sqrt{1 - v^2/c^2}}\left(\frac{l_0}{v} - \frac{vl_0}{c^2}\right) = \frac{l_0}{v}\sqrt{1 - v^2/c^2}$$

(b) Similarly in the rest frame of A, B, the coordinates of A at other times are

$$A' : \left(t,\ -l_0\sqrt{1 - v^2/c^2} + vt,\ 0,\ 0\right)$$

A' is opposite to A at time $t(A) = \dfrac{l_0}{v}\sqrt{1 - v^2/c^2}$

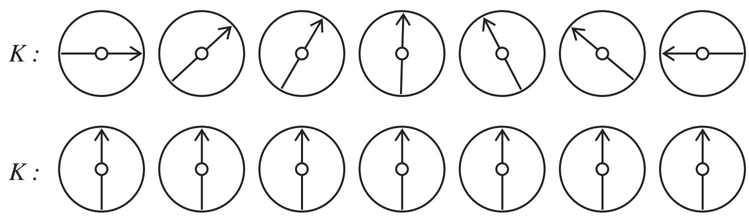
The corresponding time in the frame in which A', B' are rest is

$$t(A') = \gamma t(A) = \frac{l_0}{v}$$

1.354 By Lorentz transformation $\quad t' = \dfrac{1}{\sqrt{1 - v^2/c^2}}\left(t - \dfrac{vx}{c^2}\right)$

So at time $t = 0$, $\quad t' = \dfrac{vx}{c^2}\dfrac{1}{\sqrt{1 - v^2/c^2}}$

If $x > 0, t' < 0$ and if $x < 0, t' > 0$, so we get the diagram given below "in terms of the K-clocks".



The situation in terms of the K' clock is reversed.

1.355 Suppose $x\,(t)$ is the locus of points in the frame K at which the readings of the clocks of both reference system are permanently identical, then by Lorentz transformation

$$t' = \frac{1}{\sqrt{1 - V^2/c^2}}\left(t - \frac{Vx(t)}{c^2}\right) = t$$

On differentiating $x(t) = \dfrac{c^2}{V}\left(1 - \sqrt{1 - V^2/c^2}\right)$

$$= \frac{c}{\beta}(1 - \sqrt{1 - \beta^2})\ \left(\text{where } \beta = \frac{V}{c}\right)$$

Let $\beta = \tan h\theta$, $0 \leq \theta < \infty$. Then,

$$x(t) = \frac{c}{\tan h\theta}(1 - \sqrt{1 - \tan h^2\theta}) = c\,\frac{\cos h\theta}{\sin h\theta}\left(1 - \frac{1}{\cos h\,\theta}\right)$$

$$= c\frac{\cos h\,\theta - 1}{\sin h\theta} = c\sqrt{\frac{\cos h\,\theta - 1}{\cos h\theta + 1}} = c\tan h\,\frac{\theta}{2} \leq V$$

($\tan h\theta$ is a monotonically increasing function of θ.)

1.356 We can take the coordinates of the two events to be

$$A : (0, 0, 0, 0); \quad B : (\Delta t, a, 0, 0)$$

For B to be the effect and A to be cause we must have $\Delta t > \dfrac{|a|}{c}$.

In the moving frame, the coordinates of A and B become

$$A : (0, 0, 0, 0);\ B\text{:} \left[\gamma\left(\Delta t - \frac{aV}{c^2}\right), \gamma(a - V\,\Delta t), 0, 0\right] \left(\text{where } \gamma = \frac{1}{\sqrt{1-(V^2/c^2)}}\right)$$

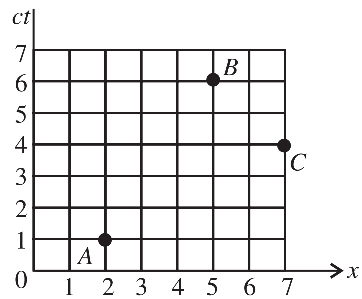
Since,

$$(\Delta t')^2 - \frac{a'^2}{c^2} = \gamma^2\left[\left(\Delta t - \frac{aV}{c^2}\right)^2 - \frac{1}{c^2}(a - V\,\Delta t)^2\right] = (\Delta t)^2 - \frac{a^2}{c^2} > 0$$

we must have $\Delta t' > \dfrac{|a|'}{c}$.

1.357 (a) The four-dimensional interval between A and B (assuming $\Delta y = \Delta z = 0$) is:

$$5^2 - 3^2 = 16 \text{ units}$$



Therefore the time interval between these two events in the reference frame in which the events occurred at the same place is

$$c(t'_B - t'_A) = \sqrt{16} = 4 \text{ m}$$

or $$t'_B - t'_A = \frac{4}{c} = \frac{4}{3} \times 10^{-8} \text{ s}$$

(b) The four dimensional interval between A and C (assuming $\Delta y = \Delta z = 0$) is:

$$3^2 - 5^2 = -16$$

So the distance between the two events in the frame in which they are simultaneous is 4 units = 4 m.

1.358 By the velocity addition formula

$$v'_x = \frac{v_x - V}{1 - Vv_x/c^2}$$

$$v'_y = \frac{v_y\sqrt{1 - V^2/c^2}}{1 - v_xV/c^2}$$

and $$v' = \sqrt{v_x'^2 + v_y'^2} = \frac{\sqrt{(v_x - V)^2 + v_y^2(1 - V^2/c^2)}}{1 - v_x V/c^2}.$$

1.359 (a) By definition, the velocity of approach is

$$v_{\text{approach}} = \frac{dx_1}{dt} - \frac{dx_2}{dt} = v_1 - (-v_2) = v_1 + v_2 = 1.25c$$

in the reference frame K.

(b) The relative velocity is obtained by the transformation law

$$v_r = \frac{v_1 - (-v_2)}{1 - \dfrac{v_1(-v_2)}{c^2}} = \frac{v_1 + v_2}{1 + \dfrac{v_1 v_2}{c^2}}$$

$$= 0.19c \quad (\text{where } c \text{ is the velocity of light})$$

1.360 The velocity of one of the rods in the reference frame fixed to the other rod is

$$V = \frac{v + v}{1 + v^2/c^2} = \frac{2v}{1 + \beta^2}$$

The length of the moving rod in this frame is

$$l = l_0\sqrt{1 - \frac{4v^2/c^2}{(1 + \beta^2)^2}} = l_0\frac{1 - \beta^2}{1 + \beta^2}$$

1.361 The approach velocity is defined by

$$\mathbf{V}_{\text{approach}} = \frac{d\mathbf{r}_1}{dt} - \frac{d\mathbf{r}_2}{dt} = \mathbf{v}_1 - \mathbf{v}_2$$

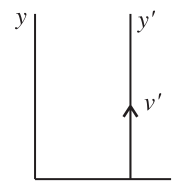
in the laboratory frame.

So, $$V_{\text{approach}} = \sqrt{v_1^2 + v_2^2}$$

On the other hand, the relative velocity can be obtained by using the velocity addition formula and has the components

$$\left[-v_1,\ v_2\sqrt{1 - \left(\frac{v_1^2}{c^2}\right)}\right] \quad \text{so,} \quad V_r = \sqrt{v_1^2 + v_2^2 - \frac{v_1 v_2}{c^2}}$$

1.362 The components of the velocity of the unstable particle in the frame K are



$$\left(V,\ v'\sqrt{1-\frac{V^2}{c^2}},\ 0\right)$$

So the velocity relative to K is

$$\sqrt{V^2+v'^2-\frac{v'^2V^2}{c^2}}$$

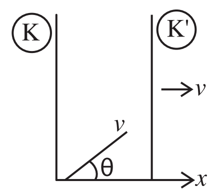
The life time in this frame dilates to

$$\Delta t_0\Big/\sqrt{1-\frac{V^2}{c^2}-\frac{v'^2}{c^2}+\frac{v'^2V^2}{c^4}}$$

and the distance traversed is

$$s=\Delta t_0\,\frac{\sqrt{V^2+v'^2-(v'^2V^2)/c^2}}{\sqrt{1-V^2/c^2}\,\sqrt{1-v'^2/c^2}}$$

1.363 In the frame K' the components of the velocity of the particle are



$$v'_x=\frac{v\cos\theta-V}{1-\dfrac{vV\cos\theta}{c^2}}$$

$$v'_y=\frac{v\sin\theta\,\sqrt{1-V^2/c^2}}{1-\dfrac{vV\cos\theta}{c^2}}$$

Hence, $$\tan\theta'=\frac{v'_y}{v'_x}=\frac{v\sin\theta}{v\cos\theta-V}\sqrt{1-V^2/c^2}$$

1.364 In K' the coordinates of A and B are

$$A:(t',\,0,\,-v'\,t',\,0);\quad B:(t',\,l,\,-v'\,t',\,0)$$

After performing Lorentz transformation to the frame K we get

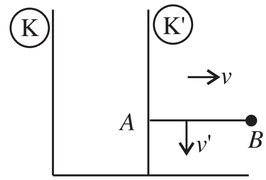
$$A:\ t=\gamma t';\quad B:\ t=\gamma\left(t'+\frac{Vl}{c^2}\right)$$

$$\begin{aligned} x&=\gamma\,Vt'; & x&=\gamma\,(l+Vt')\\ y&=v'\,t'; & y&=-v'\,t'\\ z&=0; & z&=0 \end{aligned}$$

By translating $t'\to t'-\dfrac{Vl}{c^2}$, we can write the coordinates of B as $B:\ t=\gamma t'$.

$$x=\gamma l\left(1-\frac{V^2}{c^2}\right)+Vt'\gamma=l\sqrt{1-\frac{v'^2}{c^2}}+Vt'\gamma$$

$$y = -v'\left(t' - \frac{Vl}{c^2}\right),\ z = 0$$



Thus, $$\Delta x = l\sqrt{1 - \left(\frac{V^2}{c^2}\right)},\ \Delta y = \frac{v'\,Vl}{c^2}$$

Hence, $$\tan\theta' = \frac{v'\,V}{c^2\sqrt{1 - (V/c)^2}}$$

1.365 $$-\frac{t}{\mathbf{v}} - \frac{t + dt}{\mathbf{v} + \mathbf{w}dt}\,\mathrm{K}$$

(a) In K the velocities at time t and $t + dt$ are respectively v and $v + wdt$ along x-axis which is parallel to the vector $\mathbf{V}$. In the frame K' moving with velocity $\mathbf{V}$ with respect to K, the velocities are, respectively,

$$\frac{v - V}{1 - vV/c^2} \text{ and } \frac{v + wdt - V}{1 - (v + wdt)\,V/c^2}$$

The latter velocity is written as

$$\frac{v - V}{1 - vV/c^2} + \frac{wdt}{1 - vV/c^2} + \frac{v - V}{(1 - vV/c^2)}\,\frac{wV}{c^2}dt$$

$$= \frac{v - V}{1 - vV/c^2} + \frac{wdt\,(1 - V^2/c^2)}{(1 - vV/c^2)^2}$$

Also by Lorentz transformation

$$dt' = \frac{dt - Vdx/c^2}{\sqrt{1 - V^2/c^2}} = dt\,\frac{1 - vV/c^2}{\sqrt{1 - V^2/c^2}}$$

Thus the acceleration in the K' frame is

$$w' = \frac{dv'}{dt'} = \frac{w}{(1 - vV/c^2)^3}\left(1 - \frac{V^2}{c^2}\right)^{3/2}$$

(b) In the K frame, the velocities of the particle at the time t and $t + dt$ are, respectively $(0, v, 0)$ and $(0, v + wdt, 0)$, where $\mathbf{V}$ is along x-axis.

In the K' frame the velocities are $\left(-V,\, v\sqrt{1-V^2/c^2},\, 0\right)$

and $\left(-V,\, (v+wdt)\sqrt{1-V^2/c^2},\, 0\right)$, respectively.

Thus the acceleration

$$w' = \frac{wdt\sqrt{(1-V^2/c^2)}}{dt'} = w\left(1-\frac{V^2}{c^2}\right) \quad \text{along the } y\text{-axis}$$

$$\left(\text{We have used } dt' = \frac{dt}{\sqrt{1-V^2/c^2}}.\right)$$

1.366 In the instantaneous rest frame $v = V$ and

$$w' = \frac{w}{(1-V^2/c^2)^{3/2}} \quad \text{(from solution problem 1.365(a))}$$

So,

$$= \frac{dv}{(1-V^2/c^2)^{3/2}} = w'\,dt$$

w' is constant by assumption. Thus, integration gives

$$v = \frac{w't}{\sqrt{1+(w't/c)^2}}$$

Integrating once again

$$x = \frac{c^2}{w'}\left(\sqrt{1+\left(\frac{w't}{c}\right)^2} - 1\right)$$
$$= 0.91 \text{ light year}$$

The percentage difference of rocket velocity from velocity of light is given by

$$\frac{c-v}{c} = \frac{1}{2}\left(\frac{c}{w't}\right)^2 = 0.47\%$$

1.367 The boost time τ_0 in the reference frame fixed to the rocket is related to the time τ elapsed on the Earth by

$$\tau_0 = \int_0^{\tau}\sqrt{1-\frac{v^2}{c^2}}\,dt = \int_0^{\tau}\left[1-\frac{(w't/c)^2}{1+(w't/c)}\right]^2 dt$$

$$= \int_0^{\tau} \frac{dt}{\sqrt{1 + (w' t/c)}} = \frac{c}{w'} \int_0^{w't/c} \frac{d\xi}{\sqrt{1 + \xi^2}} = \frac{c}{w'} \ln \left[\frac{w'\tau}{c} + \sqrt{1 + \left(\frac{w'\tau}{c} \right)^2} \right]$$

$$= 3.5 \text{ months} \quad \text{(on substituting values)}$$

1.368

$$m = \frac{m_0}{\sqrt{1 - \beta^2}}$$

For

$$\beta \approx 1, \frac{m}{m_0} \approx \frac{1}{\sqrt{2\,(1 - \beta)}} = \frac{1}{\sqrt{2\eta}}$$

$$\approx 70 \quad \text{(on substituting values)}$$

1.369 We define the density ρ in the frame K is such a way that $\rho\ dx\ dy\ dz$ is the rest mass dm_0 of the element. That is $\rho\ dxdydz = \rho_0 dx_0 dy_0 dz_0$, where ρ_0 is the proper density, dx_0, dy_0, dz_0 are the dimensions of the element in the rest frame K_0. Now,

$$dy = dy_0,\ dz = dz_o,\ dx = dx_0 \sqrt{1 - v^2/c^2}$$

if the frame K is moving with velocity v relative to the frame K_0. Thus

$$\rho = \frac{\rho_0}{\sqrt{1 - v^2/c^2}}$$

Defining η by $\quad \rho = \rho_0\,(1 + \eta)$

We get

$$1 + \eta = \frac{1}{\sqrt{1 - v^2/c^2}} \quad \text{or} \quad \frac{v^2}{c^2} = 1 - \frac{1}{(1 + \eta)^2} = \frac{\eta(2 + \eta)}{1(1 + \eta)^2}$$

or

$$v = c\sqrt{\frac{\eta(2 + \eta)}{(1 + \eta)^2}} = \frac{c\sqrt{\eta(2 + \eta)}}{1 + \eta}$$

$$= 0.6c \quad \text{(where } c \text{ is the velocity of light)}$$

1.370 We have

$$\frac{m_0 v}{\sqrt{1 - v^2/c^2}} = p \quad \text{or} \quad \frac{m_0}{\sqrt{1 - v^2/c^2}} = \sqrt{{m_0}^2 + \frac{p^2}{c^2}}$$

or

$$1 - \frac{v^2}{c^2} = \frac{m_0^2 c^2}{m_0^2 c^2 + p^2} = 1 - \frac{p^2}{p^2 + m_0^2 c^2}$$

or

$$v = \frac{c_{\mathrm{p}}}{\sqrt{p^2 + m_o^2 c^2}} = \frac{c}{\sqrt{1 + (m_o C/p)^2}}$$

So, $\dfrac{c - v}{c} = \left[1 - \left(1 + \left(\dfrac{m_0 c}{p}\right)^2\right)^{-1/2}\right] \times 100\ \% = 0.44\%$ (on substituting values)

1.371 By definition of η,

$$\frac{m_0 v}{\sqrt{1 - v^2/c^2}} = \eta m_0 v \quad \text{or} \quad 1 - \frac{v^2}{c^2} = \frac{1}{\eta^2}$$

or $$v = c\sqrt{1 - \frac{1}{\eta^2}} = \frac{c}{\eta}\sqrt{\eta^2 - 1} = \frac{1}{2}c\sqrt{3} \text{ (on substituting for } \eta)$$

1.372 The work done is equal to change in kinetic energy which is different in the two cases. Classically, i.e., in non-relativistic mechanics, the change in kinetic energy is

$$\frac{1}{2} m_0 c^2 ((0.8)^2 - (0.6)^2) = \frac{1}{2} m_0 c^2\, 0.28 = 0.14\ m_0 c^2$$

Relativistically, it is

$$\frac{m_0 c^2}{\sqrt{1 - (0.8)^2}} - \frac{m_0 c^2}{\sqrt{1 - (0.6)^2}} = \frac{m_0 c^2}{0.6} - \frac{m_0 c^2}{0.8} = m_0 c^2 (1.666 - 1.250)$$

$$= 0.416\ m_0 c^2 = 0.42\ m_0\ c^2$$

1.373 $$\frac{m_0 c^2}{\sqrt{1 - \dfrac{v^2}{c^2}}} = 2 m_0 c^2$$

or $$\sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{2} \quad \text{or} \quad 1 - \frac{v^2}{c^2} = \frac{1}{4}$$

or $$\frac{v}{c} = \frac{\sqrt{3}}{2}, \text{ i.e.,} \quad v = c\,\frac{\sqrt{3}}{2} = 2.6 \times 10^8 \text{ m/s}$$

1.374 Relativistically,

$$\frac{T}{m_0 c^2} = \left(\frac{1}{\sqrt{1 - \beta^2}} - 1\right) = \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4$$

So, $$\beta_{\text{rel}}^2 \cong \frac{2T}{m_0 c^2} - \frac{3}{4}(\beta_{\text{rel}}^2)^2 = \frac{2\ T}{m_0 c^2} - \frac{3}{4}\left(\frac{2T}{m_0 c^2}\right)^2$$

Thus, $$-\beta_{\text{rel}} = \left[\frac{2T}{m_0 c^2} - 3\,\frac{T^2}{m_0^2 c^4}\right]^{1/2} = \sqrt{\frac{2T}{m_0 c^2}}\left(1 - \frac{3}{4}\,\frac{T}{m_0 c^2}\right)$$

But classically, $\beta_{cl} = \sqrt{\dfrac{2T}{m_0 c^2}}$

so, $$\frac{\beta_{rel} - \beta_{cl}}{\beta_{cl}} = \frac{3}{4}\frac{T}{m_0 c^2} = \varepsilon$$

Hence if, $\dfrac{T}{m_0 c^2} < \dfrac{4}{3}\varepsilon \approx 0.013$, then the velocity β is given by the classical formula with an error less than ε.

1.375 From the formula

$$E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}}, \quad p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}}$$

we find $E^2 = c^2 p^2 + m_0^2 c^4$ or $(m_0 c^2 + T)^2 = c^2 p^2 + m_0^2 c^4$

or $T(2\, m_0 c^2 + T) = c^2 p^2$ i.e. $p = \dfrac{1}{c}\sqrt{T(2m_0 c^2 + T)} = 1.09$ GeV/**c**

1.376 Let the total force exerted by the beam on the target surface be F and the power liberated there be P. Then, using the result of the previous problem, we see

$$F = Np = \frac{N}{c}\sqrt{T(T + 2m_0 c^2)} = \frac{I}{ec}\sqrt{T(T + 2m_0 c^2)}$$

since, $I = Ne$, N being the number of particles striking the target per second. Also,

$$P = N\left(\frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} - m_0 c^2\right) = \frac{I}{e}T$$

These will be, respectively, equal to the pressure and power developed per unit area of the target if I is current density.

1.377 In the frame fixed to the sphere, the momentum transferred to the elastically scattered particle is

$$\frac{2mv}{\sqrt{1 - v^2/c^2}}$$

The density of the moving element is $n\dfrac{1}{\sqrt{1 - v^2/c^2}}$ (from solution of problem 1.369) and the momentum transferred per unit time per unit area is the pressure, given by

$$p = \frac{2mv}{\sqrt{1 - v^2/c^2}} n \frac{1}{\sqrt{1 - v^2/c^2}} \cdot v \frac{2mnv^2}{1 - v^2/c^2}$$

In the frame fixed to the gas, when the sphere hits a stationary particle, the latter recoils with a velocity

$$= \frac{v + v}{1 + v^2/c^2} = \frac{2v}{1 + v^2/c^2}$$

The momentum transferred is $\dfrac{\dfrac{m \cdot 2v}{1 + v^2/c^2}}{\sqrt{1 - \dfrac{4v^2/c^2}{(1 - v^2/c^2)^2}}} = \dfrac{2mv}{1 - v^2/c^2}$

and the pressure is $\dfrac{2mv}{1 - v^2/c^2} \cdot n \cdot v = \dfrac{2mnv^2}{1 - v^2/c^2}$

1.378 The equation of motion is

$$\frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1 - v^2/c^2}}\right) = F$$

Integrating, $\dfrac{v/c}{\sqrt{1 - v^2/c^2}} = \dfrac{\beta}{\sqrt{1 - \beta^2}} = \dfrac{Ft}{m_0 c}$ (using $v = 0$ for $t = 0$)

$$\frac{\beta^2}{1 - \beta^2} = \left(\frac{Ft}{m_0 c}\right)^2 \quad \text{or} \quad \beta^2 = \frac{(Ft)^2}{(Ft)^2 + (m_0 c)^2} \quad \text{or} \quad v = \frac{Fct}{\sqrt{(m_0 c)^2 + (Ft)^2}}$$

or $x = \displaystyle\int \frac{Fct\, dt}{\sqrt{F^2 t^2 + m_0^2 c^2}} = \frac{c}{F} \int \frac{\xi\, d\xi}{\sqrt{\xi^2 + (m_0 c)^2}} = \frac{c}{F}\sqrt{F^2 t^2 + m_0^2 c^2} + \text{constant}$

or using $x = 0$ at $t = 0$, we get, $x = \sqrt{c^2 t^2 + \left(\dfrac{m_0 c^2}{F}\right)^2} - \dfrac{m_0 c^2}{F}$

1.379 Since, $x = \sqrt{a^2 + c^2 t^2}$

so, $$\dot{x} = v = \frac{c^2 t}{a^2 + c^2 t^2}$$

or $$\frac{v}{\sqrt{1 - v^2/c^2}} = \frac{c^2 t}{a}$$

Thus, $$\frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1 - v^2/c^2}}\right) = \frac{m_0 c^2}{a} = F$$

1.380 $$\mathbf{F} = \frac{d}{dt}\left(\frac{m_0 \mathbf{v}}{\sqrt{1 - v^2/c^2}}\right) = m_0 \frac{\dot{\mathbf{v}}}{\sqrt{1 - v^2/c^2}} + m_0 \frac{v}{c^2}\, \mathbf{v} \cdot \dot{\mathbf{v}} \frac{1}{(1 - v^2/c^2)^{3/2}}$$

Thus, $$\mathbf{F}_\perp = m_0 \mathbf{w} \sqrt{1 - \beta^2},\ \mathbf{w} = \dot{\mathbf{v}},\ \mathbf{w}_\perp \mathbf{v}$$

$$\mathbf{F}_\parallel = m_0 \frac{\mathbf{w}}{(1 - \beta^2)^{3/2}},\ \mathbf{w} = \dot{\mathbf{v}},\ \mathbf{w}_\parallel \mathbf{v}$$

1.381 By definition,

$$E = m_0 \frac{c^2}{\sqrt{1 - v_x^2/c^2}} = \frac{m_0 c^3 dt}{ds},\ p_x = m_0 \frac{v_x}{\sqrt{1 - v^2/c^2}} = \frac{c m_0 dx}{dx}$$

where $ds^2 = c^2\, dt^2 - dx^2$ is the invariant interval ($dy = dz = 0$).

Thus, $$p_x' = c m_0 \frac{dx'}{ds} = c m_0 \gamma \frac{(dx - V dt)}{ds} = \frac{p_x - VE/c^2}{\sqrt{1 - V^2/c^2}}$$

$$E' = m_0 c^3 \frac{dt'}{ds} - c^3\, m_0 \gamma \frac{(dt - V d_x/c^2)}{ds} = \frac{E - V p_x}{\sqrt{1 - \dfrac{V^2}{c^2}}}$$

1.382 For a photon moving in the x direction

$$\varepsilon = c p_x$$

and $$p_y = p_z = 0$$

In the moving frame, $$\varepsilon' = \frac{1}{\sqrt{1 - \beta^2}}\left(\varepsilon - V\frac{\varepsilon}{c}\right) = \varepsilon \sqrt{\frac{1 - V/c}{1 + V/c}}$$

Note that $$\varepsilon' = \frac{\varepsilon}{2} \quad \text{if,} \quad \frac{1}{4} = \frac{1 - \beta}{1 + \beta} \quad \text{or} \quad \beta = \frac{3}{5},\ V = \frac{3c}{5}$$

1.383 As before

$$E = m_0 c^3 \frac{dt}{ds}$$

$$p_x = m_0 c \frac{dx}{ds}$$

Similarly $\qquad p_y = m_0 c \dfrac{dy}{ds}, \quad p_z = m_0 c \dfrac{dz}{ds}$

Then $\quad E^2 - c^2p^2 = E^2 - c^2\,(p_x^2 + p_y^2 + p_z^2)$

$$= m_0^2 c^4 \frac{(c^2dt^2 - dx^2 - dy^2 - dz^2)}{ds^2} = m_0^2 c^4 \text{ is invariant}$$

1.384 (b) In the C.M. frame, the total momentum is zero, Thus

$$\frac{V}{c} = \frac{cp_{1x}}{E_1 + E_2} = \frac{\sqrt{T(T + 2m_0c^2)}}{T + 2m_0c^2} = \sqrt{\frac{T}{T + 2m_0c^2}}$$

$$V = c\sqrt{\frac{T}{T + 2m_0c^2}} = 2.12 \times 10^8 \text{ m/s}$$

where we have used the result of problem 1.375.

(a) Then

$$\frac{1}{\sqrt{1 - V^2/c^2}} = \frac{1}{\sqrt{1 - \dfrac{T}{T + 2m_0c^2}}} = \sqrt{\frac{T + 2m_0c^2}{2m_0c^2}}$$

Total energy in the C.M. frame is

$$\frac{2m_0c^2}{\sqrt{1 - V^2/c^2}} = 2m_0c^2\sqrt{\frac{T + 2m_0c^2}{2m_0c^2}} = \sqrt{2m_0c^2\,(T + 2m_0c^2)} = \tilde{T} + 2m_0c^2$$

So, $\qquad \tilde{T} = 2m_0c^2\left(\sqrt{1 + \dfrac{T}{2m_0c^2}} - 1\right) = 777 \text{ MeV}$

Also $\quad 2\sqrt{c^2\tilde{p}^2 + mc^4} = \sqrt{2m_0^2c^2(T + 2m_0c^2)} \Rightarrow 4c^2\tilde{p}^2 = 2m_0c^2T$

or $\qquad \tilde{p} = \sqrt{\dfrac{1}{2}\,m_0T} = 940 \text{ MeV/c}$

1.385 $M_0c^2 = \sqrt{E^2 - c^2p^2}$

$$\sqrt{(2m_0c^2 + T)^2 - T(2m_0c^2 + T)} = \sqrt{2m_0c^2\,(2m_0c^2 + T)} = c\sqrt{2m_0\,(2m_0c^2 + T)}$$

Also $\quad cp = \sqrt{T(T + 2m_0c^2)},\ v = \dfrac{c^2p}{E} = c\sqrt{\dfrac{T}{T + 2m_0c^2}}$

1.386 Let T' = kinetic energy of a proton striking another stationary particle of the same rest mass. Then, combined kinetic energy in the C.M. frame

$$= 2m_0c^2\left(\sqrt{1 + \frac{T'}{2m_0c^2}} - 1\right) = 2T, \left(\frac{T}{m_0c^2} + 1\right)^2 = 1 + \frac{T'}{2m_0c^2}$$

$$\frac{T'}{2m_0c^2} = \frac{T(2m_0c^2 + T)}{m_0^2c^4} \quad \text{or} \quad T' = \frac{2T(T + 2m_0c^2)}{m_0c^2}$$

1.387 We have

$$E_1 + E_2 + E_3 = m_0c^2, \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = 0$$

Hence, $$(m_0c^2 - E_1)^2 - c^2\,\mathbf{p}_1^2 = (E_2 + E_3)^2 - (\mathbf{p}_2 + \mathbf{p}_3)^2c^2$$

$$\text{L.H.S.} = (m_0^2\,c^4 - E_1)^2 - c^2\mathbf{p}_1^2 = (m_0^2 + m_1^2)\,c^4 - 2m_0c^2E_1$$

The R.H.S. is an invariant. We can evaluate it in any frame. Choose the C.M. frame of the particles 2 and 3.

In this frame, R.H.S. $= (E_2' + E_3')^2 = (m_2 + m_3)^2c^4$

Thus, $$(m_0^2 + m_1^2)\,c^4 - 2m_0c^2E_1 = (m_2 + m_3)^2c^4$$

or $$2m_0c^2E_1 \le \left\{m_0^2 + m_1^2 - (m_2 + m_3)^2\right\}c^4 \quad \text{or} \quad E_1 \le \frac{m_0^2 + m_1^2 - (m_2 + m_3)^2}{2m_0}\,c^2$$

1.388 The velocity of ejected gases is u relative to the rocket. In an Earth centered frame it is

$$\frac{v - u}{1 - vu/c^2}$$

in the direction of the rocket. The momentum conservation equation then reads

$$(m + dm)\,(v + dv) + \frac{v - u}{1 - uv/c^2}(-dm) = mv$$

or $$mdv - \left(\frac{v - u}{1 - uv/c^2} - v\right) dm = 0$$

Here, $-dm$ is the mass of the ejected gases. So,

$$mdv - \frac{-u + uv/c^2}{1 - uv^2/c^2}\,dm = 0 \quad \text{or} \quad mdv + u\left(1 - \frac{v^2}{c^2}\right) dm = 0$$

(neglecting $1 - uv/c^2$, since u is non-relativistic).

Integrating, using $\beta = \dfrac{v}{c}$, we get

$$\int \frac{d\beta}{1 - \beta^2} + \frac{u}{c}\int \frac{dm}{m} = 0$$

or $$\ln \frac{1 + \beta}{1 - \beta} + \frac{u}{c} \ln m = \text{constant}$$

The constant $= \dfrac{u}{c} \ln m_0$, since $\beta = 0$ initially.

Thus, $$\frac{1 - \beta}{1 + \beta} = \left(\frac{m}{m_0}\right)^{u/c} \quad \text{or} \quad \beta = \frac{1 - (m/m_o)^{u/c}}{1 + (m/m_o)^{u/c}}$$

THERMODYNAMICS AND MOLECULAR PHYSICS

PART 2

2.1 Equation of the Gas State. Processes

2.1 Let m_1 and m_2 be the masses of the gas in the vessel before and after the gas is released. Hence mass of the gas released,

$$\Delta m = m_1 - m_2$$

Now from ideal gas equation

$$p_1 V = m_1 \frac{R}{M} T_0 \quad \text{and} \quad p_2 V = m_2 \frac{R}{M} T_0$$

as V and T are same before and after the release of the gas.

So, $$(p_1 - p_2)V = (m_1 - m_2)\frac{R}{M} T_0 = \Delta m \frac{R}{M} T_0$$

or $$\Delta m = \frac{(p_1 - p_2)VM}{RT_0} = \frac{\Delta p VM}{RT_0} \tag{1}$$

We also know $$p = \rho \frac{R}{M} T \quad \text{so,} \quad \frac{M}{RT_0} = \frac{\rho}{p_0} \tag{2}$$

(where p_0 = standard atmospheric pressure and T_0 = 273 K).

From Eqs. (1) and (2), we get

$$\Delta m = \rho V \frac{\Delta p}{p_0} = 1.3 \times 30 \times \frac{0.78}{1} = 30 \text{ g}$$

2.2 Let V is volume of each vessel. For the vessel which contained the ideal gas,

$$p_1 V = \nu_1 R T_1$$

When heated, some gas, passes into the evacuated vessel till pressure difference becomes Δp. Let p_1' and p_2' be the pressure on the two sides of the valve. Then

$$p_1' V = \nu_1' R T_2$$

and $$p_2' V = \nu_2' R T_2 = (\nu_1 - \nu_1') R T_2$$

Putting the values of ν_1 and ν_2 from the first two equations

$$p_2' V = \left(\frac{p_1 V}{RT_1} - \frac{p_1' V}{RT_2}\right) RT_2 \quad \text{or} \quad p_2' = \left(\frac{p_1}{T_1} - \frac{p_1'}{T_2}\right) T_2$$

But, $$p_1' - p_2' = \Delta p$$

So, $$p_2' = \left(\frac{p_1}{T_1} - \frac{p_2' + \Delta p}{T_2}\right) T_2$$

$$= \frac{p_1 T_2}{T_1} - p_2' - \Delta p$$

or $$p_2' = \frac{1}{2}\left(\frac{p_1 T_2}{T_1} - \Delta p\right) = 0.08 \text{ atm}$$

2.3 Let the mixture contain ν_1 and ν_2 moles of H_2 and He, respectively. If molecular weights of H_2 and He are M_1 and M_2, then their respective masses in the mixture are

$$m_1 = \nu_1 M_1 \quad \text{and} \quad m_2 = \nu_2 M_2$$

Therefore, for the total mass of the mixture we get,

$$m = m_1 + m_2 \quad \text{or} \quad m = \nu_1 M_1 + \nu_2 M_2 \tag{1}$$

Also, if ν is the total number of moles of the mixture in the vessels, then we know,

$$\nu = \nu_1 + \nu_2 \tag{2}$$

Solving Eqs. (1) and (2) for ν_1 and ν_2, we get

$$\nu_1 = \frac{(\nu M_2 - m)}{M_2 - M_1}, \; \nu_2 = \frac{m - \nu M_1}{M_2 - M_1}$$

Therefore, we get

$$m_1 = M_1 \frac{(\nu M_2 - m)}{M_2 - M_1} \quad \text{and} \quad m_2 = M_2 \frac{(m - \nu M_1)}{M_2 - M_1}$$

or $$\frac{m_1}{m_2} = \frac{M_1}{M_2}\frac{(\nu M_2 - m)}{(m - \nu M_1)}$$

One can also express the above result in terms of the effective molecular weight M of the mixture, defined as

$$M = \frac{m}{\nu} = m\frac{RT}{pV}$$

Thus, $$\frac{m_1}{m_2} = \frac{M_1}{M_2} \cdot \frac{M_2 - M}{M - M_1} = \frac{1 - M/M_2}{M/M_1 - 1}$$

Using the data and table, we get

$$M = 3.0 \text{ g} \quad \text{and} \quad \frac{m_1}{m_2} = 0.50$$

2.4 We know, for the mixture, N_2 and CO_2 (being regarded as ideal gases, their mixture too behaves like an ideal gas), $p_0V = \nu RT$, where, ν is the total number of moles of of the gases (mixture) present, p_0 is the pressure and V is the volume of the vessel.

If ν_1 and ν_2 are number of moles of N_2 and CO_2, respectively present in the mixture, then $\nu = \nu_1 + \nu_2$.

Now number of moles of N_2 and CO_2 is, by definition, given by

$$\nu_1 = \frac{m_1}{M_1} \quad \text{and} \quad \nu_2 = \frac{m_2}{M_2}$$

where, m_1 is the mass of N_2 (molecular weight $= M_1$) in the mixture and m_2 is the mass of CO_2 (molecular weight $= M_2$) in the mixture.

Therefore density of the mixture is given by

$$\rho = \frac{m_1 + m_2}{V} = \frac{m_1 + m_2}{(\nu RT/p_0)}$$

$$= \frac{p_0}{RT}\frac{m_1 + m_2}{\nu_1 + \nu_2} = \frac{p_0(m_1 + m_2)M_1M_2}{RT(m_1M_2 + m_2M_1)}$$

$$= 1.5 \text{ kg/m}^3 \text{ (on substituting values)}$$

2.5 (a) The mixture contains ν_1, ν_2 and ν_3 moles of O_2, N_2 and CO_2, respectively. Then the total number of moles of the mixture

$$\nu = \nu_1 + \nu_2 + \nu_3$$

We know, ideal gas equation for the mixture

$$pV = \nu RT \quad \text{or} \quad p = \frac{\nu RT}{V}$$

or
$$p = \frac{(\nu_1 + \nu_2 + \nu_3)RT}{V} = 1.968 \text{ atm (on substituting values)}$$

(b) Mass of oxygen (O_2) present in the mixture: $m_1 = \nu_1 M_1$

Mass of nitrogen (N_2) present in the mixture: $m_2 = \nu_2 M_2$

Mass of carbon dioxide (CO_2) present in the mixture: $m_3 = \nu_3 M_3$

So, mass of the mixture:

$$m = m_1 + m_2 + m_3 = \nu_1 M_1 + \nu_2 M_2 + \nu_3 M_3$$

Molecular mass of the mixture

$$M = \frac{\text{mass of the mixture}}{\text{total number of moles}}$$

$$= \frac{\nu_1 M_1 + \nu_2 M_2 + \nu_3 M_3}{\nu_1 + \nu_2 + \nu_3} = 36.7 \text{ g/mol (on substituting values)}$$

2.6 Let p_1 and p_2 be the pressure in the upper and lower part of the cylinder, respectively at temperature T_0. At the equilibrium position for the piston:

$$p_1 S + mg = p_2 S$$

or $$p_1 + \frac{mg}{S} = p_2 \quad \text{(where } m \text{ is the mass of the piston)}$$

But, $$p_1 = \frac{RT_0}{\eta V_0} \quad \text{(where } V_0 \text{ is the initial volume of the lower part)}$$

So, $$\frac{RT_0}{\eta V_0} + \frac{mg}{S} = \frac{RT_0}{V_0} \Rightarrow \frac{mg}{S} = \frac{RT_0}{V_0}\left(1 - \frac{1}{\eta}\right) \tag{1}$$

Let T' be the sought temperature and at this temperature the volume of the lower part becomes V', then according to the problem the volume of the upper part becomes $\eta' V$.

Hence, $$\frac{mg}{S} = \frac{RT}{V'}\left(1 - \frac{1}{\eta'}\right) \tag{2}$$

From Eqs. (1) and (2),

$$\frac{RT_0}{V_0}\left(1 - \frac{1}{\eta}\right) = \frac{RT''}{V'}\left(1 - \frac{1}{\eta'}\right) \quad \text{or} \quad T' = \frac{T_0(1 - 1/\eta)V'}{V_0(1 - 1/\eta')} \tag{3}$$

As, the total volume must be constant, so

$$V_0(1 + \eta) = V'(1 + \eta') \quad \text{or} \quad V' = \frac{V_0(1 + \eta)}{1 + \eta'}$$

Putting the value of V' in Eq. (3), we get

$$T = \frac{T_0(1 - 1/\eta)V_0 \dfrac{(1 + \eta)}{(1 + \eta')}}{V_0(1 - 1/\eta')}$$

$$= \frac{T_0(\eta^2 - 1)\eta'}{(\eta'^2 - 1)\eta} = 0.42 \text{ kK}$$

2.7 Let ρ_1 be the density after the first stroke. The mass remains constant.

So,
$$V\rho = (V + \Delta V)\,\rho_1 \Rightarrow \rho_1 = \frac{V\rho}{(V + \Delta V)}$$

Similarly, if ρ_2 is the density after second stroke
$$V\rho_1 = (V + \Delta V)\,\rho_2$$

or
$$\rho_2 = \left(\frac{V}{V + \Delta V}\right)\rho_1 = \left(\frac{V}{V + \Delta V}\right)^2 \rho_0$$

In this way after nth stroke,
$$\rho_n = \left(\frac{V}{V + \Delta V}\right)^n \rho_0$$

Since pressure $\propto$ density,
$$p_n = \left(\frac{V}{V + \Delta V}\right)^n p_0 \text{ (because temperature is constant)}$$

It is required by p_n/p_0 to be $1/\eta$,

So,
$$\frac{1}{\eta} = \left(\frac{V}{V + \Delta V}\right)^n \quad \text{or} \quad \eta = \left(\frac{V + \Delta V}{V}\right)^n$$

Hence,
$$n = \frac{\ln \eta}{\ln\,(1 + \Delta V/V)}$$

2.8 From the ideal gas equation
$$p = \frac{m}{M}\frac{RT}{V}$$
$$\frac{dp}{dt} = \frac{RT}{MV}\frac{dm}{dt} \tag{1}$$

In each stroke, volume v of the gas is ejected, where v is given by
$$v = \frac{V}{m_N}\,[m_{N-1} - m_N]$$

In case of continuous ejection, if m_{N-1} corresponds to mass of gas in the vessel at time t, then m_N is the mass at time $t + \Delta t$, where Δt, is the time in which volume v of the gas has come out.

The rate of evacuation is therefore $v/\Delta t$, i.e.,
$$C = \frac{v}{\Delta t} = -\frac{V}{m(t + \Delta t)} \cdot \frac{m(t + \Delta t) - m(t)}{\Delta t}$$

In the limit $\Delta t \to 0$, we get
$$C = \frac{V}{m}\frac{dm}{dt} \tag{2}$$

From Eqs. (1) and (2)

$$\frac{dp}{dt} = -\frac{C}{V}\frac{mRT}{MV} = -\frac{C}{V}p \quad \text{or} \quad \frac{dp}{p} = -\frac{C}{V}dt$$

Integrating we get,
$$\int_p^{p_0} \frac{dp}{p} = -\frac{C}{V}\int_t^0 dt$$

or
$$\ln\frac{p}{p_0} = -\frac{C}{V}t$$

Thus,
$$p = p_0 e^{-Ct/V}$$

2.9 Let ρ be the instantaneous density, then instantaneous mass $= V\rho$. In a short interval dt the volume is increased by Cdt.

So, $V\rho = (V + Cdt)(\rho + d\rho)$ (because mass remains constant in a short interval dt).

So,
$$\frac{d\rho}{\rho} = -\frac{C}{V}dt$$

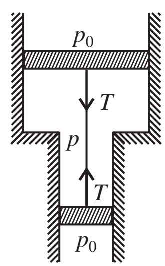
Since pressure $\propto$ density,

$$\frac{dp}{p} = -\frac{C}{V}dt$$

or
$$\int_{p_1}^{p_2} -\frac{dp}{p} = \frac{C}{V}t$$

or
$$t = \frac{V}{C}\ln\frac{p_1}{p_2} = \frac{V}{C}\ln\frac{1}{\eta} = 1.0 \text{ min}$$

2.10 The physical system consists of one mole of gas confined in the smooth vertical tube. Let m_1 and m_2 be the masses of upper and lower pistons and S_1 and S_2 be their respective areas.



For the lower piston

$$pS_2 + m_2 g = p_0 S_2 + T$$

or
$$T = (p - p_0)\, S_2 + m_2 g \tag{1}$$

Similarly for the upper piston

$$p_0 S_1 + T + m_1 g = pS_1$$

or
$$T = (p - p_0)S_1 - m_1 g \tag{2}$$

From Eqs. (1) and (2)

$$(p - p_0)(S_1 - S_2) = (m_1 + m_2)\, g$$

or $$(p - p_0)\,\Delta S = mg$$

So, $$p = \frac{mg}{\Delta S} + p_0 = \text{constant}$$

From the gas law, $pV = \nu RT$ or $p\Delta V = \nu R\Delta T$ (because p is constant) .

So, $$\left(p_0 + \frac{mg}{\Delta S}\right)\Delta Sl = R\Delta T$$

Hence, $$\Delta T = \frac{1}{R}\,(p_0\Delta S + mg)l$$
$$= 0.9 \text{ K}$$

2.11 (a) Given that $p = p_0 - \alpha V^2 = p_0 - \alpha(RT/p)^2$ (as $V = RT/p$ for one mole of gas).

Thus, $$T = \frac{1}{R\sqrt{\alpha}}\, p\sqrt{p_0 - p} = \frac{1}{R\sqrt{\alpha}}\sqrt{p_0 p^2 - p^3} \tag{1}$$

For T_{max}, $$\frac{d}{dp}(p_0 p^2 - p^3) = 0$$

which yields, $$p = 2/3\; p_0$$

Hence, using this value of p in Eq. (1), we get

$$T_{max} = \frac{1}{R\sqrt{\alpha}}\cdot\frac{2}{3}\,p_0\sqrt{p_0 - \frac{2}{3}\,p_0} = \frac{2}{3}\left(\frac{p_0}{R}\right)\sqrt{\frac{p_0}{3\alpha}}$$

(b) Given that $$p = p_0 e^{-\beta V} = p_0 e^{-\beta RT/p}$$

So, $$\frac{\beta RT}{p} = \ln\frac{p_0}{p} \quad \text{and} \quad T = \frac{p}{\beta R}\ln\frac{p_0}{p} \tag{2}$$

For T_{max}, the condition is $dT/dp = 0$, which yields $p = p_0/e$.

Hence using this value of p in Eq. (2), we get

$$T_{max} = \frac{p_0}{e\beta R}$$

2.12 Given that $T = T_0 + \alpha V^2 = T_0 + \alpha(R^2T^2/p^2)$ (as $V = RT/p$ for one mole of gas).

So, $$p = \sqrt{\alpha}\; RT\,(T - T_0)^{-1/2} \tag{1}$$

For $P_{\min}$, $\qquad \dfrac{dp}{dT} = 0$, which gives

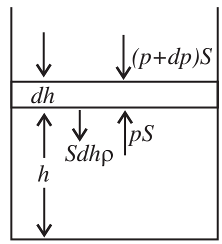
$$T = 2T_0 \tag{2}$$

From Eqs. (1) and (2), we get

$$p_{\min} = \sqrt{\alpha}\, R2T_0\, (2T_0 - T_0)^{-1/2} = 2R\sqrt{\alpha T_0}$$

2.13 Consider a thin layer at a height h and thickness dh. Let p and $dp + p$ be the pressures on the two sides of the layer. The mass of the layer is $Sdh\rho$. Equating vertical downward force to the upward force acting on the layer, we get

$$Sdh\rho g + (p + dp)S = pS$$



So, $$\frac{dp}{dh} = -\rho g \tag{1}$$

But, $$p = \frac{\rho}{M}RT$$

So, $$dp = \frac{\rho R}{M}\, dT$$

or $$-\frac{\rho R}{M}\, dT = \rho g dh \quad \text{(using Eq. 1)}$$

So, $$\frac{dT}{dh} = -\frac{gM}{R} = -34 \text{ K/km}$$

That means, temperature of air drops by 34 K at a height of 1 km above bottom.

2.14 From the previous problem we have,

$$\frac{dp}{dh} = -\rho g \tag{1}$$

But, from $p = C\rho^n$ (where C is a constant), we get

$$dp/d\rho = Cn\rho^{n-1} \tag{2}$$

We have from gas law,

$$p = \rho\frac{R}{M}T \quad \text{or} \quad C\rho^n = \rho\frac{R}{M}\cdot T$$

So, $$T = \frac{M}{R}\, C\rho^{n-1} \quad \text{(using Eq. 2)}$$

Thus, $$\frac{dT}{d\rho} = \frac{M}{R}\cdot C\,(n-1)\,\rho^{n-2} \tag{3}$$

But, $$\frac{dT}{dh} = \frac{dT}{d\rho}\cdot\frac{d\rho}{dp}\cdot\frac{dp}{dh}$$

Substituting from Eqs. (1), (2) and (3) we get

$$\frac{dT}{dh} = \frac{M}{R}\, C\,(n-1)\,\rho^{n-2}\,\frac{1}{Cn\rho^{n-1}}\,(-\rho g) = \frac{-Mg\,(n-1)}{nR}$$

2.15 We have, $dp = -\rho g dh$ and from gas law $\rho = (M/RT)p$.

Thus, $$\frac{dp}{p} = -\frac{Mg}{RT}\,dh$$

Integrating, we get

$$\int_{p_0}^{p} \frac{dp}{p} = -\frac{Mg}{RT}\int_0^h dh$$

or $\ln\dfrac{p}{p_0} = -\dfrac{Mg}{RT}h$ (where p_0 is the pressure at the surface of the Earth)

So, $$p = p_0 e^{-Mgh/RT}$$

Under standard conditions, $p_0 = 1$ atm, $T = 273$ K

Pressure at a height of 5 km $= 1 \times e^{-28\times 9.81\times 5000/8.314\times 273} = 0.5$ atm

Pressure in a mine at a depth of 5 km $= 1 \times e^{-28\times 9.81\times(-5000)/8.314\times 273} = 2$ atm

2.16 We have $dp = -\rho g dh$ and from gas law $p = (\rho/M)RT$.

Thus, $dp = \dfrac{d\rho}{M}RT$ (at constant temperature)

So, $$\frac{d\rho}{\rho} = \frac{gM}{RT}dh$$

Integrating within limits

$$\int_{\rho_0}^{\rho}\frac{d\rho}{\rho} = \int_0^h \frac{gM}{RT}dh$$

or $$\ln\frac{\rho}{\rho_0} = \frac{gM}{RT}h$$

So, $\rho = \rho_0 e^{-Mgh/RT}$ and $h = -\dfrac{RT}{Mg}\ln\dfrac{\rho}{\rho_0}$

(a) Given that $T = 273$ K and $\rho_0/\rho = e$.

Thus, $$h = -\frac{RT}{Mg} \ln e^{-1}$$

$$= \frac{RT}{Mg} = 8 \text{ km.}$$

(b) Given that $T = 273$ K and $(\rho_0 - \rho)/\rho_0 = 0.01$ or $\rho/\rho_0 = 0.99$.

Thus, $$h = -\frac{RT}{Mg} \ln \frac{\rho}{\rho_0} = 0.09 \text{ km (on substituting values)}$$

2.17 From the Barometric formula, we have

$$p = p_0 e^{-Mgh/RT}$$

and from gas law $$\rho = \frac{pM}{RT}$$

So, at constant temperature from these two equations

$$\rho = \frac{Mp_0}{RT} e^{-Mgh/RT} = \rho_0 e^{-Mgh/RT} \qquad (1)$$

Eq. (1) shows that density varies with height in the same manner as pressure. Let us consider the mass element of the gas contained in the column.

$$dm = \rho(Sdh) = \frac{Mp_0}{RT} e^{-Mgh/RT} Sdh$$

Hence the sought mass,

$$m = \frac{Mp_0 S}{RT} \int_0^h e^{-Mgh/RT}\, dh = \frac{p_0 S}{g} (1 - e^{-Mgh/RT})$$

2.18 As the gravitational field is constant the centre of gravity and the centre of mass (C.M.) are same. The location of C.M.

$$h = \frac{\int_0^\infty h\,dm}{\int_0^\infty dm} = \frac{\int_0^\infty h\rho\,dh}{\int_0^\infty \rho\,dh}$$

But from Barometric formula and gas law $\rho = \rho_0 e^{-Mgh/RT}$

So,
$$h = \frac{\int_0^\infty h(e^{-Mgh/RT})dh}{\int_0^\infty (e^{-Mgh/RT})dh} = \frac{RT}{Mg}$$

2.19 (a) We know that the variation of pressure with height of a fluid is given by

$$dp = -\rho g dh$$

But from gas law
$$p = \frac{\rho}{M}RT \quad \text{or} \quad \rho = \frac{pM}{RT}$$

From these two equations

$$dp = -\frac{pMg}{RT}\,dh \tag{1}$$

or
$$\frac{dp}{p} = \frac{-Mgdh}{RT_0(1 - ah)}$$

Integrating we get,
$$\int_{p_0}^{p} \frac{dp}{p} = \frac{-Mg}{RT_0}\int_0^h \frac{dh}{(1 - ah)}$$

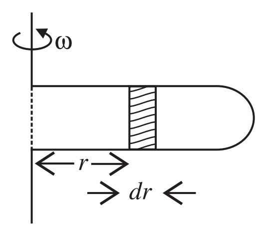
or
$$\ln\frac{p}{p_0} = \ln(1 - ah)^{Mg/aRT_0}$$

Hence,
$$p = p_0(1 - ah)^{Mg/aRT_0} \left(\text{obviously } h < \frac{1}{a}\right).$$

(b) Proceed up to Eq. (1) of part (a), and then put $T = T_0(1 + ah)$ and proceed further in the same fashion to get

$$p = \frac{p_0}{(1 + ah)^{Mg/aRT_0}}$$

2.20 Let us consider the mass element of the gas (thin layer) in the cylinder at a distance r from its open end as shown in the figure.



Using Newton's second law for the element

$$F_n = mw_n$$
$$(p + dp)\,S - pS = (\rho\,S\,dr)\,\omega^2 r$$

or $$dp = \rho\omega^2 r dr = \frac{pM}{RT}\omega^2 r dr$$

So, $$\frac{dp}{p} = \frac{M\omega^2}{RT}\, r dr$$

or $$\int_{p_0}^{p} \frac{dp}{p} = \frac{M\omega^2}{RT}\int_0^r r dr$$

Thus, $$\ln\frac{p}{p_0} = \frac{M\omega^2}{2RT} r^2 \quad \text{or} \quad p = p_0 e^{M\omega^2 r^2/2RT}$$

2.21 For an ideal gas law

$$p = \frac{\rho}{M} RT$$

On substitution, we get $$p = 0.082 \times 300 \times \frac{500}{44}\ \text{atm} = 279.5\ \text{atm}$$

From Van der Waal gas equation

$$\left(p + \frac{\nu^2 a}{V^2}\right)(V - \nu b) = \nu\ RT \quad (\text{where } V = \nu V_M)$$

or $$p = \frac{\nu RT}{V - \nu b} - \frac{a\nu^2}{V^2} = \frac{mRT/M}{V - mb/M} - \frac{am^2}{V^2 M^2}$$

$$= \frac{\rho RT}{M - \rho b} - \frac{a\rho^2}{M^2} = 79.2\ \text{atm}$$

2.22 (a) $p = \left[\frac{RT}{V_M - b} - \frac{a}{V_M^2}\right](1 + \eta) = \frac{RT}{V_M}$, (where V_M is the molar volume)

(The pressure is less for a Van der Waal gas than for an ideal gas.)

or $$\frac{a(1 + \eta)}{V_M^2} = RT\left[\frac{-1}{V_M} + \frac{-1 + \eta}{V_M - b}\right] = RT\frac{\eta V_M + b}{V_M(V_M - b)}$$

or $$T = \frac{a(1 + \eta)(V_M - b)}{RV_M(\eta V_M + b)}$$

$$= \frac{1.35 \times 1.1 \times (1 - 0.039)}{0.082 \times (0.139)} \approx 125\ \text{K}$$

(b) The corresponding pressure is

$$p = \frac{RT}{V_M - b} - \frac{a}{V_M^2} = \frac{a(1+\eta)}{V_M(\eta V_M + b)} - \frac{a}{V_M^2}$$

$$= \frac{a}{V_M^2}\frac{(V_M + \eta V_M - \eta V_M - b)}{(\eta V_M + b)} = \frac{a}{V_M^2}\frac{(V_M - b)}{(V_M + b)}$$

$$= \frac{1.35}{1} \times \frac{0.961}{0.139} \approx 9.3 \text{ atm}$$

2.23 Since, $p_1 = RT_1 \dfrac{1}{V-b} - \dfrac{a}{V^2}$ and $p_2 = RT_2 \dfrac{1}{V-b} - \dfrac{a}{V^2}$

So, $$p_2 - p_1 = \frac{R(T_2 - T_1)}{V - b}$$

or $$V - b = \frac{R(T_2 - T_1)}{p_2 - p_1} \Rightarrow b = V - \frac{R(T_2 - T_1)}{p_2 - p_1}$$

Also, $$p_1 = T_1 \frac{p_2 - p_1}{T_2 - T_1} - \frac{a}{V^2}$$

$\Rightarrow$ $$\frac{a}{V^2} = \frac{T_1(p_2 - p_1)}{T_2 - T_1} - p_1 = \frac{T_1 p_2 - p_1 T_2}{T_2 - T_1}$$

or $$a = V^2 \frac{T_1 p_2 - p_1 T_2}{T_2 - T_1}$$

Using $T_1 = 300$ K, $p_1 = 90$ atm, $T_2 = 350$ K, $p_2 = 110$ atm, $V = 0.250$, we get $a = 1.87$ atm l^2/mol^2, and $b = 0.045$ l/mol.

2.24 $$p = \frac{RT}{V - b} - \frac{a}{V^2} - V\left(\frac{\partial p}{\partial V}\right)_T = \frac{RTV}{(V-b)^2} - \frac{2a}{V^2}$$

or $$\kappa = \frac{-1}{V}\left(\frac{\partial V}{\partial p}\right)_T$$

$$= \left[\frac{RTV^3 - 2a(V-b)^2}{V^2(V-b)^2}\right]^{-1} = \frac{V^2(V-b)^2}{[RTV^3 - 2a(V-b)^2]}$$

2.25 For an ideal gas, $\kappa_0 = V/RT$.

Now, $$\kappa = \frac{(V-b)^2}{RTV}\left\{1 - \frac{2a(V-b)^2}{RTV^3}\right\}^{-1} = \kappa_0\left(1 - \frac{b}{V}\right)^2\left\{1 - \frac{2a}{RTV}\left(1 - \frac{b}{V}\right)^2\right\}^{-1}$$

$$= \kappa_0\left\{1 - \frac{2b}{V} + \frac{2a}{RTV}\right\} \text{ (to leading order in } a, b)$$

Now, $$\kappa > \kappa_0 \quad \text{if} \ \frac{2a}{RTV} > \frac{2b}{V} \quad \text{or} \quad T < \frac{a}{bR}$$

If a and b do not vary much with temperature, then the effect at high temperature is clearly determined by b and its effect is repulsive so compressibility is less.

2.2 The First Law of Thermodynamics. Heat Capacity

2.26 Internal energy of air, treating it as an ideal gas is given by

$$U = \frac{m}{M} C_V T = \frac{m}{M} \frac{R}{\gamma - 1} T = \frac{pV}{\gamma - 1} \tag{1}$$

(using ($C_V = R/(\gamma - 1)$, since $C_p - C_V = R$ and $C_p/C_V = \gamma$).

Thus at constant pressure $U =$ constant, because the volume of the room is a constant. Putting the value of $p = p_{\text{atm}}$ and V in Eq. (1), we get $U = 10$ M J.

2.27 From energy conservation

$$U_i + \frac{1}{2} (\nu M)\ v^2 = U_f$$

or $$\Delta U = (1/2)\nu M v^2 \tag{1}$$

Using, $$U = \frac{\nu RT}{\gamma - 1} \text{ (from previous problem)} \tag{2}$$

We get $$\Delta U = \frac{\nu R}{\gamma - 1} \Delta T$$

Hence from Eqs. (1) and (2),

$$\Delta T = \frac{M v^2 (\gamma - 1)}{2R}$$

2.28 On opening the valve, the air will flow from the vessel at higher pressure to the vessel at lower pressure till both vessels have the same air pressure. If this air pressure is p, the total volume of the air in the two vessels will be $(V_1 + V_2)$. Also if ν_1 and ν_2 be the number of moles of air initially in the two vessels, we have

$$p_1 V_1 = \nu_1 R T_1 \text{ and } p_2 V_2 = \nu_2 R T_2 \tag{1}$$

After the air is mixed, the total number of moles are $(v_1 + v_2)$ and the mixture is at temperature T.

Hence, $$p\,(V_1 + V_2) = (\nu_1 + \nu_2) RT \tag{2}$$

Let us look at the two portions of air as one single system. Since this system is contained in a thermally insulated vessel, no heat exchange is involved in the process. That is, total heat transfer for the combined system $Q = 0$.

Moreover, this combined system does not perform mechanical work either. The walls of the containers are rigid and there are no pistons, etc to be pushed, looking at the total system, we know $A = 0$.

Hence, internal energy of the combined system does not change in the process. Initially, energy of the combined system is equal to the sum of internal energies of the two portions of air:

$$U_i = U_1 + U_2 = \frac{\nu_1 RT_1}{\gamma - 1} + \frac{\nu_2 RT_2}{\gamma - 1} \tag{3}$$

Final internal energy of $(\nu_1 + \nu_2)$ moles of air at temperature T is given by

$$U_f = \frac{(\nu_1 + \nu_2) RT}{\gamma - 1} \tag{4}$$

Therefore, $U_i = U_f$ implies

$$T = \frac{\nu_1 T_1 + \nu_2 T_2}{\nu_1 + \nu_2} = \frac{p_1 V_1 + p_2 V_2}{(p_1 V_1/T_1) + (p_2 V_2/T_2)} = T_1 T_2 \frac{p_1 V_1 + p_2 V_2}{p_1 V_1 T_2 + p_2 V_2 T_1}$$

From Eq. (2) therefore, final pressure is given by

$$p = \frac{\nu_1 + \nu_2}{V_1 + V_2} RT = \frac{R}{V_1 + V_2}(\nu_1 T_1 + \nu_2 T_2) = \frac{p_1 V_1 + p_2 V_2}{V_1 + V_2}$$

This process in an example of free adiabatic expansion of ideal gas.

2.29 By the first law of thermodynamics, $Q = \Delta U + A$.

Here $A = 0$, as the volume remains constant.

So, $$Q = \Delta U = \frac{\nu R}{\gamma - 1}\Delta T$$

From gas law, $p_0 V = \nu RT_0$

So, $$\Delta U = -\frac{p_0 V \Delta T}{T_0(\gamma - 1)} = -0.25 \text{ kJ}$$

Hence amount of heat lost $= -\Delta U = 0.25$ kJ

2.30 By the first law of thermodynamics, $Q = \Delta U + A$.

But, $$\Delta U = \frac{p\Delta V}{\gamma - 1} = \frac{A}{\gamma - 1} \text{ (as } p \text{ is constant)}$$

So, $$Q = \frac{A}{\gamma - 1} + A = \frac{\gamma \cdot A}{\gamma - 1} = \frac{1.4}{1.4 - 1} \times 2 = 7 \text{ J}$$

2.31 Under an isobaric process,

$$A = p\Delta V = R\Delta T \text{ (as } \nu = 1) = 0.6 \text{ kJ}.$$

From the first law of thermodynamics,

$$\Delta U = Q - A = Q - R\Delta T = 1 \text{ kJ}$$

Again increment in internal energy, $\Delta U = \dfrac{R\Delta T}{\gamma - 1}$ (for $\nu = 1$)

Thus, $$Q - R\Delta T = \frac{R\Delta T}{\gamma - 1} \quad \text{or} \quad \gamma = \frac{Q}{Q - R\Delta T} = 1.6$$

2.32 Given $\nu = 2$ moles of the gas. In the first phase, under isochoric process, $A_1 = 0$, therefore from gas law if pressure is reduced n times, the temperature, i.e., new temperature becomes T_0/n.

Now from first law of thermodynamics,

$$Q_1 = \Delta U_1 = \frac{\nu R\Delta T}{\gamma - 1}$$

$$= \frac{\nu R}{\gamma - 1}\left(\frac{T_0}{n} - T_0\right) = \frac{\nu R T_0 (1 - n)}{n(\gamma - 1)}$$

During the second phase (under isobaric process), $A_2 = p\Delta V = \nu R \Delta T$.

Thus from first law of thermodynamics,

$$Q_2 = \Delta U_2 + A_2 = \frac{\nu R\Delta T}{\gamma - 1} + \nu R \Delta T$$

$$= \frac{\nu R\left(T_0 - \dfrac{T_0}{n}\right)\gamma}{\gamma - 1} = \frac{\nu R T_0 (n - 1)\gamma}{n(\gamma - 1)}$$

Hence the total amount of heat absorbed,

$$Q = Q_1 + Q_2 = \frac{\nu R T_0 (1 - n)}{n(\gamma - 1)} + \frac{\nu R T_0 (n - 1)\gamma}{n(\gamma - 1)}$$

$$= \frac{\nu R T_0 (n - 1)}{n(\gamma - 1)}(-1 + \gamma) = \nu R T_0\left(1 - \frac{1}{n}\right) = 2.5 \text{ kJ (on substituting values)}$$

2.33 Number of moles in the mixture $\nu = \nu_1 + \nu_2$

At a certain temperature, $U = U_1 + U_2$ or $\nu C_V = \nu_1 C_{V_1} + \nu_2 C_{V_2}$

Thus, $$C_V = \frac{\nu_1 C_{V_1} + \nu_2 C_{V_2}}{\nu} = \frac{\left(\nu_1 \dfrac{R}{\gamma_1 - 1} + \nu_2 \dfrac{R}{\gamma_2 - 1}\right)}{\nu}$$

Similarly, $$C_p = \frac{\nu_1 C_{p_1} + \nu_2 C_{p_2}}{\nu}$$

$$= \frac{\nu_1 \gamma_1 C_{V_1} + \nu_2 \gamma_2 C_{V_2}}{\nu} = \frac{\left(\nu_1 \dfrac{\gamma_1 R}{\gamma_1 - 1} + \nu_2 \dfrac{\gamma_2 R}{\gamma_2 - 1}\right)}{\nu}$$

Thus, $$\gamma = \frac{C_p}{C_V} = \frac{\nu_1 \dfrac{\gamma_1}{\gamma_1 - 1} R + \nu_2 \dfrac{\gamma_2}{\gamma_2 - 1} R}{\nu_1 \dfrac{R}{\gamma_1 - 1} + \nu_2 \dfrac{R}{\gamma_2 - 1}}$$

$$= \frac{\nu_1 \gamma_1 (\gamma_2 - 1) + \nu_2 \gamma_2 (\gamma_1 - 1)}{\nu_1 (\gamma_2 - 1) + \nu_2 (\gamma_1 - 1)}$$

$$= 1.33 \text{ (on substituting values)}$$

2.34 From the previous problem,

$$C_V = \frac{\nu_1 \dfrac{R}{\gamma_1 - 1} + \nu_2 \dfrac{R}{\gamma_2 - 1}}{\nu_1 + \nu_2} = 15.2 \text{ J/mol K (on substituting values)}$$

and $$C_p = \frac{\nu_1 \dfrac{\gamma_1 R}{\gamma_1 - 1} + \nu_2 \dfrac{\gamma_2 R}{\gamma_2 - 1}}{\nu_1 + \nu_2} = 23.85 \text{ J/mol K (on substituting values)}$$

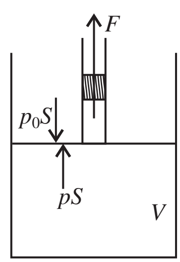
Now molar mass of the mixture $(M) = \dfrac{\text{Total mass}}{\text{Total number of moles}} = \dfrac{20 + 7}{(1/2) + (1/4)} = 36$

Hence, $$c_V = \frac{C_V}{M} = 0.42 \text{ J/gK} \quad \text{and} \quad c_p = \frac{C_p}{M} = 0.66 \text{ J/gK}$$

2.35 Let S be the area of the piston and F be the force exerted by the external agent.

Then, $F + pS = p_0 S$ (see figure) at an arbitrary instant of time. Here p is the pressure at the instant the volume is V. (Initially the pressure inside is p_0). Work done by the agent,

$$A = \int_{V_0}^{\eta V_0} F\, dx$$



$$= \int_{V_0}^{\eta V_0} (p_0 - p) S \cdot dx = \int_{V_0}^{\eta V_0} (p_0 - p) dV$$

$$= p_0(\eta - 1)V_0 - \int_{V_0}^{\eta V_0} p dV = p_0(\eta - 1)V_0 - \int_{V_0}^{\eta V_0} \nu RT \cdot \frac{dV}{V}$$

$$= (\eta - 1)\, p_0 V_0 - \nu RT \ln \eta = (\eta - 1)\, \nu RT - \nu RT \ln \eta$$

$$= \nu RT(\eta - 1 - \ln \eta) = RT(\eta - 1 - \ln \eta) \text{ (for } \nu = 1 \text{ mole)}$$

2.36 Let the agent move the piston to the right by x. In equilibrium position,

$$p_1 S + F_{agent} = p_2 S \quad \text{or} \quad F_{agent} = (p_2 - p_1)S$$

Work done by the agent in an infinitesimal change dx is

$$F_{agent} \cdot dx = (p_2 - p_1)\, Sdx = (p_2 - p_1)dV$$

By applying $pV =$ constant, for the two parts,

$$p_1 (V_0 + Sx) = p_0 V_0 \quad \text{and} \quad p_2 (V_0 - Sx) = p_0 V_0$$

So, $$p_2 - p_1 = \frac{p_0 V_0\, 2Sx}{V_0^2} - S^2x^2 = \frac{2p_0 V_0 V}{V_0^2 - V^2} \quad \text{(where } Sx = V\text{)}$$

When the volume of the left end is η times the volume of the right end,

$$(V_0 - V) = \eta (V_0 - V) \quad \text{or} \quad V = \frac{\eta - 1}{\eta + 1} V_0$$

$$A = \int_0^{v} (p_2 - p_1)dV = \int_0^{V} \frac{2p_0 V_0 V}{V_0^2 - V^2}\, dV = -p_0 V_0 \left[\ln (V_0^2 - V^2)\right]_0^V$$

$$= -p_0 V_0 \left[\ln(V_0^2 - V^2) - \ln V_0^2\right]$$

$$= -p_0 V_0 \left[\ln\left\{V_0^2 - \left(\frac{\eta - 1}{\eta + 1}\right)^2 V_0^2\right\} - \ln V_0^2\right]$$

$$= -p_0 V_0 \left(\ln \frac{4\eta}{(\eta + 1)^2}\right) = p_0 V_0 \ln \frac{(\eta + 1)^2}{4\eta}$$

2.37 In the isothermal process, heat transfer to the gas is given by

$$Q_1 = \nu RT_0 \ln \frac{V_2}{V_1} = \nu\, RT_0 \ln \eta \left(\text{for } \eta = \frac{V_2}{V_1} = \frac{p_1}{p_2}\right)$$

In the isochoric process, $A = 0$. Thus heat transfer to the gas is given by

$$Q_2 = \Delta U = \nu C_V \Delta T = \frac{\nu R}{\gamma - 1} \Delta T \left(\text{for } C_V = \frac{R}{\gamma - 1}\right)$$

But, $$\frac{p_2}{p_1} = \frac{T_0}{T} \Rightarrow T = T_0\frac{p_1}{p_2} = \eta T_0 \left(\text{for } \eta = \frac{p_1}{p_2}\right)$$

or $$\Delta T = \eta T_0 - T_0 = (\eta - 1)T_0$$

So, $$Q_2 = \frac{\nu R}{\gamma - 1}.(\eta - 1)T_0$$

Thus, net heat transfer to the gas is

$$Q = \nu RT_0 \ln \eta + \frac{\nu R}{\gamma - 1}.(\eta - 1)T_0$$

or $$\frac{Q}{\nu RT_0} = \ln \eta + \frac{\eta - 1}{\gamma - 1} \Rightarrow \frac{Q}{\nu RT_0} - \ln \eta = \frac{\eta - 1}{\gamma - 1}$$

or $$\gamma = 1 + \frac{\eta - 1}{\dfrac{Q}{\nu RT_0} - \ln \eta} = 1 + \frac{6 - 1}{\left(\dfrac{80 \times 10^3}{3 \times 8.314 \times 273}\right) - \ln 6} = 1.4$$

2.38 (a) From ideal gas law, $p = (\nu R/V)T = kT$ (where $k = \nu R/V$).

For isochoric process, $V =$ constant, thus $p = kT$, represents a straight line passing through the origin and its slope is equal to k.

For isobaric process $p =$ constant, thus on p–T curve, it is a horizontal straight line parallel to T-axis, if T is along horizontal (or x-axis).

For isothermal process, $T =$ constant, thus on p–T curve, it represents a vertical straight line if T is taken along horizontal (or x-axis).

For an adiabatic process $T^{\gamma}p^{1-\gamma} =$ constant.

On differentiating, we get

$$(1 - \gamma)\, p^{-\gamma}dp\, T^{\gamma} + \gamma p^{1-\gamma} \cdot T^{\gamma-1} \cdot dT = 0$$

So, $$\frac{dp}{dT} = \left(\frac{\gamma}{1 - \gamma}\right)\left(\frac{p^{1-\gamma}}{p^{-\gamma}}\right)\left(\frac{T^{\gamma-1}}{T^{\gamma}}\right) = \left(\frac{\gamma}{\gamma - 1}\right)\frac{p}{T}$$

The approximate plots of isochoric, isobaric, isothermal, and adiabatic processes are drawn in the answer sheet.

(b) As p is not considered variable, we have from ideal gas law

$$V = \frac{\nu R}{p}T = k'T\left(\text{where } k' = \frac{\nu R}{p}\right)$$

On V–T co-ordinate system let us, take T along x-axis.

For an isochoric process, $V =$ constant, thus $k' =$ constant and $V = k'T$ obviously represents a straight line passing through the origin of the coordinate system and k' is its slope.

For isothermal process $T =$ constant. Thus on the stated coordinate system it represents a straight line parallel to the V-axis.

For an adiabatic process, $TV^{\gamma-1} =$ constant.

On differentiating, we get

$$(\gamma - 1)V^{\gamma-2}\,dVT + V^{\gamma-1}dT = 0$$

So, $$\frac{dV}{dT} = -\left(\frac{1}{\gamma - 1}\right)\cdot\frac{V}{T}$$

The approximate plots of isochoric, isobaric, isothermal and adiabatic processes are drawn in the answer sheet.

2.39 (a) According to T–p relation in adiabatic process, $T^{\gamma} = kp^{\gamma-1}$ (where $k =$ constant).

Therefore, $$(T_2/T_1)^{\gamma} = (p_2/p_1)^{\gamma-1}$$

So, $$T^{\gamma}/T_0^{\gamma} = \eta^{\gamma-1} \text{ (for } \eta = p_2/p_1)$$

Hence, $$T = T_0 \cdot \eta^{\gamma-1/\gamma} = 290 \times 10^{(1.4-1)/1.4} = 0.56 \text{ kK}$$

(b) Using the solution of part (a), sought work done

$$A = \frac{\nu R\Delta T}{\gamma - 1} = \frac{\nu RT_0}{\gamma - 1}(\eta^{(\gamma-1)/\gamma} - 1) = 5.61 \text{ kJ (on substituting values)}$$

2.40 Let (p_0, V_0, T_0) be the initial state of the gas.

We know that work done by the gas in adiabatic process is

$$A_{\text{adia}} = \frac{-\nu R\Delta T}{\gamma - 1}$$

But from the equation $TV^{\gamma-1} =$ constant, we get $\Delta T = T_0\,(\eta^{\gamma-1} - 1)$.

Thus, $$A_{\text{adia}} = \frac{-\nu RT_0(\eta^{\gamma-1} - 1)}{\gamma - 1}$$

On the other hand, work done by the gas in isothermal process is

$$A_{\text{iso}} = \nu RT_0 \ln(1/\eta) = -\nu RT_0 \ln \eta$$

Thus, $$\frac{A_{\text{adia}}}{A_{\text{iso}}} = \frac{\eta^{\gamma-1} - 1}{(\gamma - 1)\ln \eta} = \frac{5^{0.4} - 1}{0.4 \times \ln 5} = 1.4$$

2.41 Since here the piston is conducting and it is moved slowly, the temperature on the two sides increases and is maintained at the same value.

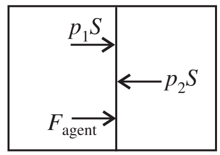
Elementary work done by the agent = work done in compression – work done in expansion, i.e., $dA = p_2dV - p_1dV = (p_2 - p_1)\,dV$, where p_1 and p_2 are pressures at any instant of the gas on expansion and compression side, respectively.

From the gas law $p_1(V_0 + Sx) = \nu RT$ and $p_2(V_0 - Sx) = \nu RT$, for each section (x is the displacement of the piston towards section 2).

So, $$p_2 - p_1 = \nu RT \frac{2Sx}{V_0^2 - S^2x^2} = \nu RT \cdot \frac{2V}{V_0^2 - V^2} \quad (\text{as } Sx = V)$$

So, $$dA = \nu RT \frac{2V}{V_0^2 - V^2}\, dV$$

Also, from the first law of thermodynamics



$$dA = -dU = -2\nu \frac{R}{\gamma - 1}\, dT \quad (\text{as } dQ = 0)$$

So, work done on the gas $= -dA = 2\nu \dfrac{R}{\gamma - 1}\, dT$

Thus, $$2\nu \frac{R}{\gamma - 1}\, dT = \nu RT \frac{2V \cdot dV}{V_0^2 - V^2}$$

or $$\frac{dT}{T} = \gamma - 1 \frac{VdV}{V_0^2 - V^2}$$

When the left end is η times the volume of the right end,

$$(V_0 + V) = \eta(V_0 - V) \quad \text{or} \quad V = \frac{\eta - 1}{\eta + 1} V_0$$

On integrating, $$\int_{T_0}^{T} \frac{dT}{T} = (\gamma - 1) \int_0^V \frac{VdV}{V_0^2 - V^2}$$

or $$\ln \frac{T}{T_0} = (\gamma - 1)\left[-\frac{1}{2} \ln\left(V_0^2 - V^2\right)\right]_0^V$$

$$= -\frac{\gamma - 1}{2}\left[\ln\left(V_0^2 - V^2\right) - \ln V_0^2\right]$$

$$= \frac{\gamma - 1}{2}\left[\ln V_0^2 - \ln V_0^2\left\{1 - \left(\frac{\eta - 1}{\eta + 1}\right)^2\right\}\right] = \frac{\gamma - 1}{2} \mathrm{In} \frac{(\eta + 1)^2}{4\eta}$$

Hence, $$T = T_0 \left(\frac{(\eta + 1)^2}{4\eta}\right)^{\frac{\gamma - 1}{2}}$$

2.42 From energy conservation as in the derivation of Bernoulli's theorem it reads

$$\frac{p}{\rho} + \frac{1}{2} \mathrm{v}^2 + gz + u + Q_d = \text{constant} \tag{1}$$

In Eq. (1), u is the internal energy per unit mass and in this case is the thermal energy per unit mass of the gas. As the gas vessel is thermally insulated $Q_d = 0$. Just inside the vessel,

$$u = \frac{C_V T}{M} = \frac{RT}{M(\gamma - 1)} \quad \text{also} \quad \frac{p}{\rho} = \frac{RT}{M}$$

Inside the vessel $v = 0$ and just outside $p = 0$, and $u = 0$. In general gz is not very significant for gases.

Thus applying Eq. (1) just inside and outside the hole, we get

$$\frac{1}{2}v^2 = \frac{p}{\rho} + u$$

$$= \frac{RT}{M} + \frac{RT}{M(\gamma-1)} = \frac{\gamma RT}{M(\gamma-1)}$$

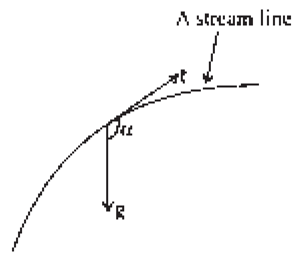
Hence, $$v^2 = \frac{2\gamma RT}{M(\gamma - 1)}$$

or $$v = \sqrt{\frac{2\gamma RT}{M(\gamma - 1)}} = 3.22 \text{ km/s}$$

Alternate:

From Eulerian equation

$$-\nabla p + \rho \mathbf{g} = \rho \mathbf{a}$$



In projection form along a stream line for steady flow, it becomes,

$$\frac{-\partial p}{\partial s} + g\cos\alpha = \rho v \frac{\partial v}{\partial s}$$

In this problem,

$$\frac{-dp}{ds} = \rho v \frac{dv}{ds} \text{ (only because } \alpha \text{ becomes } 90°)$$

Experiments show, when the particles move sufficiently fast it can be assumed that the process is adiabatic. Then the connection between the pressure and the density is given by the relation,

$$\frac{p}{\rho^\gamma} = \frac{p_i}{\rho_i^\gamma} = \text{constant}$$

where γ is the adiabatic exponent dependent on the nature of the gas and ρ_i is the density of the gas inside the vessel. The adiabatic relation is a consequence of the fact that during the expansion of a gas there is no heat exchange with the surrounding, that is there is no loss or gain of heat.

On finding the density as a function of the pressure and substituting, we obtain the equation

$$-p^{-1/\gamma}\frac{dp}{ds} = \frac{\rho_i}{p_i^{1/\gamma}}\,v\,\frac{dv}{ds}$$

This equation should be integrated along the stream tube. Since the pressure in the vessel is p_i and the pressure outside the vessel is p_0, we must integrate from p_i to p_0 for the pressure and from zero to $v = v_0$ (the outlet velocity) for the velocity.

Thus,
$$-\int_{p_i}^{p_0} p^{-1/\gamma}\,dp = \frac{\rho_i}{p_i^{1/\gamma}}\int_0^{v_0} v\,dv$$

On performing the integration we find the outlet velocity,

$$v = \sqrt{\frac{2\gamma}{\gamma-1}\,\frac{p_i}{\rho_i}\left[1-\left(\frac{p_0}{p_i^{1/\gamma}}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

According the problem $p_0 = 0$ because outside is vacuum.

Hence,
$$v = \sqrt{\frac{2\gamma}{\gamma-1}\,\frac{p_i}{\rho_i}} = \sqrt{\frac{2\gamma}{\gamma-1}\,\frac{RT}{M}} = 3.3 \text{ km/s (on substituting values)}$$

Note: *The velocity here is the velocity of hydrodynamic flow of the gas into vacuum. This requires that the diameter of the hole is not too small ($D >$ mean free path l). In the opposite case ($D \ll l$), the flow is called effusion. Then the above result does not apply and kinetic theory methods are needed.*

2.43 The differential work done by the gas,

$$dA = pdV = \frac{\nu RT^2}{a}\left(-\frac{a}{T^2}\right)dT = -\nu RdT \text{ (as } pV = \nu RT \text{ and } V = a/T)$$

So,
$$A = -\int_T^{T+\Delta T} \nu RdT = -\nu R\Delta T$$

From the first law of thermodynamics

$$Q = \Delta U + A = \frac{\nu R}{\gamma-1}\Delta T - \nu R\Delta T$$

$$= \nu R\Delta T\cdot\frac{2-\gamma}{\gamma-1} = R\Delta T\cdot\frac{2-\gamma}{\gamma-1} \text{ (for } \nu = 1 \text{ mole)}$$

2.44 According to the problem: $A \propto U$ or $dA = aU$ (where a is proportionality constant).

So,
$$pdV = \frac{a\nu RdT}{\gamma-1} \tag{1}$$

From ideal gas law, $pV = \nu RT$, which on differentiating gives

$$pdV + Vdp = \nu RdT \tag{2}$$

Thus, from Eqs. (1) and (2)

$$pdV = \frac{a}{\gamma - 1}(pdV + Vdp)$$

or $$pdV\left(\frac{a}{\gamma - 1} - 1\right) + \frac{a}{\gamma - 1} Vdp = 0$$

or $$pdV(k - 1) + kVdp = 0 \text{ (where } k = \frac{a}{\gamma - 1} = \text{another constant)}$$

or $$pdV\frac{k - 1}{k} + Vdp = 0$$

or $$npdV + Vdp = 0 \left(\text{where } \frac{k - 1}{k} = n = \text{ratio}\right)$$

Dividing both the sides by pV, we get

$$n\frac{dV}{V} + \frac{dp}{p} = 0$$

On integrating we get,

$$n \ln V + \ln p = \ln C \quad \text{(where } C \text{ is constant)}$$

or $$\ln (pV^n) = \ln C \quad \text{or} \quad pV^n = C$$

2.45 In the polytropic process work done by the gas is given by,

$$A = \frac{\nu R\left[T_i - T_f\right]}{n - 1}$$

(where T_i and T_f are initial and final temperatures of the gas like in an adiabatic process).

Also, $$\Delta U = \frac{\nu R}{\gamma - 1}(T_f - T_i)$$

By the first law of thermodynamics,

$$Q = \Delta U + A$$
$$= \frac{\nu R}{\gamma - 1}(T_f - T_i) + \frac{\nu R}{n - 1}(T_i - T_f)$$
$$= (T_f - T_i)\nu R\left[\frac{1}{\gamma - 1} - \frac{1}{n - 1}\right] = \frac{\nu R[n - \gamma]}{(n - 1)(\gamma - 1)}\Delta T$$

According to definition of molar heat capacity when number of moles $\nu = 1$ and $\Delta T = 1$, then $Q =$ molar heat capacity.

Here, $$C_n = \frac{R(n-\gamma)}{(n-1)(\gamma-1)} < 0 \text{ for } 1 < n < \gamma$$

2.46 Let the process be polytropic, according to the law $pV^n =$ constant.

Thus, $$p_f V_f^n = p_i\, V_i^n$$

or $$(p_i/p_f) = \beta$$

So, $$\alpha^n = \beta \quad \text{or} \quad \ln\beta = n\ln\alpha \quad \text{or} \quad n = \frac{\ln\beta}{\ln\alpha}$$

In the polytropic process molar heat capacity is given by

$$C_n = \frac{R\,(n-\gamma)}{(n-1)(\gamma-1)} = \frac{R}{\gamma-1} - \frac{R}{n-1}$$

$$= \frac{R}{\gamma-1} - \frac{R\ln\alpha}{\ln\beta\,\ln\alpha}\left(\text{where } n = \frac{\ln\beta}{\ln\alpha}\right)$$

So, $$C_n = \frac{8.314}{1.66-1} - \frac{8.314\ln 4}{\ln 8 - \ln 4} = -42 \text{ J/mol K}$$

2.47 (a) Increment of internal energy for ΔT becomes

$$\Delta U = \frac{\nu R\Delta T}{\gamma-1} = \frac{R\Delta T}{\gamma-1} = -324 \text{ J (as } \nu = 1 \text{ mole)}$$

From first law of thermodynamics

$$Q = \Delta U + A = \frac{R\Delta T}{\gamma-1} - \frac{R\Delta T}{n-1} = 0.11 \text{ kJ}$$

(b) Sought work done,

$$A_n = \int p\,dV = \int_{V_i}^{V_f} \frac{k}{V^n}\,dV \left(\text{where } pV^n = k = p_i V_i^n = p_f V_f^n\right)$$

$$= \frac{k}{1-n}\left(V_f^{1-n} - V_i^{1-n}\right) = \frac{\left(p_f V_f^n\, V_f^{1-n} - p_i V_i^n\, V_i^{1-n}\right)}{1-n}$$

$$= \frac{p_f V_f - p_i V_i}{1-n} = \frac{\nu R(T_f - T_i)}{1-n}$$

$$= \frac{\nu R\Delta T}{n-1} = -\frac{R\Delta T}{n-1} = 0.43 \text{ kJ (as } \nu = 1 \text{ mole)}$$

2.48 Law of the process is, $p = \alpha V$ or $pV^{-1} = \alpha$, so the process is polytropic with index $n = -1$. As $p = \alpha V$ so, $p_i = \alpha V_0$ and $p_f = \alpha\eta\, V_0$.

(a) Increment of the internal energy is given by

$$\Delta U = \frac{\nu R}{\gamma - 1}[T_f - T_i] = \frac{p_f V_f - p_i V_i}{\gamma - 1}$$

Putting $p_i = \alpha V_0$ and $p_f = \alpha \eta V_0$, we get

$$\Delta U = \frac{\alpha V_0^2(\eta^2 - 1)}{\gamma - 1}$$

(b) Work done by the gas is given by

$$A = \frac{p_i V_i - p_f V_f}{n - 1} = \frac{\alpha V_0^2 - \alpha\eta V_0 \cdot \eta V_0}{-1 - 1}$$

$$= \frac{\alpha V_0^2(1 - \eta^2)}{-2} = \frac{1}{2}\alpha V_0^2(\eta^2 - 1)$$

(c) Molar heat capacity is given by

$$C_n = \frac{R(n - \gamma)}{(n - 1)(\gamma - 1)} = \frac{R(-1 - \gamma)}{(-1 - 1)(\gamma - 1)} = \frac{R}{2} \cdot \frac{\gamma + 1}{\gamma - 1}$$

2.49 (a) $\Delta U = \frac{\nu R}{\gamma - 1}\Delta T$ and $Q = \nu C_n \Delta T$, where C_n is the molar heat capacity in the process. It is given that $Q = -\Delta U$.

So,
$$C_n \Delta T = -\frac{R}{\gamma - 1}\Delta T \quad \text{or} \quad C_n = -\frac{R}{\gamma - 1}$$

(b) By the first law of thermodynamics,

$$dQ = dU + dA$$

or
$$2dQ = dA \quad (\text{as } dQ = -dU)$$

$$2\nu C_n dT = pdV \quad \text{or} \quad \frac{2R\nu}{\gamma - 1} dT + pdV = 0$$

So,
$$\frac{2Rv}{\gamma - 1} dT + \frac{\nu RT}{V} dV = 0 \quad \text{or} \quad \frac{2dT}{(\gamma - 1)T} + \frac{dV}{V} = 0$$

or
$$\frac{dT}{T} + \frac{\gamma - 1}{2}\frac{dV}{V} = 0 \Rightarrow TV^{(\gamma-1)/2} = \text{constant}$$

(c) We know $C_n = \frac{(n - \gamma)R}{(n - 1)(\gamma - 1)}$

But from part (a), we have $C_n = -R/(\gamma - 1)$.

Thus, $$-\frac{R}{\gamma - 1} = \frac{(n-\gamma)R}{(n-1)(\gamma-1)}$$

which yields $$n = \frac{1+\gamma}{2}$$

From part (b), we know $TV^{(\gamma-1)/2}$ = constant.

So, $$\frac{T_0}{T} = \left(\frac{V}{V_0}\right)^{(\gamma-1)/2} = \eta^{(\gamma-1)/2} \text{ (where } T \text{ is the final temperature)}$$

Work done by the gas for one mole is given by

$$A = \frac{R(T_0 - T)}{n-1} = \frac{2RT_0}{\gamma - 1}\left[1 - \eta^{(1-\gamma/2)}\right]$$

2.50 Given $p = aT^\alpha$ (for one mole of gas).

So, $$pT^{-\alpha} = a \quad \text{or} \quad p\left(\frac{pV}{R}\right)^{-\alpha} = a$$

$$\Rightarrow \quad p^{1-\alpha}V^{-\alpha} = aR^{-\alpha} \quad \text{or} \quad pV^{\alpha/(\alpha-1)} = \text{constant}$$

Here polytropic exponent $n = \alpha/(\alpha - 1)$.

(a) In the polytropic process for one mole of gas,

$$A = \frac{R\Delta T}{1-n} = \frac{R\Delta T}{1 - [\alpha/(\alpha-1)]} = R\Delta T(1-\alpha)$$

(b) Molar heat capacity is given by

$$C = \frac{R}{\gamma-1} - \frac{R}{n-1} = \frac{R}{\gamma-1} - \frac{R}{[\alpha/(\alpha-1)]-1}$$
$$= \frac{R}{\gamma-1} + R(\alpha-1)$$

2.51 Given $U = aV^\alpha$

or $$\nu C_V T = aV^\alpha \quad \text{or} \quad \nu C_V \frac{pV}{\nu R} = aV^\alpha$$

$$\Rightarrow \quad aV^\alpha \cdot \frac{R}{C_V} \cdot \frac{1}{pV} = 1 \quad \text{or} \quad V^{\alpha-1} \cdot p^{-1} = \frac{C_V}{Ra}$$

or $$pV^{1-\alpha} = Ra/C_V = \text{constant} = a(\gamma-1) \text{ [as } C_V = R/(\gamma-1)]$$

So polytropic index $n = 1 - \alpha$.

(a) Work done by the gas is given by

$$A = \frac{-\nu R \Delta T}{n-1} \quad \text{and} \quad \Delta U = \frac{\nu R \Delta T}{\gamma - 1}$$

Hence, $$A = \frac{-\Delta U(\gamma - 1)}{n-1} = \frac{\Delta U(\gamma - 1)}{\alpha} \quad (\text{as } n = 1 - \alpha)$$

By the first law of thermodynamics

$$Q = \Delta U + A$$
$$= \Delta U + \frac{\Delta U(\gamma - 1)}{\alpha} = \Delta U\left[1 + \frac{\gamma - 1}{\alpha}\right]$$

(b) Molar heat capacity is given by

$$C = \frac{R}{\gamma - 1} - \frac{R}{n-1} = \frac{R}{\gamma - 1} - \frac{R}{1 - \alpha - 1}$$
$$= \frac{R}{\gamma - 1} + \frac{R}{\alpha} \quad (\text{as } n = 1 - \alpha)$$

2.52 (a) By the first law of thermodynamics

$$dQ = dU + dA = \nu C_V dT + p dV$$

Molar specific heat according to definition is given by

$$C = \frac{dQ}{\nu dT} = \frac{\nu C_V dT + p dV}{\nu dT}$$
$$= \frac{\nu C_V dT + (\nu RT/V) dV}{\nu dT} = C_V + \frac{RT}{V}\frac{dV}{dT}$$

We have $$T = T_0 e^{\alpha V}$$

On differentiating, $dT = \alpha T_0 e^{\alpha V} \cdot dV$

So, $$\frac{dV}{dt} = \frac{1}{\alpha T_0 e^{\alpha V}}$$

Hence, $$C = C_V + \frac{RT}{V} \cdot \frac{1}{\alpha T_0 e^{\alpha V}} = C_V + \frac{RT_0 e^{\alpha V}}{\alpha V T_0 e^{\alpha V}} = C_V + \frac{R}{\alpha V}$$

(b) For the process, $p = p_0 e^{\alpha V}$.

So, $$p = RT/V = p_0 e^{\alpha V}$$

or $$T = (p_0/R) e^{\alpha V} \cdot V$$

So, $$C = C_V + \frac{RT}{V}\frac{dV}{dT} = C_V + p_0 e^{\alpha V} \cdot \frac{R}{p_0 e^{\alpha V}(1 + \alpha V)} = C_V + \frac{R}{1 + \alpha V}$$

2.53 (a) Using solution of Problem 2.52,

$$C = C_V + \frac{RT}{V}\frac{dV}{dT} = C_V + \frac{pdV}{dT} \text{ (for one mole of gas)}$$

We have, $p = p_0 + \frac{\alpha}{V'}$ or $\frac{RT}{V} = p_0 + \frac{\alpha}{V'} \Rightarrow RT = p_0 V + \alpha$

Therefore, $RdT = p_0 dV$

So, $dV/dT = R/p_0$

Hence, $C = C_V + \left(p_0 + \frac{\alpha}{V}\right)\cdot\frac{R}{p_0} = \frac{R}{\gamma - 1} + \left(1 + \frac{\alpha}{p_0 V}\right)$

$$R = \frac{R\gamma}{\gamma - 1} + \frac{\alpha R}{p_0 V}$$

(b) Work done is given by

$$A = \int_{V_1}^{V_2}\left(p_0 + \frac{\alpha}{V}\right)dV = p_0(V_2 - V_1) + \alpha \ln\frac{V_2}{V_1}$$

$$\Delta U = C_V(T_2 - T_1) = C_V\left(\frac{p_2 V_2}{R} - \frac{p_1 V_1}{R}\right) \text{ (for one mole)}$$

$$= \frac{R}{(\gamma - 1)R}(p_2 V_2 - p_1 V_1)$$

$$= \frac{1}{\gamma - 1}\left[(p_0 + \alpha V_2) V_2 - \left(p_0 + \frac{\alpha}{V_1}\right) V_1\right] = \frac{p_0(V_2 - V_1)}{\gamma - 1}$$

By the first law of thermodynamics,

$$Q = \Delta U + A$$

$$= \frac{p_0(V_2 - V_1)}{(\gamma - 1)} + p_0(V_2 - V_1) + \alpha \ln\frac{V_2}{V_1}$$

$$= \frac{\gamma p_0 (V_2 - V_1)}{(\gamma - 1)} + \alpha \ln\frac{V_2}{V_1}$$

2.54 (a) From solution of Problem 2.52, heat capacity is given by

$$C = C_V + \frac{RT}{V}\frac{dV}{dT}$$

We have $T = T_0 + \alpha V \Rightarrow V = \frac{T}{\alpha} - \frac{T_0}{\alpha}$

On differentiating, we get

$$\frac{dV}{dT} = \frac{1}{\alpha}$$

Hence, $$C = C_V + \frac{RT}{V}\cdot\frac{1}{\alpha} = \frac{R}{\gamma - 1} + \frac{R(T_0 + \alpha V)}{V}\cdot\frac{1}{\alpha}$$

$$= \frac{R}{\gamma - 1} + R\left(\frac{T_0}{\alpha V} + 1\right) = \frac{\gamma R}{\gamma - 1} + \frac{RT_0}{\alpha V} = C_V + \frac{RT}{\alpha V} = C_p + \frac{RT_0}{\alpha V}$$

(b) Given that $T = T_0 + \alpha V$ and as $T = pV/R$ for one mole of an ideal gas, so

$$p = \frac{R}{V}(T_0 + \alpha V) = \frac{RT_0}{V} + \alpha R$$

Now, $$A = \int_{V_1}^{V_2} p dV = \int_{V_1}^{V_2}\left(\frac{RT_0}{V} + \alpha R\right) dV \quad \text{(for one mole)}$$

$$= RT_0 \ln\frac{V_2}{V_1} + \alpha(V_2 - V_1)R$$

Also, $$\Delta U = C_V(T_2 - T_1)$$

$$= C_V[T_0 + \alpha V_2 - (T_0 + \alpha V_1)] = \alpha C_V(V_2 - V_1)$$

By the first law of thermodynamics

$$Q = \Delta U + A$$

$$= \frac{\alpha R}{\gamma - 1}(V_2 - V_1) + RT_0 \ln\frac{V_2}{V_1} + \alpha R(V_2 - V_1)$$

$$= \alpha R(V_2 - V_1)\left[1 + \frac{1}{\gamma - 1}\right] + RT_0 \ln\frac{V_2}{V_1}$$

$$= \alpha C_p(V_2 - V_1) + RT_0 \ln\frac{V_2}{V_1}$$

2.55 Heat capacity is given by

$$C = C_V + \frac{RT}{V}\frac{dV}{dT}$$

(a) Given that

$$C = C_V + \alpha T$$

So, $$C_V + \alpha T = C_V + \frac{RT}{V}\frac{dV}{dT} \quad \text{or} \quad \frac{\alpha}{R}\,dT = \frac{dV}{V}$$

Integrating both sides, we get

$$\frac{\alpha}{R}T = \ln V + \ln C_0 = \ln VC_0 \text{ (where } C_0 \text{ is a constant)}$$

or $$V \cdot C_0 = e^{-\alpha T/R}$$

Hence, $$V \cdot e^{-\alpha T/R} = \frac{1}{C_0} = \text{constant}$$

(b) According to the problem $C = C_V + \beta V$.

But, $$C = C_V + \frac{RT}{V}\frac{dV}{dT}$$

So, $$\frac{RT}{V}\frac{dV}{dT} = \beta V$$

or $$\frac{R}{\beta}V^{-2}\,dV = \frac{dT}{T}$$

Integrating both sides, we get

$$-\frac{R}{\beta}V^{-1} = \ln T + \ln C_0 = \ln T \cdot C_0 \text{ (where } C_0 \text{ is a constant)}$$

So, $$\ln T \cdot C_0 = -\frac{R}{\beta V}$$

or $$T \cdot C_0 = e^{-R/\beta V}$$

Hence, $$Te^{R/\beta V} = \frac{1}{C_0} = \text{constant}$$

(c) Given that $$C = C_V + ap \quad \text{and} \quad C = C_V + \frac{RT}{V}\frac{dV}{dT}$$

So, $$C_V + ap = C_V + \frac{RT}{V}\frac{dV}{dT} \Rightarrow ap = \frac{RT}{V}\frac{dV}{dT}$$

or $$a\frac{RT}{V} = \frac{RT}{V}\frac{dV}{dT} \text{ (as } p = \frac{RT}{V} \text{ for one mole of gas)}$$

or $$dV/dT = a \Rightarrow dV = a\,dT \quad \text{or} \quad dT = dV/a$$

So, $$T = \frac{V}{a} + \text{ constant} \quad \text{or} \quad V - aT = \text{constant}$$

2.56 (a) By the first law of thermodynamics

$$A = Q - \Delta U$$

or $$A = CdT - C_V dT = (C - C_V)\, dT \text{ (for one mole)}$$

Given that $C = \alpha/T$

So, $$A = \int_{T_0}^{\eta T_0} \left(\frac{\alpha}{T} - C_V\right) dT = \alpha \ln \frac{\eta T_0}{T_0} - C_V(\eta T_0 - T_0)$$

$$= \alpha \ln \eta - C_V T_0 (\eta - 1)$$

$$= \alpha \ln \eta - \frac{RT_0}{\gamma - 1}(\eta - 1)$$

(b) Given that $$C = +\frac{dQ}{dT} = \frac{RT}{V}\frac{dV}{dT} + C_V$$

and $C = \alpha/T$

So, $$C_V + \frac{RT}{V}\frac{dV}{dT} = \frac{\alpha}{T}$$

or $$\frac{R}{\gamma - 1}\frac{1}{RT} + \frac{dV}{V} = \frac{\alpha}{RT^2}\, dT$$

or $$\frac{dV}{V} = \frac{\alpha}{RT^2} dT - \frac{1}{\gamma - 1} \cdot \frac{dT}{T}$$

or $$(\gamma - 1)\frac{dV}{V} = \frac{\alpha(\gamma - 1)}{RT^2} dT - \frac{dT}{T}$$

Integrating both sides, we get

$$(\gamma - 1) \ln V = -\frac{\alpha(\gamma - 1)}{RT} - \ln T + \ln K$$

or $$\ln\left[V^{\gamma - 1} \cdot \frac{T}{K}\right] = \frac{-\alpha(\gamma - 1)}{RT}$$

$$\ln\left[V^{\gamma - 1} \cdot \frac{pV}{RK}\right] = \frac{-\alpha(\gamma - 1)}{pV}$$

or $$pV^{\gamma}/RK = e^{-\alpha(\gamma - 1)/pV}$$

or $$pV^{\gamma} e^{\alpha(\gamma - 1)/pV} = RK = \text{constant}$$

2.57 The work done is

$$A = \int_{V_1}^{V_2} p\,dV = \int_{V_1}^{V_2} \left(\frac{RT}{V-b} - \frac{a}{V^2}\right) dV$$

$$= RT \ln \frac{V_2 - b}{V_1 - b} + a\left(\frac{1}{V_2} - \frac{1}{V_1}\right)$$

2.58 (a) The increment in the internal energy is

$$\Delta U = \int_{V_1}^{V_2} \left(\frac{\partial U}{\partial V}\right)_T dV$$

But from the second law of thermodynamics

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial s}{\partial V}\right)_T - p = T\left(\frac{\partial p}{\partial T}\right)_V - p$$

On the other hand

$$p = \frac{RT}{V-b} - \frac{a}{V^2}$$

or $$T\left(\frac{\partial p}{\partial T}\right)V = \frac{RT}{V-b} \quad \text{and} \quad \left(\frac{\partial U}{\partial V}\right)_T = \frac{a}{V^2}$$

So, $$\Delta U = a\left(\frac{1}{V_1} - \frac{1}{V_2}\right) = 0.11 \text{ kJ (on substituting values)}$$

(b) From the first law of thermodynamics

$$Q = A + \Delta U = RT \ln \frac{V_2 - b}{V_1 - b} = 3.8 \text{ kJ (on substituting values)}$$

2.59 (a) From the first law, for an adiabatic process, $dQ = dU + pdV = 0$.

From the previous problem

$$dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV = C_V dT + \frac{a}{V^2} dV$$

So, $$0 = C_V dT + \frac{RTdV}{V-b}$$

This equation can be integrated if we assume that C_V and b are constant, then

$$\frac{R}{C_V}\frac{dV}{V-b} + \frac{dT}{T} = 0$$

or $$\ln T + \frac{R}{C_V} \ln (V-b) = \text{constant} \quad \text{or} \quad T(V-b)^{R/C_V} = \text{constant}$$

(b) We use
$$dU = C_V dT + \frac{a}{V^2}\, dV$$

Now,
$$dQ = C_V dT + \frac{RT}{V-b} dV$$

So along constant p,
$$C_p = C_V + \frac{RT}{V-b}\left(\frac{\partial V}{\partial T}\right)_p$$

Thus,
$$C_p - C_V = \frac{RT}{V-b}\left(\frac{\partial V}{\partial T}\right)_p$$

But,
$$p = \frac{RT}{V-b}\,\frac{a}{V^2}$$

On differentiating,
$$0 = \left(-\frac{RT}{(V-b)^2} + \frac{2a}{V^2}\right)\left(\frac{\partial V}{\partial T}\right)_P + \frac{R}{V-b}$$

or
$$T\left(\frac{\partial V}{\partial T}\right)_P = \frac{RT/V-b}{\dfrac{RT}{(V-b)^2} - \dfrac{2a}{V^3}} = \frac{V-b}{1 - \dfrac{2a(V-b)^2}{RTV^3}}$$

and
$$C_p - C_V = \frac{R}{1 - \dfrac{2a(V-b)^2}{RTV^3}}$$

2.60 From the first law

$$Q = U_f - U_i + A = 0 \text{ (as the vessels are thermally insulated)}$$

As this is free expansion, $A = 0$, so, $U_f = U_i$.

But,
$$U = \nu C_V T - \frac{a\nu^2}{V}$$

So,
$$C_V(T_f - T_i) = \left(\frac{a}{V_1 + V_2} - \frac{a}{V_1}\right)\nu = \frac{-aV_2\nu}{V_1(V_1 + V_2)}$$

or
$$\Delta T = \frac{-a(\gamma - 1)V_2\nu}{RV_1(V_1 + V_2)}$$

On substitution we get, $\Delta T = -3$ K.

2.61 From the first law

$$Q = U_f - U_i + A = U_f - U_i \text{ (as } A = 0 \text{ in free expansion).}$$

So at constant temperature,

$$Q = \frac{-av^2}{V_2} - \left(-\frac{av^2}{V_1}\right) = av^2\,\frac{V_2 - V_1}{V_1 \cdot V_2}$$

$$= 0.33 \text{ kJ (on substituting values)}$$

2.3 Kinetic Theory of Gases. Boltzmann's Law and Maxwell's Distribution

2.62 From the formula $p = nkT$

$$n = \frac{p}{kT} = \frac{4 \times 10^{-15} \times 1.01 \times 10^5}{1.38 \times 10^{23} \times 300} \text{ m}^{-3}$$

$$= 1 \times 10^{11} \text{ per m}^3 = 10^5 \text{ cm}^{-3}$$

Mean distance between molecules

$$(10^{-5} \text{ cm}^3)^{1/3} = 10^{1/3} \times 10^{-2} \text{ cm} = 0.2 \text{ mm}$$

2.63 After dissociation each N_2 molecule becomes two N-atoms and so contributes, 2×3 degrees of freedom. Thus the number of moles becomes

$$\frac{m}{M}(1 + \eta)$$

and $$p = \frac{mRT}{MV}(1 + \eta) = 1.9 \text{ atm (on substituting values)}$$

(Here M is the molecular weight, in grams of N_2.)

2.64 Let n_1 = number density of He atoms, n_2 = number density of N_2 molecules.

Then, $$\rho = n_1 m_1 + n_2 m_2 \tag{1}$$

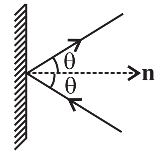
(where m_1 and m_2 are masses of helium and nitrogen molecules).

Also, $$p = (n_1 + n_2)\,kT \tag{2}$$

From Eqs. (1) and (2), we get

$$n_1 = \frac{\left(\dfrac{p}{kT} - \dfrac{\rho}{m_2}\right)}{\left(1 - \dfrac{m_1}{m_2}\right)} = 1.6 \times 10^{-19} \text{ cm}^{-3}$$

2.65 Number of nitrogen molecules coming to strike the differential area ds of the wall per unit time is given by



$$\frac{dN}{dt} = n|(\mathbf{v}\cdot d\mathbf{S})| = (nv\,dS\cos\theta)$$

where $n = N/V$ = number density or concentration of N_2 molecules in the beam.

The change in linear momentum of each of N_2 molecule is

$$\Delta\mathbf{p}_{\text{molecule}} = 2mv\cos\theta\,\mathbf{n} \text{ (see figure)}$$

So $\Delta\mathbf{p}$ for all the molecules striking the wall per unit time, i.e., the force exerted by the wall on incoming N_2 molecules is

$$(nvds\cos\theta)\Delta\mathbf{p}_{\text{molecule}} = 2nv^2\cos^2\theta\,dS\,\mathbf{n}$$

Therefore from Newton's law, the force exerted by N_2 molecules on the wall is

$$2nmv^2\cos^2\theta\;dS\,(-\mathbf{n})$$

Thus the sought pressure i.e. normal force per unit differential is

$$\frac{dF_{\text{normal}}}{ds} = 2nmv^2\cos^2\theta$$

2.66 From the formula

$$v = \sqrt{\frac{\gamma p}{\rho}} \quad \text{or} \quad \gamma = \frac{\rho v^2}{p}$$

If i = number of degrees of freedom of the gas, then

$$C_p = C_V + RT \quad \text{and} \quad C_V = \frac{i}{2}RT$$

$$\gamma = \frac{C_p}{C_V} = 1 + \frac{2}{i} \quad \text{or} \quad i = \frac{2}{\gamma - 1}$$

$$= \frac{2}{(\rho v^2/p) - 1} = 5 \text{ (on substituting values)}$$

2.67 $v_{\text{sound}} = \sqrt{\dfrac{\gamma p}{\rho}} = \sqrt{\dfrac{\gamma RT}{M}}$ and $v_{\text{rms}} = \sqrt{\dfrac{3kT}{m}} = \sqrt{\dfrac{3RT}{M}}$

So,
$$\frac{v_{\text{sound}}}{v_{\text{rms}}} = \sqrt{\frac{\gamma}{3}} = \sqrt{\frac{i+2}{3i}}$$

(a) For monoatomic gases $i = 3$, therefore

$$\frac{v_{\text{sound}}}{v_{\text{rms}}} = \sqrt{\frac{5}{9}} = 0.75$$

(b) For rigid diatomic molecules $i = 5$, therefore

$$\frac{v_{\text{sound}}}{v_{\text{rms}}} = \sqrt{\frac{7}{15}} = 0.68$$

2.68 For a general non-collinear, non-planar molecule, mean energy

$$= \frac{3}{2}kT\,(\text{translational}) + \frac{3}{2}kT\,(\text{rotational}) + (3N-6)kT\,(\text{vibrational})$$

$$= (3N-3)kT\,(\text{per molecule})$$

For linear molecules, mean energy

$$= \frac{3}{2}kT\,(\text{translational}) + kT\,(\text{rotational}) + (3N-5)kT\,(\text{vibrational})$$

$$= \left(3N - \frac{5}{2}\right) kT\,(\text{per molecule})$$

Translational energy corresponds to fraction $\frac{1}{2(N-1)}$ and $\frac{1}{2N-5/3}$ in non-linear and linear molecules, respectively.

2.69 (a) A diatomic molecule has 3 translational, 2 rotational and one vibrational degrees of freedom. The corresponding energy per mole is

$$\frac{3}{2}\,RT\,(\text{for translational}) + 2 \times \frac{1}{2}\,RT\,(\text{for rotational}) + 1 \times RT\,(\text{for vibrational})$$

$$= \frac{7}{2}RT$$

Thus, $$C_V = \frac{7}{2}R \quad \text{and} \quad \gamma = \frac{C_p}{C_V} = \frac{9}{7}$$

(b) For linear N-atomic molecules, energy per mole

$$= \left(3N - \frac{5}{2}\right)RT \;\; (\text{as before})$$

So, $$C_V = \left(3N - \frac{5}{2}\right)R \quad \text{and} \quad \gamma = \frac{6N-3}{6N-5}$$

(c) For non-collinear N-atomic molecules

$$C_V = 3(N-1)\,\text{R (as before from Problem 2.68)}$$

and $$\gamma = \frac{3N-2}{3N-3} = \frac{N-2/3}{N-1}$$

2.70 In the isobaric process, work done is $A = pdV = RdT$ (per mole).

On the other hand heat transferred is $Q = C_p\, dT$.

Now $C_p = (3N - 2)\,R$ for non-collinear molecules and $C_p = [3N - (3/2)]\,R$ for linear molecules.

Thus,
$$\frac{A}{Q} = \begin{cases} \dfrac{1}{3N-2} & \text{non-collinear} \\ \dfrac{1}{3N-(3/2)} & \text{linear} \end{cases}$$

For monoatomic gases, $C_p = 5/2$ and $A/Q = 2/5$

2.71 Given specific heats c_p, c_V (per unit mass), then

$$M(c_p - c_V) = R$$

or
$$M = \frac{R}{c_p - c_V} = 32 \text{ g/mol (on substituting values)}$$

Also,
$$\gamma = \frac{c_p}{c_V} = \frac{2}{i} + 1$$

or
$$i = \frac{2}{(c_p/c_V) - 1} = \frac{2c_V}{c_p - c_V} = 5 \text{ (on substituting values)}$$

2.72 (a) Given,
$$C_p = 29 \text{ J/K mol} = \frac{29}{8.3}R$$

$$C_V = \frac{20.7}{8.3}R, \quad \gamma = \frac{29}{20.7} = 1.4 = \frac{7}{5}$$

$$i = 2\left(\frac{C_p}{R} - 1\right) = 5$$

(b) In the process, $pT =$ constant and $(T^2/V) =$ constant.

So,
$$2\frac{dT}{T} - \frac{dV}{V} = 0$$

Thus,
$$CdT = C_V\, dT + pdV = C_V dT + \frac{RT}{V}\,dV = C_V dT + \frac{2RT}{T}\,dT$$

or
$$C = C_V + 2R = \left(\frac{29}{8.3}\right)R \Rightarrow C_V = \frac{12.4}{8.3}R = \frac{3}{2}R$$

Hence,
$$i = 2\left[\frac{C}{R} + \frac{1}{(n-1)}\right] = 3 \text{ (monoatomic)}$$

(where $n = 1/2$ is the polytropic index).

2.73 We know that,

$$\frac{1}{R}C_V = \frac{3}{2}\nu_1 + \frac{5}{2}\nu_2$$

(since a monoatomic gas has $C_V = (3/2)R$ and a diatomic gas has $C_V = (5/2)R$)

and $$\frac{1}{R}C_p = \frac{3}{2}\nu_1 + \frac{5}{2}\nu_2 + \nu_1 + \nu_2$$

Hence, $$\gamma = \frac{C_p}{C_V} = \frac{5\,\nu_1 + 7\,\nu_2}{3\,\nu_1 + 5\,\nu_2}$$

2.74 The internal energy of the molecules is

$$U = \frac{1}{2}mN< (\mathbf{u} - \mathbf{v})^2 > = \frac{1}{2}mN<u^2 - v^2>$$

(where $\mathbf{v}$ = velocity of the vessel, N = number of molecules, each of mass m).

When the vessel is stopped, internal energy becomes $(1/2)mN<u^2>$.

So, there is an increase in internal energy of $\Delta U = (1/2)mNv^2$. This will lead to a rise in temperature of

$$\Delta T = \frac{(1/2)mNv^2}{(i/2)R}$$

$$= \frac{mNv^2}{iR}$$

Since there is no flow of heat, this change of temperature will lead to an excess pressure, given by

$$\Delta p = \frac{R\Delta T}{V} = \frac{mNv^2}{iV}$$

and finally $$\frac{\Delta p}{p} = \frac{Mv^2}{iRT} = 2.2\%$$

(where M = molecular weight of N_2, i = number of degrees of freedom of N_2 = 5).

2.75 (a) From the equipartition theorem

$$\bar{\varepsilon} = \frac{3}{2}kT = 6 \times 10^{-21}\text{ J} \quad \text{and} \quad v_{\text{rms}} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3RT}{M}} = 0.47\text{ km/s}$$

(b) In equilibrium, the mean kinetic energy of the droplet will be equal to that of a molecule.

So, $$\frac{1}{2}\frac{\pi}{6}d^3\,\rho v_{\text{rms}}^2 = \frac{3}{2}kT \quad \text{or} \quad v_{\text{rms}} = 3\sqrt{\frac{2kT}{\pi d^3 \rho}} = 0.15\text{ m/s}$$

2.76 Given that, $i = 5$, $C_V = 5/2\,R$, $\gamma = 7/5$.

So, $$v'_{rms} = \sqrt{\frac{3RT}{M}} = \frac{1}{\eta} v_{rms} = \frac{1}{\eta}\sqrt{\frac{3RT}{M}} \quad \text{or} \quad T' = \frac{1}{\eta^2}\,T$$

Now, in an adiabatic process

$$TV^{\gamma-1} = TV^{2/i} = \text{constant} \quad \text{or} \quad VT^{i/2} = \text{constant}$$

So, $$V'\left(\frac{1}{\eta^2}T\right)^{i/2} = VT^{i/2} \quad \text{or} \quad V'\eta^{-i} = V \quad \text{or} \quad V' = \eta^i V$$

The gas must be expanded η^i times, i.e., 7.6 times.

2.77 Here, $$C_V = \frac{5}{2}\frac{m}{M}R \quad (\text{as } i = 5)$$

(where m = mass of the gas, M = molecular weight of the gas).

If v_{rms} increases η times, the temperature will have increased η^2 times. This will require (neglecting expansion of the vessels), a heat flow of amount

$$\frac{5}{2}\frac{m}{M}R(\eta^2 - 1)\,T = 10 \text{ kJ}$$

2.78 The root mean square angular velocity is given by

$$\frac{1}{2}I\omega^2 = 2 \times \frac{1}{2}kT \,(\text{for 2 degrees of rotations})$$

or $$\omega = \sqrt{\frac{2kT}{I}} = 6.3 \times 10^{12} \text{ rad/s}$$

2.79 Under compression, the temperature will rise, as

$$TV^{\gamma-1} = \text{constant} \quad \text{or} \quad TV^{2/i} = \text{constant}$$

So, $$T'(\eta^{-1}\,V_0)^{2/i} = T_0 V_0^{2/i} \quad \text{or} \quad T' = \eta^{+2/i}\,T_0$$

So mean kinetic energy of rotation per molecule in the compressed state is given by

$$kT' = kT_0\eta^{2/i} = 0.72 \times 10^{-20} \text{ J}$$

2.80 Number of collisions is given by

$$\frac{1}{4}n\langle v\rangle = \nu$$

Now, $$\frac{\nu'}{\nu} = \frac{n'\langle v'\rangle}{n\,\langle v\rangle} = \frac{1}{\eta}\sqrt{\frac{T'}{T}}$$

(When the gas is expanded η times, n decreases by a factor η.)

Also $$T'(\eta V)^{2/i} = TV^{2/i} \text{ or } T' = \eta^{2/i}\, T$$

So, $$\frac{v'}{v} = \frac{1}{\eta}\eta^{-1/i} = \eta^{-(i+1)/i}$$

i.e., collisions decrease by a factor $\eta^{(i+1)/i}$ (where $i = 5$ h).

2.81 In a polytropic process pV^n = constant, where n is the polytropic index. For this process

$$pV^n = \text{constant} \quad \text{or} \quad TV^{n-1} = \text{constant}$$

So, $$\frac{dT}{T} + (n-1)\frac{dV}{V} = 0$$

Then, $$dQ = CdT = dU + pdV = C_V dT + pdV$$
$$= \frac{i}{2}RdT + \frac{RT}{V}dV = \frac{i}{2}RdT - \frac{1}{n-1}RdT = \left(\frac{i}{2} - \frac{1}{n-1}\right)RdT$$

Now, $$C = R \quad \text{so} \quad \frac{i}{2} - \frac{1}{n-1} = 1$$

or $$\frac{1}{n-1} = \frac{i}{2} - 1 = \frac{i-2}{2} \quad \text{or} \quad n = \frac{i}{i-2}$$

Now, $$\frac{v'}{v} = \frac{n'}{n}\frac{<v'>}{<v>} = \frac{1}{\eta}\sqrt{\frac{T'}{T}} = \frac{1}{\eta}\left(\frac{V}{V'}\right)^{\frac{n-1}{2}}$$
$$= \frac{1}{\eta}\left(\frac{1}{\eta}\right)^{1/(i-2)} = \left(\frac{1}{\eta}\right)^{(i-1)/(i-2)}$$
$$= (\eta)^{-(i-1)/(i-2)} \text{ times} = \frac{1}{2.52} \text{ times}$$

2.82 If α is the polytropic index, then

$$pV^\alpha = \text{constant}, \; TV^{\alpha-1} = \text{constant}$$

Now, $$\frac{v'}{v} = \frac{n'<v'>}{n\;<v>} = \frac{V}{V'}\sqrt{\frac{T'}{T}} = \frac{VT^{-1/2}}{V'T'^{-1/2}} = 1$$

Hence, $$\frac{1}{\alpha-1} = -\frac{1}{2} \quad \text{or} \quad \alpha = -1$$

Then, $$C = \frac{iR}{2} + \frac{R}{2} = 3\,R$$

2.83 $v_p = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2RT}{M}} = \sqrt{\frac{2p}{\rho}} = 0.45$ km/s

$v_{avg} = \sqrt{\frac{8}{\pi}\frac{p}{\rho}} = 0.51$ km/s and $v_{rms} = \sqrt{\frac{3p}{\rho}} = 0.55$ km/s

2.84 (a) The formula is

$$df(u) = \frac{4}{\sqrt{\pi}} u^2 e^{-u^2} du \text{ (where } u = v/v_p)$$

Now, $\text{Prob}\left(\left|\frac{v - v_p}{v_p}\right| < \delta\eta\right) = \int_{1-\delta\eta}^{1+\delta\eta} df(u)$

$$= \frac{4}{\sqrt{\pi}} e^{-1} \times 2\,\delta\eta = \frac{8}{\sqrt{\pi}e}\delta\eta = 0.0166 = 1.66\%$$

(b) $\text{Prob}\left(\left|\frac{v - v_{rms}}{v_{rms}}\right| < \delta\eta\right) = \text{Prob}\left(\left|\frac{v}{v_p} - \frac{v_{rms}}{v_p}\right| < \delta\eta\frac{v_{rms}}{v_p}\right)$

$$= \text{Prob}\left(\left|u - \sqrt{\frac{3}{2}}\right| < \sqrt{\frac{3}{2}}\delta\eta\right)$$

$$= \int_{\sqrt{\frac{3}{2}} - \sqrt{\frac{3}{2}}\delta\eta}^{\sqrt{\frac{3}{2}} + \sqrt{\frac{3}{2}}\delta\eta} \frac{4}{\pi} u^2 e^{-u^2} du$$

$$= \frac{4}{\sqrt{\pi}} \times \frac{3}{2} e^{-3/2} \times 2\sqrt{\frac{3}{2}}\,\delta\eta$$

$$= \frac{12\sqrt{3}}{\sqrt{2\pi}} e^{-3/2}\delta\eta = 0.0185 = 1.85\%$$

2.85 (a) $v_{rms} - v_p = \left(\sqrt{3} - \sqrt{2}\right)\sqrt{\frac{kT}{m}} = \Delta v$

$$T = \frac{m}{k}\left(\frac{\Delta v}{(\sqrt{3} - \sqrt{2})}\right)^2 = 384 \text{ K}$$

(b) Clearly v is the most probable speed at this temperature.

So, $\sqrt{\frac{2kT}{m}} = v$ or $T = \frac{mv^2}{2k} = 342$ K

2.86 (a) We have,

$$\frac{v_1^2}{v_p^2}e^{-v_1^2/v_p^2} = \frac{v_2^2}{v_p^2}e^{-v_2^2/v_p^2} \Rightarrow \left(\frac{v_1}{v_2}\right)^2 = e^{v_1^2 - v_2^2/v_p^2} \text{ or } v_p^2 = \frac{2kT}{m} = \frac{v_1^2 - v_2^2}{(\ln v_1^2/v_2^2)}$$

So,
$$T = \frac{m(v_1^2 - v_2^2)}{2\,k \ln v_1^2/v_2^2} = 330 \text{ K}$$

(b) $F(v) = \dfrac{4}{\sqrt{\pi}}\dfrac{v^2}{v_p^2}e^{-v^2/v_p^2} \times \dfrac{1}{v_p}\left(\dfrac{1}{v_p}\text{comes from } F(v)dv = df(u),\ du = \dfrac{dv}{v_p}\right)$

Thus,
$$\frac{v^2}{v_{p_1}^3}e^{-v^2/v^2p_1} = \frac{v^2}{v_{p_2}^2} = \frac{2kT_0}{m},\ v_{p_2}^2 = \frac{2kT_0}{m}\eta$$

Now,
$$e^{-\frac{mv^2}{2kT_0}\left(1-\frac{1}{\eta}\right)} = \frac{1}{\eta^{3/2}} \text{ or } \frac{mv^2}{2kT_0}\left(1 - \frac{1}{\eta}\right) = \frac{3}{2}\ln\eta$$

Thus,
$$v = \sqrt{\frac{3kT_0}{m}}\sqrt{\frac{\ln\eta}{1 - 1/\eta}}$$

2.87 $v_{p_N} = \sqrt{\dfrac{2kT}{m_N}} = \sqrt{\dfrac{2RT}{M_N}},\ v_{p_O} = \sqrt{\dfrac{2RT}{M_O}}$

$$v_{p_N} - v_{p_O} = \Delta v = \sqrt{\frac{2RT}{M_N}}\left(1 - \sqrt{\frac{M_N}{M_O}}\right)$$

$$T = \frac{M_N(\Delta v)^2}{2R\left(1 - \sqrt{\frac{M_N}{M_O}}\right)^2} = \frac{m_N(\Delta v)^2}{2k\left(1 - \sqrt{\frac{m_N}{m_O}}\right)^2} = 363 \text{ K}$$

2.88
$$\frac{v^2}{v_{p_H}^3}e^{-v^2/v^2_{p_H}} = \frac{v^2}{v_{p_{He}}^3}e^{-v^2/v^2_{p_{He}}} \text{ or } e^{v^2\left(\frac{m_{He}}{2kT} - \frac{m_H}{2kT}\right)} = \left(\frac{m_{He}}{m_H}\right)^{3/2}$$

$$v^2 = 3kT\frac{\ln m_{He}/m_H}{m_{He} - m_H}$$

On substitution, we get, $v = 1.60$ km/s.

2.89
$$dN(v) = \frac{N\,4}{\sqrt{\pi}}\frac{v^2dv}{v_p^3}e^{-v^2/v^2_{p_H}}$$

For a given range v to $v + dv$, $(dN(v))$ is maximum when

$$\frac{\delta}{\delta v_p}\frac{dN(v)}{Nv^2dv} = 0 = \left(-3v_p^{-4} + \frac{2v^2}{v_p^6}\right)e^{-v^2/v_p^2}$$

or $$v^2 = \frac{3}{2}\, v_p^2 = \frac{3kT}{m}$$

Thus, $$T = \frac{1}{3}\frac{mv^2}{k}$$

2.90 Given $$d^3v = 2\pi v_\perp\, dv_\perp\, dv_x$$

Thus, $$dn(v) = N\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m}{2kT}(v_x^2 + v_\perp^2)}\, 2\pi v_\perp\, dv_\perp\, dv_x$$

2.91 $$<v_x> = 0 \text{ (by symmetry)}$$

$$<|v_x|> = \frac{\int\limits_0^\infty |v_x| e^{-mv_x^2/2kT}\, dv_x}{\int\limits_0^\infty e^{-mv_x^2/2kT}\, dv_x} = \frac{\int\limits_0^\infty v_x e^{-mv_x^2/2kT}\, dv_x}{\int\limits_0^\infty e^{-mv_x^2/2kT}\, dv_x}$$

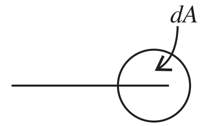
$$= \frac{\sqrt{\frac{2kT}{m}}\int\limits_0^\infty u e^{-u^2} du}{\int\limits_0^\infty e^{-u^2} du} = \frac{\sqrt{\frac{2kT}{m}}\int\limits_0^\infty \frac{1}{2} e^{-x} dx}{\int\limits_0^\infty e^{-x}\frac{dx}{2\sqrt{x}}}$$

$$= \frac{\sqrt{\frac{2kT}{m}}\,\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)} = \sqrt{\frac{2kT}{m\pi}}$$

2.92 $$<v_x^2> = \frac{\int\limits_0^\infty v_x^2\, e^{-mv_x^2/2kT}\, dv_x}{\int\limits_0^\infty e^{-mv_x^2/2kT}\, dv_x} = \frac{\frac{2kT}{m}\int\limits_0^\infty x e^{-x}\frac{dx}{2\sqrt{x}}}{\int\limits_0^\infty e^{-x}\frac{dx}{2\sqrt{x}}} = \frac{\frac{2kT}{m}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{kT}{m}$$

2.93 Here vdA is equal to number of molecules hitting an area dA of the wall per second

i.e., $$vdA = \int\limits_0^\infty dN(v_x)\, v_x dA$$



or
$$v = \int_0^\infty n\left(\frac{m}{2\pi kT}\right)^{1/2} e^{-mv_x^2/2kT} v_x dv_x$$

$$= \int_0^\infty \frac{n}{\sqrt{\pi}}\left(\frac{2kT}{m}\right)^{1/2} e^{-u^2} u du$$

$$= \frac{1}{2} n\sqrt{\frac{2kT}{m\pi}} = n\sqrt{\frac{kT}{2m\pi}} = \frac{1}{4} n\langle v\rangle \quad \left(\text{where } \langle v\rangle = \sqrt{\frac{8kT}{m\pi}}\right)$$

2.94 Let the number of molecules per unit volume with x component of velocity in the range v_x to $v_x + dv_x$, be

$$dn(v_x) = n\left(\frac{m}{2\pi kT}\right)^{1/2} e^{-mv_x^2/2kT} dv_x$$

Then,
$$p = \int_0^\infty 2mv_x \cdot v_x dn\,(v_x)$$

$$= \int_0^\infty 2mv_x^2 n\left(\frac{m}{2\pi kT}\right)^{1/2} e^{-mv_x^2/2kT} dv_x$$

$$= 2mn\frac{1}{\sqrt{\pi}}\frac{2kT}{m}\int_0^\infty u^2 e^{-u^2} du$$

$$= \frac{4}{\sqrt{\pi}} nkT \int_0^\infty xe^{-x}\frac{dx}{2\sqrt{x}} = nkT$$

2.95
$$\langle 1/v\rangle = \int_0^\infty \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv_x^2/2kT} 4\pi v^2 dv \frac{1}{v}$$

$$= \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi\frac{1}{2}\frac{2kT}{m}\int_0^\infty e^{-x} dx$$

$$= 2\left(\frac{m}{2\pi kT}\right)^{1/2} = \left(\frac{2m}{\pi kT}\right)^{1/2} = \left(\frac{16}{\pi^2}\frac{m\pi}{8kT}\right)^{1/2} = \frac{4}{\pi\langle v\rangle}$$

2.96 $dN(v) = N\left(\dfrac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} 4\pi v^2 dv = dN(\varepsilon) = \dfrac{dN(\varepsilon)}{d\varepsilon} d\varepsilon$

or
$$\frac{dN(\varepsilon)}{d\varepsilon} = N\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2RT} 4\pi v^2 \frac{dv}{d\varepsilon}$$

Now, $\varepsilon = \frac{1}{2} mv^2$ so $\frac{dv}{d\varepsilon} = \frac{1}{mv}$

or $$\frac{dN(\varepsilon)}{d\varepsilon} = N\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\varepsilon/kT}\, 4\pi \sqrt{\frac{2\varepsilon}{m}}\, \frac{1}{m}$$

$$= N \frac{2}{\sqrt{\pi}} (kT)^{-3/2}\, e^{-\varepsilon/kT}\, \varepsilon^{1/2}$$

i.e., $$dN(\varepsilon) = N \frac{2}{\sqrt{\pi}} (kT)^{-3/2}\, e^{-\varepsilon/kT}\, \varepsilon^{1/2}\, d\varepsilon$$

The most probable kinetic energy is given by

$$\frac{d}{d\varepsilon} \frac{dN(\varepsilon)}{d\varepsilon} = 0$$

or $$\frac{1}{2} \varepsilon^{-1/2} e^{-\varepsilon/kT} - \frac{\varepsilon^{1/2}}{kT} e^{\varepsilon/kT} = 0 \quad \text{or} \quad \varepsilon = \frac{1}{2} kT = \varepsilon_p$$

The corresponding velocity is $v = \sqrt{\frac{kT}{m}} \neq v_p$

2.97 The mean kinetic energy is

$$<\varepsilon> = \frac{\int_0^\infty \varepsilon^{3/2}\, e^{-\varepsilon/kT} d\varepsilon}{\int_0^\infty \varepsilon^{1/2}\, e^{-\varepsilon/kT} d\varepsilon} = kT \frac{\Gamma(5/2)}{\Gamma(3/2)} = \frac{3}{2} kT$$

Thus,

$$\frac{\delta N}{N} = \int_{\frac{3}{2}kT(1-\delta\eta)}^{\frac{3}{2}(1+\delta\eta)kT} \frac{2}{\sqrt{\pi}} (kT)^{-3/2}\, e^{-\varepsilon/kT} \varepsilon^{1/2}\, d\varepsilon$$

$$= \frac{2}{\sqrt{\pi}}\, e^{-3/2} \left(\frac{3}{2}\right)^{3/2} 2\delta\eta = 3\sqrt{\frac{6}{\pi}}\, e^{-3/2}\, \delta\eta$$

If $\delta\eta = 1\%$, the fraction of molecules is 0.9%.

2.98 $$\frac{\Delta N}{N} = \frac{2}{\sqrt{\pi}} (kT)^{-3/2} \int_{\varepsilon_0}^\infty \sqrt{\varepsilon}\, e^{-\varepsilon/kT}\, d\varepsilon$$

$$= \frac{2}{\sqrt{\pi}} (kT)^{-3/2} \sqrt{\varepsilon_0} \int_{\varepsilon_0}^\infty e^{-\varepsilon/kT} d\varepsilon \quad (\varepsilon_0 \gg kT)$$

$$= \frac{2}{\sqrt{\pi}} (kT)^{-3/2} \sqrt{\varepsilon_0}\, kT\, e^{-\varepsilon_0/kT} = 2\sqrt{\frac{\varepsilon_0}{\pi kT}}\, e^{-\varepsilon_0/kT}$$

(In evaluating the integral, we have taken out $\sqrt{\varepsilon}$ as $\sqrt{\varepsilon_0}$ since the integral is dominated by the lower limit.)

2.99 (a) $F(v) = Av^3 e^{-mv^2/2kT}$

For the most probable value of the velocity

$$\frac{dF(v)}{dv} = 0 \quad \text{or} \quad 3\,Av^2 e^{-mv^2/2kT} - Av^3\,\frac{2mv}{2kT}\,e^{-mv^2/2kT} = 0$$

So,
$$v_p = \sqrt{\frac{3kT}{m}}$$

This should be compared with the value $v_p = \sqrt{\frac{2kT}{m}}$ for the Maxwellian distribution.

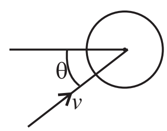
(b) In terms of energy, $\varepsilon = \frac{1}{2}mv^2$

$$\begin{aligned} F(\varepsilon) &= Av^3\; e^{-mv^2/2kT}\,\frac{dv}{d\varepsilon} \\ &= A\left(\frac{2\varepsilon}{m}\right)^{3/2} e^{-\varepsilon/kT}\,\frac{1}{\sqrt{2m\varepsilon}} = A\,\frac{2\varepsilon}{m^2}\;e^{-\varepsilon/kT} \end{aligned}$$

From this the probable energy comes out as follows:

$$F(\varepsilon) = 0 \Rightarrow \frac{2A}{m^2}\left(e^{-\varepsilon/kT} - \frac{\varepsilon}{kT}\,e^{-\varepsilon/kT}\right) = 0 \quad \text{or} \quad \varepsilon_p = kT$$

2.100 The number of molecules reaching a unit area of wall at angle between θ and $\theta + d\theta$ to its normal per unit time is



$$\begin{aligned} dv &= \int_{v=0}^{v=\infty} dn(v)\,\frac{d\Omega}{4\pi}\;v\cos\theta \\ &= \int_0^{\infty} n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT}\;v^3\;dv\,\sin\theta\,\cos\theta\;d\theta \times 2\pi \\ &= n\left(\frac{2kT}{m\pi}\right)^{1/2}\int_0^{\infty} e^{-x}\,x dx\,\sin\theta\,\cos\theta\;d\theta = n\left(\frac{2kT}{m\pi}\right)^{1/2}\;\sin\theta\,\cos\theta\;d\theta \end{aligned}$$

2.101 The number of molecules reaching per unit area of the wall with velocities in the interval v to $v + dv$ per unit time is

$$dv = \int_{\theta=0}^{\theta=\pi/2} dn(v)\,\frac{d\Omega}{4\pi}\,v\cos\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2} n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} v^3 \, dv \sin\theta \cos\theta \, d\theta \times 2\pi$$

$$= n\pi\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} v^3 dv$$

2.102 If the force exerted is $\boldsymbol{F}$, then by the law of variation of concentration with height

$$n(Z) = n_0 \, e^{-FZ/kT}$$

So, $$\eta = e^{F\Delta h/kT} \quad \text{or} \quad F = \frac{kT \ln \eta}{\Delta h} = 9 \times 10^{-20} \text{ N}$$

2.103 Here, $F = \dfrac{\pi}{6} d^3 \Delta\rho g = \dfrac{RT \ln\eta}{N_a h}$ or $N_a = \dfrac{6RT \ln\eta}{\pi d^3 g \Delta\rho h}$

Given $T = 290$ K, $\eta = 2$, $h = 4 \times 10^{-5}$ m, $d = 4 \times 10^{-7}$m, $g = 9.8$ m/s^2, $\Delta\rho = 0.2 \times 10^3$ kg/m^3 and $R = 8.31$ J/K.

Hence, $$N_a = \frac{6 \times 8.31 \times 290 \times \ln 2}{\pi \times 64 \times 9.8 \times 200 \times 4} \times 10^{26} = 6.36 \times 10^{23} \text{ mol}^{-1}$$

2.104 $$\eta = \frac{\text{concentration of } H_2}{\text{concentration of } N_2} = \eta_0 \frac{e^{-M_{N_2} gh/RT}}{e^{-M_{H_2} gh/RT}} = \eta_0 e^{(-M_{N_2} + M_{H_2}) gh/RT}$$

$$\frac{\eta}{\eta_0} = e^{(-M_{N_2} + M_{H_2}) gh/RT} = 1.39$$

2.105 We know that

$$n_1(h) = n_1 e^{-m_1 gh/kt} \quad \text{and} \quad n_2(h) = n_2 e^{-m_2 gh \; kt}$$

They are equal at a height h, where $n_1/n_2 = e^{gh(m_1 - m_2)/kT}$

or $$h = \frac{kT}{g} \frac{\ln n_1 - \ln n_2}{m_1 - m_2}$$

2.106 At a temperature T the concentration $n(z)$ varies with height according to

$$n(z) = n_0 e^{-mgz/kT}$$

This means that particles per unit area of the base of cylinder are

$$\int_0^\infty n(z) \, dz$$

$$= \int_0^\infty n_0 e^{-mgz/kT}\, dz = \frac{n_0 kT}{mg}$$

Clearly this cannot change. Thus $n_0 kT = p_0$ = pressure at the bottom of the cylinder, must not change with change of temperature.

2.107 $$<U> = \frac{\int_0^\infty mgz\, e^{-mgz/kt} dz}{\int_0^\infty e^{-mgz/kT} dz} = kT \frac{\int_0^\infty x e^{-x} dx}{\int_0^\infty e^{-x} dx} = kT \frac{\Gamma(2)}{\Gamma(1)} = kT$$

When there are many kinds of molecules, this formula holds for each kind and the average energy is given by

$$<U> = \frac{\sum F_i kT}{\sum f_i} = kT$$

where $f_i \propto$ fractional concentration of each kind of molecule at the ground level.

2.108 The constant acceleration is equivalent to a pseudo force wherein a concentration gradient is set up. Then

$$e^{-M_A wl/RT} = 1 - \eta$$

or $$w = -\frac{RT \ln(1-\eta)}{M_A l} \approx \frac{\eta RT}{M_A l} \approx 70 \text{ g}$$

2.109 In a centrifuge rotating with angular velocity ω about an axis, there is a centrifugal acceleration ω_r^2, where r is the radial distance from the axis. In a fluid if there are suspended colloidal particles they experience an additional force. If m is the mass of each particle then its volume is m/ρ and the excess force on this particle is $m/\rho\,(\rho - \rho_0)\omega^2 r$ outward, corresponding to a potential energy of $-(m/2\rho)(\rho-\rho_0)\omega^2 r^2$.

This gives rise to a concentration variation, given by

$$n(r) = n_0 \exp\left(+\frac{m}{2\rho\, kT}(\rho - \rho_0)\,\omega^2 r^2\right)$$

Thus, $$\frac{n(r_2)}{n(r_1)} = \eta = \exp\left(+\frac{M}{2\rho\, RT}(\rho - \rho_0)\omega^2(r_2^2 - r_1^2)\right)$$

(where $\dfrac{m}{k} = \dfrac{M}{R}$ and $M = N_A m$ is the molecular weight).

Thus, $$M = \frac{2\rho RT \ln \eta}{(\rho - \rho_0)\omega^2(r_2^2 - r_1^2)}$$

2.110 The potential energy associated with each molecule is $-(1/2)m\,\omega^2 r^2$ and there is a concentration variation, given by

$$n\,(r) = n_0 \exp\left(\frac{m\omega^2 r^2}{2kT}\right) = n_0 \exp\left(\frac{M\omega^2 r^2}{2RT}\right)$$

Thus, $$\eta = \exp\left(\frac{M\omega^2 l^2}{2RT}\right) \quad \text{or} \quad \omega = \sqrt{\frac{2RT}{Ml^2}\ln \eta}$$

Using $M = 12 + 32 = 44$ g, $l = 100$ cm, $R = 8.31 \times 10^7$ erg/K, $T = 300$ K, we get,

$$\omega = 280 \text{ rad/s}$$

2.111 Here, the concentration variation is

$$n(r) = n_0 \exp\left(-\frac{ar^2}{kT}\right)$$

(a) The number of molecules located at the distance between r and $r + dr$ is

$$4\,\pi r^2 dr\, n(r) = 4\pi n_0 \exp\left(-\frac{ar^2}{kT}\right) r^2 dr$$

(b) r_p is given by $\frac{d}{dr} r^2\, n(r) = 0$ or $2r - \frac{2ar^3}{kT} = 0$ or $r_p = \sqrt{\frac{kT}{a}}$

(c) The fraction of molecules lying between r and $r + dr$ is

$$\frac{dN}{N} = \frac{4\pi r^2 dr n_0 \exp\,(-ar^2/kT)}{\int_0^\infty 4\pi r^2 dr n_0 \exp\,(ar^2/kT)}$$

$$\int_0^\infty 4\pi r^2 dr \exp\left(-\frac{ar^2}{kT}\right) = \left(\frac{kT}{a}\right)^{3/2} 4\pi \int_0^\infty x \frac{dx}{2\sqrt{x}} \exp\,(-x)$$

$$= \left(\frac{kT}{a}\right)^{3/2} 2\pi\Gamma\left(\frac{3}{2}\right) = \left(\frac{\pi kT}{a}\right)^{3/2}$$

Thus, $$\frac{dN}{N} = \left(\frac{a}{\pi kT}\right)^{3/2} 4\pi r^2 dr \exp\left(\frac{-ar^2}{kT}\right)$$

(d) $$dN = N\left(\frac{a}{\pi kT}\right)^{3/2} 4\pi r^2 dr \exp\left(\frac{-ar^2}{kT}\right)$$

So, $$n(r) = N\left(\frac{a}{\pi kT}\right)^{1/2} \exp\left(\frac{-ar^2}{kT}\right)$$

When T decreases η times, $n(0) = n_0$ will increase $\eta^{3/2}$ times.

2.112 (a) Let $$U = ar^2 \quad \text{or} \quad r = \sqrt{\frac{U}{a}}$$

So, $$dr = \sqrt{\frac{1}{a}}\frac{dU}{2\sqrt{U}} = \frac{dU}{2\sqrt{aU}}$$

So, $$dN = n_0 4\pi \frac{U}{a}\frac{dU}{2\sqrt{aU}}\exp\left(\frac{U}{kT}\right)$$

$$= 2\pi n_0\, a^{-3/2}\, U^{1/2} \exp\left(\frac{U}{kT}\right) dU$$

(b) The most probable value of U is given by

$$\frac{d}{dU}\left(\frac{dN}{dU}\right) = 0 \left(\frac{1}{2\sqrt{U}} - \frac{U^{1/2}}{kT}\right)\exp\left(\frac{-U}{kT}\right) \quad \text{or} \quad U_p = \frac{1}{2}kT$$

From 2.111 (b), the potential energy at the most probable distance is kT.

2.4 The Second Law of Thermodynamics. Entropy

2.113 The efficiency is given by

$$\eta = \frac{T_1 - T_2}{T_1} \quad (\text{for } T_1 > T_2)$$

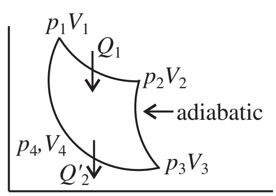
Now, in the two cases the efficiencies are

$$\eta_h = \frac{T_1 + \Delta T - T_2}{T_1 + \Delta T} \quad (T_1 \text{ increased})$$

$$\eta_l = \frac{T_1 - T_2 + \Delta T}{T_1} \quad (T_2 \text{ decreased})$$

Thus, $$\eta_h < n_1$$

2.114 (a) For H_2, $\gamma = 7/5$. From the figure



$$p_1V_1 = p_2V_2,\ p_3V_3 = p_4V_4$$

and $$p_2V_2^\gamma = p_3V_3^\gamma,\ p_1V_1^\gamma = p_4V_4^\gamma$$

Define n by $V_3 = n\, V_2$.

Then $$p_3 = p_2 n^{-\gamma}$$

So, $$p_4V_4 = p_3V_3 = p_2V_2 n^{1-\gamma} = p_1V_1 n^{1-\gamma}$$

since, $$p_4V_4^\gamma = p_1V_1^\gamma \quad \text{so,} \quad V_4^{1-\gamma} = V_1^{1-\gamma}\, n^{1-\gamma} \text{ or } V_4 = nV_1$$

Also, $$Q_1 = p_2 V_2 \ln \frac{V_2}{V_1}, Q_2' = p_3 V_3 \ln \frac{V_3}{V_4} n^{1-\gamma} = p_2 V_2 \ln \frac{V_3}{V_4}$$

Finally $$\eta = 1 - \frac{Q_2'}{Q_1} = 1 - n^{1-\gamma} = 0.242$$

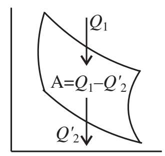
(b) Define n by $$p_3 = p_2/n$$

Then, $$p_2 V_2^\gamma = \frac{p_2}{n} V_3{}^\gamma \quad \text{or} \quad V_3 = n^{1/\gamma} V_2$$

So we get the formulae here by substituting n by $n^{1/\gamma}$ in the previous case.

Then, $$\eta = 1 - n^{(1/\gamma)-1} = 1 - n^{-(2/7)} = 0.18$$

2.115 Used as a refrigerator, the refrigerating efficiency of a heat engine is given by

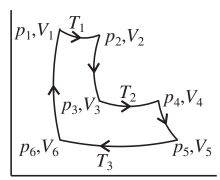


$$\varepsilon = \frac{Q_2'}{A} = \frac{Q_2'}{Q_1 - Q'_2} = \frac{Q_2'/Q_1}{1 - \dfrac{Q_2'}{Q_1}} = \frac{1-\eta}{\eta} = 9$$

(where η is the efficiency of the heat engine).

2.116 Given $$V_2 = nV_1, V_4 = nV_3$$

$$Q_1 = \text{Heat taken at the upper temperature}$$
$$= RT_1 \ln n + RT_2 \ln n = R(T_1 + T_2) \ln n$$



Now, $$T_1 V_2^{\gamma-1} = T_2 V_3^{\gamma-1}$$

or $$V_3 = \left(\frac{T_1}{T_2}\right)^{1/\gamma-1} V_2$$

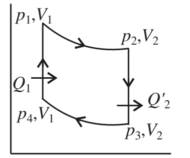
Similarly, $$V_5 = \left(\frac{T_2}{T_3}\right)^{1/\gamma-1} V_4,$$

$$V_6 = \left(\frac{T_1}{T_3}\right)^{1/\gamma-1} V_1$$

Thus, heat ejected at the lower temperature

$$Q_2 = -RT_3 \ln \frac{V_6}{V_5}$$
$$= -RT_3 \ln \left(\frac{T_1}{T_2}\right)^{1/\gamma-1} \frac{V_1}{V_4} = -RT_3 \text{ in} \left(\frac{T_1}{T_2}\right)^{1/\gamma-1} \frac{V_2}{n^2 V_3}$$
$$= -RT_3 \ln \left(\frac{T_1}{T_2}\right)^{1/\gamma-1} \frac{1}{n^2} \left(\frac{T_1}{T_2}\right)^{-1/\gamma-1} = 2RT_3 \ln n$$

Thus, $$\eta = 1 - \frac{2T_3}{T_1 + T_2}$$



2.117 Since, $$Q'_2 = C_V(T_2 - T_3) = \frac{C_V}{R}V_2(p_2 - p_3)$$

and $$Q_1 = \frac{C_V}{R}V_1(p_1 - p_4)$$

So, $$\eta = 1 - \frac{(V_2\ p_2 - p_3)}{V_1\ (p_1 - p_4)}$$

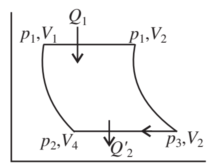
On the other hand,

$$p_1V_1^\gamma = p_2V_2^\gamma, p_3V_2^\gamma = p_4V_1^\gamma \text{ also, } V_2 = nV_1$$

Thus, $$p_1 = p_2n^\gamma \quad \text{and} \quad p_4 = p_3n^\gamma$$

So, $$\eta = 1 - n^{1-\gamma} \quad (\text{where } \gamma = 7/5 \text{ for } N_2)$$

Therefore, $$\eta = 0.602 = 60.2\%$$



2.118 Since, $$Q_1 = \frac{C_p}{R}\ p_1(V_2 - V_1)$$

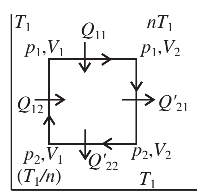
and $$Q'_2 = \frac{C_p}{R}\ p_2(V_3 - V_4)$$

So, $$\eta = 1 - \frac{p_2(V_3 - V_4)}{p_1(V_2 - V_1)}$$

Now, $p_1 = np_2$ or $V_3 = n^{1/\gamma}V_2$

$$p_2V_4^\gamma = p_1V_1^\gamma \text{ or } V_4 = n^{1/\gamma}V_1$$

So, $$\eta = 1 - \frac{1}{n}\cdot n^{1/\gamma} = 1 - n^{(1/\gamma)-1}$$



2.119 Since the absolute temperature of the gas rises n times both in the isochoric heating and the isobaric expansion, we have

$$p_1 = np_2 \text{ and } V_2 = nV_1$$

Heat taken is $$Q_1 = Q_{11} + Q_{12}$$

where, $$Q_{11} = C_P(n - 1)T_1$$

and $$Q_{12} = C_VT_1\left(1 - \frac{1}{n}\right)$$

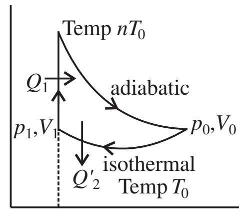
Heat rejected is $$Q'_2 = Q'_{21} + Q'_{22}$$

where, $$Q'_{21} = C_VT_1(n - 1)\,Q'_{22} = C_p\,T_1\left(1 - \frac{1}{n}\right)$$

Thus, $$\eta = 1 - \frac{Q'_2}{Q_1} = 1 - \frac{C_V(n-1) + C_p(1-1/n)}{C_p(n-1) + C_V(1-1/n)}$$

$$= \left(1 - \frac{\eta - 1 + \gamma(1-1/n)}{\gamma(n-1) + (1-1/n)}\right) = 1 - \frac{1 + \dfrac{\gamma}{n}}{\gamma + \dfrac{1}{n}} = 1 - \frac{n+\gamma}{1+n\gamma}$$

2.120 (a) Here, $p_2 = np_1,\ p_1 V_1 = p_0 V_0,$



and $np_1 V_1^\gamma = p_0 V_0^\gamma$

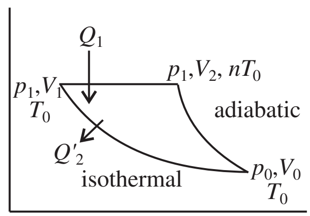
Also, $Q'_2 = RT_0 \ln \dfrac{V_0}{V_1},\ \ Q_1 = C_V T_0\,(n-1)$

But, $nV_1^{\gamma-1} = V_0^{\gamma-1}$ or $V_1 = V_0 n^{-1/\gamma-1}$

$$Q'_2 = RT_0 \ln n^{-1/\gamma-1} \frac{RT_0}{\gamma-1} \ln n$$

Thus, $$\eta = 1 - \frac{\ln n}{n-1} \quad \left(\text{on using } C_V = \frac{R}{\gamma-1}\right)$$

(b) Here, $V_2 = nV_1,\ \ p_1 V_1 = p_0 V_0$



and $p_1(n\ V_1)^\gamma = p_0 V_0^\gamma$

i.e., $n^\gamma V_1^{\gamma-1} = V_1^{\gamma-1}$ or $V_1 = n^{\gamma/\gamma-1} V_0$

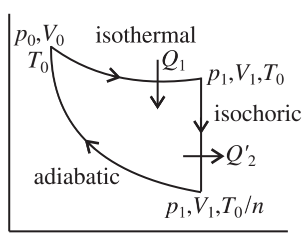
Also, $Q_1 = C_p T_0\,(n-1),$

$$Q'_2 = RT_0 \ln \frac{V_0}{V_1}$$

or $$Q'_2 = RT_0 \ln n^{\gamma/(\gamma-1)} = \frac{R\gamma}{\gamma-1} T_0 \ \ln n = C_p T_0 \ln n$$

Thus, $$\eta = 1 - \frac{\ln n}{n-1}$$

2.121 Here the isothermal process proceeds at the maximum temperature instead of at the minimum temperature of the cycle as in Problem 2.120.



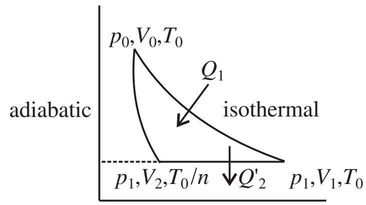
(a)

(a) Here $p_1 V_1 = p_0 V_0,\ \ p_2 = \dfrac{p_1}{n}$

and $p_2 V_1^\gamma = p_0 V_0^\gamma$ or $p_1\ V_1^\gamma = np_0 V_0^\gamma$

i.e., $$V_1^{\gamma-1} = nV_0^{\gamma-1} \quad \text{or} \quad V_1 = n^{1/\gamma-1}\, V_0$$

$$Q_2' = C_V T_0\left(1 - \frac{1}{n}\right),\ Q_1 = RT_0 \ln \frac{V_1}{V_0} = \frac{RT_0}{\gamma - 1} \ln n = C_V T_0 \ln n$$



(b)

Thus, $$\eta = 1 - \frac{Q_2'}{Q_1} = 1 - \frac{n-1}{n \ln n}$$

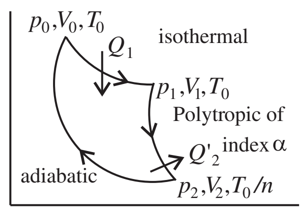
(b) Here, $V_2 = \dfrac{V_1}{n},\ p_0 V_0 = p_1 V_1$

$$p_0 V_0^\gamma = p_1 V_2^\gamma = p_1 n^{-\gamma} V_1^\gamma = V_0^{\gamma-1}\, n^{-\gamma}\, V_1^{\gamma-1} \quad \text{or} \quad V_1 = n^{(\gamma/\gamma-1)}\, V_0$$

$$Q_2' = C_p T_0\left(1 - \frac{1}{n}\right),\ Q_1 = RT_0 \ln \frac{V_1}{V_0} = \frac{R\gamma}{\gamma - 1}\, T_0 \ln n = C_p T_0 \ln n$$

Thus, $$\eta = 1 - \frac{n-1}{n \ln n}$$

2.122 The section from (p_1, V_1, T_0) to $(p_2, V_2, T_0/n)$ is a polytropic process of index α. We shall assume that the corresponding specific heat C is positive.



Here, $dQ = CdT = C_V dT + pdV$

Now, $pV^\alpha = \text{constant}$ or $TV^{\alpha-1} = \text{constant}$

So, $$pdV = \frac{RT}{V}\, dV = -\frac{R}{\alpha - 1}\, dT$$

Then, $$C = C_V - \frac{R}{\alpha - 1} = R\left(\frac{1}{\gamma - 1} - \frac{1}{\alpha - 1}\right)$$

We have $$p_1 V_1 = RT_0,\ \ p_2 V_2 = \frac{RT_0}{n} = \frac{p_1 V_1}{n}$$

$$p_0 V_0 = p_1 V_1 = np_2 V_2,\ \ p_0 V_0^\gamma = p_2 V_2^\gamma$$

$$p_1 V_1^\alpha = p_2 V_2^\alpha \quad \text{or} \quad V_0^{\gamma-1} = \frac{1}{n}\, V_2^{\gamma-1} \quad \text{or} \quad V_2 = V_0 n^{1/(\gamma-1)}$$

$$V_1^{\alpha-1} = \frac{1}{n}\, V_2^{\alpha-1} \quad \text{or} \quad V_1 = n^{-1/(\alpha-1)}\, V_2 = n^{1/(\gamma-1)-1/(\alpha-1)}\, V_0$$

Now, $$Q_2' = CT_0\left(1 - \frac{1}{n}\right),\ Q_1 = RT_0 \ln \frac{V_1}{V_0} = RT_0\left(\frac{1}{\gamma - 1} - \frac{1}{\alpha - 1}\right) \ln n = CT_0 \ln n$$

Thus, $$\eta = 1 - \frac{n-1}{n \ln n}$$

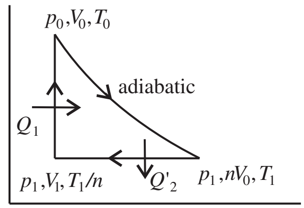
2.123 (a) Here, $Q_2' = C_p\left(T_1 - \dfrac{T_1}{n}\right) = C_pT_1\left(1 - \dfrac{1}{n}\right)$, $Q_1 = C_V\left(T_0 - \dfrac{T_1}{n}\right)$

Along the adiabatic line

$$T_0V_0^{\gamma-1} = T_1(nV_0)^{\gamma-1} \quad \text{or} \quad T_0 = T_1 n^{\gamma-1}$$

So, $$Q_1 = C_V\frac{T_1}{n}(n^\gamma - 1)$$

Thus, $$\eta = 1 - \frac{\gamma(n-1)}{n^{\gamma-1}}$$



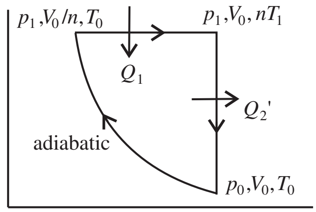
(*a*)

(b) Here $Q_2' = C_V(nT_1 - T_0)$, $Q_1 = C_p \cdot T_1(n-1)$

Along the adiabatic line $TV^{\gamma-1} = $ constant.

So, $$T_0V_0^{\gamma-1} = T_1\left(\frac{V_0}{n}\right)^{\gamma-1} \text{ or } T_1 = n^{\gamma-1}\,T_0$$

Thus, $$\eta = 1 - \frac{n^\gamma - 1}{\gamma n^{\gamma-1}(n-1)}$$



(*b*)

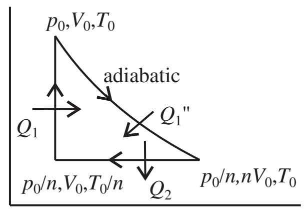
2.124 (a) Here, $Q_2' = C_pT_0\left(1 - \dfrac{1}{n}\right)$, $Q_1'' = RT_0 \ln n$

and $$Q_1' = C_VT_0\left(1 - \frac{1}{n}\right)$$

Now, $$Q_1 = Q_1' + Q_1''$$

So, $$\eta = 1 - \frac{Q_2'}{Q_1} = 1 - \frac{C_p(1 - 1/n)}{C_V(1 - 1/n) + R\ln n}$$

$$= 1 - \frac{\gamma}{1 + \dfrac{R}{C_V}\dfrac{n\ln n}{n-1}} = 1 - \frac{\gamma(n-1)}{n - 1 + (\gamma - 1)n\ln n}$$



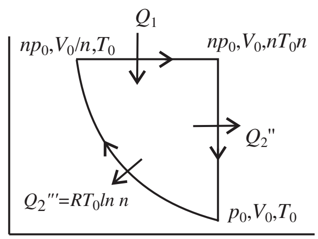
(*a*)

(b) Here, $Q_1 = C_pT_0(n-1)$, $Q_2'' = C_VT_0(n-1)$,

and $$Q_2''' = RT_0 \ln n$$

Now, $$Q_2' = Q_2'' + Q_2'''$$

So, $$\eta = 1 - \frac{Q_2'}{Q_1} = 1 - \frac{n - 1 + (\gamma - 1)\ln n}{\gamma(n-1)}$$



(*b*)

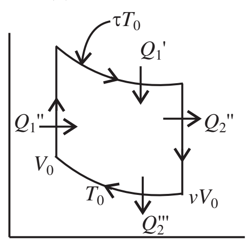
2.125 We have

$$Q_1' = \tau RT_0 \ln \nu,\ Q_2' = C_VT_0(\tau - 1)$$

and $$Q_2' = RT_0 \ln \nu,\ Q_1'' = C_VT_0(\tau - 1)$$

as well as $$Q_1 = Q_1' + Q_1'$$

and $$Q_2' = Q_2'' + Q_2'''$$



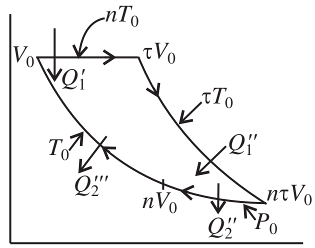
So, $\eta = 1 + \dfrac{Q_2'}{Q_1}$

$$= 1 - \frac{C_V(\tau - 1) + R\ln\nu}{C_V(\tau - 1) + \tau R\ln\nu}$$

$$= 1 - \frac{\dfrac{\tau - 1}{\gamma - 1} + \ln\nu}{\dfrac{\tau - 1}{\gamma - 1} + \tau\ln\nu} = \frac{(\tau - 1)\ln\nu}{\tau\ln\nu + \dfrac{\tau - 1}{\gamma - 1}}$$

2.126 Here, $Q_1'' = C_p T_0(\tau - 1)$, $Q_1' = \tau R T_0 \ln n$

and $Q_2'' = C_p T_0(\tau - 1)$, $Q_2''' = RT_0\ln n$



In addition, we have

$$Q_1 = Q_1' + Q_1''$$

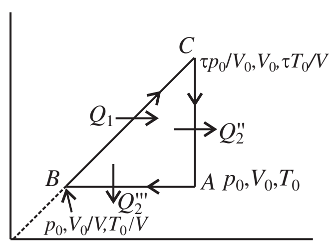
and $$Q_2' = Q_2'' + Q_2'''$$

So, $\eta = 1 - \dfrac{Q_2'}{Q_1} = 1 - \dfrac{C_p(\tau - 1) + R\ln n}{C_p(\tau - 1) + \tau R\ln n}$

$$= 1 - \frac{\tau - 1 + \left(1 - \dfrac{1}{\gamma}\right)\ln n}{\tau - 1 + \left(1 - \dfrac{1}{\gamma}\right)\tau\ln n} = \frac{(\tau - 1)\ln n}{\tau\ln n + \dfrac{\gamma(\tau - 1)}{\gamma - 1}}$$

2.127 Because of the linearity of the section BC, whose equation is

$$\frac{p}{p_0} = \frac{\nu V}{V_0} \times (\equiv p = \alpha V)$$



We have $\dfrac{\tau}{\nu} = v$ or $\nu = \sqrt{\tau}$

Here, $Q_2'' = C_V T_0(\sqrt{\tau} - 1)$,

$$Q_2''' = C_p T_0\left(1 - \frac{1}{\sqrt{\tau}}\right) = C_p\frac{T_0}{\sqrt{\tau}}(\sqrt{\tau} - 1)$$

Thus, $Q_2' = Q_2'' + Q_2''' = \dfrac{RT_0}{\gamma - 1}(\sqrt{\tau} - 1)\left(1 + \dfrac{\gamma}{\sqrt{\tau}}\right)$

Along BC, the specific heat C is given by

$$CdT = C_V dT + pdV = C_V dT + d\left(\frac{1}{2}\alpha V^2\right) = \left(C_V + \frac{1}{2}R\right)dT$$

Thus, $Q_1 = \dfrac{1}{2}RT_0\dfrac{\gamma + 1}{\gamma - 1}\dfrac{\tau - 1}{\sqrt{\tau}}$

Finally $\eta = 1 - \dfrac{Q_2'}{Q_1} = 1 - 2\dfrac{\sqrt{\tau} + \gamma}{\sqrt{\tau} + 1}\dfrac{1}{\gamma + 1} = \dfrac{(\gamma - 1)(\sqrt{\tau} - 1)}{(\gamma + 1)(\sqrt{\tau} + 1)}$

2.128 We write Claussius inequality in the form

$$\int \frac{đ_1 Q}{T} - \int \frac{đ_2 Q}{T} \leq 0$$

where $đ_1 Q$ is the heat transferred to the system but $đ_2 Q$ is heat rejected by the system, both are positive and this explains the minus sign before $đ_2 Q$.

In this inequality $T_{\max} > T > T_{\min}$ and we can write

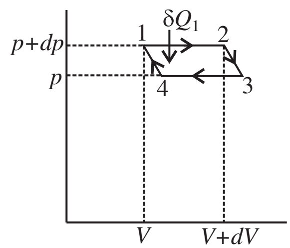
$$\int \frac{đ_1 Q}{T_{\max}} - \int \frac{đ_2 Q}{T_{\min}} < 0$$

Thus,
$$\frac{Q_1}{T_{\max}} < \frac{Q_2'}{T_{\min}} \text{ or } \frac{T_{\min}}{T_{\max}} < \frac{Q_2'}{Q_1}$$

or
$$\eta = 1 - \frac{Q_2'}{Q_1} < 1 - \frac{T_{\min}}{T_{\max}} = \eta_{\text{carnot}}$$

2.129 We consider an infinitesimal carnot cycle with isothermal process at temperatures $T + dT$ and T. Let δA be the work done in the cycle and δQ, be the heat received at the higher temperature. Then by Carnot's theorem

$$\frac{\delta A}{\delta Q_1} = \frac{dT}{T}$$



On the other hand,

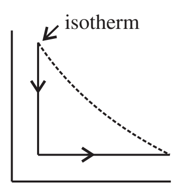
$$\delta A = dp dV = \left(\frac{\partial p}{\partial T}\right)_V dT dV$$

while,
$$\delta Q_1 = dU_{12} + p dV = \left[\left(\frac{\partial U}{\partial V}\right)_T + p\right] dV$$

Hence,
$$\left(\frac{\partial U}{\partial V}\right)_T + p = T\left(\frac{\partial p}{\partial T}\right)_V$$

2.130 (a) In an isochoric process, the entropy change will be

$$\Delta S = \int_{T_i}^{T_f} \frac{C_V dT}{T} = C_V \ln \frac{T_f}{T_i} = C_V \ln n = \frac{R \ln n}{\gamma - 1}$$



For carbon dioxide $\gamma = 1.30$, so

$$\Delta S = 19.2 \text{ J/K mol}$$

(b) For an isobaric process, entropy change will be

$$\Delta S = C_p \ln \frac{T_f}{T_i} = C_p \ln n = \frac{\gamma R \ln n}{\gamma - 1}$$
$$= 25 \text{ J/K mol}$$

2.131 In an isothermal expansion, entropy change is

$$\Delta S = \nu R \ln \frac{V_f}{V_i}$$

So,
$$\frac{V_f}{V_i} = e^{\Delta S/\nu R} = 2.0 \text{ times}$$

2.132 The entropy change depends on the final and initial states only, so we can calculate it directly along the isotherm, it is $\Delta S = 2\,R \ln n = 20$ J/K, assuming that the final volume is n times the initial volume.

2.133 If the initial temperature is T_0 and volume is V_0 then in an adiabatic expansion,

$$TV^{\gamma-1} = T_0 V_0^{\gamma-1}$$

So,
$$T = T_0 n^{1-\gamma} = T_1 \quad (\text{where } n = V_1/V_0)$$

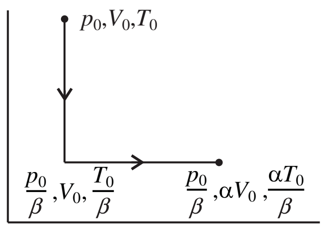
V_1 being the volume at the end of the adiabatic process. There is no entropy change in this process. Next the gas is compressed isobarically and the net entropy change is

$$\Delta S = \left(\frac{m}{M} C_p\right) \ln \frac{T_f}{T_1}$$

But,
$$\frac{V_1}{T_1} = \frac{V_0}{T_f} \quad \text{or} \quad T_f = T_1 \frac{V_0}{V_1} = T_0 n^{-\gamma}$$

So,
$$\Delta S = \left(\frac{m}{M} C_p\right) \ln \frac{1}{n} = -\frac{m}{M} C_p \ln n = -\frac{m}{M} \frac{R\gamma}{\gamma - 1} \ln n = -9.7 \text{ J/K}$$

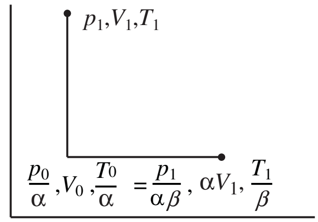
2.134 The entropy change depends on the initial and final state only so can be calculated for any process. We choose to evaluate the entropy change along the pair of lines shown in the figure. Then



$$\Delta S = \int_{T_0}^{T_0/\beta} \frac{C_V\, dT}{T} + \int_{T_0/\beta}^{\alpha T_0/\beta} \nu C_p \frac{dT}{T}$$
$$= (-C_V \ln \beta + C_p \ln \alpha)\, \nu = \frac{\nu R}{\gamma - 1} (\gamma \ln \alpha - \ln \beta)$$
$$\approx -11 \text{ J/K}$$

2.135 To calculate the required entropy difference we only have to calculate the entropy difference for a process in which the state of the gas in vessel 1 is changed to that in vessel 2.

$$\Delta S = \nu\left(\int_{T_1}^{T_1/\alpha\beta} C_V \frac{dT}{T} + \int_{T_1/\alpha\beta}^{T_1/\beta} C_p \frac{dT}{T}\right)$$



$$= \nu(C_p \ln \alpha - C_V \ln \alpha\beta)$$

$$= \nu\left(R \ln \alpha - \frac{R}{\gamma - 1} \ln \beta\right) = \nu\, R\, \left(\ln \alpha - \frac{\ln \beta}{\gamma - 1}\right)$$

$$\Delta S = 0.85 \text{ J/K}$$

(using $\gamma = 5/3$, $\alpha = 2$, $\beta = 1.5$ and $\nu = 1.2$).

2.136 For the polytropic process with index n, $pV^n =$constant.
Along this process (as in solution of Problem 2.122),

$$C = R\left(\frac{1}{\gamma - 1} - \frac{1}{n - 1}\right) = \frac{n - \gamma}{(\gamma - 1)(n - 1)} \cdot R$$

So,
$$\Delta S = \int_{T_0}^{\tau T_0} C \frac{dT}{T} = \frac{n - \gamma}{(\gamma - 1)(n - 1)} R \ln \tau$$

2.137 The process in question may be written as

$$\frac{p}{p_0} = \alpha \frac{V}{V_0}$$

(where α is a constant and p_0, V_0 are some reference values).

For this process (from solution of Problem 2.127), the specific heat is

$$C = C_V + \frac{1}{2} R = R\left(\frac{1}{\gamma - 1} + \frac{1}{2}\right) = \frac{1}{2} R \frac{\gamma + 1}{\gamma - 1}$$

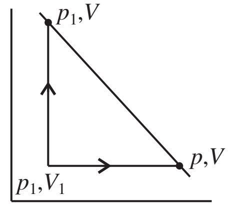
Along the line volume increases α times then so does the pressure. The temperature must then increase α^2 times. Thus,

$$\Delta S = \int_{T_0}^{\alpha^2 T_0} \nu C \frac{dT}{T} = \frac{\nu R}{2} \frac{\gamma + 1}{\gamma - 1} \ln \alpha^2 = \nu R \frac{\gamma + 1}{\gamma - 1} \ln \alpha$$

$$\Delta S = 46.1 \text{ J/K}.$$

(using $\nu = 2$, $\gamma = \dfrac{5}{3}$ and $\alpha = 2$)

2.138 Let p_1, V_1 be a reference point on the line and let (p, V) be any other point.



Then, $$p = p_0 - \alpha V$$

The entropy difference

$$\Delta S = S(p, V) - S(p_1, V_1)$$
$$= C_V \ln \frac{p}{p_1} + C_p \ln \frac{V}{V_1}$$
$$= C_V \ln \frac{p_0 - \alpha V}{p_1} + C_p \ln \frac{V}{V_1}$$

For an extremum of ΔS

$$\frac{\partial \Delta S}{\partial V} = \frac{-\alpha C_V}{p_0 - \alpha V} + \frac{C_p}{V_1} = 0$$

or $$C_p(p_0 - \alpha V) - \alpha V C_V = 0$$

or $$\gamma(p_0 - \alpha V) - \alpha V = 0 \quad \text{or} \quad V = V_m = \frac{\gamma p_0}{\alpha(\gamma + 1)}$$

This gives a maximum of ΔS because $\dfrac{\partial^2 \Delta S}{\partial\ V^2} < 0$

Note that a maximum of ΔS is a maximum of $S(p,V)$.

2.139 For the process, $$S = aT + C_V \ln T$$

and specific heat $$C = T\frac{dS}{dT} = aT + C_V$$

On the other hand $dQ = CdT = C_V dT + pdV$ for an ideal gas.

Thus, $$pdV = \frac{RT}{V} dV = aTdT$$

or $$\frac{RdV}{a\ V} = dT \quad \text{or} \quad \frac{R}{a} \ln V + \text{constant} = T$$

Using $$T = T_0 \text{ when } V = V_0$$

we get $$T = T_0 + \frac{R}{a} \ln \frac{V}{V_0}$$

2.140 For a Van der Waal gas

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT$$

The entropy change along an isotherm can be calculated from

$$\Delta S = \int_{V_1}^{V_2} \left(\frac{\partial S}{\partial V}\right)_T dV$$

It follows from solution of Problem 2.129 that

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V = \frac{R}{V - b}$$

(assuming a, b to be known constants).

Thus,
$$\Delta S = R \ln \frac{V_2 - b}{V_1 - b}$$

2.141 We use,
$$\Delta S = \int_{V_1, T_1}^{V_2, T_2} dS(V, T) = \int_{T_1}^{T_2} \left(\frac{\partial S}{\partial T}\right)_{V_1} dT + \int_{V_1}^{V_2} \left(\frac{\partial S}{\partial V}\right)_{T = T_2} dV$$

$$= \int_{T_1}^{T_2} \frac{C_V dT}{T} + \int_{V_1}^{V_2} \frac{R}{V - b} dV = C_V \ln \frac{T_2}{T_1} + R \ln \frac{V_2 - b}{V_1 - b}$$

(assuming CV, a, b to be known constants).

2.142 We can take $S \to 0$ as $T \to 0$.

Then,
$$S = \int_0^T C \frac{dT}{T} = \int_0^T a T^2 dT = \frac{1}{3} a T^3$$

2.143
$$\Delta S = \int_{T_1}^{T_2} \frac{C dT}{T} = \int_{T_1}^{T_2} \frac{m(a + bT)}{T} dT = mb\,(T_2 - T_1) + ma \ln \frac{T_2}{T_1}$$

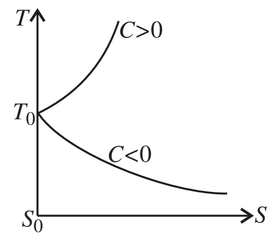
$= 2.0$ kJ (on substituting values)

2.144 Here,
$$T = aS^n \quad \text{or} \quad S = \left(\frac{T}{a}\right)^{\frac{1}{n}}$$

Then,
$$C = T \frac{1}{n} \frac{T^{\frac{1}{n} - 1}}{a^{1/n}} = \frac{S}{n}$$

Clearly,
$$C < 0 \text{ if } n < 0$$

2.145 We know,



$$S - S_0 = \int_{T_0}^{T} \frac{CdT}{T} = C \ln \frac{T}{T_0}$$

(assuming C to be a known constant).

Then, $$T = T_0 \exp\left(\frac{S - S_0}{C}\right)$$

2.146 (a) $C = T\dfrac{dS}{dT} = -\dfrac{\alpha}{T}$

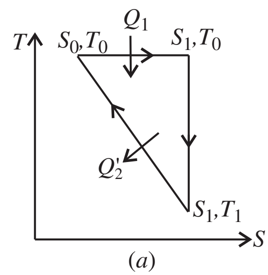
(b) $Q = \displaystyle\int_{T_1}^{T_2} CdT = \alpha \ln \frac{T_1}{T_2}$

(c) $W = \Delta Q - \Delta U$

$$= \alpha \ln \frac{T_1}{T_2} + C_V(T_1 - T_2)$$

Since for an ideal gas C_V is constant and $\Delta U = C_V(T_2 - T_1)$ (as U does not depend on V).

2.147 (a) We have from the definition



$$Q = \int TdS = \text{area under the curve}$$

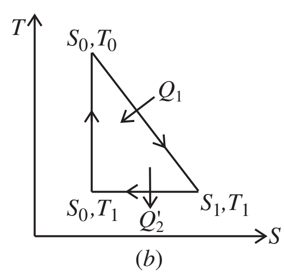
$$Q_1 = T_0(S_1 - S_0)$$

$$Q_2' = \frac{1}{2}(T_0 + T_1)(S_1 - S_0)$$

Thus, using $T_1 = \dfrac{T_0}{n}$

We get $\eta = 1 - \dfrac{T_0 + T_1}{2T_0} = 1 - \dfrac{1 + 1/n}{2} = \dfrac{n-1}{2n}$

(b) Here, $Q_1 = \dfrac{1}{2}(S_1 - S_0)(T_1 + T_0)$



$$Q_2' = T_1(S_1 - S_0)$$

$$\eta = 1 - \frac{2T_1}{T_1 + T_0} = \frac{T_0 - T_1}{T_0 + T_1} = \frac{n-1}{n+1}$$

2.148 In this case, known as free expansion, no work is done and no heat is exchanged. So internal energy must remain unchanged, i.e., $U_f = U_i$. For an ideal gas this implies constant temperature, i.e., $T_f = T_i$. The process is irreversible but the entropy change can be calculated by considering a reversible isothermal process. Then, as before

$$\Delta S = \int \frac{dQ}{T} = \int_{V_1}^{V_2} \frac{pdV}{T} = \nu R \ln n = 20.1 \text{ J/K}$$

2.149 The process consists of two parts. The first part is a free expansion in which $U_f = U_i$. The second part is an adiabatic compression in which work done results in change of internal energy. Obviously,

$$0 = U_F - U_f + \int_{V_f}^{V_0} pdV \quad \text{and} \quad V_f = 2V_0$$

Now in the first part

$$p_f = \frac{1}{2} p_0 \text{ an } V_f = 2V_0$$

(because there is no change of temperature).

In the second part,

$$p\, V^\gamma = \frac{1}{2} p_0 \, (2V_0)^\gamma = 2^{\gamma-1} p_0 V_0^\gamma$$

$$\int_{2V_0}^{V_0} pdV = \int_{2V_0}^{V_0} \frac{2^{\gamma-1} p_0 V_0^\gamma}{V^\gamma}\, dV = \left[\frac{2^{\gamma-1} p_0 V_0^\gamma}{-\gamma+1}\, V^{1-\gamma}\right]_{2V_0}^{V_0}$$

$$= 2^{\gamma-1} p_0 V_0^\gamma V_0^{-\gamma+1} \frac{2^{\gamma-1}-1}{\gamma-1}$$

$$= \frac{RT_0}{\gamma-1}(1 - 2^{\gamma-1})$$

Thus, $$\Delta U = U_F - U_i = \frac{RT_0}{\gamma-1}(2^{\gamma-1} - 1)$$

The entropy change $\Delta S = \Delta S_{\text{I}} + \Delta S_{\text{II}}$.

$\Delta S_{\text{I}} = R \ln 2$ and $\Delta S_{\text{II}} = 0$ as the process is reversible adiabatic. Thus, $\Delta S = R \ln 2$.

2.150 In all adiabatic processes, by virtue of first law of thermodynamics

$$Q = U_f - U_i + A = 0$$

Thus, $$U_f = U_i - A$$

For a slow process,

$$A' = \int_{V_0}^{V} p dV \quad \text{(where for a quasistatic adiabatic process } pV^{\gamma} = \text{constant)}$$

On the other hand for a fast process, the external work done is $A'' < A'$. In fact $A'' = 0$ for free expansion. Thus U'_f (slow) $< U''_f$ (fast).

Since U depends on temperature only, if $T'_f < T''_f$, consequently $p''_f > p'_f$ from the ideal gas equation $pV = RT$.

Thus the pressure will be higher after the fast expansion.

2.151 Let $V_1 = V_0$, $V_2 = \eta v_0$.

Since the temperature is the same, the required entropy change can be calculated by considering isothermal expansion of the gas in either parts into the whole vessel.

Thus,
$$\begin{aligned}\Delta S &= \Delta S_{\text{I}} + \Delta S_{\text{II}} \\ &= \nu_1 R \ln \frac{V_1 + V_2}{V_1} + \nu_2 R \ln \frac{V_1 + V_2}{V_2} \\ &= \nu_1 R \ln (1 + n) + \nu_2 R \ln \left(\frac{1 + n}{n}\right) \\ &= 5.1 \text{ J/K}\end{aligned}$$

2.152 Let c_1 be the specific heat of copper and c_2 be the specific heat of water.

Then,
$$\begin{aligned}\Delta S &= \int_{7+273}^{T_0} \frac{c_2 m_2 dT}{T} - \int_{T_0}^{97+273} \frac{m_1 c_1 dT}{T} \\ &= m_2 c_2 \ln \frac{T_0}{280} - m_1 c_1 \ln \frac{370}{T_0}\end{aligned}$$

T_0 can be found from,

$$c_2 m_2 (T_0 - 280) = m_1 c_1 (370 - T_0) \quad \text{or} \quad T_0 = \frac{280 m_2 c_2 + 370 m_1 c_1}{c_2 m_2 + m_1 c_1}$$

Using $c_1 = 0.39$ J/g K, $c_2 = 4.18$ J/g K,

$$T_0 \approx 300 \text{ K and } \Delta S = 28.4 - 24.5 \cong 3.9 \text{ J/K}$$

2.153 For an ideal gas the internal energy depends on temperature only. We can consider the process in question to be one of simultaneous free expansion, then the total energy $U = U_1 + U_2$.

Since $U_1 = C_V T_1,\ U_2 = C_V T_2$ so, $U = 2C_V(T_1 + T_2)/2$ and $(T_1 + T_2)/2$ is the final temperature.

The entropy change is obtained by considering isochoric process because in effect, the gas remains confined to its vessel.

So, $$\Delta S = \int_{T_1}^{(T_1+T_2)/2} \frac{C_V\,dT}{T} - \int_{(T_1+T_2)/2}^{T_2} C_V \frac{dT}{T} = C_V \ln \frac{(T_1 + T_2)^2}{4T_1T_2}$$

Since, $$(T_1 + T_2)^2 = (T_1 - T_2)^2 + 4T_1T_2,$$

We get $$\Delta S > 0$$

2.154 (a) Each atom has probability 1/2 to be in either compartment. Thus $P = 2^{-N}$.

(b) Typical atomic velocity at room temperature is $\sim 10^5$ cm/s so it takes an atom 10^{-5} s to cross the vessel. This is the relevant time scale for our problem. Let $T = 10^{-5}$ s, then in time T there will be t/T crossing or arrangements of the atoms. This will be large enough to produce the given arrangement if

$$\frac{t}{\tau} 2^{-N} \approx 1 \quad \text{or} \quad N \approx \frac{\ln t/\tau}{\ln 2} \approx 75$$

2.155 The statistical weight is

$$N_{C_{N/2}} = \frac{N!}{N/2\ !\dfrac{N}{2}\ !} = \frac{10 \times 9 \times 8 \times 7 \times 6}{8 \times 4 \times 3 \times 2} = 252$$

The probability distribution is

$$N_{C_{N/2}} 2^{-N} = 252 \times 2^{-10} = 24.6\%$$

2.156 The probabilities that the half A contains n molecules is

$$N_{C_n} \times 2^{-N} = \frac{N!}{n\,!(N - n)\,!} 2^{-N};\ \frac{1}{32}, \frac{5}{32}, \frac{10}{32}, \frac{5}{32}, \frac{1}{32}, \text{ respectively.}$$

2.157 The probability of one molecule being confined to the marked volume is

$$P = \frac{V}{V_0}$$

We can choose this molecule in many $(N_{C_1}$ ways. The probability that n molecules get confined to the marked volume is clearly

$$N_{C_n} P^n (1 - P)^{N-n} = \frac{N!}{n\,!(N - n)\,!} P^n (1 - P)^{N-n}$$

2.158 In a sphere of diameter d, the number of molecules are

$$N = \frac{\pi d^3}{6} n_0$$

where n_0 = Loschmidt's number = number of molecules per unit volume (1 cc) under NTP.

The relative fluctuation in this number is

$$\frac{\partial N}{N} = \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}} = \eta$$

or $$\frac{1}{\eta^2} = \frac{\pi}{6} d^3 n_0 \quad \text{or} \quad d^3 = \frac{6}{\pi n_0 \eta^2} \Rightarrow d = \left(\frac{6}{\pi \eta^2 n_0}\right)^{1/3} = 0.41\ \mu\text{m}$$

The average number of molecules in this sphere is

$$\frac{1}{\eta^2} = 10^6$$

2.159 For a monoatomic gas, $C_V = 3/2R$ per mole.

The entropy change in the process is

$$\Delta S = S - S_0 = \int_{T_0}^{T_0+\Delta T} C_V \frac{dT}{T} = \frac{3}{2} R \ln\left(1 + \frac{\Delta T}{T_0}\right)$$

Now from the Boltzmann equation, $S = k \ln \Omega$.

So, $$\frac{\Omega}{\Omega_0} = e^{(S-S_0)/k} = \left(1 + \frac{\Delta T}{T_0}\right)^{\frac{3N_A}{2}} = \left(1 + \frac{1}{300}\right)^{\frac{3\times 6}{2}\times 10^{23}} = 10^{1.31\times 10^{21}}$$

Thus the statistical weight increases by this factor.

2.5 Liquids. Capillary Effects

2.160 (a) $$\Delta p = \alpha\left(\frac{1}{d/2} + \frac{1}{d/2}\right) = \frac{4\alpha}{d}$$

$$= \frac{4 \times 490 \times 10^{-3}}{1.5 \times 10^{-6}} \frac{N}{m^2} = 1.307 \times 10^6 \frac{N}{m^2} = 13 \text{ atm}$$

(b) The soap bubble has two surfaces.

So, $$\Delta p = 2\alpha\left(\frac{1}{d/2} + \frac{1}{d/2}\right) = \frac{8\alpha}{d}$$

$$= \frac{8 \times 45}{3 \times 10^{-3}} = 1.2 \times 10^{-3} \text{ atm}$$

2.161 The pressure just inside the hole will be less than the outside pressure by $4\alpha/d$. This can support a height h of Hg where,

$$\rho\, gh = \frac{4\alpha}{d} \quad \text{or} \quad h = \frac{4\alpha}{\rho g d}$$

So, $$h = \frac{4 \times 490 \times 10^{-3}}{13.6 \times 10^{3} \times 9.8 \times 70 \times 10^{-6}} = \frac{200}{13.6 \times 70} \approx 0.21 \text{ m of Hg}$$

2.162 By Boyle's law

$$\left(p_0 + \frac{8\alpha}{d}\right)\frac{4\pi}{3}\left(\frac{d}{2}\right)^3 = \left(\frac{p_0}{n} + \frac{8\alpha}{\eta d}\right)\frac{4\pi}{3}\left(\frac{\eta d}{2}\right)^3$$

or $$p_0\left(1 - \frac{\eta^3}{n}\right) = \frac{8\alpha}{d}(\eta^2 - 1)$$

Thus, $$\alpha = \frac{1}{8}\, p_0 d\left(1 - \frac{\eta^3}{n}\right)\Big/(\eta^2 - 1)$$

2.163 The pressure has terms due to hydrostatic pressure and capillarity and they add as

$$p = p_0 + \rho g h + \frac{4\alpha}{d}$$

$$= \left(1 + \frac{5 \times 9.8 \times 10^3}{10^5} + \frac{4 \times 0.73 \times 10^{-3}}{4 \times 10^{-6}} \times 10^{-5}\right) \text{atm} = 2.22 \text{ atm}$$

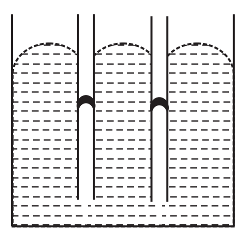
2.164 By Boyle's law

$$\left(p_0 + hg\rho + \frac{4\alpha}{d}\right)\frac{\pi}{6}d^3 = \left(p_0 + \frac{4\alpha}{nd}\right)\frac{\pi}{6}n^3 d^3$$

or $$hg\rho - p_0(n^3 - 1) = \frac{4\alpha}{d}(n^2 - 1)$$

or $$h = \left[p_0(n^3 - 1) + \frac{4\alpha}{d}(n^2 - 1)\right]\Big/ g\rho = 4.98 \text{ m of water.}$$

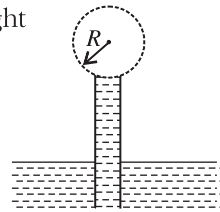
2.165 Clearly



$$\Delta h \rho g = 4\alpha|\cos\theta|\left(\frac{1}{d_1} - \frac{1}{d_2}\right)$$

$$\Delta h = \frac{4\alpha|\cos\theta|(d_2 - d_1)}{d_1 d_2 \rho g} = 11 \text{ mm}$$

2.166 In a capillary with diameter $d = 0.5$ mm, water will rise to a height



$$h = \frac{2\alpha}{\rho g r} = \frac{4\alpha}{\rho g d}$$

$$= \frac{4 \times 73 \times 10^{-3}}{10^3 \times 9.8 \times 0.5 \times 10^{-3}} = 59.6 \text{ mm}$$

Since this is greater than the height (= 25 mm) of the tube, a meniscus of radius R will be formed at the top of the tube, where

$$R = \frac{2\alpha}{\rho g h} = \frac{2 \times 73 \times 10^{-3}}{10^3 \times 9.8 \times 25 \times 10^{-3}} \approx 0.6 \text{ mm}$$

2.167 Initially the pressure of air in the capillary is p_0 and it's length is l. When submerged under water, the pressure of air in the portion above water must be $p_0 + 4\alpha/d$, since the level of water inside the capillary is the same as the level outside. Thus by Boyle's law

$$\left(p_0 + \frac{4\alpha}{d}\right)(l - x) = p_0 l$$

or $$\frac{4\alpha}{d}(l - x) = p_0 x$$

or $$x = \frac{l}{1 + p_0 d/4\alpha} = 1.4 \text{ cm (on substituting values)}$$

2.168 We have by Boyle's law

$$\left(p_0 - \rho g h + \frac{4\alpha \cos\theta}{d}\right)(l - h) = p_0 l$$

or $$\frac{4\alpha \cos\theta}{d} = \rho g h + \frac{p_0 h}{l - h}$$

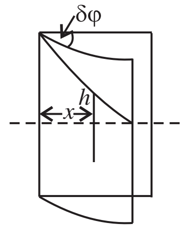
Hence, $$\alpha = \left(\rho g h + \frac{p_0 h}{l - h}\right)\frac{d}{4\cos\theta}$$

2.169 Suppose the liquid rises to a height h. Then the total energy of the liquid in the capillary is

$$E(h) = \frac{\pi}{4}\left(d_2^2 - d_1^2\right) h \times \rho g \times \frac{h}{2} - \pi(d_2 + d_1)\,\alpha h$$

Minimising E, we get $$h = \frac{4\alpha}{\rho g(d_2 - d_1)} = 6 \text{ cm}$$

2.170 Let h be the height of the water level at a distance x from the edge. Then the total energy of water in the wedge above the level outside is



$$E = \int x\delta\varphi \cdot dx \cdot h \cdot \rho g \frac{h}{2} - 2\int dx \cdot h \cdot \alpha \cos\theta$$

$$= \int dx \frac{1}{2} x\rho g\delta\varphi \left(h^2 - 2\frac{2\alpha \cos\theta}{x\rho g\delta\varphi} h \right)$$

$$= \int dx \frac{1}{2} x\rho g\delta\varphi \left[\left(h - \frac{2\alpha \cos\theta}{x\rho g\delta\varphi} \right)^2 - \frac{4\alpha^2 \cos^2\theta}{x^2\rho^2 g^2 \delta\varphi^2} \right.$$

This is minimum when $h = \dfrac{2\alpha \cos\theta}{x\rho g\delta\varphi}$

2.171 From the equation of continuity

$$\frac{\pi}{4} d^2 \cdot v = \frac{\pi}{4}\left(\frac{d}{n}\right)^2 \cdot V \quad \text{or} \quad V = n^2 v$$

We then apply Bernoulli's theorem

$$\frac{p}{\rho} + \frac{1}{2}v^2 + \Phi = \text{constant}$$

The pressure p differs from the atmospheric pressure by capillary effects. At the upper section, $p = p_0 + 2\alpha/d$ neglecting the curvature in the vertical plane.

Thus, $$\frac{p_0 + 2\alpha/d}{\rho} + \frac{1}{2}v^2 + gl = \frac{p_0 + 2n\alpha/d}{\rho} + \frac{1}{2}n^4 v^2$$

or $$v = \sqrt{\frac{2gl - (4\alpha/\rho d)(n-1)}{n^4 - 1}}$$

Finally, the liquid coming out per second is,

$$V = \frac{1}{4}\pi d^2 \sqrt{\frac{2gl - (4\alpha/\rho d)(n-1)}{n^4 - 1}} = 0.9 \text{ cm}^3/\text{s (on substituting values)}$$

2.172 The radius of curvature of the drop is R_1 at the upper end of the drop and R_2 at the lower end. Then the pressure inside the drop is $p_0 + (2\alpha/R_1)$ at the top end and $p_0 + (2\alpha/R_2)$ at the bottom end. Hence,

$$p_0 + \frac{2\alpha}{R_1} = p_0 + \frac{2\alpha}{R_2} + \rho g h \quad \text{or} \quad \frac{2\alpha(R_2 - R_1)}{R_1 R_2} = \rho g h$$

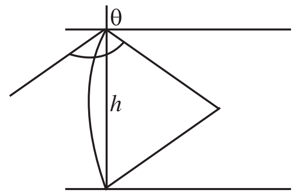
To a first approximation $R_1 \approx R_2 \approx h/2$.

So, $$R_2 - R_1 \approx 1/8\rho g h^3/\alpha \approx 0.20 \text{ mm}$$

and if $$h = 2.3 \text{ mm}, \alpha = 73 \text{ mN/m}$$

2.173 We must first calculate the pressure difference inside the film from that outside. This is given by

$$p = \alpha\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$



Here, $2r_1|\cos\theta| = h$ and $r_2 \approx R$, where R is the radius of the tablet and can be neglected. Thus the total force exerted by mercury drop on the upper glass plate is typically,

$$\frac{2\pi R^2\alpha|\cos\theta|}{h}$$

We should put h/n for h because the tablet is compressed n times. Then since Hg is nearly incompressible, $\pi R^2 h$ = constant, so $R \to R\sqrt{n}$. Thus, total force

$$= \frac{2\pi R^2\alpha|\cos\theta|}{h}n^2$$

Part of the force is needed to keep the Hg in the shape of a tablet rather than in the shape of infinitely thin sheet. This part can be calculated being putting $n = 1$ above. Thus

$$mg + \frac{2\pi R^2\alpha|\cos\theta|}{h} = \frac{2\pi R^2\alpha|\cos\theta|}{h}n^2$$

or $$m = \frac{2\pi R^2\alpha|\cos\theta|}{hg}(n^2 - 1) = 0.7 \text{ kg}$$

2.174 The pressure inside the film is less than that outside by an amount

$$p = \alpha\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

(where r_1 and r_2 are the principal radii of curvature of the meniscus).

One of these is small, being given by $h = 2r_1\cos\theta$ while the other is large and will be ignored.

Then, $$F \approx \frac{2A\cos\theta}{h}\alpha$$

(where A = area of the water film between the plates)

Now, $$A = \frac{m}{\rho h}$$

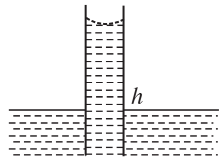
So, $$F = \frac{2m\alpha}{\rho h^2} \text{ [when } \theta \text{ (the angle of contact) = 0]}$$

$$= 1.0 \text{ N (on substituting values)}$$

2.175 This is analogous to the previous problem except that, $A = \pi R^2$.

So, $$F = \frac{2\pi R^2 \alpha}{h} = 0.6 \text{ kN}$$

2.176 The energy of the liquid between the plates is



$$E = ldh\rho g \frac{h}{2} - 2\,\alpha l h = \frac{1}{2}\rho g l d h^2 - 2\alpha l h$$

$$= \frac{1}{2}\rho g l d \left(h - \frac{2\alpha}{\rho g d} \right)^2 - \frac{2\alpha^2 l}{\rho g d}$$

This energy is minimum when, $h = 2\alpha/\rho g d$ and the minimum potential energy is then

$$E_{\min} = -\frac{2\alpha^2 l}{\rho g d}$$

The force of attraction between the plates can be obtained from this as

$$F = \frac{-\partial E_{\min}}{\partial d} = -\frac{2\alpha^2 l}{\rho g d^2} \quad \text{(minus sign means the force is attractive)}$$

Thus, $$F = \frac{\alpha l h}{d} = 13 \text{ N}$$

2.177 Suppose the radius of bubble is x at some instant. Then the pressure inside is $p_0 + 4\alpha/x$. The flow through the capillary is given by Poiseuille's equation,

$$Q = \frac{\pi r^4}{8\eta l}\frac{4\alpha}{x} = -4\pi x^2 \frac{dx}{dt}$$

Integrating we get,

$$\frac{\pi r^4 \alpha}{2\eta l} t = \pi (R^4 - x^4)$$

(where we have used the fact that $t = 0$ and $x = R$).

The life time of the bubble corresponding to $x = 0$ is

$$t = \frac{2\eta l R^4}{\alpha r^4}$$

2.178 If the liquid rises to a height h, the energy of the liquid column becomes

$$E = \rho g \pi r^2 h \cdot \frac{h}{2} - 2\pi r h \alpha = \frac{1}{2}\rho g \pi \left(rh - 2\frac{\alpha}{\rho g} \right)^2 - \frac{2\pi \alpha^2}{\rho g}$$

This is minimum when $rh = 2\alpha/\rho g$ and that is relevant height to which water must rise.

At this point,
$$E_{\min} = -\frac{2\pi\alpha^2}{\rho g}$$

Since $E = 0$ in the absence of surface tension, heat liberated must be

$$Q = \frac{2\pi\alpha^2}{\rho g}$$

2.179 (a) The free energy per unit area being α,

$$F = \pi\alpha d^2 = 3\ \mu\text{J (on substituting values)}$$

(b) $F = 2\pi\alpha d^2$ because the soap bubble has two surfaces.
Substitution gives, $F = 10\ \mu$J.

2.180 When two mercury drops each of diameter d merge, the resulting drop has diameter d_1

where,
$$\frac{\pi}{6}d_1^3 = \frac{\pi}{6}d^3 \times 2 \quad \text{or} \quad d_1 = 2^{1/3}d$$

The increase in free energy is

$$\Delta F = \pi\ 2^{2/3}\ d^2\ \alpha - 2\pi d^2\ \alpha = 2\pi d^2\ \alpha\ (2^{-1/3} - 1) = -1.43\ \mu\text{J}$$

2.181 Work must be done to stretch the soap film and compress the air inside. The former is simply $2\alpha \times 4\pi R^2 = 8\pi R^2\ \alpha$, there being two sides of the film. To get the latter we note that the compression is isothermal and work done is

$$-\int_{V_i=V_0}^{V_f=V} p\,dV$$

where,
$$V_0 p_0 = \left(p_0 + \frac{4\alpha}{R}\right)\cdot V \quad \text{and} \quad V = \frac{4\pi}{3}R^3$$

or
$$V_0 = \frac{pV}{p_0} \quad \text{and} \quad p = p_0 + \frac{4\alpha}{R}$$

and minus sign is needed because we are calculating work done on the system. Thus since pV remains constant, the work done is

$$pV\ln\frac{V_0}{V} = p\ V\ln\frac{p}{p_0}$$

So,
$$A' = 8\pi\ R^2\ \alpha + p\ V\ln\frac{p}{p_0}$$

2.182 When heat is given to a soap bubble, the temperature of the air inside rises and the bubble expands but unless the bubble bursts, the amount of air inside does not change. Further we shall neglect the variation of the surface tension with temperature. Then from the gas equations

$$\left(p_0 + \frac{4\alpha}{r}\right)\frac{4\pi}{3} r^3 = \nu RT \quad (\nu = \text{constant})$$

Differentiating we get,

$$\left(p_0 + \frac{8\alpha}{3r}\right) 4\pi r^2 dr = \nu R dT$$

or
$$dV = 4\pi r^2 dr = \frac{\nu R dT}{p_0 + \dfrac{8\alpha}{3r}}$$

Now from the first law of thermodynamics

$$đQ = \nu C dT = \nu C_V dT + \frac{\nu R dT}{p_0 + 8\alpha/3r}\left(p_0 + \frac{4\alpha}{r}\right)$$

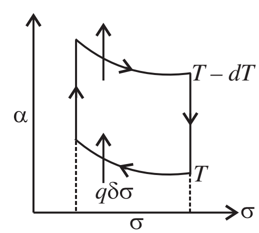
or
$$C = C_V + R\frac{p_0 + 4\alpha/r}{p_0 8\alpha/3r}$$

using $C_p = C_V + R$, we get

$$C = C_p + \frac{1/2\ R}{1 + 3p_0 r/8\alpha}$$

2.183 Consider an infinitesimal Carnot cycle with isotherms at $T - dT$ and T. Let A be the work done during the cycle. Then

$$A = [\alpha(T - dT) - \alpha(T)]\,\delta\sigma = -\frac{d\alpha}{dT}\,dT\,\delta\sigma$$



(where $\delta\sigma$ is the change in the area of film; we are considering only one surface here).

Then by Carnot theorem

$$\eta = \frac{A}{Q_1} = \frac{dT}{T}$$

or
$$\frac{-\dfrac{d\alpha}{dT}dT\,\delta\sigma}{q\,\delta\sigma} = \frac{dT}{T} \quad \text{or} \quad q = -T\frac{d\alpha}{dT}$$

2.184 As before we can calculate the heat required. It is taking into account two sides of the soap film. So,

$$\delta q = -T\frac{d\alpha}{dT}\delta\sigma \times 2$$

Thus, $$\Delta S = \frac{\delta q}{T} = -2\frac{d\alpha}{dT}\delta\sigma$$

Now, $$\Delta F = 2\alpha\delta\sigma$$

So, $$\Delta U = \Delta F + T\Delta S = 2\left(\alpha - T\frac{d\alpha}{dT}\right)\delta\sigma$$

2.6 Phase Transformations

2.185 The condensation takes place at constant pressure and temperature and the work done is $A = p\Delta V$, where ΔV is the volume of the condensed vapour phase. It is

$$p\Delta V = \frac{\Delta m}{M}RT = 1.2\ \text{J}$$

(where $M = 18$ g is the molecular weight of water).

2.186 The specific volume of water (the liquid) will be written as V'_l. Since $V'_{\text{vap}} >> V'_l$, most of the weight is due to water. Thus if m_l is mass of the liquid and m_{vap} that of the vapour then

$$m = m_l + m_{\text{vap}}$$

$$V = m_l V'_l + m_{\text{vap}} V'_{\text{vap}} \quad \text{or} \quad V - mV'_l = m_{\text{vap}}\left(V'_{\text{vap}} - V'_l\right)$$

So, $$m_{\text{vap}} = \frac{V - mV'_l}{V'_{\text{vap}} - V'_l} = 20\ \text{g}$$

Its volume is $V'_{\text{vap}} = 1.0$ l

2.187 The volume of the condensed vapour was originally $V_0 - V$ at temperature $T = 373$ K. Its mass will be given by

$$p_0\left(V_0 - V\right) = \frac{m}{M}RT \quad \text{or} \quad m = \frac{Mp_0(V_0 - V)}{RT} = 2\ \text{g}$$

(where p_0 = standard atmospheric pressure).

2.188 We let V'_l = specific volume of liquid. $V'_{vap} = NV'_l$ = specific volume of vapour.

Let V = original volume of the vapour.

Then, $$M\frac{pV}{RT} = m_l + m_{vap} = \frac{V}{NV'_l} \quad \text{or} \quad \frac{V}{n} = (m_l + Nm_{vap})\, V'_l$$

So, $$(N-1)\, m_l\, V'_l = V\left(1 - \frac{1}{n}\right) = \frac{V}{n}(n-1) \quad \text{or} \quad \eta = \frac{m_l V'_l}{V/n} = \frac{n-1}{N-1}$$

In the case when the final volume of the substance corresponds to the midpoint of a horizontal portion of the isothermal line in the pV diagram, the final volume must be $(1+N)\, V'_l/2$ per unit mass of the substance. Of this, the volume of the liquid is $V'_l/2$ per unit total mass of the substance.

Thus, $$\eta = \frac{1}{1+N}$$

2.189 From the first law of thermodynamics

$$\Delta U + A = Q = mq$$

(where q is the specific latent heat of vaporization).
So, increment in entropy is given by

$$\Delta S = \frac{Q}{T} = \frac{mq}{T} = 6.0 \text{ kJ/K (on substituting values)}$$

Now, $$A = p\,(V'_{vap} - V'_l)\, m = m\frac{RT}{M}$$

Thus, $$\Delta U = m\left(q - \frac{RT}{M}\right)$$

For water, this gives $\approx 2.08 \times 10^6$ J.

2.190 Some of the heat used in heating water to the boiling temperature $T = 100°\text{C} = 373$ K. The remaining heat $= Q - mc\Delta T$ (here c = specific heat of water, $\Delta T = 100$ K) is used to create vapour. If the piston rises to a height h then the volume of vapour will be $\approx Sh$ (neglecting water). Its mass will be

$$\frac{p_0 Sh}{RT} \times M$$

Heat of vapourization will be

$$\frac{p_0 Sh\, Mq}{RT}$$

To this must be added the work done in creating the saturated vapour $= p_0 Sh$.

Thus, $$Q - mc\Delta T \approx p_0 Sh\left(1 + \frac{qM}{RT}\right) \quad \text{or} \quad h \approx \frac{Q - mc\Delta T}{p_0 S\left(1 + \frac{qM}{RT}\right)} = 20 \text{ cm}$$

2.191 A quantity $mc(T - T_0)/q$ of saturated vapour must condense to heat the water to bolling point $T = 373$ K. (Here $c =$ specific heat of water, $T_0 = 295$ K = initial water temperature.)

The work done in lowering the piston will then be

$$\frac{mc(T - T_0)}{q} \times \frac{RT}{M} = 25 \text{ J}$$

Since work done per unit mass of the condensed vapour is $pV = RT/M$.

2.192 Given

$$\Delta P = \frac{\rho_{\text{vap}}}{\rho_l}\frac{2\alpha}{r} = \frac{\rho_{\text{vap}}}{\rho_l} \times \frac{4\alpha}{d} = \eta\, p_{\text{vap}}$$

$$= \eta P_{\text{vap}} = \eta\,\frac{(m/M)\,RT}{V_{\text{vap}}} = \frac{\eta RT}{M}\,\rho_{\text{vap}}$$

or $$d = \frac{4\alpha M}{\rho_l RT\eta}$$

For water, $\alpha = 73$ dynes/cm, $M = 18$ g, $\rho_l = 1$ g/cc and $T = 300$ K. Using these values and with $\eta = 0.01$, we get $d \approx 0.2\ \mu$m.

2.193 At equilibrium the number of "liquid" molecules evaporating must equal the number of "vapour" molecules condensing. By kinetic theory, this number is

$$\eta \times \frac{1}{4}\, n < v > = \eta \times \frac{1}{4}\, n \times \sqrt{\frac{8}{\pi}\frac{kT}{m}}$$

The required mass is

$$\mu = m \times \eta \times n \times \sqrt{\frac{kT}{2\pi m}} = \eta\, nkT\sqrt{\frac{m}{2\pi kt}}$$

$$= \eta\, p_0\sqrt{\frac{M}{2\pi RT}} = 0.35 \text{ g/cm}^2$$

where p_0 is standard atmospheric pressure, $T = 373$ K and M = molecular weight of water.

2.194 Here we must assume that μ is also the rate at which the tungsten filament loses mass when in an atmosphere of its own vapour at this temperature and that η (of the previous problem) ≈ 1. Then

$$p = \mu\sqrt{\frac{2\pi RT}{M}} = 0.9 \text{ nPa}$$

where p = pressure of the saturated vapor.

2.195 From the Van der Waal's equation

$$p = \frac{RT}{V - b} - \frac{a}{V^2}$$

(where V = volume of one gram mole of the substances).

For water $V = 18$ cm^3 per mol $= 1.8 \times 10^{-2}$ l mol^{-1} and $a = 5.47$ atm l^2/mol^2.

If molecular attraction vanishes, the equation will be

$$p' = \frac{RT}{V - b} \quad \text{(for the same specific volume)}$$

Thus,

$$\Delta p = \frac{a}{V^2}\,\text{m} = \frac{5.47}{1.8 \times 1.8} \times 10^4 \text{ atm} \approx 1.7 \times 10^4 \text{ atm}$$

2.196 The internal pressure being a/V^2, the work done in condensation is

$$\int_{V_l}^{V_g} \frac{a}{V^2}\,dV = \frac{a}{V_l} - \frac{a}{V_g} \approx \frac{a}{V_l}$$

This by assumption is Mq, M being the molecular weight, q being specific talent heat of vaporization and V_l, V_g being the molar volumes of the liquid and gas, respectively

Thus, $$p_i = \frac{a}{V_l^2} = \frac{Mq}{V_l} = \rho q$$

(where ρ is the density of the liquid).

For water $$p_i = 2 \times 10^4 \text{ atm}$$

2.197 The Van der Waal's equation can be written as (for one mole)

$$p\,(V) = \frac{RT}{V - b} - \frac{a}{V^2}$$

At the critical point $\left(\frac{\partial p}{\partial V}\right)_T$ and $\left(\frac{\partial^2 p}{\partial V^2}\right)_T$ vanish. Thus,

$$0 = \frac{RT}{(V - b)^2} - \frac{2a}{V^3} \quad \text{or} \quad \frac{RT}{(V - b)^2} = \frac{2a}{V^3}$$

$$0 = \frac{2RT}{(V - b)^3} - \frac{6a}{V^4} \quad \text{or} \quad \frac{RT}{(V - b)^3} = \frac{3a}{V^4}$$

Solving these simultaneously, we get on division

$$V - b = \frac{2}{3}\,V,\ V = 3\,b \approx V_{MCr}$$

This is the critical molar volume. Putting this back, we get

$$\frac{RT_{Cr}}{4b^2} = \frac{2a}{27b^3} \quad \text{or} \quad T_{Cr} = \frac{8a}{27bR}$$

Finally $$p_{cr} = \frac{RT_{Cr}}{V_{MCr} - b} - \frac{a}{V_{MCr}^2} = \frac{4a}{27b^2} - \frac{a}{9b^2} = \frac{a}{27b^2}$$

From these we see that

$$\frac{p_{Cr}V_{MCr}}{RT_{Cr}} = \frac{a/9b}{8a/27b} = \frac{3}{8}$$

2.198 We have, $$\frac{p_{Cr}}{RT_{Cr}} = \frac{a/27b^2}{8a/27b} = \frac{1}{8b}$$

Thus, $$b = R\frac{T_{Cr}}{8p_{Cr}} = \frac{0.082 \times 304}{73 \times 8} = 0.043 \text{ l/mol}$$

and $$\frac{(RT_{Cr})^2}{p_{Cr}} = \frac{64a}{27} \quad \text{or} \quad a = \frac{27}{64}\frac{(RT_{Cr})^2}{p_{Cr}} = 3.59 \text{ atm l}^2\text{/mol}$$

2.199 Specific volume is molar volume divided by molecular weight. Thus

$$V'_{Cr} = \frac{V_{MCr}}{M} = \frac{3RT_{Cr}}{8Mp_{Cr}} = \frac{3 \times 0.082 \times 562}{8 \times 78 \times 47}\text{l/g} = 4.71 \text{ cm}^3\text{/g}$$

2.200 We have, $$\left(p + \frac{a}{V_M^2}\right)(V_M - b) = RT$$

or $$\frac{p + a/V_M^2}{p_{Cr}} \times \frac{V_M - b}{V_{MCr}} = \frac{8}{3}\frac{T}{T_{Cr}}$$

or $$\left(\pi + \frac{a}{p_{Cr}V_M^2}\right) \times \left(\nu - \frac{b}{V_{MCr}}\right) = \frac{8}{3}\tau$$

$$\left(\text{where } \pi = \frac{p}{p_{Cr}}, \nu = \frac{V_M}{V_{MCr}}, \tau = \frac{T}{T_{Cr}}\right)$$

or $$\left(\pi + \frac{27b^2}{V_M^2}\right)\left(\nu - \frac{1}{3}\right) = \frac{8}{3}\tau \quad \text{or} \quad \left(\pi + \frac{3}{\nu^2}\right)\left(\nu - \frac{1}{3}\right) = \frac{8}{3}\tau$$

When $\pi = 12$ and $\nu = 1/2$, we get

$$\tau = \frac{3}{8} \times 24 \times \frac{1}{6} = \frac{3}{2}$$

2.201 (a) The critical volume V_{MCr} is the maximum volume in the liquid phase and the minimum volume in the gaseous. Thus,

$$V_{\max} = \frac{1000}{18} \times 3 \times 0.030 \text{ l} = 5 \text{ l}$$

(b) The critical pressure is the maximum possible pressure in the vapour phase in equilibrium with liquid phase. Thus,

$$p_{\max} = \frac{a}{27b^2} = \frac{5.47}{27 \times 0.03 \times 0.03} = 225 \text{ atm}$$

2.202 We have,
$$T_{Cr} = \frac{8}{27}\frac{a}{bR} = \frac{8}{27} \times \frac{3.62}{0.043 \times 0.082} \approx 304 \text{ K}$$

$$\rho_{Cr} = \frac{M}{3b} = \frac{44}{3 \times 43} \text{ g/cm}^3 = 0.34 \text{ g/cm}^3$$

2.203 The vessel is such that either vapour or liquid of mass m occupies it at critical point. Then its volume will be

$$V_{Cr} = \frac{m}{M} V_{MCr} = \frac{3}{8}\frac{RT_{Cr}}{p_{Cr}}\frac{m}{M}$$

The corresponding volume in liquid phase at room temperature is

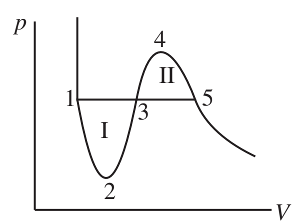
$$V = \frac{m}{\rho}$$

(where ρ = density of liquid ether at room temperature).

Thus,
$$\eta = \frac{V}{V_{Cr}} = \frac{8Mp_{Cr}}{3RT_{Cr}\rho} \approx 0.254$$

(using the given data and ρ = 720 g/l).

2.204 We apply the following relation, at constant T to the cycle 1234531



$$T\oint dS = \oint dU + \oint pdV$$

Here,
$$dS = \oint dU = 0$$

So,
$$\oint pdV = 0$$

This implies that the areas I and II are equal. This reasoning is inapplicable to the cycle 1231, for example. This cycle is irreversible because it involves the irreversible transition from a single phase to a two-phase state at the point 3.

2.205 When a portion of super-cool water turns into ice some heat is liberated, which should heat it upto ice point. Neglecting the variation of specific heat of water, the fraction of water turning into ice is clearly

$$f = \frac{c|t|}{q} = 0.25$$

(where c = specific heat of water and q = latent heat of fusion of ice).

Clearly $f = 1$ at $t = -80°\text{C}$.

2.206 From the Clausius–Clapeyron (C - C) equation

$$\frac{dT}{dp} = \frac{T(V_2' - V_1')}{q_{12}}$$

Here, q_{12} is the specific latent heat absorbed in $1 \rightarrow 2$ and 1 = solid, 2 = liquid.

$$\Delta T = \frac{T(V_2' - V_1')}{q_{12}} \Delta p = -\frac{273 \times 0.091}{333} \times 1 \text{ atm cm}^3 \text{ K/J}$$

$$1 \frac{\text{atm} \times \text{cm}^3}{\text{J}} = \frac{10^5 \text{N/m}^2 \times 10^{-6} \text{m}^3}{\text{J}} = 10^{-1} \text{ Nm/J}, \ \Delta T = 0.0075 \text{ K}$$

2.207 Here 1 = liquid, 2 = steam.

So, $$\Delta T = \frac{T(V_2' - V_1')}{q_{12}} \Delta p$$

or $$V_2' \approx \frac{q_{12}}{T} \frac{\Delta T}{\Delta p} = \frac{2250}{373} \times \frac{0.9}{3.2} \times 10^{-3} \text{ m}^3/\text{g} = 1.7 \text{ m}^3/\text{kg}$$

2.208 From C-C equation

$$\frac{dp}{dT} = \frac{q_{12}}{T(V_2' - V_1')} = \frac{q_{12}}{TV_2'}$$

Assuming the saturated vapour to be ideal gas, we can write

$$\frac{1}{V_2'} = \frac{mp}{RT}$$

Thus, $$\Delta p = \frac{Mq}{RT^2} p_0 \Delta T$$

and $$p = p_0\left(1 + \frac{Mq}{RT^2}\Delta T\right) \approx 1.04 \text{ atm}$$

2.209 From *C-C* equation, neglecting the volume of the liquid

$$\frac{dp}{dT} \approx \frac{q_{12}}{TV'_2} \approx \frac{Mq}{RT^2} p \quad (\text{as } q = q_{12})$$

or $$\frac{dp}{p} = \frac{Mq}{RT}\frac{dT}{T}$$

Now, $$pV = \frac{m}{M} RT \quad \text{or} \quad m = \frac{MpV}{RT} \quad (\text{for an ideal gas})$$

So, $$\frac{dm}{m} = \frac{dp}{p} - \frac{dT}{T} \; (V \text{ is constant } = \text{ specific volume})$$

$$= \left(\frac{Mq}{RT} - 1\right)\frac{dT}{T} = \left(\frac{18 \times 2250}{8.3 \times 373} - 1\right) \times \frac{1.5}{375} \approx 4.85\%$$

2.210 From *C-C* equation

$$\frac{dp}{dT} \approx \frac{q}{TV_2'} \approx \frac{Mq}{RT^2} p$$

Integrating $$\ln p = \text{constant} - \frac{Mq}{RT}$$

So, $$p = p_0 \exp\left[\frac{Mq}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right]$$

This is reasonable for $|T - T_0| << T_0$, and far below critical temperature.

2.211 As before from solution of Problem 2.206, the lowering of melting point is given by

$$\Delta T = -\frac{C\Delta V'}{q} p$$

The superheated ice will then melt in part. The fraction that will melt is

$$\eta = \frac{CT\Delta V'}{q^2} p = 0.03$$

where C is the specific heat capacity of ice, $T = 273$ K and q is the specific heat of melting.

2.212 (a) The equations of the transition lines are

$$\log p = 9.05 - \frac{1800}{T} \quad \text{(solid-gas)}$$

$$= 6.78 - \frac{1310}{T} \quad \text{(liquid-gas)}$$

At the triple point they intersect. Thus,

$$2.27 = \frac{490}{T_{tr}} \text{ or } T_{tr} = \frac{490}{2.27} = 216 \text{ K}$$

The corresponding p_{tr} is 5.14 atm.

In the formula $\log p = a - b/T$, we compare b with the corresponding term in the equation in Problem 2.210. Then

$$\ln p = a \times 2.303 - \frac{2.303\ b}{T} \Rightarrow 2.303 = \frac{Mq}{R}$$

or $$q_{\text{solid-gas}} = \frac{2.303 \times 1800 \times 8.31}{44} = 783 \text{ J/g}$$

$$q_{\text{liquid-gas}} = \frac{2.303 \times 1310 \times 8.31}{44} = 570 \text{ J/g}$$

Finally $$q_{\text{solid-liquid}} = 213 \text{ J/g (on subtraction)}$$

2.213 $$\Delta S = \int_{T_1}^{T_2} mc\frac{dT}{T} + \frac{mq}{T_2} \approx m\left(c \ln \frac{T_2}{T_1} + \frac{q}{T_2}\right)$$

$$= 10^3\left(4.18 \ln \frac{373}{283} + \frac{2250}{373}\right) \approx 7.2 \text{ kJ/K}$$

2.214 $$\Delta S \approx \frac{q_m}{T_1} + c \ln \frac{T_2}{T} + \frac{q_v}{T_2}$$

$$= \frac{333}{273} + 4.18 \ln \frac{373}{283} \approx 8.56 \text{ J/K}$$

2.215 Specific heat of copper, c = 0.39 J/gK. Suppose all the ice does not melt, then

heat rejected $= 90 \times 0.39\ (90 - 0) = 3159$ J

and heat gained by ice $= 50 \times 2.09 \times 3 + x \times 333$

This gives, $$x = 8.5 \text{ g}$$

The hypothesis is correct and final temperature will be $T = 273$ K.

Hence change in entropy of copper piece

$$\Delta S = mc \ln \frac{T}{T_1} = mc \ln \frac{273}{363} = -10 \text{ J/K}$$

2.216 (a) Here $t_2 = 60°\text{C}$. Suppose the final temperature is $t°\text{C}$. Then,

heat lost by water $= m_2\, c\,(t_2 - t)$

heat gained by ice $= m_1 q_m + m_1\, c\,(t - t_1)$ (if all the ice melts)

In this case, $m_1 q_m = m_2 \times 4.18\,(60 - t)$ (for $m_1 = m_2$)

So the final temperature will be 0°C and only some ice will melt.

Then $100 \times 4.18\,(60) = m_1' \times 333$

or $m_1' = 75.3\ g =$ amount of ice that will melt

Finally $\Delta S = 75.3 \times \dfrac{333}{273} + 100 \times 4.18 \ln \dfrac{273}{333}$

$$\Delta S = \frac{m'_1 q_m}{T_1} + m_2 c \ln \frac{T_1}{T_2}$$

$$= m_2 c \frac{(T_2 - T_1)}{T_1} - m_2 c \ln \frac{T_2}{T_1}$$

$$= m_2 c \left[\frac{T_2}{T_1} - 1 - \ln \frac{T_2}{T_1}\right] = 8.8 \text{ J/K}$$

(b) If $m_2 c t_2 > m_1 q_m$, then all the ice will melt as one can check and the final temperature can be obtained like this

$$m_2 c\,(T_2 - T) = m_1 q_m + m_1 c\,(T - T_1)$$

$$(m_2 T_2 + m_1 T_1)\, c - m_1 q_m = (m_1 + m_2)\, cT$$

or $$T = \frac{m_2 T_2 + m_1 T_1 - \dfrac{m_1 q_m}{c}}{m_1 + m_2} \approx 280 \text{ K}$$

and $$\Delta S = \frac{m_1 q}{T_1} + c\left(m_1 \ln \frac{T}{T_1} - m_2 \ln \frac{T_2}{T}\right) = 19 \text{ J/K}$$

2.217 $$\Delta S = -\frac{m q_2}{T_2} - mc \ln \frac{T_2}{T_1} + \frac{M q_{\text{ice}}}{T_1}$$

where, $mq_{\text{ice}} = m\,(q_2 + c\,(T_2 - T_1))$

So, $$\Delta S = mq_2\left(\frac{1}{T_1} - \frac{1}{T_2}\right) + mc\left(\frac{T_2}{T_1} - 1 - \ln\frac{T_2}{T_1}\right)$$

$$= 0.2245 + 0.2564 = 0.48 \text{ J/K}$$

2.218 When heat dQ is given to the vapour its temperature will change by dT, pressure by dp and volume by dV, it being assumed that the vapour remains saturated.

Then by C-C equation,

$$\frac{dp}{dT} = \frac{q}{TV'}\,(V'_{\text{vap}} >> V'_l) \quad \text{or} \quad dp = \frac{q}{TV'}\,dT$$

On the other hand, $$pV' = \frac{RT}{M}$$

So, $$pdV' + V'dp = \frac{RdT}{M}$$

Hence, $$pdV' = \left(\frac{R}{M} - \frac{q}{T}\right)dT$$

Finally $$dQ = CdT = dU + pdV'$$

$$= C_v dT + \left(\frac{T}{M} - \frac{q}{T}\right)dT = C_p dT - \frac{q}{T}\,dT$$

(C_p, C_V refer to unit mass here).

Thus, $$C = C_p - \frac{q}{T}$$

For water, $$C_p = \frac{R\gamma}{\gamma - 1}\cdot\frac{1}{M} \quad (\text{with } \gamma = 1.32 \text{ and } M = 18)$$

So, $C_p = 1.90$ J/g K

and $C = -4.13$ J/g K $= -74$ J/mol K

2.219 The required entropy change can be calculated along a process in which the water is heated from T_1 to T_2 and then allowed to evaporate. The entropy change for this is

$$\Delta S = C_p \ln\frac{T_2}{T_1} + \frac{qM}{T_2}$$

where $q =$ specific latent heat of vaporization.

2.7 Transport Phenomena

2.220 (a) The fraction of gas molecules which traverses distances exceeding the mean free path without collision is just the probability to traverse the distance $s = \lambda$ without collision.

Thus,
$$P = e^{-1} = \frac{1}{e} = 0.37$$

Hence the sought fraction $\eta = 0.37$.

(b) This probability is

$$P = e^{-1} - e^{-2} = 0.23$$

Hence the sought fraction $\eta = 0.23$.

2.221 From the formula $\dfrac{1}{\eta} = e^{\Delta l/\lambda}$ or $\lambda = \dfrac{\Delta l}{\ln \eta}$.

2.222 (a) Let $P(t)$ = probability of no collision in the interval $(0, t)$. Then

$$P(t + dt) = P(t)(1 - \alpha dt)$$

or
$$\frac{dP}{dt} = -\alpha P(t) \quad \text{or} \quad P(t) = e^{-\alpha t}$$

where we have used $P(0) = 1$.

(b) The mean interval between collision is also the mean interval of no collision. Then,

$$<t> = \frac{\int_0^\infty te^{-\alpha t}dt}{\int_0^\infty e^{-\alpha t}dt} = \frac{1}{\alpha}\frac{\Gamma(2)}{\Gamma(1)} = \frac{1}{\alpha}$$

2.223 (a) $\lambda = \dfrac{1}{\sqrt{2}\pi d^2 n} = \dfrac{kT}{\sqrt{2}\pi d^2 p}$

$$= \frac{1.38 \times 10^{-23} \times 273}{\sqrt{2}\pi\,(0.37 \times 10^{-9})^2 \times 10^5} = 6.2 \times 10^{-8}\,\text{m}$$

$$\tau = \frac{\lambda}{<v>} = \frac{6.2 \times 10^{-8}}{454} s = 0.136 \text{ ns}$$

(b) $$\lambda = 6.2 \times 10^{6} \text{ m}$$

$$\tau = 1.36 \times 10^{4} \text{s} = 3.8 \text{ h}$$

2.224 The mean distance between molecules is of the order

$$\left(\frac{22.4 \times 10^{-3}}{6.0 \times 10^{23}}\right)^{1/3} = \left(\frac{224}{6}\right)^{1/3} \times 10^{-9} \text{ m} \approx 3.34 \times 10^{-9} \text{ m}$$

This is about 18.5 times smaller than the mean free path calculated in Problem 2.223(a) above.

2.225 We know that the Van der Waal's constant b is four times the molecular volume. Thus,

$$b = 4N_A \frac{\pi}{6} d^3 \quad \text{or} \quad d = \left(\frac{3d}{2\pi\ N_A}\right)^{1/3}$$

Hence, $$\lambda = \left(\frac{kT_0}{\sqrt{2}\pi P_0}\right)\left(\frac{2\pi N_A}{3b}\right)^{2/3} = 84 \text{ nm (on substituting values)}$$

2.226 The velocity of sound in N_2 is

$$\sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma RT}{M}}$$

So, $$\frac{1}{\nu} = \sqrt{\frac{\gamma RT_0}{M}} = \frac{RT_0}{\sqrt{2}\pi\ d^2 p_0 N_A}$$

$$= 5.5 \text{ GHz (on substituting values)}$$

2.227 (a) $\lambda > l$ if $p < \dfrac{kT}{\sqrt{2}\pi\ d^2 l}$

≈ 0.7 Pa for O_2

(b) The corresponding molecular concentration n is obtained by dividing by kT and is 1.84×10^{20} per m^3 = 1.84×10^{14} per cm^3 and the corresponding mean distance is given by

$$\frac{l}{n^{1/3}} = \frac{10^{-2}}{(0.184)^{1/3} \times 10^{5}} = 1.8 \times 10^{-7} \text{ m} \approx 0.18\ \mu\text{m}$$

2.228 (a)
$$v = \frac{1}{\tau} = \frac{1}{\lambda/<v>} = \frac{<v>}{\lambda}$$
$$= \sqrt{2}\pi d^2\, n <v> = 0.74 \times 10^{10}\ \text{s}^{-1}$$

where $$n = \frac{p_0}{kT_0} \text{ and } <v> = \sqrt{\frac{8\ RT}{\pi M}}$$

(b) Total number of collisions is
$$\frac{1}{2}\, nv = 1.0 \times 10^{29}\ \text{s}^{-1}\ \text{cm}^{-3}$$

Note, the factor 1/2, which come because when two molecules collide we must not count it twice.

2.229 (a)
$$\lambda = \frac{1}{\sqrt{2}\pi d^2 n}$$

d is a constant and n is a constant for an isochoric process, so λ is constant for an isochoric process.
$$v = \frac{<v>}{\lambda} = \sqrt{\frac{8RT}{M\pi}}\Big/\lambda$$

i.e., $$v \propto \sqrt{T}$$

(b)
$$\lambda = \frac{1}{\sqrt{2}\pi d^2}\frac{kT}{P} \propto T \quad \text{(for an isobaric process)}$$
$$v = \frac{<v>}{\lambda} \propto \frac{\sqrt{T}}{T} = \frac{1}{\sqrt{T}} \quad \text{(for an isobaric process)}$$

2.230 (a) In an isochoric process λ is constant and
$$v \propto \sqrt{T} \propto \sqrt{pV} \propto \sqrt{p} \propto \sqrt{n}$$

so v increases n times

(b) $\lambda = \dfrac{kT}{\sqrt{2}\pi d^2 p}$ must decrease n times in an isothermal process and v must increase n times because $<v>$ is constant in an isothermal process.

2.231 (a)
$$\lambda \propto \frac{1}{n} \geq \frac{1}{N/V} = \frac{V}{N}$$

Thus, $$\lambda \propto V \text{ and } v \propto \frac{T^{1/2}}{V}$$

But in an adiabatic process $TV^{\gamma-1}$ = constant, so $TV^{2/5}$ = constant (as $\gamma = 7/5$ here).

or $$T^{1/2} \propto V^{-1/5}$$

Thus, $$v \propto V^{-6/5}$$

(b) Given

$$\lambda \propto \frac{T}{p}$$

But,

$$p\left(\frac{T}{p}\right)^{\gamma} = \text{constant}$$

or

$$\frac{T}{p} \propto p^{-1/\gamma} \quad \text{or} \quad T \propto p^{1-1/\gamma}$$

Thus,

$$\lambda \propto p^{-1/\gamma} = p^{-5/7}$$

When,

$$v = \frac{<v>}{\lambda} \propto \frac{p}{\sqrt{T}} \propto p^{1/2+1/2\gamma} = p^{\gamma+1/2\gamma} = p^{6/7}$$

(c) Given $\lambda \propto V$

But, $TV^{2/5} = \text{constant}$ or $V \propto T^{-5/2}$

Thus,

$$\lambda \propto T^{-5/2}$$

$$v \propto \frac{T^{1/2}}{V} \propto T^3$$

2.232 In the polytropic process of index n, $pV^n = \text{constant}$, $TV^{n-1} = \text{constant}$ and $p^{1-n}\, T^n = \text{constant}$.

(a) When $\lambda \propto V$,

$$v \propto \frac{T^{1/2}}{V} = V^{1-n/2}\ V^{-1} = V^{-n+1/2}$$

(b) When $\lambda \propto \frac{T}{p}$,

$$T^n \propto p^{n-1} \text{ or } T \propto p^{1-1/n}$$

So,

$$\lambda \propto p^{-1/n}$$

$$v = \frac{<v>}{\lambda} \propto \frac{p}{\sqrt{T}} \propto p^{1-1/2+1/2n} = p^{(n+1)/2n}$$

(c) When $\lambda \propto \frac{T}{p}$,

$$p \propto T^{n/n-1}$$

$$\lambda \propto T^{1-n/n-1} = T^{-1/n-1} = T^{1/1-n}$$

$$v \propto \frac{p}{\sqrt{T}} \propto T^{(n/n-1)-(1/2)} = T^{(n+1)/2(n-1)}$$

2.233 (a) The number of collision between the molecules in a unit volume is

$$\frac{1}{2}n\nu = \frac{1}{\sqrt{2}}\pi d^2\ n^2 <v> \propto \frac{\sqrt{T}}{V^2}$$

This remains constant is polytropic process , pV^{-3} = constant.

Using solution of Problem 2.122, the molar specific heat for the polytropic process, pV^{α}= constant, is

$$C = R\left(\frac{1}{\gamma - 1} - \frac{1}{\alpha - 1}\right)$$

Thus, $$C = R\left(\frac{1}{\gamma - 1} + \frac{1}{4}\right) = R\left(\frac{5}{2} + \frac{1}{4}\right) = \frac{11}{4}R$$

It can also be written as $\frac{1}{4}R(1 + 2i)$ (where $i = 5$)

On substituting values $C = 23$ J/K mol.

(b) In this case $\frac{\sqrt{T}}{V}$ = constant and so pV^{-1} = constant.

So, $$C = R\left(\frac{1}{\gamma - 1} + \frac{1}{2}\right) = R\left(\frac{5}{2} + \frac{1}{2}\right) = 3R$$

which can also be written as $\frac{R}{2}(i + 1)$ (where $i = 5$).

On substituting values $C = 29$ J/K mol.

2.234 We can assume that all molecules incident on the hole, leak out. Then,

$$-dN = -d(nV) = \frac{1}{4}n <v> S\,dt$$

or $$dn = -n\frac{dt}{4v/S<v>} = -n\frac{dt}{\tau}$$

Integrating, $$n = n_0\, e^{-t/\tau}$$

Hence, $$<v> = \sqrt{\frac{8RT}{\pi M}}$$

2.235 If the temperature of the compartment 2 is η times more than that of compartment 1, it must contain $1/\eta$ times less number of molecules since pressure must be the same when the big hole is open. If M = mass of the gas in 1 than the mass of the gas is 2 must be M/η. So immediately after the big hole is closed,

$$n_1^0 = \frac{M}{mV} \quad \text{and} \quad n_2^0 = \frac{M}{mV\eta}$$

where m = mass of each molecule and n_1^0, n_2^0 are concentrations in 1 and 2. After the big hole is closed the pressures will differ and concentration will become n_1 and n_2, where

$$n_1 + n_2 = \frac{M}{mV\eta}(1 + \eta)$$

On the other hand

$$n_1 <v_1> = n_2<v_2>, \quad \text{i.e.,} \quad n_1 = \sqrt{\eta}\, n_2$$

Thus, $$n_2\left(1 + \sqrt{\eta}\right) = \frac{m}{mV\eta}(1 + \eta) = n_2^0(1 + \eta)$$

So, $$n_2 = n_2^0 \frac{1 + \eta}{1 + \sqrt{\eta}}$$

2.236 We know

$$\eta = \frac{1}{3} <v> \lambda\rho = \frac{1}{3} <v> \frac{1}{\sqrt{2}\pi d^2}\, m\, \alpha \sqrt{T}$$

Thus η changing α times implies T changing α^2 times.

On the other hand

$$D = \frac{1}{3} <v> \lambda = \frac{1}{3}\sqrt{\frac{8kT}{\pi m}}\, \frac{kT}{\sqrt{2}\pi d^2 p}$$

Thus D changing β times means $\dfrac{T^{3/2}}{p}$ changing β times.

So p must change $\dfrac{\alpha^3}{\beta}$ times $= 2$ times (on substituting values).

2.237 $D \propto \dfrac{\sqrt{T}}{n} \propto V\sqrt{T}$ and $\eta \propto \sqrt{T}$

(a) D will increase n times, and η will remain constant if T is constant.

(b) $D \propto \dfrac{T^{3/2}}{p} \propto \dfrac{(pV)^{3/2}}{p} = p^{1/2}\, V^{3/2}$

Also, $$\eta \propto \sqrt{pV}$$

Thus D will increase $n^{3/2}$ times, η will increase $n^{1/2}$ times, if p is constant.

2.238 $D \propto V\sqrt{T}$ and $\eta \propto \sqrt{T}$

In an adiabatic process

$$TV^{\gamma-1} = \text{constant} \quad \text{or} \quad T \propto V^{1-\gamma}$$

Now V is decreased $1/n$ times.

Thus,
$$D \propto V^{3-\gamma/2} = \left(\frac{1}{n}\right)^{3-\gamma/2} = \left(\frac{1}{n}\right)^{4/5}$$

$$\eta \propto V^{1\gamma/2} = \left(\frac{1}{n}\right)^{-1/5} = n^{1/5}$$

So D decreases $n^{4/5}$ times and η increase $n^{1/5}$ times ≈ 6.3 and 1.6 times, respectively.

2.239 (a) $D \propto V\sqrt{T} \propto \sqrt{pV^3}$

Thus D remains constant in the process $pV^3 = \text{constant}$. So polytropic index of the process $n = 3$.

(b) $\eta \propto \sqrt{T} \propto \sqrt{pV}$

So η remains constant in the isothermal process. $pV = \text{constant}$, $n = 1$, here.

(c) Heat conductivity $\kappa = \eta C_V$ and C_V is a constant for the ideal gas.

Thus $n = 1$ here also.

2.240
$$\eta = \frac{1}{3}\sqrt{\frac{8kT}{\pi m}}\,\frac{m}{\sqrt{2}\pi d^2} = \frac{2}{3}\sqrt{\frac{mkT}{\pi^3}}\,\frac{1}{d^2}$$

or
$$d = \left(\frac{2}{3\,\eta}\right)^{1/2}\left(\frac{mkT}{\pi^3}\right)^{1/4} = \left(\frac{2}{3 \times 18.9} \times 10^6\right)^{1/2}\left(\frac{4 \times 8.31 \times 273 \times 10^{-3}}{\pi^3 \times 36 \times 10^{46}}\right)^{1/4}$$

$$= 10^{-10}\left(\frac{2}{3 \times 18.9}\right)^{1/2}\left(\frac{4 \times 83.1 \times 273}{\pi^3 \times 0.36}\right)^{1/4} \approx 0.178 \text{ nm}$$

2.241
$$\kappa = \frac{1}{3}\langle v \rangle \lambda \rho c_V$$

$$= \frac{1}{3}\sqrt{\frac{8kT}{m\pi}}\,\frac{1}{\sqrt{2}\pi d^2 n}\,mn\,\frac{C_V}{M}$$

c_V is the specific heat capacity which is C_V/M. Now C_V is the same for all monoatomic gases such as He and A.

Thus,
$$\kappa \propto \frac{1}{\sqrt{M}d^2}$$

or
$$\frac{\kappa_{\text{He}}}{\kappa_A} = 8.7 = \frac{\sqrt{M_A}\,d_A^2}{\sqrt{M_{\text{He}}}\,d_{\text{He}}^2} = \sqrt{10}\,\frac{d_A^2}{d_{\text{He}}^2}$$

$$\frac{d_A}{d_{\text{He}}} = \sqrt{\frac{8.7}{\sqrt{10}}} = 1.658 \approx 1.7$$

2.242 In this case

$$N_1 \frac{r_2^2 - r_1^2}{r_1^2 r_2^2} = 4\pi\eta\omega$$

or
$$N_1 \frac{2R\Delta R}{R^4} \approx 4\pi\eta\omega \quad \text{or} \quad N_1 = \frac{2\pi\eta\omega R^3}{\Delta R}$$

To decrease N_1 n times, η must be decreased n times. Now η does not depend on pressure until the pressure is so low that the mean free path equals, say, $1/2\ \Delta R$. Then the mean free path is fixed and η decreases with pressure. The mean free path equals $1/2\ \Delta R$, when

$$\frac{1}{\sqrt{2}\pi d^2 n_0} = \Delta R \quad (\text{where } n_0 = \text{concentration})$$

Corresponding pressure is
$$p_0 = \frac{\sqrt{2}kT}{\pi d^2 \Delta R}$$

The sought pressure is n times less, given by

$$p = \frac{\sqrt{2}kT}{\pi d^2 n \Delta R} = 70.7 \times \frac{10^{-23}}{10^{-18} \times 10^{-3}} \approx 0.71 \text{ Pa}$$

The answer is qualitative and depends on the choice $1/2\ \Delta R$ for the mean free path.

2.243 We neglect the moment of inertia of the gas in a shell. Then the moment of friction forces on a unit length of the cylinder must be a constant as a function of r.

So,
$$2\pi r^3 \eta \frac{d\omega}{dr} = N_1 \quad \text{or} \quad \omega(r) = \frac{N_1}{4\pi\eta}\left(\frac{1}{r_1^2} - \frac{1}{r^2}\right)$$

and
$$\omega = \frac{N_1}{4\pi\eta}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) \quad \text{or} \quad \eta = \frac{N_1}{4\pi\omega}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$

2.244 We consider two adjoining layers. The angular velocity gradient is ω/h. So the moment of the frictional force is

$$N = \int_0^a r 2\pi r dr \eta r \frac{\omega}{h} = \frac{\pi \eta a^4 \omega}{2h}$$

2.245 In the ultra-rarefied gas we must determine η by taking $\lambda = (1/2)h$.

Then, $$\eta = \frac{1}{3}\sqrt{\frac{8kT}{m\pi}} \times \frac{1}{2}h \times \frac{mp}{kT} = \frac{1}{3}\sqrt{\frac{2M}{\pi RT}}hp$$

So, $$N = \frac{1}{3}\omega a^4 p \sqrt{\frac{\pi M}{2RT}}$$

2.246 Take an infinitesimal section of length dx and apply Poiseuille's equation to this.

Then, $$\frac{dV}{dt} = \frac{-\pi a^4}{8\eta}\frac{\partial p}{\partial x}$$

From the formula, $$pV = RT \cdot \frac{m}{M}$$

$$pdV = \frac{RT}{M}dm$$

or $$\frac{dm}{dt} = \mu = -\frac{\pi a^4 M}{8\eta RT}\frac{pdp}{dx}$$

This equation implies that if the flow is isothermal, then

$$p\frac{dp}{dx} = \frac{|p_2^2 - p_1^2|}{2l} = \text{constant}$$

Thus, $$\mu = \frac{\pi a^4 M}{16\eta RT}\frac{|p_2^2 - p_1^2|}{l}$$

2.247 Let T be the temperature of the interface. Then heat flowing from left is equal to the heat flowing into right in equilibrium.

Thus, $$\kappa_1 = \frac{T_1 - T}{l_1} = \kappa_2 = \frac{T - T_2}{l_2} \quad \text{or} \quad T = \frac{(\kappa_1 T_1)/l_1 + (\kappa_2 T_2)/l_2}{\kappa_1/l_1 + \kappa_2/l_2}$$

2.248 We have

$$\kappa_1\frac{T_1 - T}{l_1} = \kappa_2\frac{T - T_2}{l_2} = \kappa\frac{T_1 - T_2}{l_1 - l_2}$$

or using the previous result

$$\frac{\kappa_1}{l_1}\left(T_1 - \frac{\dfrac{\kappa_1 T_1}{l_1} + \dfrac{\kappa_2 T_2}{l_2}}{\dfrac{\kappa_1}{l_1} + \dfrac{\kappa_2}{l_2}}\right) = \kappa \frac{T_1 - T_2}{l_1 + l_2}$$

or
$$\frac{\kappa_1}{l_1}\frac{\dfrac{\kappa_2}{l_2}(T_1 - T_2)}{\dfrac{\kappa_1}{l_1} + \dfrac{\kappa_2}{l_2}} = \kappa \frac{T_1 - T_2}{l_1 + l_2} \Rightarrow \kappa = \frac{l_1 + l_2}{\dfrac{l_1}{\kappa_1} + \dfrac{l_2}{\kappa_2}}$$

2.249 By definition the heat flux per unit area is

$$\dot{Q} = -K\frac{dT}{dx} = -\alpha\frac{d}{dx}\ln T = \text{constant} = +\frac{\alpha}{l}\ln T_1/T_2$$

Integrating $\quad \ln T = \dfrac{x}{l}\ln\dfrac{T_2}{T_1} + \ln T_1$

(where T_1 = temperature at the end $x = 0$)

So, $$T = T_1\left(\frac{T_2}{T_1}\right)^{x/l} \quad \text{and} \quad \dot{Q} = \frac{\alpha}{l}\ln T_1/T_2$$

2.250 Suppose the chunks have temperatures T_1, T_2 at time t and $T_1 - dT_1$, $T_2 + dT_2$ at time $dt + t$.

Then, $\quad C_1 dT_1 = C_2 dT_2 = \dfrac{\kappa S}{l}(T_1 - T_2)\, dt$

Thus, $\quad d\Delta T = -\dfrac{\kappa S}{l}\left(\dfrac{1}{C_1} + \dfrac{1}{C_2}\right)\Delta T\, dt$ (where $\Delta T = T_1 - T_2$)

Hence, $\quad \Delta T = (\Delta T)_0 e^{-t/\tau}\left(\text{where } \dfrac{1}{\tau} = \dfrac{\kappa \cdot S}{l}\left(\dfrac{1}{C_1} + \dfrac{1}{C_2}\right)\right)$

2.251
$$\dot{Q} = \kappa\frac{\partial T}{\partial x} = -A\sqrt{T}\frac{\partial T}{\partial x} \quad (\text{where } A = \text{constant})$$

$$= -\frac{2}{3}A\frac{\partial T^{3/2}}{\partial x}$$

$$= \frac{2}{3}A\frac{(T_1^{3/2} - T_2^{3/2})}{l}$$

Thus, $\quad T^{3/2} = \text{constant} - \dfrac{x}{l}(T_1^{3/2} - T_2^{3/2})$

Using $T = T_1$, at $x = 0$, we get

$$T^{3/2} = T_1^{3/2} + \frac{x}{l}(T_2^{3/2} - T_1^{3/2}) \quad \text{or} \quad \left(\frac{T}{T_1}\right)^{3/2} = 1 + \frac{x}{l}\left(\left(\frac{T_2}{T}\right)^{3/2} - 1\right)$$

$$T = T_1\left[1 + \frac{x}{l}\left\{\left(\frac{T_2}{T_1}\right)^{3/2} - 1\right\}\right]^{2/3}$$

where x is the distance from the plate maintained at the temperature T_1.

2.252 $\kappa = \frac{1}{3}\sqrt{\frac{8RT}{\pi M}}\frac{1}{\sqrt{2}\pi d^2 n}\, mn\,\frac{R(i/2)}{M} = \frac{R^{3/2} i T^{3/2}}{3\pi^{3/2} d^2\sqrt{MN_A}}$

Then from the previous problem

$$q = \frac{2iR^{3/2}(T_2^{3/2} - T_1^{3/2})}{9\pi^{3/2} d^2\sqrt{MN_A}\, l},$$

$i = 3$ here and d is the effective diameter of helium atom.

2.253 At this pressure and average temperature (= 27°C = 300 K)

$$T = \frac{(T_1 + T_2)}{2}$$

$$\lambda = \frac{1}{\sqrt{2}\pi d^2}\frac{\kappa T}{p} = 2330 \times 10^{-5}\text{ m} = 23.3\text{ mm} \gg 5.0\text{ mm} = l$$

The gas is ultra-thin and we write $\lambda = (1/2)\, l$ here.

Then, $\qquad q = \kappa\frac{dT}{dx} = \kappa\frac{T_2 - T_1}{l}$

where, $\qquad \kappa = \frac{1}{3}\langle v\rangle \times \frac{1}{2} l \times \frac{MP}{RT} \times \frac{R}{\gamma - 1} \times \frac{1}{M} = \frac{p\langle v\rangle}{6T(\gamma - 1)} l$

and $\qquad q = \frac{p\langle v\rangle}{6T(\gamma - 1)}(T_2 - T_1) = 22\text{ W/m}^2$ (on substituting values)

where, $\langle v\rangle = \sqrt{\frac{8RT}{M\pi}}$.

We have used $T_2 - T_1 \ll \frac{T_2 + T_1}{2}$ here.

2.254 At equilibrium, $2\pi r\kappa\frac{dT}{dr} = -A = $ constant.

So, $\qquad T = B - \frac{A}{2\pi\kappa}\ln r$

But $T = T_1$, when $r = R_1$ and $T = T_2$, when r $= R_2$.

From this we find $T = T_1 + \dfrac{T_2 - T_1}{\ln R_2/R_1} \ln \dfrac{r}{R_1}$

2.255 At equilibrium $4\pi r^2 \kappa \dfrac{dT}{dr} = -A = \text{constant}$

$$T = B + \frac{A}{4\pi\kappa}\frac{1}{r}$$

Using $T = T_1$ when $r = R_1$ and $T = T_2$ when $r = R_2$,

$$T = T_1 + \frac{T_2 - T_1}{(1/R_2) - (1/R_1)}\left(\frac{1}{r} - \frac{1}{R_1}\right)$$

2.256 The heat flux vector is $-\kappa \nabla T$ and its divergence equals w. Thus,

$$\nabla^2 T = -\frac{w}{\kappa}$$

or $\qquad \dfrac{1}{r}\left(\dfrac{\partial}{\partial r}\right)\left(r\dfrac{\partial T}{\partial r}\right) = -\dfrac{w}{\kappa}$ in cylindrical coordinates.

or $$T = B + A \ln r - \frac{w}{2\kappa} r^2$$

Since T is finite at $r = 0$, $A = 0$. Also $T = T_0$ at $r = R$

So, $$B = T_0 + \frac{w}{4\kappa} R^2$$

Thus, $$T = T_0 + \frac{w}{4\kappa}(R^2 - r^2)$$

where r is the distance from the axis of the wire also called axial radius.

2.257 Here again

$$\nabla^2 T = -\frac{w}{\kappa}$$

So in spherical polar coordinates,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = -\frac{w}{\kappa} \quad \text{or} \quad r^2\frac{\partial T}{\partial r} = -\frac{w}{\kappa} \quad \text{or} \quad r^2\frac{\partial T}{\partial r} = -\frac{w}{3\kappa} r^3 + A$$

or $$T = B - \frac{A}{r} - \frac{w}{6\kappa} r^2$$

Again $$A = 0 \quad \text{and} \quad B = T_0 + \frac{w}{6\kappa} R^2$$

So finally, $$T = T_0 + \frac{w}{6\kappa} \times (R^2 - r^2)$$

ELECTRODYNAMICS

PART 3

3.1 Constant Electric Field in Vacuum

3.1 $F_{ele}(\text{for electrons}) = \dfrac{q^2}{4\pi\varepsilon_0 r^2}$ and $F_{gr} = \dfrac{\gamma m^2}{r^2}$

Thus, $$\frac{F_{ele}}{F_{gr}}(\text{for electrons}) = \frac{q^2}{4\pi\varepsilon_0 \gamma m^2}$$

$$= \frac{(1.602 \times 10^{-19}\ \text{C})^2}{\left(\dfrac{1}{9 \times 10^9}\right) \times 6.67 \times 10^{-11}\text{m}^3/(\text{kg}\cdot\text{s}^2) \times (9.11 \times 10^{-31}\ \text{kg})^2} = 4 \times 10^{42}$$

Similarly $$\frac{F_{ele}}{F_{gr}}(\text{for proton}) = \frac{q^2}{4\pi\varepsilon_0 \gamma m^2}$$

$$= \frac{(1.602 \times 10^{-19}\ \text{C})^2}{\left(\dfrac{1}{9 \times 10^9}\right) \times 6.67 \times 10^{-11}\ \text{m}^3/\ (\text{kg}\cdot\text{s}^2) \times (1.672 \times 10^{-27}\ \text{kg})^2} = 1 \times 10^{36}$$

For $F_{ele} = F_{gr}$,

$$\frac{q^2}{4\pi\varepsilon_0 r^2} = \frac{\gamma m^2}{r^2} \quad \text{or} \quad \frac{q}{m} = \sqrt{4\pi\varepsilon_0 \gamma}$$

$$= \sqrt{\frac{6.67 \times 10^{-11}\text{m}^3\ (\text{kg}\cdot\text{s}^2)}{9 \times 10^9}} = 0.86 \times 10^{-10}\text{C/kg}$$

3.2 Total number of atoms in the sphere of mass 1 gram $= \dfrac{1}{63.54} \times 6.023 \times 10^{23}$.

So the total nuclear charge $\lambda = \dfrac{6.023 \times 10^{23}}{63.54} \times 1.6 \times 10^{-19} \times 29.$

Now the charge on the sphere = total nuclear charge − total electronic charge

$$= \frac{6.023 \times 10^{23}}{63.54} \times 1.6 \times 10^{-19} \times \frac{29 \times 1}{100} = 4.298 \times 10^{2}\,\text{C}$$

Hence, force of interaction between these two spheres,

$$F = \frac{1}{4\pi\varepsilon_0} \cdot \frac{[4.398 \times 10^{2}]^{2}}{1^2}\,\text{N} = 9 \times 10^{9} \times 10^{4} \times 19.348\,\text{N} = 1.74 \times 10^{15}\,\text{N}$$

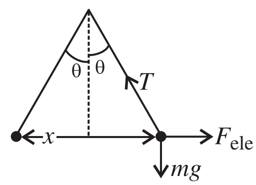
3.3 Let the balls be deviated by an angle θ from the vertical when separation between them equals x.

Applying Newton's second law of motion for any one of the spheres, we get

$$T\cos\theta = mg \qquad (1)$$

and

$$T\sin\theta = F_{\text{ele}} \qquad (2)$$



From the Eqs. (1) and (2)

$$\tan\theta = \frac{F_{\text{ele}}}{mg} \qquad (3)$$

But from the figure

$$\tan\theta = \frac{x}{2\sqrt{l^2 - \left(\frac{x}{2}\right)^2}} \cong \frac{x}{2l} \text{ as } x << l \qquad (4)$$

From Eqs. (3) and (4)

$$F_{\text{ele}} = \frac{mg\,x}{2l} \quad \text{or} \quad \frac{q^2}{4\pi\varepsilon_0 x^2} = \frac{mg\,x}{2l}$$

Thus

$$q^2 = \frac{2\pi\varepsilon_0 mgx^3}{l} \qquad (5)$$

Differentiating Eq. (5) with respect to time

$$2q\frac{dq}{dt} = \frac{2\pi\varepsilon_0 mg}{l}\,3x^2\,\frac{dx}{dt}$$

According to the problem $\frac{dx}{dt} = v = \frac{a}{\sqrt{x}}$ $\left(\text{approach velocity is } \frac{dx}{dt}\right)$

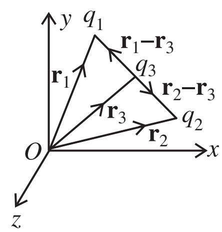
so,

$$\left(\frac{2\pi\varepsilon_0 mg}{l}x^3\right)^{1/2}\frac{dq}{dt} = \frac{3\pi\varepsilon_0 mg}{l}\,x^2\,\frac{a}{\sqrt{x}}$$

Hence, $$\frac{dq}{dt} = \frac{3}{2} a \sqrt{\frac{2\pi\varepsilon_0 mg}{l}}$$

Note: *If the plane of figure is x-y plane and point of suspension is O, Eq. (3) can be obtained more easily using net torque equal to zero about OZ-axis, i.e.,* $\tau_{0Z} = O$.

3.4 Let us choose coordinate axes as shown in the figure and fix three charges, q_1, q_2 and q_3 having position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$, respectively.



Now, for the equilibrium of q_3

$$\frac{q_2 q_3 (\mathbf{r}_2 - \mathbf{r}_3)}{|\mathbf{r}_2 - \mathbf{r}_3|^3} + \frac{q_1 q_3 (\mathbf{r}_1 - \mathbf{r}_3)}{|\mathbf{r}_1 - \mathbf{r}_3|^3} = 0$$

or $$\frac{q_2}{|\mathbf{r}_2 - \mathbf{r}_3|^2} = \frac{q_1}{|\mathbf{r}_1 - \mathbf{r}_3|^2}$$

because $$\frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|} = -\frac{\mathbf{r}_1 - \mathbf{r}_3}{|\mathbf{r}_1 - \mathbf{r}_3|}$$

So, $$\sqrt{q_2}\,(\mathbf{r}_1 - \mathbf{r}_3) = \sqrt{q_1}\,(\mathbf{r}_3 - \mathbf{r}_2)$$

or $$\mathbf{r}_3 = \frac{\sqrt{q_2}\,\mathbf{r}_1 + \sqrt{q_1}\,\mathbf{r}_2}{\sqrt{q_1} + \sqrt{q_2}}$$

Also for the equilibrium of q_1

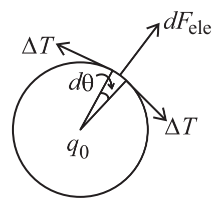
$$\frac{q_3 (\mathbf{r}_3 - \mathbf{r}_1)}{|\mathbf{r}_3 - \mathbf{r}_1|^3} + \frac{q_2 (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} = 0$$

or $$q_3 = \frac{-q_2}{|\mathbf{r}_2 - \mathbf{r}_1|^2}|\mathbf{r}_1 - \mathbf{r}_3|^2$$

Substituting the value of $\mathbf{r}_3$, we get

$$q_3 = \frac{-q_1 q_2}{(\sqrt{q_1} + \sqrt{q_2})^2}$$

3.5 When the charge q_0 is placed at the center of the ring, the wire gets stretched. The component of extra tension ΔT towards the center will balance the electric force dF_{ele} due to the charge q_0 on a differential part of the ring which subtends angle $d\theta$ at the center, having the charge



$$dq = \left(\frac{q}{2\pi}\right) d\theta$$

$$2\Delta T \sin\frac{d\theta}{2} = dF_{\text{ele}}$$

As $d\theta$ is very small

$$2\Delta T \sin\frac{d\theta}{2} \approx T d\theta$$

Using $$dF_{ele} = \frac{q_0 dq}{4\pi\varepsilon_0 r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q_0}{r^2}\left(\frac{q}{2\pi}\right)d\theta$$

$$\Delta T d\theta = \frac{1}{4\pi\varepsilon_0}\frac{q_0}{r^2}\left(\frac{q}{2\pi}\right)d\theta$$

Hence, $$\Delta T = \frac{qq_0}{8\pi^2\varepsilon_0 r^2}$$

3.6 Sought field strength $\mathbf{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{|(\mathbf{r}-\mathbf{r}_0)|^3}\cdot(\mathbf{r}-\mathbf{r}_0) = 2.7\mathbf{i} - 3.6\mathbf{j}$ kV/m

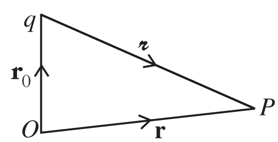
So, $$|\mathbf{E}| = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\mathbf{r}-\mathbf{r}_0|^2}$$

$$= 4.5 \text{ kV/m (on substituting values)}$$

3.7 Electric field strength due to a point charge (q) at a field point (P) in vacuum is given by

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{|(\mathbf{r}-\mathbf{r}_0)|^3}$$



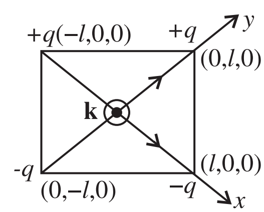
Taking into account the expresion for E, let us fix the coordinate system by taking the point of intersection of the diagonals as the origin and let $\mathbf{k}$ be the unit vector directed normally, emerging from the plane of figure.

Hence the sought field strength:

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{l\mathbf{i} + x\mathbf{k}}{(l^2+x^2)^{3/2}} + \frac{-q}{4\pi\varepsilon_0}\frac{l(-\mathbf{i}) + x\mathbf{k}}{(l^2+x^2)^{3/2}}$$

$$+ \frac{-q}{4\pi\varepsilon_0}\frac{l\mathbf{j} + x\mathbf{k}}{(l^2+x^2)^{3/2}} + \frac{q}{4\pi\varepsilon_0}\frac{l(-\mathbf{j}) + x\mathbf{k}}{(l^2+x^2)^{3/2}}$$

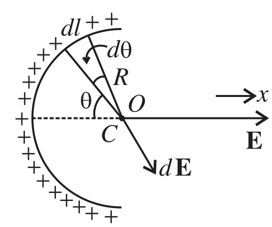
$$= \frac{q}{4\pi\varepsilon_0(l^2+x^2)^{3/2}}[2\,l\mathbf{i} - 2l\mathbf{j}]$$



Thus, $$E = \frac{ql}{\sqrt{2}\pi\varepsilon_0(l^2+x^2)^{3/2}}$$

3.8 From the symmetry of the problem the sought field strength

$$E = \int dE_x$$



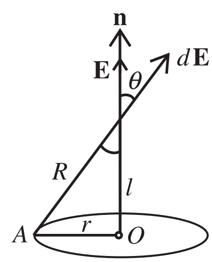
where the projection of field strength along x-axis due to an elemental charge is

$$dE_x = \frac{dq\cos\theta}{4\pi\varepsilon_0 R^2} = \frac{q\,R\cos\theta\, d\theta}{4\pi^2\varepsilon_0 R^3}$$

Hence, $$E = \frac{q}{4\pi^2\varepsilon_0 R^2}\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta = \frac{q}{2\pi^2\varepsilon_0 R^2}$$

$$= 0.10 \text{ kV/m (on substituting values)}$$

3.9 It can be easily shown that vector **E** in this case must be directed along the axis of the ring (see figure). Let us isolate differential linear charge element dq on the ring in the vicinity of point A. We write the expression for the component dE_n of the field created by this element at a field point P as



$$dE_n = \frac{1}{4\pi\varepsilon_0}\frac{dq}{R^2}\cos\theta = \frac{dq}{4\pi\varepsilon_0}\frac{l}{(r^2 + l^2)^{3/2}}$$

(where **n** is the unit vector perpendicular to the plane of ring or along the axis of the ring).

Hence, $$E = \int dE_n = \frac{1}{4\pi\varepsilon_0}\frac{l}{(r^2 + l^2)^{3/2}}\int dq = \frac{q}{4\pi\varepsilon_0}\frac{l}{(r^2 + l^2)^{3/2}}$$

It can be seen that for $l \gg r$, the field $E \approx q/4\pi\varepsilon_0 l^2$, i.e., at large distances the system behaves as a point charge.

From the expression of $E = E(l)$, the electric field is zero at the center of the ring $(l = 0)$ and also zero when l is very large. Hence, there must be a value of l for which the electric field is maximum. This value of l can be obtained by setting the first derivative of E to zero as follows

$$\frac{dE}{dl} = 0 \quad \text{or} \quad \frac{q\,(r^2 - 2l^2)}{4\pi\varepsilon_0\,(r^2 + l^2)^{5/2}} = 0$$

which yields, $$l = \frac{r}{\sqrt{2}}$$

$$E_{\max} = \frac{q}{6\sqrt{3}\pi\varepsilon_0 r^2}$$

3.10 The electric potential at a distance x from the given ring is given by

$$\varphi(x) = \frac{q}{4\pi\varepsilon_0 x} - \frac{q}{4\pi\varepsilon_0\,(R^2 + x^2)^{1/2}}$$

Hence, the field strength along x-axis (which is the net field strength in our case) is

$$E_x = -\frac{d\varphi}{dx} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{x^2} - \frac{qx}{(R^2 + x^2)^{3/2}}\right]$$

$$= \frac{\dfrac{q}{4\pi\epsilon_0}x^3\left[\left(1 + \dfrac{R^2}{x^2}\right)^{3/2} - 1\right]}{x^2(R^2 + x^2)^{3/2}}$$

$$= \frac{\frac{q}{4\pi\varepsilon_0} x^3 \left[1 + \frac{3}{2}\frac{R^2}{x^2} + \frac{3}{8}\frac{R^4}{x^4} + \cdots\right]}{x^2\,(R^2 + x^2)^{3/2}}$$

Neglecting the higher power of R/x, as $x \gg R$

$$E = \frac{3qR^2}{8\pi\varepsilon_0 x^4}$$

Note: *Instead of $\varphi(x)$, we may write $E(x)$ directly using solution of problem 3.9.*

3.11 From the solution of Problem 3.9, the electric field strength due to ring at a point on its axis (say x-axis) at distance x from the center of the ring is given by

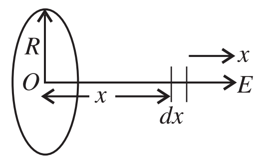
$$E(x) = \frac{qx}{4\pi\varepsilon_0 (R^2 + x^2)^{3/2}}$$

and from symmetry, **E** at every point on the axis is directed along the x-axis (see figure). Let us consider an element dx on thread which carries the charge $\lambda\, dx$. The electric force experienced by the element in the field of ring

$$dF = (\lambda\, dx)\, E(x) = \frac{\lambda\, qx\, dx}{4\pi\varepsilon_0 (R^2 + x^2)^{3/2}}$$

Thus the sought interaction

$$F = \int_0^\infty \frac{\lambda\, qx\, dx}{4\,\pi\varepsilon_0 (R^2 + x^2)^{3/2}}$$



On integrating we get

$$F = \frac{\lambda q}{4\pi\varepsilon_0 R}$$

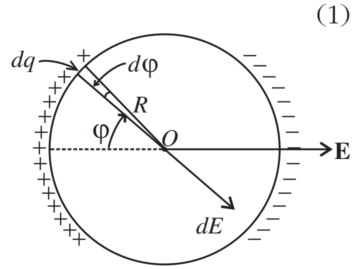
3.12 (a) The given charge distribution is shown in the figure. The symmetry of this distribution implies that vector **E** at the point O is directed to the right, and its magnitude is equal to the sum of the projection onto the direction of **E** of vectors **dE** from elementary charges dq. The projection of vector **dE** onto vector **E** is

$$dE \cos\varphi = \frac{1}{4\pi\varepsilon_0}\frac{dq}{R^2}\cos\varphi$$

where $\qquad dq = \lambda R d\varphi = \lambda_0 R \cos\varphi\, d\varphi. \qquad (1)$

Integrating Eq. (1) over φ between 0 and 2π we find the magnitude of the vector **E** as

$$E = \frac{\lambda_0}{4\pi\varepsilon_0 R}\int_0^{2\pi} \cos^2\varphi\, d\varphi = \frac{\lambda_0}{4\varepsilon_0 R}$$

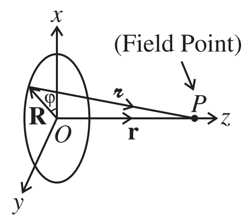


It should be noted that this integral is evaluated in the most simple way if we take into account that $<\cos^2\varphi> = 1/2$. Then

$$\int_0^{2\pi} \cos^2\varphi\, d\varphi = <\cos^2\varphi> 2\pi = \pi$$

(b) Let us take differential length element of the ring at an azimuthal angle φ from the x-axis, the element subtends an angle $d\varphi$ at the center, and carries charge $dq = \lambda R d\varphi = (\lambda_0 \cos\varphi) R d\varphi$.

Taking the plane of ring as x-y plane and center of the ring as origin O, locations of field point $\mathbf{r}$, of charge element $\mathbf{R}$ and of field point relative to charge element $\boldsymbol{\mathscr{r}}$ are shown in the figure.



Electric field strength at the field point due to considered charge element

$$d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\mathscr{r}^3}\boldsymbol{\mathscr{r}}$$

Using $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{R} = x\mathbf{k} - (R\cos\varphi\,\mathbf{i} + R\sin\varphi\,\mathbf{j})$ and $\mathscr{r}^3 = (R^2 + x^2)^{3/2}$

$$d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{(\lambda_0\cos\varphi)\, R d\varphi}{(R^2 + x^2)^{3/2}}\{x\mathbf{k} - (R\cos\varphi\,\mathbf{i} + R\sin\varphi\,\mathbf{j})\}$$

So, sought net electric field strength

$$\mathbf{E} = \int d\mathbf{E}$$

$$= \frac{\lambda_0 R}{4\pi\varepsilon_0 (R^2 + x^2)^{3/2}}\left[x\mathbf{k}\int_0^{2\pi}\cos\varphi\, d\varphi - R\mathbf{i}\int_0^{2\pi}\cos^2\varphi\, d\varphi - R\mathbf{j}\int_0^{2\pi}\sin 2\varphi\, d\varphi\right]$$

Taking into account

$$\int_0^{2\pi}\cos\varphi\, d\varphi = 0, \quad \int_0^{2\pi}\sin 2\varphi\, d\varphi = 0 \quad \text{and} \quad \int_0^{2\pi}\cos^2\varphi\, d\varphi = \pi$$

We get $$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{\pi\lambda_0 R^2(-\mathbf{i})}{(R^2 + x^2)^{3/2}}$$

Hence, $$E = \frac{\lambda_0 R^2}{4\varepsilon_0 (R^2 + x^2)^{3/2}}$$

For $x \gg R$,

$$E_x = \frac{p}{4\pi\varepsilon_0 x^3} \quad \text{where} \quad p = \lambda_0 \pi R^2$$

Alternate:

Take an element S at an azimuthal angle φ from the x-axis, the element subtending an angle $d\varphi$ at the center. The elementary field at P due to the element is

$$\frac{\lambda_0 \cos\varphi d\varphi R}{4\pi\varepsilon_0 (x^2 + R^2)} \quad \text{(along } SP \text{ with components)}$$

$$\frac{\lambda_0 \cos\varphi d\varphi\ R}{4\pi\varepsilon_0 (x^2 + R^2)} \times \{\cos\theta \text{ along } OP, \sin\theta \text{ along } OS\}$$

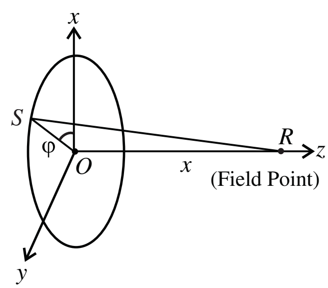
where $$\cos\theta = \frac{x}{(x^2 + R^2)^{1/2}}$$

and $$\sin\theta = \frac{R}{(x^2 + R^2)^{1/2}}$$

The component along OP vanishes on integration as $\int_0^{2\pi} \cos\varphi d\varphi = 0$.

The component along OS can be broken into the parts along Ox and Oy and given by

$$\frac{\lambda_0 R^2 \cos\varphi d\varphi}{4\pi\varepsilon_0 (x^2 + R^2)^{3/2}} \times \{\cos\varphi \text{ along } Ox, \sin\varphi \text{ along } Oy\}$$



On integration, the part along Oy vanishes.

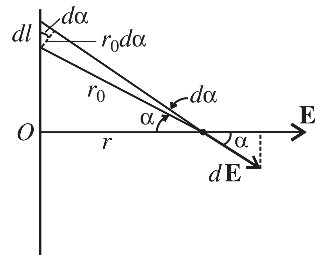
So, $$E_x = \frac{\lambda_0 R^2}{4\pi\varepsilon_0 (x^2 + R^2)^{3/2}} \int_0^{2R} \cos^2\varphi d\varphi$$

as $$\int_0^{2R} \cos^2\varphi \sin\varphi d\varphi = 0$$

Finally $$E = E_x = \frac{\lambda_0 R^2}{4\varepsilon_0 (x^2 + R^2)^{3/2}}$$

3.13 (a) It is clear from symmetry considerations that vector **E** must be directed as shown in the figure. This shows the way of solving this problem: we must find the component dE_r of the field created by the element dl of the rod, having the charge dq and then integrate the result over all the elements of the rod. In this case

$$dE_r = dE\cos\alpha = \frac{1}{4\pi\epsilon_0}\frac{\lambda dl}{r_0^2}\cos\alpha$$



where $\lambda = q/2a$ is the linear charge density. Let us reduce this equation to the form convenient for integration. Figure shows that $dl\cos\alpha = r_0 d\alpha$ and $r_0 = r/\cos\alpha$.

Consequently,

$$dE_r = \frac{1}{4\pi\varepsilon_0}\frac{\lambda r_0 d\alpha}{r_0^2} = \frac{\lambda}{4\pi\varepsilon_0 r}\cos\alpha\, d\alpha$$

This expression can be easily integrated to yield

$$E = \frac{\lambda}{4\pi\varepsilon_0 r}2\left(\int_0^{\alpha_0}\cos\alpha\, d\alpha\right) = \frac{\lambda}{4\pi\varepsilon_0 r}(2\sin\alpha_0)$$

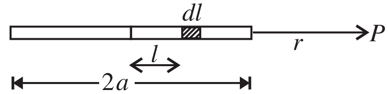
where α_0 is the maximum value of the angle α.

$$\sin\alpha_0 = \frac{a}{\sqrt{a^2 + r^2}}$$

Thus,
$$E = \frac{q/2a}{4\pi\varepsilon_0 r}\left(2\frac{a}{\sqrt{a^2 + r^2}}\right)$$
$$= \frac{q}{4\pi\varepsilon_0 r\sqrt{a^2 + r^2}}$$

Note that in this case also $E \approx \dfrac{q}{4\pi\varepsilon_0 r^2}$ for $r \gg a$, as for the field of a point charge.

(b) Let, us consider the element of length dl at a distance l from the center of the rod, as shown in the figure.



Then field at P, due to this element is given by

$$dE = \frac{\lambda dl}{4\pi\varepsilon_0 (r - l)^2}$$

if the element lies on the side as shown in the figure,

and
$$dE = \frac{\lambda dl}{4\pi\varepsilon_0 (r + l)^2}$$

if the element lies on the other side.

Hence, $$E = \int dE = \int_0^a \frac{\lambda\, dl}{4\pi\varepsilon_0\,(r - l)^2} + \int_0^a \frac{\lambda\, dl}{4\pi\varepsilon_0\,(r + l)^2}$$

On integrating and putting $\lambda = \dfrac{q}{2a}$, we get

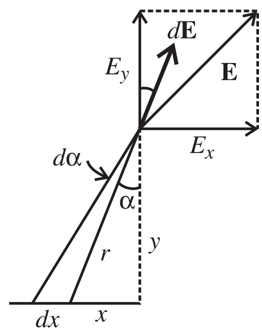
$$E = \frac{q}{4\pi\varepsilon_0}\frac{1}{(r^2 - a^2)}$$

For $r \gg a$, $$E \approx \frac{q}{4\pi\varepsilon_0 r^2}$$

3.14 The problem is reduced to finding E_x and E_y, which are the projections of $\mathbf{E}$, under the assumption $\lambda > 0$ (see figure).

Let us start with E_x. The contribution to E_x from the charge element of the segment dx is

$$dE_x = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx}{r^2}\sin\alpha \qquad (1)$$



Let us reduce this expression to the form convenient for integration. Thus,

$$dx = \frac{r\, d\alpha}{\cos\alpha},\ r = \frac{y}{\cos\alpha}$$

Then, $$dE_x = \frac{\lambda}{4\pi\varepsilon_0 y}\sin\alpha\ d\alpha$$

Integrating this expression over α between 0 and $\pi/2$, we find

$$E_x = \frac{\lambda}{4\pi\varepsilon_0 y}$$

In order to find the projection E_y it is sufficient to recall that dE_y differs from dE_x in that $\sin\alpha$ in Eq. (1) is simply replaced by $\cos\alpha$.

This gives

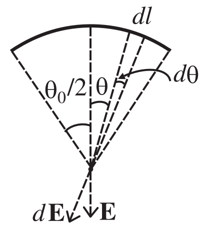
$$dE_y = (\lambda\cos\alpha\ d\alpha)/4\,\pi\varepsilon_0 y \quad \text{and} \quad E_y = \lambda/4\pi\varepsilon_0 y$$

We have obtained an interesting result that $E_x = E_y$ independently of y, i.e., $\mathbf{E}$ is oriented at the angle of 45° to the rod. The modulus of $\mathbf{E}$ is

$$\mathbf{E} = \sqrt{E_x^2 + E_y^2} = \frac{\lambda\sqrt{2}}{4\pi\varepsilon_0 y}$$

3.15 For Fig. (a): Using the solution of Problem 3.14, the net electric field strength at the point O due to straight parts of the thread equals zero. For the curved part (arc), let us derive a general expression i.e. let us calculate the field strength at the center of arc of radius R and linear charge density λ and which subtends angle θ_0 at the center.

From the symmetry the sought field strength will be directed along the bisector of the angle θ_0 and is given by



$$E = \int_{-\theta_0/2}^{+\theta_0/2} \frac{\lambda\,(R d\theta)}{4\pi\varepsilon_0 R^2} \cos\theta = \frac{\lambda}{2\pi\varepsilon_0 R} \sin\frac{\theta_0}{2}$$

In our problem $\theta_0 = \pi/2$, thus the field strength due to the turned part at the point

$$E_0 = \frac{\sqrt{2}\lambda}{4\pi\varepsilon_0 R}$$

which is also the sought result. For Fig. (b): Using the solution of Problem 3.14 (a), net field strength at O due to straight parts equals

$$\sqrt{2}\left(\frac{\sqrt{2}\,\lambda}{4\pi\varepsilon_0 R}\right) = \frac{\lambda}{2\pi\varepsilon_0 R} \quad \text{(directed vertically downward)}$$

Now using the solution of Problem 3.15 (a) field strength due to the given curved part (semi-circle) at the point O becomes

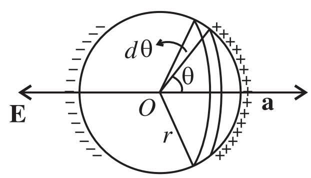
$$\frac{\lambda}{2\pi\varepsilon_0 R} \quad \text{(directed vertically upward)}$$

Hence the sought net field strength becomes zero.

3.16 Given charge distribution on the surface $\sigma = \mathbf{a} \cdot \mathbf{r}$ is shown in the figure. Symmetry of this distribution implies that the sought $\mathbf{E}$ at the center O of the sphere is opposite to $\mathbf{a}$. For the ring element

$$dq = \sigma\ (2\pi r \sin\theta)\ r d\theta = (\mathbf{a} \cdot \mathbf{r})$$

$$2\pi r^2 \sin\theta\ d\theta = 2\pi a r^3 \sin\theta \cos\theta\ d\theta$$



Again from symmetry, field strength due to any ring element $d\mathbf{E}$ is also opposite to $\mathbf{a}$, i.e., $d\mathbf{E} \uparrow\downarrow \mathbf{a}$. Hence,

$$d\mathbf{E} = \frac{dq r \cos\theta}{4\pi\varepsilon_0\,(r^2 \sin^2\theta + r^2 \cos^2\theta)^{3/2}} \frac{-\mathbf{a}}{a} \quad \text{(using the result of Problem 3.9)}$$

$$= \frac{(2\pi a r^3 \sin\theta \cos\theta\ d\theta)\ r\cos\theta}{4\pi\varepsilon_0 r^3} \frac{(-\mathbf{a})}{a}$$

$$= \frac{-\mathbf{a}\, r}{2\varepsilon_0} \sin\theta \cos^2\ d\theta$$

Thus, $\quad \mathbf{E} = \displaystyle\int d\mathbf{E} = \frac{(-\mathbf{a})r}{2\varepsilon_0} \int_0^{\pi} \sin\theta \cos^2\theta\ d\theta$

Integrating, we get

$$\mathbf{E} = \frac{\mathbf{a}\, r}{2\varepsilon_0} \frac{2}{3} = -\frac{\mathbf{a}\, r}{3\varepsilon_0}$$

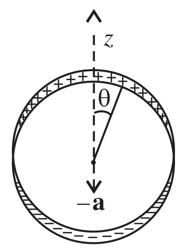
3.17 We start from two charged spherical balls each of radius R with equal and opposite charge densities $+\rho$ and $-\rho$. The center of the balls are at $+\mathbf{a}/2$ and $-\mathbf{a}/2$, respectively, so the equation of their surfaces are

$$\left|\mathbf{r} - \frac{\mathbf{a}}{2}\right| = R \quad \text{or} \quad r - \frac{a}{2}\cos\theta \cong R$$

and
$$r + \frac{a}{2}\cos\theta \cong R$$

considering a to be small. The distance between the two surfaces in the radial direction at angle is $\theta \mid a\cos\theta \mid$ and does not depend on the azimuthal angle. It is seen from the diagram that the surface of the sphere has in effect a surface density $\sigma = \sigma_0 \cos\theta$ when $\sigma_0 = \rho a$.

Inside any uniformly charged spherical ball, the field is radial and has the magnitude given by Gauss's theorem



$$4\pi r^2 E = \frac{4\pi}{3} r^3 \rho/\varepsilon_0$$

or
$$E = \frac{\rho r}{3\varepsilon_0}$$

In vector notation, using the fact that V must be measured from the center of the ball, we get, for the present case

$$\mathbf{E} = \frac{\rho}{3\varepsilon_0}\left(\mathbf{r} - \frac{\mathbf{a}}{2}\right) - \frac{\rho}{3\varepsilon_0}\left(\mathbf{r} + \frac{\mathbf{a}}{2}\right)$$

$$= -\rho\,\mathbf{a}/3\varepsilon_0 = -\frac{\sigma_0}{3\varepsilon_0}\mathbf{k}$$

where $\mathbf{k}$ is the unit vector along the polar axis from which θ is measured.

3.18 Let us consider an elemental spherical shell of thickness dr. Thus surface charge density of the shell $\sigma = \rho dr = (\mathbf{a}\cdot\mathbf{r})\, dr$.

Thus using the solution of Problem 3.16, field strength due to this spherical shell

$$d\mathbf{E} = -\frac{\mathbf{a}r}{3\varepsilon_0}\, dr$$

Hence, the sought field strength

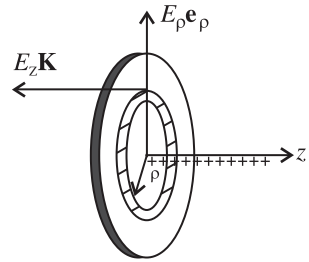
$$\mathbf{E} = -\frac{\mathbf{a}}{3\varepsilon_0}\int_0^R r\,dr = -\frac{\mathbf{a}R^2}{6\varepsilon_0}$$

3.19 Due to the line charge

$$\mathbf{E} = \frac{\lambda}{4\pi\varepsilon_0\rho}(-\mathbf{k}) + \frac{\lambda}{4\pi\varepsilon_0\rho}(\mathbf{e}_\rho) \text{ (see solution of Problem 3.14)}$$

Distribution of field **E** allows us to take a finite ring element of radius $0 < \rho < R$ and thickness $d\rho$. Hence area vector of the taken ring element $d\mathbf{S} = (2\pi\rho d\rho)(-\mathbf{k})$.

Now flux across the considered ring element

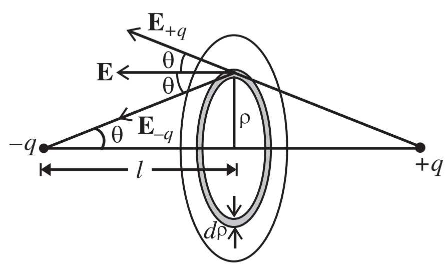


$$d\Phi = \mathbf{E}\cdot d\mathbf{S} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{-\mathbf{k}}{\rho} + \frac{\mathbf{e}_\rho}{\rho}\right]\cdot 2\pi\rho d\rho\,(-\mathbf{k})$$

$$d\Phi = +\frac{2\pi\lambda\, d\rho}{4\pi\varepsilon_0} \text{ (because } \mathbf{e}_\rho \perp \mathbf{k})$$

So, $$\Phi = \int d\Phi = \frac{\lambda R}{2\varepsilon_0}$$

3.20 Take the line joining the point charges as z-axis, In this problem at a normal distance ρ (from z-axis) **E** is the same and is directed towards positive z-axis. So, symmetry of the field **E** allows us to take a finite ring element of radius ρ and thickness $d\rho$.



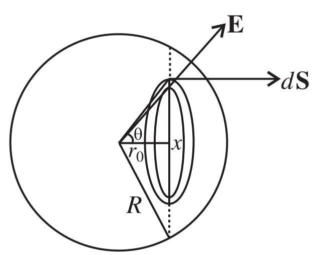
E at all the locations of taken ring element is

$$\mathbf{E} = \frac{2q}{4\pi\varepsilon_0(l^2+\rho^2)}\cos\theta\,\mathbf{k} = \frac{2ql}{4\pi\varepsilon_0(l^2+\rho^2)^{3/2}}\mathbf{k}$$

So, $$d\Phi = \mathbf{E}\cdot d\mathbf{S} = \frac{2ql}{4\pi\varepsilon_0(l^2+\rho^2)^{3/2}}\mathbf{k}\cdot 2\pi\rho d\rho\,\mathbf{k} = \frac{ql\rho d\rho}{\varepsilon_0(l^2+\rho^2)^{3/2}}$$

Hence, $$\Phi = \int d\Phi = \frac{ql}{\varepsilon_0}\int_0^R \frac{\rho d\rho}{(l^2+\rho^2)^{3/2}} = \frac{q}{\varepsilon_0}\left[1 - \frac{l}{\sqrt{l^2+R^2}}\right]$$

It can also be solved by considering a ring element or by using solid angle.



3.21 From Gauss' theorem, electric field strength at an inside point at a distance r from the center of uniformly charged ball of volume density ρ is $\mathbf{E} = \dfrac{\rho\mathbf{r}}{3\varepsilon_0}$. So **E** is not uniform over the given section (a disk of radius $\sqrt{R^2 - r_0^2}$) of the ball. Thus the section is considered to be made up of ring elements whose radii varies from zero to $\sqrt{R^2 - r_0^2}$. Let us consider a ring element of radius x and thickness dx, as shown in the figure, so that magnitude of **E** takes same value over the considered ring element.

Flux over the considered ring element

$$d\Phi = \mathbf{E} \cdot d\mathbf{S} = E_r dS \cos\theta$$

Using $E_r = \rho r/3\varepsilon_0$, $ds = 2\pi x dx$, and $\cos\theta = r_0/r$, we get

$$d\Phi = \frac{\rho r}{3\varepsilon_0} 2\pi x\, dx \frac{r_0}{r} = \frac{\rho r_0}{3\varepsilon_0} 2\pi x dx$$

Hence, sought flux

$$\Phi = \frac{2\pi\rho\, r_0}{3\varepsilon_0} \int_0^{\sqrt{R^2 - r_0^2}} x dx = \frac{2\pi\rho r_0}{3\varepsilon_0} \frac{(R^2 - r_0^2)}{2} = \frac{\pi\rho r_0}{3\varepsilon_0} (R^2 - r_0^2)$$

3.22 The field at P due to each of the threads at A and B are is the same having the magnitude

$$E_0 = \frac{\lambda}{2\pi\varepsilon_0 (x^2 + l^2/4)^{1/2}} \quad \text{(directed along } AP \text{ and } BP\text{)}$$

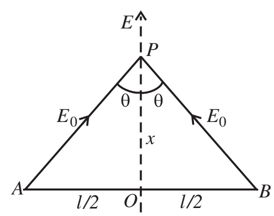
The resultant is along OP, given by

$$E = 2E_0 \cos\theta$$

$$= \frac{\lambda x}{\pi\varepsilon_0 (x^2 + l^2/4)}$$

$$= \frac{\lambda}{\pi\varepsilon_0 \left[x + \frac{l^2}{4x} - 2 \cdot \frac{l}{2\sqrt{x}} \cdot \sqrt{x} + l \right]}$$

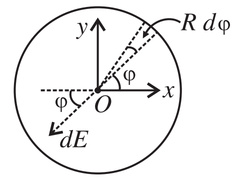
$$= \frac{\lambda}{\pi\varepsilon_0 \left[\left(\sqrt{x} - \frac{1}{2\sqrt{x}} \right)^2 + l \right]}$$



This is the maximum when $x = \frac{l}{2}$ and $E = E_{\max} = \frac{\lambda}{\pi\varepsilon_0 l}$.

3.23 Take a section of the cylinder perpendicular to its axis through the point where the electric field is to be calculated. (All points on the axis are equivalent.) Consider an element S with azimuthal angle φ. The length of the element is $Rd\varphi$, R being the radius of cross section of the cylinder. The element itself is a section of an infinite strip. The electric field at O due to this strip is

$$\frac{\sigma_0 \cos\varphi \, (R d\varphi)}{2\pi\varepsilon_0 R} \text{ along } SO$$



This can be resolved into

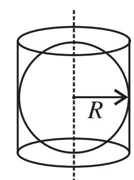
$$\frac{\sigma_0 \cos\varphi \; d\varphi}{2\pi\varepsilon_0} \begin{Bmatrix} \cos\varphi \text{ along } Ox \\ \sin\varphi \text{ along } yO \end{Bmatrix}$$

On integration the component along yO vanishes. What remains is

$$\int_0^{2x} \frac{\sigma_0 \cos^2\varphi d\varphi}{2\pi\varepsilon_0} = \frac{\sigma_0}{2\varepsilon_0}$$

along xO, i.e., along the direction $\varphi = \pi$.

3.24 Since the field is axis-symmetric (as the field of a uniformly charged filament), we conclude that the flux through the sphere of radius R is equal to the flux through the lateral surface of a cylinder having the same radius and the height $2R$, as shown in the figure.



Now, $$\Phi = \int \mathbf{E} \cdot d\mathbf{S} = E_\rho S$$

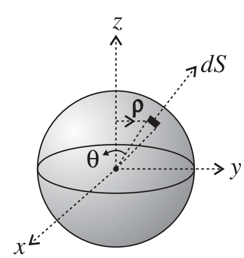
But, $$E_\rho = \frac{a}{R}$$

Thus, $$\Phi = \frac{a}{R} S = \frac{a}{R} 2\pi \; R \cdot 2R = 4\pi a R$$

Alternate:

The distribution of electric field is given by

$$\mathbf{E} = \frac{a\,(x\mathbf{i} + y\mathbf{j})}{x^2 + y^2} = \frac{a\boldsymbol{\rho}}{\rho^2}$$



where $\boldsymbol{\rho} = x\mathbf{i} + y\mathbf{j}$, which is the vector component of position vector $\mathbf{r}$ normal to the z-axis. Let us take a differential area element dS on the surface of the given sphere of radius R, as shown in the figure.

Then, $$d\Phi = \mathbf{E} \cdot d\mathbf{S} = \frac{a\boldsymbol{\rho}}{\rho^2} \cdot d\mathbf{S} = \frac{a}{\rho} dS \sin\theta = \frac{a \, dS}{\rho}\left(\frac{\rho}{R}\right) = \frac{a}{R} dS$$

So, $$\Phi = \frac{a}{R} \int dS = \frac{a}{R}(4\pi R^2) = 4\pi a R$$

3.25 (a) Let us consider a sphere of radius $r < R$, then charge enclosed by the considered sphere

$$q_{\text{enclosed}} = \int_0^r 4\pi r^2 dr\rho = \int_0^r 4\pi r^2 \rho_0 \left(1 - \frac{r}{R}\right) dr \tag{1}$$

Now, applying Gauss' theorem

$$E_r 4\pi r^2 = \frac{q_{\text{enclosed}}}{\varepsilon_0}$$

(where E_r is the projection of electric field along the radial line)

$$= \frac{\rho_0}{\varepsilon_0} \int_0^r 4\pi r^2 \left(1 - \frac{r}{R}\right) dr$$

or
$$E_r = \frac{\rho_0 r}{3\varepsilon_0}\left[1 - \frac{3r}{4R}\right]$$

For a point outside the sphere $r > R$, so

$$q_{\text{enclosed}} = \int_0^R 4\pi r^2 dr\, \rho_0 \left(1 - \frac{r}{R}\right) \text{ (as there is no charge outside the ball)}$$

Again from Gauss' theorem

$$E_r 4\pi r^2 = \int_0^R \frac{4\pi r^2 dr\, \rho_0 \left(1 - \frac{r}{R}\right)}{\varepsilon_0}$$

or
$$E_r = \frac{\rho_0}{r^2 \varepsilon_0}\left[\frac{R^3}{3} - \frac{R^4}{4R}\right] = \frac{\rho_0 R^3}{12\, r^2 \varepsilon_0}$$

(b) As magnitude of electric field decreases with increasing r for $r > R$, field will be maximum for $r < R$. Now, for E_r to be maximum

$$\frac{d}{dr}\left(r - \frac{3r^2}{4R}\right) = 0 \quad \text{or} \quad 1 - \frac{3r}{2R} = 0 \quad \text{or} \quad r = r_m = \frac{2R}{3}$$

Hence,
$$E_{\max} = \frac{\rho_0 R}{9\varepsilon_0}$$

3.26 Let the charge carried by the sphere be q, then using Gauss' theorem for a spherical surface having radius $r > R$, we can write

$$E\,4\pi r^2 = \frac{q_{\text{enclosed}}}{\varepsilon_0} = \frac{q}{\varepsilon_0} + \frac{1}{\varepsilon_0}\int_R^r \frac{\alpha}{r}\,4\pi r^2 dr$$

On integrating, we get

$$E\,4\pi r^2 = \frac{(q - 2\pi\alpha R^2)}{\varepsilon_0} + \frac{4\pi\alpha r^2}{2\varepsilon_0}$$

The intensity E does not depend on r when the expression in the parentheses is equal to zero. Hence,

$$q = 2\pi\alpha R^2 \quad \text{and} \quad E = \frac{\alpha}{2\varepsilon_0}$$

3.27 Let us consider a spherical layer of radius r and thickness dr, having its center coinciding with the center of the system. Then using Gauss' theorem for this surface,

$$E_r\,4\pi r^2 = \frac{q_{\text{enclosed}}}{\varepsilon_0} = \int_0^r \frac{\rho dV}{\varepsilon_0}$$

$$= \frac{1}{\varepsilon_0}\int_0^r \rho_0 e^{-\alpha r3}\,4\pi r^2 dr$$

After integration

$$E_r 4\pi r^2 = \frac{\rho\,4\pi}{3\varepsilon_0\alpha}\,[1 - e^{-\alpha r^3}]$$

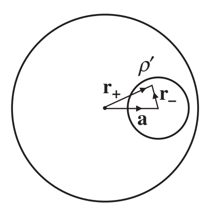
or
$$E_r = \frac{\rho_0}{3\varepsilon_0\alpha r^2}\,[1 - e^{-\alpha r^3}]$$

now when $\alpha r^3 \gg 1$,
$$E_r \approx \frac{\rho_0}{3\varepsilon_0\alpha r^2}$$

and when $\alpha r^3 \ll 1$,
$$E_r \approx \frac{\rho_0 r}{3\varepsilon_0}$$

3.28 Using Gauss' theorem we can easily show that the electric field strength within a uniformly charged sphere is

$$\mathbf{E} = \left(\frac{\rho}{3\varepsilon_0}\right)\mathbf{r}$$



The cavity, in our problem, may be considered as the superposition of two balls, one with the charge density ρ and the other with $-\rho$.

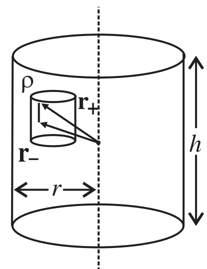
Let P be a point inside the cavity such that its position vector with respect to the center of cavity be $\mathbf{r}_-$ and with respect to the center of the ball be $\mathbf{r}_+$. Then from the principle of superposition, field inside the cavity, at an arbitrary point P is given by

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-$$
$$= \frac{\rho}{3\varepsilon_0}(\mathbf{r}_+ - \mathbf{r}_-) = \frac{\rho}{3\varepsilon_0}\mathbf{a}$$

Note: The *expression* obtained for ***E*** *shows that it is valid regardless of the ratio between the radii of the sphere and the distance between their* centers.

3.29 Let us consider a cylindrical Gaussian surface of radius r and height h inside an infinitely long charged cylinder with charge density ρ. Now from Gauss' theorem

$$\mathbf{E}_r 2\pi rh = \frac{q_{\text{enclosed}}}{\varepsilon_0}$$



(where E_r is the field inside the cylinder at a distance r from its axis)

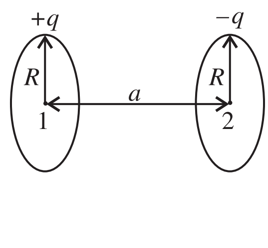
or $$E_r\, 2\pi rh = \frac{\rho\pi r^2 h}{\varepsilon_0} \quad\Rightarrow\quad E_r = \frac{\rho r}{2\varepsilon_0}$$

Now, using the method of Problem 3.28, field at a point P inside the cavity is

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho}{2\varepsilon_0}(\mathbf{r}_+ - \mathbf{r}_-) = \frac{\rho}{2\varepsilon_0}\mathbf{a}$$

3.30 The arrangement of the rings is as shown in the figure. Now, potential at the point 1 is equal to sum of potential at 1 and due to the ring 1 and potential at 1 due to the ring 2. So,

$$\varphi_1 = \frac{q}{4\pi\varepsilon_0 R} + \frac{-q}{4\pi\varepsilon_0 (R^2 + a^2)^{1/2}}$$



Similarly, the potential at point 2

$$\varphi_2 = \frac{-q}{4\pi\varepsilon_0 R} + \frac{q}{4\pi\varepsilon_0 (R^2 + a^2)^{1/2}}$$

Hence, the sought potential difference

$$\varphi_1 - \varphi_2 = \Delta\varphi = 2\left(\frac{q}{4\pi\varepsilon_0 R}\right) + \frac{-q}{4\pi\varepsilon_0 (R^2 + a^2)^{1/2}}$$
$$= \frac{q}{2\pi\varepsilon_0 R}\left(1 - \frac{1}{\sqrt{1 + (a/R)^2}}\right)$$

3.31 We know from Gauss' theorem that the electric field due to an infinitely long straight wire, at a perpendicular distance r from it is

$$E_r = \frac{\lambda}{2\pi\varepsilon_0 r}.$$

So, the work done is

$$\int_1^2 E_r dr = \int_x^{\eta x} \frac{\lambda}{2\pi\varepsilon_0 r}\, dr$$

(where x is perpendicular distance from the thread by which point 1 is removed from it).

Hence, $$\Delta\varphi_{12} = \frac{\lambda}{2\pi\varepsilon_0} \ln \eta$$

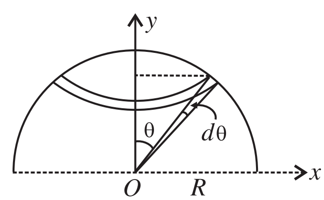
$$= 5 \text{ kV (on substituting values)}$$

3.32 Let us consider a ring element as shown in the figure. Then the charge carried by the element, $dq = (2\pi R \sin\ \theta)\ Rd\theta\ \sigma$.

Hence, the potential due to the considered element at the center of the hemisphere is

$$d\varphi = \frac{1}{4\pi\varepsilon_0}\frac{dq}{R} = \frac{2\ \pi\sigma R \sin\theta d\theta}{4\pi\varepsilon_0} = \frac{\sigma R}{2\varepsilon_0}\sin\theta d\theta$$

So potential due to the whole hemisphere is



$$\varphi = \frac{R\sigma}{2\varepsilon_0}\int_0^{\pi/2} \sin\theta\ d\theta = \frac{\sigma R}{2\varepsilon_0}$$

Now from the symmetry of the problem, net electric field of the hemisphere is directed towards the negative y-axis. So, we have

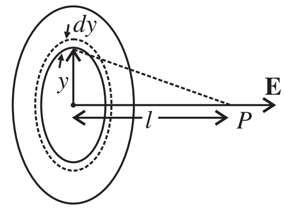
$$dE_y = \frac{1}{4\pi\varepsilon_0}\frac{dq\cos\theta}{R^2} = \frac{\sigma}{2\varepsilon_0}\sin\theta\cos\theta\ d\theta$$

Thus, $$E = E'_y = \frac{\sigma}{2\varepsilon_0}\int_0^{\pi/2}\sin\theta\cos\theta\ d\theta = \frac{\sigma}{4\varepsilon_0}\int_0^{\pi/2}\sin 2\theta d\theta = \frac{\sigma}{4\varepsilon_0} \text{ (along } yO)$$

3.33 Let us consider an elementary ring of thickness dy and radius y as shown in the figure. Then potential at a point P, at distance l from the center of the disk is

$$d\varphi = \frac{\sigma\ 2\pi y dy}{4\pi\varepsilon_0 (y^2 + l^2)^{1/2}}$$

Hence potential due to the whole disk is given by



$$\varphi = \int_0^R \frac{\sigma\, 2\pi y dy}{4\pi\varepsilon_0\,(y^2 + l^2)^{1/2}} = \frac{\sigma l}{2\varepsilon_0}\left(\sqrt{1 + (R/l)^2} - 1\right)$$

From symmetry

$$E = E_l = -\frac{d\varphi}{dl}$$

$$= -\frac{\sigma}{2\varepsilon_0}\left[\frac{2l}{2\sqrt{R^2 + l^2}} - 1\right] = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{1}{\sqrt{1 + (R/l)^2}}\right]$$

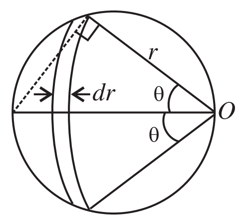
when $l \rightarrow 0$, $\quad \varphi = \dfrac{\sigma R}{2\varepsilon_0}$ and $E = \dfrac{\sigma}{2\varepsilon_0}$

when $l \gg R$, $\quad \varphi \approx \dfrac{\sigma R^2}{4\varepsilon_0 l}$ and $E \approx \dfrac{\sigma R^2}{4\varepsilon_0 l^2}$

3.34 By definition the potential in the case of a surface charge distribution is defined by

$$\varphi = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\, dS}{r} \tag{1}$$

In order to simplify integration, we shall choose the area element $\mathbf{dS}$ in the form of a part of the ring of radius r and width dr (see figure). Then $dS = 2\theta r\, dr$, $r = 2R\cos\theta$ and $dr = -2R\sin\theta\, d\theta$. After substituting these expresion into integral in Eq. (1), we obtain the expression for φ at the point O as



$$\varphi = -\frac{\sigma R}{\pi\varepsilon_0}\int_{\pi/2}^{0} \theta \sin\theta\, d\theta$$

We integrate by parts, denoting $\theta = u$ and $\sin\theta\, d\theta = dv$ to get

$$\int \theta \sin\theta\, d\theta = -\theta\cos\theta + \int \cos\theta\, d\theta$$

$$= -\theta\cos\theta + \sin\theta$$

which gives -1 after substituting the limits of integration. As a result, we obtain

$$\varphi = \frac{\sigma R}{\pi\varepsilon_0}$$

3.35 In accordance with the problem $\varphi = \mathbf{a}\cdot\mathbf{r}$. Thus from the equation $\mathbf{E} = -\nabla\varphi$, we get

$$\mathbf{E} = -\left[\frac{\partial}{\partial x}(a_x x)\,\mathbf{i} + \frac{\partial}{\partial_y}(a_y y)\,\mathbf{j} + \frac{\partial}{\partial_x}(a_z z)\,\mathbf{k}\right] = -[a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\mathbf{k}] = -\mathbf{a}$$

3.36 (a) Given, $\varphi = a(x^2 - y^2)$

So, $\mathbf{E} = -\nabla\varphi = -2a\,(x\,\mathbf{i} - y\,\mathbf{j})$

The sought shape of field lines is as shown in the figure (a) of answer sheet, assuming $a > 0$.

(b) Since, $\varphi = axy$

So, $\mathbf{E} = -\nabla\varphi = -ay\,\mathbf{i} - ax\,\mathbf{j}$

The plot of field lines is as shown in the figure (b) of answer sheet.

3.37 Given, $\varphi = a\,(x^2 + y^2) + bz^2$

So, $\mathbf{E} = -\nabla\varphi = -[2\,ax\,\mathbf{i} + 2ay\,\mathbf{j} + 2\,bz\,\mathbf{k}]$

Hence, $|\mathbf{E}| = 2\sqrt{a^2\,(x^2 + y^2) + b^2z^2}$

Shape of the equipotential surface:

Put $\rho = x\,\mathbf{i} + y\,\mathbf{j}$ or $\rho^2 = x^2 + y^2$

Then the equipotential surface has the equation

$$a\rho^2 + bz^2 = \text{constant} = \varphi$$

If $a > 0$, $b > 0$, then $\varphi > 0$ and the equation of the equipotential surface is

$$\frac{\rho}{\varphi/a} + \frac{z^2}{\varphi/b} = 1$$

which is an ellipse in ρ, z coordinates. In three dimensions, the surface is an ellipsoid of revolution with semi-axes $\sqrt{\varphi/a}$ and $\sqrt{\varphi/b}$.

If $a > 0$, $b < 0$, then φ can be ≥ 0. If $\varphi > 0$ then the equation is

$$\frac{\rho^2}{\varphi/a} - \frac{z^2}{\varphi/|b|} = 1$$

This is a single cavity hyperboloid of revolution about z axis.

If $\varphi = 0$, then the equation can be written as

$$a\rho^2 - |b|\,z^2 = 0$$

or $$z = \pm\sqrt{\frac{a}{|b|}}\,\rho$$

This is the equation of a right circular cone.

If $\varphi < 0$, then the equation can be written as

$$|b|\, z^2 - a\rho^2 = |\varphi|$$

or
$$\frac{z^2}{|\varphi|/|b|} - \frac{\rho^2}{|\varphi|/a} = 1$$

This is a two cavity hyperboloid of revolution about z-axis.

3.38 From Gauss' theorem, intensity at a point inside the sphere, at a distance r from the center is given by, $E_r = \dfrac{\rho r}{3\varepsilon_0}$ and outside it, is given by $E_r = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r^2}$.

(a) Potential at the center of the sphere

$$\varphi_0 = \int_0^\infty \mathbf{E}\cdot d\mathbf{r} = \int_0^R \frac{\rho r}{3\varepsilon_0}\, dr + \int_R^\infty \frac{q}{4\pi\varepsilon_0 r^2}\, dr = \frac{\rho}{3\varepsilon}\frac{R^2}{2} + \frac{q}{4\pi\varepsilon_0 R}$$

$$= \frac{q}{8\pi\varepsilon_0 R} + \frac{q}{4\pi\varepsilon_0 R} = \frac{3q}{8\pi\varepsilon_0 R}\left(\text{as } \rho = \frac{3q}{4\pi R^3}\right)$$

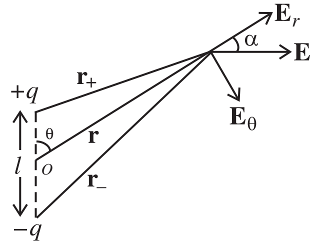
(b) Now, potential at any point inside the sphere, at a distance r from it's center

$$\varphi(r) = \int_r^R \frac{\rho}{3\varepsilon_0}\, r\, dr + \int_r^\infty \frac{q}{4\pi\varepsilon_0}\frac{dr}{r^2}$$

On integration, $\varphi(r) = \dfrac{3q}{8\varepsilon_0 R}\left[1 - \dfrac{r^2}{3R^2}\right] = \varphi_0\left[1 - \dfrac{r^2}{3R^2}\right]$

3.39 Let two charges $+q$ and $-q$ be separated by a distance l. Then electric potential at a point at distance $r \gg l$ from this dipole

$$\varphi(r) = \frac{+q}{4\pi\varepsilon_0 r_+} + \frac{-q}{4\pi\varepsilon_0 r_-} = \frac{q}{4\pi\varepsilon_9}\left(\frac{r_- - r_+}{r_+\, r_-}\right) \qquad (1)$$



But $\qquad r_- - r_+ \approx l\cos\theta$ and $r_+\, r_- \approx r^2 \qquad (2)$

From Eqs. (1) and (2),

$$\varphi(r) = \frac{ql\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}\ \varphi = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\varepsilon_0 r^3}$$

where **p** is the electric moment vector.

Now, $$E_r = \frac{-\partial\varphi}{\partial r} = \frac{2\,p\cos\theta}{4\pi\varepsilon_0 r^3}$$

and $$E_\theta = -\frac{\partial\varphi}{r\partial\theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$$

So, $$E = \sqrt{E_r^2 + E_\theta^2} = \frac{p}{4\pi\varepsilon_0 r^3}\sqrt{4\cos^2\theta + \sin^2\theta} = \frac{p}{4\pi\varepsilon_0 r^3}\sqrt{1 + 3\cos^2\theta}$$

3.40 From the results obtained in the previous problem

$$E_r = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3} \quad \text{and} \quad E_\theta = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$$

From the given figure in the probem book, it is clear that,

$$E_z = E_r\cos\theta - E_\theta\sin\theta = \frac{p}{4\pi\varepsilon_0 r^3}(3\cos^2\theta - 1)$$

and $$E_\perp = E_r\sin\theta + E_\theta\cos\theta = \frac{3\,p\sin\theta\cos\theta}{4\pi\varepsilon_0 r^3}$$

When, $\mathbf{E}_\perp \mathbf{p}$, $|\mathbf{E}| = \mathbf{E}_\perp$ and $E_z = 0$

So, $$3\cos^2\theta = 1 \text{ and } \cos\theta = \frac{1}{\sqrt{3}}$$

Thus $\mathbf{E}_\perp \mathbf{p}$ at the points located on the lateral surface of the cone, with its axis coinciding with the direction of z-axis and semi vertex angle $\theta = \cos^{-1} 1/\sqrt{3}$.

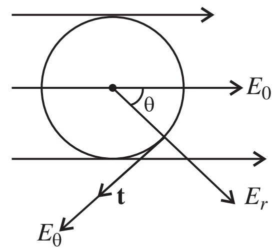
3.41 Let us assume that the dipole is at the center of the one equipotential surface which is spherical (see figure). On an equipotential surface, the net electric field strength along the tangent of it becomes zero. Thus,

$$-E_0\sin\theta + E_\theta = 0 \quad \text{or} \quad -E_0\sin\theta + \frac{p\sin\theta}{4\pi\varepsilon_0 r^3} = 0$$

where E is electric field generated by the dipole.

Hence, $$r = \left(\frac{p}{4\pi\varepsilon_0 E_0}\right)^{1/3}$$

Alternate:



Potential at the point, near the dipole is given by

$$\varphi = \frac{\mathbf{p}\cdot\mathbf{r}}{4\pi\varepsilon_0 r^3} - \mathbf{E}_0\cdot\mathbf{r} + \text{constant},$$

$$= \left(\frac{p}{4\pi\varepsilon_0 r^3} - E_0\right)\cos\theta + \text{constant}$$

For φ to be constant,

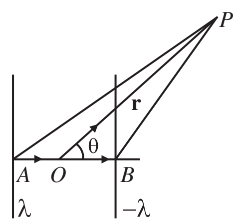
$$\frac{p}{4\pi\varepsilon_0 r^3} - E_0 = 0 \quad \text{or} \quad \frac{p}{4\pi\varepsilon_0 r^3} = E_0$$

Thus, $$r = \left(\frac{p}{4\pi\varepsilon_0 E_0}\right)^{1/3}$$

3.42 Let P be a point, at distance $r \gg l$ and at an angle θ to the vector $\mathbf{l}$ (see figure). Thus $\mathbf{E}$ at P

$$= \frac{\lambda}{2\pi\varepsilon_0}\frac{\mathbf{r}+\frac{\mathbf{l}}{2}}{\left|\mathbf{r}+\frac{\mathbf{l}}{2}\right|^2} - \frac{\lambda}{2\pi\varepsilon_0}\frac{\mathbf{r}-\frac{\mathbf{l}}{2}}{\left|\mathbf{r}-\frac{\mathbf{l}}{2}\right|^2}$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{r+\mathbf{l}/2}{r^2+\frac{l^2}{4}+rl\cos\theta} - \frac{\mathbf{r}-\mathbf{l}/2}{r^2+\frac{l^2}{4}-rl\cos\theta}\right]$$



$$= \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{\mathbf{l}}{r^2} - \frac{2l\mathbf{r}}{r^3}\cos\theta\right)$$

Hence, $$E = |\mathbf{E}| = \frac{\lambda l}{2\pi\varepsilon_0 r^2} \quad (\text{for } r \gg l)$$

Also, $$\varphi = \frac{\lambda}{2\pi\varepsilon_0}\ln|\mathbf{r}+\mathbf{l}/2| - \frac{\lambda}{2\pi\varepsilon_0}\ln|\mathbf{r}-\mathbf{l}/2|$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{r^2+rl\cos\theta+l^2/4}{r^2-rl\cos\theta+l^2/4} = \frac{\lambda l\cos\theta}{2\pi\varepsilon_0 r} \quad (\text{for } r \gg l)$$

3.43 The potential can be calculated by superposition. Choose the plane of the upper ring as $x = l/2$ and that of the lower ring as $x = -l/2$.

Then, $$\varphi = \frac{q}{4\pi\varepsilon_0[R^2+(x-l/2)^2]^{1/2}} - \frac{q}{4\pi\varepsilon_0[R^2+(x+l/2)^2]^{1/2}}$$

$$\approx \frac{q}{4\pi\varepsilon_0 [R^2 + x^2 - lx]^{1/2}} - \frac{q}{4\pi\varepsilon_0 [R^2 + x^2 + lx]^{1/2}}$$

$$\approx \frac{q}{4\pi\varepsilon_0 (R^2 + x^2)^{1/2}}\left(1 + \frac{lx}{2(R^2 + x^2)}\right) - \frac{q}{4\pi\varepsilon_0 (R^2 + x^2)^{1/2}}\left(1 - \frac{lx}{2(R^2 + x^2)}\right)$$

$$\approx \frac{qlx}{4\pi\varepsilon_0 (R^2 + x^2)^{3/2}}$$

For $|x| \gg R$, $\varphi \approx \dfrac{ql}{4\pi\varepsilon_0 x^2}$.

The electric field is

$$E = -\frac{\partial \varphi}{\partial x}$$
$$= -\frac{ql}{4\pi\varepsilon_0 (R^2 + x^2)^{3/2}} + \frac{3}{2}\frac{ql}{(R^2 + x^2)^{5/2}\, 4\pi\varepsilon_0} \times 2x = \frac{ql\,(2x^2 - R^2)}{4\pi\varepsilon_0 (R^2 + x^2)^{5/2}}$$

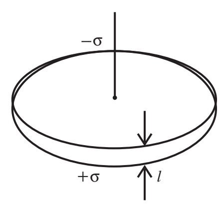
For $|x| \gg R$, $E \approx \dfrac{ql}{2\pi\varepsilon_0 x^3}$.

The plot is as shown in the answer sheet.

3.44 The field of a pair of oppositely charged sheets with holes can by superposition be reduced to that of a pair of uniform opposite charged sheets and disks with opposite charges. Now the charged sheets do not contribute any field outside them. Thus, using the result of the previous problem

$$\varphi = \int_0^R \frac{(-\sigma)\, l\, 2\pi r dr\, x}{4\pi\varepsilon_0 (r^2 + x^2)^{3/2}} = -\frac{\sigma x l}{4\varepsilon_0}\int_{x^2}^{R^2+x^2} \frac{dy}{y^{3/2}} \quad \text{(putting } y = x^2 + r^2\text{)}$$

$$= \frac{\sigma x l}{2\varepsilon_0 \sqrt{R^2 + x^2}}$$



$$E_x = -\frac{\partial \varphi}{\partial x} = -\frac{\sigma l}{2\varepsilon_0}\left[\frac{1}{\sqrt{R^2 + x^2}} - \frac{x^2}{(R^2 + x^2)^{3/2}}\right] = -\frac{\sigma l R^2}{2\varepsilon_0 (R^2 + x^2)^{3/2}}$$

The plot is as shown in the answer sheet.

3.45 For $x > 0$, we can use the result as given above and write

$$\varphi \approx \pm\frac{\sigma l}{2\varepsilon_0}\left(1 - \frac{|x|}{(R^2 + x^2)^{1/2}}\right)$$

for the solution that vanishes at α. There is a discontinuity in potential for $|x| = 0$. The solution for negative x is obtained by $\sigma \to -\sigma$. Thus,

$$\varphi = -\frac{\sigma l x}{2\varepsilon_0 (R + x^2)^{1/2}} + \text{constant}$$

Hence, ignoring the jump

$$E = -\frac{\partial \varphi}{\partial x} = \frac{\sigma l R^2}{2\varepsilon_0 (R^2 + x^2)^{3/2}}$$

For large $|x|$,

$$\varphi \approx \pm \frac{p}{4\pi\varepsilon_0 x^2} \quad \text{and} \quad E \approx \frac{p}{2\pi\varepsilon_0 |x|^3} \text{ (where } p = \pi R^2 \sigma l)$$

3.46 In cylindrical coordinates r, θ, z,

$$E_r = \frac{\lambda}{2\pi\varepsilon_0 r}, \quad E_\theta = 0, \ E_\varphi = 0$$

and
$$\mathbf{F} = p\frac{\partial \mathbf{E}}{\partial l}$$

or
$$(\mathbf{p} \cdot \nabla)\mathbf{E}$$

(a) For $\mathbf{p}$ along the thread, $\mathbf{F} = 0$

as $\mathbf{E}$ does not change as the point of observation is moved along the thread.

(b) For $\mathbf{p}$ along $\mathbf{r}$
$$\mathbf{F} = p_r \frac{\partial (E_r \mathbf{e}_r)}{\partial r} = p\left(\frac{\partial E_r}{\partial r}\right)\mathbf{e}_r \left(\text{because } \frac{\partial \mathbf{e}_r}{\partial r} = 0\right)$$

$$= \frac{\lambda p}{2\pi\varepsilon_0 r^2}\mathbf{e}_r = -\frac{\lambda \mathbf{p}}{2\pi\varepsilon_0 r^2}$$

(c) For $\mathbf{p}$ along $\mathbf{e}_\theta$
$$\mathbf{F} = p_\theta \frac{\partial (E_r \mathbf{e}_r)}{r\partial \theta} = p\frac{\partial}{r\partial\theta}\left(\frac{\partial}{2\pi\varepsilon_0 r}\mathbf{e}_r\right)$$

$$= \frac{p\lambda}{2\pi\varepsilon_0 r^2}\frac{\partial \mathbf{e}_r}{\partial\theta} = \frac{p\lambda}{2\pi\varepsilon_0 r^2}\mathbf{e}_\theta = \frac{\mathbf{p}\lambda}{2\pi\varepsilon_0 r^2}\left(\text{because } \frac{\partial \mathbf{e}_r}{\partial\theta} = \mathbf{e}_\theta\right)$$

3.47 Force on a dipole of moment p is given by

$$F = \left|p\frac{\partial \mathbf{E}}{\partial l}\right|$$

In our problem, field due to a dipole at a distance l is given by

$$|\mathbf{E}| = \frac{p}{2\pi\varepsilon_0 l^3}$$

Hence, the force of interaction

$$F = \frac{3p^2}{2\pi\varepsilon_0 l^4} = 2.1 \times 10^{-16}\text{N}$$

3.48 Given, $\quad - d\varphi = \mathbf{E}\cdot d\mathbf{r} = a\,(y\,dx + xdy) = ad\,(xy)$

On integrating, $\quad \varphi = -a\,xy + \text{constant}$

3.49 Given, $\quad -d\varphi = \mathbf{E}\cdot d\mathbf{r} = [2\,axy\mathbf{i} + 2(x^2-y^2)\mathbf{j}]\cdot[dx\,\mathbf{i} + dy\,\mathbf{j}]$

or $\quad d\varphi = 2a\,x\,y\,dx + a\,(x^2 - y^2)dy = ad(x^2y) - ay^2\,dy$

On integrating, $\quad \varphi = ay\left(\frac{y^2}{3} - x^2\right) + \text{constant}$

3.50 Given,

$$- d\varphi = \mathbf{E}\cdot d\mathbf{r} = (ay\mathbf{i} + (ax + bz)\mathbf{j} + by\mathbf{k})\cdot(dx\mathbf{i} + dy\mathbf{j} + dz\,\mathbf{k})$$
$$= a\,(y\,dx + ax\,dy) + b\,(zdy + ydz) = ad\,(xy) + bd\,(yz)$$

On integrating, $\quad \varphi = -\,(a\,xy + b\,yz) + \text{constant}$

3.51 Field intensity along x-axis

$$E_x = -\frac{\partial\varphi}{\partial x} = 3\,ax^2$$

Then using Gauss' theorem in differential form

$$\frac{\varphi E_x}{\varphi x} = \frac{\rho(x)}{\varepsilon_0} \quad \text{so,} \quad \rho\,(x) = 6a\,\varepsilon_0 x$$

3.52 In the space between the plates, we have by the Poisson equation

$$\frac{\partial^2\varphi}{\partial x^2} = -\frac{\rho_0}{\varepsilon_0}$$

or $$\varphi = -\frac{\rho_0}{2\varepsilon_0}x^2 + Ax + B$$

where ρ_0 is the constant space charge density between the plates.

We can choose $\quad \varphi\,(0) = 0 \;$ so, $\quad B = 0$

Then $$\varphi(d) = \Delta\varphi = Ad - \frac{\rho_0 d^2}{2\varepsilon_0} \quad \text{or} \quad A = \frac{\Delta\varphi}{d} + \frac{\rho_0 d}{2\varepsilon_0}$$

Now, $$E = \frac{\partial \varphi}{\partial x} = \frac{\rho_0}{\varepsilon_0}x - A = 0 \quad (\text{for } x = 0)$$

if $$A = \frac{\Delta\varphi}{d} + \frac{\rho_0 d}{2\varepsilon_0} = 0$$

then, $$\rho_0 = -\frac{2\varepsilon_0 \Delta\varphi}{d^2}$$

Also, $$E(d) = \frac{\rho_0 d}{\varepsilon_0}$$

3.53 As $\varphi = \varphi(r)$, so field intensity is along the radial line and is given by

$$E_r = -\frac{\partial\varphi}{\partial r} = -2ar \tag{1}$$

Due to spherical symmetry of field of $\mathbf{E}$, electric flux over a spherical Gaussian surface of radius r is $4\pi r^2 E_r$.

From Gauss' theorem,

$$4\pi r^2 E_r = \frac{q}{\varepsilon_0}$$

On differenting,

$$4\pi\, d(r^2 E_r) = \frac{dq}{\varepsilon_0}$$

where dq is the charge contained between the sphere of radii r and $r + dr$.

Therefore, $$dq = \rho \cdot 4\pi r^2 dr$$

So, $$4\pi\left(r^2 dE_r + 2rE_r\, dr\right) = \frac{\rho \cdot 4\pi r^2\, dr}{\varepsilon_0}$$

$$\frac{\partial E_r}{\partial r} + \frac{2}{r}E_r = \frac{\rho}{\varepsilon_0}$$

$$-2a + \frac{2}{r}(-2ar) = \frac{\rho}{\varepsilon_0}$$

Hence, $$\rho = -6\varepsilon_0 a$$

Alternate:

Here $\mathbf{E} = -\nabla\varphi = -2ar\,\mathbf{e}_r = E_r\,\mathbf{e}_r$, where $\mathbf{e}_r$ is unit vector towards radius vector and $E_r = -2ar$.

From Gauss' law in the differential form

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) = \frac{\rho}{\varepsilon_0}$$

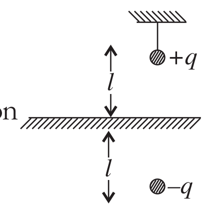
$$-\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\, 2ar) = \frac{\rho}{\varepsilon_0} \quad (\text{using } E_r = -2ar)$$

Hence, $$\rho = -6\varepsilon_0 a$$

3.2 Conductors and Dielectrics in an Electric Field

3.54 When the ball is charged, for the equilibrium of ball, electric force on it must counter balance the excess spring force exerted on the ball due to the extension in the spring. Thus, $$F_{\text{ele}} = F_{\text{spring}}$$

or $$\frac{q^2}{4\pi\varepsilon_0\,(2l)^2} = \kappa x$$

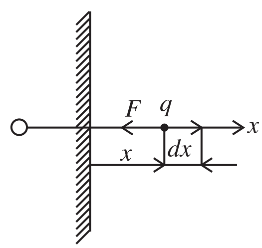


(the force on the charge q might be considered to arise from attraction by the electrical image),

or $$q = 4/\sqrt{\pi\varepsilon_0\kappa x}$$

which is the sought charge on the sphere.

3.55 By definition, the work of this force done upon an elementary displacement dx (see figure) is given by

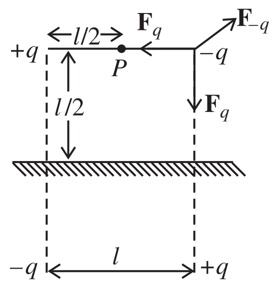


$$dA = F_x dx = -\frac{q^2}{4\pi\varepsilon_0\,(2x)^2}\,dx,$$

where the expression for the force is obtained with the help of the image method, Integrating this equation over x between l and ∞, we find

$$A = -\frac{q^2}{16\,\pi\varepsilon_0}\int_l^\infty \frac{dx}{x^2} = -\frac{q^2}{16\pi\varepsilon_0 l}$$

3.56 (a) Using the concept of electrical image, it is clear that the magnitude of the force acting on each charge,



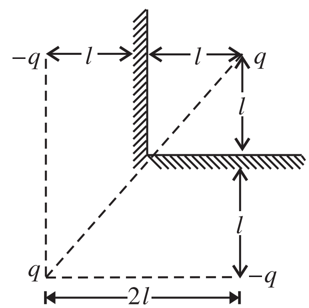
$$|\mathbf{F}| = \sqrt{2}\,\frac{q^2}{4\pi\varepsilon_0 l^2} - \frac{q^2}{4\pi\varepsilon_0 (\sqrt{2}\,l)^2}$$

$$= \frac{q^2}{8\pi\varepsilon_0 l^2}(2\sqrt{2} - 1)$$

(b) Also, from the figure, magnitude of electrical field strength at P

$$E = 2\left(1 - \frac{1}{5\sqrt{5}}\right)\frac{q}{\pi\varepsilon_0 l^2}$$

3.57 Using the concept of electrical image, it is easily seen that the force on the charge q is



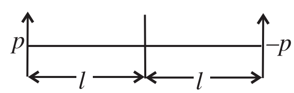
$$F = \frac{\sqrt{2}\,q^2}{4\pi\varepsilon_0 (2l)^2} + \frac{(-q)^2}{4\pi\varepsilon_0 (2\sqrt{2}\,l)^2}$$

$$= \frac{(2\sqrt{2} - 1)\,q^2}{32\pi\varepsilon_0 l^2} \text{ (It is an attractive force.)}$$

3.58 Using the concept of electrical image, force on the dipole **p**,

$$\mathbf{F} = p\,\frac{\partial \mathbf{E}}{\partial l}$$

(where **E** is field at the location of **p** due to (−**p**))



or $$|\mathbf{F}| = p\left|\frac{\partial \mathbf{E}}{\partial l}\right| = \frac{3p^2}{32\pi\varepsilon_0 l^4}$$

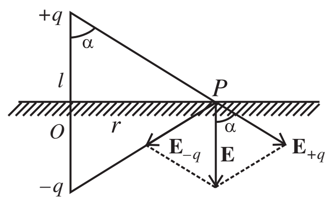
as $$|\mathbf{E}| = \frac{p}{4\,\pi\varepsilon_0 (2l)^3}$$

3.59 To find the surface charge density, we must know the electric field at the point P (see figure) which is at a distance r from the point O.

Using the image mirror method, the field at P,

$$E = 2E\cos\alpha = 2\,\frac{q}{4\pi\varepsilon_0 x^2}\,\frac{1}{x} = \frac{ql}{2\pi\varepsilon_0 (l^2 + r^2)^{3/2}}$$

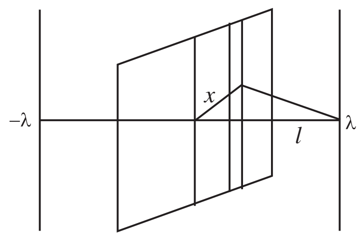
Now from Gauss' theorem, the surface charge density on conductor is connected with the electric field near its surface (in vaccum) through the relation $\sigma = \varepsilon_0 E_n$, where E_n is the projection of $\mathbf{E}$ onto the outward normal $\mathbf{n}$ (with respect to the conductor).



As our field strength $\mathbf{E} \uparrow\downarrow \mathbf{n}$, so

$$\sigma = -\varepsilon_0 E = -\frac{ql}{2\pi(l^2 + r^2)^{3/2}}$$

3.60 (a) The force F_1 on unit length of the thread is given by



$$F_1 = \lambda E_1$$

where E_1 is the field at the thread due to image charge given by

$$E_1 = \frac{-\lambda}{2\pi\varepsilon_0 (2l)}$$

Thus, $$F_1 = -\frac{\lambda^2}{4\pi\varepsilon_0 l}$$

Here, minus sign means that the force is one of attraction.

(b) There is an image thread with charge density $-\lambda$ behind the conducting plane. We calculate the electric field on the conductor considering the thread and its image. It is

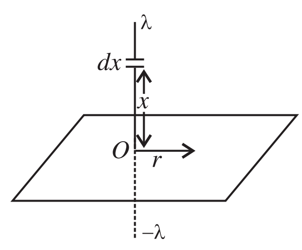
$$E(x) = E_n(x) = \frac{\lambda l}{\pi\varepsilon_0 (x^2 + l^2)}$$

Thus, $$\sigma(x) = \varepsilon_0 E_n = \frac{\lambda l}{\pi (x^2 + l^2)}$$

3.61 (a) At O

$$E_n(O) = 2\int_l^\infty \frac{\lambda\, dx}{4\pi\varepsilon_0 x^2} = \frac{\lambda}{2\pi\varepsilon_0 l}$$

So, $$\sigma(O) = \varepsilon_0 E_n = \frac{\lambda}{2\pi l}$$



(b)
$$E_n(r) = 2\int_l^\infty \frac{\lambda dx}{4\pi\varepsilon_0 (x^2 + r^2)} \frac{x}{(x^2 + r^2)^{1/2}} = \frac{\lambda}{2\pi\varepsilon_0} \int_l^\infty \frac{x\,dx}{(x^2 + r^2)^{3/2}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0} \int_{l^2+r^2}^\infty \frac{dy}{y^{3/2}} \quad \text{(putting } y = x^2 + r^2\text{)}$$

$$= \frac{\lambda}{2\pi\varepsilon_0\sqrt{l^2 + r^2}}$$

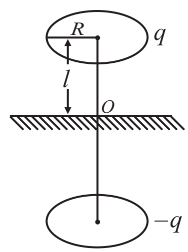
Hence,
$$\sigma(r) = \varepsilon_0 E_n = \frac{\lambda}{2\pi\sqrt{l^2 + r^2}}$$

3.62 (a) It can be easily seen that in accordance with the image method, a charge $-q$ must be located on a similar ring but on the other side of the conducting plane (see figure) at the same perpendicular distance. From the solution of Problem 3.9, electric field strength on axis of a ring of radius R at a distance l from it center is

$$\frac{ql}{4\pi\varepsilon_0 (R^2 + l^2)^{3/2}}$$

In our case this expression must be doubled. Hence, net field strength at point O

$$\mathbf{E} = 2\frac{ql}{4\pi\varepsilon_0(R^2 + l^2)^{3/2}}(-\mathbf{n})$$



where $\mathbf{n}$ is unit vector outward normal to the conducting plane.

Now,
$$E_n = \frac{\sigma}{\varepsilon_0}$$

Hence,
$$\sigma = \frac{-ql}{2\pi(R^2 + l^2)^{3/2}}$$

where minus sign indicates that the induced charge is opposite in sign to that of charge $q > 0$.

(b) The net field strength at the center of the ring is vector sum of field strengths setup by the charges q and $-q$ and is given by

$$\mathbf{E} = 0 + \frac{q(2l)}{4\pi\varepsilon_0 [R^2 + (2l)^2]^{3/2}}(-\mathbf{n}) \quad \text{(using result of Problem 3.9)}$$

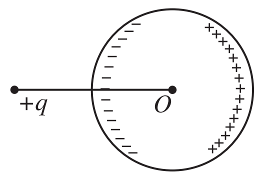
The potential at the center of the ring is equal to the algebraic sum of the potentials at this point created by the charges q and $-q$, given by

$$\varphi = \frac{q}{4\pi\varepsilon_0 R} - \frac{q}{4\pi\varepsilon_0\sqrt{R^2 + (2l)^2}}$$
$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R} - \frac{q}{\sqrt{R^2 + 4l^2}}\right)$$

3.63 Potential φ is the same for all the points of the sphere. Thus we can calculate its value at the center O of the sphere, because only for this point, it can be calculated in the most simple way.

So,
$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{l} + \varphi' \tag{1}$$

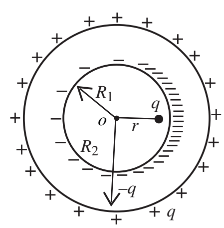
where the first term is the potential of the charge q, while the second is the potential due to the charges induced on the surface of the sphere. But since all induced charges are at the same distance equal to the radius of the circle from the point C and the total induced charge is equal to zero, so $\varphi' = 0$.



Thus Eq. (1) is reduced to the form,

$$\varphi = \frac{1}{4\pi\varepsilon_0}\frac{q}{l}$$

3.64 Point charge q is not placed at the center of the sphere, so the induced negative charge $-q$ on the inner surface is not distributed uniformly. The left over charge $+q$ on the outer surface of the conductor distributes itself uniformly so that the field strength due to the left over charge at inside point of the conductor becomes zero. Hence, the potential at O is given by



$$\varphi_0 = \frac{q}{4\pi\varepsilon_0 r} + \frac{\int(-\sigma)\,ds}{4\pi\varepsilon_0 R_1} + \frac{q}{4\pi E_0 R_2}$$

$$\varphi_0 = \frac{q}{4\pi\varepsilon_0 r} + \frac{(-q)}{4\pi\varepsilon_0 R_1} + \frac{q}{4\varepsilon_0\pi R_2}$$

It should be noticed that the potential can be found in such a simple way only at O, since all the induced charges on the inner surface are at the same distance from this point, and their distribution (which is unknown to us) does not play any role.

3.65 Potential at the inside sphere is given by

$$\varphi_a = \frac{q_1}{4\pi\varepsilon_0 a} + \frac{q_2}{4\pi\varepsilon_0 b}$$

Obviously, $\varphi_a = 0$ for $q_2 = -\dfrac{b}{a}q_1$ (1)

When $r \geq b$,

$$\varphi_r = \frac{q_1}{4\pi\varepsilon_0 r} + \frac{q_2}{4\pi\varepsilon_0} = \frac{q_1}{4\pi\varepsilon_0}\left(1-\frac{b}{a}\right)\Big/ r \quad \text{(using Eq. 1)}$$

And when $r \leq b$,

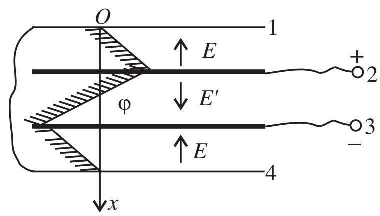
$$\varphi_r = \frac{q_1}{4\pi\varepsilon_0 r} + \frac{q_2}{4\pi\varepsilon_0 b} = \frac{q_1}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{a}\right)$$

3.66 (a) As the metallic plates 1 and 4 are isolated and connected by means of a conductor, $\varphi_1 = \varphi_4$. Plates 2 and 3 have the same amount of positive and negative charges and due to induction, plates 1 and 4 are respectively negatively and positively charged and in addition to it all the four plates are located at a small but equal distance d relative to each other. Hence, electric field strength between 1–2 and 3–4 is the same (say $\mathbf{E}$). Let $\mathbf{E}'$ be the field strength between the plates 2 and 3, which is directed from 2 to 3. Hence $\mathbf{E}' \uparrow\downarrow \mathbf{E}$ (see figure).

According to the problem

$$E'd = \Delta\varphi = \varphi_2 - \varphi_3 \qquad (1)$$

In addition,



$$\varphi_1 - \varphi_4 = (\varphi_1 - \varphi_2) + (\varphi_2 - \varphi_3) + (\varphi_3 - \varphi_4)$$

or $\quad 0 = -Ed + \Delta\varphi - Ed$

or $\quad \Delta\varphi = 2Ed \Rightarrow E = \dfrac{\Delta\varphi}{2d} = \dfrac{E'}{2} \qquad (2)$

Distribution of potential φ is also shown in the figure.

(b) Since $E \propto \sigma$, we can state according to Eq. (2) for part (a), that the charge on rge on the plate 2 is divided into two parts; such that 1/3rd of it lies on the upper side and 2/3rd on its lower face.

Thus charge density of upper face of plate 2 or of plate 1 or plate 4 and lower face of 3 is

$$\sigma = \varepsilon_0 E = \frac{\varepsilon_0 \Delta\varphi}{2d}$$

and charge density of lower face of 2 or upper face of 3 is

$$\sigma' = \varepsilon_0 E' = \varepsilon_0 \frac{\Delta\varphi}{d}$$

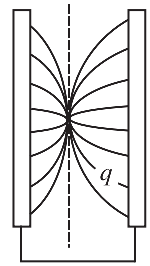
Hence, the net charge density of plate 2 or 3 becomes

$$\sigma + \sigma' = \frac{3\varepsilon_0 \Delta\varphi}{2d}$$

which is obvious from the argument.

3.67 The problem of point charge between two conducting planes is more easily tackled (if we want only the total charge induced on the planes) if we replace the point charge by a uniformly charged plane sheet.

Let σ be the charge density on this sheet and E_1, E_2 be the outward electric field on the on the two sides of this sheet.



Then,
$$E_1 + E_2 = \frac{\sigma}{\varepsilon_0}$$

The conducting planes will be assumed to be grounded.

Then,
$$E_1 x = E_2 (l - x)$$

Hence,
$$E_1 = \frac{\sigma}{l\varepsilon_0}(l - x) \quad \text{and} \quad E_2 = \frac{\sigma}{l\varepsilon_0} x$$

This means that the induced charge densities on the plane conductors are

$$\sigma_1 = -\frac{\sigma}{l}(l - x) \quad \text{and} \quad \sigma_2 = -\frac{\sigma}{l} x$$

Hence,
$$q_1 = -\frac{q}{l}(l - x) \quad \text{and} \quad q_2 = -\frac{q}{l} x$$

3.68 Near the conductor

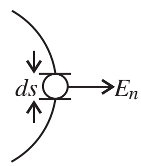
$$E = E_n \frac{\sigma}{\varepsilon_0}$$

This field can be written as the sum of two parts E_1 and E_2. E_1 is the electric field due to an infinitesimal area dS. Very near it

$$E_1 = \pm\frac{\sigma}{2\varepsilon_0}$$

The remaining part contributes

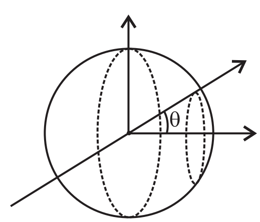
$$E_2 = \frac{\sigma}{2\varepsilon_0} \quad \text{(on both sides)}$$



In calculating the force on the element dS, we drop E_1 (because it is a self-force). Thus,

$$\frac{dF}{dS} = \sigma \frac{\sigma}{2\varepsilon_0} = \frac{\sigma^2}{2\varepsilon_0}$$

3.69 The total force on the hemisphere is



$$F = \int_0^{\pi/2} \frac{\sigma^2}{2\varepsilon_0}. \cos\theta \cdot 2\pi R \sin\theta R \, d\theta$$

$$= \frac{2\pi R^2 \sigma^2}{2\varepsilon_0} \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta$$

$$= \frac{2\pi R^2}{2\varepsilon_0} \times \frac{1}{2} \times \left(\frac{q}{4\pi R^2}\right)^2 = \frac{q^2}{32\pi\varepsilon_0 R^2}$$

$= 0.5$ kN (on substituting values)

3.70 We know that the force acting on the area element dS of a conductor is,

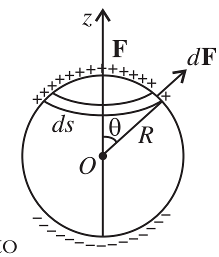
$$d\mathbf{F} = \frac{\sigma^2}{2\varepsilon_0} d\mathbf{S} \tag{1}$$

If follows from symmetry considerations that the resultant force $\mathbf{F}$ is directed along the z-axis, and hence, it can be represented as the sum (integral) of the projection of elementary forces given in Eq. (1) onto the z-axis, so

$$dF_z = dF \cos\theta \tag{2}$$

For simplicity, let us consider an element area $dS = 2\pi R \sin\theta \; d\theta$ (see figure).

Now, Eq. (2) takes the form



$$dF_z = \frac{\pi\sigma^2 R^2}{\varepsilon_0} \sin\theta\cos\theta\, d\theta$$

$$= -\left(\frac{\pi\sigma_0^2 R^2}{\varepsilon_0}\right)\cos^3\theta\, d\cos\theta$$

Integrating this expression over the half sphere, i.e., with respect to $\cos\theta$ between 1 and 0, we obtain

$$F = F_z = \frac{\pi\sigma_0^2 R^2}{4\varepsilon_0}$$

3.71 The total polarization is $P = (\varepsilon - 1)\varepsilon_0 E$. This must be equal to $n_0 p/N$, where n_0 is the concentration of water molecules. Thus,

$$N = \frac{n_0 P}{(\varepsilon - 1)\varepsilon_0 E} = 2.93 \times 10^3 \text{ (on substituting values)}$$

3.72 From the general formula in vector form

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{(3\mathbf{p}\cdot\mathbf{e}_r)\mathbf{e}_r - \mathbf{p}}{r^3}$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\mathbf{p}}{l^3} \quad (\text{where } r = l \text{ and } \mathbf{r} \uparrow\uparrow \mathbf{p})$$

This will cause the induction of a dipole moment

$$\mathbf{p}_{\text{ind}} = \beta \frac{1}{4\pi\varepsilon_0} \frac{2\mathbf{p}}{l^3} \times \varepsilon_0$$

Thus, the force

$$|\mathbf{F}| = \frac{\beta}{4\pi} \frac{2p}{l^3} \frac{\delta}{\delta l} \frac{1}{4\pi\varepsilon_0} \frac{2p}{l^3} = \frac{3\beta p^2}{4\pi^2\varepsilon_0 l^7}$$

3.73 The electric field E at distance x from the center of the ring is

$$E(x) = \frac{qx}{4\pi\varepsilon_0(R^2 + x^2)^{3/2}}$$

The induced dipole moment is

$$p = \beta\varepsilon_0 E = \frac{q\,\beta x}{4\pi(R^2 + x^2)^{3/2}}$$

The force on this molecule is

$$F = p\frac{\partial}{\partial x} E = \frac{q\beta x}{4\pi(R^2 + x^2)^{3/2}} \frac{q}{4\pi\varepsilon_0} \frac{\partial}{\partial x} \frac{x}{(R^2 + x^2)^{3/2}} = \frac{q^2\beta}{16\pi^2\varepsilon_0} \frac{x(R^2 - 2x^2)}{(R^2 + x^2)}$$

This vanishes for $x = \dfrac{\pm R}{\sqrt{2}}$ (apart from $x = 0$, $x = \infty$).

It is maximum when

$$\frac{\partial}{\partial x} \frac{x(R^2 - x^2 \times 2)}{(R^2 + x^2)^4} = 0$$

or $$(R^2 - 2x^2)(R^2 + x^2) - 4x^2(R^2 + x^2) - 8x^2(R^2 - 2x^2) = 0$$

or $$R^4 - 13x^2R^2 + 10x^4 = 0 \quad \text{or} \quad x^2 = \frac{R^2}{20}(13 \pm \sqrt{129})$$

or $$x = \frac{R}{\sqrt{20}}\sqrt{13 \pm \sqrt{129}} \text{ (on either side).}$$

Plot of $F_x(x)$ is as shown in the answer sheet.

3.74 Inside the ball

$$\mathbf{D}(\mathbf{r}) = \frac{q}{4\pi} \frac{\mathbf{r}}{r^3} = \varepsilon\varepsilon_0\mathbf{E}$$

Also $$\varepsilon_0\mathbf{E} + \mathbf{P} = \mathbf{D} \quad \text{or} \quad \mathbf{P} = \frac{\varepsilon - 1}{\varepsilon}\mathbf{D} = \frac{\varepsilon - 1}{\varepsilon} \frac{q}{4\pi} \frac{\mathbf{r}}{r^3}$$

Also, $$q' = -\oint \mathbf{P} \cdot d\mathbf{S} = -\frac{\varepsilon - 1}{\varepsilon} \frac{q}{4\pi} \int d\Omega = \frac{\varepsilon - 1}{\varepsilon} q$$

3.75 $D_{\text{diel}} = \varepsilon\varepsilon_0 E_{\text{diel}} = D_{\text{conductor}} = \sigma$ or $E_{\text{diel}} = \dfrac{\sigma}{\varepsilon\varepsilon_0}$

$$P_n = (\varepsilon - 1)\varepsilon_0 E_{\text{diel}} = \frac{\varepsilon - 1}{\varepsilon}\sigma$$

$$\sigma' = -P_n = -\frac{\varepsilon - 1}{\varepsilon}\sigma$$

This is the surface density of bound charges.

3.76 From the solution of the Problem 3.74, charge on the interior surface of the conductor is given by

$$q'_{\text{in}} = -\frac{\varepsilon - 1}{\varepsilon}\int \sigma\, dS = -\frac{\varepsilon - 1}{\varepsilon}q$$

Since the dielectric as a whole is neutral, there must be a total charge equal to

$$q'_{\text{outer}} = +\frac{\varepsilon - 1}{\varepsilon}\ q$$

on the outer surface of the dielectric.

3.77 (a) Positive extraneous charge is distributed uniformly over the internal surface layer. Let σ_0 be the surface density of the charge.

Clearly, for $r < a$ $\qquad E = 0$

By Gauss' theorem:

For $a < r$, $\qquad \varepsilon_0 E \times 4\pi r^2 = 4\pi a^2 \sigma_0$

or $\qquad E = \dfrac{\sigma_0}{\varepsilon_0\,\varepsilon}, \left(\dfrac{a}{r}\right)^2 \quad (a < r < b)$

For $r > b$, similarly

$$E = \frac{\sigma_0}{\varepsilon_0}\left(\frac{a}{r}\right)^2$$

Now, $\qquad E = -\dfrac{\partial\varphi}{\partial r}$

So by integration from infinity, where $\varphi(\infty) = 0$, we get

$$\varphi = \frac{\sigma_0 a^2}{\varepsilon_0 r} \quad (r > b)$$

For $a < r < b$, $\qquad \varphi = \dfrac{\sigma_0 a^2}{\varepsilon\varepsilon r} + B \quad$ (where B is a constant)

By continuity, $\varphi = \frac{\sigma_0 a^2}{\varepsilon_0 \varepsilon}\left(\frac{1}{r} - \frac{1}{b}\right) + \frac{\sigma_0 a^2}{\varepsilon_0 b}$

For $r < a$, $\varphi = A = \text{constant}$

By continuity, $\varphi = \frac{\sigma_0 a^2}{\varepsilon_0 \varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{\sigma_0 a^2}{\varepsilon_0 b}$

(b) Positive extraneous charge is distributed uniformly over the internal volume of the dielectric. Let ρ_0 = volume density of the charge in the dielectric, for $a < r < b$. Then,

$$E = 0 \quad (\text{for } r < a)$$

and $$\varepsilon_0\, \varepsilon\, 4\pi r^2 E = \frac{4\pi}{3}(r^3 - a^3)\,\rho_0 \quad (\text{for } a < r < b)$$

or $$E = \frac{\rho_0}{3\varepsilon_0\varepsilon}\left(r - \frac{a^3}{r^2}\right)$$

$$E = \frac{4\pi}{3}\frac{(b^3 - a^3)\rho_0}{\varepsilon_0 4\pi r^2} \quad (\text{for } r > b)$$

or $$E = \frac{(b^3 - a^3)\rho_0}{3\varepsilon_0 r^2} \quad (\text{for } r > b)$$

By integration,

$$\varphi = \frac{(b^3 - a^3)}{3\varepsilon_0 r} \quad (\text{for } r > b)$$

or $$\varphi = B - \frac{\rho_0}{3\varepsilon_0\varepsilon}\left(\frac{r^2}{2} + \frac{a^3}{r}\right) \quad (\text{for } a < r < b)$$

By continuity,

$$\frac{b^3 - a^3}{3\varepsilon_0 b}\rho_0 = B - \frac{\rho_0}{3\varepsilon_0\varepsilon}\left(\frac{b^2}{2} + \frac{a^3}{b}\right)$$

or $$B = \frac{\rho_0}{3\varepsilon_0\varepsilon}\left\{\frac{\varepsilon(b^3 - a^3)}{b} + \left(\frac{b^2}{2} + \frac{a^3}{b}\right)\right\}$$

Finally $$\varphi = B - \frac{\rho_0}{3\varepsilon_0\varepsilon}\left\{\left(\frac{a^2}{2} + a^2\right) = B - \frac{\rho_0 a^2}{2\varepsilon_0\varepsilon} \quad (\text{for } r < a)\right.$$

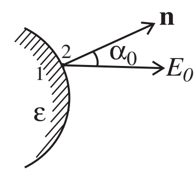
On the basis of obtained expressions, $E(r)$ and $(\varphi)(r)$ can be plotted as shown in the answer sheet.

3.78 Let us take points 1 and 2 just left and right of the interface. Point 2 can be treated like A given in the problem book. Continuity of tangential component of **E** at the interface gives

$$E_{1t} = E_{2t}$$

$$E_{1t} = E_0 \sin \alpha_0$$

As there is no free charge at the interface, so normal component of **D** has continuity at the interface.



Hence,
$$D_{1n} = D_{2n}$$

$$\varepsilon_0 \varepsilon E_{1n} = \varepsilon_0 E_0 \cos \alpha_0$$

$$E_{1n} = \frac{E_0 \cos \alpha_0}{\varepsilon}$$

So,
$$E_1 = \sqrt{E_{1t}^2 + E_{1n}^2}$$

$$\tan \alpha = \frac{E_{1t}}{E_{1n}} = \varepsilon \tan \alpha_0$$

From the boundary condition of polarization vector **P**, we get

$$P_{2n} - P_{1n} = -\sigma'$$

$$0 - P_{1n} = -\sigma'$$

$$P_{1n} = \sigma'$$

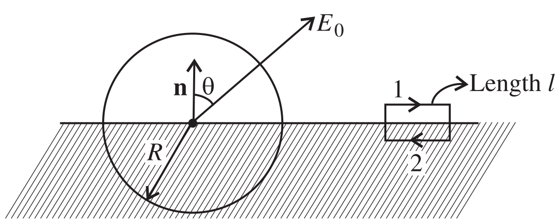
$$\varepsilon_0(\varepsilon - 1) E_{1n} = \sigma'$$

$$\varepsilon_0 (\varepsilon - 1) \frac{E_0 \cos \alpha_0}{\varepsilon} = \sigma'$$

$$\sigma' = \frac{\varepsilon_0 (\varepsilon - 1)}{\varepsilon} E_0 \cos \alpha_0$$

3.79 From the previous problem, we know

$$\sigma' = \varepsilon_0 \frac{\varepsilon - 1}{\varepsilon} E_0 \cos \theta$$



(a) Then, $\oint \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} Q = \pi R^2 E_0 \cos \theta \, \frac{\varepsilon - 1}{\varepsilon}$

(b) $$\oint \mathbf{D} \cdot d\mathbf{r} = (D_{1t} - D_{2t})\, l = (\varepsilon_0 E_0 \sin\theta - \varepsilon\varepsilon_0 E_0 \sin\theta)$$

$$l = -(\varepsilon - 1)\,\varepsilon_0 E_0 l \sin\theta$$

3.80 (a) $$\text{div } D = \frac{\partial D_x}{\partial x} = \rho \quad \text{and} \quad D = \rho l$$

$$E_x = \frac{\rho l}{\varepsilon\varepsilon_0} \quad (\text{for } l < d)$$

and $$E_x = \frac{\rho d}{\varepsilon_0} \text{ constant (for } l > d)$$

Also, $$\varphi(x) = -\frac{\rho l^2}{2\varepsilon\rho\varepsilon_0} \quad (\text{for } l < d) \quad \text{and} \quad \varphi(x) = A - \frac{\rho l d}{\varepsilon_0} \quad (\text{for } l > d)$$

By continuity, $$\varphi(x) = \frac{\rho d}{\varepsilon_0}\left(d - \frac{d}{2\varepsilon} - 1\right)$$

On the basis of obtained expressions, $E_x(x)$ and $\varphi(x)$ can be plotted as shown in the figure of answer sheet.

(b) $$\rho' = -\text{div } \mathbf{P} = -\text{div } (\varepsilon - 1)\,\varepsilon_0 \boldsymbol{E} = -\rho\,\frac{(\varepsilon - 1)}{\varepsilon}$$

$$\sigma' = P_{1n} - P_{2n} \text{ (where } n \text{ is the normal from 1 to 2)}$$

$$= P_{1n} \,(\mathbf{P}_2 = 0, \text{ as 2 is vacuum})$$

$$= (\rho d - \rho d/\varepsilon) = \rho d\,\frac{\varepsilon - 1}{\varepsilon}$$

3.81 (a) $$\text{div } \mathbf{D} = \frac{l}{r^2}\frac{\partial}{\partial r}(r^2 D_r) = \rho$$

$$r^2 D_r = \rho\,\frac{r^3}{3} + A D_r = \frac{1}{3}\rho r + \frac{A}{r^2},\ r < R$$

$$A = 0 \text{ as } D_r \neq \infty \text{ at } r = 0,$$

Thus, $$E_r = \frac{\rho r}{3\varepsilon\varepsilon_0}$$

For $r > R$, $$D_r = \frac{B}{r^2}$$

By continuity of D_r at $r = R$,

$$B = \frac{\rho r^3}{3}$$

So, $$E_r = \frac{\rho R^3}{3\varepsilon_0 r^2} \quad (\text{for } r > R)$$

$$\varphi = \frac{\rho R^3}{3\varepsilon_0 r} \quad (\text{for } r > R) \quad \text{and} \quad \varphi = -\frac{\rho r^2}{6\varepsilon\varepsilon_0} + C \quad (\text{for } r < R)$$

By continuity of φ $C = +\dfrac{\rho R^2}{3\varepsilon_0} + \dfrac{\rho R^2}{6\varepsilon\varepsilon_0}$

See answer sheet for graphs of $E(r)$ and $\varphi(r)$.

(b) $\rho' = \operatorname{div} \mathbf{P} = -\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left\{\dfrac{r^3}{3}\rho\left(1 - \dfrac{1}{\varepsilon}\right)\right\} = -\dfrac{\rho(\varepsilon - 1)}{\varepsilon}$

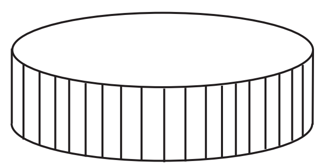
$\sigma' = P_{1r} - P_{2r} = P_{1r} = \dfrac{1}{3}\rho R\left(1 - \dfrac{1}{\varepsilon}\right)$

3.82 Because there is a discontinuity in polarization at the boundary of the dielectric disk, a bound surface charge appears, which is the source of the electric field inside and outside the disk. We have for the electric field at the origin,

$$\mathbf{E} = -\int \frac{\sigma' \, dS}{4\pi\varepsilon_0 r^3} \mathbf{r}$$

where $\mathbf{r}$ = radius vector to the origin from the element dS. $\sigma' = P_n = P\cos\theta$ on the curved surface (as $P_n = 0$ on the flat surface). Here θ = angle between $\mathbf{r}$ and $\mathbf{P}$. By symmetry, $\mathbf{E}$ will be parallel to $\mathbf{P}$.

Thus, $$E \cong -\int_0^{2\pi} \frac{P\cos\theta \; Rd\theta \cdot \cos\theta}{4\pi\varepsilon_0 r^2} \cdot d$$



where, $r = R$, if $d \ll R$.

So, $$E \cong -\frac{Pd}{4\varepsilon_0 R}$$

and $$\mathbf{E} = -\frac{\mathbf{P}d}{4\varepsilon_0 R}$$

3.83 Since there are no free extraneous charges anywhere

$$\operatorname{div} \mathbf{D} = \frac{\partial D_x}{\partial x} = 0 \quad \text{or} \quad D_x = \text{constant}$$

But $\quad D_x = 0$ at $\infty \quad$ so, $\quad D_x = 0$, everywhere

Thus, $$\mathbf{E} = -\frac{\mathbf{P}_0}{\varepsilon_0}\left(1 - \frac{x^2}{d^2}\right) \quad \text{or} \quad E_x = -\frac{P_0}{\varepsilon_0}\left(1 - \frac{x^2}{d^2}\right)$$

So, $$\varphi = \frac{P_0 x}{\varepsilon_0} - \frac{P_0 x^3}{3\varepsilon_0 d^2} + \text{constant}$$

Hence, $$\varphi(+d) - \varphi(-d) = \frac{2P_0 d}{\varepsilon_0} - \frac{2P_0 d^3}{3d^2\varepsilon_0} = \frac{4P_0 d}{3\varepsilon_0}$$

3.84 (a) We have $$D_1 = D_2 \quad \text{or} \quad \varepsilon E_2 = E_1$$

Also, $$E_1\frac{d}{2} + E_2\frac{d}{2} = E_0\, d \;\Rightarrow\; E_1 + E_2 = 2E_0$$

Hence, $$E_2 = \frac{2E_0}{\varepsilon + 1} \quad \text{and} \quad E_1 = \frac{2\varepsilon E_0}{\varepsilon + 1}$$

and $$D_1 = D_2 = \frac{2\varepsilon\varepsilon_0 E_0}{\varepsilon + 1}$$

(b) We have $$D_1 = D_2 \quad \text{or} \quad \varepsilon E_2 = E_1 = \frac{\sigma}{\varepsilon_0} = E_0$$

Thus, $$E_1 = E_0 \quad \text{and} \quad E_2 = \frac{E_0}{\varepsilon}$$

and $$D_1 = D_2 = \varepsilon_0 E_0$$

3.85 (a) Constant voltage across the plates implies

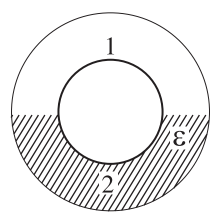
$$E_1 = E_2 = E_0,\; D_1 = \varepsilon_0 E_0,\; D_2 = \varepsilon_0\varepsilon E_0$$

(b) Constant charge across the plates implies

$$E_1 = E_2 \;\; D_1 = \varepsilon_0 E_1,\; D_2 = \varepsilon\,\varepsilon_0 E_2 = \varepsilon D_1$$

So, $$E_1(1 + \varepsilon) = 2\,E_0 \quad \text{or} \quad E_1 = E_2 = \frac{2E_0}{\varepsilon + 1}$$

3.86 At the interface of the dielectric and vacuum,



$$E_{1t} = E_{2t}$$

The electric field must be radial and

$$E_1 = E_2 = \frac{A}{\varepsilon_0\varepsilon r^2} \quad (\text{for } a < r < b)$$

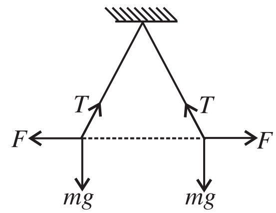
Now, $$q = \frac{A}{R^2}(2\pi R^2) + \frac{A}{\varepsilon R^2}(2\pi R^2)$$

$$= A\left(1 + \frac{1}{\varepsilon}\right)2\pi$$

or $$E_1 = E_2 = \frac{q}{2\pi\varepsilon_0 r^2(1 + \varepsilon)}$$

3.87 In air the forces are as shown. In K–oil,

$$F \to F' = F/\varepsilon \quad \text{and} \quad mg \to mg\left(1 - \frac{\rho_0}{\rho}\right)$$



where ρ_0 is the density of K–oil and ρ that of the material of which the balls are made.

Since the inclinations do not change

$$\frac{1}{\varepsilon} = 1 - \frac{\rho_0}{\rho}$$

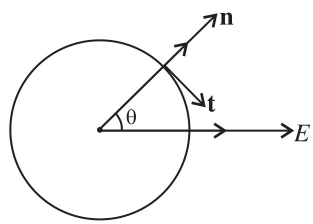
or $$\frac{\rho_0}{\rho} = 1 - \frac{1}{\varepsilon} = \frac{\varepsilon - 1}{\varepsilon}$$

or $$\rho = \rho_0 \frac{\varepsilon}{\varepsilon - 1}$$

$$= 1.6 \text{ g/cm}^3 \text{ (on substituting values)}$$

3.88 Within the ball, the electric field can be resolved into normal and tangential components,

$$E_n = E\cos\theta \quad \text{and} \quad E_t = E\sin\theta$$



Then, $$D_n = \varepsilon\varepsilon_0 E\cos\theta$$

and $$P_n = (\varepsilon - 1)\,\varepsilon_0 E\cos\theta$$

or $$\sigma' = (\varepsilon - 1)\,\varepsilon_0 E\cos\theta$$

So, $$\sigma_{\max} = (\varepsilon - 1)\,\varepsilon_0 E,$$

$$= 3.5 \text{ nC/m}^2 \text{ (on substituting values)}$$

and total charge of one sign,

$$q' = \int_0^1 (\varepsilon - 1)\,\varepsilon_0 E\cos\theta\; 2\pi R^2 d(\cos\theta)$$

$$= \pi R^2 \varepsilon_0 (\varepsilon - 1)\,E = 10 \text{ pC (on substituting values)}$$

(Since we are interested in the total charge of one sign, we must integrate $\cos\theta$ from 0 to 1 only).

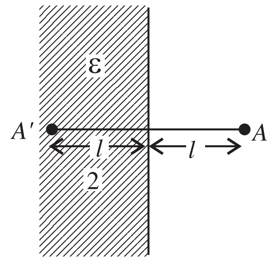
3.89 The charge is at A in the medium 1 and has an image point at A' in the medium 2. The electric field in the medium 1 is due to the actual charge q at A and the image charge q' at A'. The electric field in 2 is due to a corrected charge q'' at A. Thus, on the boundary between 1 and 2

$$E_{1n} = \frac{q'}{4\pi\varepsilon_0 r^2}\cos\theta - \frac{q}{4\pi\varepsilon_0 r^2}\cos\theta$$

$$E_{2n} = \frac{-q''}{4\pi\varepsilon_0 r^2}\cos\theta$$

$$E_{1t} = \frac{q'}{4\pi\varepsilon_0 r^2}\sin\theta + \frac{q}{4\pi\varepsilon_0 r^2}\sin\theta$$

$$E_{2t} = \frac{q''}{4\pi\varepsilon_0 r^2}\sin\theta$$



The boundary conditions are

$$D_{1n} = D_{2n} \text{ and } E_{1t} = E_{2t}$$

$$\varepsilon q'' = q - q'$$

$$q'' = q + q'$$

So, $$q'' = \frac{2q}{\varepsilon + 1} \quad \text{and} \quad q' = -\frac{\varepsilon - 1}{\varepsilon + 1} q$$

(a) The surface density of the bound charge on the surface of the dielectric is

$$\sigma' = P_{2n} = D_{2n} - \varepsilon_0 E_{2n} = (\varepsilon - 1)\,\varepsilon_0 E_{2n}$$

$$= -\frac{\varepsilon - 1}{\varepsilon + 1}\frac{q}{2\pi r^2}\cos\theta = -\frac{\varepsilon - 1}{\varepsilon + 1}\frac{ql}{2\pi r^3}$$

(b) Total bound charge is

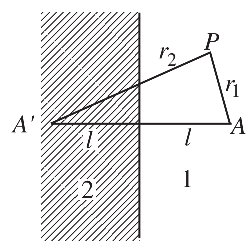
$$-\frac{\varepsilon - 1}{\varepsilon + 1} q \int_0^\infty \frac{l}{2\pi\,(l^2 + x^2)^{3/2}}\, 2\pi x\, dx = -\frac{\varepsilon - 1}{\varepsilon + 1} q$$

3.90 The force on the point charge q is due to the bound charges. This can be calculated from the field at this charge after extracting out the self field. This image field is

$$E_{\text{image}} = \frac{\varepsilon - 1}{\varepsilon + 1}\frac{q}{4\pi\varepsilon_0\,(2l)^2}$$

Thus, $$F = \frac{\varepsilon - 1}{\varepsilon + 1}\frac{q^2}{16\pi\varepsilon_0 l^2}$$

3.91 Given $$\mathbf{E}_P = \frac{q\mathbf{r}_1}{4\pi\varepsilon_0 r_1^3} + \frac{q'\mathbf{r}_2}{4\pi r_2^3\varepsilon_0} \quad (\text{for } P \text{ in } 1)$$



and $$\mathbf{E}_P = \frac{q''\,\mathbf{r}_1}{4\pi\varepsilon_0 r_1^3} \quad (\text{for } P \text{ in } 2)$$

where $$q'' = \frac{2q}{\varepsilon + 1} \quad \text{and} \quad q' = q'' - q$$

In the limit $\mathbf{l} \to 0$

$$\mathbf{E}_\text{P} = \frac{(q + q')\mathbf{r}}{4\pi\varepsilon_0 r^3} = \frac{q\mathbf{r}}{2\pi\varepsilon_0(1 + \varepsilon)\, r^3} \quad (\text{in either part})$$

Thus, $$E_\text{P} = \frac{q}{2\pi\varepsilon_0\,(1 + \varepsilon) r^2}$$

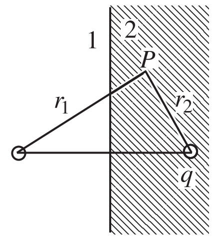
$$\varphi = \frac{q}{2\pi\varepsilon_0 (1+\varepsilon)r}$$

$$D = \frac{q}{2\pi\varepsilon_0 (1+\varepsilon)\, r^2} \times \begin{cases} 1 \text{ in vacuum} \\ \varepsilon \text{ in dielectric} \end{cases}$$

3.92

$$\mathbf{E}_P = \frac{q\mathbf{r}_2}{4\pi\varepsilon_0 \varepsilon r_2^3} + \frac{q'\mathbf{r}_1}{4\pi\varepsilon_0 r_1^3} \quad \text{(for } P \text{ in 2)}$$

$$\mathbf{E}_P = \frac{q''\mathbf{r}_2}{4\pi\varepsilon_0 r_2^3} \quad \text{(for } P \text{ in 1)}$$



Using the boundary conditions,

$$E_{1n} = \varepsilon\, E_{2n},\ E_{1t} = E_{2t}$$

This implies

$$q - \varepsilon q' = q'' \quad \text{and} \quad q + \varepsilon q' = \varepsilon q''$$

So,
$$q'' = \frac{2q}{\varepsilon + 1} \quad \text{and} \quad q' = \frac{\varepsilon - 1}{\varepsilon + 1}\frac{q}{\varepsilon}$$

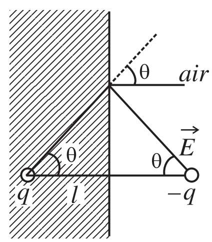
Then, as earlier,

$$\sigma' = \frac{ql}{2\pi r^3} \cdot \left(\frac{\varepsilon - 1}{\varepsilon + 1}\right) \cdot \frac{1}{\varepsilon}$$

3.93 To calculate the electric field, first we note that an image charge will be needed to ensure that the electric field on the metal boundary is normal to the surface.

The image charge must have magnitude $-q/\varepsilon$, so that the tangential component of the electric field may vanish. Now,

$$E_n = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\varepsilon r^2}\right) 2\cos\theta = \frac{ql}{2\pi\varepsilon_0 \varepsilon r^3}$$



Then,
$$P_n = D_n - \varepsilon_0 E_n = \frac{(\varepsilon - 1)\, ql}{2\pi\varepsilon r^3} = \sigma'$$

This is the density of bound charge on the surface.

3.94 Since the condenser plates are connected,

$$E_1 h + E_2 (d - h) = 0$$

and
$$P + \varepsilon_0 E_1 = \varepsilon_0 E_2$$

or
$$E_1 + \frac{P}{\varepsilon_0} = E_2$$

Thus,
$$E_2 d - \frac{Ph}{\varepsilon_0} = 0 \quad \text{or} \quad E_1 = \frac{Ph}{\varepsilon_0 d}$$

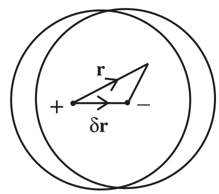
$$E_2 = -\frac{P}{\varepsilon_0}\left(1 - \frac{h}{d}\right)$$

3.95 Given $\mathbf{P} = \alpha\mathbf{r}$, where $\mathbf{r}$ = distance from the axis. The space density of charges is given by, $\rho' = -\text{div}\,\mathbf{P} = -2\alpha$. On using these, we get

$$\text{div}\,\mathbf{r} = \frac{1}{r}\;\frac{\partial(\mathbf{r}\cdot\mathbf{r})}{\partial r} = 2$$

3.96 In a uniformly charged sphere

$$\mathbf{E} = \frac{\rho_0}{3\varepsilon_0}\,\mathbf{r}$$



The total electric field is

$$\mathbf{E} = \frac{1}{3\varepsilon_0}\,\rho_0\mathbf{r} - \frac{1}{3\varepsilon_0}\,(\mathbf{r} - \delta\mathbf{r})\,\rho_0$$

$$= \frac{1}{3\varepsilon_0}\,\rho_0\,\delta\mathbf{r} = -\frac{\mathbf{P}}{3\varepsilon_0} \quad (\text{where } \rho\,\delta\mathbf{r} = -\mathbf{P})$$

The potential outside is

$$\varphi = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{\mathbf{r}} - \frac{Q}{|\,\mathbf{r} - \delta\mathbf{r}\,|}\right) = \frac{\mathbf{P}_0\cdot\mathbf{r}}{4\pi\varepsilon_0\mathbf{r}^3} \quad (\text{for } r > R)$$

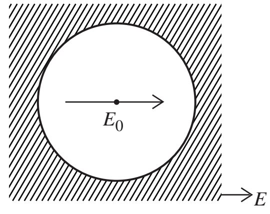
where $\mathbf{P}_0 = -\frac{4\pi}{3}R^3\rho_0\partial\mathbf{r}$, is the total dipole moment.

3.97 The electric field $\mathbf{E}_0$ in a spherical cavity in a uniform dielectric of permittivity ε is related to the far away field $\mathbf{E}$, in the following manner. Imagine the cavity to be filled up with the dielectric. Then there will be a uniform field $\mathbf{E}$ everywhere and a polarization $\mathbf{P}$, given by $\mathbf{P} = (\varepsilon - 1)\,\varepsilon_0\mathbf{E}$.

Now take out the sphere making the cavity, the electric field inside the sphere will be

$$-\frac{\mathbf{P}}{3\varepsilon_0}$$

By superposition,
$$\mathbf{E}_0 - \frac{\mathbf{P}}{3\varepsilon_0} = \mathbf{E}$$



or $$\mathbf{E}_0 = \mathbf{E} + \frac{1}{3}(\varepsilon - 1)\,\mathbf{E} = \frac{1}{3}(\varepsilon + 2)\,\mathbf{E}$$

3.98 By superposition the field $\mathbf{E}$ inside the ball is given by

$$\mathbf{E} = \mathbf{E}_0 - \frac{\mathbf{P}}{3\varepsilon_0}$$

On the other hand, if the sphere is not too small, then the macroscopic equation $\mathbf{P} = (\varepsilon - 1)\,\varepsilon_0\mathbf{E}$ must hold. Thus,

$$\mathbf{E}\left(1 + \frac{1}{3}(\varepsilon - 1)\right) = \mathbf{E}_0 \quad \text{or} \quad \mathbf{E} = \frac{3\mathbf{E}_0}{\varepsilon + 2}$$

Also, $$\mathbf{P} = 3\varepsilon_0\frac{\varepsilon - 1}{\varepsilon + 2}\mathbf{E}_0$$

3.99 This is to be handled by the same trick as in Problem 3.96. We have effectively a two dimensional situation. For a uniform cylinder full of charge, with charge density ρ_0, the electric field E at an inside point is along the (cylindrical) radius vector $\mathbf{r}$ and is given by

$$\mathbf{E} = \frac{1}{2\varepsilon_0}\rho\mathbf{r}$$

Since $$\operatorname{div}\mathbf{E} = \frac{l}{r}\frac{\partial}{\partial r}(rE_r) = \frac{\rho}{\varepsilon_0} \quad \text{hence,} \quad E_r = \frac{\rho}{2\varepsilon_0}\,r$$

Therefore the polarized cylinder can be thought of as two equal and opposite charge distributions displaced with respect to each other. So,

$$\mathbf{E} = \frac{1}{2\varepsilon_0}\rho\mathbf{r} - \frac{1}{2\varepsilon_0}\rho\,(\mathbf{r} - \delta\mathbf{r}) = \frac{1}{2\varepsilon_0}\rho\,\delta\mathbf{r} = -\frac{\mathbf{P}}{2\varepsilon_0}$$

using $\mathbf{P} = -\rho\delta\mathbf{r}$ (direction of electric dipole moment vector being from the negative charge to positive charge).

3.100 As in solution of Problem 3.98, we can write

$$\mathbf{E} = \mathbf{E}_0 - \frac{\mathbf{P}}{2\varepsilon_0}$$

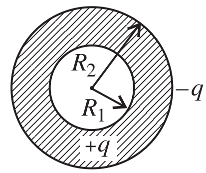
Also, $$\mathbf{P} = (\varepsilon - 1)\,\varepsilon_0\mathbf{E}$$

So, $$\mathbf{E}\left(\frac{\varepsilon + 1}{2}\right) = \mathbf{E}_0$$

or $$\mathbf{E} = \frac{2\mathbf{E}_0}{\varepsilon + 1} \quad \text{and} \quad \mathbf{P} = 2\,\varepsilon_0\frac{\varepsilon - 1}{\varepsilon + 1}\mathbf{E}_0$$

3.3 Electric Capacitance. Energy of an Electric Field

3.101 Let us impart an imaginary charge q to the conductor, then



$$\varphi_+ - \varphi_- = \int_{R_1}^{R_2} \frac{q}{4\pi\varepsilon_0 \varepsilon r^2}\, dr + \int_{R_2}^{\infty} \frac{q}{4\pi\varepsilon_0 r^2}\, dr$$

$$= \frac{q}{4\pi\varepsilon_0\varepsilon}\left[\frac{1}{R_1} - \frac{1}{R_2}\right] + \frac{q}{4\pi\varepsilon_0}\frac{1}{R_2}$$

$$= \frac{q}{4\pi\varepsilon_0\varepsilon}\left[\frac{(\varepsilon - 1)}{R_2} + \frac{1}{R_1}\right]$$

Hence the sought capacitance,

$$C = \frac{q}{\varphi_+ - \varphi_-} = \frac{q 4\pi\varepsilon_0\varepsilon}{q\left[\frac{(\varepsilon - 1)}{R_2} + \frac{1}{R_1}\right]} = \frac{4\pi\varepsilon_0\varepsilon R_1}{(\varepsilon - 1)\frac{R_1}{R_2} + 1}$$

3.102 From the symmetry of the problem, the voltage across each capacitor $\Delta\varphi = \xi/2$ and charge on each capacitor $q = C\xi/2$ in the absence of dielectric.

Now when the dielectric is filled up in one of the capacitors, the equivalent capacitance of the system is given by

$$C'_0 = \frac{C\varepsilon}{1 + \varepsilon}$$

and the potential difference across the capacitor, which is filled with dielectric is

$$\Delta\varphi' = \frac{q'}{\varepsilon C} = \frac{C\varepsilon}{(1 + \varepsilon)}\frac{\xi}{C\varepsilon} = \frac{\xi}{(1 + \varepsilon)}$$

But $\varphi \propto E$.

So, as φ decreases $1/2\,(1 + \varepsilon)$ times, the field strength also decreases by the same factor and flow of charge

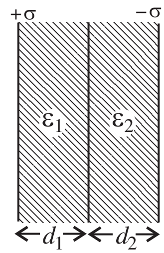
$$\Delta q = q' - q$$

$$= \frac{C\varepsilon}{(1 + \varepsilon)}\xi - \frac{C}{2}\xi = \frac{1}{2}C\xi\frac{(\varepsilon - 1)}{(\varepsilon + 1)}$$

3.103 (a) Since it is a series combination of two capacitors,

$$\frac{1}{C} = \frac{d_1}{\varepsilon_0\varepsilon_1 S} + \frac{d_2}{\varepsilon_0\varepsilon_2 S} \quad \text{or} \quad C = \frac{\varepsilon_0 S}{(d_1/\varepsilon_1) + (d_2/\varepsilon_2)}$$

(b) Let σ be the initial surface charge density, then density of bound charge on the boundary plane is given by

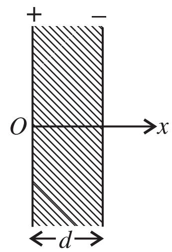


$$\sigma' = \sigma\left(1 - \frac{1}{\varepsilon_1}\right) - \sigma\left(1 - \frac{1}{\varepsilon_2}\right) = \sigma\left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)$$

But, $$\sigma = \frac{q}{S} = \frac{CV}{S} = \frac{\varepsilon_0 S \varepsilon_1 \varepsilon_2}{\varepsilon_2 d_1 + \varepsilon_1 d_2}\frac{V}{S}$$

So, $$\sigma' = \frac{\varepsilon_0 V(\varepsilon_1 - \varepsilon_2)}{\varepsilon_2 d_1 + \varepsilon_1 d_2}$$

3.104 (a) We point the x-axis towards the right and place the origin on the left hand side plate. The left hand side plate is assumed to be positively charged. Since ε varies linearly, we can write,



$$\varepsilon(x) = a + bx$$

where a and b can be determined from the boundary condition. We have,

$$\varepsilon = \varepsilon_1 \text{ at } x = 0 \quad \text{and} \quad \varepsilon = \varepsilon_2 \text{ at } x = d$$

Thus, $$\varepsilon(x) = \varepsilon_1 + \left(\frac{\varepsilon_2 - \varepsilon_1}{d}\right)x$$

Now potential difference between the plates

$$\varphi_+ - \varphi_- = \int_0^d \mathbf{E} \cdot d\mathbf{r} = \int_0^d \frac{\sigma}{\varepsilon_0 \varepsilon(x)}\, dx$$

$$= \int_0^d \frac{\sigma}{\varepsilon_0\left(\varepsilon_1 + \dfrac{\varepsilon_2 - \varepsilon_1}{d}x\right)}\, dx = \frac{\sigma d}{(\varepsilon_2 - \varepsilon_1)\,\varepsilon_0} \ln \frac{\varepsilon_2}{\varepsilon_1}$$

Hence, the sought capacitance,

$$C = \frac{\sigma S}{\varphi_+ - \varphi_-} = \frac{(\varepsilon_2 - \varepsilon_1)\,\varepsilon_0 S}{(\ln \varepsilon_2/\varepsilon_1)\, d}$$

(b) Since, $$D = \frac{q}{S} \quad \text{and} \quad P = \frac{q}{S} - \frac{q}{S\varepsilon(x)}$$

and the space density of bound charge is

$$\rho' = -div\, P = -\frac{q(\varepsilon_2 - \varepsilon_1)}{Sd\varepsilon^2(x)}$$

3.105 Let us impart an imaginary charge q to the conductor. Now, the potential difference between the plates will be

$$\varphi_+ - \varphi_- = \int_{R_1}^{R_2} \mathbf{E} \cdot d\mathbf{r}$$

$$= \int_{R_1}^{R_2} \frac{q}{4\pi\varepsilon_0 a/r} \frac{1}{r^2}\, dr = \frac{q}{4\pi\varepsilon_0 a} \ln R_2/R_1$$

Hence, the sought capacitance

$$C = \frac{q}{\varphi_+ - \varphi_-} = \frac{q 4\pi\ \varepsilon_0 a}{q \ln R_2/R_1} = \frac{4\ \pi\varepsilon_0 a}{\ln R_2/R_1}$$

3.106 Let λ be the linear charge density.

Then, $$E_{1m} = \frac{\lambda}{2\pi\varepsilon_0 R_1 \varepsilon_1} \quad (1)$$

and $$E_{2m} = \frac{\lambda}{2\pi\varepsilon_0 R_2 \varepsilon_2} \quad (2)$$

The breakdown in either case will occur at the smaller value of r for a simultaneous breakdown of both dielectrics. From Eqs. (1) and (2)

$$E_{1m} R_1 \varepsilon_1 = E_{2m} R_2 \varepsilon_2$$

which is the sought relationship.

3.107 Let λ be the linear charge density, then, the sought potential difference

$$\varphi_+ - \varphi_- = \int_{R_1}^{R_2} \frac{\lambda}{2\pi\varepsilon_0\ \varepsilon_1 r}\, dr + \int_{R_2}^{R_3} \frac{\lambda}{2\pi\varepsilon_0\ \varepsilon_2 r}\, dr$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{1}{\varepsilon_1} \ln R_2/R_1 + \frac{1}{\varepsilon_2} \ln R_3/R_2\right]$$

As, $E_1 R_1 \varepsilon_1 < E_2 R_2 \varepsilon_2$, so

$$\frac{\lambda}{2\pi\varepsilon_0} = E_1 R_1 \varepsilon_1$$

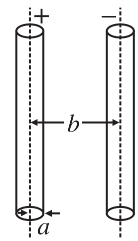
is the maximum acceptable value, and for values greater than $E_1 R_1 \varepsilon_1$, dielectric breakdown will take place.

Hence, the maximum potential difference between the plates

$$\varphi_+ - \varphi_- = E_1 R_1 \varepsilon_1 \left[\frac{1}{\varepsilon_1} \ln R_2/R_1 + \frac{1}{\varepsilon_2} \ln R_3/R_2\right]$$

$$= E_1 R_1 \left[\ln R_2/R_1 + \frac{\varepsilon_1}{\varepsilon_2} \ln R_3/R_2\right]$$

3.108 Let us suppose that linear charge density of the wires be λ, then the potential difference will be $\varphi_+ - \varphi_- = \varphi - (-\varphi) = 2\varphi$. The intensity of the electric field created by one of the wires at a distance x from its axis can be easily found with the help of the Gauss' theorem as



$$E = \frac{\lambda}{2\pi\varepsilon_0 x}$$

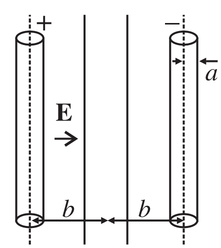
Then, $$\varphi = \int_a^{b-a} E\,dx = \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{b-a}{a}$$

Hence, capacitance per unit length is given by

$$\frac{\lambda}{\varphi_+ - \varphi_-} = \frac{\lambda}{2\varphi} = \frac{\pi\varepsilon_0}{\ln (b-a)/a} \cong \frac{\pi\varepsilon_0}{\ln b/a} \quad (\text{as } b \gg a)$$

3.109 The field in the regions between the conducting plane and the wire can be obtained by using an oppositely charged wire as an image on the other side.

Then the potential difference between the wire and the plane is



$$\Delta\varphi = \int \mathbf{E} \cdot d\mathbf{r}$$

$$= \int_a^b \left[\frac{\lambda}{2\pi\varepsilon_0 r} + \frac{\lambda}{2\pi\varepsilon_0 (2b - r)}\right] dr$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{b}{a} - \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{b}{2b - a}$$

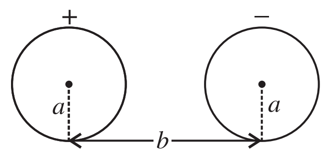
$$= \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{2b - a}{a}$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \ln \frac{2b}{a} \qquad (\text{as } b \gg a)$$

Hence, the sought mutual capacitance of the system per unit length of the wire

$$\frac{\lambda}{\Delta\varphi} = \frac{2\pi\varepsilon_0}{\ln 2b/a}$$

3.110 When $b \gg a$, the charge distribution on each spherical conductor is practically unaffected by the presence of the other conductor. Then, the potential φ_+ (φ_-) on the positive (respectively negative) charged conductor is

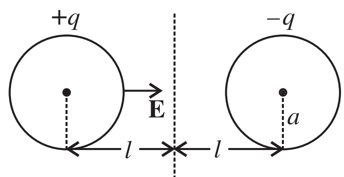


$$+\frac{q}{4\pi\varepsilon_0\varepsilon a}\left(-\frac{q}{4\pi\varepsilon_0\varepsilon a}\right)$$

Thus, $\varphi_+ - \varphi_- = \dfrac{q}{2\pi\varepsilon\varepsilon_0 a}$ and $C = \dfrac{q}{\varphi_+ - \varphi_-} = 2\pi\varepsilon_0\varepsilon a$

Note: *If we require terms which depend on a/b, we have to take account of distribution of charge on the conductors.*

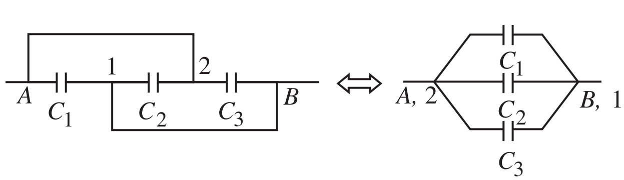
3.111 Let us apply the method of image. Then the potential difference between the positively charged sphere and the conducting plane is one half the nominal potential difference between the sphere and its image and is defined as



$$\Delta\varphi = \frac{1}{2}(\varphi_+ - \varphi_-) \cong \frac{q}{4\pi\varepsilon_0 a}$$

Thus, $$C = \frac{q}{\Delta\varphi} \approx 4\pi\varepsilon_0 a \quad (\text{for } l \gg a)$$

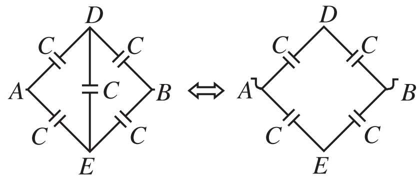
3.112 (a) Since $\varphi_1 = \varphi_B$ and $\varphi_2 = \varphi_A$. The arrangement of capacitors shown in the problem is equivalent to the arrangement shown in the figure.



Hence, the capacitance between A and B is

$$C_0 = C_1 + C_2 + C_3$$

(b) From the symmetry of the problem, there is no potential difference between D and E. So, the combination reduces to a simple arrangement shown in the figure.

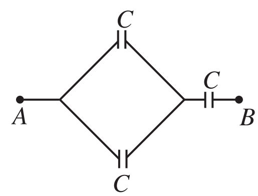


Hence, the net capacitance between **A** and **B** is

$$C_0 = \frac{C}{2} + \frac{C}{2} = C$$

3.113 (a) In the given arrangement, we have three capacitors of equal capacitance $C = \varepsilon_0 S/d$ and the first and third plates are at the same potential.

Hence, we can resolve the network into a simple form using series and parallel grouping of capacitors, as shown in the figure.



Thus, the equivalent capacitance is

$$C_0 = \frac{(C + C)\,C}{(C + C) + C} = \frac{2}{3}\,C = \frac{2}{3}\,\frac{\varepsilon_0 S}{d}$$

(b) Let us imagine that plates 1 and 2 have the charges $+\,q$ and $-\,q$ and these are distributed to the other plates using charge conservation and electric induction (see figure).

Since the potential difference between the plates 1 and 2 is zero,

$$-\frac{q_1}{C} + \frac{q_2}{C} - \frac{q_1}{C} = 0 \quad \left(\text{where } C = \frac{\varepsilon_0 S}{d}\right)$$

or $$q_2 = 2q_1$$

The potential difference between A and B

$$\varphi = \varphi_A - \varphi_B = \frac{q_2}{C}$$

Hence, the sought capacitance

$$C_0 = \frac{q}{\varphi} = \frac{q_1 + q_2}{q_2/C} = \frac{3q_1}{2q_1/C} = \frac{3}{2}\,C = \frac{3\varepsilon_0 S}{2d}$$

3.114 The amount of charge that the capacitor of capacitance C_1 can withstand would be $q_1 = C_1 V_1$ and similarly the charge that the capacitor of capacitance C_2 can withstand will be $q_2 = C_2 V_2$. But in a series combination, charge on both the capacitors will be same, so, $q_{\max}$ that the combination can withstand $= C_1 V_1$.

Also, $C_1 V_1 < C_2 V_2$, from the numerical data given.

Now, net capacitance of the system

$$C_0 = \frac{C_1 C_2}{C_1 + C_2}$$

Hence, $$V_{max} = \frac{q_{max}}{C_0} = \frac{C_1 V_1}{C_1 C_2 / C_1 + C_2} = V_1\left(1 + \frac{C_1}{C_2}\right) = 9 \text{ kV}$$

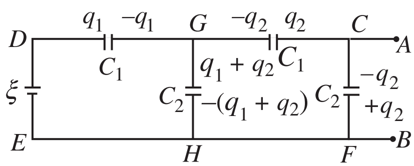
3.115 Let us distribute the charges, as shown in the figure. Now, we know that in a closed circuit, $-\,\Delta\varphi = 0$.

So, in the loop *DCFED*,

$$\frac{q_1}{C_1} - \frac{q_2}{C_1} - \frac{q_2}{C_2} = \xi \quad \text{or} \quad q_1 = C_1\left[\xi + q_2\left(\frac{1}{C_1} + \frac{1}{C_2}\right)\right] \tag{1}$$

Again in the loop *DGHED*,

$$\frac{q_1}{C_1} + \frac{q_1 + q_2}{C_2} = \xi \tag{2}$$



Using Eqs. (1) and (2), we get

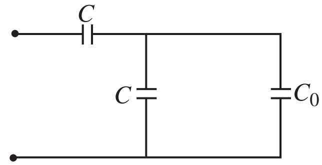
$$q_2\left[\frac{1}{C_1} + \frac{3}{C_2} + \frac{C_1}{C_2^2}\right] = -\frac{\xi C_1}{C_2}$$

Now, $$\varphi_A - \varphi_B = \frac{-q_2}{C_2} = \frac{\xi}{C_2^2/C_1}\frac{1}{\left[\frac{1}{C_1} + \frac{3}{C_2} + \frac{C_1}{C_2^2}\right]}$$

or $$\varphi_A - \varphi_B = \frac{\xi}{\left[\frac{C_2^2}{C_1^2} + \frac{3C_2}{C_1} + 1\right]} = \frac{\xi}{\eta^2 + 3\eta + 1} = 10 \text{ V}$$

3.116 The infinite circuit may be reduced to the circuit shown in the figure where C_0 is the net capacitance of the combination.

So, $$\frac{1}{C + C_0} + \frac{1}{C} = \frac{1}{C_0}$$



Solving the quadratic equation,

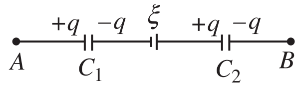
$$CC_0 + C_0^2 - C^2 = 0$$

we get,

$$C_0 = \frac{(\sqrt{5} - 1)}{2}\,\mathrm{C} \quad \text{(taking only + } ve \text{ value as } C_0 \text{ cannot be negative)}$$

$$= 0.62\ \mathrm{C}$$

3.117 Let us make the charge distribution as shown in the figure.



Now, $$\varphi_A - \varphi_B = \frac{q}{C_1} - \xi + \frac{q}{C_2}$$

or $$q = \frac{(\varphi_A - \varphi_B) + \xi}{C_1 + C_2}\, C_1 C_2$$

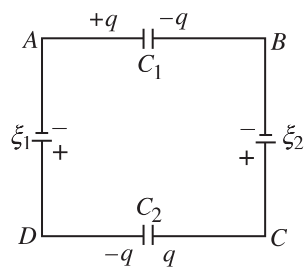
Hence, voltage across the capacitor C_1

$$= \frac{q}{C_1} = \frac{(\varphi_A - \varphi_B) + \xi}{C_1 + C_2}\, C_2 = 10\ \mathrm{V}$$

and voltage across the capacitor, C_2

$$= \frac{q}{C_2} = \frac{(\varphi_A - \varphi_B) + \xi}{C_1 + C_2}\, C_1 = 5\ \mathrm{V}$$

3.118 Let $\xi_2 > \xi_1$, then using $-\Delta\varphi = 0$ in the closed circuit (see figure),



$$\frac{-q}{C_1} + \xi_2 - \frac{q}{C_2} - \xi_1 = 0$$

or $$q = \frac{(\xi_2 - \xi_1)\, C_1 C_2}{(C_1 + C_2)}$$

Hence the potential difference (P.D.) across the left and right plates of capacitors,

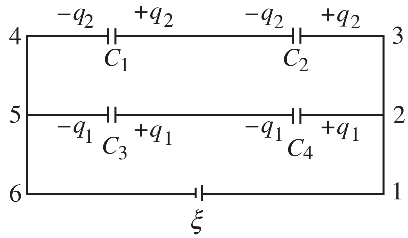
$$\varphi_1 = \frac{q}{C_1} = \frac{(\xi_2 - \xi_1)\, C_2}{C_1 + C_2}$$

and similarly, $$\varphi_2 = \frac{-q}{C_2} = \frac{(\xi_1 - \xi_2)\, C_1}{C_1 + C_2}$$

3.119 Using the solution of the foregoing problem, the amount of charge on each capacitor is given by

$$|q| = \frac{|\xi_2 - \xi_1|\, C_1 C_2}{C_1 + C_2}$$

3.120 Make the charge distribution as shown in the figure. In the circuit 12561, using $-\Delta\varphi = 0$



$$\frac{q_1}{C_4} - \frac{q_1}{C_3} - \xi = 0 \quad \text{or} \quad q_1 = \frac{\xi C_3 C_4}{C_3 + C_4}$$

and in the circuit 13461,

$$\frac{q_2}{C_2} + \frac{q_2}{C_1} - \xi = 0 \quad \text{or} \quad q_2 = \frac{\xi\, C_1 C_2}{C_1 + C_2}$$

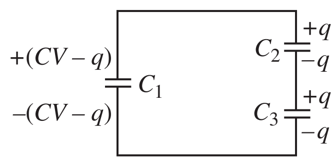
Now, $$\varphi_A - \varphi_B = \frac{q_2}{C_1} - \frac{q_1}{C_3}$$

$$= \xi\left[\frac{C_2}{C_1 + C_2} - \frac{C_4}{C_3 + C_4}\right] = \xi\left[\frac{C_2 C_3 - C_1 C_4}{(C_1 + C_2)(C_3 + C_4)}\right]$$

It becomes zero, when

$$(C_2 C_3 - C_1 C_4) = 0 \quad \text{or} \quad \frac{C_1}{C_2} = \frac{C_3}{C_4}$$

3.121 Let us assume the charge q flows through the connecting wires, then at the state of equilibrium, charge distribution will be as shown in the figure. In the closed circuit, using $-\Delta\varphi = 0$, we get



$$-\frac{(C_1 V - q)}{C_1} + \frac{q}{C_2} + \frac{q}{C_3} = 0$$

or $$q = \frac{V}{(1/C_1 + 1/C_2 + 1/C_3)} = 0.06 \text{ mC}$$

3.122 Initially, charge on the capacitor C_1 or C_2, is given by

$$q = \frac{\xi C_1 C_2}{C_1 + C_2}$$

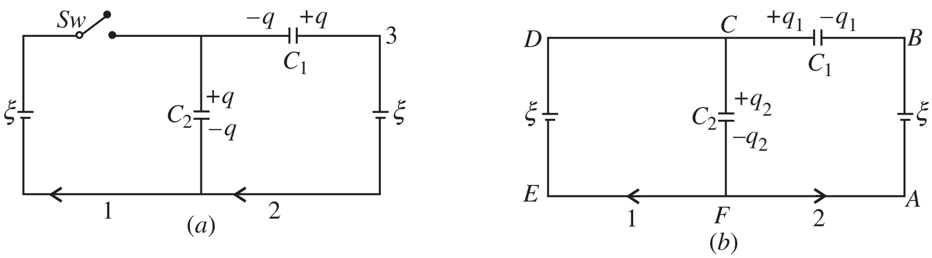
since they are in a series combination Fig. (a).

When the switch is closed, in the circuit *CDEFC*, from $-\Delta\varphi = 0$ Fig. (b).

$$\xi - \frac{q_2}{C_2} = 0 \quad \text{or} \quad q_2 = C_2\,\xi \tag{1}$$

And in the closed loop *BCFAB*, from $-\Delta\varphi = 0$

$$\frac{-q_1}{C_1} + \frac{q_2}{C_2} - \xi = 0 \tag{2}$$



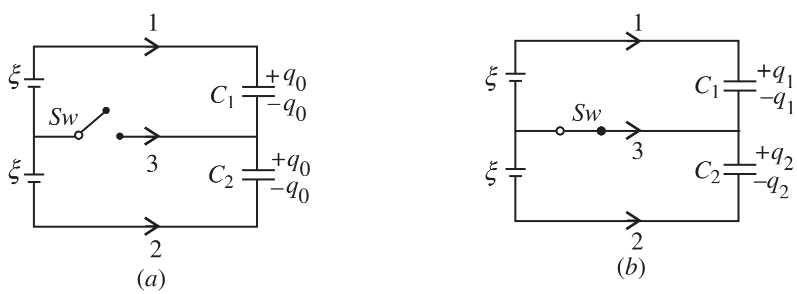
From Eqs. (1) and (2), $q_1 = 0$.

Now, charge flowing through section 1 $= (q_1 + q_2) - 0 = C_2\,\xi$

and charge flowing through section $2 = -q_1 - q = \dfrac{\xi C_1 C_2}{C_1 + C_2}$

3.123 When the switch is open Fig. (a),

$$q_0 = \frac{2\,\xi C_1 C_2}{C_1 + C_2}$$



and when the switch is closed Fig. (b),

$$q_1 = \xi C_1 \quad \text{and} \quad q_2 = \xi C_2$$

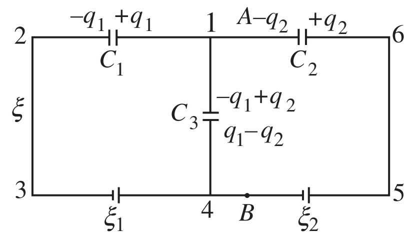
Hence, the flow of charge, due to the shorting of switch

through section $1 = q_1 - q_0 = \xi C_1 \left[\dfrac{C_1 - C_2}{C_1 + C_2}\right] = -24\ \mu C$

through section $2 = -q_2 - (q_0) = \xi C_2 \left[\dfrac{C_1 - C_2}{C_1 + C_2}\right] = -36\ \mu C$

and through the section $3 = q_2 - (q_2 - q_1) - 0 = \xi\,(C_2 - C_1) = 60\ \mu C$

3.124 First of all, make the charge distribution as shown in the figure.



In the loop 12341, using $-\Delta\varphi = 0$

$$\frac{q_1}{C_1} - \xi_1 + \frac{q_1 - q_2}{C_3} = 0 \qquad (1)$$

Similarly in the loop 61456, using $-\Delta\varphi = 0$

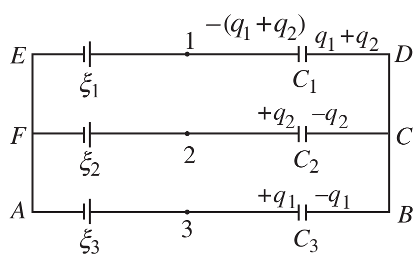
$$\frac{q_2}{C_2} + \frac{q_2 - q_1}{C_3} - \xi_2 = 0 \qquad (2)$$

From Eqs. (1) and (2) we have

$$q_2 - q_1 = \frac{\xi_2 C_2 - \xi_1 C_1}{\dfrac{C_2}{C_3} + \dfrac{C_1}{C_3} + 1}$$

Hence, $\quad \varphi_A - \varphi_B = \dfrac{q_2 - q_1}{C_3} = \dfrac{\xi_2 C_2 - \xi_1 C_1}{C_1 + C_2 + C_3}$

3.125 In the loop *ABDEA*, using $-\Delta\varphi = 0$



$$-\xi_3 + \frac{q_1}{C_3} + \frac{q_1 + q_2}{C_1} + \xi_1 = 0 \qquad (1)$$

Similarly in the loop *CDEFC*

$$\frac{q_1 + q_2}{C_1} + \xi_1 - \xi_2 + \frac{q_2}{C_2} = 0 \qquad (2)$$

Solving Eqs. (1) and (2), we get

$$q_1 + q_2 = \frac{\xi_2 C_2 - \xi_1 C_2 - \xi_1 C_3 + \xi_3 C_3}{\dfrac{C_3}{C_1} + \dfrac{C_2}{C_1} + 1}$$

Now, $$\varphi_1 - \varphi_0 = \varphi_1 = -\frac{q_1 + q_2}{C_1} \text{ (as } \varphi_0 = 0)$$

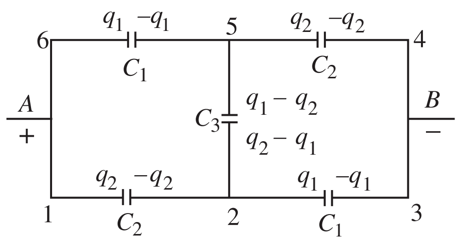
$$= \frac{\xi_1(C_2 + C_3) - \xi_2 C_2 - \xi_3 C_3}{C_1 + C_2 + C_3}$$

Using symmetry, $$\varphi_2 = \frac{\xi_2(C_1 + C_3) - \xi_1 C_1 - \xi_3 C_3}{C_1 + C_2 + C_3}$$

and $$\varphi_3 = \frac{\xi_3(C_1 + C_2) - \xi_1 C_1 - \xi_2 C_2}{C_1 + C_2 + C_3}$$

The answers have wrong sign in the answer sheet.

3.126 Taking advantage of the symmetry of the problem, we can make a diagram for charge distribution as shown in the figure.



In the loop, 12561, using $-\Delta\varphi = 0$

$$\frac{q_2}{C_2} + \frac{q_2 - q_1}{C_3} - \frac{q_1}{C_1} = 0$$

or $$\frac{q_1}{q_2} = \frac{C_1(C_3 + C_2)}{C_2(C_1 + C_3)} \tag{1}$$

Now, capacitance of the network

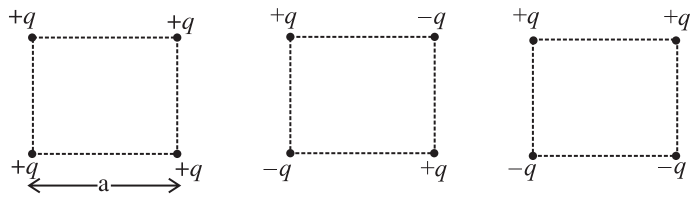
$$C_0 = \frac{q_1 + q_2}{\varphi_A - \varphi_B} = \frac{q_1 + q_2}{q_2/C_2 + q_1/C_1}$$

$$= \frac{(1 + q_1/q_2)}{\left(\frac{1}{C_2} + \frac{q_1}{q_2 C_1}\right)} \tag{2}$$

From Eqs. (1) and (2)

$$C_0 = \frac{2C_1C_2 + C_3(C_1 + C_2)}{C_1 + C_2 + 2C_3}$$

3.127 (a) Interaction energy of any two point charges q_1 and q_2 is given by $\frac{q_1 q_2}{4\pi\varepsilon_0 r}$, where r is the separation between the charges.



(a) (b) (c)

Hence, interaction energy of the system is given by

$$U_a = 4\frac{q^2}{4\pi\varepsilon_0 a} + 2\frac{q^2}{4\pi\varepsilon_0(\sqrt{2}\,a)}$$

$$U_b = 4\frac{-q^2}{4\pi\varepsilon_0\, a} + 2\frac{q^2}{4\pi\varepsilon_0\,(\sqrt{2}\,a)}$$

and $$U_c = 2\frac{q^2}{4\pi\varepsilon_0 a} - \frac{2q^2}{4\pi\varepsilon_0\, a} - \frac{2q^2}{4\pi\varepsilon_0\,(\sqrt{2}\,a)} = -\frac{\sqrt{2}\,q^2}{4\pi\varepsilon_0\, a}$$

3.128 Since the chain is of infinite length, any two charges of the same sign will occur symmetrically to any other charges of the opposite sign. So, interaction energy of each charge with all the others is given by

$$U = -2\frac{q^2}{4\pi\varepsilon_0\, a}\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\dots \text{ up to } \infty\right] \quad (1)$$

But, $$\ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \dots \text{up to } \infty$$

Putting $x = 1$, we get $$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} + \dots \text{ up to } \infty \quad (2)$$

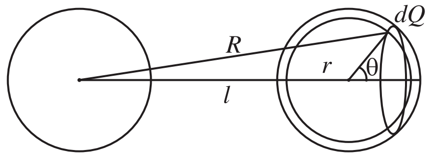
From Eqs. (1) and (2),

$$U = \frac{-2q^2 \ln 2}{4\pi\varepsilon_0 a}$$

3.129 Using electrical image method, interaction energy of the charge q with those induced on the plane is given by

$$U = \frac{-q^2}{4\pi\varepsilon_0\,(2l)} = -\frac{q^2}{8\pi\varepsilon_0 l}$$

3.130 Consider the interaction energy of one of the balls (say 1) and thin spherical shell of the other. This interaction energy can be written as



$$\int \varphi\, dq$$

$$= \int \frac{q_1}{4\pi\varepsilon_0 R} \rho_2(r)\, 2\pi r^2 \sin\theta \, d\theta \, dr$$

$$= \int_0^{\pi} \frac{\rho_2(r)\, q_1 r^2 \sin\theta \, d\theta \, dr}{2\,\varepsilon_0 (l^2 + r^2 + 2lr\cos\theta)^{1/2}} = \frac{q_1 r}{2\varepsilon_0 l}\, dr \int_{l-r}^{l+r} dx\, \rho_2(r)$$

$$= \frac{q_1 r}{2\varepsilon_0 l}\, dr\, \xi\, 2r\, \rho_2(r) \cdot 2 = \frac{q_1}{4\pi\varepsilon_0 l} 4\,\pi r^2\, dr\, \rho_2(r)$$

Then, finally integrating

$$U_{\text{int}} = \frac{q_1 q_2}{4\pi\varepsilon_0 l} \quad \text{where } q_2 = \int_0^{\infty} 4\pi r^2 \rho_2(r)\, dr$$

3.131 Charge contained in the capacitor of capacitance C_1 is $q = C_1\varphi$ and the energy, stored in it is given by

$$U_i = \frac{q^2}{2C_1} = \frac{1}{2} C_1 \varphi^2$$

Now, when the capacitors are connected in parallel, the equivalent capacitance of the system $C = C_1 + C_2$ and hence, energy stored in the system is

$$U_f = \frac{C_1^{\,2}\, \varphi^2}{2(C_1 + C_2)} \quad \text{(since charge is conserved during the process)}$$

So, increment in the energy

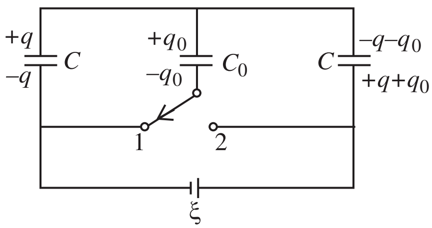
$$\Delta U = \frac{C_1^{\,2}\, \varphi^2}{2} \left(\frac{1}{C_1 + C_2} - \frac{1}{C_1} \right) = \frac{-C_2 C_1 \varphi^2}{2\,(C_1 + C_2)} = -0.03 \text{ mJ}$$

3.132 The charge on the condensers in position 1 is as shown in Fig. (a).

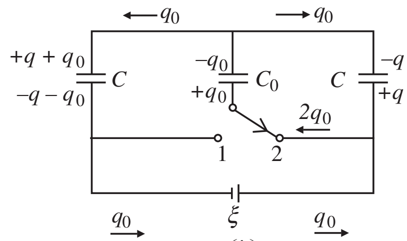
Here, $$\frac{q}{C} = \frac{q_0}{C_0} = \frac{q + q_0}{C + C_0}$$

and $$(q + q_0)\left(\frac{1}{C + C_0} + \frac{1}{C}\right) = \xi \quad \text{or} \quad q + q_0 = \frac{C(C + C_0)}{C_0 + 2C}$$

Hence, $$q = \frac{C^2\xi}{C_0 + 2C} \quad \text{and} \quad q_0 = \frac{CC_0\xi}{C_0 + 2C}$$



(a)



(b)

After the switch is thrown to position 2, the charges change as shown in Fig. (b). A charge q_0 has flown in the right loop through the two condensers and a charge q_0 through the cell, and because of the symmetry of the problem there is no change in the energy stored in the condensers. Thus,

$$H\text{ (Heat produced)} = \text{Energy delivered by the cell}$$

$$= \Delta q\xi = q_0\xi = \frac{CC_0\,\xi^2}{C_0 + 2C}$$

3.133 Initially, the charge on the right plate of the capacitor, $q = C(\xi_1 - \xi_2)$ and finally, when switched to the position 2, charge on the same plate of capacitor is $q' = C\xi_1$.

So, $$\Delta q = q' = C\xi_2$$

Now, from energy conservation,

$$\Delta U + \text{Heat liberated} = A_{\text{cell}}$$

(where ΔU is the electrical energy),

or $$\frac{1}{2}C\xi_1^2 - \frac{1}{2}C(\xi_1 - \xi_2)^2 + \text{Heat liberated} = \Delta q\,\xi_1$$

as only the cell with e.m.f. ξ_1 is responsible for redistribution of the charge.

So, $$C\xi_1\xi_2 - \frac{1}{2}C\xi_2^2 + \text{Heat liberated} = C\xi_2\xi_1$$

Hence, heat liberated $= \dfrac{1}{2}C\xi_2^2$.

3.134 Self energy of each shell is given by $\frac{q\varphi}{2}$, where φ is the potential of the shell, created only by the charge q on it.

Hence, self energies of the shells 1 and 2 are:

$$W_1 = \frac{q_1^{\ 2}}{8\pi\varepsilon_0 R_1} \quad \text{and} \quad W_2 = \frac{q_2^2}{8\ \pi\varepsilon_0 R_2}$$

The interaction energy between the charged shells equals charge q of one shell, multiplied by the potential φ created by other shell, at the point of location of charge q.

So,
$$W_{12} = q_1 \frac{q_2}{4\pi\varepsilon_0 R_2} = \frac{q_1 q_2}{4\pi\varepsilon_0 R_2}$$

Hence, total energy of the system

$$U = W_1 + W_2 + W_{12}$$
$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1^2}{2R_1} + \frac{q_2^2}{2R_2} + \frac{q_1 q_2}{R_2}\right]$$

3.135 Electric fields inside and outside the sphere is obtained with the help of Gauss' theorem.

$$E_1 = \frac{qr}{4\pi\varepsilon_0 r^2} \quad (\text{for } r \le R) \quad \text{and} \quad E_2 = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2} \quad (\text{for } r > R)$$

Sought self energy of the ball

$$U = W_1 + W_2$$
$$= \int_0^R \frac{\varepsilon_0 E_1^2}{2} 4\pi r^2 dr + \int_R^\infty \frac{\varepsilon_0 E_2^2}{2} 4\pi r^2 dr = \frac{q^2}{8\pi\varepsilon_0 R}\left(\frac{1}{5} + 1\right)$$

Hence,
$$U = \frac{3q^2}{4\pi\varepsilon_0\ 5R} \quad \text{and} \quad \frac{W_1}{W_2} = \frac{1}{5}$$
$$= \frac{3q^2}{20\pi\varepsilon_0 R}$$

3.136 For a spherical layer,

$$\int \frac{1}{2}\varepsilon_0 \varepsilon E^2\ dV = \int \frac{1}{2}\varepsilon\varepsilon_0\ E^2\ 4\pi r^2 dr$$

To find the electrostatic energy inside the dielectric layer, we have to integrate the above expression in the limit [a, b]. We get,

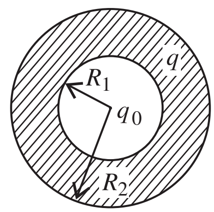
$$U = \frac{1}{2}\varepsilon_0\varepsilon \int_a^b \left(\frac{q}{4\pi\varepsilon_0\varepsilon r^2}\right)^2 4\pi r^2\ dr = \frac{q^2}{8\pi\varepsilon_0\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right] = 27 \text{ mJ (on substituting values)}$$

3.137 Since the field is conservative, the total work done by the field force

$$A = U_i - U_f = \frac{1}{2} q(\varphi_1 - \varphi_2)$$

$$= \frac{1}{2} \frac{q^2}{4\pi\varepsilon_0} \left[\frac{1}{R_1} - \frac{1}{R_2}\right] = \frac{q^2}{8\pi\varepsilon_0} \left[\frac{1}{R_1} - \frac{1}{R_2}\right]$$

3.138 Initially, energy of the system, $U_i = W_1 + W_{12}$ where, W_1 is the self energy and W_{12} is the mutual energy.



So, $$U_i = \frac{1}{2} \frac{q^2}{4\pi\varepsilon_0 R_1} + \frac{q q_0}{4\pi\varepsilon_0 R_1}$$

and on expansion, energy of the system is given by

$$U_f = W'_1 + W'_{12}$$

$$= \frac{1}{2} \frac{q^2}{4\pi\varepsilon_0 R_1} + \frac{q q_0}{4\pi\varepsilon_0 R_1}$$

Now, work done by the field force, A equals the decrease in the electrical energy,

i.e., $$A = (U_i - U_f) = \frac{q(q_0 + q/2)}{4\pi\varepsilon_0} \left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

Alternate:

The work of electric forces is equal to the decrease in electric energy of the system,

$$A = U_i - U_f$$

In order to find the difference $U_i - U_f$, we note that upon expansion of the shell, the electric field and hence the energy localized in it, changed only in the hatched spherical layer consequently (see figure).

$$U_i - U_f = \int_{R_2}^{R_2} \frac{\varepsilon_0}{2} \left(E_1^2 - E_2^2\right) \cdot 4\pi r^2 dr$$

where E_1 and E_2 are the field intensities (in the hatched region at a distance r from the center of the system) before and after the expansion of the shell. By using Gauss' theorem, we find

$$E_1 = \frac{1}{4\pi\varepsilon_0} \frac{q + q_0}{r^2} \quad \text{and} \quad E_2 = \frac{1}{4\pi\varepsilon_0} \frac{q_0}{r^2}$$

As a result of integration, we obtain

$$A = \frac{q\,(q_0 + q/2)}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

3.139 Energy of the charged sphere of radius r can be obtained from the equation

$$U = \frac{1}{2}\, q\,\varphi = \frac{1}{2}\, q\,\frac{q}{4\pi\varepsilon_0 r} = \frac{q^2}{8\pi\varepsilon_0 r}$$

If the radius of the shell changes by dr, then work done is

$$4\,\pi r^2\, F_u dr = -dU = \frac{q^2}{8\pi\varepsilon_0 r^2}$$

Thus sought force per unit area,

$$F_u = \frac{q^2}{4\pi r^2\,(8\pi\varepsilon_0 r^2)} = \frac{(4\pi r^2\,\sigma)^2}{4\pi r^2 \times 8\,\pi\varepsilon_0 r^2} = \frac{\sigma^2}{2\varepsilon_0}$$

3.140 Initially, there will be induced charges of magnitude $-q$ and $+q$ on the inner and outer surfaces of the spherical layer, respectively. Hence, the total electrical energy of the system is the sum of self energies of spherical shells having radii a and b, and their mutual energies including the point charge q.

$$U_i = \frac{1}{2}\,\frac{q^2}{4\pi\varepsilon_0\, b} + \frac{1}{2}\,\frac{(-q)^2}{4\pi\varepsilon_0\, a} + \frac{-qq}{4\pi\varepsilon_0\, a} + \frac{qq}{4\pi\varepsilon_0 b} + \frac{-qq}{4\pi\varepsilon_0 b}$$

or $$U_i = \frac{q^2}{8\pi\varepsilon_0}\left[\frac{1}{b} - \frac{1}{a}\right]$$

Finally, charge q is at infinity, hence $U_f = 0$.

Now, work done by the agent = increment in the energy

$$= U_f - U_i = \frac{q^2}{8\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

3.141 (a) Sought work is equivalent to the work performed against the electric field created by one plate held at rest, and moving the other plate away. Therefore the required work,

$$A_{\text{agent}} = qE\,(x_2 - x_1)$$

where $E = \sigma/2\varepsilon_0$ is the intensity of the field created by one plate at the location of other.

So, $$A_{\text{agent}} = q\,\frac{\sigma}{2\varepsilon_0}\,(x_2 - x_1) = \frac{q^2}{2\varepsilon_0 S}\,(x_2 - x_1)$$

Alternate:

$$A_{\text{ext}} = \Delta U \text{ (as field is potential)}$$
$$= \frac{q^2}{2\varepsilon_0 S} x_2 - \frac{q^2}{2\varepsilon_0 S} x = \frac{q^2}{2\varepsilon_0 S}(x_2 - x_1)$$

(b) When voltage is kept constant, the force acting on each plate of capacitor will depend on the distance between the plates.

So, elementary work done by agent, in its displacement over a distance dx, relative to the other is given by

$$dA = -F_x dx$$

But, $$F_x = -\left(\frac{\sigma(x)}{2\varepsilon_0}\right) S\sigma(x) \text{ and } \sigma(x) = \frac{\varepsilon_0 V}{x}$$

Hence, $$A = \int dA = \int_{x_1}^{x_2} \frac{1}{2}\varepsilon_0 \frac{SV^2}{x^2}\,dx = \frac{\varepsilon_0 S V^2}{2}\left[\frac{1}{x_1} - \frac{1}{x_2}\right]$$

Alternate:

From energy conservation,

$$U_f - U_i = A_{\text{cell}} + A_{\text{agent}}$$

or $$\frac{1}{2}\frac{\varepsilon_0 S}{x_2}V^2 - \frac{1}{2}\frac{\varepsilon_0 S}{x_1}V^2 = \left[\frac{\varepsilon_0 S}{x_2} - \frac{\varepsilon_0 S}{x_1}\right]V^2 + A_{\text{agent}}$$

(as $A_{\text{cell}} = (q_f - q_i)V = (C_f - C_i)\,V^2$).

So, $$A_{\text{agent}} = \left[\frac{1}{x_1} - \frac{1}{x_2}\right]$$

3.142 (a) When a metal plate of thickness ηd is inserted inside the capacitor, the capacitance of the system becomes

$$C_0 = \frac{\varepsilon_0 S}{d(1-\eta)}$$

Initially, charge on the capacitor,

$$q_0 = C_0 V = \frac{\varepsilon_0 S V}{d(1-\eta)}$$

Finally, capacitance of the capacitor,

$$C = \frac{\varepsilon_0 S}{d}$$

As the source is disconnected, charge on the plates will remain same during the process. From energy conservation,

$$U_f - U_i = A_{\text{agent}} \text{ (as the cell does no work)}$$

or $$\frac{1}{2}\frac{q_0^2}{C} - \frac{1}{2}\frac{q_0^2}{C_0} = A_{\text{agent}}$$

Hence, $$A_{\text{agent}} = \frac{1}{2}\left[\frac{\varepsilon_0 SV}{d(1-\eta)}\right]^2\left[\frac{1}{C} - \frac{(1-\eta)}{C}\right] = \frac{1}{2}\frac{CV^2\eta}{(1-\eta)^2} = 1.5 \text{ mJ}$$

(b) Initially, capacitance of the system is given by

$$C_0 = \frac{C\varepsilon}{\eta(1-\varepsilon)+\varepsilon} \quad \text{(since the two capacitors are in series)}$$

So, charge on the plate $q_0 = C_0 V$.

Capacitance of the capacitor, after the glass plate has been removed equals C.

From energy conservation,

$$\begin{aligned} A_{\text{agent}} &= U_f - U_i \\ &= \frac{1}{2}q_0^2\left[\frac{1}{C} - \frac{1}{C_0}\right] = \frac{1}{2}\frac{CV^2\varepsilon\eta\,(\varepsilon-1)}{[\varepsilon - \eta(\varepsilon-1)]^2} = 0.8 \text{ mJ} \end{aligned}$$

3.143 When the capacitor which is immersed in water is connected to a constant voltage source, it gets charged. Suppose σ_0 is the free charge density on the condenser plates. Because water is a dielectric, bound charges also appear in it. Let σ' be the surface density of bound charges. (Because of homogeneity of the medium and uniformity of the field, when we ignore edge effects no volume density of bound charges exists.) The electric field due to free charges is only σ_0/ε_0; that due to bound charges is σ'/ε_0 and the total electric field is $\sigma_0/\varepsilon\varepsilon_0$. Recalling that the sign of bound charges is opposite of the free charges, we have

$$\frac{\sigma_0}{\varepsilon\varepsilon_0} = \frac{\sigma_0}{\varepsilon_0} - \frac{\sigma'}{\varepsilon_0} \quad \text{or} \quad \sigma' = \left(\frac{\varepsilon-1}{\varepsilon}\right)\sigma_0$$

Because of the field that exists due to the free charges (not the total field; the field due to the bound charges must be excluded for this purpose as they only give rise to self energy effects), there is a force attracting the bound charges to the nearby plates. This force is

$$\frac{1}{2}\sigma'\frac{\sigma_0}{\varepsilon_0} = \frac{(\varepsilon-1)}{2\,\varepsilon\varepsilon_0}\sigma_0^2 \quad \text{(per unit area)}$$

The factor $1/2$ needs an explanation. Normally the force on a test charge is qE in an electric field E. But if the charge itself is produced by the electric filed then the force must be constructed bit by bit and is

$$F = \int_0^E q(E')\, dE'$$

if $q(E') \propto E'$, then we get

$$F = \frac{1}{2}\, q(E)\, E$$

This factor of 1/2 is well known. For example, the energy of a dipole of moment $\mathbf{p}$ in an electric field $\mathbf{E}_0$ is $-\,\mathbf{p}\cdot\mathbf{E}_0$, while the energy per unit volume of a linear dielectric in an electric field is $-1/2\,\mathbf{P}\cdot\mathbf{E}_0$ where $\mathbf{P}$ is the polarization vector (i.e., dipole moment per unit volume). Now the force per unit area manifests itself as excess pressure of the liquid.

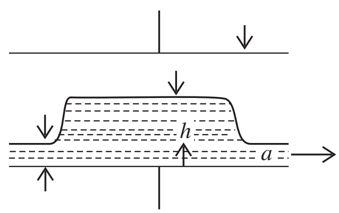
Noting that $$\frac{V}{d} = \frac{\sigma_0}{\varepsilon\varepsilon_0}$$

We get, $$\Delta p = \frac{\varepsilon_0\varepsilon\,(\varepsilon - 1)V^2}{2d^2}$$

Using $\varepsilon = 81$ for water, gives $\Delta p = 7.17$ kPa $= 0.07$ atm.

3.144 One way of solving this problem will be exactly as in the previous case, so let us try an alternative method based on energy. Suppose the liquid rises by the distance h. Then let us calculate the extra energy of the liquid as a sum of the polarization energy and the ordinary gravitational energy. The latter is

$$\frac{1}{2}\,h\cdot\rho\, g\,\cdot\, Sh = \frac{1}{2}\,\rho g S h^2$$



If σ is the free charge surface density on the plate, the bound charge density is, from the previous problem,

$$\sigma' = \frac{\varepsilon - 1}{\varepsilon}\,\sigma$$

This is also the volume density of induced dipole moment, i.e., polarization. Then the energy is, as before

$$-\frac{1}{2}\,\mathbf{p}\cdot\mathbf{E}_0 = -\frac{1}{2}\cdot\sigma'\,E_0 = \frac{-1}{2}\cdot\sigma'\,\frac{\sigma}{\varepsilon_0} = \frac{-(\varepsilon - 1)\sigma^2}{2\varepsilon_0\varepsilon}$$

where $\mathbf{p}$ is polarization vector and the total polarization energy is

$$-S\,(a + h)\,\frac{(\varepsilon - 1)\,\sigma^2}{2\varepsilon_0\varepsilon}$$

Then, total energy is

$$U(h) = -S(a + h)\frac{(\varepsilon - 1)\,\sigma^2}{2\,\varepsilon_0\varepsilon} + \frac{1}{2}\rho g S h^2$$

The actual height to which the liquid rises is determined from the formula

$$\frac{dU}{dh} = U'(h) = 0$$

This gives
$$h = \frac{(\varepsilon - 1)\,\sigma^2}{2\varepsilon_0\varepsilon\rho g}$$

Alternate:

To find the stable height of the liquid, the process of its rise should be assumed to be very slow (in equilibrium). From energy conservation in differential form,

$$dU_{\text{ele}} + dU_{\text{gr}} = 0$$

So,
$$\frac{dU_{\text{ele}}(h)}{dh} + \frac{dU_{\text{gr}}(h)}{dh} = 0$$

$$\frac{d}{dh}\left(\frac{q^2}{2C}\right) + \frac{d}{dh}\left(mg\frac{h}{2}\right) = 0$$

Using $m = \rho S h$ and $q = \sigma S =$ constant, where S is area of each of the plates,

$$\frac{q^2}{2}\frac{d}{dh}\left(\frac{1}{C}\right) + \frac{d}{dh}\left(\rho S h g\frac{h}{2}\right) = 0$$

But,
$$\frac{1}{C} = \frac{a}{\varepsilon\varepsilon_0 S} + \frac{h}{\varepsilon\varepsilon_0 S} + \frac{d - h}{\varepsilon_0 S}$$
(where d is separation between the plates)

So,
$$\frac{d}{dh}\left(\frac{1}{C}\right) = \frac{d}{dh}\left[\frac{h}{\varepsilon\varepsilon_0 S} + \frac{d - h}{\varepsilon_0 S}\right]$$
(because initial liquid height a is constant)

$$\frac{d}{dh}\left(\frac{1}{C}\right) = \frac{d}{dh}\left[\frac{\varepsilon d - (\varepsilon - 1)h}{\varepsilon_0 S\varepsilon}\right] = -\frac{(\varepsilon - 1)}{\varepsilon\varepsilon_0 S}$$

Hence,
$$-\frac{q^2}{2}\frac{(\varepsilon - 1)}{\varepsilon\varepsilon_0 S} + (\rho g S)\,h = 0$$

$$\frac{\sigma^2}{2}\frac{(\varepsilon - 1)}{\varepsilon_0} = \rho g h$$

$$h = \frac{\sigma^2(\varepsilon - 1)}{2\rho g\varepsilon\varepsilon_0}$$

Note: *Assuming that the lower plate is just immersed, the answer remains the same.*

3.145 We know that the energy of the capacitor,

$$U = \frac{q^2}{2C}$$

Hence, from
$$F_x = \left.\frac{\partial U}{\partial x}\right|_{q\,=\,\text{constant}}$$

we have,

$$F_x = \frac{q^2}{2}\frac{\partial C}{\partial x}\bigg/C^2 \tag{1}$$

Now, since $d \ll R$, the capacitance of the given capacitor can be calculated by the formula of a parallel plate capacitor. Therefore, if the dielectric is introduced upto a depth x and the length of the capacitor is l, we have,

$$C = \frac{2\pi\varepsilon_0\,\varepsilon R x}{d} + \frac{2\pi R\varepsilon_0\,(l - x)}{d} \tag{2}$$

From Eqs. (1) and (2), we get,

$$F_x = \varepsilon_0\,(\varepsilon - 1)\,\frac{\pi R V^2}{d}$$

3.146 When the capacitor is kept at a constant potential difference V, the work performed by the moment of electrostatic forces between the plates, when the inner movable plate is rotated by an angle $d\varphi$, equals the increase in the potential energy of the system. This comes about because when changes are made, charges flow from the battery to keep the potential constant and the amount of the work done by these charges is twice in magnitude and opposite in sign to the change in the energy of the capacitor. Thus,

$$N_z = \frac{\partial U}{\partial \varphi} = \frac{1}{2}\,V^2\,\frac{\partial C}{\partial \varphi}$$

Now the capacitor can be thought of as made up of two parts (with and without the dielectric) in parallel.

Thus,
$$C = \frac{\varepsilon_0 R^2 \varphi}{2d} + \frac{\varepsilon_0\varepsilon(\pi - \varphi)\,R^2}{2d}$$

as the area of a sector of angle φ is $1/2\,R^2\,\varphi$. Differentiation then gives

$$N_z = -\frac{(\varepsilon - 1)\,\varepsilon_0 R^2 V^2}{4d}$$

The negative sign of N_z indicates that the moment of the force is acting clockwise (i.e., trying to suck in the dielectric).

3.4 Electric Current

3.147 The convection current is

$$I = \frac{dq}{dt} \tag{1}$$

here, $dq = \lambda dx$, where λ is the linear charge density.

But, from the Gauss' theorem, electric field at the surface of the cylinder

$$E = \frac{\lambda}{2\pi\varepsilon_0 a}$$

Hence, substituting the value of λ and subsequently of dq in Eq. (1), we get

$$I = \frac{2E\,\pi\varepsilon_0 a dx}{dt}$$

$$= 2\pi\varepsilon_0 Eav \quad \left(\text{as } \frac{dx}{dt} = v\right)$$

$$= 0.5\ \mu\text{A (on substituting values)}$$

3.148 Since $d \ll r$, the capacitance of the given capacitor can be calculated using the formula for a parallel plate capacitor. Therefore, if water (permittivity ε) is introduced up to a height x and the capacitor is of length l, we have

$$C = \frac{\varepsilon\varepsilon_0 \cdot 2\pi rx}{d} + \frac{\varepsilon_0(l - x)\,2\pi r}{d} = \frac{\varepsilon_0 2\pi r}{d}(\varepsilon x + l - x)$$

Hence charge on the plate at that instant, $q = CV$.

Again we know that the electric current intensity,

$$I = \frac{dq}{dt} = \frac{d\,(CV)}{dt}$$

$$= \frac{V\varepsilon_0\, 2\pi r}{d}\,\frac{d(\varepsilon x + l - x)}{dt} = \frac{V 2\pi r\varepsilon_0}{d}(\varepsilon - 1)\frac{dx}{dt}$$

But, $$\frac{dx}{dt} = v$$

So, $$I = \frac{2\pi r\varepsilon_0(\varepsilon - 1)V}{d}\,v \approx 0.11\ \mu\text{A}$$

3.149 We have,

$$R_t = R_0 (1 + \alpha t) \qquad (1)$$

where R_t and R_0 are resistances at t°C and 0°C, respectively and α is the mean temperature coefficient of resistance.

So, $\quad R_1 = R_0(1 + \alpha_1 t)$ and $R_2 = \eta R_0(1 + \alpha_2 t)$

(a) In case of series combination, $R = \Sigma R_i$

So,
$$R = R_1 + R_2 = R_0\,[(1 + \eta) + (\alpha_1 + \eta\alpha_2)\, t]$$
$$= R_0\,(1 + \eta)\left[1 + \frac{\alpha_1 + \eta\alpha_2}{1 + \eta}\, t\right] \qquad (2)$$

Comparing Eqs. (1) and (2), we conclude that temperature coefficient of resistance of the circuit

$$\alpha = \frac{\alpha_1 + \eta\alpha_2}{1 + \eta}$$

(b) In parallel combination

$$R = \frac{R_0(1 + \alpha_1 t)\, R_0\, \eta\, (1 + \alpha_2 t)}{R_0(1 + \alpha_1 t) + \eta\, R_0\, (1 + \alpha_2 t)} = R'(1 + \alpha' t), \text{ where } R' = \frac{\eta R_0}{1 + \eta}$$

Now, neglecting the terms proportional to the product of temperature coefficients, as they have very small values, we get

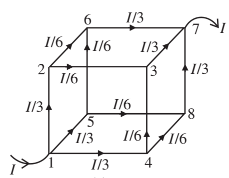
$$\alpha' = \frac{\eta\alpha_1 + \alpha_2}{1 + \eta}$$

3.150 Using the property of symmetry, the currents are flowing as shown in the figures.

(a) In the figures, Ohm's law is applied between 1 and 7 via 1487 (say). So,

$$IR_{\text{eq}} = \frac{I}{3}R + \frac{I}{6}R + \frac{I}{3}R = \frac{5}{6}RI$$

Thus, $\quad R_{\text{eq}} = \dfrac{5R}{6}$



(a)

(b) Between 1 and 2 from the loop 14321,

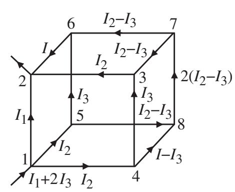
$$I_1 R = 2I_2 R + I_3 R \quad \text{or} \quad I_1 = I_3 + 2I_2$$

From the loop 48734,

$$(I_2 - I_3)\, R + 2(I_2 - I_3)\, R + (I_2 - I_3)\, R = I_3 R$$

or $\quad 4\,(I_2 - I_3) = I_3 \Rightarrow I_3 = \dfrac{4}{5} I_2$

So, $\quad I_1 = \dfrac{14}{5} I_2$



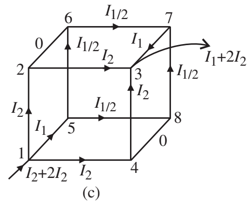
(b)

Then, $$(I_1 + 2I_2)\, R_{eq} = \frac{24}{5}\, I_2 R_{eq}$$

$$= I_1 R = \frac{14}{5}\, I_2 R$$

or $$R_{eq} = \frac{7}{12}\, R$$

(c) Between 1 and 3 from the loop 15621



(c)

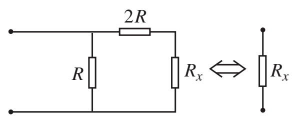
$$I_2 R = I_1 R + \frac{I_1}{2} R \quad \text{or} \quad I_2 = 3\,\frac{I_1}{2}$$

Then, $$(I_1 + 2I_2) R_{eq} = 4 I_1 R_{eq}$$

$$= I_2 R + I_2 R = 3 I_1 R$$

Hence, $$R_{eq} = \frac{3}{4}\, R$$

3.151 Total resistance of the circuit will be independent of the number of cells,



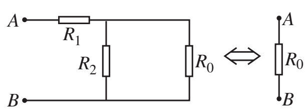
if $$R_x = \frac{(R_x + 2R)R}{R_x + 2R + R}$$

or $$R_x^2 + 2RR_x - 2R^2 = 0$$

On solving and rejecting the negative root of the quadratic equation, we have

$$R_x = R\,(\sqrt{3} - 1)$$

3.152 Let R_0 be the resistance of the network.



Then, $$R_0 = \frac{R_0 R_2}{R_0 + R_2} + R_1$$

or $$R_0^2 - R_0 R_1 - R_1 R_2 = 0$$

On solving we get,

$$R_0 = \frac{R_1}{2}\left(1 + \sqrt{1 + 4\,\frac{R_2}{R_1}}\right) = 6\Omega$$

3.153 Suppose that the voltage V is applied between the points A and B, then

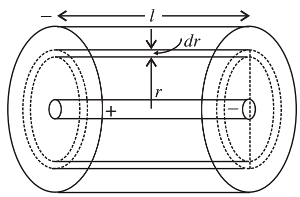
$$V = IR = I_0 R_0$$

where R is the resistance of the whole grid, I is the current through the grid and I_0 is the current through the segment AB. Now from symmetry, $I/4$ is the part of the current flowing through all the four wire segments, meeting at the point A and similarly the amount of current flowing through the wires meeting at B is also $I/4$. Thus a current $I/2$ flows through the conductor AB, i.e.,

$$I_0 = \frac{I}{2}$$

Hence,
$$R = \frac{R_0}{2}$$

3.154 Let us imagine a thin cylindrical layer of inner and outer radii r and $r + dr$, respectively. As lines of current at all the points of this layer are perpendicular to it, such a layer can be treated as a cylindrical conductor of thickness dr and cross-sectional area $2\pi rl$. So, we have,



$$dR = \rho \frac{dr}{S(r)} = \rho \frac{dr}{2\pi rl}$$

and integrating between the limit $[a, b]$, we get

$$R = \frac{\rho}{2\pi l} \ln \frac{b}{a}$$

3.155 Let us imagine a thin spherical layer of inner and outer radii r and $r + dr$. Lines of current at all the points of this layer are perpendicular to it and therefore such a layer can be treated as a spherical conductor of thickness dr and cross sectional area $4\pi r^2$. So,

$$dR = \rho \frac{dr}{4\pi r^2}$$

and integrating between the limits $[a, b]$, we get,

$$R = \frac{\rho}{4\pi}\left[\frac{1}{a} - \frac{1}{b}\right]$$

Now, for $b \to \infty$, we have
$$R = \frac{\rho}{4\pi a}$$

3.156 In this case, resistance of the medium is given by

$$R = \frac{\rho}{4\pi}\left[\frac{1}{a} - \frac{1}{b}\right] \quad \text{(where } \rho \text{ is the resistivity of the medium)}$$

The current is given by

$$I = \frac{\varphi}{R} = \frac{\varphi}{\frac{\rho}{4\pi}\left[\frac{1}{a} - \frac{1}{b}\right]} \tag{1}$$

Also,
$$I = \frac{-dq}{dt} = -\frac{d(C\varphi)}{dt} = -C\frac{d\varphi}{dt} \quad \text{(as capacitance is constant)} \tag{2}$$

Equating Eqs. (1) and (2), we get

$$\frac{\varphi}{\frac{\rho}{4\pi}\left[\frac{1}{a}-\frac{1}{b}\right]} = -C\frac{d\varphi}{dt}$$

or
$$-\int \frac{d\varphi}{\varphi} = \frac{\Delta t}{\frac{C\rho}{4\pi}\left[\frac{1}{a}-\frac{1}{b}\right]}$$

or
$$\text{In } \eta = \frac{\Delta t\, 4\pi ab}{C\rho(b-a)}$$

Hence, resistivity of the medium

$$\rho = \frac{4\pi\Delta t\, ab}{C(b-a)\ln\eta}$$

3.157 Let us imagine that the balls have charge $+q$ and $-q$, respectively. The electric field strength at the surface of a ball will be determined only by its own charge and the charge can be considered to be uniformly distributed over the surface, because the other ball is at an infinite distance. Magnitude of the field strength is given by

$$E = \frac{q}{4\pi\varepsilon_0 a^2}$$

So, current density

$$j = \frac{1}{\rho}\frac{q}{4\pi\varepsilon_0 a^2}$$

and electric current

$$I = \int \mathbf{j}\cdot d\mathbf{S} = jS = \frac{q}{\rho 4\pi\varepsilon_0 a^2}\cdot 4\pi a^2 = \frac{q}{\rho\varepsilon_0}$$

But, potential difference between the balls

$$\varphi_+ - \varphi_- = 2\frac{q}{4\pi\varepsilon_0 a}$$

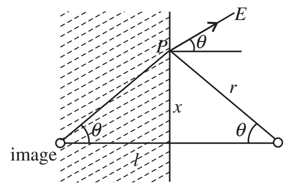
Hence, the sought resistance

$$R = \frac{\varphi_+ - \varphi_-}{I} = \frac{2q/4\pi\varepsilon_0 a}{q/\rho\varepsilon_0} = \frac{\rho}{2\pi a}$$

3.158 (a) The potential in the unshaded region beyond the conductor has the potential of the given charge and its image has the form

$$\varphi = A\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

where r_1, r_2 are the distances of the point from the charge and its image, respectively. The potential has been taken to be zero on the conducting plane and on the ball.



$$\varphi \approx A\left(\frac{1}{a} - \frac{1}{2l}\right) = V$$

So, $A \approx Va$. In this calculation, the condition $a \ll l$, is used to ignore the variation of φ over the ball.

The electric field at P can be calculated similarly. The charge on the ball is

$$Q = 4\pi\varepsilon_0 Va$$

and $$E_P = \frac{Va}{r^2} 2\cos\theta = \frac{2alV}{r^3} \text{ then } j = \frac{1}{\rho}E = \frac{2alV}{\rho r^3}$$

(b) The total current flowing into the conducting plane is

$$I = \int_0^\infty 2\pi x dx j = \int_0^\infty 2\pi x dx \frac{2alV}{\rho(x^2 + l^2)^{3/2}}$$

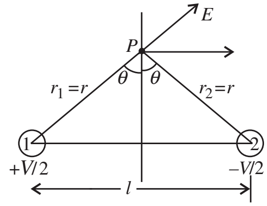
Putting $y = x^2 + l^2$, we get

$$I = \frac{2\pi alV}{\rho}\int_{l^2}^\infty \frac{dy}{y^{3/2}} = \frac{4\pi aV}{\rho}$$

Hence, $$R = \frac{V}{I} = \frac{\rho}{4\pi a}$$

3.159 (a) The wires can be assumed to be perfect conductors, so the resistance is entirely due to the medium. If the wire is of length L, the resistance R of the medium is $\propto 1/L$ because different sections of the wire are connected in parallel (by the medium) rather than in series. Thus if R_1 is the resistance per unit length of the wire, then $R = R_1/L$. Unit of R_1 is ohm-meter.

The potential at a point P is by symmetry and superposition for $(l \gg a)$, is given by



$$\varphi \cong \frac{A}{2}\ln\frac{r_1}{a} - \frac{A}{2}\ln\frac{r_2}{a}$$
$$= \frac{A}{2}\ln\frac{r_1}{r_2}$$

Then, $\varphi_1 = \frac{V}{2} = \frac{A}{2} \ln \frac{a}{l}$ (for the potential of 1)

or $A = \frac{-V}{\ln l/a}$

and $\varphi = -\frac{V}{2 \ln l/a} \ln r_1/r_2$

We then calculate the field at a point P which is equidistant from 1 and 2 and at a distance r from both:

$$E = \frac{V}{2 \ln l/a}\left(\frac{1}{r}\right) \times 2 \sin\theta$$

$$= \frac{Vl}{2 \ln l/a} \frac{1}{r^2}$$

and $j = \sigma E = \frac{1}{\rho} \frac{Vl}{2r^2 \ln l/a}$

(b) Near either wire $E = \frac{V}{2 \ln l/a} \frac{1}{a}$

and $j = \sigma E = \frac{1}{\rho} \frac{V}{2 \ln l/a}$

Then $I = \frac{V}{R} = L \frac{V}{R_1} = j 2\pi aL$

which gives $R_1 = \frac{\rho}{\pi} \ln l/a$

3.160 Let us imagine that the plates of the capacitor have the charges $+q$ and $-q$.

The capacitance of the network

$$C = \frac{q}{\varphi} = \frac{\varepsilon\varepsilon_0 \int E_n dS}{\varphi} \tag{1}$$

Now, electric current

$$I = \int \mathbf{j} \cdot d\mathbf{S} = \int \sigma E_n dS, \text{ as } \mathbf{j} \uparrow\uparrow \mathbf{E} \tag{2}$$

Hence, using Eq. (1) in (2), we get

$$I = \frac{C\varphi}{\varepsilon\varepsilon_0}\sigma = \frac{C\varphi}{\rho\varepsilon\varepsilon_0} = 1.5\ \mu\text{A}$$

3.161 Let us imagine that the conductors have the charges $+q$ and $-q$. As the medium is a poor conductor, the surfaces of the conductors are equipotential and the field configuration is same as in the absence of the medium.

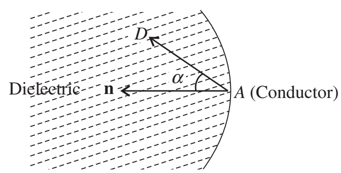
Let us surround, for example, the positively charged conductor, by a closed surface S, just containing the conductor.

Then, $$R = \frac{\varphi}{I} = \frac{\varphi}{\int \mathbf{j}\cdot d\mathbf{S}} = \frac{\varphi}{\int \sigma E_n dS} \quad (\text{as } \mathbf{j} \uparrow\uparrow \mathbf{E})$$

and $$C = \frac{q}{\varphi} = \frac{\varepsilon\varepsilon_0 \int E_n dS}{\varphi}$$

So, $$RC = \frac{\varepsilon\varepsilon_0}{\sigma} = \rho\varepsilon\varepsilon_0$$

3.162 The dielectric ends in a conductor. It is given that, on one side (the dielectric side) the electric displacement D is as shown in figure. Within the conductor, at any point A, there can be no normal component of electric field. For if there were such a field, a current will flow towards depositing charge there which in turn will set up countering electric field causing the normal component to vanish. Then by Gauss' theorem, we can easily derive $\sigma = D_n = D\cos\alpha$, where σ is the surface charge density it A.



The tangential component is determined from the circulation theorem as

$$\oint \mathbf{E}\cdot d\mathbf{r} = 0$$

It must be continuous across the surface of the conductor. Thus, inside the conductor there is a tangential electric field of magnitude

$$\frac{D\sin\alpha}{\varepsilon_0\varepsilon} \text{ at } A$$

This implies a current, by Ohm's law, of density

$$j = \frac{E}{\rho} = \frac{D\sin\alpha}{\varepsilon_0\varepsilon\rho}$$

3.163 The resistance of a layer of the medium, of thickness dx and at a distance x from the first plate of the capacitor is given by

$$dR = \frac{1}{\sigma(x)}\frac{dx}{S} \tag{1}$$

Now, since σ varies linearly with the distance from the plate, it may be represented as, $\sigma = \sigma_1 + \left(\frac{\sigma_2 - \sigma_1}{d}\right)x$, at a distance x from any one of the plates.

From Eq. (1) $$dR = \frac{1}{\sigma_1 + \left(\frac{\sigma_2 - \sigma_1}{d}\right)x} \frac{dx}{S}$$

or $$R = \frac{1}{S}\int_0^d \frac{dx}{\sigma_1 + \left(\frac{\sigma_2 - \sigma_1}{d}\right)x} = \frac{d}{S(\sigma_2 - \sigma_1)} \ln \frac{\sigma_2}{\sigma_1}$$

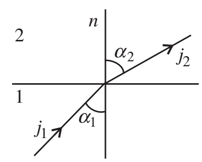
Hence, $$I = \frac{V}{R} = \frac{SV(\sigma_2 - \sigma_1)}{d \ln \frac{\sigma_2}{\sigma_1}} = 5 \text{ nA}$$

3.164 By charge conservation current density j, leaving the medium 1 must enter the medium 2.

Thus, $$j_1 \cos\alpha_1 = j_2 \cos\alpha_2$$

Another relation follows from

$$E_{1t} = E_{2t}$$



which is a consequence of

$$\oint \mathbf{E} \cdot d\mathbf{r} = 0.$$

Thus, $$\frac{1}{\sigma_1} j_1 \sin\alpha_1 = \frac{1}{\sigma_2} j_2 \sin\alpha_2$$

or $$\frac{\tan\alpha_1}{\sigma_1} = \frac{\tan\alpha_2}{\sigma_2}$$

or $$\frac{\tan\alpha_1}{\tan\alpha_2} = \frac{\sigma_1}{\sigma_2}$$

3.165 The electric field in conductor 1 is

$$E_1 = \frac{\rho_1 I}{\pi R^2}$$

and that in 2 is $$E_2 = \frac{\rho_2 I}{\pi R^2}$$

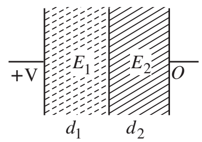
Applying Gauss' theorem to a small cylindrical pill-box at the boundary,

$$-\frac{\rho_1 I}{\pi R^2} dS + \frac{\rho_2 I}{\pi R^2} dS = \frac{\sigma dS}{\varepsilon_0}$$

Thus, $$\sigma = \varepsilon_0 (\rho_2 - \rho_1) \frac{1}{\pi R^2}$$

and charge at the boundary $= \varepsilon_0 (\rho_2 - \rho_1) I$

3.166 We have, $E_1 d_1 + E_2 d_2 = V$ and by current conservation



$$\frac{1}{\rho_1} E_1 = \frac{1}{\rho_2} E_2$$

Thus, $$E_1 = \frac{\rho_1 V}{\rho_1 d_1 + \rho_2 d_2}$$

$$E_2 = \frac{\rho_2 V}{\rho_1 d_1 + \rho_2 d_2}$$

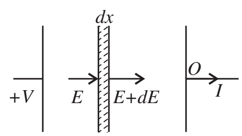
At the boundary between the two dielectrics,

$$\sigma = D_2 - D_1 = \varepsilon_0 \varepsilon_2 E_2 - \varepsilon_0 \varepsilon_1 E_1$$

$$= \frac{\varepsilon_0 V}{\rho_1 d_1 + \rho_2 d_2} (\varepsilon_2 \rho_2 - \varepsilon_1 \rho_1)$$

3.167 By current conservation

$$\frac{E(x)}{\rho(x)} = \frac{E(x) + dE(x)}{\rho(x) + d\rho(x)} = \frac{dE(x)}{d\rho(x)}$$



Solving we get,

$$E(x) = C\rho(x) = \frac{I\rho(x)}{A}$$

Hence, charge induced in the slice per unit area is given by

$$d\sigma = \varepsilon_0 \frac{I}{A} [\{\varepsilon(x) + d\varepsilon(x)\}\{\rho(x) + d\rho(x)\} - \varepsilon(x)\rho(x)] = \varepsilon_0 \frac{I}{A} d[\varepsilon(x)\rho(x)]$$

Thus, $$dQ = \varepsilon_0 I d[\varepsilon(x)\rho(x)]$$

Hence total charge induced obtained by integration is

$$Q = \varepsilon_0 I (\varepsilon_2 \rho_2 - \varepsilon_1 \rho_1)$$

3.168 As in the previous problem,

$$E(x) = C\rho(x) = C(\rho_0 + \rho_1 x)$$

where $$\rho_0 + \rho_1 d = \eta \rho_0 \quad \text{or} \quad \rho_1 = \frac{(\eta - 1)\rho_0}{d}$$

By integration, $V = \int_0^d C\rho\,(x)\;dx = C\rho_0 d\left(1 + \frac{\eta - 1}{2}\right) = \frac{1}{2}\,C\rho_0 d\,(\eta + 1)$

Thus, $$C = \frac{2\;V}{\rho_0 d\,(\eta + 1)}$$

Thus volume density of charge present in the medium

$$= \frac{dQ}{Sdx} = \varepsilon_0\;dE\,(x)/dx$$
$$= \frac{2\varepsilon_0 V}{\rho_0 d\,(\eta + 1)} \times \frac{(\eta - 1)\;\rho_0}{d} = \frac{2\varepsilon_0\;V(\eta - 1)}{(\eta + 1)\;d^2}$$

3.169 (a) Consider a cylinder of unit length and divide it into shells of radius r and thickness dr. Different sections are parallel. For a typical section

$$d\left(\frac{1}{R_1}\right) = \frac{2\pi r\;dr}{(\alpha/r^2)} = \frac{2\pi r^3 dr}{\alpha}$$

Integrating, $$\frac{1}{R_1} = \frac{\pi R^4}{2\alpha} = \frac{S^2}{2\pi\alpha}$$

or $$R_1 = \frac{2\pi\alpha}{S^2} \quad (\text{where } S = \pi R^2)$$

(b) Suppose the electric field inside is $E_z = E_0$ (z-axis is along the axis of the conductor). This electric field cannot depend on r in steady conditions when other components of E are absent, otherwise it would violate the circulation theorem

$$\oint \mathbf{E} \cdot d\mathbf{r} = 0$$

The current through a section between radii (r and $r + dr$) is

$$2\pi r dr \frac{1}{\alpha/r^2}\;E = 2\pi r^3 dr \frac{E}{\alpha}$$

Thus, $$I = \int_0^R 2\pi r^3 dr \frac{E}{\alpha} = \frac{\pi R^4 E}{2\alpha}$$

Hence, $$E = \frac{2\;\alpha\pi I}{S^2} \qquad (\text{where } S = \pi R^2)$$

3.170 The formula is,

$$q = CV_0\,(1 - e^{-t/RC})$$

or $$V = \frac{q}{C} = V_0\,(1 - e^{-t/RC}) \quad \text{or} \quad \frac{V}{V_0} = 1 - e^{-t/RC}$$

or $$e^{-t/RC} = 1 - \frac{V}{V_0} = \frac{V_0 - V}{V_0}$$

Hence, $$t = RC \ln \frac{V_0}{V_0 - V} = RC \ln 10, \text{ if } V = 0.9\ V_0$$

Thus, $$t = 0.6\ \mu\text{s}$$

3.171 The charge decays according to the formula

$$q = q_0 e^{-t/RC}$$

Here, $$RC = \text{mean life} = \text{half-life}/\ln 2$$

So, half life $$T = RC \ln 2$$

But, $$C = \frac{\varepsilon\varepsilon_0 A}{d} \quad \text{and} \quad R = \frac{\rho d}{A}$$

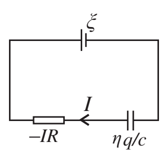
Hence, $$\rho = \frac{T}{\varepsilon\varepsilon_0 \ln 2} = 1.4 \times 10^{13}\ \Omega\text{m}$$

3.172 Suppose q is the charge at time t. Initially $q = C\xi$, at $t = 0$.

Then at time t, from Loop rule

$$IR = \frac{\eta q}{C} - \xi \tag{1}$$

Differentiating Eq. (1) with respect to time and considering that $I = -dq/dt$ ($-$sign because charge decreases)



$$R\frac{dI}{dt} = -\frac{\eta}{C} I$$

i.e., $$\frac{dI}{I} = \frac{-\eta}{RC} \cdot dt$$

Integration of this equation gives

$$\ln \frac{I}{I_0} = \frac{-\eta t}{RC}$$

So, $$I = I_0 e^{-\eta t/RC}$$

where I_0 is determined by condition of Eq. (1).
Indeed we can write

$$I_0 R = \frac{\eta q_0}{C} - \xi$$

where $q_0 = C\xi$ is the charge of the capacitor before its capacitance has charged.

Therefore, $$I_0 = (\eta - 1)\frac{\xi}{R}$$

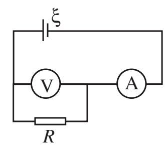
Finally, $$I = -\frac{dq}{dt} = \frac{\xi(\eta - 1)}{R}e^{-\eta t/RC}$$

3.173 Let r = internal resistance of the battery. We shall take the resistance of the ammeter to be = 0 and that of voltmeter to be G.

Initially, $$V = \xi - Ir,\ I = \frac{\xi}{r + G}$$

So, $$V = \xi\frac{G}{r + G} \qquad (1)$$

After the voltmeter is shunted



$$\frac{V}{\eta} = \xi - \frac{\xi r}{r + \frac{RG}{R + G}} \quad \text{(Voltmeter)} \qquad (2)$$

and $$\frac{\xi}{r + \frac{RG}{R + G}} = \eta\frac{\xi}{r + G} \quad \text{(Ammeter)} \qquad (3)$$

From Eqs. (2) and (3) we have

$$\frac{V}{\eta} = \xi - \frac{\eta r\xi}{r + G} \qquad (4)$$

From Eqs. (1) and (4)

$$\frac{G}{\eta} = r + G - \eta r \quad \text{or} \quad G = \eta r$$

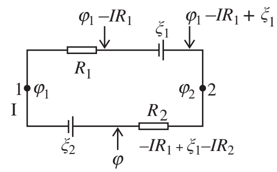
Then Eq. (1) gives the required reading

$$\frac{V}{n} = \frac{\xi}{\eta + 1}$$
$$= 2.0 \text{ V (on substituting values)}$$

3.174 Assume the current flow to be as shown in figure. Then potentials are as shown. Thus,

$$\varphi_1 = \varphi_1 - IR_1 + \xi_1 - IR_2 - \xi_2$$

or $$I = \frac{\xi_1 - \xi_2}{R_1 + R_2}$$



and $$\varphi_2 = \varphi_1 - IR_1 + \xi_1$$

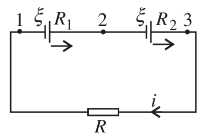
So, $$\varphi_1 - \varphi_2 = -\xi_1 + \frac{\xi_1 - \xi_2}{R_1 + R_2} R_1$$

$$= -\frac{(\xi_1 R_2 + \xi_2 R_1)}{(R_1 + R_2)}$$

$$= -4V$$

3.175 Let us consider the current I, flowing through the circuit, as shown in the figure.

Applying loop rule for the circuit, $-\Delta\varphi = 0$.



So, $$-2\xi + IR_1 + IR_2 + IR = 0$$

or $$2\xi = I(R_1 + R_2 + R)$$

or $$I = \frac{2\xi}{R + R_1 + R_2}$$

Now, if $$\varphi_1 - \varphi_2 = 0$$

$$-\xi + IR_1 = 0$$

or $$\frac{2\xi R_1}{R + R_1 + R_2} = \xi \quad \text{and} \quad 2R_1 = R_2 + R + R_1$$

or $$R = R_1 - R_2 \quad \text{(which is not possible as } R_2 > R_1\text{)}$$

Thus, $$\varphi_2 - \varphi_3 = -\xi + IR_2 = 0$$

or $$\frac{2\xi R_2}{R + R_1 + R_2} = \xi$$

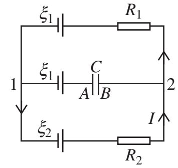
So, $$R = R_2 - R_1$$

which is the required resistance.

3.176 (a) Current, $$I = \frac{N\xi}{NR} = \frac{N\alpha R}{NR} = \alpha \quad (\text{as } \xi = \alpha R)$$

(b) $$\varphi_A - \varphi_B = n\xi - nIR = n\alpha R - n\alpha R = 0$$

3.177 Since the capacitor is fully charged, no current flows through it. So, current is given by



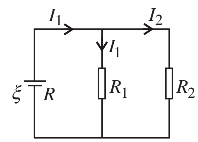
$$I = \frac{\xi_2 - \xi_1}{R_1 + R_2} \quad (\text{as } \xi_2 > \xi_1)$$

Hence, $$\varphi_A - \varphi_B = \xi_1 - \xi_2 + IR_2$$

$$= \xi_1 - \xi_2 + \frac{\xi_2 - \xi_1}{R_1 + R_2} R_2$$

$$= \frac{(\xi_1 - \xi_2)R_1}{R_1 + R_2} = -0.5 \text{ V}$$

3.178 Let us make the current distribution, as shown in the figure.



Current $I = \dfrac{\xi}{R + \dfrac{R_1 R_2}{R_1 + R_2}}$ (using loop rule)

So, current through the resistor R_1,

$$I_1 = \frac{\xi}{R + \dfrac{R_1 R_2}{R_1 + R_2}} \frac{R_2}{R_1 + R_2}$$

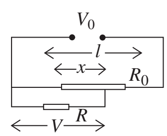
$$= \frac{\xi R_2}{RR_1 + RR_2 + R_1 R_2} = 1.2 \text{ A}$$

and similarly, current through the resistor R_2,

$$I_2 = \frac{\xi}{R + \dfrac{R_1 R_2}{R_1 + R_2}} \frac{R_1}{R_1 + R_2} = \frac{\xi R_1}{RR_1 + R_1 R_2 + RR_2} = 0.8 \text{ A}$$

3.179 Total resistance

$$= \frac{l - x}{l} R_0 + \frac{R \cdot \dfrac{xR_0}{l}}{R + \dfrac{xR_0}{l}}$$



$$= \frac{l - x}{l} R_0 + \frac{xRR_0}{lR + xR_0}$$

$$= R_0 \left[\frac{l - x}{l} + \frac{xR}{lR + xR_0} \right]$$

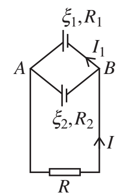
Then $$V = V_0 \frac{xR}{lR + xR_0} \bigg/ \left(1 - \frac{x}{l} + \frac{xR}{xR_0 + lR} \right)$$

$$= V_0 Rx \bigg/ \left\{ lR + R_0 x \left(1 - \frac{x}{l} \right) \right\}$$

For $R \gg R_0$, $$V \cong V_0 \frac{x}{l}$$

3.180 Let us connect a load of resistance R between the points A and B (see figure).

From the loop rule, $\Delta\varphi = 0$, so



$$IR = \xi_1 - I_1 R_1 \quad (1)$$

and

$$IR = \xi_2 - (I - I_1) R_2 \quad (2)$$

Solving Eqs. (1) and (2), we get

$$I = \frac{\xi_1 R_2 + \xi_2 R_1}{R_1 + R_2} \bigg/ R + \frac{R_1 R_2}{R_1 + R_2} = \frac{\xi_0}{R + R_0}$$

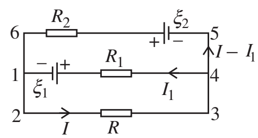
where

$$\xi_0 = \frac{\xi_1 R_2 + \xi_2 R_1}{R_1 + R_2} \text{ and } R_0 = \frac{R_1 R_2}{R_1 + R_2}$$

Thus one can replace the given arrangement of the cells by a single cell having the e.m.f. ξ_0 and internal resistance R_0.

3.181 Make the current distribution, as shown in the figure.

Now, in the loop 12341, applying $-\Delta\varphi = 0$, we get



$$IR + I_1 R_1 + \xi_1 = 0 \quad (1)$$

and in the loop 23562,

$$IR - \xi_2 + (I - I_1) R_2 = 0 \quad (2)$$

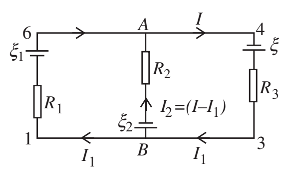
Solving Eqs. (1) and (2), we obtain current through the resistance R,

$$I = \frac{(\xi_2 R_1 - \xi_1 R_2)}{RR_1 + RR_2 + R_1 R_2} = 0.02\,\text{A}$$

and it is directed from left to the right.

3.182 At first indicate the currents in the branches using charge conservation (which also includes the point rule).

In the loops $1BA61$ and $B34AB$, from the loop rule using of $-\Delta\varphi = 0$, we get, respectively



$$-\xi_2 + (I - I_1) R_2 + \xi_1 - I_1 R_1 = 0 \quad (1)$$

$$IR_3 + \xi_3 - (I - I_1) R_2 + \xi_2 = 0 \quad (2)$$

On solving Eqs. (1) and (2), we obtain

$$I_1 = \frac{(\xi_1 - \xi_2) R_3 + R_2(\xi_1 + \xi_3)}{R_1 R_2 + R_2 R_3 + R_3 R_1}$$

$$\approx 0.06\text{ A}$$

Thus,

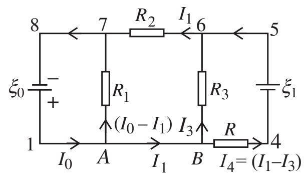
$$\varphi_A - \varphi_B = \xi_2 - I_2 R_2 = 0.9\,\text{V}$$

3.183 Indicate the currents in all the branches using charge conservation as shown in the figure. Applying loop rule and using $-\Delta\varphi = 0$ in the loops $1A781$, $1B681$ and $B456B$, respectively, we get

$$\xi_0 = (I_0 - I_1)R_1 \qquad (1)$$

$$I_3R_3 + I_1R_2 - \xi_0 = 0 \qquad (2)$$

$$(I_1 - I_3)R - \xi - I_3 R_3 = 0 \qquad (3)$$



Solving Eqs. (1), (2) and (3), we get the sought current

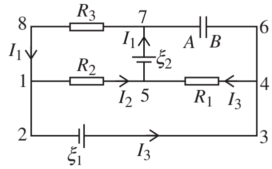
$$(I_1 - I_3) = \frac{\xi(R_2 + R_3) + \xi_0 R_3}{R(R_2 + R_3) + R_2R_3}$$

3.184 Indicate the currents in all the branches using charge conservation as shown in the figure.

Applying the loop rule and using $-\Delta\varphi = 0$ in the loops 12341 and 15781, we get

$$-\xi_1 + I_3R_1 - (I_1 - I_3)R_2 = 0 \qquad (1)$$

$$-\xi_2 + (I_1 - I_3)R_2 + I_1R_3 = 0 \qquad (2)$$



Solving Eqs. (1) and (2), we get

$$I_3 = \frac{\xi_1(R_2 + R_3) + \xi_2R_2}{R_1R_2 + R_2R_3 + R_3R_1}$$

Hence, the sought potential difference

$$\varphi_A - \varphi_B = \xi_2 - I_3R_1$$

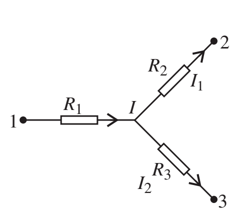
$$= \frac{\xi_2R_3(R_1 + R_2) - \xi_1R_1(R_2 + R_3)}{R_1R_2 + R_2R_3 + R_3R_1} = -1\text{V (on substituting values)}$$

3.185 Let us distribute the currents in the paths as shown in the figure.

Now, $\qquad \varphi_1 - \varphi_2 = IR_1 + I_1R_2 \qquad (1)$

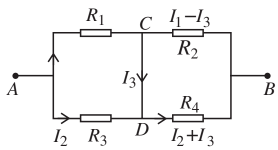
and $\qquad \varphi_1 - \varphi_3 = IR_1 + (I - I_1)R_3 \qquad (2)$

Simplifying Eqs. (1) and (2) we get



$$I = \frac{R_3(\varphi_1 - \varphi_2) + R_2(\varphi_1 - \varphi_3)}{R_1R_2 + R_2R_3 + R_3R_1} = 0.2\text{A}$$

3.186 Distribution of current is as shown in figure. From Kirchhoff's second law



$$I_1R_1 = I_2R_3$$
$$I_1R_1 + (I_1 - I_3)R_2 = V$$
$$I_2R_3 + (I_3 + I_2)\,R_4 = V$$

Eliminating I_2, we get

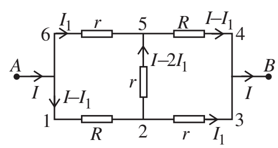
$$I_1(R_1 + R_2) - I_3R_2 = V$$
$$I_1\frac{R_1}{R_3}(R_3 + R_4) + I_3R_4 = V$$

Hence, $$I_3\left[R_4(R_1 + R_2) + \frac{R_1R_2}{R_3}(R_3 + R_4)\right] = V\left[(R_1 + R_2) - \frac{R_1}{R_3}(R_3 + R_4)\right]$$

$$I_3 = \frac{[R_3(R_1 + R_2) - R_1(R_3 + R_4)]V}{R_3R_4(R_1 + R_2) + R_1R_2(R_3 + R_4)}$$

On substitution, we get $I_3 = 1.0$ A from C to D.

3.187 From the symmetry of the problem, current flow is indicated, as shown in the figure.



Now, $\varphi_A - \varphi_B = I_1r + (I - I_1)R$ (1)

In the loop 12561, from $-\Delta\varphi = 0$

$$(I - I_1)\,R + (I - 2I_1)\,r - I_1r = 0$$

or $$I_1 = \frac{(R + r)I}{3r + R} \qquad (2)$$

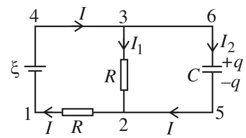
Equivalent resistance between the terminals A and B using Eqs. (1) and (2), is given by

$$R_0 = \frac{\varphi_A - \varphi_B}{I} = \frac{\left[\frac{R + r}{3r + R}(r - R) + R\right]I}{I} = \frac{r(3R + r)}{3r + R}$$

3.188 Let, at any moment of time, charge on the plates be $+q$ and $-q$, respectively, then voltage across the capacitor

$$\varphi = q/C \qquad (1)$$

Now, from charge conservation,



$$I = I_1 + I_2, \text{ where } I_2 = \frac{dq}{dt} \qquad (2)$$

In the loop 65146, using $-\Delta\varphi = 0$

$$\frac{q}{C} + \left(I_1 + \frac{dq}{dt}\right)R - \xi = 0 \qquad (3)$$

using Eq. (1) and Eq. (2).

In the loop 25632, using $-\Delta\varphi = 0$

$$-\frac{q}{C} + I_1 R = 0 \quad \text{or} \quad I_1 R = \frac{q}{C} \qquad (4)$$

Using Eq. (4) in Eq. (3)

$$\frac{dq}{dt} R = \xi_1 - \frac{2q}{C} \quad \text{or} \quad \frac{dq}{\xi - \frac{2q}{C}} = \frac{dt}{R} \qquad (5)$$

On integrating Eq. (5) between suitable limits,

$$\frac{\int_0^q dq}{\xi - \frac{2q}{C}} = \frac{1}{R}\int_0^t dt \quad \text{or} \quad -\frac{C}{2} \ln \frac{\xi - \frac{2q}{C}}{\xi} = \frac{t}{R}$$

Thus, $$\frac{q}{C} = V = \frac{1}{2}\xi(1 - e^{2t/RC})$$

3.189 (a) Since current I is a linear function of time, and at $t = 0$ and Δt, it equals I_0 and zero, respectively, it may be represented as

$$I = I_0\left(1 - \frac{t}{\Delta t}\right)$$

Thus, $$q = \int_0^{\Delta t} I\, dt = \int_0^{\Delta t} I_0\left(1 - \frac{t}{\Delta t}\right)dt = \frac{I_0 \Delta t}{2}$$

So, $$I_0 = \frac{2q}{\Delta t}$$

Hence, $$I = \frac{2q}{\Delta t}\left(1 - \frac{t}{\Delta t}\right)$$

The heat generated is given by

$$H = \int_0^{\Delta t} I^2 R dt = \int_0^{\Delta t} \left[\frac{2q}{\Delta t}\left(1 - \frac{t}{\Delta t}\right)\right]^2 R\, dt = \frac{4q^2 R}{3\Delta t}$$

(b) Obviously the current through the coil is given by

$$I = I_0 \left(\frac{1}{2}\right)^{t/\Delta t}$$

Then, $$q = \int_0^{\infty} I dt = \int_0^{\infty} I_0 2^{-t/\Delta t}\, dt = \frac{I_0 \Delta t}{\ln 2}$$

So, $$I_0 = \frac{q \ln 2}{\Delta t}$$

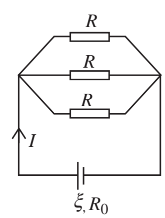
Hence, heat generated in the circuit in the time interval $t\,[0, \infty]$

$$H = \int_0^{\infty} I^2 R dt = \int_0^{\infty} \left[\frac{q \ln 2}{\Delta t}\, 2^{-t/\Delta t}\right]^2 R dt = -\frac{q^2 \ln 2}{2\Delta t} R$$

3.190 The equivalent circuit may be drawn as shown in the figure.

Resistance of the network $= R_0 + (R/3)$.

Let us assume that e.m.f. of the cell is ξ, then current



$$I = \frac{\xi}{R_0 + (R/3)}$$

Now the thermal power generated in the circuit

$$P = I^2 R/3 = \frac{\xi}{(R_0 + (R/3))^2}(R/3)$$

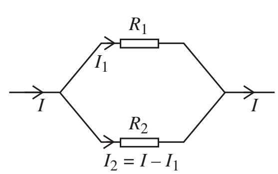
For P to be maximum,

$$\frac{dP}{dR} = 0, \text{ which yields}$$

$$R = 3R_0$$

3.191 We assume current conservation but not Kirchhoff's second law. Then thermal power dissipated is

$$P(I_1) = I_1^2 R_1 + (I - I_1)^2 R_2$$
$$= I_1^2 (R_1 + R_2) - 2II_1 R_2 + I^2 R_2$$
$$= [R_1 + R_2]\left[I_1 - \frac{R_2}{R_1 + R_2} I\right]^2 + I^2 \frac{R_1 R_2}{R_1 + R_2}$$



The resistances being positive, we see that the power dissipated is minimum when

$$I_1 = I\frac{R_2}{R_1 + R_2}$$

This corresponds to the usual distribution of currents over resistances joined in parallel.

3.192 Let internal resistance of the cell be r, then

$$V = \xi - Ir \tag{1}$$

where I is the current in the circuit. We know that thermal power generated in the battery is,

$$Q = I^2 r \tag{2}$$

Putting r from Eq. (1) in Eq. (2), we obtain

$$Q = (\xi - V)I = 0.6\,\text{W}$$

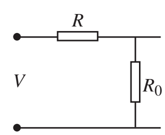
In a battery, work is done by electric forces whose origin lies in the chemical processes going on inside the cell. The work so done is stored and used in the electric circuit outside. Its magnitude just equals the power used in the electric circuit. We can say net power developed by the electric force is

$$P = -IV = -2.0\text{ W}$$

Minus sign means that power is generated and not consumed.

3.193 As far as the motor is concerned, the power delivered is displaced and can be represented by a load R_0. Thus,

$$I = \frac{V}{R + R_0}$$



and
$$P = I^2 R_0 = \frac{V^2 R_0}{(R_0 + R)^2}$$

This is maximum when $R_0 = R$ and the current I is then,

$$I = \frac{V}{2R}$$

The maximum power delivered is

$$\frac{V^2}{4R} = P_{\text{max}}$$

The power input is $\dfrac{V^2}{R + R_0}$ and its value when P is maximum is $\dfrac{V^2}{2R}$.

The efficiency then is $\dfrac{1}{2} = 50\%$.

3.194 If the wire diameter decreases by δ, then by the information given, power input

$$P = \frac{V^2}{R} = H \text{ (heat lost through the surface)}$$

Now, $H \propto (1 - \delta)$ like the surface area and $R \propto (1 - \delta)^{-2}$.

So, $$\frac{V^2}{R_0}(1 - \delta)^2 = A(1 - \delta) \quad \text{or} \quad V^2(1 - \delta) = \text{constant}$$

But, $$V \propto 1 + \eta$$

So, $$(1 + \eta)^2(1 - \delta) = \text{constant} = 1$$

Thus, $$\delta = 2\eta = 2\%$$

3.195 The equation of heat balance is

$$\frac{V^2}{R} - k(T - T_0) = C\frac{dT}{dt}$$

Put $$T - T_0 = \xi$$

So, $$C\dot{\xi} + k\xi = \frac{V^2}{R} \quad \text{or} \quad \dot{\xi} + \frac{k}{C}\xi = \frac{V^2}{CR}$$

or $$\frac{d}{dt}\xi e^{kt/C} = \frac{V^2}{CR}e^{kt/C}$$

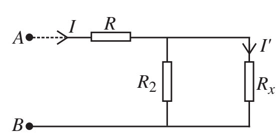
or $$\xi e^{kt/C} = \frac{V^2}{kR}e^{kt/C} + A$$

where A is a constant. Clearly,

$$\xi = 0 \text{ at } t = 0, \text{ so } A = -\frac{V^2}{kR}$$

Hence, $$T = T_0 + \frac{V^2}{kR}(1 - e^{-kt/C})$$

3.196 Let $\varphi_A - \varphi_B = \varphi$.



Now, thermal power generated in the resistance R_x

$$P = I'^2 R_x = \left[\frac{\varphi}{R_1 + \dfrac{R_2 R_x}{R_2 + R_x}} \frac{R_2}{R_2 + R_x}\right]^2 R_x$$

For P to be independent of R_x

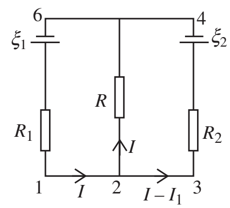
$$\frac{dP}{dR_x} = 0, \text{ which yields}$$

$$R_x = \frac{R_1 R_2}{R_1 + R_2} = 12\,\Omega$$

3.197 Indicate the current flow in the circuit as shown in the figure.

Applying loop rule in the closed loop 12561 and using $-\Delta\varphi = 0$, we get

$$I_1 R + I R_1 = \xi_1 \qquad (1)$$



and in the loop 23452,

$$(I - I_1)R_2 - I_1 R = -\xi_2 \qquad (2)$$

Solving Eqs. (1) and (2), we get

$$I_1 = \frac{\xi_1 R_2 + \xi_2 R_1}{RR_1 + R_1 R_2 + RR_2}$$

So, thermal power generated in the resistance R is given by

$$P = I_1^2\, R = \left[\frac{\xi_1 R_2 + \xi_2 R_1}{RR_1 + R_1 R_2 + RR_2}\right]^2 R$$

For P to be maximum,

$$\frac{dP}{dR} = 0, \text{ which yields}$$

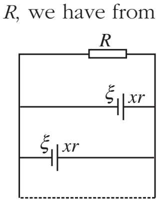
$$R = \frac{R_1 R_2}{R_1 + R_2}$$

Hence,
$$P_{\max} = \frac{(\xi_1 R_2 + \xi_2 R_1)^2}{4R_1 R_2\,(R_1 + R_2)}$$

3.198 Suppose that there are x number of cells connected in series, in each of the n parallel groups, then

$$nx = N \quad \text{or} \quad x = N/n \qquad (1)$$

Now, for any one of the loop, consisting of x cells and the resistor R, we have from the loop rule



$$IR + \frac{I}{n}\,xr - x\xi = 0$$

So,
$$I = \frac{x\xi}{R + xr/n} = \frac{N/n\;\xi}{R + Nr/n^2} \quad \text{(using Eq. 1)}$$

Heat generated in the resistor R, is given by

$$Q = I^2 R = \left(\frac{Nn\xi}{n^2 R + NR}\right)^2 R$$

and for Q to be maximum,

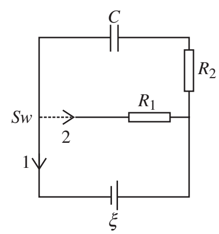
$$\frac{dQ}{dn} = 0, \text{ which yields}$$

$$n = \sqrt{\frac{Nr}{R}} = 3$$

3.199 When switch 1 is closed, maximum charge accumulated on the capacitor is given by

$$q_{\max} = C\xi \tag{1}$$

and when switch 2 is closed at any arbitrary instant of time,



$$(R_1 + R_2)\left(-\frac{dq}{dt}\right) = \frac{q}{c}$$

(because the capacitor is discharging),

or $$\int_{q_{\max}}^{q} \frac{1}{q}\, dq = -\frac{1}{(R_1 + R_2)C} \int_0^t dt$$

Integrating, we get

$$\ln \frac{q}{q_{\max}} = \frac{-t}{(R_1 + R_2)\, C} \quad \text{or} \quad q = q_{\max}\, e^{\frac{-t}{(R_1+R_2)C}} \tag{2}$$

Differentiating with respect to time, we get

$$I(t) = \frac{dq}{dt} = q_{\max}\, e^{\frac{-t}{(R_1+R_2)C}} \left(-\frac{1}{(R_1 + R_2)C}\right)$$

or $$I(t) = \frac{C\xi}{(R_1 + R_2)C}\, e^{\frac{-t}{(R_1+R_2)C}}$$

Negative sign is ignored, as we are not interested in the direction of the current.

Thus, $$I\,(t) = \frac{\xi}{(R_1 + R_2)}\, e^{\frac{-t}{(R_1+R_2)C}} \tag{3}$$

When the switch (Sw) is at the position 1, the charge (maximum) accumulated on the capacitor is, $q = C\xi$.

When the switch (Sw) is thrown to position 2, the capacitor starts discharging and as a result, the electric energy stored in the capacitor totally turns into heat energy through the resistors R_1 and R_2 (during a very long interval of time). Thus from the energy conservation, the total heat liberated through the resistors is given by

$$H = U_i = \frac{q^2}{2C} = \frac{1}{2} C\xi^2$$

During the process of discharging of the capacitor, the current through the resistors R_1 and R_2 is the same at all moments of time, thus

$$H_1 \propto R_1 \text{ and } H_2 \propto R_2$$

So, $$H_1 = \frac{HR_1}{(R_1 + R_2)} \text{ (as } H = H_1 + H_2)$$

Hence, $$H_1 = \frac{1}{2} \frac{CR_1}{R_1 + R_2} \xi^2$$

$$= 60 \text{ mJ (on substituting values)}$$

3.200 When the plate is absent, the capacity of the condenser is

$$C = \frac{\varepsilon_0 S}{d}$$

When the plate is present, the capacity is

$$C' = \frac{\varepsilon_0 S}{d(1 - \eta)} = \frac{C}{1 - \eta}$$

(a) The energy increment is clearly

$$\Delta U = \frac{1}{2} CV^2 - \frac{1}{2} C' V^2 = \frac{C\eta}{2(1 - \eta)} V^2$$

(b) The charge on the plate is

$$q_i = \frac{CV}{1 - \eta} \text{ (initially) and } q_f = CV \text{ (finally)}$$

A charge $CV\eta/1 - \eta$ has flown through the battery, charging it and withdrawing $CV^2\eta/1 - \eta$ units of energy from the system into the battery. The energy of the capacitor has decreased by just half of this. The remaining half, i.e., (1/2) $CV^2\eta/1 - \eta$ is the work done by the external agent in withdrawing the plate. This ensures conservation of energy.

3.201 Initially, capacitance of the system $= C\varepsilon$.

So, initial energy of the system $U_i = \frac{1}{2}(C\varepsilon)V^2$

and final energy of the capacitor $U_f = \frac{1}{2}CV^2$

Hence capacitance energy increment

$$\Delta U = \frac{1}{2}CV^2 - \frac{1}{2}(C\varepsilon)V^2 = -\frac{1}{2}CV^2(\varepsilon - 1) = -0.5 \text{ mJ}$$

From energy conservation

$$\Delta U = A_{\text{cell}} + A_{\text{agent}}$$

as there is no heat being released.

But, $$A_{\text{cell}} = (C_f - C_i)V^2 = (C - C\varepsilon)V^2$$

Hence, $$A_{\text{agent}} = \Delta U - A_{\text{cell}}$$

$$= \frac{1}{2}C(\varepsilon - 1)V^2 = 0.5 \text{ mJ}$$

3.202 If C_0 is the initial capacitance of the condenser before water rises in it then

$$U_i = \frac{1}{2}C_0V^2, \text{ where } C_0 = \frac{\varepsilon_0 2l\pi R}{d}$$

(R is the mean radius and l is the length of the capacitor plates.)

Suppose the liquid rises to a height h in it. Then capacitance of the condenser is

$$C = \frac{\varepsilon\varepsilon_0 h 2\pi R}{d} + \frac{\varepsilon(l - h)\,2\pi R}{d} = \frac{\varepsilon_0 2\pi R}{d}(l + (\varepsilon - 1)\,h)$$

and energy of the capacitor and the liquid (including both gravitational and electrostatic contributions) is

$$\frac{1}{2}\frac{\varepsilon_0 2\pi R}{d}(l + (\varepsilon - 1)\,h)V^2 + \rho g\,(2\pi R\,hd)\,\frac{h}{2}$$

If the capacitor were not connected to a battery this energy would have to be minimized. But the capacitor is connected to the battery and, in effect the potential energy of the whole system has to be minimized. Suppose we increase h by δh. Then the energy the capacitor and the liquid increases by

$$\delta h\left(\frac{\varepsilon_0\,2\pi R}{2d}(\varepsilon - 1)\,V^2 + \rho g\,(2\pi Rd)\,h\right)$$

and that of the cell diminishes by the quantity A_{cell} which is the product of charge flown and V

$$\delta h \frac{\varepsilon_0 (2\pi R)}{d} (\varepsilon - 1) V^2$$

In equilibrium, the two must balance, so

$$\rho g d h = \frac{\varepsilon_0 (\varepsilon - 1) V^2}{2d}$$

Hence, $$h = \frac{\varepsilon_0 (\varepsilon - 1) V^2}{2\rho g d^2}$$

3.203 (a) Let us imagine a thin spherical layer with inner and outer radii r and $r + dr$, respectively. Lines of current at all the points of this layer are perpendicular to it and therefore such a layer can be treated as a spherical conductor of thickness dr and cross sectional area $4\pi r^2$, Now, we know that resistance,

$$dR = \rho \frac{dr}{S(r)} = \rho \frac{dr}{4\pi r^2} \tag{1}$$

Integrating Eq. (1) between the limits,

$$\int_0^R dR = \int_a^b \rho \frac{dr}{4\pi r^2} \quad \text{or} \quad R = \frac{\rho}{4\pi}\left[\frac{1}{a} - \frac{1}{b}\right] \tag{2}$$

Capacitance of the network,

$$C = \frac{4\pi\varepsilon_0\varepsilon}{\left[\frac{1}{a} - \frac{1}{b}\right]} \tag{3}$$

and $$q = C\varphi \left[\begin{array}{l}\text{where } q \text{ is the charge} \\ \text{at any arbitrary moment}\end{array}\right] \tag{4}$$

also, $$\varphi = \left(\frac{-dq}{dt}\right) R, \text{ as capacitor is discharging.} \tag{5}$$

From Eqs. (2), (3), (4) and (5), we get

$$q = \frac{4\pi\varepsilon_0\varepsilon}{\left[\frac{1}{a} - \frac{1}{b}\right]} \frac{\left[-\frac{dq}{dt}\right]\rho\left[\frac{1}{a} - \frac{1}{b}\right]}{4\pi} \quad \text{or} \quad \frac{dq}{q} = \frac{dt}{\rho\varepsilon\varepsilon_0}$$

Integrating $$\int_{q_0}^{q} -\frac{dq}{q} = \frac{1}{\rho\varepsilon_0\varepsilon}\int_0^t dt = \frac{dt}{\rho\varepsilon\varepsilon_0}$$

Hence, $$q = q_0 e^{-t/\rho\varepsilon_0\varepsilon}$$

(b) From energy conservation, heat generated during the spreading of the charge

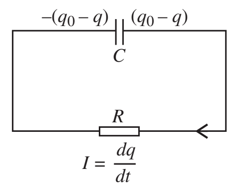
$$H = U_i - U_f \text{ (because } A_{\text{cell}} = 0)$$
$$= \frac{1}{2}\frac{q_0^2}{4\pi\varepsilon_0\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right] - 0 = \frac{q_0^2}{8\pi\varepsilon_0\varepsilon}\,\frac{b-a}{ab}$$

3.204 (a) Let, at any moment of time, charge on the plates be $(q_0 - q)$ then, current through the resistor

$$I = -\frac{d(q_0 - q)}{dt}$$

because the capacitor is discharging.

or $$I = \frac{dq}{dt}$$



Now, applying loop rule in the circuit

$$IR - \frac{q_0 - q}{C} = 0$$

or $$\frac{dq}{dt}R = \frac{q_0 - q}{C}$$

or $$\frac{dq}{q_0 - q} = \frac{1}{RC}\,dt$$

At $t = 0$, $q = 0$ and at $t = \tau$, $q = q$.

So, $$\ln\frac{q_0 - q}{q_0} = \frac{-\tau}{RC}$$

Thus, $$q = q_0\,(1 - e^{-\tau/RC}) = 0.18 \text{ mC}$$

(b) Amount of heat generated = decrease in capacitance energy

$$= \frac{1}{2}\frac{q_0^2}{C} - \frac{1}{2}\frac{\left[q_0 - q_0\,(1 - e^{-\tau/RC})\right]^2}{C}$$
$$= \frac{1}{2}\frac{q_0^2}{C}[1 - e^{-2\tau/RC}] = 82 \text{ mJ}$$

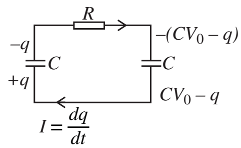
3.205 Let at any moment of time, charge flown be q, then current $I = dq/dt$.

Applying loop rule in the circuit and using $\Delta\varphi = 0$, we get

$$\frac{dq}{dt}R - \frac{(CV_0 - q)}{C} + \frac{q}{C} = 0$$

or $$\frac{dq}{CV_0 - 2q} = \frac{1}{RC}\,dt$$

So, $$\ln \frac{CV_0 - 2q}{CV_0} = -2\frac{t}{RC} \quad (\text{for } 0 \le t \le t)$$



or $$q = \frac{CV_0}{2}(1 - e^{-2t/RC})$$

Hence, $$I = \frac{dq}{dt} = \frac{CV_0}{2}\frac{2}{RC}e^{-2t/RC} = \frac{V_0}{R}e^{-2t/RC}$$

Now, heat liberated will be given by

$$Q = \int_0^\infty I^2 R dt = \frac{V_0^2}{R^2} R \int_0^\infty e^{-4t/RC} dt = \frac{1}{4} CV_0^2$$

3.206 In a rotating frame, to first order in $\boldsymbol{\omega}$, the main effect is a Coriolis force given by $2m(\mathbf{v} \times \boldsymbol{\omega})$. This unbalanced force will cause electrons to react by setting up a magnetic field $\mathbf{B}$ so that the magnetic force $e(\mathbf{v} \times \mathbf{B})$ balances the Coriolis force.

Thus, $$-\frac{e}{2m}\mathbf{B} = \boldsymbol{\omega} \quad \text{or} \quad \mathbf{B} = -\frac{2m}{e}\boldsymbol{\omega}$$

The flux associated with this is

$$\Phi = N\pi r^2 B = N\pi r^2 \frac{2m}{e}\omega$$

where $N = l/2\pi r$ is the number of turns of the ring. If ω changes (and there is time for the electron to rearrange) then, B also changes and so does Φ. An e.m.f. will be induced and a current will flow, which is given by

$$I = N\pi r^2 \frac{2m}{e}\omega/R$$

The total charge flowing through the ballastic galvanometer, as the ring is stopped, is

$$q = N\pi r^2 \Big/ \frac{2m}{e}\omega/R$$

So, $$\frac{e}{m} = \frac{2N\pi r^2\omega}{qR} = \frac{l\omega r}{qR}$$

$$= 1.8 \times 10^{11}\ \text{C/kg (on substituting values)}$$

3.207 Let n_0 be the total number of electrons then, total momentum of electrons,

$$p = n_0 m_e v_d \quad (\text{where } v_d \text{ is drift velocity}) \qquad (1)$$

Now, $$I = \rho S_x v_d = \frac{n_0 e}{V} S_x v_d = \frac{ne}{l} v_d \qquad (2)$$

Here S_x = cross sectional area, ρ = electron charge density, V = volume of sample. From Eqs. (1) and (2)

$$p = \frac{m_e}{e} Il = 0.40 \ \mu \text{Ns}$$

3.208 By definition $nev_d = j$, where v_d is the drift velocity, n is number density of electrons.

Then $$\tau = \frac{l}{v_d} = \frac{nel}{j}$$

So distance actually travelled

$$S = \langle v \rangle \tau = \frac{nel \langle v \rangle}{j} = 10^7 \text{ m}$$

($\langle v \rangle$ = mean velocity of thermal motion of an electron.)

3.209 (a) Let n be the volume density of electrons, then from $I = \rho S v_d$, we get

$$I = neS|\langle \mathbf{v} \rangle| = neS\frac{l}{t}$$

So, $$t = \frac{neSl}{I} = 3 \text{ Ms}$$

(b) Sum of electric forces is given by

$$|(nv)\, e\mathbf{E}\ | = |\, n\, Sle\rho\mathbf{j}| \quad \text{(where } \rho \text{ is resistivity of the material)}$$

$$= nSle\rho\frac{I}{S} = nel\,\rho I = 1.0 \text{ MN}$$

3.210 From Gauss' theorem, field strength at a surface of a cylindrical shape equals, $\lambda/2\pi\varepsilon_0 r$, where λ is the linear charge density.

Now, $$eV = \frac{1}{2} m_e v^2 \quad \text{or} \quad v = \sqrt{\frac{2eV}{m_e}} \qquad (1)$$

Also, $$dq = \lambda dx \quad \text{so,} \quad \frac{dq}{dt} = \lambda\frac{dx}{dt}$$

or $$I = \lambda v \quad \text{or} \quad \lambda = \frac{I}{v} = \frac{I}{\sqrt{\frac{2eV}{m_e}}} \quad \text{(using Eq. 1)}$$

Hence, $$E = \frac{I}{2\pi\varepsilon_0 r}\sqrt{\frac{m_e}{2eV}} = 32 \text{ V/m}$$

For the point inside the solid charged cylinder, applying Gauss' theorem

$$2\pi rhE = \pi r^2 h \frac{q}{\varepsilon_0 \pi R^2 l}$$

or
$$E = \frac{q/l}{2\pi\varepsilon_0 R^2}\, r = \frac{\lambda r}{2\pi\varepsilon_0 R^2}$$

So, from
$$E = -\frac{d\varphi}{dr}$$

we get
$$\int_{\varphi_1}^{\varphi_2} - \, d\varphi = \int_0^R \frac{\lambda}{2\pi\varepsilon_0 R^2}\, r dr$$

or
$$\varphi_1 - \varphi_2 = \frac{\lambda}{2\pi\varepsilon_0 R^2}\left[\frac{r^2}{2}\right]_0^R = \frac{\lambda}{4\pi\varepsilon_0}$$

Hence,
$$\varphi_1 - \varphi_2 = \frac{I}{4\pi\varepsilon_0}\sqrt{\frac{m_e}{2eV}} = 0.80 \text{ V}$$

3.211 (a) Between the plates

$$\varphi = ax^{4/3}$$

or
$$\frac{\partial\varphi}{\partial x} = a \times \frac{4}{3}x^{l/3}$$

$$\frac{\partial^2\varphi}{\partial x^2} = \frac{4}{9}\, ax^{-2/3} = \frac{-\rho}{\varepsilon_0}$$

or
$$\rho = -\frac{4\varepsilon_0 a}{9}\, x^{-2/3}$$

(b) Let the charge on the electron be $-e$.

Then
$$\frac{1}{2}\, mv^2 - e\varphi = \text{constant} = 0$$

(as the electron is initially emitted with negligible energy),

or
$$v^2 = \frac{2e\varphi}{m} \quad \text{and} \quad v = \sqrt{\frac{2e\varphi}{m}}$$

So,
$$j = -\rho v = \frac{4\varepsilon_0 a}{9}\sqrt{\frac{2\varphi e}{m}}\, x^{-2/3} = \frac{4}{9}\,\varepsilon_0 a^{3/2}\sqrt{\frac{2e}{m}}$$

(j is measured from the anode to cathode, hence the $-ve$ sign.)

3.212 We know that,
$$E = \frac{V}{d}$$

So by the definition of the mobility

$$v^+ = u_0^+ \frac{V}{d}, \quad v^- = u_0^- - \frac{V}{d}$$

and
$$j = (n_+ u_0^+ + n_- u_0^-) \frac{eV}{d}$$

The negative ions move towards the anode and the positive ions towards the cathode and the total current is the sum of the currents due to them.

On the other hand, in equilibrium

$$n_+ = n_-$$

So,
$$n_+ = n_- = \frac{I}{S} \Big/ (u_0^+ + u_0^-) \frac{eV}{d}$$

$$= \frac{Id}{eVS\,(u_0^+ + u_0^-)} = 2.3 \times 10^8 \text{ cm}^{-3}$$

3.213 Velocity = mobility × field

or
$$v = u \frac{V_0}{l} \sin \omega t \quad (\text{which is positive for } 0 \leq \omega t \leq \pi)$$

So, the maximum displacement in one direction is

$$x_{\max} = \int_0^{\pi} u \frac{V_0}{l} \sin \omega t dt = \frac{2uV_0}{l\,\omega}$$

At $\omega = \omega_0$, $x_{\max} = l$ so, $\frac{2uV_0}{l\omega_0} = l$

Thus,
$$u = \frac{\omega_0 l^2}{2V_0}$$

3.214 When the current is saturated, all the ions produced reach the plate.

Then,
$$\dot{n}_i = \frac{I_{\text{sat}}}{eV} = 6 \times 10^9 \text{ cm}^{-3}\text{ s}^{-1}$$

(Both positive ions and negative ions are counted here.)

The equation of balance is

$$\frac{dn}{dt} = \dot{n}_i - rn^2$$

The first term on the right is the production rate and the second term is the recombination rate which by the usual statistical arguments is proportional to n^2 (no of positive ions × no. of $-ve$ ions). In equilibrium

$$\frac{dn}{dt} = 0$$

So,
$$n_{\text{eq}} = \sqrt{\frac{\dot{n}_i}{r}} = 6 \times 10^7 \text{ cm}^{-3}$$

3.215 Initially, $$n = n_0 = \sqrt{\dot{n}_i/r}$$

Since we can assume that long exposure to the ionizer has caused equilibrium to be set up. After the ionizer is switched off,

$$\frac{dn}{dt} = -rn^2$$

or $$rdt = -\frac{dn}{n^2} \quad \text{or} \quad rt = \frac{1}{n} + \text{constant}$$

But, $$n = n_0 \text{ at } t = 0 \quad \text{so,} \quad rt = \frac{1}{n} - \frac{1}{n_0}$$

The concentration will decrease by a factor η when

$$rt_0 = \frac{1}{n_0/\eta} - \frac{1}{n_0} = \frac{\eta - 1}{n_0}$$

or $$t_0 = \frac{\eta - 1}{\sqrt{r\,\dot{n}_i}} = 13 \text{ ms}$$

3.216 Ions produced will cause charge to decay. Clearly,

$$\eta CV = \text{decrease of charge} = \dot{n}_i\, eAdt = \frac{\varepsilon_0 A}{d} V\eta$$

or $$t = \frac{\varepsilon_0 V\eta}{\dot{n}_i ed^2} = 4.6 \text{ days}$$

Note that n_i here, is the number of ion pairs produced.

3.217 If ν = number of electrons moving to the anode at distance x, then

$$\frac{dv}{dx} = \alpha v \quad \text{or} \quad v = v_0\, e^{\alpha x}$$

Assuming saturation, $$I = ev_0 e^{\alpha d}$$

3.218 Since the electrons are produced uniformly through the volume, the total current attaining saturation is clearly,

$$I = \int_0^d e(\dot{n}_i\, Adx)\, e^{\alpha x} = e\dot{n}_i A\left(\frac{e^{\alpha d} - 1}{\alpha}\right)$$

Thus, $$j = \frac{I}{A} = e\dot{n}_i\left(\frac{e^{\alpha d} - 1}{\alpha}\right)$$

3.5 Constant Magnetic Field. Magnetics

3.219 (a) From Biot-Savart's law

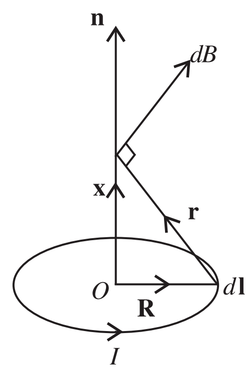
$$d\mathbf{B} = \frac{\mu_0}{4\pi} I \frac{d\mathbf{l} \times \mathbf{r}}{r^3}$$

So,
$$d\mathbf{B} = \frac{\mu_0}{4\pi} I \frac{(Rd\theta)R}{R^3} (\text{as } d\mathbf{l} \perp \mathbf{r})$$

From symmetry

$$B = \int dB = \int_0^{2\pi} \frac{\mu_0}{4\pi}\frac{I}{R}\, d\theta = \frac{\mu_0}{2}\frac{I}{R} = 6.3\,\mu\text{T}$$

(b) From Biot-Savart's law



$$\mathbf{B} = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l} \times \mathbf{r}}{r^3} \quad (\text{here } \mathbf{r} = \mathbf{x} - \mathbf{R})$$

So,
$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I}{r^3}\left[\oint d\mathbf{l} \times \mathbf{x} - \oint d\mathbf{l} \times \mathbf{R}\right]$$

Since $\mathbf{x}$ is a constant vector and $|\mathbf{R}|$ is also constant, so,

$$\oint d\mathbf{l} \times \mathbf{x} = \left(\oint d\mathbf{l}\right) \times \mathbf{x} = 0 \left(\text{because} \oint d\mathbf{l} = \mathbf{0}\right)$$

and
$$-\oint d\mathbf{l} \times \mathbf{R} = \oint R\,dl\,\mathbf{n} = \mathbf{n}R\oint dl = 2\pi R^2 \mathbf{n}$$

Here $\mathbf{n}$ is a unit vector perpendicular to the plane containing the current loop (see figure) and in the direction of $\mathbf{x}$.

Thus, we get
$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{2\pi R^2 I}{(x^2 + R^2)^{3/2}}\mathbf{n}$$
$$= 2.3\,\mu\text{T}$$

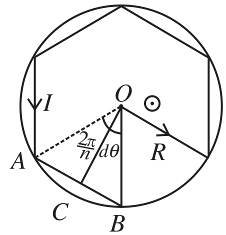
3.220 Since $\angle AOB = 2\pi/n, OC$ or perpendicular distance of any segment from center equals $R\cos\pi/n$. Now magnetic induction at O, due to the right current carrying element AB

$$= \frac{\mu_0}{4\pi}\frac{I}{R\cos\pi/n} 2\sin\frac{\pi}{n}$$

From Biot–Savart's law, the magnetic field at O due to any section such as AB is perpendicular to the plane of the figure and has the magnitude

$$B = \int \frac{\mu_0}{4\pi} I \frac{dx}{r^2}\cos\theta = \int_{-\frac{\pi}{n}}^{\frac{\pi}{n}} \frac{\mu_0 I}{4\pi}\frac{R\cos\pi/n\ \sec^2\theta\, d\theta}{R^2\cos 2\pi/n\sec^2\theta} cos\theta = \frac{\mu_0 I}{4\pi}\frac{1}{R\cos\pi/n} 2\sin\frac{\pi}{n}$$

As there are n number of sides and magnetic induction vectors, due to each side at O, are equal in magnitude and direction. So



$$B_0 = \frac{\mu_0}{4\pi} \frac{nI}{R\cos\frac{\pi}{n}} 2\sin\frac{\pi}{n} \cdot n$$

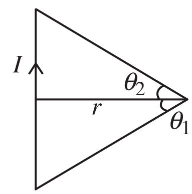
$$= \frac{\mu_0}{2\pi} \frac{nI}{R} \tan\frac{\pi}{n}$$

For $n \to \infty$

$$B_0 = \frac{\mu_0}{2} \frac{I}{R} \lim_{n\to\infty} \left(\frac{\tan\frac{\pi}{n}}{\pi/n} \right)$$

$$= \frac{\mu_0}{2} \frac{I}{R}$$

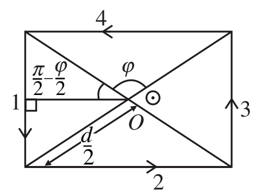
3.221 We know that magnetic induction due to a straight current carrying wire at any point, at a perpendicular distance from it is given by



$$B = \frac{\mu_0}{4\pi} \frac{I}{r} (\sin\theta_1 + \sin\theta_2) \text{ (as shown in figure)}$$

Here, $$B_1 = B_3 = \frac{\mu_0}{4\pi} \frac{I}{(d/2)\sin\frac{\varphi}{2}} \left\{ \cos\frac{\varphi}{2} + \cos\frac{\varphi}{2} \right\}$$

and $$B_2 = B_4 = \frac{\mu_0}{4\pi} \frac{I}{(d/2)\cos\frac{\varphi}{2}} \left(\sin\frac{\varphi}{2} + \sin\frac{\varphi}{2} \right)$$



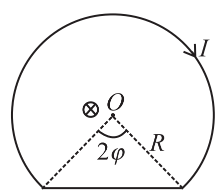
Hence, the magnitude of total magnetic induction at O,

$$B_0 = B_1 + B_2 + B_3 + B_4$$

$$= \frac{\mu_0}{4\pi} \frac{4I}{d/2} \left[\frac{\cos\frac{\varphi}{2}}{\sin\frac{\varphi}{2}} + \frac{\sin\frac{\varphi}{2}}{\cos\frac{\varphi}{2}} \right]$$

$$= \frac{4\mu_0 I}{\pi d \sin\varphi} = 0.10 \text{ mT}$$

3.222 Magnetic induction due to the arc segment at O is given by



$$B_{arc} = \frac{\mu_0}{4\pi} \frac{I}{R} (2\pi - 2\varphi)$$

and magnetic induction due to the line segment at O is given by

$$B_{\text{line}} = \frac{\mu_0}{4\pi} \frac{I}{R \cos \varphi} [2 \sin \varphi]$$

So, total magnetic induction

$$B_0 = B_{\text{arc}} + B_{\text{line}} = \frac{\mu_0}{2\pi} \frac{I}{R} [\pi - \varphi + \tan \varphi] = 28 \ \mu\text{T}$$

3.223 (a) From the Biot-Savart law

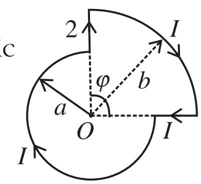
$$d\mathbf{B} = \frac{\mu_0}{4\pi} I \frac{(d\mathbf{l} \times \mathbf{r})}{r^3}$$

So, magnetic field induction due to the arc of radius a at O is given by

$$B_a = \frac{\mu_0}{4\pi} \frac{I}{a} (2\pi - \varphi) \text{ (directed inward)}$$

and magnetic field induction due to the arc of radius b at O is given by

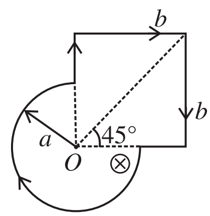
$$B_b = \frac{\mu_0}{4\pi} \frac{I}{a} \varphi \text{ (directed inward)}$$



Due to each of straight current segments 1 and 2, magnetic induction is zero, because $d\mathbf{l} \parallel \mathbf{r}$.

Hence, $$B_0 = \frac{\mu_0}{4\pi} I \left[\frac{2\pi - \varphi}{a} + \frac{\varphi}{b} \right]$$

(b) Due to each of straight currents directed towards (away) center, magnetic induction vector at the center is zero. From the symmetry, direction of magnetic induction vector due to rest of two straight wires and the arc current, all are directed normally into the plane of the figure. Magnetic induction due to each of straight current is



$$\frac{\mu_0}{4\pi} \frac{I}{b} \sin 45°$$

and due to arc current is

$$\frac{\mu_0}{4\pi} \frac{I}{a} \left(\frac{3\pi}{2} \right)$$

Hence, the total magnetic induction at O is given by

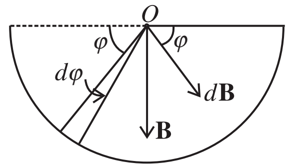
$$B_0 = \frac{\mu_0}{4\pi} \frac{I}{a} \left(\frac{3\pi}{2} \right) + 2 \times \frac{\mu_0}{4\pi} \frac{I}{b} \sin 45° = \frac{\mu_0}{4\pi} \frac{I}{n} \left[\frac{3\pi}{2a} + \frac{\sqrt{2}}{b} \right]$$

3.224 The thin walled tube with a longitudinal slit can be considered equivalent to a full tube and a strip carrying the same current density in the opposite direction. Inside the tube, the former does not contribute so the total magnetic field is simply due to the strip. It is

$$B = \frac{\mu_0}{2\pi} \frac{(I/2\pi R) \ h}{r} = \frac{\mu_0}{4\pi^2} \frac{Ih}{Rr}$$

where r is the distance of the field point from the strip.

3.225 First of all let us find out the direction of vector **B** at point *O*. For this purpose, we divide the entire conductor into elementary fragments with current *dI*. It is obvious that the sum of any two symmetric fragments gives a resultant along **B** as shown in the figure and consequently, vector **B** will also be directed as shown.



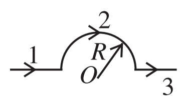
So, $$|\mathbf{B}| = \int dB \sin\varphi$$

$$= \int \frac{\mu_0}{2\pi R}\, dI \sin\varphi$$

$$= \int_0^{\pi} \frac{\mu_0}{2\pi^2 R} I \sin\varphi\, d\varphi \quad \left(as\ dI = \frac{I}{\pi}\, d\varphi\right)$$

Hence, $$B = \frac{\mu_0 I}{\pi^2 R}$$

3.226 From symmetry, for Fig. (a)

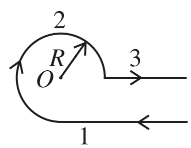


(*a*)

$$B_0 = B_1 + B_2 + B_3$$

$$= 0 + \frac{\mu_0}{4\pi}\frac{I}{R}\pi + 0 = \frac{\mu_0}{4}\frac{I}{R}$$

From symmetry, for Fig. (b)



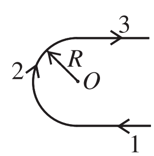
(*b*)

$$B_0 = B_1 + B_2 + B_3$$

$$= \frac{\mu_0}{4\pi}\frac{I}{R} + \frac{\mu_0}{2\pi}\frac{I}{R}\frac{3\pi}{2} + 0$$

$$= \frac{\mu_0}{4\pi}\frac{I}{R}\left[1 + \frac{3\pi}{2}\right]$$

From symmetry, for Fig. (c)



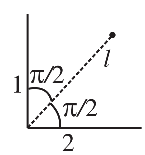
(*c*)

$$B_0 = B_1 + B_2 + B_3$$

$$= \frac{\mu_0}{4\pi}\frac{I}{R} + \frac{\mu_0}{4\pi}\frac{I}{R}\pi + \frac{\mu_0}{4\pi}\frac{I}{R}$$

$$= \frac{\mu_0}{4\pi}\frac{I}{R}(2 + \pi)$$

3.227 $$\mathbf{B}_0 = \mathbf{B}_1 + \mathbf{B}_2$$



or $$|\mathbf{B}_0| = \frac{\mu_0 I}{4\pi l}\sqrt{2} = 2.0\ \mu\text{T (using Problem 3.221)}$$

3.228 (a)
$$\mathbf{B}_0 = \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3$$
$$= \frac{\mu_0}{4\pi}\frac{I}{R}(-\mathbf{k}) + \frac{\mu_0}{4\pi}\frac{I}{R}\pi(-\mathbf{i}) + \frac{\mu_0}{4\pi}\frac{I}{R}(-\mathbf{k})$$
$$= -\frac{\mu_0}{4\pi}\frac{I}{R}[2\mathbf{k} + \pi\,\mathbf{i}]$$

So, $$|\mathbf{B}_0| = \frac{\mu_0}{4\,\pi}\frac{I}{R}\sqrt{\pi^2 + 4} = 0.30\ \mu\text{T}$$

(b)
$$\mathbf{B}_0 = \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3$$
$$= \frac{\mu_0}{4\pi}\frac{I}{R}(-\mathbf{k}) + \frac{\mu_0}{4\pi}\frac{I}{R}\pi(-\mathbf{i}) + \frac{\mu_0}{4\pi}\frac{I}{R}(-\mathbf{i})$$
$$= -\frac{\mu_0}{4\pi}\frac{I}{R}[\mathbf{k} + (\pi + 1)\,\mathbf{i}]$$

So, $$|\mathbf{B}_0| = \frac{\mu_0}{4\pi}\frac{I}{R}\sqrt{1 + (\pi + 1)^2} = 0.34\ \mu\text{T}$$

(c) Here, using the law of parallel resistances

$$I_1 + I_2 = I \quad \text{and} \quad \frac{I_1}{I_2} = \frac{1}{3}$$

So, $$\frac{I_1 + I_2}{I_2} = \frac{4}{3}$$

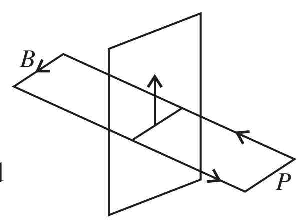
Hence, $$I_2 = \frac{3}{4}I \quad \text{and} \quad I_1 = \frac{1}{4}I$$

Thus, $$\mathbf{B}_0 = \frac{\mu_0}{4\pi}\frac{I}{R}(-\mathbf{k}) + \frac{\mu_0}{4\pi}\frac{I}{R}(-\mathbf{j}) + \left[\frac{\mu_0}{4\pi}\left(\frac{3\pi}{2}\right)\frac{I_1}{R}(-\mathbf{i}) + \frac{\mu_0}{4\pi}\frac{(\pi/2)\,I_2}{R}\mathbf{i}\right]$$
$$= \frac{-\mu_0}{4\pi}\frac{I}{R}(\mathbf{j} + \mathbf{k}) + 0$$

Thus, $$|\mathbf{B}_0| = \frac{\mu_0}{4\pi}\frac{\sqrt{2}I}{R} = 0.11\ \mu\text{T}$$

3.229 (a) We apply circulation theorem as shown in the figure. The current is vertically upwards in the plane and the magnetic field is horizontal and parallel to the plane.

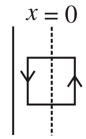
Then, $$\oint \mathbf{B}\cdot d\mathbf{l} = 2\,Bl = \mu_0\,Il \quad \text{or} \quad B = \frac{\mu_0 I}{2}$$



(b) Each plane contributes $\mu_0 I/2$ between the planes and outside the plane that is cancelled out.

Thus, $$B = \begin{cases} \mu_0 I \text{ between the plane} \\ 0 \text{ outside} \end{cases}$$

3.230 We assume that the current flows perpendicular to the plane of the paper, then by circulation theorem, inside the plate



$$2Bdl = \mu_0\,(2\,x\,dl)\,\mathrm{j}$$

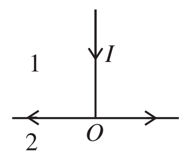
or $$B = \mu_0\,xj, |x| \le d$$

Outside the plate, $$2Bdl = \mu_0\,(2d\,dl)\,j$$

or $$B = \mu_0\,dj\,|x| \ge d$$

3.231 It is easy to convince oneself that in both the regions 1 and 2, there can only be a circular magnetic field (i.e., the component B_φ). Any radial field in region 1 or any B_z away from the current plane will imply a violation of Gauss' law of magnetostatics. B_φ must obviously be symmetrical about the straight wire. Then in 1,

$$B_\varphi\,2\pi r = \mu_0 I$$



or $$B_\varphi = \frac{\mu_0}{2\pi}\frac{I}{r}$$

In 2, $$B_\varphi \cdot 2\pi r = 0 \quad \text{or} \quad B_\varphi = 0$$

3.232 Along the axis of loop,

$$B = \frac{\mu_0 I R^2}{2\,(R^2 + x^2)^{3/2}} = B_x \text{ (along the axis)}$$

Thus, $$\int \mathbf{B}\cdot d\mathbf{r} = \int_{-\infty}^{\infty} B_x dx = \frac{\mu_0 I R^2}{2}\int_{-\infty}^{\infty}\frac{dx}{(R^2 + x^2)^{3/2}}$$

$$= \frac{\mu_0\,IR^2}{2}\int_{-\pi/2}^{\pi/2}\frac{R\sec^2\theta d\theta}{R^3\sec^3\theta} \quad \text{(on putting } x = R\tan\theta\text{)}$$

$$= \mu_0 I\,\frac{1}{2}\int_{-\pi/2}^{\pi/2}\cos\theta\;d\,\theta = \mu_0 I$$

The physical interpretation of this result is that $\int_{-\infty}^{\infty} B_x\,dx$ can be thought of as the circulation of B over a closed loop by imaging that the two ends of the axis are connected by a line at infinity (e.g., a semicircle of infinite radius).

3.233 By circulation theorem, inside the conductor

$$B_\varphi 2\pi r = \mu_0 j_z \pi r^2 \quad \text{or} \quad B_\varphi = \mu_0 j_z r/2$$

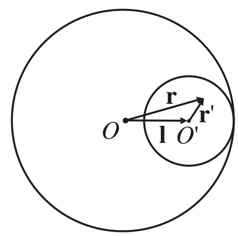
i.e.,
$$\mathbf{B} = \frac{1}{2}\mu_0 \mathbf{j} \times \mathbf{r}$$

Similarly, outside the conductor

$$B_\varphi 2\pi r = \mu_0 j_z \pi R^2 \quad \text{or} \quad B_\varphi = \frac{1}{2}\mu_0 j_z \frac{R^2}{r}$$

So,
$$\mathbf{B} = \frac{1}{2}\mu_0 (\mathbf{j} \times \mathbf{r}) \frac{R^2}{r^2}$$

2.234 The given current, which will be assumed uniform, arises due to a negative current flowing in the cavity superimposed on the true current everywhere including the cavity. Then from the previous problem, by superposition,



$$\mathbf{B} = \frac{\mu_0}{2}[\mathbf{j} \times \mathbf{r}] - \frac{\mu_0}{2}[\mathbf{j} \times \mathbf{r}'] = \frac{\mu_0}{2}[\mathbf{j} \times (\mathbf{r} - \mathbf{r}')]$$

$$\mathbf{B} = \frac{\mu_0}{2}(\mathbf{j} \times \mathbf{l})$$

If $\mathbf{l}$ vanishes so that the cavity is concentric with the conductor, then there is no magnetic field in the cavity.

3.235 By circulation theorem

$$B_\varphi \cdot 2\pi r = \mu_0 \int_0^r j(r') \times 2\pi r' \, dr'$$

Using $B_\varphi = br^\alpha$, inside the stream,

$$br^{\alpha+1} = \mu_0 \int_0^r j(r') r' \, dr'$$

So by differentiation,

$$(\alpha + 1) br^\alpha = \mu_0 j(r) r$$

Hence,
$$j(r) = \frac{b(\alpha + 1)}{\mu_0} r^{\prime\prime-1}$$

3.236 On the surface of the solenoid there is a surface current density, given by

$$\mathbf{j}_s = nI\mathbf{e}_\varphi$$

Then, $$\mathbf{B} = -\frac{\mu_0}{4\pi}\, nI \int R\, d\varphi\, dz \frac{\mathbf{e}_\varphi \times \mathbf{r_0}}{r_0^3}$$

where $\mathbf{r}_0$ is the vector from the current element to the field point, which is the center of the solenoid,

Now, $$-\,\mathbf{e}_\varphi \times \mathbf{r}_0 = R\mathbf{e}_z$$

$$r_0 = (z^2 + R^2)^{1/2}$$

Thus, $$B = B_z = \frac{\mu_0 nI}{4\pi} \times 2\pi R^2 \int_{-l/2}^{l/2} \frac{dz}{(R^2 + z^2)^{3/2}}$$

$$= \frac{1}{2}\mu_0\, nI \int_{-\tan^{-1}\frac{1}{2R}}^{+\tan^{-1}\frac{1}{2R}} \cos\alpha\, d\alpha \text{ (on putting } z = R\tan\alpha)$$

$$= \mu_0 nI \sin\alpha = \mu_0\, nI \frac{l/2}{\sqrt{(l/2)^2 + R^2}}$$

$$= \mu_0\, nI/\sqrt{1 + \left(\frac{2R}{l}\right)^2}$$

3.237 (a) We proceed exactly as in the previous problem. Then the magnetic induction on the axis at a distance x from one end is clearly,

$$B = \frac{\mu_0\, nl}{4\pi} \times 2\pi R^2 \int_0^\infty \frac{dz}{[R^2 + (z - x)^2]^{3/2}} = \frac{1}{2}\,\mu_0 nI\, R^2 \int_x^\infty \frac{dz}{(z^2 + R^2)^{3/2}}$$

$$= \frac{1}{2}\mu_0\, nI \int_{\tan^{-1}\frac{x}{R}}^{\pi/2} \cos\theta\, d\theta = \frac{1}{2}\,\mu_0 nI\left(1 - \frac{x}{\sqrt{x^2 + R^2}}\right)$$

$x > 0$ means that the field point is outside the solenoid. B then decreases with (x). $x < 0$ means that the point gets more and more inside the solenoid. B then increases with (x) and eventually becomes constant, equal to $\mu_0 nI$. The $B - x$ graph is as given in the answer sheet.

(b) We have,

$$\frac{B_0 - \delta B}{B_0} = \frac{1}{2}\left[1 - \frac{x_0}{\sqrt{R^2 + {x_0}^2}}\right] = 1 - \eta$$

or $$-\frac{x_0}{\sqrt{R^2 + x_0{}^2}} = 1 - 2\eta$$

Since η is small ($\approx 1\%$), x_0 must be negative.

Thus, $$x_0 = -|x_0|$$

and $$\frac{|x_0|}{\sqrt{R^2 + |x_0|^2}} = 1 - 2\eta$$

$$|x_0|^2 = (1 - 4\eta + 4\eta^2)(R^2 + |x_0|^2)$$

$$0 = (1 - 2\eta)^2 R^2 - 4\eta(1 - \eta)|x_0|^2$$

or $$|x_0| = \frac{(1 - 2\eta)R}{2\sqrt{\eta(1 - \eta)}}$$

$$\approx 5\text{R}$$

3.238 If the strip is tightly wound, it must have a pitch of h. This means that the current will flow obliquely, partly along $\mathbf{e}_\varphi$ and partly along $\mathbf{e}_z$. Obviously, the surface current density is

$$\mathbf{j}_s = \frac{I}{h}\left[\sqrt{1 - (h/2\pi R)^2}\,\mathbf{e}_\varphi + \frac{h}{2\pi R}\mathbf{e}_z\right]$$

By comparison with the case of a solenoid and a hollow straight conductor, we see that field inside the coil ($r > R$) is

$$= \mu_0 \frac{I}{h}\sqrt{1 - (h/2\pi R)^2}$$

(Cf. $B = \mu_0 nI$)

For the field outside the coil ($r < R$), only the other term contributes, so

$$B_\varphi \times 2\pi r = \mu_0 \frac{I}{h} \times \frac{h}{2\pi R} \times 2\pi R$$

or $$B_\varphi = \frac{\mu_0}{4\pi} \cdot \frac{2I}{r}$$

$$= 0.3\,\text{mT}$$

Note: *Surface current density is defined as current flowing normally across a unit length over a surface.*

3.239 Suppose a is the radius of cross-section of the core. The winding has a pitch $2\pi R/N$, so the surface current density is

$$\mathbf{j}_s \cong \frac{I}{2\pi R/N}\,\mathbf{e}_1 + \frac{I}{2\pi a}\,\mathbf{e}_2$$

where $\mathbf{e}_1$ is a unit vector along the cross section of the core and $\mathbf{e}_2$ is a unit vector along its length.

The magnetic field inside the cross section of the core is due to first term above, and is given by

$$B_\varphi \cdot 2\pi R = \mu_0 NI$$

(NI is total current due to the above surface current (first term).)

Thus, $$B_\varphi = \mu_0\, NI/2\pi R.$$

The magnetic field at the center of the core can be obtained from the basic formula:

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{\mathbf{j}_s \times \mathbf{r}_0}{r_0^3}\, dS$$

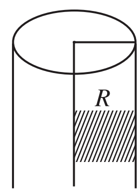
and is due to the second term.

So, $$\mathbf{B} = B_z\mathbf{e}_z = \mathbf{e}_z\frac{\mu_0}{4\pi}\frac{I}{2\pi a}\int \frac{1}{R^3} Rd\varphi \times 2\pi a$$

or $$B_z = \frac{\mu_0 I}{2R}$$

The ratio of the two magnetic fields, $\eta = \dfrac{N}{\pi} = 8 \times 10^2$

3.240 We need the flux through the shaded area.
Now by Ampere's theorem,



$$B_\varphi 2\pi r = \mu_0 \frac{I}{\pi R^2}\cdot \pi r^2$$

or $$B_\varphi = \frac{\mu_0}{2\pi} I \frac{r}{R^2}$$

The flux through the shaded region is

$$\Phi_1 = \int_0^R 1 \cdot dr B_\varphi(r)$$

$$= \int_0^R dr \frac{\mu_0}{2\pi} I \frac{r}{R^2} = \frac{\mu_0}{4\pi} I$$

$$= 1.0\,\mu\text{Wb/m}$$

3.241 Using solution of Problem 3.237, the magnetic field is given by,

$$B = \frac{1}{2}\mu_0 nI\left(1 - \frac{x}{\sqrt{x^2 + R^2}}\right)$$

At the end, $$B = \frac{1}{2}\mu_0 nI = \frac{1}{2}B_0 \quad (\text{where } B_0 = \mu_0 nI)$$

is the field deep inside the solenoid. Thus,

$$\Phi = \frac{1}{2}\mu_0 nIS = \frac{\Phi}{2} \quad \text{where } \Phi = \mu_0 nIS$$

is the flux of the vector B through the cross-section deep inside the solenoid.

3.242 $$B_\varphi 2\pi r = \mu_0 NI$$

or $$B_\varphi = \frac{\mu_0 NI}{2\pi R}$$

Then, $$\Phi = \int_b^a B_\varphi h dr \ (\text{for } a \leq \ r \leq b)$$

$$= \frac{\mu_0}{4\pi} 2NIh \ln \eta \quad \left(\text{where } \eta = \frac{b}{a}\right)$$

$$= 8\,\mu\text{Wb}$$

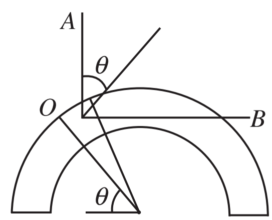
3.243 Magnetic moment of a current loop is given by $p_m = nIS$, where n is the number of turns and S is the cross sectional area. In our problem,

$$n = 1,\ S = \pi R^2 \text{ and } B = \frac{\mu_0}{2}\frac{I}{R}$$

So, $$p_m = \frac{2BR}{\mu_0}\pi R^2 = \frac{2\pi BR^3}{\mu_0} = 30 \text{ mA m}^2$$

3.244 Take an element of length $r\,d\theta$ containing $\frac{N}{\pi r}\cdot r d\theta$ turns. Its magnetic moment is

$$\frac{N}{\pi} d\theta \cdot \frac{\pi}{4} d^2 I$$



normal to the plane of cross section. We resolve it along OA and OB. The moment along OA integrates to

$$p_m = \int_0^\pi \frac{N}{4} d^2 I \cos\theta \ d\theta = 0$$

while that along OB gives

$$p_m = \int_0^\pi \frac{Nd^2 I}{4} \sin\theta \ d\theta = \frac{1}{2} Nd^2 I = 0.5 \text{ Am}^2$$

3.245 (a) From Biot-Savart's law, the magnetic induction due to a circular current carrying wire loop at its center is given by

$$B_r = \frac{\mu_0}{2r} I$$

The plane spiral is made up of concentric circular loops, having different radii, varying from a to b. Therefore, the total magnetic induction at the center,

$$B_0 = \int \frac{\mu_0}{2r} IdN \tag{1}$$

where $\frac{\mu_0}{2r} I$ is the contribution of one turn of radius r and dN is the number of turns in the interval $(r, r + dr)$, i.e.,

$$dN = \frac{N}{b - a} dr$$

Substituting in Eq. (1) and integrating the result over r between a and b, we obtain

$$B_0 = \int_a^b \frac{\mu_0 I}{2r} \frac{N}{(b - a)} dr = \frac{\mu_0 IN}{2(b - a)} \ln \frac{b}{a} = 7\ \mu\text{T}$$

(b) The magnetic moment of a turn of radius r is $p_m = I\pi r^2$ and for all turns,

$$p = \int p_m dN = \int_a^b I\pi r^2 \frac{N}{b - a} dr = \frac{\pi IN (b^3 - a^3)}{3\,(b - a)}$$

$$= 15\ \text{mA m}^2$$

3.246 (a) Let us take a ring element of radius r and thickness dr. Then charge on the ring element

$$dq = \sigma\, 2\pi r dr$$

and current due to this element

$$dI = \frac{(\sigma 2\pi r dr)\, \omega}{2\pi} = \sigma\, \omega r\, dr$$

So, magnetic induction at the center, due to this element

$$dB = \frac{\mu_0}{2} \frac{dI}{r}$$

and hence, from symmetry

$$B = \int dB = \int_0^R \frac{\mu_0 \sigma \omega r dr}{r} = \frac{\mu_0}{2} \sigma \omega R$$

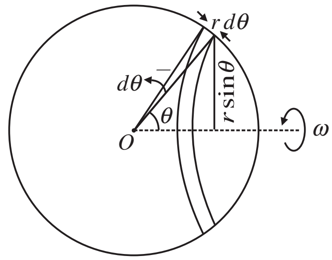
(b) Magnetic moment of the element

$$dp_m = (dI)\pi r^2 = \sigma \omega dr \pi r^2 = \sigma \pi \omega r^3\, dr$$

Hence, the sought magnetic moment

$$p_m = \int dp_m = \int_0^R \sigma \pi \omega r^3 dr = \sigma \omega \pi \frac{R^4}{4}$$

3.247 Since only the outer surface of the sphere is charged, consider the element as a ring, as shown in the figure.



The equivalent current due to the ring element

$$dI = \frac{\omega}{2\pi}\ (2\pi r \sin\theta\ \ r d\theta)\ \sigma \qquad (1)$$

and magnetic induction due to this loop element at the center of the sphere O is given by

$$dB = \frac{\mu_0}{4\pi}\, dI \frac{2\pi r\ \sin\theta r \sin\theta}{r^3} = \frac{\mu_0}{4\pi}\, dI\ \frac{\sin^2\theta}{r}$$

Hence, the total magnetic induction due to the sphere at the center O is given by

$$B = \int dB = \int_0^{\pi/2} \frac{\mu_0}{4\pi}\ \frac{\omega}{2\pi}\ \frac{2\pi r^2\ \sin\theta d\theta\ \sin^2\theta\ \sigma}{r} \qquad \text{(using Eq. 1)}$$

Hence, $$B = \int_0^{\pi/2} \frac{\mu_0 \sigma\omega r}{4\ \pi}\ \sin^3\theta\ d\theta = \frac{2}{3}\,\mu_0\sigma\omega r = 29\ \text{pT}$$

3.248 The magnetic moment must clearly be along the axis of rotation. Consider a volume element dV. It contains a charge $3q/4\pi R^3\, dV$. The rotation of the sphere causes this charge to revolve around the axis and constitute a current given by

$$\frac{3q}{4\pi R^3}\, dV \times \frac{\omega}{2\pi}$$

Its magnetic moment will be

$$\frac{3q}{4\pi R^3}\, dV \times \frac{\omega}{2\pi} \times \pi r^2 \sin^2\theta$$

So the total magnetic moment is

$$p_m = \int_0^R \int_0^\pi \frac{3q}{2R^3} r^2 \sin\theta\ d\theta \times \frac{\omega r^2 \sin^2\theta}{2} dr$$

$$= \frac{3q}{2R^3} \times \frac{\omega}{2} \times \frac{R^5}{5} \times \frac{4}{3} = \frac{1}{5}\ qR^2\ \omega$$

The mechanical moment is

$$M = \frac{2}{5}\ mR^2\ \omega \quad \text{so,} \quad \frac{p_m}{M} = \frac{q}{2\,m}$$

3.249 Because of polarization, a space charge is present within the cylinder. Its density is

$$\rho_p = -\text{div}\,\mathbf{P} = -2\alpha$$

Since the cylinder as a whole is neutral, a surface charge density σ_p, must be present on the surface of the cylinder also. This has the magnitude (algebraically)

$$\sigma_p \times 2\pi R = 2\alpha\pi R^2 \quad \text{or} \quad \sigma_p = \alpha R$$

When the cylinder rotates, currents are set up which give rise to magnetic fields. The contribution of ρ_p and σ_p can be calculated separately and then added.

For the surface charge the current is (for a particular element),

$$\sigma R \times 2\pi R\, dx \times \frac{\omega}{2\pi} = \alpha R^2 \omega\, dx$$

Its contribution to the magnetic field at the center is

$$\frac{\mu_0 R^2 (\alpha R^2\, \omega\, dx)}{2(x^2 + R^2)^{3/2}}$$

and the total magnetic field is

$$B_s = \int_{\infty}^{-\infty} \frac{\mu_0 R^2 (\alpha R^2\, \omega\, dx)}{2(x^2 + R^2)^{3/2}} = \frac{\mu_0 \alpha R^4 \omega}{2} \int_{-\infty}^{\infty} \frac{dx}{(x^2 + R^2)^{3/2}}$$

$$= \frac{\mu_0 \alpha R^4 \omega}{2} \times \frac{2}{R^2} = \mu_0 \alpha R^2 \omega$$

For the volume charge density, consider a circle of radius r, radial thickness dr and length dx.

The current is

$$-2\alpha \times 2\pi r\, dr\, dx \times \frac{\omega}{2\pi} = -2\, \alpha r dr\, \omega\, dx$$

The total magnetic field due to the volume charge distribution is

$$B_v = -\int_0^R dr \int_{-\infty}^{\infty} dx\, 2\pi\, r\omega \frac{\mu_0 r^2}{2(x^2 + r^2)^{3/2}} = -\int_0^R \alpha\mu_0\, \omega r^3 dr \int_{-\infty}^{\infty} dx\, (x^2 + r^2)^{3/2}$$

$$= -\int_0^R \alpha\mu_0\, \omega r \times 2 = -\mu_0\, \alpha\, \omega\, R^2 \text{ so, } B = B_s + B_v = 0$$

3.250 Force of magnetic interaction, $\mathbf{F}_{\text{mag}} = e\,(\mathbf{v} \times \mathbf{B})$

where, $$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{e(\mathbf{v} \times \mathbf{r})}{r^3}$$

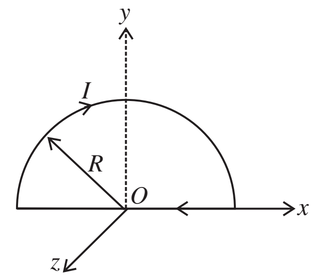
So, $$\mathbf{F}_{\text{mag}} = \frac{\mu_0}{4\pi}\frac{e^2}{r^3}[\mathbf{v}\times(\mathbf{v}\times\mathbf{r})]$$

$$= \frac{\mu_0}{4\pi}\frac{e^2}{r^3}[(\mathbf{v}\cdot\mathbf{r})\times\mathbf{v} - (\mathbf{v}\cdot\mathbf{v})\times\mathbf{r}] = \frac{\mu_0}{4\pi}\frac{e^2}{r^3}(-v^2\mathbf{r})$$

and $$\mathbf{F}_{\text{ele}} = e\mathbf{E} = e\frac{1}{4\pi\varepsilon_0}\frac{e\mathbf{r}}{|\mathbf{r}|^3}$$

Hence, $$\frac{|\mathbf{F}_{\text{mag}}|}{|\mathbf{F}_{\text{ele}}|} = -v^2\mu_0\varepsilon_0 = \left(\frac{v}{c}\right)^2 = 1.00\times10^{-6}$$

3.251 (a) The magnetic field at O is only due to the curved path, as for the line element, $d\mathbf{l}\uparrow\uparrow\mathbf{r}$.

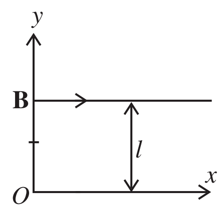


Hence, $$\mathrm{B} = \frac{\mu_0 I}{4\pi R}\pi(-\mathbf{k}) = \frac{\mu_0 I}{4R}(-\mathbf{k})$$

Thus, $$\mathbf{F}_u = IB(-\mathbf{j}) = \frac{\mu_0 I^2}{4R}(-\mathbf{j})$$

So, $$F_u = \frac{\pi_0 I^2}{4R} = 0.20\,\text{mN/m}$$

(b) In this part, magnetic induction $\boldsymbol{B}$ at O will be effective only due to the two semi infinite segments of wire. Hence,



$$\mathbf{B} = 2\cdot\frac{\mu_0 I}{4\pi\left(\frac{l}{2}\right)}\sin\frac{\pi}{2}(-\mathbf{k})$$

$$= \frac{\mu_0 I}{\pi l}(-\mathbf{k})$$

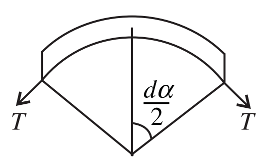
Thus force per unit length,

$$F_u = \frac{\mu_0 l^2}{\pi l}(-\mathbf{i})$$

$$= 0.13\text{ mN/m}$$

3.252 Each element of length $dl = Rd\alpha$ experiences a force $BI\,Rd\alpha$, where $d\alpha$ is the angle subtended by the element at the center. This causes a tension T in the wire.

For equilibrium, ampere force $BI\,Rd\alpha$ must be balanced by the radial component of tensile forces, which is



$$2T\sin(d\alpha/2) = Td\alpha$$

So, $$Td\alpha = BI\,Rd\alpha,$$

Hence, $$T = BIR$$

Thus, the wire experiences a stress

$$\frac{BIR}{\pi d^2/4}$$

This must equal the breaking stress σ_m for rupture. Thus,

$$B_{\max} = \frac{\pi d^2 \sigma_m}{4IR}$$
$$= 8 \text{ kT}$$

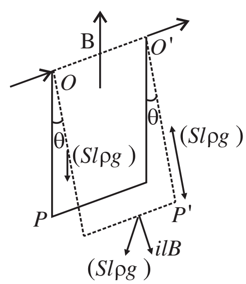
3.253 The Ampere forces on the sides OP and $O'P'$ at their mid points are directed along the same line, in opposite directions and have equal values, hence the net force as well as the net torque of these forces about the axis OO' is zero. The Ampere force on the segment PP' and the corresponding moment of this force about the axis OO' is effective and deflecting in nature.

The weight of each segment develops restoring torque. Let the length of each side be l and ρ be the density of the material then, mass of each segment $m = Sl\rho$. The Ampere force on the segment PP' is IlB, from the formula of Ampere force on a straight wire in a uniform magnetic field $\mathbf{F} = I(\mathbf{l} \times \mathbf{B})$.

In equilibrium the deflecting torque must be equal to the restoring torque. So,

$$Fl \cos\varphi = 2mg(l/2) \sin\varphi + mgl \sin\varphi$$

$$I\, lB\, (l \cos\varphi) = 2S\rho g l^2 \sin\varphi$$



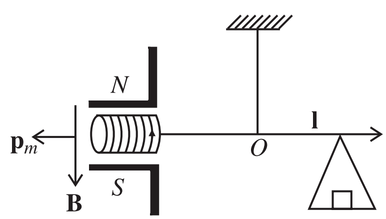
Hence, $$B = \frac{2S\rho g}{I} \tan\varphi$$
$$= 10 \text{ mT}$$

3.254 We know that the torque acting on a magnetic dipole

$$\mathbf{N} = \mathbf{p}_m \times \mathbf{B}$$

But, $\mathbf{p}_m = IS\mathbf{n}$, where $\mathbf{n}$ is the normal on the plane of the loop and is directed in the direction of advancement of a right handed screw, if we rotate the screw in the sense of current in the loop.

On passing current through the coil, the torque acting on the magnetic dipole is counterbalanced by the moment of additional weight about O.



Hence, the direction of current in the loop must be in the direction shown in the figure.

$$\mathbf{p}_m \times \mathbf{B} = -\mathbf{l} \times \Delta m\mathbf{g}$$

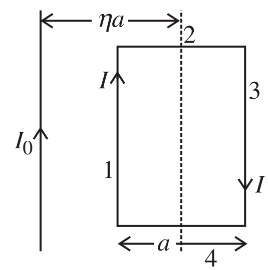
or $$NI\,SB = \Delta\ mgl$$

So, $$B = \frac{\Delta\, mgl}{NIS} = 0.4\ \text{T (on substituting values).}$$

3.255 (a) As is clear from the condition, Ampere's forces on the sides 2 and 4 are equal in magnitude but opposite in direction. Hence, the net effective force on the frame is the resultant of the forces experienced by sides 1 and 3.

Now, the Ampere force on side 1,

$$F_1 = \frac{\mu_0}{2\pi} \frac{II_0}{\left(\eta - \frac{1}{2}\right)}$$



and that on side 3,

$$F_3 = \frac{\mu_0}{2\pi} \frac{I_0 I}{\left(\eta + \frac{1}{2}\right)}$$

So, the resultant force on the frame $= F_1 - F_3$ (as they are opposite in nature).

$$= \frac{2\mu_0\, II_0}{\pi(4\eta^2 - 1)} = 0.40\ \mu\text{N}$$

(b) Work done in turning the frame through some angle,

$$A = \int I d\,\Phi = I(\Phi_f - \Phi_i)$$

where Φ_f is the flux through the frame in final position and Φ_i in the initial position.

Here, $|\Phi_f| = |\Phi_i| = \Phi$ and $\Phi_i = -\Phi_f$

So, $\Delta\Phi = 2\,\Phi$ and $A = I\,2\,\Phi$

Hence, $A = 2I \int \mathbf{B} \cdot d\mathbf{S}$

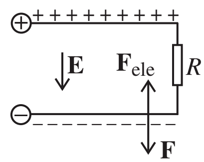
$$= 2I \int_{a(\eta-\frac{1}{2})}^{a(\eta+\frac{1}{2})} \frac{\mu_0}{2\,\pi} \frac{I_0 a}{r} dr = \frac{\mu_0 II_0 a}{\pi} \ln\left(\frac{2\eta + 1}{2\eta - 1}\right)$$

$$= 0.10\ \mu\text{J}$$

3.256 There are excess surface charges on each wire (irrespective of whether the current is flowing through them or not). Hence, in addition to the magnetic force $\mathbf{F}_{\text{mag}}$, we must take into account the electric force $\mathbf{F}_{\text{ele}}$. Suppose that an excess charge λ corresponds to a unit length of the wire, then electric force exerted per unit length of the wire by other wire can be found with the help of Gauss' theorem.

$$F_{ele} = \lambda E = \lambda \frac{1}{4\pi\varepsilon_0} \frac{2\lambda}{l} = \frac{2\lambda^2}{4\pi\varepsilon_0 l} \tag{1}$$

where l is the distance between the axes of the wires. The magnetic force acting per unit length of the wire can be found with the help of the theorem on circulation of vector **B**



$$F_{mag} = \frac{\mu_0}{4\pi} \frac{2I^2}{l}$$

where I is the current in the wire. (2)

Now, from the relation, $\lambda = C\varphi$, where C is the capacitance of the wires per unit length, as given in solution Problem 3.108 and $\varphi = IR$. Substituting we get,

$$\lambda = \frac{\pi\varepsilon_0}{\ln\eta} IR \quad \text{or} \quad \frac{I}{\lambda} = \frac{\ln\eta}{\pi\varepsilon_0 R} \tag{3}$$

Dividing Eq. (2) by (1) and then substituting the value of $\frac{I}{\lambda}$ from Eq. (3), we get,

$$\frac{F_m}{F_e} = \frac{\mu_0}{\varepsilon_0} \frac{(\ln\eta)^2}{\pi^2 R^2}$$

The resultant force of interaction vanishes when this ratio equals unity. This is possible when $R = R_0$, where

$$R_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{\ln\eta}{\pi} = 0.36 \text{ k}\Omega$$

3.257 Using solution of Problem 3.225, the magnetic field due to the conductor with semicircular cross-section is given by,

$$B = \frac{\mu_0 I}{\pi^2 R}$$

Then
$$\frac{\partial F}{\partial l} = BI = \frac{\mu_0 I^2}{\pi^2 R}$$

3.258 We know that Ampere's force per unit length on a wire element in a magnetic field is given by

$$d\mathbf{F}_n = I(\mathbf{n} \times \mathbf{B}) \tag{1}$$

where **n** is the unit vector along the direction of current.

Now, let us take an element of the conductor I_2, as shown in the figure. This wire element is in the magnetic field, produced by the current I_1, which is directed normally into the sheet of the paper and its magnitude is given by

$$|\mathbf{B}| = \frac{\mu_0 I_1}{2\pi r} \tag{2}$$

From Eqs. (1) and (2)

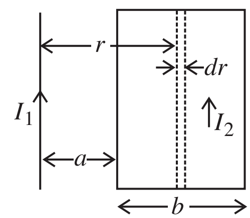
$$d\mathbf{F}_n = \frac{I_2}{b}\, dr\,(\mathbf{n} \times \mathbf{B}) \quad \text{(because the current through the element} = I_2/b\, dr)$$

So, $$d\mathbf{F}_n = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{b}\frac{dr}{r} \quad (\text{towards left, as } \mathbf{n} \perp \mathbf{B})$$

Hence the magnetic force on the conductor:

$$\mathbf{F}_n = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{b}\int_a^{a+b}\frac{dr}{r} \text{ (towards left)}$$

$$= \frac{\mu_0}{2\pi}\frac{I_1 I_2}{b}\ln\frac{a+b}{a} \text{ (towards left)}$$

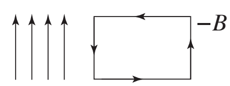


Then according to the Newton's third law, the magnitude of sought magnetic interaction force

$$= \frac{\mu_0 I_1 I_2}{2\pi b}\ln\frac{a+b}{a}$$

3.259 By the circulation theorem $B = \mu_0 I$, where I = current per unit length flowing along the plane perpendicular to the paper. Currents flow in the opposite paper. Currents flow in the opposite sense in the two planes and produce the given field B by superposition. The field due to one of the plates is just $1/2\,B$. The force on the plate is $1/2\ B \times I \times$ length $\times$ breadth $= B^2/2\mu_0$ per unit area.

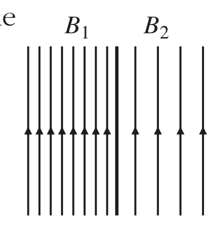
(Recall the formula $F = BIl$ on a straight wire.)



3.260 In Fig. (a), the external field must be $B_1 + B_2/2$, which when superposed with the internal field $B_1 - B_2/2$ (of opposite sign on the two sides of the plate), must give the actual field.

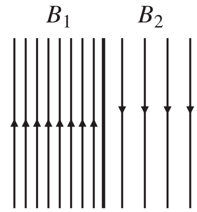
Now $$\frac{B_1 - B_2}{2} = \frac{1}{2}\mu_0 I$$

or $$I = \frac{B_1 - B_2}{\mu_0}$$



(a)

Thus, $$F = \frac{B_1^2 - B_2^2}{2\mu_0}$$



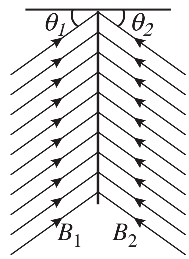
(b)

In Fig. (b), the external field must be $B_1 - B_2/2$ upward with an internal field, $B_1 + B_2/2$, upward on the left and downward on the right. Thus,

$$I = \frac{B_1 + B_2}{\mu_0} \quad \text{and} \quad F = \frac{B_1{}^2 - B_2{}^2}{2\mu_0}$$

In Fig. (c), the boundary condition from Gauss' theorem is, $B_1 \cos\theta_1 = B_2 \cos\theta_2$. Also $(B_1 \sin\theta_1 + B_2 \sin\theta_2) = \mu_0 I$, where $I =$ current per unit length. The external field parallel to the plate must be

$$\frac{B_1 \sin\theta_1 - B_2 \sin\theta_2}{2}$$



(c)

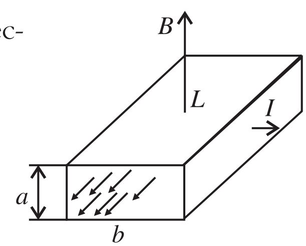
The perpendicular component $B_1 \cos\theta_1$, does not matter since the corresponding force is tangential.

Thus, $$F = \frac{B_1^2 \sin^2\theta_1 - B_2^2 \sin^2\theta_2}{2\mu_0} \text{ per unit area}$$

$$= \frac{B_1^2 - B_2^2}{2\mu_0} \text{ per unit area}$$

The direction of the current in the plane conductor is perpendicular to the paper and beyond the drawing.

3.261 The current density is I/aL, where L is the length of the section. The difference in pressure produced must be



$$\Delta p = \frac{I}{aL} \times B \times (abL)/ab = \frac{IB}{a} = 0.5 \text{ kPa}$$

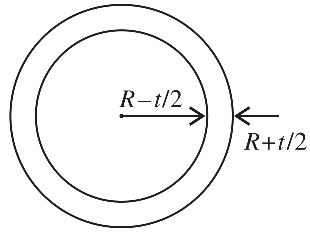
3.262 Let $t =$ thickness of the wall of the cylinder. Then,

$$J = I/2\pi Rt \text{ (along } z \text{ axis)}$$

The magnetic field due to this at distance such that $r\,(R - t/2 < r < R + t/2)$, is given by

$$B_\varphi (2\pi r) = \mu_0 \frac{I}{2\pi RT} \pi \{r^2 - (R - t/2)^2\}$$

$$B_\varphi = \frac{\mu_0 I}{4\pi Rrt} \{r^2 - (R - t/2)^2\}$$



Now, $$\mathbf{F} = \int \mathbf{J} \times \mathbf{B}\, dV$$

and $$p = \frac{F_r}{2\pi RL}$$

$$= \frac{1}{2\pi RL} \int_{R-\frac{t}{2}}^{R+\frac{t}{2}} \frac{\mu_0 I^2}{8\pi^2 R^2 t^2 r} \left\{ r^2 - \left(R - \frac{t}{2}\right)^2 \right\} \times 2\pi r L dr$$

$$= \frac{\mu_0 I^2}{8\pi^2 R^3 t^2} \int_{R-\frac{t}{2}}^{R+\frac{t}{2}} \left\{ r^2 - \left(R - \frac{t}{2}\right) \right\}^2 dr$$

$$= \frac{\mu_0 I^2}{8\pi^2 R^3 t^2} \left[\frac{\left(R + \frac{t}{2}\right)^3 - \left(R - \frac{t}{2}\right)^3}{3} - \left(R - \frac{t}{2}\right)^2 t \right]$$

$$= \frac{\mu_0 I^2}{8\pi^2 R^3 t} [Rt + 0\,(t^2)] \approx \frac{\mu_0 I^2}{8\pi^2 R^2}$$

3.263 When self-forces are involved, a typical factor of 1/2 comes into play. For example, the force on a current carrying straight wire in a magnetic induction B is BIl. If the magnetic induction B is due to the current itself then the force can be written as

$$F = \int_0^I B\,(I')\, dI'\, l$$

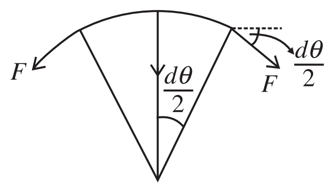
If $B\,(I') \propto I'$, then this becomes, $F = \frac{1}{2} B\,(I)\, Il.$

In the present case, $B(I) = \mu_0 nI$ and this acts on nI ampere turns per unit length, so, pressure is given by

$$p = \frac{F}{\text{Area}} = \frac{1}{2} \mu_0 n \frac{I \times nI \times 1 \times l}{1 \times l} = \frac{1}{2} \mu_0 n^2 I^2$$

3.264 The magnetic induction B in the solenoid is given by $B = \mu_0 nI$. The force on an element $dl = R d\alpha$ of the current carrying conductor is

$$dF = \frac{1}{2} \mu_0 n\, II\, R d\alpha = \frac{1}{2} \mu_0\, nI^2 R d\alpha$$



this is radially outwards. The factor 1/2 is explained above. For equilibrium, it should be balanced by the radial component of the longitudinal tensile forces, $T d\alpha$.

Hence,
$$T = \frac{1}{2}\mu_0 n I^2 R$$

This equals F_{lim} when, $I = I_m = \sqrt{\frac{2\,F_{\text{lim}}}{\mu_0 n R}}$

Note that F_{lim}, here, is actually a force and not a stress.

3.265 Resistance of the liquid between the plates $= \frac{\rho d}{S}$

Voltage between the plates $= Ed = vBd$

Current through the plates $= \frac{vBd}{R + \rho d/S}$

Power generated in the external resistance R, is given by

$$P = \frac{v^2B^2R}{\left(R + \frac{\rho d}{S}\right)^2} = \frac{v^2B^2d^2}{\left(\sqrt{R} + \frac{\rho d}{S\sqrt{R}}\right)^2} = \frac{v^2B^2d^2}{\left[\left\{R^{1/4} - \left(\frac{\rho d}{s\sqrt{R}}\right)^{1/2}\right\}^2 + 2\sqrt{\frac{\rho d}{S}}\right]^2}$$

This is maximum when

$$R = \frac{\rho D}{S} \text{ and } P_{\max} = \frac{v^2B^2Sd}{4\rho}$$

3.266 The electrons in the conductor are drifting with a speed of

$$v_d = \frac{J}{ne} = \frac{I}{\pi R^2 ne}$$

where e = magnitude of the charge on the electron, n = concentration of the conduction electrons.

The magnetic field inside the conductor due to this current is given by

$$B_\varphi\,(2\pi r) = \pi r^2 \frac{I}{\pi R^2}\mu_0 \quad \text{or} \quad B_\varphi = \frac{\mu_0}{2\pi}\frac{Ir}{R^2}$$

A radial electric field vB_φ must come into being in equilibrium. Its potential difference is

$$\Delta\varphi = \int_0^R \frac{I}{\pi R^2 ne}\frac{\mu_0}{2\pi}\frac{Ir}{R^2}\,dr$$

$$= \frac{I}{\pi R^2 ne}\left(\frac{\mu_0 I}{2\pi R^2}\right)\frac{1}{2} R^2$$

$$= \frac{\mu_0 I^2}{4\pi^2 R^2 ne}$$

$$= 2 \text{ pV}$$

3.267 Here, $v_d = \frac{E}{B}$ and $j = nev_d$

So, $n = \frac{jB}{eE} = \frac{200 \times 10^4 \text{ A/m}^2 \times 1 \text{ T}}{1.6 \times 10^{-19} \text{ C} \times 5 \times 10^{-4} \text{ V/m}}$

$$= 2.5 \times 10^{28} \text{ per m}^3 = 2.5 \times 10^{22} \text{ cm}^{-3}$$

Atomic weight of Na being 23 and its density ≈ 1, molar volume is 23 cm^3. Thus, number of atoms per unit volume is

$$\approx \frac{6 \times 10^{23}}{23} = 2.6 \times 10^{22} \text{ cm}^{-3}$$

Thus, there is almost one conduction electron per atom.

3.268 By definition, mobility $= \dfrac{\text{Drift velocity}}{\text{Electric field component causing this drift}}$

or $\mu = \frac{v_d}{E_L}$

On the other hand, $E_T = v_d B = \frac{E_L}{\eta}$ (as given)

So, $\mu = \frac{1}{\eta B} = 3.2 \times 10^{-3} \text{ m}^2/(\text{V}\cdot\text{s})$

3.269 Due to the straight conductor, $B_\varphi = \frac{\mu_0 I}{2\pi r}$

We use the formula, $\mathbf{F} = (\mathbf{p}_m \cdot \nabla)\,\mathbf{B}$

(a) The vector $\mathbf{p}_m$ is parallel to the straight conductor, so

$$\mathbf{F} = p_m \frac{\partial}{\partial Z}\mathbf{B} = 0$$

because neither the direction nor the magnitude of **B** depends on z.

(b) The vector $\mathbf{p}_m$ is oriented along the radius vector **r**, so

$$\mathbf{F} = p_m \frac{\partial}{\partial r}\mathbf{B}$$

The direction of **B** at $r + dr$ is parallel to the direction at r. Thus only the φ component of **F** will survive.

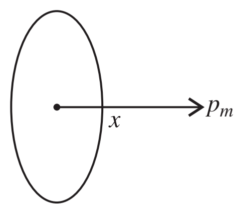
$$F_\varphi = p_m \frac{\partial}{\partial r} \frac{\mu_0 I}{2\pi r} = -\frac{\mu_0 I p_m}{2\pi r^2}$$

(c) The vector $\mathbf{p}_m$ coincides in direction with the magnetic field, produced by the conductor carrying current I.

$$\mathbf{F} = p_m \frac{\partial}{r\partial\varphi} \frac{\mu_0 I}{2\pi} \mathbf{e}_\varphi = \frac{\mu_0 I p_m}{2\pi r^2} \frac{\partial \mathbf{e}_\varphi}{\partial \varphi}$$

So, $$\mathbf{F} = -\frac{\mu_0 I p_m}{2\pi r^2} \mathbf{e}_r, \text{ as, } \frac{\partial \mathbf{e}_\varphi}{\partial_\varphi} = -\mathbf{e}_r$$

3.270 We have, $$F_x = p_m \frac{\partial B_x}{\partial x}$$



But, $$B_x = \frac{\mu_0 I}{4\pi} \int \frac{R dl}{(x^2 + R^2)^{3/2}} = \frac{\mu_0 I R^2}{2(x^2 + R^2)^{3/2}}$$

So, $$F = \frac{\mu_0}{4\pi} \frac{I \cdot 2\pi R^2}{(x^2 + R^2)^{5/2}} \frac{3}{2} \cdot 2x \cdot p_m = \frac{\mu_0}{4\pi} \frac{6\pi R^2 I p_m x}{(x^2 + R^2)^{5/2}}$$

3.271
$$F = p_{2m} \frac{\partial}{\partial l} \left[\frac{(\mu_0\, 3\mathbf{p}_{1m} \cdot \mathbf{r})}{4\pi} \frac{\mathbf{r} - \mathbf{p}_{1m} r^2}{r^5} \right]$$

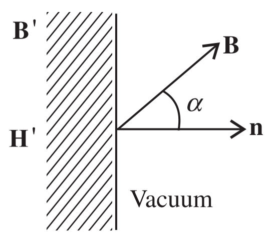
$$= p_{2m} \frac{\partial}{\partial l} \left[\frac{\mu_0}{2\pi} \frac{p_{1m}}{l^3} \right] = \frac{-3}{2} \frac{\mu_0\, p_{1m}\, p_{2m}}{\pi l^4} = 9 \text{ nN}$$

3.272 From solution of Problem 3.270, for $x \gg R$,

$$B_x = \frac{\mu_0 I' R^2}{2x^3}$$

or $$I' \approx \frac{2B_x x^3}{\mu_0 R^2} = \frac{2 \times 3 \times 10^{-5} T \times (10^{-1} \text{ m})^3}{1.26 \times 10^{-6} \times (10^{-2} \text{ m})^4} \approx 0.5 \text{ kA}$$

3.273
$$B'_n = B \cos\alpha,$$



$$H'_t = \frac{1}{\mu_0} B \sin\alpha,$$

$$B'_t = \mu B \sin\alpha$$

So, $$B' = B\sqrt{\mu^2 \sin^2\alpha + \cos^2\alpha}$$

3.274 (a) $\oint \mathbf{H} \cdot d\mathbf{S} = \oint \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{J} \right) \cdot d\mathbf{S} = -\oint \mathbf{J} \cdot d\mathbf{S}$, since $\oint \mathbf{B} \cdot d\mathbf{S} = 0$.

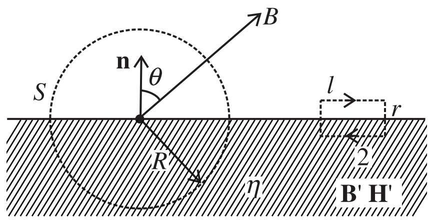
Now $\mathbf{J}$ is non-vanishing only in the bottom half of the sphere.

Here, $\quad B'_n = B\cos\theta,\ H'_t = \frac{1}{\mu_0} B \sin\theta,\ \ B'_t = \mu\ B\sin\theta,\ H'_n = \frac{B}{\mu\mu_0}\cos\theta$

$$J_n = \frac{B\cos\theta}{\mu_0}\left(1 - \frac{1}{\mu}\right) \text{ and } J_t = \frac{\mu - 1}{\mu_0} B \sin\theta$$

Only J_n contributes to the surface integral, so

$$-\oint \mathbf{J} \cdot d\mathbf{S} = -\oint_{\text{lower}} J \cdot d\mathbf{S} = \oint_{\text{lower}} J_n\, dS = \frac{\pi\ R^2 B\cos\theta}{\mu_0}\left(1 - \frac{1}{\mu}\right)$$



(b) $$\oint \mathbf{B} \cdot d\mathbf{r} = (B_t - B'_t)\ l = (1 - \mu)/B\ l \sin\theta$$

3.275 Inside the cylindrical wire, there is an external current of density $I/\pi R^2$. This gives a magnetic field H_φ with

$$H_\varphi 2\pi r = I\frac{r^2}{R^2} \quad \text{or} \quad H_\varphi = \frac{Ir}{2\pi R^2}$$

From this $\quad B_\varphi = \frac{\mu\mu_0 Ir}{2\pi R^2} \quad \text{and} \quad J_\varphi = \frac{\mu - 1}{2\pi}\frac{Ir}{R^2} = \frac{\chi\ Ir}{2\pi R^2}$

Hence, total volume of molecular current is

$$\oint_{r=R} \mathbf{J}_\varphi \cdot d\mathbf{r} = \int \frac{\chi I}{2\pi R} dl = \chi I$$

The surface current is obtained by using the equivalence of the surface current density to $\mathbf{J} \times \mathbf{n}$. This gives rise to a surface current density in the z-direction of

$$\frac{-\chi I}{2\pi R}$$

The total molecular surface current is

$$I_s' = -\frac{\chi I}{2\pi R}(2\pi R) = -\chi I$$

The two currents have opposite signs.

3.276 We can obtain the form of the curves required here, by qualitative arguments.

From $$\oint H \cdot d\mathbf{l} = I$$

we get, $$H(x \gg 0) = H(x \ll 0) = nI$$

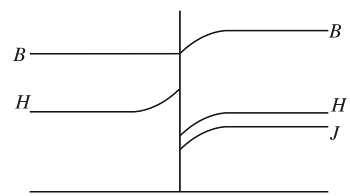
Then $$B(x \gg 0) = \mu\mu_0 nI$$

$$B(x \ll 0) = \mu_0 nI$$

Also,

$$B(x < 0) = \mu_0 H(x < 0)$$

$$J(x < 0) = 0$$



B is continuous at $x = 0$, H is not. These give the required curves as shown in the answer sheet.

3.277 The lines of B as well as H field are circles around the wire. Thus,

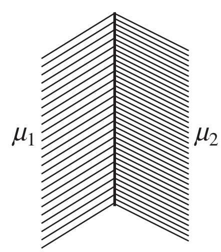
$$H_1 \pi r + H_2 \pi r = I \quad \text{or} \quad H_1 + H_2 = \frac{I}{\pi r}$$

Also, $$\mu_0 \mu_1 H_1 = \mu_2 H_2 \mu_0 = B_1 = B_2 = B$$

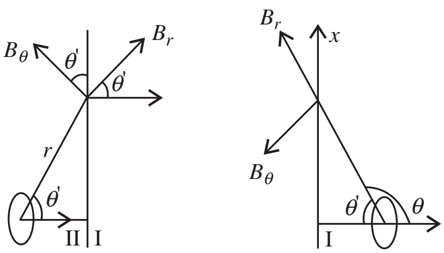
Thus, $$H_1 = \frac{\mu_2}{\mu_1 + \mu_2}\frac{I}{\pi r}$$

$$H_2 = \frac{\mu_1}{\mu_1 + \mu_2}\frac{I}{\pi r}$$

and $$B = \mu_0 \frac{\mu_1 \mu_2}{\mu_1 + \mu_2}\frac{I}{\pi r}$$



3.278 The medium I is vacuum and contains a circular current carrying coil with current I. The medium II is magnetic with permeability μ. The boundary is the plane $z = 0$ and the coil is in the plane $z = l$. To find the magnetic induction, we note that the effect of the magnetic medium can be written as due to an image coil in II as far as the medium I is concerned. On the other hand, the induction in $\boldsymbol{I}$ can be written as due to the coil in I, carrying a different current. It is sufficient to consider the far away fields and ensure that the boundary conditions are satisfied there. Now for actual coil in medium I,



$$B_r = -\frac{2p_m \cos\theta'}{r^3} \cdot \left(\frac{\mu_0}{4\pi}\right) \quad \text{and} \quad B_\theta = \frac{p_m \sin\theta'}{r^3}\left(\frac{\mu_0}{4\pi}\right)$$

So, $$B_2 = \frac{\mu_0\, p_m}{4\pi}(2\cos^2\theta' - \sin^2\theta') \text{ and } B_x = \frac{\mu_0\, p_m}{4\pi}(-3\sin\theta' \cos\theta')$$

where, $p_m = I(\pi a^2)$, a = radius of the coil.

Similarly due to the image coil,

$$B_z = \frac{\mu_0\, P'_m}{4\pi}(2\cos^2\theta' - \sin^2\theta'),\ B_x = \frac{\mu_0\, P'_m}{4\pi}(3\sin\theta' \cos\theta'),\ P'_m = I'(\pi a^2)$$

As far as medium II is concerned, we write similarly

$$B_z = \frac{\mu_0\, p''_m}{4\pi}(2\cos^2\theta' - \sin^2\theta'),\ B_x = \frac{\mu_0\, p''m}{4\pi}(-3\sin\theta' \cos\theta'),\ p''_m = I''(\pi a^2)$$

The boundary conditions are

$$p_m + P'_m = p''_m \ (\text{from } B_{1n} = B_{2n})$$

$$-p_m + P'_m = -\frac{1}{\mu} p''_m \quad (\text{from } H_{1t} = H_{2t})$$

Thus, $$I'' = \frac{2\mu}{\mu + 1} I \quad \text{and} \quad I' = \frac{\mu - 1}{\mu + 1} I$$

In the limit when the coil is on the boundary, the magnetic field everywhere can be obtained by taking the current to be

$$\frac{2\mu}{\mu + 1} I$$

Thus, $$\mathbf{B} = \frac{2\mu}{\mu + 1}\mathbf{B}_0$$

3.279 We use the fact that within an isolated uniformly magnetized ball, $\mathbf{H}' = -\mathbf{J}/3$, $\mathbf{B}' = 2\mu_0 \mathbf{J}/3$, where $\mathbf{J}$ is the magnetization vector. Then, in a uniform magnetic field with induction $\mathbf{B}_0$, we have by superposition,

$$\mathbf{B}_{\text{in}} = \mathbf{B}_0 + \frac{2\mu_0 \mathbf{J}}{3},\ \mathbf{H}_{\text{in}} = \frac{\mathbf{B}_0}{\mu_0} - \frac{\mathbf{J}}{3}$$

or $$\mathbf{B}_{in} + 2\mu_0 \mathbf{H}_{in} = 3\mathbf{B}_0$$

Also, $$\mathbf{B}_{in} = \mu\mu_0\, \mathbf{H}_{in}$$

Thus, $$\mathbf{H}_{in} = \frac{3\mathbf{B}_0}{\mu_0\,(\mu + 2)} \quad \text{and} \quad \mathbf{B}_{in} = \frac{3\,\mu\mathbf{B}_0}{\mu + 2}$$

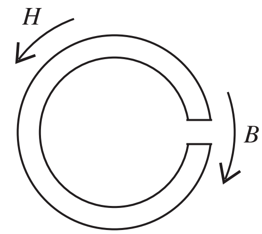
3.280 The coercive force H_c is just the magnetic field within the cylinder. This is by circulation theorem,

$$H_C = \frac{NI}{l} = 6 \text{ kA/m}$$

(from $\oint \mathbf{H}\cdot d\mathbf{r} = I$, total current, considering a rectangular contour).

3.281 We use, $$\oint \mathbf{H}\cdot d\mathbf{l} = 0$$

Neglecting the fringing of the lines of force, we write this as



$$H\,(\pi d - b) + \frac{B}{\mu_0}\,b = 0$$

or $$H \approx \frac{-Bb}{\mu_0 \pi d} = 101 \text{ A/m}$$

The sense of H is opposite to B.

3.282 Here, $\oint \mathbf{H}\cdot d\mathbf{l} = NI$ or $H\,(2\pi R) + \dfrac{Bb}{\mu_0} = NI$ so, $H = \dfrac{NI\mu_0 - Bb}{2\pi R\mu_0}$

Hence, $$\mu = \frac{B}{\mu_0 H} = \frac{2\pi RB}{\mu_0 NI - Bb} = 3700$$

3.283 One has to draw the graph of $\mu = \dfrac{B}{\mu_0 H}$ versus H from the given graph. The μ–H graph starts out horizontally, and then rises steeply at about $H = 0.04$ kA/m before falling again. It is easy to check that $\mu_{max} \approx 10{,}000$.

3.284 From the theorem on circulation of vector **H**

$$H\pi d + \frac{Bb}{\mu_0} = NI \quad \text{or} \quad B = \frac{\mu_0 NI}{b} - \frac{\mu_0 \pi d}{b} H = (1.51 - 0.987)\,H$$

where B is in Tesla and H in kA/m. Besides, B and H are interrelated as in the Fig. 3.76 of the problem book. Thus we have to solve for B, H graphically by simultaneously drawing the two curves (the hysterisis curve and the straight line) and find the point of intersection. It is at $H \approx 0.26$ kA/m, $B = 1.25$ T.

Then, $$\mu = \frac{B}{\mu_0 H} \approx 4000$$

3.285 From the formula,

$$\mathbf{F} = (\mathbf{p}_m \cdot \nabla)\,\mathbf{B} \rightarrow \mathbf{F} = \int (\mathbf{J} \cdot \nabla)\,\mathbf{B}\,dV$$

Thus, $$\mathbf{F} = \frac{\chi}{\mu\mu_0}\int (\mathbf{B} \cdot \nabla)\,\mathbf{B}\,dV$$

Since **B** is predominantly along the x-axis,

so, $$F_x = \frac{\chi}{\mu\,\mu_0}\int B_x \frac{\partial B_x}{\partial x}\,Sdx = \frac{\chi S}{2\mu\,\mu_0}\int_{x=0}^{x=L} dB_x^2 = -\frac{\chi SB^2}{2\mu\,\mu_0} = \frac{\chi SB^2}{2\mu_0}$$

3.286 The force in question is

$$\mathbf{F} = (\mathbf{p}_m \cdot \nabla)\,\mathbf{B} = \frac{\chi BV}{\mu\,\mu_0}\,\frac{dB}{dx}$$

Since B is essentially in the x-direction,

so, $$F_x \approx \frac{\chi V}{2\mu_0}\,\frac{dB^2}{dx} = \frac{\chi B_0^2 V}{2\mu_0}\,\frac{d}{dx}\,(e^{-2ax^2}) = -4\,ax\,e^{-2ax^2}\,\frac{\chi B_0^2}{2\mu_0}\,V$$

This is maximum when its derivative vanishes, i.e.,

$$16a^2x^2 - 4a = 0 \quad \text{or} \quad x_m = \frac{1}{\sqrt{4a}}$$

The maximum force is

$$F_{\max} = 4a\,\frac{1}{\sqrt{4a}}\,e^{-1/2}\,\frac{\chi B_0^2 V}{2\mu_0} = \frac{\chi B_0^2 V}{\mu_0}\sqrt{\frac{a}{e}}$$

So, $$\chi = \frac{\mu_0\,F_{\max}\sqrt{\dfrac{e}{a}}}{VB_0^2} = 3.6 \times 10^{-4}$$

3.287 The force is given by

$$F_x = (\mathbf{p}_m \cdot \nabla)\,B_x = \frac{\chi BV}{\mu\,\mu_0}\,\frac{dB}{dx} \approx \frac{\chi V}{2\mu_0}\,\frac{dB^2}{dx}$$

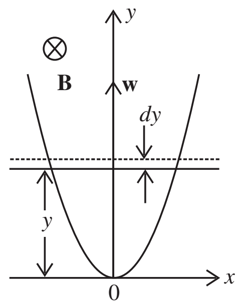
This force is attractive and an equal force must be applied for balance. The work done by applied forces is

$$A = \int_{x=0}^{x=L} -\,F_x dx = \frac{\chi V}{2\mu_0}\,(-B^2)_{x=0}^{x=L} \approx \frac{\chi VB^2}{2\mu_0}$$

3.6 Electromagnetic Induction. Maxwell's Equations

3.288 Taking the direction of vector **B** normally inward and direction on normal **n** to the plane pointing outward. Here,

$$d\Phi = \mathbf{B}\cdot d\mathbf{S} = -2Bxdy$$



Since $$y = ax^2 \quad \text{or} \quad x = \sqrt{\frac{y}{a}}$$

So, $$d\Phi = -2B\sqrt{\frac{y}{a}}dy$$

From Faraday's law of electromagnetic induction,

$$\xi_{in} = -\frac{d\Phi}{dt}$$

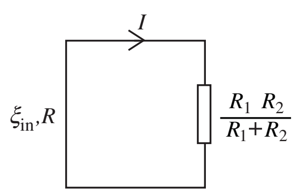
Hence, $$\xi_{in} = 2B\sqrt{\frac{y}{a}}\frac{dy}{dt}$$

$$= By\sqrt{\frac{8w}{a}} \quad \left(\text{using } \frac{dy}{dt} = \sqrt{2wy}\right)$$

From Lenz's law, and right handed screw rule, the induced current and hence the induced e.m.f. in the loop is anticlockwise.

3.289 Let us assume **B** is directed into the plane of the loop. Then the motional e.m.f.

$$\xi_{in} = \int (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l} = vBl$$



and is directed in the sense of $(\mathbf{v}\times\mathbf{B})$.

So, $$I = \frac{\xi_{in}}{R + \frac{R_1R_2}{R_1+R_2}} = \frac{Bvl}{R+R_u}\left(\text{where } R_u = \frac{R_1R_2}{R_1+R_2}\right)$$

as R_1 and R_2 are in parallel connections.

3.290 (a) As the metal disk rotates, any free electron also rotates with it with the same angular velocity ω, and that is why an electron must have an acceleration of $\omega^2 r$ directed towards the disk's center, where r is the separation of the electron from the center of the disk. We know from Newton's second law that if a particle has some acceleration then there must be a net effective force on it in the direction of acceleration. We also know that a charged particle can be influenced by two fields-electric and magnetic. In our problem, magnetic field is absent hence there must be an electric field near an electron and is directed opposite to the acceleration of the electron.

If E is the electric field strength at a distance r from the center of the disk, we have from Newton's second law

$$F_n = mw_n$$

$$eE = mr\omega^2$$

or
$$E = \frac{m\omega^2 r}{e}$$

and the potential difference is given by

$$\varphi_{\text{cen}} - \varphi_{\text{rim}} = \int_0^a \mathbf{E} \cdot d\mathbf{r} = \int_0^a \frac{m\omega^2 r}{e} dr \text{ as } \mathbf{E} \uparrow\downarrow d\mathbf{r}$$

Thus,
$$\varphi_{\text{cen}} - \varphi_{\text{rim}} = \Delta\varphi = \frac{m\omega^2}{e}\frac{a^2}{2} = 3.0 \text{ nV}$$

(b) When field **B** is present, by definition of motional e.m.f.

$$\varphi_1 - \varphi_2 = \int_1^2 -(\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r}$$

Hence the sought potential difference,

$$\varphi_{\text{cen}} - \varphi_{\text{rim}} = \int_0^a -vBdr = \int_0^a -\omega rBdr \ \ (\text{as } v = \omega r)$$

Thus,
$$\varphi_{\text{rim}} - \varphi_{\text{cen}} = \varphi = \frac{1}{2}\omega Ba^2 = 20 \text{ mV}$$

In general $\omega < eB/m$, so we can neglect the effect discussed in part (a) here.

3.291 By definition,

$$\mathbf{E} = -(\mathbf{v} \times \mathbf{B})$$

So,
$$\int_A^C \mathbf{E} \cdot d\mathbf{r} = \int_A^C -(\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r} = \int_0^d -vB\,dr$$

But, $v = \omega r$, where r is the perpendicular distance of the point from A.

Hence,
$$\int_A^C \mathbf{E} \cdot d\mathbf{r} = \int_0^d -\omega Brdr = -\frac{1}{2}\omega Bd^2 = -10 \text{ mV}$$

This result can be generalized to a wire AC of arbitary planar shape. We have

$$\int_A^C \mathbf{E}\cdot d\mathbf{r} = -\int_A^C (\mathbf{v}\times\mathbf{B})\cdot dr$$

$$= -\int_A^C ((\omega\times\mathbf{r})\times\mathbf{B})\cdot d\mathbf{r}$$

$$= -\int_A^C (\mathbf{B}\cdot\mathbf{r}\boldsymbol{\omega} - \mathbf{B}\cdot\boldsymbol{\omega}\mathbf{r})\cdot d\mathbf{r}$$

$$= -\frac{1}{2}B\omega\, d^2$$

$$= -10 \text{ mV}$$

d being AC and $\boldsymbol{r}$ being measured from A.

3.292 Magnitude of flux at any moment of time,

$$|\Phi(t)| = \mathbf{B}\cdot\mathbf{S} = B\left(\frac{1}{2}R^2\varphi\right)$$

where φ is the sector angle, enclosed by the field.

Now, magnitude of induced e.m.f. is given by,

$$\xi_{\text{in}} = \left|\frac{d\Phi_t}{dt}\right| = \left|\frac{BR^2}{2}\frac{d\varphi}{dt}\right| = \frac{BR^2}{2}\omega$$

where ω is the angular velocity of the disk. But as it starts rotating from rest at $t = 0$ with an angular acceleration β, its angular velocity $\omega(t) = \beta t$. So,

$$\xi_{\text{in}} = \frac{BR^2}{2}\beta t$$

According to Lenz's law the first half cycle current in the loop is in anticlockwise direction, and in subsequent half cycle it is in clockwise direction.

Thus, $$\xi_{\text{in}} = (-1)^n\frac{BR^2}{2}\beta t \quad \text{(where } n \text{ is number of half revolutions)}$$

The plot $\xi_m(t)$, where $t_n = \sqrt{2\pi n/\beta}$, is as shown in the answer sheet.

3.293 Field, due to the current carrying wire in the region, right to it, is directed into the plane of the paper and its magnitude is given by

$$B = \frac{\mu_0}{2\pi r}\frac{I}{r}$$

where r is the perpendicular distance from the wire.

As B is same along the length of the rod thus, motional e.m.f.

$$\xi_{in} = \left| \int_1^2 (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} \right| = vBl$$

and it is directed in the sense of $(\mathbf{v} \times \mathbf{B})$.

So, current (induced) in the loop

$$I_{in} = \frac{\xi_{in}}{R} = \frac{1}{2}\frac{\mu_0 lvI}{\pi Rr}$$

3.294 Field due to the current carrying wire at a perpendicular distance x from it is given by

$$B(x) = \frac{\mu_0}{2\pi}\frac{I}{x}$$

There will be no induced e.m.f. in the segments (2) and (4) as, $\mathbf{v} \uparrow\uparrow d\mathbf{l}$. The magnitude of e.m.f. induced in 1 and 3, will be

$$\xi_1 = vB(a+x)\,a = v\left(\frac{\mu_0}{4\pi}\frac{2I}{a+x}\right)a = \frac{\mu_0 avI}{2\pi(a+x)}$$

$$\xi_2 = vB(x)a = v\left(\frac{\mu_0}{4\pi}\frac{2I}{x}\right)a = \frac{\mu_0 av}{2\pi}\frac{I}{x}$$

respectively, and their sense will be in the direction of $(\mathbf{v} \times \mathbf{B})$.

So, e.m.f. induced in the network

$$= \xi_1 - \xi_2 \text{ (as } \xi_1 > \xi_2\text{)}$$

$$= \frac{av\mu_0 I}{2\pi}\left[\frac{1}{x} - \frac{1}{a+x}\right] = \frac{va^2\mu_0 I}{2\pi x(a+x)}$$

3.295 As the rod rotates, motional e.m.f. is induced in it

$$\xi = \int_0^a (\omega r)\,B\,dr = \frac{1}{2}a^2 B\omega$$

The net current in the conductor is then

$$\frac{\xi(t) - \frac{1}{2}a^2 B\omega}{R}$$

A magnetic force will then act on the conductor of magnitude BI per unit length. Its direction will be normal to B and the rod and its torque will be

$$\int_0^a \left(\frac{\xi(t) - \frac{1}{2} a^2 B\omega}{R} \right) dx\, Bx$$

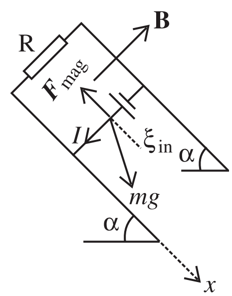
Obviously both magnetic and mechanical torque acting on the C.M. of the rod must be equal but opposite in sense. Then for equilibrium at constant ω,

$$\frac{\xi(t) - \frac{1}{2} a^2 B\omega}{R} \frac{Ba^2}{2} = \frac{1}{2} mga \sin \omega t$$

or $$\xi(t) = \frac{1}{2} a^2 B\omega + \frac{mg\,R}{aB} \sin \omega t = \frac{1}{2aB} (a^3 B^2 \omega + 2mg\,R \sin \omega t)$$

(The answer given in the book is incorrect dimensionally.)

3.296 The rod behaves like a cell of e.m.f. $= vBl$, where v is the velocity of the rod at an arbitrary instant of time. From Lenz's law, current in the loop is in clockwise direction.



From Newton's second law for the rod, $F_x = mw_x$

or $$mg \sin \alpha - IlB = mw$$

For steady state, acceleration of the rod must be equal to zero.

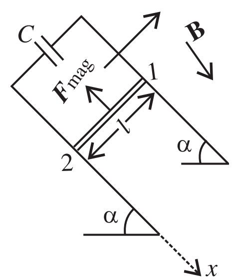
Hence, $$mg \sin \alpha = IlB \qquad (1)$$

But, $$I = \frac{\xi_{in}}{R} = \frac{vBl}{R} \qquad (2)$$

From Eqs. (1) and (2)

$$v = \frac{mg \sin \alpha\, R}{B^2 l^2}$$

3.297 From Lenz's law, the current through the copper bar is directed from 1 to 2 or in other words, the induced current in the circuit is in clockwise direction.



Potential difference across the capacitor plates

$$\frac{q}{C} = \xi_{in} \quad \text{or} \quad q = C\xi_{in}$$

Hence, the induced current in the loop

$$I = \frac{dq}{dt} = C \frac{d\xi_{in}}{dt}$$

But the variation of magnetic flux through the loop is caused by the movement of the bar.

So, the induced e.m.f. $\xi_{in} = Blv$

On differentiation $\frac{d\xi_{in}}{dt} = Bl\frac{dv}{dt} = Blw$

Hence, $I = C\frac{d\xi}{dt} = CBlw$

Now, the forces acting on the bars are the weight and the Ampere's force, where $F_{amp} = IlB\,(CB\,lw)\,B = Cl^2B^2w$.

From Newton's second law, for the rod,

$$F_x = mw_x$$

or $mg\sin\alpha - Cl^2B^2w = mw$

Hence, $$w = \frac{mg\sin\alpha}{Cl^2B^2 + m} = \frac{g\sin\alpha}{1 + \frac{l^2B^2C}{m}}$$

3.298 Flux of **B**, at an arbitrary moment of time t is given by

$$\Phi_t = \mathbf{B}\cdot\mathbf{S} = B\frac{\pi a^2}{2}\cos\omega t$$

From Faraday's law, induced e.m.f.

$$\xi_{in} = -\frac{d\Phi}{dt}$$

$$= -\frac{d\left(B\pi\frac{a^2}{2}\cos\omega t\right)}{dt} = \frac{B\pi a^2\omega}{2}\sin\omega t$$

and induced current

$$I_{in} = \frac{\xi_{in}}{R} = \frac{B\pi a^2}{2R}\,\omega\,\sin\omega t$$

Now, thermal power generated in the circuit at the moment at time $= t$

$$P(t) = \xi_{in}\times I_{in} = \left(\frac{B\pi a^2\omega}{2}\right)^2\frac{1}{R}\sin^2\omega t$$

and mean thermal power generated is given by

$$< P > = \frac{\left[\frac{B\pi a^2\omega}{2}\right]^2\frac{1}{2}\int\limits_0^T\sin^2\omega t\,dt}{\int\limits_0^T dt} = \frac{1}{2R}\left(\frac{B\pi a^2\omega}{2}\right)^2$$

Note: *The calculation of ξ_{in}, which can also be checked by using motional e.m.f. is correct even though the conductor is not a closed semicircle, because the flux linked to the rectangular part containing the resistance R is not changing. The answer given in the book is off by a factor* 1/4.

3.299 The flux through the coil changes sign. Initially it is BS per turn. Finally it is $-BS$ per turn. Now if flux is Φ at an intermediate state, then the current at that moment will be

$$I = \frac{-N\dfrac{d\Phi}{dt}}{R}$$

So, charge that flows during a sudden turning of the coil is

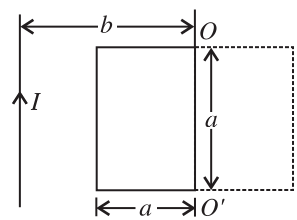
$$q = \int I\,dt = -\frac{N}{R}[\Phi - (-\Phi)]$$

$$= 2N\frac{BS}{R}$$

Hence, $$B = \frac{1}{2}\frac{qR}{NS} = 0.5 \text{ T (on substituting values)}$$

3.300 According to Ohm's law and Faraday's law of induction, the current I_0 appearing in the frame, during its rotation, is determined by the formula

$$I_0 = -\frac{d\Phi}{dt} = -L\frac{dI_0}{dt}$$



Hence, the required amount of electricity (charge) is

$$q = \int I_0\,dt$$

$$= -\frac{1}{R}\int(d\Phi + L\,dI_0)$$

$$= -\frac{1}{R}(\Delta\Phi + L\,\Delta I_0)$$

Since the frame has been stopped after rotation, the current in it vanishes, and hence $\Delta I_0 = 0$. We have to find the increment of the flux $\Delta\Phi$ through the frame ($\Delta\Phi = \Phi_2 - \Phi_1$).

Let us choose the normal **n** to the plane of the frame, for instance, so that in the final position, **n** is directed behind the plane of the figure (along **B**).

Then it can be easily seen that in the final position, $\Phi_2 > 0$, while in the initial position, $\Phi_1 < 0$ (the normal is opposite to **B**), and $\Delta\varphi$ turns out to be simply equal to the flux through the surface bounded by the final and initial positions of the frame:

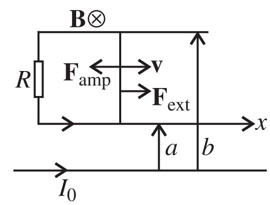
$$\Delta\Phi = \Phi_2 + |\Phi_1| = \int_{b-a}^{b+a} B\,a dr$$

where B is a function of r, whose form can be easily found with the help of the theorem of circulation. Finally omitting the minus sign, we obtain

$$q = \frac{\Delta\Phi}{R} = \frac{\mu_0 aI}{2\pi R} \ln \frac{b+a}{b-1}$$

3.301 As **B**, due to the straight current carrying wire, varies along the rod (connector) and enters linearly so, to make the calculations simple, **B** is made constant by taking its average value in the range $[a, b]$.

$$<B> = \frac{\int_a^b B dr}{\int_a^b dr} = \frac{\int_a^b \frac{\mu_0}{2\pi}\frac{I_0}{r} dr}{\int_a^b dr}$$



or $$<B> = \frac{\mu_0}{2\pi}\frac{I_0}{(b-a)} \ln\frac{b}{a}$$

(a) The flux of **B** changes through the loop due to the movement of the connector. According to Lenz's law, the current in the loop will be anticlockwise. The magnitude of motional e.m.f.

$$\xi_{in} = v<B>(b-a)$$

$$= \frac{\mu_0}{2\pi}\frac{I_0}{(b-a)}\left(\ln\frac{b}{a}\right)(b-a)\frac{dx}{dt} = \frac{\mu_0}{2\pi} I_0 \ln\frac{b}{a} v$$

So, induced current

$$I_{in} = \frac{\xi_{in}}{R} = \frac{\mu_0}{2\pi}\frac{I_0 v}{R}\ln\frac{b}{a}$$

(b) The force required to maintain the constant velocity of the connector must be the magnitude equal to that of Ampere's force acting on the connector, but in opposite direction.

So, $$F_{ext} = I_{in}\, l <B> = \left(\frac{\mu_0}{2\pi}\frac{I_0}{R} v \ln\frac{b}{a}\right)(b-a)\left(\frac{\mu_0}{2\pi}\frac{I_0}{(b-a)}\ln\frac{b}{a}\right)$$

$$= \frac{v}{R}\left(\frac{\mu_0}{2\pi} I_0 \ln\frac{b}{a}\right)^2$$

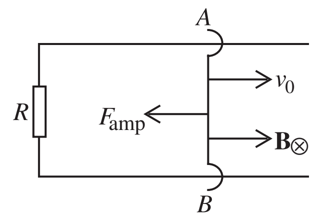
and will be directed as shown in the figure.

3.302 (a) The flux through the loop changes due to the movement of the rod AB. According to Lenz's law, current should be anticlockwise, since we have assumed that **B** is directed into the plane of the loop. The motional e.m.f. $\xi_{in}(t) = Blv$ and induced current $I_{in} = vBl/R$.

From Newton's law in projection form,

$$F_x = mw_x$$

So, $$-F_{amp} = m\frac{v\,dv}{dx}$$



But $$F_{amp} = I_{in}\,lB = \frac{vB^2l^2}{R}$$

So, $$-\frac{vB^2l^2}{R} = mv\frac{dv}{dx}$$

or $$\int_0^x dx = -\frac{mR}{B^2l^2}\int_{v_0}^{0} dv \quad \text{or} \quad x = \frac{mRv_0}{B^2l^2}$$

(b) Sum of the works done by Ampere's force and induced e.m.f. is zero, so from energy conservation

$$\Delta E + Q = 0 \text{ (where } Q \text{ is heat liberated)}$$

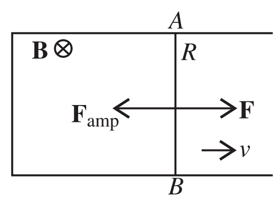
or $$\left[0 - \frac{1}{2}mv_0^2\right] + Q = 0$$

So, $$Q = \frac{1}{2}mv_0^2$$

3.303 Using the calculation done in the previous problem, Ampere's force on the connector,

$$\mathbf{F}_{amp} = \frac{vB^2l^2}{R} \text{ (directed towards left)}$$

Now from Newton's second law,



$$F - F_{amp} = m\frac{dv}{dt}$$

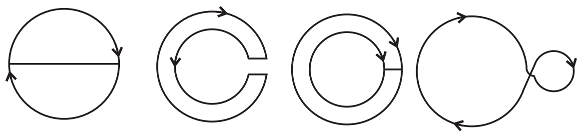
So, $$F = \frac{vB^2l^2}{R} + m\frac{dv}{dt}$$

or $$t = m\int_0^v \frac{dv}{F - \dfrac{vB^2l^2}{R}}$$

or $$\frac{t}{m} = -\frac{R}{B^2l^2}\ln\left(\frac{F - \dfrac{vB^2l^2}{R}}{F}\right)$$

Thus, $$v = (1 - e^{-tB^2l^2/Rm})\frac{RF}{B^2l^2}$$

3.304 According to Lenz's law, the sense of induced e.m.f. is such that it opposes the cause of change of flux. In our problem, magnetic field is directed away from the reader and is diminishing. So,



(*a*) (*b*) (*c*) (*d*)

In Fig. (a), in the round conductor, it is clockwise and there is no current in the connector.

In Fig. (b), in the outside conductor, it is clockwise.

In Fig. (c), in both the conductors, it is clockwise; and there is no current in the connector to obey the charge conservation.

In Fig. (d), in the left side of the figure, it is clockwise.

3.305 The loops are connected in such a way that if the current is clockwise in one, it is anticlockwise in the other. Hence the e.m.f. in loop b opposes the e.m.f. in loop a.

In the loop with side equal to a, e.m.f. is $\frac{d}{dt}(a^2B) = a^2\frac{d}{dt}(B_0 \sin\omega t)$

Similarly, e.m.f. in loop with side equal to $b = b^2 B_0 \omega \cos\omega t$.

Hence, net e.m.f. in the circuit $= (a^2 - b^2) B_0\omega \cos \omega t$, as both the e.m.f's are in opposite sense, and resistance of the circuit $= 4(a + b)\rho$.

Therefore, the amplitude of the current

$$= \frac{(a^2 - b^2) B_0\omega}{4(a + b)\rho} = 0.5 \text{ A}$$

3.306 The flat shape is made up of concentric loops, having different radii, varying from O to a. Let us consider an elementary loop of radius r, then, e.m.f. induced due to this loop is

$$\frac{-d(\mathbf{B}\cdot\mathbf{S})}{dt} = \pi r^2 B_0\omega \cos\omega t$$

and the total induced e.m.f.

$$\xi_{in} = \int_o^a (\pi r^2 B_0\omega \cos\omega t)\, dN \qquad (1)$$

where $\pi r^2\omega \cos\omega t$ is the contribution of one turn of radius r and dN is the number of turns in the interval $(r, r + dr)$.

So, $$dN = \left(\frac{N}{a}\right) dr \tag{2}$$

From Eqs. (1) and (2),

$$\xi = \int_0^a -(\pi r^2 B_0 \omega \cos\omega t)\frac{N}{a}\, dr = \frac{\pi B_0 \omega \cos\omega t N a^2}{3}$$

Maximum value of e.m.f. amplitude

$$\xi_{\max} = \frac{1}{3}\pi B_0\, \omega N a^2$$

3.307 The flux through the loop changes due to the variation in **B** with time and also due to the movement of the connector.

So, $$\xi_{in} = \left|\frac{d\,(\mathbf{B}\cdot\mathbf{S})}{dt}\right| = \left|\frac{d\,(BS)}{dt}\right| \text{ (as } \mathbf{S} \text{ and } \mathbf{B} \text{ are collinear)}$$

But, B, after t seconds of beginning of motion $= Bt$, and S becomes $=1/2\; l\, wt^2$, as connector starts moving from rest with a constant acceleration w.

So, $$\xi_{in} = \frac{3}{2} Bl\, wt^2$$

$$= 12 \text{ mV (on substituting values)}$$

3.308 We use $$B = \mu_0\, n\, I$$

Then, from the law of electromagnetic induction

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}$$

So, for $r < a$

$$E_\varphi 2\pi r = -\pi r^2 \mu_0 n\, \dot{I} \quad \text{or} \quad E_\varphi = -\frac{1}{2}\mu_0\, nr\dot{I} \;\; (\text{where } \dot{I} = dI/dt)$$

For $r > a$,

$$E_\varphi\, 2\pi r = -\pi a^2 \mu_0 n\dot{I} \quad \text{or} \quad E_\varphi = -\mu_0 n\dot{I}a^2/2r$$

The meaning of minus sign can be deduced from Lenz's law.

3.309 The e.m.f. induced in the turn is $\mu_0 n\dot{I}\,\pi\, d^2/4$. The resistance is $\pi d/S\rho$. So, the current is $\mu_0 n\dot{I}Sd/4\rho = 2$ mA, where ρ is the resistivity of copper.

3.310 The changing magnetic field will induce an e.m.f. in the ring, which is obviously equal in the two parts by symmetry (the e.m.f. induced by electromagnetic induction does not depend on resistance). The current that will flow due to this will be different in the two parts. This will cause an acceleration of charge, leading to the setting up of an electric field E which has opposite sign in the two parts. Thus,

$$\frac{\xi}{2} - \pi a E = r\dot{I} \quad \text{and} \quad \frac{\xi}{2} + \pi a E = \eta\, rI$$

where ξ is the total induced e.m.f. From this,

$$\xi = (\eta + 1) rI$$

and
$$E = \frac{1}{2\pi a}(\eta - 1) rI = \frac{1}{2\pi a}\frac{\eta - 1}{\eta + 1}\xi$$

But by Faraday's law,
$$\xi = \pi a^2 b$$

So,
$$E = \frac{1}{2} ab \frac{\eta - 1}{\eta + 1}$$

3.311 Consider the rotating frame with an instantaneous angular velocity $\omega(t)$. In this frame, a Coriolis force, $2\, mv' \times \omega(t)$ acts, which must be balanced by the magnetic force $e\mathbf{v} \times \mathbf{B}(t)$.

Thus,
$$\omega(t) = -\frac{e}{2m}\mathbf{B}(t)$$

It is assumed that ω is small and varies slowly, so ω^2 and ω can be neglected.

3.312 The solenoid has an inductance,

$$L = \mu_0 n^2 \pi b^2 I$$

where n = number of turns of the solenoid per unit length. When the solenoid is connected to the source, an e.m.f. is set up, which, because of the inductance and resistance, rises slowly according to the equation

$$RI + L\dot{I} = V$$

This has the well-known solution

$$I = \frac{V}{R}(1 - e^{-tR/L})$$

Corresponding to this current, an e.m.f. is induced in the ring. Its magnetic field $B = \mu_0 nI$, in the solenoid, produces a force per unit length given by

$$\frac{dF}{dl} = BI = \frac{\mu_0^2 n^2 \pi a^2 I I}{r}$$
$$= \frac{\mu_0^2\, \pi a^2 V^2}{r}\left(\frac{n^2}{RL}\right) e^{-tR/L}\,(1 - e^{-tR/L}),$$

acting on each segment of the ring. This force is zero initially and zero for a large t. Its maximum value is for some finite t. The maximum value of

$$e^{-tR/L}(1 - e^{-tR/L}) = \frac{1}{4} - \left(\frac{1}{2} - e^{-tR/L}\right)^2 = \frac{1}{4}$$

So,
$$\frac{dF_{max}}{dl} = \frac{\mu_0^2 \pi a^2 V^2}{r} \frac{n^2}{4RL} = \frac{\mu_0 a^2 V^2}{4rRlb^2}$$

3.313 The amount of heat generated in the loop during a small time interval dt

$$dQ = \xi^2/R\,dt, \text{ but, } \xi = -\frac{d\Phi}{dt} = 2at - a\tau$$

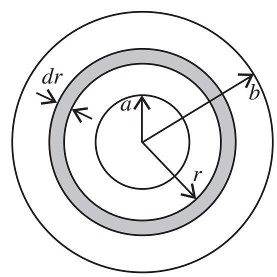
So,
$$dQ = \frac{(2at - a\tau)^2}{R} dt$$

and hence, the amount of heat generated in the loop during the time interval 0 to τ

$$Q = \int_0^\tau \frac{(2at - a\tau)^2}{R} dt = \frac{1}{3} \frac{a^2 \tau^3}{R}$$

3.314 Take an elementary ring of radius r and width dr. The e.m.f. induced in this elementary ring is $\pi r^2 \beta$. Now the conductance of this ring is

$$d\left(\frac{1}{R}\right) = \frac{h\,dr}{\rho\, 2\pi r} \quad \text{so} \quad dI = \frac{hrdr}{2\rho}\beta$$



Integrating we get the total current as

$$I = \int_0^b \frac{hrdr}{2\rho}\beta = \frac{h\beta(b^2 - a^2)}{4\rho}$$

3.315 Given $L = \mu_0 n^2 V = \mu_0 n^2 l_0 \pi R^2$, where R is the radius of the solenoid.

Thus,
$$n = \sqrt{\frac{L}{\mu_0 l_0 \pi}} \frac{1}{R}$$

So, length of the wire required is

$$l = n l_0 2\pi R = \sqrt{\frac{4\pi L l_0}{\mu_0}} = 0.10 \text{ km}$$

3.316 From the previous problem, we know that, length of the wire needed

$$l' = \sqrt{\frac{Ll\,4\pi}{\mu_0}} \quad \text{(where } l = \text{length of solenoid)}$$

Now, $$R = \frac{\rho_0 l'}{S}$$

where S = area of cross section of the wire, also $m = \rho S l'$.

Thus, $$l' = \frac{RS}{\rho_0} = \frac{Rm}{\rho\rho_0 l'} \quad \text{or} \quad l' = \sqrt{\frac{Rm}{\rho\rho_0}}$$

where ρ_0 = resistivity of copper and ρ = density of copper.

Equating, $$\frac{Rm}{\rho\rho_0} = \frac{Ll}{\mu_0/4\pi}$$

or $$L = \frac{\mu_0}{4\pi}\frac{mR}{\rho\rho_0 l}$$

3.317 The current at time t is given by

$$I(t) = \frac{V}{R}(1 - e^{-tR/L})$$

The steady state value is

$$I_0 = \frac{V}{R}$$

and $$\frac{I(t)}{I_0} = \eta = 1 - e^{-tR/L} \quad \text{or} \quad e^{-tR/L} = 1 - \eta$$

or $$t_0\frac{R}{L} = \ln\frac{1}{1-\eta} \quad \text{or} \quad t_0 = \frac{L}{R}\ \ln\frac{1}{1-\eta} = 1.49\ \text{s}$$

3.318 The time constant τ is given by

$$\tau = \frac{L}{R} = \frac{L}{\rho_0\dfrac{l_0}{S}}$$

where ρ_0 = resistively, l_0 = length of the winding wire, S = cross section of the wire.

But $$m = l\rho_0 S$$

So eliminating S, $$\tau = \frac{L}{\dfrac{\rho_0 l_0}{m/\rho l_0}} = \frac{mL}{\rho\rho_0 l_0^2}$$

From solution of Problem 3.315, $l_0 = \sqrt{\dfrac{4\pi lL}{\mu_0}}$

(Note the interchange of l and l_0 because of difference in notation here.)

Thus, $$\tau = \frac{mL}{\rho\rho_0\dfrac{4\pi}{\mu_0}Ll} = \mu_0 4\pi\frac{m}{\rho\rho_0 l} = 0.7\ \text{ms}$$

3.319 Between the cables, where $a < r < b$, the magnetic field $\mathbf{H}$ satisfies

$$H_\varphi 2\pi r = I \quad \text{or} \quad H_\varphi = \frac{I}{2\pi r}$$

So,
$$B_\varphi = \frac{\mu\mu_0 I}{2\pi r}$$

The associated flux per unit length is

$$\Phi = \int_{r=a}^{r=b} \frac{\mu\mu_0 I}{2\pi r} \times 1 \times dr = \frac{\mu\mu_0 I}{2\pi} \ln\frac{b}{a}$$

Hence, the inductance per unit length is

$$L_1 = \frac{\Phi}{I} = \frac{\mu\mu_0}{2\pi} \ln \eta \left(\text{where } \eta = \frac{b}{a}\right)$$

We get
$$L_1 = 0.26 \frac{\mu\text{H}}{\text{m}}$$

3.320 Within the solenoid

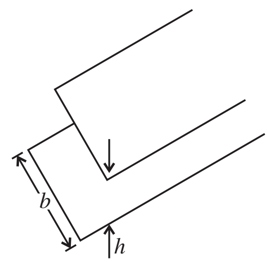
$$H_\varphi \cdot 2\pi r = NI \quad \text{or} \quad H_\varphi = \frac{NI}{2\pi r}, \ B_\varphi = \mu\mu_0 \frac{NI}{2\pi r}$$

and the flux is
$$\Phi = N\Phi_1 = N\frac{\mu\mu_0}{2\pi} NI \int_b^{a+b} \frac{a\,dr}{r}$$

Finally,
$$L = \frac{\mu\mu_0}{2\pi} N^2 a \ln\left(1 + \frac{a}{b}\right)$$

3.321 Neglecting end effects, the magnetic field B between the plates, which is mainly parallel to the plates, is

$$B = \mu_0 \frac{I}{b}$$



Thus, the associated flux per unit length of the plates is

$$\Phi = \mu_0 \frac{I}{b} \times h \times 1 = \left(\mu_0 \frac{h}{b}\right) \times I$$

So, L_1 = inductance per unit length $= \mu_0 \dfrac{h}{b} = 25$ nH/m.

3.322 For a single current carrying wire,

$$B_\varphi = \frac{\mu_0 I}{2\pi r} \quad (\text{for } r > a)$$

For the double line cable with current flowing in opposite direction in the two conductors $B_\varphi \approx \mu_0 I/\pi r$ between the cables, by superposition. The associated flux is

$$\Phi = \int_a^{d-a} \frac{\mu_0 I}{\pi} \frac{dr \times 1}{r} \approx \frac{\mu_0 I}{\pi} \ln \frac{d}{a} = \frac{\mu_0}{\pi} \ln \eta \times I \text{ per unit length}$$

Hence, $$L_1 = \frac{\mu_0}{\pi} \ln \eta$$

is the inductance per unit length.

3.323 In a superconductor there is no resistance. Hence,

$$L\frac{dI}{dt} = +\frac{d\Phi}{dt}$$

So integrating, $$I = \frac{\Delta\Phi}{L} = \frac{\pi a^2 B}{L}$$

because $$\Delta\Phi = \Phi_f - \Phi_i$$

$$\Phi_f = \pi a^2 B \text{ as } \Phi_i = 0$$

Also, the work done is

$$A = \int \xi I dt = \int I dt \frac{d\Phi}{dt} = \frac{1}{2} LI^2 = \frac{1}{2} \frac{\pi^2 a^4 B^2}{L}$$

3.324 In a solenoid, the inductance

$$L = \mu\mu_0 n^2 V = \mu\mu_0 \frac{N\ S}{l}$$

where S = area of cross-section of the solenoid, l = its length, $V = Sl$, $N = nl$ = total number of turns.

When the length of the solenoid is increased, for example, by pulling it, its inductance will decrease. If the current remains unchanged, the flux, linked to the solenoid, will also decrease. An induced e.m.f. will then come into play, which by Lenz's law will try to oppose the decrease of flux, for example, by increasing the current. In the superconducting state, the flux will not change and so, I/l = constant.

Hence, $$\frac{I}{l} = \frac{I_0}{l_0} \quad \text{or} \quad I = I_0 \frac{l}{l_0} = I_0 (1 + \eta) = 2\text{A (on substituting values)}$$

3.325 The flux linked to the ring cannot change on transition to the super-conduction state, for reasons similar to those given above. Thus, a current I must be induced in the ring, where

$$I = \frac{\Phi}{L} = \frac{\pi a^2 B}{\mu_0 a\left(\ln \frac{8a}{b} - 2\right)} = \frac{\pi a B}{\mu_0\left(\ln \frac{8a}{b} - 2\right)} = 50\,\text{A (on substituting values)}$$

3.326 We write the equation of the circuit as

$$RI + \frac{L}{\eta}\frac{dI}{dt} = \xi$$

for $t \geq 0$. The current at $t = 0$ just after inductance is changed is $I = \eta\xi/R$, so that the flux through the inductance is unchanged.

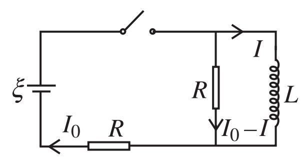
We look for a solution of the above equation in the form

$$I = A + Be^{-t/C}$$

Substituting $$C = \frac{L}{\eta R}, \qquad B = \eta - 1, \qquad A = \frac{\xi}{R}$$

We get, $$I = \frac{\xi}{R}(1 + (\eta - 1)\, e^{-\eta Rt/L}$$

3.327 In the left loop, $\xi - I_0R = (I_0 - I)R$



or $$\xi + IR = 2I_0R$$

So, $$\xi - I_0R = \xi - \frac{(\xi + IR)}{2} = \frac{\xi - I_R}{2}$$

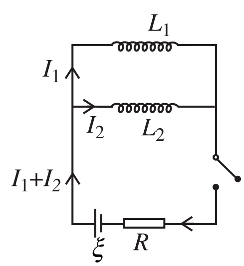
But in the bigger loop

$$\xi - I_0R = L\frac{dI}{dt}$$

or $$\frac{\xi - IR}{2} = L\frac{dI}{dt}$$

So, $$\xi - IR = L\frac{dI}{dt}$$

3.328 The equations are



$$L_1\frac{dI_1}{dt} = L_2\frac{dI_2}{dt} = \xi - R\,(I_1 + I_2)$$

Then, $$\frac{d}{dt}(L_1I_1 - L_2I_2) = 0$$

or $$L_1I_1 - L_2I_2 = \text{constant}$$

But initially at $t = 0$, $I_1 = I_2 = 0$.

So, constant must be zero and at all times. Therefore,

$$L_1 I_1 = L_2 I_2$$

In the final steady state, current must obviously be $I_1 + I_2 = \xi / R$. Thus in steady state,

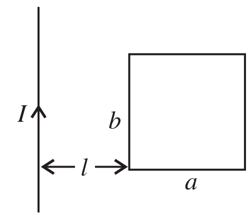
$$I_1 = \frac{\xi L_2}{R(L_1 + L_2)} \quad \text{and} \quad I_2 = \frac{\xi L_1}{R(L_1 + L_2)}$$

3.329 Here,

$$B = \frac{\mu_0 I}{2\pi r} \text{ (at a distance } r \text{ from the wire)}$$

The flux through the frame is,

$$\Phi_{12} = \int_l^{a+l} \frac{\mu_0 I}{2\pi r} b dr = \frac{\mu_0 b}{2\pi} I \ln\left(1 + \frac{a}{l}\right)$$



Thus, $$L_{12} = \frac{\Phi_{12}}{I} = \frac{\mu_0 b}{2\pi} \ln\left(1 + \frac{a}{l}\right)$$

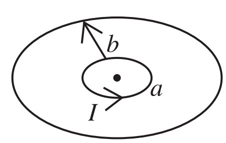
3.330 Here, $$B = \frac{\mu_0 I}{2\pi r} \quad \text{and} \quad \Phi = \mu_0 \mu \frac{l}{2\pi} \int_a^b \frac{h dr}{r} N$$

Thus, $$L_{12} = \frac{\mu \mu_0 \, hN}{2\pi} \ln \frac{b}{a}$$

3.331 The direct calculation of the flux Φ_2 is a rather complicated problem, since the configuration of the field itself is complicated. However, the application of the reciprocity theorem simplifies the solution of the problem. Indeed, let the same current I flow through loop 2. Then the magnetic flux created by this current through loop 1 can be easily found.

As the loop is very small, so, induction value of B at the center is for the whole loop. Magnetic induction at the center of the loop is

$$B = \frac{\mu_0 I}{2b}$$



So, flux through loop 1 is

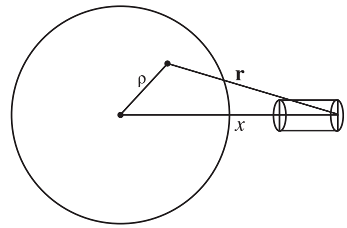
$$\Phi_{12} = \pi a^2 \frac{\mu_0 I}{2b}$$

(a) So, $$L_{12} = \frac{\Phi_{21}}{I} = \frac{1}{2} \frac{\mu_0 \pi a^2}{b}$$

(b) From reciprocity theorem,

$$\Phi_{12} = \Phi_{21}, \; \Phi_{21} = \frac{\mu_0 \pi a^2 I}{2b}$$

3.332 Let $\mathbf{p}_m$ be the magnetic moment of the magnet M. Then, the magnetic field due to this magnet is



$$\frac{\mu_0}{4\pi}\left[\frac{3\,(\mathbf{p}_m\cdot\mathbf{r})\,\mathbf{r}}{r^5}-\frac{\mathbf{p}_m}{r^3}\right]$$

The flux associated with this, when the magnet is along the axis at a distance x from the center, is

$$\Phi = \frac{\mu_0}{4\pi}\int\left[\frac{3\,(\mathbf{p}_m\cdot\mathbf{r})\,\mathbf{r}}{r^5}-\frac{\mathbf{p}_m}{r^3}\right]\cdot d\mathbf{S} = \Phi_1 - \Phi_2$$

where, $$\Phi_2 = \frac{\mu_0}{4\pi}p_m\int_0^a \frac{2\pi\rho d\,\rho}{(x^2+\rho^2)^{3/2}} = \frac{\mu_0 p_m}{2}\left(\frac{1}{x}-\frac{1}{\sqrt{x^2+a^2}}\right)$$

and $$\Phi_1 = \frac{3\mu_0 p_m x^2}{4\pi}\int_0^a \frac{2\pi\rho d\rho}{(x^2+\rho^2)^{5/2}}$$

$$= \frac{\mu_0 p_m x^2}{2}\left(\frac{1}{x^3}-\frac{1}{(x^2+a^2)^{5/2}}\right)$$

So, $$\Phi = \frac{-\mu_0 p_m a^2}{2\,(x^2+a^2)^{3/2}}$$

When the flux changes, an e.m.f. $-N\dfrac{d\Phi}{dt}$ is induced and a current $-\dfrac{N}{R}\dfrac{d\varphi}{dt}$ flows. The total charge q, flowing as the magnet is removed to infinity from $x = 0$, is

$$q = \frac{N}{R}\,\Phi\,(x=0) = \frac{N}{R}\cdot\frac{\mu_0 p_m}{2a}$$

or $$p_m = \frac{2aqR}{N\mu_0}$$

3.333 If a current I flows in one of the coils, the magnetic field at the center of the other coil is

$$B = \frac{\mu_0 a^2 I}{2(l^2+a^2)^{3/2}} = \frac{\mu_0 a^2 I}{2l^3},\text{ as } l \gg a$$

The flux associated with the second coil is then approximately $\mu_0\pi\, a^4 I/2\, l^3$.

Hence, $$L_{12} = \frac{\mu_0\pi a^4}{2l^3}$$

3.334 When the current in one of the loops is $I_1 = \alpha t$, an e.m.f. $L_{12}\,dI_1/dt = L_{12}\,\alpha$, is induced in the other loop. Then, if the current in the other loop is I_2, we must have,

$$L_2\frac{dI_2}{dt} + RI_2 = L_{12}\alpha$$

This equation has the solution,

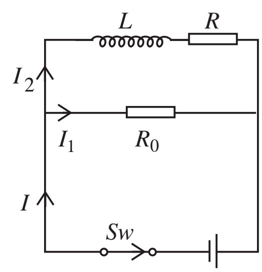
$$I_2 = \frac{L_{12}\alpha}{R}\,(1 - e^{-tR/L_2})$$

which is the required current.

3.335 Initially, after a steady current is set up, the current is flowing as shown in the figure. In steady conditions, $I_{20} = \xi/R$ and $I_{10} = \xi/R_0$.

The current in the inductance cannot change suddenly. We then have the equation,

$$L\frac{dI_2}{dt} + (R + R_0)\, I_2 = 0$$



This equation has the solution

$$I_2 = I_{20}e^{-t\,(R+R_0)/L}$$

The heat dissipated in the coil is

$$Q = \int_0^\infty I_2^2\, Rdt = I_{20}^2 R\int_0^\infty e^{-2t(R+R_0/L)}dt$$

$$= RI_{20}^2 \times \frac{L}{2(R + R_0)} = \frac{L\xi^2}{2R(R + R_0)} = 3\mu\text{J}$$

3.336 To find the magnetic field energy we recall that the flux varies linearly with current. Thus, when the flux is Φ for current I, we can write $\phi = AI$. The total energy enclosed in the field, when the current is I, is

$$W = \int \xi\, Idt = \int N\frac{d\Phi}{dt} I\, dt$$

$$= \int Nd\Phi I = \int_0^I NA\; IdI = \frac{1}{2}NAI^2 = \frac{1}{2}\,N\Phi I$$

The characteristic factor 1/2 appears in this way.

3.337 We apply circulation theorem,

$$H\cdot 2\pi b = NI \quad \text{or} \quad H = \frac{NI}{2\pi b}$$

Thus the total energy,

$$W = \frac{1}{2}\,BH\cdot 2\pi b\cdot \pi a^2 = \pi^2 a^2 bBH$$

Given N, I, b, we know H, and can find out B from the $B - H$ curve. Then W can be calculated $= 2.0$ J.

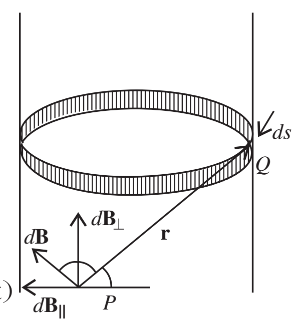
3.338 From
$$\oint H \cdot d\mathbf{r} = NI$$
$$H \cdot \pi d + \frac{B}{\mu_0} \cdot b \approx NI, \ (d \gg b)$$
Also,
$$B = \mu\mu_0 H$$
Thus,
$$H = \frac{NI}{\pi d + \mu d}$$
Since B is continuous across the gap, B is given by,
$$B = \mu\mu_0 \frac{NI}{\pi d + \mu b}$$
both in the magnetic field and the gap.

(a) $$\frac{W_{\text{gap}}}{W_{\text{mag}}} = \frac{\dfrac{B^2}{2\mu_0} \times S \times b}{\dfrac{B^2}{2\mu\mu_0} \times S \times \pi d} = \frac{\mu b}{\pi d} = 3.0$$

(b) The flux is $N \int \mathbf{B} \cdot d\mathbf{S} = N\mu\mu_0 \dfrac{NI}{\pi d + \mu b} \cdot S = \mu_0 \dfrac{SN^2 I}{b + \dfrac{\pi d}{\mu}}$

So,
$$L \approx \frac{\mu_0 SN_2}{b + \dfrac{\pi d}{\mu}}$$
Total energy
$$= \frac{B^2}{2\mu_0}\left(\frac{\pi d}{\mu} + b\right) S = \frac{1}{2} \frac{\mu_0 N^2 S}{b + \dfrac{\pi d}{\mu}} \cdot I^2 = \frac{1}{2} LI^2 = 0.15\text{H}$$
The L found in the one way is similar with the one found in the other way. Note that in calculating the flux, we do not consider the field in the gap since it is not linked to the winding. But the total energy includes that of the gap.

3.339 When the cylinder with a linear charge density λ rotates with a circular frequency ω, a surface current density (charge/length $\times$ time) of $I = \lambda\omega/2\pi$ is set up.



The direction of the surface current is normal to the plane of paper at Q and the contribution of this current to the magnetic field at P is
$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I(\mathbf{e} \times \mathbf{r})}{r^3} dS \text{ (where } \mathbf{e} \text{ is the direction of the current)}$$
In magnitude, $|\mathbf{e} \times \mathbf{r}| = r$, since $\mathbf{e}$ is normal to $\mathbf{r}$ and the direction of $d\mathbf{B}$ is as shown in the figure.

Its component, $d\mathbf{B}_{\|}$, cancels out by cylindrical symmetry. The component that remains is

$$|\mathbf{B}_{\perp}| = \frac{\mu_0}{4\pi}\int \frac{IdS}{r^2}\cos\theta = \frac{\mu_0 I}{4\pi}\int d\Omega = \mu_0 I$$

where we have used,

$$\frac{dS\cos\theta}{r^2} = d\Omega \text{ and } \int d\Omega = 4\pi$$

which is the total solid angle around any point.

The magnetic field vanishes outside the cylinder by similar argument.

The total energy per unit length of the cylinder is

$$W_1 = \frac{1}{2\mu_0}\mu_0^2\left(\frac{\lambda\omega}{2\pi}\right)^2 \times \pi a^2 = \frac{\mu_0}{8\pi}a^2\lambda^2\omega^2$$

3.340
$$w_{\text{ele}} = \frac{1}{2}\varepsilon_0 E^2 \quad \text{(for the electric field)}$$

$$w_{\text{mag}} = \frac{1}{2\mu_0}B^2 \text{ (for the magnetic field)}$$

Thus, $$\frac{1}{2\mu_0}B^2 = \frac{1}{2}\varepsilon_0 E^2$$

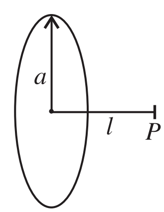
or $$E = \frac{B}{\sqrt{\varepsilon_0\mu_0}} = 3 \times 10^8 \text{ V/m}$$

3.341 The electric field at P is

$$E_P = \frac{ql}{4\pi\varepsilon_0(a^2 + l^2)^{3/2}}$$

To get the magnetic field, note that the rotating ring constitutes a current $I = q\,\omega/2\pi$, and the corresponding magnetic field at P is

$$B_P = \frac{\mu_0 a^2 I}{2(a^2 + l^2)^{3/2}}$$



Thus, $$\frac{w_{\text{ele}}}{w_{\text{mag}}} = \frac{\varepsilon_0\mu_0 E^2}{B^2} = \varepsilon_0\mu_0\left(\frac{ql \times 2}{4\pi\,\varepsilon_0\mu_0 a^2 I}\right)^2$$

$$= \frac{1}{\varepsilon_0\mu_0}\left(\frac{l}{a^2\omega}\right)^2$$

or $$\frac{w_M}{w_E} = \frac{\varepsilon_0\mu_0\omega^2 a^4}{l^2}$$

$$= 1.1 \times 10^{-15}$$

3.342 The total energy of the magnetic field is

$$\frac{1}{2}\int (\mathbf{B}\cdot\mathbf{H})\,dV = \frac{1}{2}\int \mathbf{B}\cdot\left(\frac{\mathbf{B}}{\mu_0} - \mathbf{J}\right) dV$$

$$= \frac{1}{2\mu_0}\int \mathbf{B}\cdot\mathbf{B}\,dV - \frac{1}{2}\int \mathbf{J}\cdot\mathbf{B}dV$$

The second term can be interpreted as the energy of magnetization, and has the density

$$-\frac{1}{2}\,\mathbf{J}\cdot\mathbf{B}.$$

3.343 (a) In series, the current I flows through both coils, and the total e.m.f. induced when the current changes is

$$-\,2L\frac{dI}{dt} = -L'\frac{dI}{dt}$$

or

$$L' = 2L$$

(b) In parallel, the current flowing through either coil is, $\frac{I}{2}$ and the e.m.f. induced is

$$-\,L\left(\frac{1}{2}\frac{dI}{dt}\right)$$

Since,

$$-\,L\left(\frac{1}{2}\frac{dI}{dt}\right) = -\,L'\frac{dI}{dt}$$

We get,

$$L' = \frac{1}{2}L$$

3.344 We use

$$L_1 = \mu_0 n_1^2\,V, L_2 = \mu_0 n_2^2 V$$

So,

$$L_{12} = \mu_0 n_1 n_2 V = \sqrt{L_1 L_2}$$

3.345 The interaction energy is

$$\frac{1}{2\mu_0}\int |\mathbf{B}_1 + \mathbf{B}_2|^2\,dV - \frac{1}{2\mu_0}\int |\mathbf{B}_1|^2\,dV - \frac{1}{2\mu_0}\int |\mathbf{B}_2|^2\,dV$$

$$= \frac{1}{\mu_0}\int \mathbf{B}_1\cdot\mathbf{B}_2\,dV$$

Here, if $\mathbf{B}_1$ is the magnetic field produced by the first of the current carrying loops, and $\mathbf{B}_2$ that of the second one, then the magnetic field due to both the loops will be $\mathbf{B}_1 + \mathbf{B}_2$.

3.346 We can think of the smaller coil as constituting a magnet of dipole moment,

$$p_m = \pi a^2 I_1$$

Its direction is normal to the loop and it makes an angle θ with the direction of the magnetic field, due to the bigger loop. This magnetic field is

$$B_2 = \frac{\mu_0 I_2}{2b}$$

The interaction energy has the magnitude

$$|W| = \frac{\mu_0 I_1 I_2}{2b}\pi a^2 \cos\theta$$

Its sign depends on the sense of the currents.

3.347 (a) There is a radial, outward conduction current. Let Q be the instantaneous charge on the inner sphere, then,

$$j \times 4\pi r^2 = -\frac{dQ}{dt} \quad \text{or} \quad \mathbf{j} = -\frac{1}{4\pi r^2}\frac{dQ}{dt}\,\mathbf{r}$$

On the other hand $\qquad \mathbf{j}_d = \dfrac{\partial \mathbf{D}}{\partial t} = \dfrac{d}{dt}\left(\dfrac{Q}{4\pi r^2}\,\mathbf{r}\right) = -\mathbf{j}$

where D is electric desplacement vector.

(b) At the given moment,

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 \varepsilon r^2}\,\mathbf{r}$$

and by Ohm's law $\qquad \mathbf{j} = \dfrac{\mathbf{E}}{\rho} = \dfrac{q}{4\pi\varepsilon_0\varepsilon\rho r^2}\mathbf{r}$

Then, $\qquad \mathbf{j}_d = -\dfrac{q}{4\pi\varepsilon_0\varepsilon\rho r^2}\,\mathbf{r}$

and $\qquad \oint \mathbf{j}_d \cdot d\mathbf{S} = -\dfrac{q}{4\pi\varepsilon_0\varepsilon\rho}\displaystyle\int \dfrac{dS\cos\theta}{r^2} = -\dfrac{q}{\varepsilon_0\varepsilon\rho}$

The surface integral must be $-ve$ because $\mathbf{j}_d$, being opposite of $\mathbf{j}$, is inward.

3.348 Here also we see that on neglecting edge effects, $\mathbf{j}_d = -\mathbf{j}$. Thus Maxwell's equations reduce to, div $\mathbf{B} = 0$, Curl $\mathbf{H} = 0$, $\mathbf{B} = \mu\mathbf{H}$.

A general solution of this equation is $\mathbf{B}$ = constant = $\mathbf{B}_0$. Here, $\mathbf{B}_0$ can be thought of as an extraneous magnetic field. If it is zero, $\mathbf{B} = 0$.

3.349 Given $\qquad I = I_m \sin\omega t$

We see that $\qquad j = \dfrac{I_m}{S}\sin\omega t = -j_d = -\dfrac{\partial D}{\partial t}$

or $\qquad D = \dfrac{I_m}{\omega S}\cos\omega t$

So, amplitude of electric field

$$E_m = \frac{I_m}{\varepsilon_0 \omega S} = 7\ \text{V/cm}$$

3.350 The electric field between the plates can be written as

$$E = Re\frac{V_m}{d}e^{i\omega t} \quad \text{instead of} \quad \frac{V_m}{d}\cos\omega t$$

This gives rise to a conduction current,

$$j_c = \sigma E = \text{Re}\,\frac{\sigma}{d} V_m e^{i\omega t}$$

and a displacement current,

$$j_d = \frac{\partial D}{\partial t} = \text{Re}\,\varepsilon_0 \varepsilon\, i\omega\, \frac{V_m}{d} e^{i\omega t}$$

The total current is

$$j_T = \frac{V_m}{d}\sqrt{\sigma^2 + (\varepsilon_0 \varepsilon\,\omega)^2}\,\cos(\omega t + \alpha)$$

where, $\tan\alpha = \dfrac{\sigma}{\varepsilon_0 \varepsilon \omega}$

The corresponding magnetic field is obtained by using circulation theorem,

$$H \cdot 2\pi r = \pi r^2 j_T$$

or $H = H_m \cos(\omega t + \alpha)$

where, $H_m = \dfrac{r V_m}{2d}\sqrt{\sigma^2 + (\varepsilon_0 \varepsilon \omega)^2}$

3.351 Inside the solenoid, there is a magnetic field, given by

$$B = \mu_0 n I_m \sin \omega t$$

Since this varies in time, there is an associated electric field. This is obtained by using,

$$\oint \mathbf{E} \cdot d\mathbf{r} = -\frac{d}{dt}\int_s \mathbf{B} \cdot d\mathbf{S}$$

For $r < R$, $\quad 2\pi r E = -\dot{B} \cdot \pi r^2 \quad$ or $\quad E = -\dfrac{\dot{B} r}{2}$

For $r > R$, $\quad E = -\dfrac{\dot{B} R^2}{2r}$

The associated displacement current density is

$$j_d = \varepsilon_0 \frac{\partial E}{\partial t} = \begin{bmatrix} -\varepsilon_0\, \ddot{B}\, r/2 \\ -\varepsilon_0\, \ddot{B}\, R^2/2r \end{bmatrix}$$

(The answer given in the book is dimensionally incorrect without the factor ε_0.)

3.352 In the non-relativistic limit,

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0 r^3}\,\mathbf{r}$$

(a) On a straight line coinciding with the charge path,

$$\mathbf{j}_d = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{q}{4\pi}\left[\frac{-\mathbf{v}}{r^3} - \frac{3\mathbf{r} r}{r^4}\right] \left(\text{using } \frac{d\mathbf{r}}{dt} = -\mathbf{v}\right)$$

But in this case,

$$\dot{r} = -v \text{ and } v\frac{\mathbf{r}}{r} = \mathbf{v} \quad \text{so,} \quad \mathbf{j}_d = \frac{2q\mathbf{v}}{4\pi r^3}$$

(b) In this case, $\dot{r} = 0$, as $\mathbf{r} \perp \mathbf{v}$.

Thus,

$$\mathbf{j}_d = -\frac{q\mathbf{v}}{4\pi r^3}$$

3.353 We have,

$$E_p = \frac{qx}{4\pi\varepsilon_0 (a^2 + x^2)^{3/2}}$$

then,

$$j_d = \frac{\partial D}{\partial t} = \varepsilon_0 \frac{\partial E}{\partial t} = \frac{qv}{4\pi(a^2 + x^2)^{5/2}} \, (a^2 - 2x^2)$$

This is maximum, when $x = x_m = 0$, and minimum at some other value. The maximum displacement current density is

$$(j_d)_{\max} = \frac{qv}{4\pi a^3}$$

To check this, we calculate

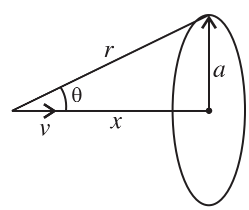
$$\frac{\partial j_d}{\partial x} = \frac{qv}{4\pi}[(-4x(a^2 + x^2) - 5x(a^2 - 2x^2)]$$

This vanishes for $x = 0$ and for $x = \sqrt{\frac{3}{2}}\, a$. The latter is easily shown to be a smaller local minimum (negative maximum).

3.354 We use Maxwell's equations in the form

$$\oint \mathbf{B} \cdot d\mathbf{r} = \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \int \mathbf{E} \cdot d\mathbf{S}$$

when the conduction current vanishes at the site.



We know that,

$$\int \mathbf{E} \cdot d\mathbf{S} = \frac{q}{4\pi\varepsilon_0} \int \frac{d\mathbf{S} \cdot \mathbf{r}}{r^2}$$

$$= \frac{q}{4\pi\varepsilon_0} \int d\Omega = \frac{q}{4\pi\varepsilon_0} 2\pi(1 - \cos\theta)$$

where $2\pi\,(1 - \cos\theta)$ is the solid angle, formed by the disk like surface, at the charge.

Thus,

$$\oint \mathbf{B} \cdot d\mathbf{r} = 2\pi a B = \frac{1}{2}\mu_0 q \cdot \sin\theta \cdot \dot{\theta}$$

On the other hand, $\quad x = a \cot\theta$

Differentiating and using $$\frac{dx}{dt} = -v$$

we get, $$v = a \operatorname{cosec}^2\theta\,\dot{\theta}$$

Thus, $$B = \frac{\mu_0\, qvr \sin\theta}{4\pi r^3}$$

This can be written as

$$\mathbf{B} = \frac{\mu_0 q\,(\mathbf{v}\times\mathbf{r})}{4\pi r^3}$$

and $$\mathbf{H} = \frac{q}{4\pi}\,\frac{\mathbf{v}\times\mathbf{r}}{r^3}$$

(The sense has to be checked independently.)

3.355 (a) If $\mathbf{B} = \mathbf{B}(t)$, then,

$$\text{Curl }\mathbf{E} = \frac{-\partial\mathbf{B}}{\partial t} \neq 0$$

So, **E** cannot vanish.

(b) Here also, curl $\mathbf{E} \neq 0$, so **E** cannot be uniform.

(c) Suppose for instance,

$$\mathbf{E} = \mathbf{a}\, f(t)$$

where **a** is a spatially and temporally fixed vector. Then $-\partial\mathbf{B}/dt = \text{curl}\,\mathbf{E} = 0$. Generally speaking, this contradicts the other equation, curl $\mathbf{H} = \partial\mathbf{D}/\partial t \neq 0$ because in this case the left hand side is time independent but right-hand side is dependent on time. The only exception is when $f(t)$ is a linear function. Then the uniform field **E** can be time-dependent.

3.356 From the equation

$$\text{Curl }\mathbf{H} - \frac{\partial D}{\partial t} = \mathbf{j}$$

We get on taking divergence of both sides,

$$-\frac{\partial}{\partial t}\,\text{div }\mathbf{D} = \text{div }\mathbf{j}$$

But div $\mathbf{D} = \rho$ and hence,

$$\text{div}\,\mathbf{j} + \frac{\partial\rho}{\partial t} = 0.$$

3.357 From the equation $$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

On taking divergence of both sides,

$$0 = -\frac{\partial}{\partial t}\,\text{div }\mathbf{B}$$

This is compatible with div $\mathbf{B} = 0$.

3.358 A rotating magnetic field can be represented by

$$B_x = B_0 \cos \omega t;\ B_y = B_0 \sin \omega t \text{ and } B_z = B_{zo}$$

Then, $$\text{Curl } \mathbf{E} = \frac{-\partial \mathbf{B}}{\partial t}$$

So, $$-(\text{Curl } \mathbf{E})_x = -\omega B_0 \sin \omega t = -\omega B_y$$

$$-(\text{Curl } \mathbf{E})_y = \omega B_0 \cos \omega t = \omega B_x \text{ and } -(\text{Curl } \boldsymbol{E})_z = 0$$

Hence, $$\text{Curl } \mathbf{E} = -\boldsymbol{\omega} \times \mathbf{B}$$

3.359 Consider a particle with charge e, moving with velocity $\mathbf{v}$, in frame K. It experiences a force $\mathbf{F} = e\,(\mathbf{v} \times \mathbf{B})$.

In the frame K', moving with velocity **v,** relative to K, the particle is at rest. This means that there must be an electric field $\mathbf{E}$ in K', so that the particle experiences a force given by

$$\mathbf{F}' = e\mathbf{E}' = \mathbf{F} = e\mathbf{v} \times \mathbf{B}$$

Thus, $$\mathbf{E}' = \mathbf{v} \times \mathbf{B}$$

3.360 Within the plate, there will appear a $(\mathbf{v} \times \mathbf{B})$ force, which will cause the charge inside the plate to drift, until a countervailing electric field is set up. This electric field is related to B, as $E = eB$, since v and B are mutually perpendicular, and E is perpendicular to both. The charge density $\pm\sigma$, on the force of the plate producing this electric field is given by

$$E = \frac{\sigma}{\varepsilon_0} \quad \text{or} \quad \sigma = \varepsilon_0 vB = 0.40 \text{ pC/m}^2$$

3.361 Choose $\boldsymbol{\omega} \uparrow\uparrow \mathbf{B}$ along the z–axis, and choose $\mathbf{r}$ as the cylindrical polar radius vector of a reference point (perpendicular distance from the axis). This point has the velocity

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

and experiences a $(\mathbf{v} \times \mathbf{B})$ force, which must be counterbalanced by an electric field

$$\mathbf{E} = -(\boldsymbol{\omega} \times \mathbf{r}) \times \mathbf{B} = -(\boldsymbol{\omega} \cdot \mathbf{B})\,\mathbf{r}$$

There must appear a space charge density

$$\rho = \varepsilon_0 \text{ div } \mathbf{E} = -2\,\varepsilon_0\, \boldsymbol{\omega} \cdot \mathbf{B} = -8 \text{ pC/m}^3$$

Since the cylinder as a whole is electrically neutral, the surface of the cylinder must acquire a positive charge of surface density,

$$\sigma = +\frac{2\,\varepsilon_0 (\boldsymbol{\omega} \cdot \mathbf{B})\ \pi a^2}{2\pi a} = \varepsilon_0\, a\, \boldsymbol{\omega} \cdot \boldsymbol{B} = +2\text{pC/m}^2$$

3.362 In the reference frame K' moving with the particle,

$$\mathbf{E}' \cong \mathbf{E} + \mathbf{v}_0 \times \mathbf{B} = \frac{q\mathbf{r}}{4\pi\varepsilon_0 r^3}$$

$$\mathbf{B}' \cong \mathbf{B} - \mathbf{v}_0 \times \frac{E}{c^2} = 0$$

Here, $\mathbf{v}_0$ = velocity of K' relative to the K frame, in which the particle has velocity $\mathbf{v}$. Clearly, $\mathbf{v}_0 = \mathbf{v}$. From the second equation,

$$\mathbf{B} \cong \frac{\mathbf{v} \times \mathbf{E}}{c^2} = \varepsilon_0 \mu_0 \times \frac{q}{4\pi\varepsilon_0} \frac{\mathbf{v} \times \mathbf{r}}{r^3} = \frac{\mu_0}{4\pi} \frac{q(\mathbf{v} \times \mathbf{r})}{r^3}$$

3.363 Suppose, there is only electric field $\boldsymbol{E}$ in K. Then in K', considering non-relativistic velocity $\mathbf{v}$,

$$\mathbf{E}' = \mathbf{E}, \mathbf{B}' = -\frac{\mathbf{v} \times \mathbf{E}}{c^2}$$

So, $$\mathbf{E}' \cdot \mathbf{B}' = 0$$

In the relativistic case,

$$\left.\begin{aligned} \mathbf{E}'_{\|} &= \mathbf{E}_{\|} \\ \mathbf{E}'_{\perp} &= \frac{\mathbf{E}_{\perp}}{\sqrt{1 - v^2/c^2}} \end{aligned}\right\} \quad \begin{aligned} \mathbf{B}'_{\|} &= \mathbf{B}_{\|} = 0 \\ \mathbf{B}'_{\perp} &= \frac{-\mathbf{v} \times \mathbf{E}/c^2}{\sqrt{1 - v^2/c^2}} \end{aligned}$$

Now, $$\mathbf{E}' \cdot \mathbf{B}' = \mathbf{E}_{\|} \cdot \mathbf{B}'_{\|} + \mathbf{E}'_{\perp} \qquad \mathbf{B}'_{\perp} = 0, \text{ since}$$

$$E'_{\perp} \cdot B'_{\perp} = \frac{-\mathbf{E}_{\perp} \cdot (\mathbf{v} \times \mathbf{E})/(1 - v^2/c^2)}{-\mathbf{E}_{\perp} \cdot (\mathbf{v} \times \mathbf{E}_{\perp}) \Big/ \left(1 - \frac{v^2}{c^2}\right) = 0}$$

3.364 In K, $$\mathbf{B} = b\frac{y\mathbf{i} - x\mathbf{j}}{x^2 + y^2} \quad (\text{where } b = \text{constant})$$

In K', $$\mathbf{E}' = \mathbf{v} \times \mathbf{B} = bv\frac{y\mathbf{j} - x\mathbf{i}}{x^2 + y^2} = \frac{bv\mathbf{r}}{r^2}$$

The electric field is radial ($\mathbf{r} = x\mathbf{i} + y\mathbf{j}$).

3.365 In K, $$\mathbf{E} = a\frac{\mathbf{r}}{r^2}, \mathbf{r} = (x\mathbf{i} + y\mathbf{j})$$

In K', $$\mathbf{B}' = -\frac{\mathbf{v} \times \mathbf{E}}{c^2} = \frac{a\mathbf{r} \times \mathbf{v}}{c^2 r^2}$$

The magnetic lines are circular.

3.366 In the non-relativistic limit, we neglect v^2/c^2 and write

$$\left.\begin{aligned}\mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel}\\ \mathbf{E}'_{\perp} &\cong \mathbf{E}_{\perp} + \mathbf{v}\times\mathbf{B}\end{aligned}\right\}\qquad \begin{aligned}\mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel}\\ \mathbf{B}'_{\perp} &\cong \mathbf{B}_{\perp} - \mathbf{v}\times\mathbf{E}/c^2\end{aligned}$$

These two equations can be combined to give

$$\mathbf{E}' = \mathbf{E} + \mathbf{v}\times\mathbf{B} \quad \text{and} \quad \mathbf{B}' = \mathbf{B} - \mathbf{v}\times\mathbf{E}/c^2$$

3.367 Choose $\mathbf{E}$ in the direction of the z–axis, $\mathbf{E} = (0, 0, \mathrm{E})$, The frame K' is moving with velocity $\mathbf{v} = (v\sin\alpha,\ 0,\ v\cos\alpha)$, in the $x-z$ plane. Then in the frame K',

$$\mathbf{E}'_{||} = \mathbf{E}_{||}\,\mathbf{B}'_{||} = 0$$

$$\mathbf{E}'_{\perp} = \frac{\mathbf{E}_{\perp}}{\sqrt{1 - v^2/c^2}} \quad \mathbf{B}'_{\perp} = \frac{-\,\mathbf{v}\times\ \mathbf{E}/c^2}{\sqrt{1 - v^2/c^2}}$$

The vector along $\mathbf{v}$ is $\mathbf{e} = (\sin\alpha,\ 0,\ \cos\alpha)$ and the perpendicular vector in the $x-z$ plane is $\mathbf{f} = (-\cos\alpha, 0,\ \sin\alpha)$.

(a) Thus, using $\quad \mathbf{E} = E\cos\alpha\,\mathbf{e} + E\sin\alpha\,\mathbf{f}$

We get, $\quad E'_{||} = E\cos\alpha \quad \text{and} \quad E'_{\perp} = \dfrac{E\sin\alpha}{\sqrt{1 - v^2/c^2}}$

So, $$E' = E\sqrt{\frac{1 - \beta^2\cos^2\alpha}{1 - \beta^2}} = 9\,\mathrm{k}V/\mathrm{m}$$

and $$\tan\alpha' = \frac{\tan\alpha}{\sqrt{1 - v^2/c^2}} \quad (\text{where } \beta = v/c)$$

(b) $$\mathbf{B}'_{||} = 0, \mathbf{B}'_{\perp} = \frac{\mathbf{v}\times\ \mathbf{E}/c^2}{\sqrt{1 - v^2/c^2}}$$

$$B' = \frac{\beta E\sin\alpha}{c\sqrt{1 - \beta^2}}$$

$$= 14\ \mu\mathrm{T}$$

3.368 Choose $\mathbf{B}$ in the z direction and the velocity $\mathbf{v} = (v\sin\alpha,\ 0,\ v\cos\alpha)$ in the $x-z$ plane, then in the K' frame

$$\mathbf{E}'_{||} = \mathbf{E}_{||} = 0 \qquad \mathbf{B}'_{||} = \mathbf{B}_{||}$$

$$\mathbf{E}'_{\perp} = \frac{\mathbf{v}\times\mathbf{B}}{\sqrt{1 - v^2/c^2}} \qquad \mathbf{B}'_{\perp} = \frac{\mathbf{B}_{\perp}}{\sqrt{1 - v^2/c^2}}$$

We find similarly, $$E' = \frac{c\,\beta\,B\,\sin\alpha}{\sqrt{1-\beta^2}}$$

$$B' = B\sqrt{\frac{1-\beta^2\cos^2\alpha}{1-\beta^2}}$$

$$= 1.4\ \text{nV/m}$$

and $$\tan\alpha' = \frac{\tan\alpha}{\sqrt{1-\beta^2}}$$

3.369 (a) We see that, $\mathbf{E}'\cdot\mathbf{B}' = \mathbf{E}'_{\parallel}\cdot\mathbf{B}'_{\parallel} + \mathbf{E}'_{\perp}\cdot\mathbf{B}'_{\perp}$

$$= \mathbf{E}_{\parallel}\cdot\mathbf{B}_{\parallel} + \frac{(\mathbf{E}_{\perp} + \mathbf{v}\times\mathbf{B})\cdot\left(\mathbf{B}_{\perp} - \dfrac{\mathbf{v}\times\mathbf{E}}{c^2}\right)}{1-\dfrac{v^2}{c^2}}$$

$$= \mathbf{E}_{\parallel}\cdot \mathrm{B}_{\parallel} + \frac{\mathbf{E}_{\perp}\cdot\mathbf{B}_{\perp} - (\mathbf{v}\times\mathbf{B})\cdot(\mathbf{v}\times\mathbf{E})/c^2}{1-v^2/c^2}$$

$$= \mathbf{E}_{\parallel}\cdot \mathrm{B}_{\parallel} + \frac{\mathbf{E}_{\perp}\mathbf{B}_{\perp} - (\mathbf{v}\times\mathbf{B}_{\perp})\cdot(\mathbf{v}\times\mathbf{E}_{\perp})/c^2}{1-\dfrac{v^2}{c^2}}$$

But, $$\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}\times\mathbf{D} = A\cdot C \quad B\cdot D - A\cdot DB\cdot C$$

So, $$\mathbf{E}'\cdot\mathbf{B}' = \mathbf{E}_{\parallel}\cdot\mathbf{B}_{\parallel} + \mathbf{E}_{\perp}\cdot\mathbf{B}_{\perp}\,\frac{\left(1-\dfrac{v^2}{c^2}\right)}{1-\dfrac{v^2}{c^2}} = \mathbf{E}\cdot\mathbf{B}$$

(b) We see that, $E'^2 - c^2B'^2$

$$= E'^2_{\parallel} - c^2B'^2_{\parallel} + E'^2_{\perp} - c^2B'^2_{\perp}$$

$$= E^2_{\parallel} - c^2B^2_{\parallel} + \frac{1}{1-\dfrac{v^2}{c^2}}\left[(\mathbf{E}_{\perp} + \mathbf{v}\times\mathbf{B})^2 - c^2\left(\mathbf{B}_{\perp} - \frac{\mathbf{v}\times\mathbf{E}}{c^2}\right)^2\right]$$

$$= E^2_{\parallel} - c^2B^2_{\parallel} + \frac{1}{1-\dfrac{v^2}{c^2}}\left[E^2_{\perp} - c^2B^2_{\perp} + (\mathbf{v}\times\mathbf{B}_{\perp})^2 - \frac{1}{c^2}(\mathbf{v}\times\mathbf{E}_{\perp})^2\right]$$

$$= E^2_{\parallel} - c^2B^2_{\parallel} + \frac{1}{1-\dfrac{v^2}{c^2}}\left[E^2_{\perp} - c^2B^2_{\perp}\right]\left(1-\frac{v^2}{c^2}\right) = E^2 - c^2B^2$$

(Since, $(\mathbf{v}\times\mathbf{A}_{\perp})^2 = v^2A^2_{\perp}$.)

3.370 In this case, $\mathbf{E} \cdot \mathbf{B} = 0$, as the fields are mutually perpendicular. Also,

$$E^2 - c^2B^2 = -20 \times 10^8 \left(\frac{\mathrm{V}}{\mathrm{m}}\right)^2$$

Thus, we can find a frame in which $E' = 0$, and

$$B' = \frac{1}{c}\sqrt{c^2B^2 - E^2} = B\sqrt{1 - \frac{E^2}{c^2B^2}}$$

$$= 0 \cdot 20\sqrt{1 - \left(\frac{4 \times 10^4}{3 \times 10^8 \times 2 \times 10^{-4}}\right)^2} = 0.15 \text{ mT}$$

3.371 Suppose the charge q moves in the positive direction of the x–axis of the frame K. Let us go over to the moving frame K', at whose origin the charge is at rest. We take the x and x' axes of the two frames to be coincident, and the y- and y'-axes to be parallel.

In the K' frame,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{r}'}{r'^3}$$

and this has the following components

$$E'_x = \frac{1}{4\pi\varepsilon_0}\frac{qx'}{r'^3} \quad \text{and} \quad E'_y = \frac{1}{4\pi\varepsilon_0}\frac{qy'}{r'^3}.$$

Now let us go back to the frame K. At the moment when the origins of the two frames coincide, we take $t = 0$. Then,

$$x = r\cos\theta = x'\sqrt{1 - \frac{v^2}{c^2}} \text{ and } y = r\sin\theta = y'$$

Also, $$E_x = E'_x,\ E_y = E'_y/\sqrt{1 - v^2/c^2}$$

From these equations, $$r'^2 = \frac{r^2\,(1 - \beta^2\sin^2\theta)}{1 - \beta^2}$$

$$\mathbf{E}' = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^3\,(1 - \beta^2\sin^2\theta)^{3/2}}\left[(1 - \beta^2)^{3/2}\left(x'\mathbf{i} + \frac{y'}{\sqrt{1 - \beta^2}}\mathbf{j}\right)\right]$$

$$= \frac{q\mathbf{r}\,(1 - \beta^2)}{4\pi\varepsilon_0 r^3\,(1 - \beta^2\sin^2\theta)^{3/2}}$$

3.7 Motion of Charged Particles in Electric and Magnetic Fields

3.372 Let the electron leave the negative plate of the capacitor at time $t = 0$.

As, $$E_x = -\frac{d\varphi}{dx},\ E = \frac{\varphi}{l} = \frac{at}{l}$$

therefore, the acceleration of the electron is

$$w = \frac{eE}{m} = \frac{eat}{ml}$$

or $$\frac{dv}{dt} = \frac{eat}{ml}$$

or $$\int_0^v dv = \frac{ea}{ml}\int_0^t tdt$$

or $$v = \frac{1}{2}\frac{ea}{ml}t^2 \tag{1}$$

But from, $$s = \int vdt$$

$$l = \frac{1}{2}\frac{ea}{ml}\int_0^t t^2 dt$$

or $$l = \frac{eat^3}{6\ ml}$$

or $$t = \left(\frac{6\ ml^2}{ea}\right)^{1/3}$$

Putting the value of t in Eq. (1), we get

$$v = \frac{1}{2}\frac{ea}{ml}\left(\frac{6\ ml^2}{ea}\right)^{2/3} = \left(\frac{9}{2}\frac{ale}{m}\right)^{1/3} = 16\ \text{km/s}$$

3.373 The electric field inside the capacitor varies with time as, $E = at$. Hence, electric force on the proton, $F = eat$ and subsequently, acceleration of the proton is

$$w = \frac{eat}{m}$$

Now, if t is the time elapsed during the motion of the proton between the plates, then $t = l/v_\parallel$, as no acceleration is effective in this direction. (Here $v_\parallel$ is velocity along the length of the plate.)

From kinematics, $$\frac{dv_\perp}{dt} = w$$

So, $$\int_0^{v_\perp} dv_\perp = \int_0^t w\,dt$$

as initially, the component of velocity in the direction $\perp$ to plates was zero.

or $$v_\perp = \int_0^t \frac{ea}{m}\frac{t^2}{2m} = \frac{ea}{2m}\frac{l^2}{v_\parallel^2}$$

Now, $$\tan\alpha = \frac{v_\perp}{v_\parallel} = \frac{eal^2}{2mv_\parallel^3}$$

$$= \frac{eal^2}{2m\left(\frac{2eV}{m}\right)^{\frac{3}{2}}}, \text{ as } v_\parallel = \left(\frac{2eV}{m}\right)^{\frac{1}{2}} \text{ (from energy conservation)}$$

$$= \frac{al^2}{4}\sqrt{\frac{m}{2eV^3}}$$

3.374 The equation of motion is

$$\frac{dv}{dt} = v\frac{dv}{dx} = \frac{q}{m}(E_0 - ax)$$

Integrating, $$\frac{1}{2}v^2 - \frac{q}{m}(E_0 x - \frac{1}{2}ax^2) = \text{constant}$$

But initially, $v = 0$ when $x = 0$, so "constant" $= 0$.

Thus, $$v^2 = \frac{2q}{m}\left(E_0 x - \frac{1}{2}ax^2\right)$$

Thus, $v = 0$, again for $$x = x_m = \frac{2E_0}{a}$$

The corresponding acceleration is

$$\left(\frac{dv}{dt}\right)_{x_m} = \frac{q}{m}(E_0 - 2E_0) = -\frac{qE_0}{m}$$

3.375 From the law of relativistic conservation of energy

$$\frac{m_0 c^2}{\sqrt{1 - (v^2/c^2)}} - e\,Ex = m_0 c^2$$

as the electron is at rest ($v = 0$ for $x = 0$) initially.

Thus clearly, $$T = eEx$$

On the other hand, $$\sqrt{1-(v^2/c^2)} = \frac{m_0c^2}{m_0c^2 + eEx}$$

or $$\frac{v}{c} = \frac{\sqrt{(m_0c^2 + eEx)^2 - m_0^2c^4}}{m_0c^2 + eEx}$$

or $$ct = \int c dt = \int \frac{(m_0c^2 + eEx)\, dx}{\sqrt{(m_0c^2 + eEx)^2 - m_0^2c^4}}$$

$$= \frac{1}{2\, eE}\int \frac{dy}{\sqrt{y - m_0^2c^4}} = \frac{1}{eE}\sqrt{(m_0c^2 + eEx)^2 - m_0^2c^4} + \text{constant}$$

The "constant" = 0, at $t = 0$, for $x = 0$.

So, $$ct = \frac{1}{eE}\sqrt{(m_0c^2 + eEx)^2 - m_0^2\, c^4}$$

Finally, using $T = eEx$, we get

$$ce \cdot Et_0 = \sqrt{T(T + 2m_0c^2)}$$

or $$t_0 = \frac{\sqrt{T(T + 2m_0c^2)}}{eEc} = 3.0 \text{ ns}$$

3.376 As before, $T = eEx$. Now in linear motion,

$$\frac{d}{dt}\frac{m_0v}{\sqrt{1-v^2/c^2}} = \frac{m_0w}{\sqrt{1-v^2/c^2}} + \frac{m_0w}{(1-v^2/c^2)^{3/2}}\frac{v}{c^2}\, w$$

$$= \frac{m_0}{(1-v^2/c^2)^{3/2}}\, w = \frac{(T + m_0c^2)^3}{m_0^2c^6}\, w = eE$$

So, $$w = \frac{eEm_0^2c^6}{(T + m_0c^2)^3} = \frac{eE}{m_0}\left(1 + \frac{T}{m_0c^2}\right)^{-3}$$

3.377 The equations are

$$\frac{d}{dt}\left(\frac{m_0v_x}{\sqrt{1-(v^2/c^2)}}\right) = 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{m_0v_y}{\sqrt{1-v^2/c^2}}\right) = eE$$

Hence, $$\frac{v_x}{\sqrt{1-v^2/c^2}} = \text{constant} = \frac{v_0}{\sqrt{1-(v_0^2/c^2)}}$$

Also, by energy conservation

$$\frac{m_0 c^2}{\sqrt{1-(v^2/c^2)}} = \frac{m_0 c^2}{\sqrt{1-(v_0^2/c^2)}} + eEy$$

Dividing, $$v_x = \frac{v_0 \varepsilon_0}{\varepsilon_0 + eEy},\ \varepsilon_0 = \frac{m_0 c^2}{\sqrt{1-(v_0^2/c^2)}}$$

Also, $$\frac{m_0}{\sqrt{1-(v^2/c^2)}} = \frac{\varepsilon_0 + eEy}{c^2}$$

Thus, $$(\varepsilon_0 + eEy)v_y = c^2 eEt + \text{constant}$$

This "constant" = 0 as $v_y = 0$ at $t = 0$.

Integrating again,

$$\varepsilon_0 y + \frac{1}{2} eEy^2 = \frac{1}{2} c^2 Et^2 + \text{constant}$$

Again, this "constant" = 0, as $y = 0$, at $t = 0$.

Thus, $$(ce\,Et)^2 = (eyE)^2 + 2\varepsilon_0 eEy + \varepsilon_0^2 - \varepsilon_0^2$$

or $$ceEt = \sqrt{(\varepsilon_0 + eEy)^2 - \varepsilon_0^2}$$

or $$\varepsilon_0 + eEy = \sqrt{\varepsilon_0^2 + c^2 e^2 E^2 t^2}$$

Hence, $$v_x = \frac{v_0 \varepsilon_0}{\sqrt{\varepsilon_0^2 + c^2 e^2 E^2 t^2}} \quad \text{also,} \quad v_y = \frac{c^2 eEt}{\sqrt{\varepsilon_0^2 + c^2 e^2 E^2 t^2}}$$

and $$\tan\theta = \frac{v_y}{v_x} = \frac{eEt}{m_0 v_0}\sqrt{1-(v_0^2/c^2)}$$

3.378 From the figure,

$$\sin\alpha = \frac{d}{R} = \frac{dqB}{mv}$$

as radius of the arc $R = mv/qB$, where v is the velocity of the particle when it enters into the field. From initial condition of the problem

$$qV = \frac{1}{2} mv^2 \quad \text{or} \quad v = \sqrt{\frac{2qV}{m}}$$

Hence, $$\sin\alpha = \frac{dqB}{m\sqrt{\frac{2qV}{m}}} = dB\sqrt{\frac{q}{2\,mV}}$$

and $$\alpha = \sin^{-1}\left(dB\sqrt{\frac{q}{2\,mV}}\right) = 30^\circ \text{ (on substituting values)}$$

3.379 (a) For motion along circle, the magnetic force acted on the particle will provide the centripetal force necessary for its circular motion,

i.e., $$\frac{mv^2}{r} = evB \quad \text{or} \quad v = \frac{eBr}{m} = 100 \text{ km/s}$$

and the period of revolution, $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi r}{v} = \dfrac{2\pi m}{eB} = 6.5\ \mu\text{s}$

(b) Generally, $\dfrac{d\mathbf{p}}{dt} = \mathbf{F}$

But, $$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}\frac{m_0\mathbf{v}}{\sqrt{1-(v^2/c^2)}} = \frac{m_0\dot{\mathbf{v}}}{\sqrt{1-(v^2/c^2)}} + \frac{m_0}{(1-(v^2/c^2))^{3/2}}\frac{\mathbf{v}\,(\mathbf{v}\cdot\dot{\mathbf{v}})}{c^2}$$

For transverse motion, $\mathbf{v}\cdot\dot{\mathbf{v}} = 0$ so,

$$\frac{d\mathbf{p}}{dt} = \frac{m_0\dot{\mathbf{v}}}{\sqrt{1-(v^2/c^2)}} = \frac{m_0}{\sqrt{1-(v^2/c^2)}}\frac{v^2}{r}$$

Thus, $$\frac{m_0v^2}{r\sqrt{1-v^2/c^2}} = Bev \quad \text{or} \quad \frac{v/c}{\sqrt{1-(v^2/c^2)}} = \frac{Ber}{m_0c}$$

or $$\frac{v}{c} = \frac{Ber}{\sqrt{B^2e^2r^2 + m_0^2c^2}}$$

Finally, $$T = \frac{2\pi r}{v} = \frac{2\pi m_0}{eB\sqrt{1-v^2/c^2}} = \frac{2\pi}{cBe}\sqrt{B^2e^2r^2 + m_0^2c^2} = 4.1\text{ ns}$$

3.380 (a) As before, $p = B\,qr$

(b) $$T = \sqrt{c^2p^2 + m_0^2\,c^4} = \sqrt{c^2B^2q^2r^2 + m_0^2c^4}$$

(c) $$w = \frac{v^2}{r} = -\frac{c^2}{r\,[1 + (m_0c/Bqr)^2]}$$

using the result for v from the previous problem.

3.381 From solution of Problem 3.279,

$$T = \frac{2\pi\varepsilon}{c^2eB}\ (\text{relativistic}),\ T_0 = \frac{2\pi m_0c^2}{c^2eB}\ \ (\text{non-relativistic})$$

Here, $$m_0c^2/\sqrt{1-v^2/c^2} = \varepsilon$$

Thus, $$\delta T = \frac{2\pi T}{c^2\ eB},\ (T = \text{K.E.})$$

Now, $$\frac{\delta T}{T_0} = \eta = \frac{T}{m_0c^2} \quad \text{so,} \quad T = \eta\ m_0c^2$$

3.382 The given potential difference is not large enough to cause significant deviations from the non-relativistic formula. So,

$$T = eV = \frac{1}{2} mv^2$$

Thus, $$v = \sqrt{\frac{2eV}{m}}$$

So, $$v_{||} = \sqrt{\frac{2eV}{m}} \cos\alpha \quad v_{\perp} = \sqrt{\frac{2eV}{m}} \sin\alpha$$

Now, $$\frac{mv_{\perp}^2}{r} = Bev_{\perp} \quad \text{or} \quad r = \frac{mv_{\perp}}{Be}$$

and $$T = \frac{2\pi r}{v_{\perp}} = \frac{2\pi m}{Be}$$

Pitch $$p = v_{||} T = \frac{2\pi m}{Be}\sqrt{\frac{2eV}{m}} \cos\alpha = 2\pi\sqrt{\frac{2mV}{eB^2}} \cos\alpha = 2.0 \text{ cm}$$

3.383 The charged particles will traverse a helical trajectory and will be focused on the axis after traversing a number of turns. Thus,

$$\frac{1}{v_0} \approx n \cdot \frac{2\pi m}{qB_1} = (n+1)\frac{2\pi m}{qB_2}$$

So, $$\frac{n}{B_1} = \frac{n+1}{B_2} = \frac{1}{B_2 - B_1}$$

Hence, $$\frac{1}{v_0} = \frac{2\pi m}{q(B_2 - B_1)}$$

or $$\frac{l^2}{2\, qV/m} = \frac{(2\pi)^2}{(B_2 - B_1)^2} \times \frac{1}{(q/m)^2}$$

or $$\frac{q}{m} = \frac{8\pi^2 V}{l^2 (B_2 - B_1)^2}$$

3.384 Let us take the point A as the origin O and the axis of the solenoid as z-axis. At an arbitrary moment of time let us resolve the velocity of electron into its two rectangular components, $\mathbf{v}_{||}$ along the axis and $\mathbf{v}_{\perp}$ to the axis of solenoid. We know the magnetic force does no work, so the kinetic energy as well as the speed of the electron $|\mathbf{v}_{\perp}|$ will remain constant in the x-y plane. Thus $\mathbf{v}_{\perp}$ can change only its direction as shown in the problem figure, $\mathbf{v}_{||}$ will remain constant as it is parallel to $\mathbf{B}$. Thus at time $= t$,

$$v_x = v_{\perp} \cos\omega t = v \sin\alpha \cos\omega t$$

$$v_y = v_{\perp} \sin\omega t = v \sin\alpha \sin\omega t$$

and $$v_z = v\cos\alpha \quad (\text{where } \omega = eB/m)$$

As at $t = 0$, we have $x = y = z = 0$, so the equation of motion of the electron is

$$z = v\cos\alpha t$$

$$x = \frac{v\sin\alpha}{\omega}\sin\omega t$$

$$y = \frac{v\sin\alpha}{\omega}(\cos\omega t - 1)$$

The equation of the helix.

On the screen $z = l$, so $t = \dfrac{l}{v\cos\alpha}$

Then, $$r^2 = x^2 + y^2 = \frac{2v^2\sin^2\alpha}{\omega^2}\left(1 - \cos\frac{\omega l}{v\cos\alpha}\right)$$

$$r = \frac{2v\sin\alpha}{\omega}\left|\sin\frac{\omega l}{2v\cos\alpha}\right| = 2\,\frac{mv}{eB}\sin\alpha\left|\sin\frac{leB}{2\,mv\cos\alpha}\right|$$

3.385 Choose the wire along the z–axis, and the initial direction of the electron, along the x–axis. Then the magnetic field in the $x–z$ plane is along the y–axis and outside the wire it is

$$B = B_y = \frac{\mu_0 I}{2\pi x} \quad (\text{as } B_x = B_z = 0, \text{ if } y = 0)$$

The motion must be confined to the $x - z$ plane. Then the equations of motion are

$$\frac{d}{dt}mv_x = -ev_x B_y$$

$$\frac{d(mv_z)}{dt} = +\,ev_x B_y$$

Multiplying the first equation by v_x and the second by v_z and then adding, we get

$$v_x\frac{dv_x}{dt} + v_z\frac{dv_z}{dt} = 0$$

or $$v_x^2 + v_z^2 = v_0^2,\ \text{say} \quad \text{or} \quad v_z = \sqrt{v_0^2 - v_x^2}$$

Then, $$v_x\frac{dv_x}{dx} = -\frac{e}{m}\sqrt{v_0^2 - v_x^2}\,\frac{\mu_0 I}{2\pi x}$$

or $$-\frac{v_x dv_x}{\sqrt{v_0^2 - v_x^2}} = \frac{\mu_0 Ie}{2\pi m}\frac{dx}{x}$$

Integrating, $$\sqrt{v_0^2 - v_x^2} = \frac{\mu_0 Ie}{2\pi m}\ln\frac{x}{a}$$

on using, $v_x = v_0$, if $x = a$ (i.e., initially).

Now, $$v_x = 0, \quad \text{when} \quad x = x_m$$

So, $$x_m = ae^{v_0/b}, \quad \text{where} \quad b = \frac{\mu_0 Ie}{2\pi m}$$

3.386 Inside the capacitor, the electric field follows the $1/r$ law, and so the potential can be written as

$$\varphi = \frac{V \ln r/a}{\ln b/a} \quad \text{and} \quad E = \frac{-V}{\ln b/a}\frac{1}{r},$$

Here r is the distance from the axis of the capacitor.

Also, $$\frac{mv^2}{r} = \frac{qV}{\ln b/ar}\frac{1}{r}$$

or $$mv^2 = \frac{qV}{\ln b/a}$$

On the other hand, $mv = qBr$ in the magnetic field.

Thus, $$v = \frac{V}{Br \ln b/a} \quad \text{and} \quad \frac{q}{m} = \frac{v}{Br} = \frac{V}{B^2 r^2 \ln (b/a)}$$

3.387 (a) The equations of motion are

$$m\frac{dv_x}{dt} = -qBv_z, \; m\frac{dv_y}{dt} = qE \quad \text{and} \quad m\frac{dv_z}{dt} = qv_x B$$

These equations can be solved easily.

First, $$v_y = \frac{qE}{m}t, \; y = \frac{qE}{2m}\; t'^{\,2}$$

Then, $$v_x^2 + v_z^2 = \text{constant} = v_0^2 \quad \text{(as before)}$$

In fact, $v_x = v_0 \cos\omega t$ and $v_z = v_0 \sin\omega t$ as one can check.

Integrating again and using $x = z = 0$, at $t = 0$, we get

$$x = \frac{v_0}{\omega}\sin\omega t, \; z = \frac{v_0}{\omega}(1 - \cos\omega t)$$

Thus, $$x = z = 0 \quad \text{for} \quad t = t_n = n\frac{2\pi}{\omega}$$

At that instant, $$y_n = \frac{qE}{2m} \times \frac{2\pi}{qB/m} \times n^2 \times \frac{2\pi}{qB/m} = \frac{2\pi^2 mEn^2}{qB^2}$$

(b) Also, $$\tan\alpha_n = \frac{v_x}{v_y} \quad (v_z = 0 \text{ at this moment})$$

$$= \frac{mv_0}{qEt_n} = \frac{mv_0}{qE} \times \frac{qB}{m} \times \frac{1}{2\pi n} = \frac{Bv_0}{2\pi En}$$

3.388 The equation of the trajectory is

$$x = \frac{v_0}{\omega}\sin\omega t,\ z = \frac{v_0}{\omega}(1 - \cos\omega t),\ y = \frac{qE}{2m}t^2 \quad (\text{as in Problem 3.384})$$

Now on the screen $x = l$, so

$$\sin\omega t = \frac{\omega l}{v_0} \quad \text{or} \quad \omega t = \sin^{-1}\frac{\omega l}{v_0}$$

At that moment, $$y = \frac{qE}{2m\omega^2}\left(\sin^{-1}\frac{\omega l}{v_0}\right)^2$$

So, $$\frac{\omega l}{v_0} = \sin\sqrt{\frac{2m\omega^2 y}{qE}} = \sin\sqrt{\frac{2qB^2 y}{Em}}$$

and $$z = \frac{v_0}{\omega}2\sin^2\frac{\omega t}{2} = l\tan\frac{\omega t}{2}$$

$$= l\tan\frac{1}{2}\left[\sin^{-1}\frac{\omega l}{v_0}\right] = l\tan\sqrt{\frac{qB^2 y}{2mE}}$$

For small z, $$\frac{qB^2 y}{2\ mE} = \left(\tan^{-1}\frac{z}{l}\right)^2 \approx \frac{z^2}{l^2}$$

or $$y = \frac{2mE}{qB^2 l^2}\cdot z^2$$

This is the equation of a parabola.

3.389 In crossed field, $$eE = evB \quad \text{so} \quad v = \frac{E}{B}$$

Then force exerted on the plate, $$F = \frac{I}{e} \times m\frac{E}{\mathbf{B}} = \frac{mIE}{eB} = 20\ \mu\text{N}$$

3.390 When the electric field is switched off, the path followed by the particle will be helical and pitch

$$\Delta l = v_{||}T$$

(where $v_{||}$ is the velocity of the particle, parallel to $\mathbf{B}$ and T the time period of revolution).

$$= v\cos(90 - \varphi)\,T = v\sin\varphi\ T$$

$$= v\sin\varphi\,\frac{2\pi m}{qB}\left(\text{as } T = \frac{2\pi}{qB}\right) \tag{1}$$

Now, when both the fields were present, $qE = qvB \sin(90 - \varphi)$, as no net force was effective on the system.

So, $$v = \frac{E}{B \cos\varphi} \tag{2}$$

From Eqs. (1) and (2), $\Delta l = \frac{E}{B}\frac{2\pi m}{qB}\tan\varphi = 6 \text{ cm}$

3.391 When there is no deviation,

$$-q\mathbf{E} = q(\mathbf{v} \times \mathbf{B})$$

or in scalar from, $$E = vB \ (\text{as } \mathbf{v} \perp \mathbf{B}) \quad \text{or} \quad v = \frac{E}{B} \tag{1}$$

Now, when the magnetic field is switched on, let the deviation in the field be x. Then,

$$x = \frac{1}{2}\left(\frac{qvB}{m}\right)t^2$$

where t is the time required to pass through this region.

Also, $$t = \frac{a}{v}$$

Thus, $$x = \frac{1}{2}\left(\frac{qvB}{m}\right)\left(\frac{a}{v}\right)^2 = \frac{1}{2}\frac{q}{m}\frac{a^2B^2}{E} \tag{2}$$

For the region where the field is absent, velocity in the upward direction

$$= \left(\frac{qvB}{m}\right)t = \frac{q}{m}aB \tag{3}$$

Now, $$\Delta x - x = \frac{qaB}{m}t'$$

$$= \frac{qa}{m}\frac{B^2b}{E}, \text{ where } t' = \frac{b}{v} = \frac{bB}{E} \tag{4}$$

From Eqs. (2) and (4),

$$\Delta x - \frac{1}{2}\frac{q}{m}\frac{a^2B^2}{E} = \frac{q}{m}\frac{aB^2b}{E}$$

or $$\frac{q}{m} = \frac{2E\Delta x}{aB^2(a + 2b)}$$

3.392 (a) The equation of motion is

$$m\frac{d^2\mathbf{r}}{dt^2} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Now, $$\mathbf{v} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & B \end{vmatrix} = \mathbf{i}B\dot{y} - \mathbf{j}B\dot{x}$$

So, the equation becomes,

$$\frac{dv_x}{dt} = \frac{qBv_y}{m}; \frac{dv_y}{dt} = \frac{qE}{m} - \frac{qB}{m}v_x \quad \text{and} \quad \frac{dv_z}{dt} = 0$$

(Here, $v_x = \dot{x}$, $v_y = \dot{y}$, $v_z = \dot{z}$).

The last equation is easy to integrate as v_z = constant = 0, since v_z is zero initially. Thus integrating again, z = constant = 0, and motion is confined to the x–y plane.

Method 1:

Let us put $$\xi = v_x + iv_y$$

or $$\dot{\xi} = \frac{dv_x}{dt} + i\frac{dv_y}{dt}$$

$$= \frac{qB}{m}v_y + i\left(\frac{qE}{m} - \frac{qB}{m}v_x\right)$$

$$= \omega v_y + i\left(\frac{\omega E}{B} - \omega v_x\right) \left(\text{where } \omega = \frac{q\mathrm{B}}{m}\right)$$

$$= \omega v_y + i\omega\frac{E}{B} - i\omega v_x$$

$$= i\omega\frac{E}{B} - i\omega(v_x + iv_y) \quad (\text{because } i^2 = -1)$$

This equation after being multiplied by $e^{i\omega t}$ can be rewritten as

$$\frac{d}{dt}(\xi\, e^{i\omega t}) = i\,\omega e^{i\omega t}\frac{E}{B}$$

and integrated to give,

$$\xi = \frac{E}{B} + Ce^{-i\omega t - i\alpha}$$

where C and α are two real constants. Taking real and imaginary parts,

$$v_x = \frac{E}{B} + C\cos(\omega t + \alpha) \quad \text{and} \quad v_y = -C\sin(\omega t + \alpha)$$

Since $v_y = 0$, when $t = 0$, we can take $\alpha = 0$, then $v_x = 0$ at $t = 0$ gives,

$$C = -\frac{E}{B}$$

and we get,

$$v_x = \frac{E}{B}(1 - \cos\omega t) \text{ and } v_y = \frac{E}{B}\sin\omega t$$

Integrating again and using $x = y = 0$, at $t = 0$, we get

$$x(t) = \frac{E}{B}\left(t - \frac{\sin\omega t}{\omega}\right),\ y(t) = \frac{E}{\omega B}(1 - \cos\omega t)$$

This is the equation of a cycloid.

Alternate of method 1, to find $v_x(t)$ and $v_y(t)$:

We have,

$$\dot{v}_x = \frac{qB}{m}v_y \quad \text{and} \quad \dot{v}_y = \frac{qE}{m} - \frac{qB}{m}v_x$$

or $$\dot{v}_x = \omega v_y \quad \text{and} \quad \dot{v}_y = \omega\left(\frac{E}{B} - v_x\right)$$

After differentiating $\dot{v}_y$ with respect to time, we get

$$\ddot{v}_y = -\omega\dot{v}_x = -\omega^2 v_y$$

At $t = 0$, $v_y = 0$, so its solution becomes

$$v_y = v_{ym}\sin\omega t \text{ (where } v_{ym} \text{ is velocity amplitude)}$$

At $t = 0$, $$\dot{v}_y = \frac{qE}{m}$$

So, $$\omega v_{ym} = \frac{\omega E}{B} \Rightarrow v_{ym} = \frac{E}{B}$$

Hence, $$v_y = \frac{E}{B}\sin\omega t$$

So $$\Delta x = \int v_y\,dt = \frac{E}{B\omega}[\cos\omega t]_0^t$$

$$= \frac{E}{B\omega}[1 - \cos\omega t]$$

because at $t = 0$, $x = y = 0$.

Now differentiating $\dot{v}_x = \omega v_y$ with respect to time, we get

$$\ddot{v}_x = \omega\dot{v}_y = \omega^2[E/B - v_x]$$

or $$\ddot{v}_x + \omega^2 v_x = \omega^2 E \tag{1}$$

This equation is the form $\ddot{x} + \omega^2 x = A$

(where A is any constant.)

or $$\ddot{x} + \omega^2\left(x - \frac{A}{\omega^2}\right) = 0$$

Putting $$x' = x - \frac{A}{\omega^2}$$

and $$\ddot{x}' = \ddot{x} \quad \text{in the solutions}$$

We get $$x' = a\cos(\omega t + \delta)$$

or $$x - \frac{A}{\omega^2} = a\cos(\omega t + \delta)$$

or $$x = a\cos(\omega t + \delta) + \frac{A}{\omega^2}$$

Thus the solution of Eq. (1) is

$$v_x = v_{xm}\cos(\omega t + \delta) + \frac{E}{B}$$

On differentiation

$$\dot{v}_x = -\omega v_{xm}\sin(\omega t + \delta)$$

At $t = 0$, $\dot{v}_x = 0$, $\delta = 0$, hence

$$v_x = v_{xm}\cos\omega t + \frac{E}{B}$$

Again at $t = 0$,

$$v_x = 0, \quad \text{so,} \quad v_{xm} = -\frac{E}{B}$$

or $$v_x = \frac{E}{B}\,[1 - \cos\omega t]$$

Alternate:

Suppose we have two inertial systems of reference: the system K and the system K' moving relative to the first system at a velocity $\mathbf{v}_0$. We know the magnitudes of the fields $\mathbf{E}$ and $\mathbf{B}$ at a certain point in space and time in the system K. If $v_0 \ll c$ the fields $\mathbf{E}'$ and $\mathbf{B}'$ at the same point in space and time in the system $\mathbf{E}'$ are given by

$$\mathbf{E} = \mathbf{E} + (\mathbf{v}_0 \times \mathbf{B}),\ \mathbf{B}' = \mathbf{B} - (\mathbf{v}_0 \times \mathbf{E})/c^2$$

The particle moves under the action of the Lorentz force. It can be easily seen that the particle always remains in the plane *xy*. Its motion can be described most easily in a certain system K' where only the magnetic field is present. Let us find this reference system. It follows from transformation $\mathbf{E}' = \mathbf{E} + (\mathbf{v}_0 \times \mathbf{B})$, $\mathbf{E}' = 0$ in a reference system that moves with a velocity $\mathbf{v}_0$ satisfying the relation $\mathbf{E} = -(\mathbf{v}_0 \times \mathbf{B})$. It is more convenient to choose the system K' whose velocity $\mathbf{v}_0$ is directed towards the positive values on the *x*-axis, since in such system the particle will move perpendicular to vector $\mathbf{B}'$ and its motion will be the simplest.

Thus in the system K' that moves to the right at a velocity $v_0 = E/B$, the field $\mathbf{E}' = 0$ and only the field $\mathbf{B}'$ is observed. In accordance with transformation $\mathbf{B}' = \mathbf{B} - (\mathbf{v}_0 \times \mathbf{E})/c^2$ and figure, we have

$$\mathbf{B}' = \mathbf{B} - (\mathbf{v}_0 \times \mathbf{E})/c^2 = \mathbf{B}(1 - v_0^2/c^2)$$

Since for a non-relativistic particle $v_0 \ll c$, we can assume that $\mathbf{B}' = \mathbf{B}$.

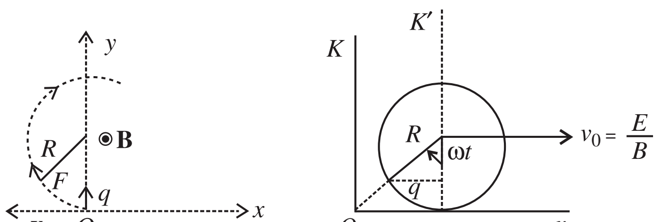
In the system K', the particle will move only in the magnetic field, it's velocity being perpendicular to this field. The equation of motion for this particle in the system K' will have the form

$$\frac{mv_0^2}{R} = qv_0 B$$

This equation is written for the instant $t = 0$, where the particle moved in the syatem K' as is shown in Fig. (a). Since the Lorentz force $\mathbf{F}$ is always perpendicular to the velocity of the particle, v_0 = constant, and it follows from above equation that in the system K' the particle will move in a circle of radius

$$R = \frac{mv_0}{qB}$$

Thus the particle moves uniformly with the velocity v_0 in a circle in the system K', which, in turn, moves uniformly to the right with the same velocity $v_0 = E/B$. This motion can be compared with the motion of the point q at the rim of a wheel Fig. (b) rolling with the angular velocity $\omega = v_0/R = qB/m$.



(*a*) (*b*)

Fig. (b) readily shows the coordinates of the particle q at the instant t are given by

$$x = v_0 t - R\sin\omega t = a(\omega t - \sin\omega t)$$

$$y = R - R\cos\omega t = a(1 - \cos\omega t)$$

where $$a = \frac{mE}{qB^2} \text{ and } \omega = \frac{qB}{m}$$

(b) The velocity is zero, when $\omega t = 2n\pi$. We see that

$$v^2 = v_x^2 + v_y^2 = \left(\frac{E}{B}\right)^2 (2 - 2\cos\,\omega t)$$

or $$v = \frac{ds}{dt} = \frac{2E}{B}\left|\sin\frac{\omega t}{2}\right|$$

The quantity inside the modulus is positive for $0 < \omega t < 2\pi$. Thus we can drop the modulus and write for the distance traversed between two successive zeroes of velocity as

$$S = \frac{4E}{\omega B}\left(1 - \cos\frac{\omega t}{2}\right)$$

Putting $\omega t = 2\pi$, we get

$$S = \frac{8E}{\omega B} = \frac{8mE}{qB^2}$$

(c) The drift velocity is in the x-direction and has the magnitude

$$\langle v_x \rangle = \left\langle \frac{E}{B}(1 - \cos\omega t) \right\rangle = \frac{E}{B}$$

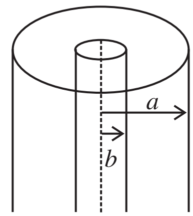
3.393 When a current I flows along the axis, a magnetic field $B_\varphi = \dfrac{\mu_0 I}{2\pi\rho}$ is set up, where $\rho^2 = x^2 + y^2$. In terms of components,

$$B_x = -\frac{\mu_0 I y}{2\pi\rho^2},\ B_y = \frac{\mu_0 I x}{2\pi\rho^2} \text{ and } B_z = 0$$

Suppose a potential difference V is set up between the inner cathode and the outer anode. This means a potential function of the form

$$\varphi = V\frac{\ln\rho/b}{\ln a/b} \quad (a > \rho > b)$$

as one can check by solving Laplace equation.



The electric field corresponding to this is

$$E_x = -\frac{Vx}{\rho^2 \ln a/b},\ E_y = -\frac{Vy}{\rho^2 \ln a/b},\ E_z = 0$$

The equations of motion are

$$\frac{d}{dt}\, mv_x = +\frac{|e|Vz}{\rho^2 \ln a/b} + \frac{|e|\mu_0 I}{2\pi\rho^2}\, x\dot{z}$$

$$\frac{d}{dt}\, mv_y = +\frac{|e|\, Vy}{\rho^2 \ln a/b} + \frac{|e|\mu_0 I}{2\pi\rho^2}\, y\dot{z}$$

and
$$\frac{d}{dt}\, mv_z = -|e|\frac{\mu_0 I}{2\pi\rho^2}(x\dot{x} + y\dot{y}) = |e|\frac{\mu_0 I}{2\pi}\frac{d}{dt}\ln\rho$$

$(-|e|)$ is the charge on the electron.

Integrating the last equations, we get

$$mv_z = -|e|\frac{\mu_0 I}{2\pi}\ln\frac{\rho}{a} = m\dot{z}$$

Since $v_z = 0$ when $\rho = a$, we now substitute this $\dot{z}$ in the other two equations to get

$$\frac{d}{dt}\left(\frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2\right)$$

$$= \left[\frac{|e|\,V}{\ln a/b} - \frac{|e|^2}{m}\left(\frac{\mu_0 I}{2\pi}\right)^2 \text{In}\,\frac{\rho}{b}\right]\cdot\frac{x\dot{x} + y\dot{y}}{\rho^2}$$

$$= \left[\frac{|e|\,V}{\ln\, a/b} - \frac{|e|^2}{m}\left(\frac{\mu_0 I}{2\pi}\right)^2 \ln\frac{\rho}{b}\right]\cdot\frac{1}{2\rho^2}\frac{d}{dt}\rho^2$$

$$= \left[\frac{|e|\,V}{\ln\, a/b} - \frac{|e|^2}{m}\left(\frac{\mu_0 I}{2\,\pi}\right)^2 \ln\frac{\rho}{a}\right]\frac{d}{dt}\ln\frac{\rho}{b}$$

Integrating and using $v^2 = 0$, at $\rho = b$, we get,

$$\frac{1}{2}mv^2 = \left[\frac{|e|\,V}{\ln\, a/b}\ln\frac{\rho}{a} - \frac{1}{2m}|e|^2\left(\frac{\mu_0 I}{2\pi}\right)^2\left(\ln\frac{\rho}{b}\right)\right]$$

The RHS must be positive, for all $a > \rho > b$. The condition for this is

$$V \geq \frac{1}{2}\frac{|e|}{m}\left(\frac{\mu_0 I}{2\pi}\right)^2 \ln\frac{a}{b}$$

3.394 This differs from the previous problem in ($a \leftrightarrow b$) and the magnetic field is along the z-direction. Thus $B_x = B_y = 0$, $B_z = B$.

Assuming as usual the charge of the electron to be $-|e|$, we write the equation of motion as

$$\frac{d}{dt} mv_x = \frac{|e| V_x}{\rho^2 \ln b/a} - |e| B\dot{y}, \; \frac{d}{dt} mv_y = \frac{|e| V_y}{\rho^2 \ln b/a} + |e| B\dot{x}$$

and $$\frac{d}{dt} mv_z = 0 => z = 0$$

The motion is confined to the plane $z = 0$. Eliminating B from the first two equations,

we get $$\frac{d}{dt}\left(\frac{1}{2} mv^2\right) = \frac{|e| V}{\ln b/a} \frac{x\dot{x} + y\dot{y}}{\rho^2}$$

or $$\frac{1}{2} mv^2 = |e| V \frac{\ln \rho/a}{\ln b/a}$$

So, as expected, since magnetic forces do not work

$$v = \sqrt{\frac{2\,|e| V}{m}} \quad (\text{at } \rho = b)$$

On the other hand, eliminating V, we also get

$$\frac{d}{dt} m (xv_y - yv_x) = |e| B (x\dot{x} + y\dot{y})$$

i.e., $$(xv_y - yv_x) = \frac{|e| B}{2m} \rho^2 + \text{constant}$$

The constant is easily evaluated, since v is zero at $\rho = a$. Thus,

$$(xv_y - yv_x) = \frac{|e| B}{2m} (\rho^2 - a^2) > 0$$

At $\rho = b$, $(xv_y - yv_x) \leq vb$. Thus,

$$vb \geq \frac{|e| B}{2m} (b^2 - a^2)$$

or $$B \leq \frac{2mb}{b^2 - a^2}\sqrt{\frac{2\ |e|\ V}{m}} \times \frac{1}{|e|}$$

or $$B \leq \frac{2b}{b^2 - a^2}\sqrt{\frac{2mV}{|e|}}$$

3.395 The equations are as in solution of Problem 3.392.

$$\frac{dv_x}{dt} = \frac{qB}{m}\,v_y,\ \frac{dv_y}{dt} = \frac{qE_m}{m}\cos\omega t - \frac{qB}{m}\,v_x \text{ and } \frac{dv_z}{dt} = 0$$

with $$\omega = \frac{qB}{m},\ \xi = v_x + iv_y$$

We get, $$\frac{d\xi}{dt} = i\frac{E_m}{B}\,\omega\cos\omega\ t - i\omega\xi$$

Or multiplying by $e^{i\omega t}$, we get

$$\frac{d}{dt}(\xi e^{i\omega t}) = i\frac{E_m}{2B}\,\omega\,(e^{2i\omega t} + 1)$$

Integrating, $$\xi e^{i\omega t} = \frac{E_m}{4B}\,e^{2i\omega t} + \frac{E_m}{2B}\,i\omega t$$

or $$\xi = \frac{E_m}{4B}(e^{i\omega t} + 2i\,\omega t e^{\,i\omega t}) + Ce^{\,i\omega t}$$

Since $$\xi = 0 \text{ at } t = 0,\ C = -\frac{E_m}{4B}$$

Thus, $$\xi = i\frac{E_m}{2B}\sin\omega t + i\ \frac{E_m}{2B}\,\omega t\ e^{i\omega t}$$

or $$v_x = \frac{E_m}{2B}\,\omega t\sin\omega t \text{ and } v_y = \frac{E_m}{2B}\sin\omega t + \frac{E_m}{2B}\,\omega t\cos\omega t$$

Integrating again,

$$x = \frac{a}{2\omega^2}(\sin\omega t - \omega t\cos\omega t),\ y = \frac{a}{2\omega}\,t\sin\omega t$$

where $a = qE_m/m$, and we have used $x = y = 0$, at $t = 0$.

The trajectory is an unwinding spiral.

3.396 We know that for a charged particle (proton) in a magnetic field,

$$\frac{mv^2}{r} = Bev \quad \text{or } mv = Ber$$

But, $$\omega = \frac{eB}{m}$$

Thus, $$E = \frac{1}{2}\, mv^2 = \frac{1}{2}\, m\omega^2 r^2$$

So, $$\Delta E = m\omega^2 r\, \Delta r = 4\pi^2\nu^2\, mr\, \Delta r$$

On the other hand $\Delta E = 2eV$, where V is the effective acceleration voltage across the dees, there being two crossings per revolution.

So, $$V \geq 2\pi^2\nu^2 mr\, \Delta r/e$$
$$= 0.10 \text{ MV}$$

3.397 (a) From $$\frac{mv^2}{r} = Bev \quad \text{or} \quad mv = Ber$$

We get, $$T = \frac{(Ber)^2}{2m} = \frac{1}{2}\, mv^2 = 12 \text{ MeV}$$

(b) From $$\frac{2\pi}{\omega} = \frac{2\pi r}{v}$$

We get, $$f_{\min} = \frac{v}{2\pi r} = \frac{1}{\pi r}\sqrt{\frac{T}{2m}} = 20 \text{ MHz}$$

3.398 (a) The total time of acceleration is

$$t = \frac{1}{2\nu} \cdot n$$

where n is the number of passages of the dees.

But, $$T = neV = \frac{B^2e^2r^2}{2m}$$

or $$n = \frac{B^2er^2}{2mV}$$

So, $$t = \frac{\pi}{eB/m} \times \frac{B^2er^2}{2mV} = \frac{\pi Br^2}{2V} = \frac{\pi^2 m\nu r^2}{eV} = 17\ \mu\text{s}$$

(b) The distance covered is

$$s = \sum \nu_n \frac{1}{2\nu}$$

But, $$v_n = \sqrt{\frac{2eV}{m}}\sqrt{n}$$

So, $$s = \sqrt{\frac{eV}{2m\nu^2}} \sum \sqrt{n} = \sqrt{\frac{eV}{2m\nu^2}} \int \sqrt{n}\, dn = \sqrt{\frac{eV}{2m\nu^2}}\, \frac{2}{3}\, n^{3/2}$$

But, $$n = \frac{B^2 e^2 r^2}{2\ eVm} = \frac{2\ \pi^2 m\nu^2 r^2}{eV}$$

Thus, $$s \approx \frac{4\pi^3 \nu^2 m r^2}{3eV} = 0.74 \text{ km}$$

3.399 In the nth orbit,

$$\frac{2\pi r_n}{v_n} = nT_0 = \frac{n}{\nu}$$

We ignore the rest mass of the electron and write $v_n \approx c$. Also $W \approx cp = cBer_n$.

Thus, $$\frac{2\pi W}{Bec^2} = \frac{n}{\nu}$$

or $$n = \frac{2\pi W\nu}{Bec^2} = 9$$

3.400 The basic condition is the relativistic equation

$$\frac{mv^2}{r} = Bqv \quad \text{or} \quad mv = \frac{m_0 v}{\sqrt{1 - v^2/c^2}} = Bqr$$

Using, $$\omega = \frac{Bq}{m}$$

We get, $$\omega = \frac{\omega_0}{\sqrt{1 + \dfrac{\omega_0^2\ r^2}{c^2}}} \quad \text{and} \quad \omega_0 = \frac{Bq}{m_0}\ r$$

The time of acceleration is

$$t = \sum_{n=1}^{N} \frac{1}{2\nu_n} = \sum_{n=1}^{N} \frac{\pi}{\omega_n} = \sum_{n}^{N} \frac{\pi W_n}{qBc^2}$$

N is the number of crossings of either Dee.

But, $$W_n = m_0 c^2 + \frac{n\Delta W}{2}$$

there being two crossings of the Dees per revolution.

So, $$t = \sum \frac{\pi m_0 c^2}{qBc^2} + \sum \frac{\pi \Delta W_n}{2qBc^2}$$

$$= N\frac{\pi}{\omega_0} + \frac{N(N+1)}{4} \frac{\pi \Delta W}{qBc^2} \cong N^2 \frac{\pi \Delta W}{4qBc^2}\ (N \gg 1)$$

Also, $$r = r_N \frac{v_N}{\omega_N} \approx \frac{c}{\pi}\frac{\partial t}{\partial N} \cong \frac{\Delta W}{2\,qBc}N$$

Hence finally, $$\omega = \frac{\omega_0}{\sqrt{1 + \dfrac{q^2 B^2}{m_0^2 c^2} \times \dfrac{\Delta W^2}{4q^2B^2c^2} N^2}}$$

$$= \frac{\omega_0}{\sqrt{1 + \dfrac{(\Delta W)^2}{4m_0{}^2 c^4} \times \dfrac{4\,qBc^2}{\pi \Delta W}t}} = \frac{\omega_0}{\sqrt{1 + at}}$$

$$a = \frac{qB\Delta W}{\pi\, m_0^2 c^2}$$

3.401 When the magnetic field is set up in the solenoid, an electric field is induced in it, which will accelerate the charged particle. If B is the rate at which the magnetic field is increasing, then

$$\pi r^2 \dot{B} = 2\pi r E \quad \text{or} \quad E = \frac{1}{2} r\dot{B}$$

Thus, $$m\frac{dv}{dt} = \frac{1}{2}\, r\dot{B}q \quad \text{or} \quad v = \frac{qBr}{2m}$$

After the field is set up, the particle will execute a circular motion of radius ρ, where

$$mv = Bq\rho \quad \text{or} \quad \rho = \frac{1}{2}\, r$$

3.402 The increment in energy per revolution is $e\Phi$, so the number of revolutions is

$$N = \frac{W}{e\Phi} = 5 \times 10^6 \text{ revolutions}$$

The distance traversed is $s = 2\pi rN = 8 \times 10^3$ km.

3.403 We know that,

$$\frac{dp}{dt} = eE = \frac{e}{2\pi r}\frac{d\Phi}{dt} = \frac{e}{2\pi r}\frac{d}{dt}\int_0^r 2\pi r'\, B(r')\, dr'$$

On the other hand, $p = B\,(r)\; er \quad (\text{where } r = \text{constant})$

So, $$\frac{dp}{dt} = er\frac{d}{dt}\,B\,(r) = er\,\dot{B}\,(r)$$

Hence, $$er\,\dot{B}(r) = \frac{e}{2\pi r}\,\pi r^2\,\frac{d}{dt}<B>$$

So, $$\dot{B}(r) = \frac{1}{2}\,\frac{d}{dt}<B>$$

This equation is most easily satisfied by taking

$$B(r_0) = \frac{1}{2} < B >$$

3.404 The condition, $B(r_0) = \frac{1}{2}<B> = \frac{1}{2}\int_0^{r_0} B\cdot 2\pi r\frac{dr}{\pi r_0^2}$

or $$B(r_0) = \frac{1}{r_0^2}\int_0^{r_0} Brdr$$

This gives r_0.

In the present case,

$$B_0 - ar_0^2 = \frac{1}{r_0^2}\int_0^{r_0} (B - ar^2)\, rdr$$

$$= \frac{1}{2}\left(B_0 - \frac{1}{2}ar_0^2\right)$$

or $$\frac{3}{4}\,ar_0^2 = \frac{1}{2}\,B_0$$

or $$r_0 = \sqrt{\frac{2B_0}{3a}}$$

3.405 The induced electric field (or eddy current field) is given by

$$E(r) = \frac{1}{2\pi r}\,\frac{d}{dt}\int_0^r 2\pi r'\,(r')\,B\,(r')\,dr'$$

Hence, $$\frac{dE}{dr} = -\frac{1}{2\pi r^2}\frac{d}{dt}\int_0^r 2\pi r'\,B\,(r')\,dr' + \frac{dB\,(r)}{dt}$$

$$= -\frac{1}{2}\,\frac{d}{dt} < B > \;+ \frac{dB(r)}{dt}$$

This vanishes for $r = r_0$ by the betatron condition, where r_0 is the radius of the equilibrium orbit.

Hence, $$\frac{dE}{dr} = \dot{B}(r_0) - \frac{1}{2}\dot{B} = 0$$

3.406 From the betatron condition,

$$\frac{1}{2}\,\frac{d}{dt}\,\langle B\rangle \;=\frac{dB}{dt}(r_0)\;=\;\frac{B}{\Delta t}$$

Thus,
$$\frac{d}{dt}\langle B\rangle \;=\;\frac{2B}{\Delta t}$$

and
$$\frac{d\Phi}{dt}\;=\;\pi r^2\frac{d\,\langle B\rangle}{dt}\;=\;\frac{2\pi r^2 B}{\Delta t}$$

So, energy increment per revolution is

$$e\frac{d\Phi}{dt}\;=\;\frac{2\pi r^2 eB}{\Delta t}\;=\;0.10\text{ keV}$$

3.407 (a) Even in the relativistic case, know that $p = Ber$.

Thus, $W = \sqrt{c^2p^2 + m_0^2c^4} - m_0c^2 = m_0c^2\left(\sqrt{1 + (Ber/m_0c)^2} - 1\right)$

(b) The distance traversed is

$$2\pi r\frac{W}{e\Phi}\;=\;2\pi r\frac{W}{2\pi r^2 eB/\Delta t}\;=\;\frac{W\Delta t}{Ber}$$

on using the result of the previous problem.

CONTENTS

OSCILLATIONS AND WAVES

PART 4

4.1 Mechanical Oscillations

4.1 (a) Given $$x = a\cos\left(\omega t - \frac{\pi}{4}\right)$$

So, $$v_x = \dot{x} = -a\omega\sin\left(\omega t - \frac{\pi}{4}\right) \quad \text{and} \quad w_x = \ddot{x} = -a\omega^2\cos\left(\omega t - \frac{\pi}{4}\right) \tag{1}$$

On the basis of obtained expressions plots, $x(t)$, $v_x(t)$ and $w_x(t)$ can be drawn as shown in the answer sheet, of the problem book.

(b) From Eq. (1)

$$v_x = -a\omega\sin\left(\omega t - \frac{\pi}{4}\right)$$

So, $$v_x^2 = a^2\omega^2\sin^2\left(\omega t - \frac{\pi}{4}\right) \tag{2}$$

But from the given law of motion,

$$x = a\cos\left(\omega t - \frac{\pi}{4}\right) \quad \text{so,} \quad x^2 = a^2\cos^2\left(\omega t - \frac{\pi}{4}\right)$$

or $$\cos^2\left(\omega t - \frac{\pi}{4}\right) = \frac{x^2}{a^2} \quad \text{or} \quad \sin^2\left(\omega t - \frac{\pi}{4}\right) = 1 - \frac{x^2}{a^2} \tag{3}$$

Using Eq. (3) in Eq. (2),

$$v_x^2 = a^2\omega^2\left(1 - \frac{x^2}{a^2}\right) \quad \text{or} \quad v_x^2 = \omega^2(a^2 - x^2) \tag{4}$$

or $$\left(\frac{v_x}{a\omega}\right)^2 + \left(\frac{x}{a}\right)^2 = 1$$

Again from Eq. (4),

$$\omega_x = -\,a\omega^2 \cos(\omega t - \pi/4) = -\omega^2 x$$

4.2 (a) From the given motion law of the particle

$$x = a\sin^2\left(\omega t - \frac{\pi}{4}\right) = \frac{a}{2}\left[1 - \cos\left(2\omega t - \frac{\pi}{2}\right)\right]$$

or $$x - \frac{a}{2} = -\frac{a}{2}\cos\left(2\omega t - \frac{\pi}{2}\right) = -\frac{a}{2}\sin 2\omega t = \frac{a}{2}\sin(2\omega t + \pi)$$

i.e., $$x - \frac{a}{2} = \frac{a}{2}\sin(2\omega t + \pi) \tag{1}$$

Now comparing this equation with the general equation of harmonic oscillations, $X = A\sin(\omega_0 t + \alpha)$, we get amplitude, $A = a/2$ and angular frequency, $\omega_0 = 2\omega$. Thus, the period of one full oscillation, $T = 2\pi/\omega_0 = \pi/\omega$.

(b) Differentiating Eq. (1) with respect to time, we get

$$v_x = a\omega\cos(2\omega t + \pi)$$

or $$v_x^2 = a^2\omega^2\cos^2(2\omega t + \pi) = a^2\omega^2\,[1 - \sin^2(2\omega t + \pi)] \tag{2}$$

Also, $$\left(x - \frac{a}{2}\right)^2 = \frac{a^2}{4}\sin^2(2\omega t + \pi) \quad \text{(using Eq. 1)}$$

so, $$4\frac{x^2}{a^2} + 1 - \frac{4x}{a} = \sin^2(2\omega t + \pi) \quad \text{or} \quad 1 - \sin^2(2\omega t + \pi) = \frac{4x}{a}\left(1 - \frac{x}{a}\right) \tag{3}$$

From Eqs. (2) and (3),

$$v_x = a^2\omega^2\frac{4x}{a}\left(1 - \frac{x}{a}\right) = 4\omega^2 x(a - x)$$

Plot of $v_x(x)$ is as shown in the answer sheet.

4.3 Let the general equation of simple harmonic motion (S.H.M.) be

$$x = a\cos(\omega t + \alpha) \tag{1}$$

So, $$v_x = -a\omega\sin(\omega t + \alpha) \tag{2}$$

Let us assume that at $t = 0$, $x = x_0$ and $v_x = v_{x_0}$.

Thus, from Eqs. (1) and (2), for $t = 0$, $x_0 = a\cos\alpha$, and $v_{x_0} = -a\omega\sin\alpha$.

Therefore, $\tan\alpha = \dfrac{-v_{x_0}}{\omega x_0}$ and $a = \sqrt{x_0^2 + \left(\dfrac{v_{x_0}}{\omega}\right)^2} = 35.35$ cm

Under our assumption, Eqs. (1) and (2) give the sought x and v_x if

$$t = 2.40 \text{ s}, \quad a = \sqrt{x_0^2 + \left(\frac{v_{x_0}}{\omega}\right)^2} \quad \text{and} \quad \alpha = \tan^{-1}\left(-\frac{v_x}{\omega x_0}\right) = -\frac{\pi}{4}$$

Putting all the given numerical values, we get

$$x = -29 \text{ cm} \quad \text{and} \quad v_x = -81 \text{ cm/s}$$

4.4 From the equation,

$$v_x^2 = \omega^2(a^2 - x^2) \text{ (see Eq. 4 of Problem 4.1)}$$

$$v_1^2 = \omega^2(a^2 - x_1^2) \quad \text{and} \quad v_2^2 = \omega^2(a^2 - x_2^2)$$

Solving these equations simultaneously, we get

$$\omega = \sqrt{\frac{(v_1^2 - v_2^2)}{(x_2^2 - x_1^2)}}, \quad a = \sqrt{\frac{(v_1^2\, x_2^2 - v_2^2\, x_1^2)}{(v_1^2 - v_2^2)}}$$

4.5 (a) When a particle starts from an extreme position, you can write the motion law as

$$x = a \cos \omega t \qquad (1)$$

(Here, x is the displacement from the equilibrium position.)

If t_1 be the time taken to cover the distance $a/2$, then from Eq. (1), we get

$$a - \frac{a}{2} = \frac{a}{2} = a\cos\omega t_1 \quad \text{or} \quad \cos\omega t_1 = \frac{1}{2} = \cos\frac{\pi}{3} \left(\text{as } t_1 < \frac{T}{4}\right)$$

Thus, $$t_1 = \frac{\pi}{3\omega} = \frac{\pi}{3(2\pi/T)} = \frac{T}{6}$$

As $x = a \cos \omega t$, so, $v_x = -a\omega \sin \omega t$

Thus, $v = |v_x| = -v_x = a\omega \sin \omega t$ (for $t \leqslant t_1 = T/6$).

Hence, sought mean velocity

$$< v > = \frac{\int v\, dt}{\int dt} = \int_0^{T/6} a\left(\frac{2\pi}{T}\right)\frac{\sin\omega t\, dt}{T/6} = \frac{3a}{T} = 0.5 \text{ m/s}$$

(b) In this case, it is easier to write the motion law in the form:

$$x = a \sin \omega t \qquad (2)$$

If t_2 is the time taken to cover the distance $a/2$, then from Eq. (2), we get

$$\frac{a}{2} = a \sin \frac{2\pi}{T} t_2 \quad \text{or} \quad \sin \frac{2\pi}{T} t_2 = \frac{1}{2} = \sin \frac{\pi}{6} \quad \left(\text{as } t_2 < \frac{T}{4}\right)$$

Thus,
$$\frac{2\pi}{T} t_2 = \frac{\pi}{6} \quad \text{or} \quad t_2 = \frac{T}{12}$$

Differentiating Eq. (2) with respect to time, we get

$$v_x = a\omega \cos \omega t = a \frac{2\pi}{T} \cos \frac{2\pi}{T} t$$

So,
$$v = |v_x| = a \frac{2\pi}{T} \cos \frac{2\pi}{T} t \quad (\text{for } t \le t_2 = T/12)$$

Hence, the sought mean velocity

$$< v > = \frac{\int v \, dt}{\int dt} = \frac{1}{(T/12)} \int_0^{T/12} a \frac{2\pi}{T} \cos \frac{2\pi}{T} t \, dt = \frac{6a}{T} = 1 \text{ m/s}$$

4.6 (a) As $x = a \sin \omega t$, so, $v_x = a\omega \cos \omega t$

Thus,
$$< v_x > = \frac{\int v_x dt}{\int dt} = \frac{\int_0^{3/8T} a\omega \cos(2\pi/T)t \, dt}{(3T/8)}$$

$$= \frac{2\sqrt{2} \, a\omega}{3\pi} \quad (\text{using } T = 2\pi/\omega)$$

(b) In accordance with the problem

$$\mathbf{v} = v_x \mathbf{i} \quad \text{so,} \quad |< \mathbf{v} >| = |< v_x >|$$

Hence, using part (a),

$$|< v >| = \left| \frac{2\sqrt{2} \, a\omega}{3\pi} \right| = \frac{2\sqrt{2} \, a\omega}{3\pi}$$

(c) We have,
$$v_x = a\omega \cos \omega t$$

So,
$$\left.\begin{aligned} v = |v_x| &= a\omega \cos \omega t, \quad \text{for } t \le \frac{T}{4} \\ &= -a\omega \cos \omega t, \quad \text{for } \frac{T}{4} \le t \le \frac{3}{8} T \end{aligned}\right]$$

Hence, $$< v > = \frac{\int v\,dt}{\int dt} = \frac{\int_0^{T/4} a\omega \cos \omega t\,dt + \int_{T/4}^{3T/8} -a\omega \cos \omega t\,dt}{3T/8}$$

Using $\omega = 2\pi/T$, and on evaluating the integral, we get

$$< v > = \frac{2(4-\sqrt{2})a\omega}{3\pi}$$

4.7 From the given motion law, $x = a \cos \omega t$, it is obvious that the time taken to cover the distance equal to the amplitude (a), starting from extreme position equals $T/4$. Now, one can write

$$t = n\frac{T}{4} + t_0 \quad \left(\text{where } t_0 < \frac{T}{4} \text{ and } n = 0, 1, 2, \ldots\right)$$

As the particle moves according to the law, $x = a \cos \omega t$, so at $n = 1, 3, 5 \ldots$ (or for odd n values) it passes through the mean position and for even numbers of n it comes to an extreme position (if $t_0 = 0$).

Case 1: When n is an odd number.

In this case, from the equation $x = \pm a \sin \omega t$, if t is counted from $nT/4$ then the distance covered in the time interval t_0 becomes,

$$s_1 = a \sin \omega t_0 = a \sin \omega\left(t - n\frac{T}{4}\right) = a \sin\left(\omega t - \frac{n\pi}{2}\right)$$

Thus, the sought distance covered for odd n is

$$s = na + s_1 = na + a \sin\left(\omega t - \frac{n\pi}{2}\right) = a\left[n + \sin\left(\omega t - \frac{n\pi}{2}\right)\right]$$

Case 2: When n is an even number.

In this case, from the equation $x = a \cos \omega t$, if it is counted from $nT/4$ then the distance covered (s_2) in the interval t_0 is given by

$$a - s_2 = a \cos \omega t_0 = a \cos \omega\left(t - n\frac{T}{4}\right) = a \cos\left(\omega t - n\frac{\pi}{2}\right)$$

or $$s_2 = a\left[1 - \cos\left(\omega t - \frac{n\pi}{2}\right)\right]$$

Hence, the sought distance for even n is

$$s = na + s_2 = na + a\left[1 - \cos\left(\omega t - \frac{n\pi}{2}\right)\right] = a\left[n + 1 - \cos\left(\omega t - \frac{n\pi}{2}\right)\right]$$

In general,

$$s = \begin{cases} a\left[n + 1 - \cos\left(\omega t - \dfrac{n\pi}{2}\right)\right], & \text{if } n \text{ is even} \\ a\left[n + \sin\left(\omega t - \dfrac{n\pi}{2}\right)\right], & \text{if } n \text{ is odd} \end{cases}$$

4.8 Obviously the motion law is of the form, $x = a \sin \omega t$ and $v_x = \omega a \cos \omega t$. Comparing $v_x = \omega a \cos \omega t$ with $v_x = 35 \cos \pi t$, we get

$$\omega = \pi, \quad a = \frac{35}{\pi}$$

Thus, $$T = \frac{2\pi}{\omega} = 2 \quad \text{and} \quad \frac{T}{4} = 0.5 \text{ s}$$

Now, $$t = 5 \times T/4 + 0.3 = 2.8 \text{ s} \quad (\text{where } T/4 = 0.5 \text{ s})$$

As $n = 5$ is odd, like in Problem 4.7, we have to find the distance covered by the particle starting from the extreme position in the time interval 0.3 s.

Thus, from the equation

$$x = a \cos \omega t = \frac{35}{\pi} \cos \pi (0.3)$$

$$\frac{35}{\pi} - s_1 = \frac{35}{\pi}[\cos \pi (0.3)] \quad \text{or} \quad s_1 = \frac{35}{\pi}[1 - \cos \pi (0.3)]$$

Hence, the sought distance

$$s = 5 \times \frac{35}{\pi} + \frac{35}{\pi}\{1 - \cos 0.3\pi\}$$

$$= \frac{35}{\pi}(6 - \cos 0.3\pi) = \frac{35}{22} \times 7(6 - \cos 54°) \cong 60 \text{ cm}$$

4.9 Since the motion is periodic, the particle repeatedly passes through any given region in the range $-a \le x \le a$. The probability that it lies in the range $(x, x + dx)$ is defined as the fraction $\Delta t/t$ (as $t \to \infty$), where Δt is the time when the particle lies in the range $(x, x + dx)$ out of the total time t. Because of periodicity, the probability is given by

$$dP = \frac{dP}{dx} dx = \frac{dt}{T} = \frac{2dx}{vT}$$

where the factor 2 is needed because the particle is in the range $(x, x + dx)$ during both upward and downward phases of its motion. Now, in a harmonic oscillator

$$v = \dot{x} = \omega a \cos \omega t = \omega\sqrt{a^2 - x^2}$$

Since $\omega T = 2\pi$ (where T is the time period)

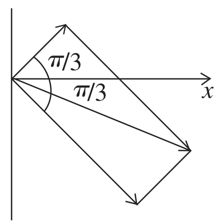
we get $$dP = \frac{dP}{dx}dx = \frac{1}{\pi}\frac{dx}{\sqrt{a^2 - x^2}}$$

Note that $$\int_{-a}^{+a} \frac{dP}{dx}dx = 1$$

So, $$\frac{dP}{dx} = \frac{1}{\pi}\frac{1}{\sqrt{a^2 - x^2}}$$

is properly normalized.

4.10 (a) We take a graph paper and choose an axis (x-axis) and an origin. Draw a vector of magnitude 3 inclined at an angle $\pi/3$ with the x-axis. Draw another vector of magnitude 8 inclined at an angle $-\pi/3$ (since $\sin(\omega t + \pi/6) = \cos(\omega t - \pi/3)$) with the x-axis. The magnitude of the resultant of both these vectors (drawn from the origin) obtained using parallelogram law is the resultant amplitude.



Clearly, $$a^2 = 3^2 + 8^2 + 2\cdot 3\cdot 8\cdot \cos\frac{2\pi}{3}$$

$$= 9 + 64 - 48 \times \frac{1}{2}$$

$$= 73 - 24 = 49$$

Thus, $a = 7$ units

(b) One can follow the same graphical method here but the result can be obtained more quickly by breaking into sines and cosines and adding:

So, $$x = \left(3 + \frac{5}{\sqrt{2}}\right)\cos\omega t + \left(6 - \frac{5}{\sqrt{2}}\right)\sin\omega t$$

$$= A\cos(\omega t + \alpha)$$

Then $$a^2 = \left(3 + \frac{5}{\sqrt{2}}\right)^2 + \left(6 - \frac{5}{\sqrt{2}}\right)^2 = 9 + 25 + \frac{30 - 60}{\sqrt{2}} + 36$$

$$= 70 - 15\sqrt{2} = 70 - 21.2$$

So, $a = 6.985 \approx 7$ units

Note: *In using the graphical method, convert all oscillations to either sines or cosines but do not use both.*

4.11 Given, $x_1 = a \cos \omega t$ and $x_2 = a \cos 2\omega t$

So, the net displacement

$$x = x_1 + x_2 = a\,(\cos \omega t + \cos 2\omega t) = a\,(\cos \omega t + 2\cos^2 \omega t - 1)$$

and $v_x = \dot{x} = a\,(-\,\omega \sin \omega t - 4\omega \cos \omega t \sin \omega t)$

For $\dot{x}$ to be maximum,

$$\ddot{x} = a\omega^2 \cos \omega t - 4\,a\omega^2 \cos^2 \omega t + 4\,a\omega^2 \sin^2 \omega t = 0$$

or $$8\cos^2 \omega t + \cos \omega t - 4 = 0$$

which is a quadratic equation for $\cos \omega t$. Solving for acceptable value

$$\cos \omega t = 0.644$$

So, $$\sin \omega t = 0.765$$

and $$v_{\max} = |v_{x_{\max}}| = +\,a\omega[0.765 + 4 \times 0.765 \times 0.644] = +2.73\,a\omega$$

4.12 We can write

$$a \cos 2.1t \cos 50.0t = \frac{a}{2}(\cos 52.1t + \cos 47.9t)$$

Thus, the angular frequencies of constituent oscillations will be

$$52.1\ \text{s}^{-1} \quad \text{and} \quad 47.9\ \text{s}^{-1}$$

To get the beat period note that the variable amplitude $a \cos 2.1t$ becomes maximum (positive or negative), when

$$2.1t = n\pi$$

Thus, the interval between two maxima is

$$\frac{\pi}{2.1} \approx 1.5\ \text{s}$$

4.13 If the frequency of A with respect to K' is ν_0 and K' oscillates with frequency $\bar{\nu}$ with respect to K, the beat frequency of the point A in the K-frame will be ν when

$$\bar{\nu} = \nu_0 \pm \nu$$

In the present case $$\bar{\nu} = 20 \text{ or } 24$$

This means $$\nu_0 = 22 \quad \text{and} \quad \nu = 2$$

Thus, beats of $2\nu = 4$ will be heard when $\bar{\nu} = 26$ or 18.

4.14 (a) From the equation $x = a \sin \omega t$

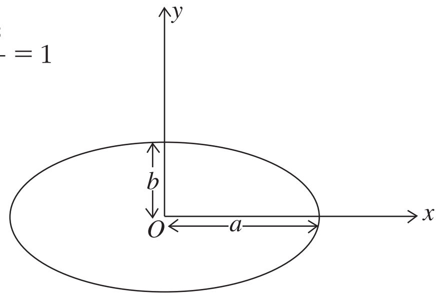
$$\sin^2 \omega t = \frac{x^2}{a^2} \quad \text{or} \quad \cos^2 \omega t = 1 - \frac{x^2}{a^2} \tag{1}$$

And from the equation $y = b \cos \omega t$

we have,
$$\cos^2 \omega t = \frac{y^2}{b^2} \tag{2}$$

From Eqs. (1) and (2), we get

$$1 - \frac{x^2}{a^2} = \frac{y^2}{b^2} \quad \text{or} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



which is the standard equation of ellipse shown in the figure.

We observe that,

at $t = 0$, $\quad x = 0 \quad$ and $\quad y = b$

and at $t = \pi/2\omega$, $\quad x = +a \quad$ and $\quad y = 0$

Thus, we observe that at $t = 0$, the point is at O (see figure) and at the following moments, the coordinate y diminishes and x becomes positive. Consequently, the motion is clockwise.

(b) As $x = a \sin \omega t$ and $y = b \cos \omega t$, so, we may write $\mathbf{r} = a \sin \omega t\, \mathbf{i} + b \cos \omega t\, \mathbf{j}$.

Thus,
$$\ddot{\mathbf{r}} = \mathbf{w} = -\omega^2 \mathbf{r}$$

4.15 (a) From the equation $x = a \sin \omega t$, we have

$$\cos \omega t = \sqrt{1 - \left(\frac{x^2}{a^2}\right)}$$

and from the equation $y = a \sin 2\omega t$, we have

$$y = 2a \sin \omega t \cdot \cos \omega t = 2x\sqrt{1 - \left(\frac{x^2}{a^2}\right)} \quad \text{or} \quad y^2 = 4x^2\left(1 - \frac{x^2}{a^2}\right)$$

(b) From the equation, $x = a \sin \omega t$

$$\sin^2 \omega t = \frac{x^2}{a^2}$$

From the equation, $y = a \cos 2\omega t$

$$y = a(1 - 2\sin^2 \omega t) = a\left(1 - 2\frac{x^2}{a^2}\right)$$

For the plots see answer sheet of the problem book.

4.16 As $$U(x) = U_0 (1 - \cos ax)$$

so, $$F_x = -\frac{dU}{dx} = -U_0\, a \sin ax$$

or $$F_x = -U_0 a(ax)$$

(because for small angle of oscillations $\sin ax \cong ax$)

or $$F_x = -U_0 a^2 x$$

But we know $F_x = -m\,\omega_0^2 x$. for small oscillations.

Thus, $$\omega_0^2 = \frac{U_0 a^2}{m} \quad \text{or} \quad \omega_0 = a\sqrt{\frac{U_0}{m}}$$

Hence, the sought time period

$$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{a}\sqrt{\frac{m}{U_0}} = 2\pi\sqrt{\frac{m}{a^2 U_0}}$$

4.17 If $$U(x) = \frac{a}{x^2} - \frac{b}{x}$$

then the equilibrium position is

$$x = x_0 \quad (\text{when } U'(x_0) = 0)$$

or $$-\frac{2a}{x_0^3} + \frac{b}{x_0^2} = 0 \Rightarrow x_0 = \frac{2a}{b}$$

Now write $$x = x_0 + y$$

Then, $$U(x) = \frac{a}{x_0^2} - \frac{b}{x_0} + (x - x_0)\,U'(x_0) + \frac{1}{2}(x - x_0)^2 U''(x_0)$$

But, $$U''(x_0) = \frac{6a}{x_0^4} - \frac{2b}{x_0^3} = \left(\frac{2a}{b}\right)^{-3}(3b - 2b) = \frac{b^4}{8a^3}$$

So, finally $$U(x) = U(x_0) + \frac{1}{2}\left(\frac{b^4}{8a^3}\right)y^2 + \ldots$$

We neglect remaining terms for small oscillations and compare with the potential energy for a harmonic oscillator, then

$$\frac{1}{2}m\omega^2 y^2 = \frac{1}{2}\left(\frac{b^4}{8a^3}\right)y^2 \quad \text{so,} \quad \omega = \frac{b^2}{\sqrt{8a^3 m}}$$

Thus, $$T = 2\pi\frac{\sqrt{8ma^3}}{b^2}$$

Note: *Equilibrium position is generally a minimum of the potential energy. Then $U'(x_0) = 0$, $U''(x_0) > 0$. The equilibrium position can in principle be a maximum but then $U''(x_0) < 0$ and the frequency of oscillations about this equilibrium position will be imaginary. The answer given in the book is incorrect both numerically and dimensionally.*

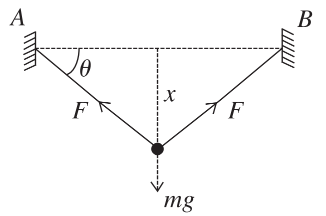
4.18 Let us locate and depict the forces acting on the ball at the position when it is at a distance x below the normal (non-deformed) position of the string. At this position, the unbalanced downward force on the ball will be $mg - 2F \sin \theta$.

By Newton's law,

$$m\ddot{x} = mg - 2F \sin \theta$$

$$= mg - 2F\theta \quad \text{(when } \theta \text{ is small)}$$

$$= mg - 2F\frac{x}{l/2} = mg - \frac{4F}{l}x$$



Thus,
$$\ddot{x} = g - \frac{4F}{ml}x = -\frac{4F}{ml}\left(x - \frac{mgl}{4F}\right)$$

Substituting $x' = x - \dfrac{mgl}{4F}$, we get

$$\ddot{x} = -\frac{4F}{ml}x'$$

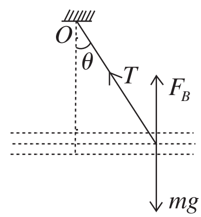
Thus,
$$T = \pi\sqrt{\frac{ml}{F}} = 0.2 \text{ s}$$

4.19 Let us depict the forces acting on the oscillating ball at an arbitrary angular position θ (see figure), relative to equilibrium position where F_B is the force of buoyancy. For the ball from the equation:

$$N_z = I\beta_z$$

(where we have taken the positive sense of z-axis in the direction of angular velocity, i.e., $\dot{\theta}$ of the ball and passes through the point of suspension of the pendulum O), we get

$$-mgl \sin\theta + F_B l \sin\theta = ml^2 \ddot{\theta} \qquad (1)$$



Using $m = (4/3)\pi r^3 \sigma$, $F_B = (4/3)\pi r^3 \rho$ and $\sin\theta \cong \theta$ (for small θ), in Eq. (1), we get

$$\ddot{\theta} = -\frac{g}{l}\left(1 - \frac{\rho}{\sigma}\right)\theta$$

Thus, the sought time period

$$T = 2\pi \frac{1}{\sqrt{g/l(1-\rho/\sigma)}} = 2\pi\sqrt{\frac{l/g}{1-1/\eta}}$$

Hence, $$T = 2\pi\sqrt{\frac{\eta l}{g(\eta-1)}} = 1.1 \text{ s}$$

4.20 Obviously for small β, the ball executes part of S.H.M. Due to the perfectly elastic collision, the velocity of ball is simply reversed. As the ball is in S.H.M. ($|\theta| < \alpha$ on the left), its motion law in differential form can be written as

$$\ddot{\theta} = -\frac{g}{l}\theta = -\omega_0^2\theta \tag{1}$$

If we assume that the ball is released from the extreme position, $\theta = \beta$ at $t = 0$, the solution of differential equation would be taken in the form

$$\theta = \beta\cos\omega_0 t = \beta\cos\sqrt{\frac{g}{l}}\,t \tag{2}$$

If t' is the time taken by the ball to go from the extreme position $\theta = \beta$ to the wall, i.e., $\theta = -\alpha$, then Eq. (2) can be rewritten as

$$-\alpha = \beta\cos\sqrt{\frac{g}{l}}\,t'$$

or $$t' = \sqrt{\frac{l}{g}}\cos^{-1}\left(-\frac{\alpha}{\beta}\right) = \sqrt{\frac{l}{g}}\left(\pi - \cos^{-1}\frac{\alpha}{\beta}\right)$$

Thus, the sought time $$T = 2t' = 2\sqrt{\frac{l}{g}}\left(\pi - \cos^{-1}\frac{\alpha}{\beta}\right)$$

$$= 2\sqrt{\frac{l}{g}}\left(\frac{\pi}{2} + \sin^{-1}\frac{\alpha}{\beta}\right) \quad (\text{because } \sin^{-1}x + \cos^{-1}x = \pi/2)$$

4.21 Imagine that the downward acceleration of the elevator car has continued for time t', then the sought time $t = \sqrt{2h/w} + t'$, where obviously $\sqrt{2h/w}$ is the time of upward acceleration of the elevator. One should note that if the point of suspension of a mathematical pendulum moves with an acceleration $\mathbf{w}$, then the time period of the pendulum will be

$$2\pi\sqrt{\frac{l}{|\mathbf{g} - \mathbf{w}|}}$$

In this problem the time period of the pendulum while it is moving upward with acceleration w becomes $2\pi\sqrt{l/(g+w)}$, and its time period, while the elevator moves downward, with the same magnitude of acceleration, becomes

$$2\pi\sqrt{\frac{l}{g-w}}$$

As the time of upward acceleration equals $\sqrt{2h/w}$, the total number of oscillations during this time is

$$\frac{\sqrt{2h/w}}{2\pi\sqrt{l/(g+w)}}$$

Thus, the indicated time

$$\frac{\sqrt{2h/w}}{2\pi\sqrt{l/(g+w)}}\cdot 2\pi\sqrt{\frac{l}{g}}=\sqrt{\frac{2h}{w}}\cdot\sqrt{\frac{(g+w)}{g}}$$

Similarly, the indicated time for the time interval t'

$$\frac{t'}{2\pi\sqrt{l\,/\,(g-w)}}2\pi\sqrt{\frac{l}{g}}=t'\sqrt{\frac{(g-w)}{g}}$$

We require that

$$\sqrt{\frac{2h}{w}}\sqrt{\frac{(g+w)}{g}}+t'\sqrt{\frac{(g-w)}{g}}=\sqrt{\frac{2h}{w}}+t'$$

or
$$t'=\sqrt{\frac{2h}{w}}\,\frac{\sqrt{g+w}-\sqrt{g}}{\sqrt{g}-\sqrt{g-w}}$$

Hence, the sought time

$$t=\sqrt{\frac{2h}{w}}+t'=\sqrt{\frac{2h}{w}}\,\frac{\sqrt{g+w}-\sqrt{g-w}}{\sqrt{g}-\sqrt{g-w}}$$

$$=\sqrt{\frac{2h}{w}}\,\frac{\sqrt{(1+\eta)}-\sqrt{1-\eta}}{1-\sqrt{1-\eta}}\quad(\text{where } \eta = w/g)$$

4.22 If the hydrometer were in equilibrium or floating, its weight will be balanced by the buoyancy force exerted on it by the fluid. During the small oscillation, let us locate the hydrometer when it is at a vertically downward distance x from its equilibrium position. Obviously, the net unbalanced force on the hydrometer is the excess buoyancy force directed upward and equals $\pi r^2 x\rho g$. Hence, for the hydrometer

$$m\ddot{x}=-\pi r^2\rho g x$$

or
$$\ddot{x}=-\frac{\pi r^2\rho g}{m}x$$

Hence, the sought time period

$$T = 2\pi\sqrt{\frac{m}{\pi r^2 \rho g}} = 2.5 \text{ s}$$

4.23 First, let us calculate the stiffness κ_1 and κ_2 of both parts of the spring. If we subject the original spring of stiffness κ having the natural length l_0 (say), to the deforming forces F–F (say) to elongate the spring by amount x, then

$$F = \kappa x \tag{1}$$

Therefore, the elongation per unit length of the spring is x/l_0. Now, let us subject one of the parts of the spring of natural length ηl_0 to the same deforming forces F–F. Then the elongation of the spring will be

$$\frac{x}{l_0}\eta l_0 = \eta x$$

Thus,
$$F = \kappa_1(\eta x) \tag{2}$$

Hence, from Eqs. (1) and (2)

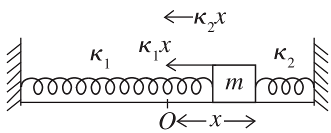
$$\kappa = \eta\kappa_1 \quad \text{or} \quad \kappa_1 = \kappa/\eta \tag{3}$$

Similarly,
$$\kappa_2 = \frac{\kappa}{1-\eta}$$

The position of the block m when both the parts of the spring are non-deformed, is its equilibrium position O. Let us displace the block m towards right or in positive x-axis by the small distance x. Let us depict the forces acting on the block when it is at a distance x from its equilibrium position (see figure) From the second law of motion in projection form, i.e., $F_x = mw_x$, we have

$$-\kappa_1 x - \kappa_2 x = m\ddot{x}$$

or
$$-\left(\frac{\kappa}{\eta} + \frac{\kappa}{1-\eta}\right)x = m\ddot{x}$$



Thus,
$$\ddot{x} = -\frac{\kappa}{m}\frac{1}{\eta(1-\eta)}x$$

Hence, the sought time period

$$T = 2\pi\sqrt{\eta(1-\eta)m/\kappa} = 0.13 \text{ s}$$

4.24 Similar to the solution of Problem 4.23, the net unbalanced force on the block m when it is at a small horizontal distance x from the equilibrium position becomes $(\kappa_1 + \kappa_2)\,x$. From $F_x = mw_x$ for the block

$$-(\kappa_1 + \kappa_2)x = m\ddot{x}$$

Thus, $$\ddot{x} = -\left(\frac{\kappa_1 + \kappa_2}{m}\right)x$$

Hence, the sought time period

$$T = 2\pi\sqrt{\frac{m}{\kappa_1 + \kappa_2}}$$

Alternate:

Let us set the block m in motion to perform small oscillations. Let us locate the block when it is at a distance x from its equilibrium position.

As the spring force is restoring conservative force and deformation of both the springs is the same, so from the conservation of mechanical energy of oscillation of the spring–block system

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\kappa_1 x^2 + \frac{1}{2}\kappa_2 x^2 = \text{constant}$$

Differentiating with respect to time

$$\frac{1}{2}m\, 2\dot{x}\ddot{x} + \frac{1}{2}(\kappa_1 + \kappa_2)2x\dot{x} = 0$$

or $$\ddot{x} = -\left(\frac{\kappa_1 + \kappa_2}{m}\right)x$$

Hence, the sought time period

$$T = 2\pi\sqrt{\frac{m}{\kappa_1 + \kappa_2}}$$

4.25 During the vertical oscillations let us locate the block at a vertically downward distance x from its equilibrium position. At this moment if x_1 and x_2 are the additional or further elongations of the upper and lower springs relative to the equilibrium position, then the net unbalanced force on the block will be $\kappa_2 x_2$ directed in upward direction. Hence,

$$-\kappa_2 x_2 = m\ddot{x} \tag{1}$$

We also have $$x = x_1 + x_2 \tag{2}$$

Since the springs are massless and initially the net force on the spring is also zero, so for the spring

$$\kappa_1 x_1 = \kappa_2 x_2 \tag{3}$$

Solving the Eqs. (1), (2) and (3) simultaneously, we get

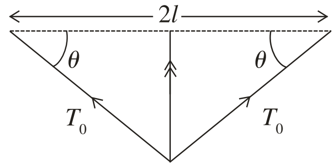
$$-\frac{\kappa_1\,\kappa_2}{\kappa_1 + \kappa_2}x = m\ddot{x}$$

Thus, $$\ddot{x} = -\frac{\kappa_1\,\kappa_2}{m(\kappa_1 + \kappa_2)}\,x$$

Hence, the sought time period

$$T = 2\pi\sqrt{m\frac{(\kappa_1 + \kappa_2)}{\kappa_1\ \kappa_2}}$$

4.26 The force F, acting on the weight deflected from the position of equilibrium is $2T_0 \sin\theta$. Since the angle θ is small, the net restoring force



$$F = 2T_0\frac{x}{l}$$

or $$F = kx \quad (\text{where } k = 2T_0/l)$$

So, by using the formula,

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \text{or} \quad \omega_0 = \sqrt{\frac{2T_0}{ml}}$$

4.27 If the mercury rises in the left arm by x, it must fall by a slanting length equal to x in the other arm. Total pressure difference in the two arms will then be

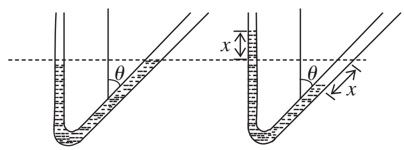
$$\rho gx + \rho gx\cos\theta = \rho gx(1 + \cos\theta)$$

This will give rise to a restoring force

$$-\rho gSx(1 + \cos\theta)$$

This must equal mass times acceleration which can be obtained from work–energy principle. The kinetic energy of the mercury in the tube is clearly $1/2\, m\dot{x}^2$. So mass times acceleration must be $m\ddot{x}$. Hence,

$$m\ddot{x} + \rho gS(1 + \cos\theta)x = 0$$



This is S.H.M. with a time period

$$T = 2\pi\sqrt{\frac{m}{\rho gS(1 + \cos\theta)}} = 0.8 \text{ s}$$

4.28 In the equilibrium position, the centre of mass (C.M.) of the rod lies midway between the two rotating wheels. Let us displace the rod horizontally by a small distance and then release it. Let us depict the forces acting on the rod when its C.M. is at distance x from its equilibrium position (see figure). Since there is no net vertical force acting on the rod, Newton's second law gives

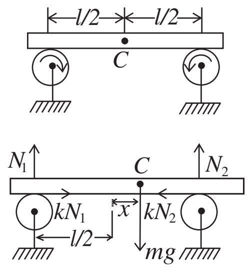
$$N_1 + N_2 = mg \tag{1}$$

For the translational motion of the rod, from the equation $F_x = mw_{cx}$, we have

$$kN_1 - kN_2 = m\ddot{x} \tag{2}$$

As the rod experiences no net torque about an axis perpendicular to the plane of the figure through the C.M. of the rod.

$$N_1\left(\frac{l}{2}+x\right)=N_2\left(\frac{l}{2}-x\right) \tag{3}$$



Solving Eqs. (1), (2) and (3) simultaneously, we get

$$\ddot{x}=-k\,\frac{2g}{l}\,x$$

Hence, the sought time period

$$T=2\pi\sqrt{\frac{l}{2kg}}=\pi\sqrt{\frac{2l}{kg}}=1.5\text{ s}$$

4.29 (a) The only force acting on the ball is the gravitational force **F**, of magnitude $\gamma\,4/3\ \pi\rho\ mr$, where γ is the gravitational constant, ρ the density of the Earth and r is the distance of the body from the centre of the Earth.

But, $g = \gamma\ 4\pi/3\ \rho R$, so the expression for **F** can be written as, $\mathbf{F} = -mg\,\mathbf{r}/R$, where R is the radius of the Earth and the equation of motion in projection form has the form, $m\ddot{x} + mg/Rx = 0$.

(b) The equation obtained above has the form of an equation of S.H.M. having the time period,

$$T=2\pi\sqrt{\frac{R}{g}}$$

Hence, the body will reach the other end of the shaft in the time

$$t=\frac{T}{2}=\pi\sqrt{\frac{R}{g}}=42\text{ min}$$

(c) From the conditions of S.H.M., the speed of the body at the centre of the Earth will be maximum, having the magnitude

$$v=R\omega=R\sqrt{\frac{g}{R}}=\sqrt{gR}=7.9\text{ km/s}$$

4.30 In the frame of point of suspension, the mathematical pendulum of mass m (say) will oscillate. In this frame, it will experience the inertial force $m\,(-\,\mathbf{w})$ in addition to the real forces during its oscillations. Therefore, in equilibrium position m is deviated by some angle, say α

$$T_0\cos\alpha = mg + mw\cos(\pi-\beta) \quad \text{and} \quad T_0\sin\alpha = mw\sin(\pi-\beta)$$

So, from these two equations

$$\tan \alpha = \frac{g - w \cos \beta}{w \sin \beta}$$

and $$\cos \alpha = \sqrt{\frac{m^2 w^2 \sin^2 \beta + (mg - mw \cos \beta)^2}{mg - mw \cos \beta}} \quad (1)$$

Let us displace the bob from its equilibrium position by some small angle and then release it. Now locate the bob at an angular position $(\alpha + \theta)$ from vertical as shown in the figure.

From the equation $N_{0_z} = I\beta_z$, we have

$$-mgl \sin(\alpha + \theta) - mw \cos(\pi - \beta) l \sin(\alpha + \theta) + mw \sin(\pi - \beta) l \cos(\alpha + \theta) = ml^2 \ddot{\theta}$$

or $$-g(\sin \alpha \cos \theta + \cos \alpha \sin \theta) - w \cos(\pi - \beta)(\sin \alpha \cos \theta + \cos \alpha \sin \theta) + w \sin \beta (\cos \alpha \cos \theta - \sin \alpha \sin \theta) = l\ddot{\theta}$$

But for small θ, $\sin \theta \cong \theta$, $\cos \theta = 1$. So,

$$-g(\sin \alpha + \cos \alpha\theta) - w \cos(\pi - \beta)(\sin \alpha + \cos \alpha\theta) + w \sin \beta (\cos \alpha - \sin \alpha\theta) = l\ddot{\theta}$$

or $$(\tan \alpha + \theta)(w \cos \beta - g) + w \sin \beta (1 - \tan \alpha \theta) = \frac{l}{\cos \alpha} \ddot{\theta} \quad (2)$$

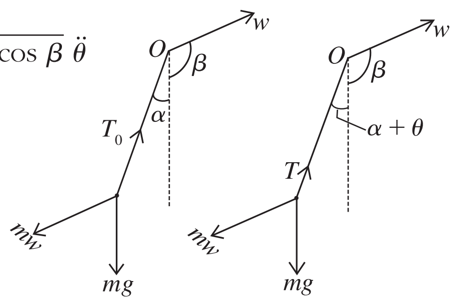
Solving Eqs. (1) and (2) simultaneously, we get

$$-(g^2 - 2wg \cos \beta + w^2)\,\theta = l\sqrt{g^2 + w^2 2wg \cos \beta}\;\ddot{\theta}$$

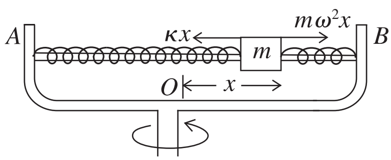
Thus, $$\ddot{\theta} = -\frac{|\mathbf{g} - \mathbf{w}|}{l}\theta$$

Hence, the sought time period

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{|\mathbf{g} - \mathbf{w}|}} = 0.8 \text{ s}$$



4.31 Obviously the sleeve performs small oscillations in the frame of rotating rod. In the rod's frame let us depict the forces acting on the sleeve along the length of the rod while the sleeve is at a small distance x towards right from its equilibrium position.



The free-body diagram of block does not contain Coriolis force, because it is perpendicular to the length of the rod. From $F_x = mw_x$ for the sleeve in the frame of rod

$$-\kappa x + m\omega^2 x = m\ddot{x}$$

or $$\ddot{x} = -\left(\frac{\kappa}{m} - \omega^2\right)x \quad (1)$$

Thus, the sought time period

$$T = \frac{2\pi}{\sqrt{\kappa/m - \omega^2}} = 0.7 \text{ s}$$

It is obvious from Eq. (1) that the sleeve will not perform small oscillations if

$$\omega \geq \sqrt{\frac{\kappa}{m}} = 10 \text{ rad/s}$$

4.32 When the bar is about to start sliding along the plank, it experiences the maximum restoring force which is being provided by the limiting friction.

Thus, $$kN = m\omega_0^2 a \quad \text{or} \quad kmg = m\omega_0^2 a$$

or $$k = \frac{\omega_0^2 a}{g} = \frac{a}{g}\left(\frac{2\pi}{T}\right)^2 = 0.4$$

4.33 The natural angular frequency of a mathematical pendulum equals $\omega_0 = \sqrt{g/l}$.

(a) The solution of S.H.M. equation in angular form will be

$$\theta = \theta_m \cos(\omega_0 t + \alpha)$$

If at the initial moment, i.e., at $t = 0$, $\theta = \theta_m$, then $\alpha = 0$. Thus, the above equation takes the form

$$\theta = \theta_m \cos \omega_0 t$$

$$= \theta_m \cos \sqrt{\frac{g}{l}}\, t = 3° \cos \sqrt{\frac{9.8}{0.8}}\, t$$

Thus, $$\theta = 3° \cos 3.5t$$

(b) The S.H.M. equation in angular form will be

$$\theta = \theta_m \sin(\omega_0 t + \alpha)$$

If at the initial moment $t = 0$, $\theta = 0$, then $\alpha = 0$. Then the above equation takes the form

$$\theta = \theta_m \sin \omega_0 t$$

Let v_0 be the velocity of the lower end of pendulum at $\theta = 0$, then from conservation of mechanical energy of oscillation

$$E_{\text{mean}} = E_{\text{extreme}} \quad \text{or} \quad T_{\text{mean}} = U_{\text{extreme}}$$

or $$\frac{1}{2} m v_0^2 = mgl(1 - \cos\theta_m)$$

Thus, $$\theta_m = \cos^{-1}\left(1 - \frac{v_0^2}{2gl}\right)$$

$$= \cos^{-1}\left[1 - \frac{(0.22)^2}{2 \times 9.8 \times 0.8}\right] = 4.5°$$

Thus, the sought equation becomes

$$\theta = \theta_m \sin \omega_0 t = 4.5° \sin 3.5t$$

(c) Let θ_0 and v_0 be the angular deviation and linear velocity at $t = 0$. As the mechanical energy of oscillation of the mathematical pendulum is conserved

$$\frac{1}{2} m v_0^2 + mgl(1 - \cos\theta_0) = mgl(1 - \cos\theta_m)$$

or
$$\frac{v_0^2}{2} = gl(\cos\theta_0 - \cos\theta_m)$$

Thus,
$$\theta_m = \cos^{-1}\left\{\cos\theta_0 - \frac{v_0^2}{2gl}\right\}$$

$$= \cos^{-1}\left\{\cos 3° - \frac{(0.22)^2}{2 \times 9.8 \times 0.8}\right\} = 5.4°$$

Then from $\theta = 5.4° \sin(3.5t + \alpha)$, we see that $\sin\alpha = 3/5.4$ and $\cos\alpha < 0$ because the velocity is directed towards the centre. Thus, $\alpha = \pi/2 + 1.0$ radians and we get the answer, $\theta = 5.4° \cos(3.5t + 1.0)$.

4.34 While the body A is at its upper extreme position, the spring is obviously elongated by the amount

$$\left(a - \frac{m_1 g}{\kappa}\right)$$

If we indicate y-axis in vertically downward direction, Newton's second law of motion in projection form, i.e., $F_y = m w_y$ for body A gives

$$m_1 g + \kappa\left(a - \frac{m_1 g}{\kappa}\right) = m_1 \omega^2 a \quad \text{or} \quad \kappa\left(a - \frac{m_1 g}{\kappa}\right) = m_1(\omega^2 a - g) \tag{1}$$

(Since at any extreme position the magnitude of acceleration of an oscillating body equals $\omega^2 a$ and is restoring in nature.)

If N is the normal force exerted by the floor on the body B, while the body A is at its upper extreme position, from Newton's second law for body B

$$N + \kappa\left(a - \frac{m_1 g}{\kappa}\right) = m_2 g$$

or
$$N = m_2 g - \kappa\left(a - \frac{m_1 g}{\kappa}\right) = m_2 g - m_1(\omega^2 a - g) \text{ (using Eq. 1)}$$

Hence, $$N = (m_1 + m_2)\,g - m_1\omega^2 a = 40\text{ N}$$

When the body A is at its lower extreme position, the spring is compressed by the distance $(a + m_1 g/\kappa)$. From Newton's second law in projection form, i.e., $F_y = mw_y$ for body A in this state

$$m_1 g - \kappa\left(a + \frac{m_1 g}{\kappa}\right) = m_1(-\omega^2 a) \quad \text{or} \quad \kappa\left(a + \frac{m_1 g}{\kappa}\right) = m_1(g + \omega^2 a) \tag{2}$$

In this case, if N' is the normal force exerted by the floor on the body B, from Newton's second law for body B, we get

$$N' = \kappa\left(a + \frac{m_1 g}{\kappa}\right) + m_2 g = m_1(g + \omega^2 a) + m_2 g \quad \text{(using Eq. 2)}$$

Hence, $$N' = (m_1 + m_2)g + m_1\omega^2 a = 60\text{ N}$$

From Newton's third law, the magnitude of sought forces are N' and N, respectively.

4.35 (a) For the block from Newton's second law in projection form $F_y = mw_y$

$$N - mg = m\ddot{y} \tag{1}$$

But from $$y = a\,(1 - \cos\omega t)$$

we get $$\ddot{y} = \omega^2 a \cos\omega t \tag{2}$$

From Eqs. (1) and (2)

$$N = mg\left(1 + \frac{\omega^2 a}{g}\cos\omega t\right) \tag{3}$$

From Newton's third law the force which the body m exerts on the block is directed vertically downward and equals

$$N' = mg\left(1 + \frac{\omega^2 a}{g}\cos\omega t\right)$$

(b) When the body m starts falling behind the plank or losing contact, $N = 0$ (because the normal reaction is the contact force). Thus, from Eq. (3)

$$mg\left(1 + \frac{\omega^2 a}{g}\cos\omega t\right) = 0$$

Hence, $$a_{\min} = \frac{g}{\omega^2} = 8\text{ cm}$$

(c) We observe that the motion takes place about the mean position $y = a$ at the initial instant $y = 0$. As shown in (b) the normal reaction vanishes at a height (g/ω^2) above the position of equilibrium and the body flies off as free body. The speed of the body at a distance (g/ω^2) from the equilibrium position is $\omega\sqrt{a^2 - (g/\omega^2)^2}$, so that the condition of the problem gives

$$\frac{\left[\omega\sqrt{a^2-(g/\omega^2)^2}\right]^2}{2g} + \frac{g}{\omega^2} + a = h$$

Hence, on solving the resulting quadratic equation and taking the positive root,

$$a = -\frac{g}{\omega^2} + \sqrt{\frac{2hg}{\omega^2}} \cong 20 \text{ cm}$$

4.36 (a) Let $y\,(t)$ = displacement of the body from the end of the unstretched position of the spring (not the equilibrium position). Then

$$m\ddot{y} = -\kappa y + mg$$

This equation has the solution in the form

$$y = A + B\cos(\omega t + \alpha)$$

If $$-m\omega^2 B\cos(\omega t + \alpha) = -\kappa[A + B\cos(\omega t + \alpha)] + mg$$

then $$\omega^2 = \frac{\kappa}{m} \quad \text{and} \quad A = \frac{mg}{\kappa}$$

and at $t = 0$, we have $y = 0$ and $\dot{y} = 0$. So,

$$-\omega B\sin\alpha = 0 \quad \text{and} \quad A + B\cos\alpha = 0$$

Since $B > 0$ and $A > 0$, we must have $\alpha = \pi$

$$B = A = \frac{mg}{\kappa}$$

and $$y = \frac{mg}{\kappa}(1 - \cos\omega t); \quad \omega = \sqrt{\kappa/m}$$

(b) Tension in the spring is

$$T = \kappa y = mg\,(1 - \cos\omega t)$$

so, $$T_{\max} = 2mg \quad \text{and} \quad T_{\min} = 0$$

4.37 In accordance with the problem

$$\mathbf{F} = -\alpha m\mathbf{r}$$

So, $$m(\ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j}) = -\alpha m(x\mathbf{i} + y\mathbf{j})$$

Thus, $$\ddot{x} = -\alpha x \quad \text{and} \quad \ddot{y} = -\alpha y$$

Hence, the solution of the differential equation $\ddot{x} = -\alpha x$ becomes

$$x = a\cos(\omega_0 t + \delta) \quad (\text{where } \omega_0^2 = \alpha) \tag{1}$$

So,
$$\dot{x} = -a\omega_0 \sin(\omega_0 t + \alpha) \tag{2}$$

From the initial conditions of the problem, $v_x = 0$ and $x = r_0$ at $t = 0$. So from Eq. (2), $\alpha = 0$, and equation takes the form

$$x = r_0 \cos \omega_0 t \quad \text{so,} \quad \cos \omega_0 t = \frac{x}{r_0} \tag{3}$$

One of the solutions of the other differential equation $\ddot{y} = -\alpha y$, becomes

$$y = a' \sin(\omega_0 t + \delta') \quad (\text{where } \omega_0^2 = \alpha) \tag{4}$$

From the initial condition, $y = 0$ at $t = 0$, so $\delta' = 0$ and Eq. (4) becomes

$$y = a' \sin \omega_0 t \tag{5}$$

Differentiating with respect to time, we get

$$\dot{y} = a'\omega_0 \cos \omega_0 t \tag{6}$$

But from the initial condition of the problem, $\dot{y} = v_0$ at $t = 0$. So, from Eq. (6)

$$v_0 = a'\omega_0 \quad \text{or} \quad a' = \frac{v_0}{\omega_0}$$

Using it in Eq. (5), we get

$$y = \frac{v_0}{\omega_0} \sin \omega_0 t \tag{7}$$

or
$$\sin \omega_0 t = \frac{\omega_0 y}{v_0}$$

Squaring and adding Eqs. (3) and (7), we get

$$\sin^2 \omega_0 t + \cos^2 \omega_0 t = \frac{\omega_0^2 y^2}{v_0^2} + \frac{x^2}{r_0^2}$$

or
$$\left(\frac{x}{r_0}\right)^2 + \alpha\left(\frac{y}{v_0}\right)^2 = 1 \quad (\text{as } \alpha = \omega_0^2)$$

4.38 (a) Since the elevator car is a translating non-inertial frame, the body m will experience an inertial force mw directed downward in addition to the real forces in the elevator's frame. From the Newton's second law in projection form $F_y = mw_y$, for the body in the frame of elevator car

$$-\kappa\left(\frac{mg}{\kappa} + y\right) + mg + mw = m\ddot{y}$$

(because the initial elongation in the spring is mg/κ).

So,
$$m\ddot{y} = -\kappa y + mw = -\kappa\left(y - \frac{mw}{\kappa}\right)$$

or
$$\frac{d^2}{dt^2}\left(y - \frac{mw}{\kappa}\right) = -\frac{\kappa}{m}\left(y - \frac{mw}{\kappa}\right) \tag{1}$$

Eq. (1) shows that the motion of the body m is S.H.M. and its solution becomes

$$y - \frac{mw}{\kappa} = a\sin\left(\sqrt{\frac{\kappa}{m}}\,t + \alpha\right) \tag{2}$$

Differentiating Eq. (2) with respect to time

$$\dot{y} = a\sqrt{\frac{\kappa}{m}}\cos\left(\sqrt{\frac{\kappa}{m}}t + \alpha\right) \tag{3}$$

Using the initial condition $y(0) = 0$ in Eq. (2), we get

$$a\sin\alpha = -\frac{mw}{\kappa}$$

and using the other initial condition $\dot{y}(0) = 0$ in Eq. (3), we get

$$a\sqrt{\frac{\kappa}{m}}\cos\alpha = 0$$

Thus,
$$\alpha = -\frac{\alpha}{2} \quad \text{and} \quad a = \frac{mw}{\kappa}$$

Hence, using these values in Eq. (2), we get

$$y = \frac{mw}{\kappa}\left(1 - \cos\sqrt{\frac{\kappa}{m}}t\right)$$

(b) Proceed upto Eq. (1) as in part (a). The solution of this differential equation will be

$$y - \frac{mw}{\kappa} = a\sin\left(\sqrt{\frac{\kappa}{m}}t + \delta\right)$$

or
$$y - \frac{\alpha t}{\kappa/m} = a\sin\left(\sqrt{\frac{\kappa}{m}}\,t + \delta\right)$$

or
$$y - \frac{\alpha t}{\omega_0^2} = a\sin(\omega_0 t + \delta) \quad \left(\text{where } \omega_0 = \sqrt{\frac{\kappa}{m}}\right) \tag{4}$$

From the initial condition that at $t = 0$, $y\,(0) = 0$, so $0 = a \sin\delta$ or $\delta = 0$. Thus, Eq. (4) takes the form

$$y - \frac{\alpha t}{\omega_0^2} = a \sin \omega_0 t \tag{5}$$

Differentiating Eq. (5) we get

$$\dot{y} - \frac{\alpha}{\omega_0^2} = a\omega_0 \cos \omega t \tag{6}$$

But from the other initial condition $\dot{y}(0) = 0$ at $t = 0$. So, from Eq. (6)

$$-\frac{\alpha}{\omega_0^2} = a\omega_0 \quad \text{or} \quad \alpha = -\frac{\alpha}{\omega_0^3}$$

Putting the value of a in Eq. (5), we get the sought $y(t)$, i.e.,

$$y - \frac{\alpha t}{\omega_0^2} = -\frac{\alpha}{\omega_0^3} \sin \omega_0 t \quad \text{or} \quad y = \frac{\alpha}{\omega_0^3}(\omega_0 t - \sin \omega_0 t)$$

4.39 There is an important difference between a rubber cord or steel coir and a spring. A spring can be pulled or compressed and in both cases obeys Hooke's law. But a rubber cord becomes loose when one tries to compress it and does not then obey Hooke's law. Thus, if we suspend a body by a rubber cord, it stretches by a distance mg/κ in reaching the equilibrium configuration. If we further stretch it by a distance Δh it will execute harmonic oscillations when released if $\Delta h \le mg/\kappa$ because only in this case will the cord remain taut and obey Hooke's law. Thus,

$$\Delta h_{\max} = \frac{mg}{\kappa} = 10 \text{ cm}$$

The energy of oscillation in this case is

$$\frac{1}{2}\kappa(\Delta h_{\max})^2 = \frac{1}{2}\frac{m^2 g^2}{\kappa} = 4.8 \text{ mJ}$$

4.40 Since the pan is of negligible mass, there is no loss of kinetic energy even though the collision is inelastic. The mechanical energy of the body m in the field generated by the joint action of both the gravity force and the elastic force is conserved, i.e., $\Delta E = 0$. During the motion of the body m from the initial to the final position (position of maximum compression of the spring) $\Delta T = 0$, and therefore $\Delta U = \Delta U_{gr} + \Delta U_{sp} = 0$ or

$$-mg(h + x) + \frac{1}{2}\kappa x^2 = 0$$

On solving the quadratic equation, we get

$$x = \frac{mg}{\kappa} \pm \sqrt{\frac{m^2 g^2}{\kappa^2} + \frac{2mgh}{\kappa}}$$

Since minus sign is not acceptable

$$x = \frac{mg}{\kappa} + \sqrt{\frac{m^2 g^2}{\kappa^2} + \frac{2mgh}{\kappa}}$$

If the body m were at rest on the spring, the corresponding position of m will be its equilibrium position and at this position the resultant force on the body m will be zero. Therefore the equilibrium compression Δx (say) due to the body m will be given by

$$\kappa \Delta x = mg \quad \text{or} \quad \Delta x = \frac{mg}{\kappa}$$

Therefore, separation between the equilibrium position and one of the extreme positions, i.e., the sought amplitude

$$a = x - \Delta x = \sqrt{\frac{m^2 g^2}{\kappa^2} + \frac{2mgh}{\kappa}} = \frac{mg}{\kappa}\sqrt{1 + \frac{2\kappa h}{mg}}$$

The mechanical energy of oscillation which is conserved, equals $E = U_{\text{extreme}}$, because at the extreme position kinetic energy becomes zero.

Although the weight of body m is a conservative force, it is not restoring in this problem, hence U_{extreme} is only concerned with the spring force. Therefore,

$$E = U_{\text{extreme}} = \frac{1}{2}\kappa a^2 = mgh + \frac{m^2 g^2}{2\kappa}$$

4.41 Unlike the previous problem, the kinetic energy of body m decreases due to the perfectly inelastic collision with the pan. Obviously, the body m comes to strike the pan with velocity $v_0 = \sqrt{2gh}$. If v be the common velocity of the "body m + pan" system due to the collision then from the conservation of linear momentum

$$mv_0 = (M + m)\, v$$

or

$$v = \frac{mv_0}{(M+m)} = \frac{m\sqrt{2gh}}{(M+m)} \tag{1}$$

At the moment the body m strikes the pan, the spring is compressed due to the weight of the pan by the amount Mg/κ. If l is the further compression of the spring due to the velocity acquired by the "pan + body m" system, then from the conservation of mechanical energy of the said system in the field generated by the joint action of both the gravity and spring forces

$$\frac{1}{2}(M+m)\, v^2 + (M+m) gl = \frac{1}{2}\kappa\left(\frac{Mg}{\kappa} + l\right)^2 - \frac{1}{2}\kappa\left(\frac{Mg}{\kappa}\right)^2$$

or

$$\frac{1}{2}(M+m)\frac{m^2 2gh}{(M+m)} + (M+m)\, gl$$

$$= \frac{1}{2}\kappa\left(\frac{Mg}{\kappa}\right)^2 + \frac{1}{2}\kappa l^2 + Mgl - \frac{1}{2}\kappa\left(\frac{Mg}{\kappa}\right)^2 \quad \text{(using Eq. 1)}$$

or $$\frac{1}{2}\kappa l^2 - mgl - \frac{m^2 gh}{(m+M)} = 0$$

Thus, $$l = \frac{mg \pm \sqrt{m^2 g^2 + \dfrac{2\,\kappa g h m^2}{M+m}}}{\kappa}$$

Since minus sign is not acceptable

$$l = \frac{mg}{\kappa} + \frac{1}{\kappa}\sqrt{m^2 g^2 + \frac{2\,\kappa m^2 gh}{(M+m)}}$$

If the oscillating "pan + body m" system were at rest, it corresponds to the equilibrium position, i.e., the spring was compressed by $(M+m)g/\kappa$, therefore the amplitude of oscillation

$$a = l - \frac{mg}{\kappa} = \frac{1}{\kappa}\sqrt{m^2 g^2 + \frac{2\kappa m^2 gh}{(M+m)}} = \frac{mg}{\kappa}\sqrt{1 + \frac{2h\kappa}{(M+m)g}}$$

The mechanical energy of oscillation which is only conserved with the restoring forces becomes $E = U_{\text{extreme}} = 1/2\ \kappa a^2$ (because spring force is the only restoring force, not the weight of the body).

Alternately $$E = T_{\text{mean}} = \frac{1}{2}(M+m)\,a^2\omega^2$$

Thus, $$E = \frac{1}{2}(M+m)a^2\left(\frac{\kappa}{M+m}\right) = \frac{1}{2}\,\kappa a^2$$

4.42 We have $$\mathbf{F} = a\,(\dot{y}\,\mathbf{i} - \dot{x}\,\mathbf{j})$$

or $$m\,(\ddot{x}\,\mathbf{i} + \ddot{y}\,\mathbf{j}) = a\,(\dot{y}\,\mathbf{i} - \dot{x}\,\mathbf{j})$$

So, $$m\ddot{x} = a\dot{y} \quad \text{and} \quad m\ddot{y} = -\,a\dot{x} \tag{1}$$

From the initial condition, $t = 0$, $\dot{x} = 0$ and $y = 0$. So, integrating equation $m\ddot{x} = a\dot{y}$, we get

$$m\dot{x} = ay \quad \text{or} \quad \dot{x} = \frac{a}{m}\,y \tag{2}$$

Using Eq. (2) in the equation $m\ddot{y} = -\,a\dot{x}$, we get

$$m\ddot{y} = -\left(\frac{a^2}{m}\right)y \quad \text{or} \quad \ddot{y} = -\left(\frac{a}{m}\right)^2 y \tag{3}$$

One of the solutions of differential Eq. (3) is

$$y = A\sin(\omega_0 t + \alpha) \quad (\text{where } \omega_0 = a/m)$$

As at $t = 0$, $y = 0$, so the solution takes the form $y = A \sin \omega_0 t$. On differentiating with respect to time

$$\dot{y} = A\omega_0 \cos \omega_0 t$$

From the initial condition of the problem, at $t = 0$, $\dot{y} = v_0$.

So, $$v_0 = A\omega_0 \quad \text{or} \quad A = v_0/\omega_0$$

Thus, $$y = (v_0/\omega_0) \sin \omega_0 t$$

Thus, from Eq. (2)

$$\dot{x} = v_0 \sin \omega_0 t \tag{4}$$

On integrating $$x = B - \frac{v_0}{\omega_0} \cos \omega_0 t \tag{5}$$

On using $x = 0$ at $t = 0$, $$B = \frac{v_0}{\omega_0}$$

Finally $$x = \frac{v_0}{\omega_0}(1 - \cos \omega_0 t) \tag{6}$$

Hence, from Eqs. (4) and (6), we get

$$\left[x - \left(\frac{v_0}{\omega_0}\right)\right]^2 + y^2 = \left(\frac{v_0}{\omega_0}\right)^2$$

which is the equation of a circle of radius (v_0/ω_0) with the centre at the point $x_0 = v_0/\omega_0$, $y_0 = 0$.

4.43 If water has frozen, the system consisting of the light rod and the frozen water in the hollow sphere constitute a compound (physical) pendulum to a very good approximation because we can take the whole system to be rigid. For such systems, the time period is given by $T_1 = 2\pi\sqrt{1/g}\sqrt{1 + k^2/l^2}$, where $k^2 = 2/5\ R^2$ and k is the radius of gyration of the sphere.

The situation is different when water is unfrozen. When dissipative forces (viscosity) are neglected, we are dealing with ideal fluids. Such fluids instantaneously respond to (unbalanced) internal stresses. Suppose the sphere with liquid water actually executes small rigid oscillations. Then the portion of the fluid above the centre of the sphere will have a greater acceleration than the portion below the centre because the linear acceleration of any element is in this case, equal to angular acceleration of the element multiplied by the distance of the element from the centre of suspension (recall that we are considering small oscillations). Then, as is obvious in a frame moving with the centre of mass, there will appear an unbalanced couple (not negated by any pseudo forces) which will cause the fluid to move rotationally so as to destroy differences in acceleration. Thus, for this case of ideal fluids, the pendulum must move in such a way that the elements of the fluid all undergo the same acceleration. This implies that we have a simple (mathematical) pendulum with the time period

$$T_0 = 2\pi\sqrt{\frac{l}{g}}$$

Thus,
$$T_1 = T_0\sqrt{1 + \frac{2}{5}\left(\frac{R}{l}\right)^2}$$

(One expects that a liquid with very small viscosity will have a time period close to T_0 while one with high viscosity will have period closer to T_1.)

4.44 Let us locate the rod at the position when it makes an angle θ from the vertical. In this problem, both the gravity and spring forces are restoring conservative forces. Thus, from the conservation of mechanical energy of oscillation of the oscillating system

$$\frac{1}{2}\frac{ml^2}{3}(\dot{\theta})^2 + mg\frac{1}{2}(1 - \cos\theta) + \frac{1}{2}\kappa(l\theta)^2 = \text{constant}$$

Differentiating with respect to time, we get

$$\frac{1}{2}\frac{ml^2}{3}2\,\dot{\theta}\,\ddot{\theta} + \frac{mgl}{2}\sin\,\theta\,\dot{\theta} + \frac{1}{2}\kappa l^2\;2\,\theta\,\dot{\theta} = 0$$

Thus, for very small θ

$$\ddot{\theta} = -\frac{3g}{2l}\left(1 + \frac{2\kappa l}{mg}\right)\theta$$

Hence,
$$\omega_0 = \sqrt{\frac{3g}{2l}\left(1 + \frac{2\kappa l}{mg}\right)}$$

4.45 (a) Let us locate the system when the threads are deviated through an angle $\alpha' < \alpha$, during the oscillations of the system (see figure). From the conservation of mechanical energy of the system

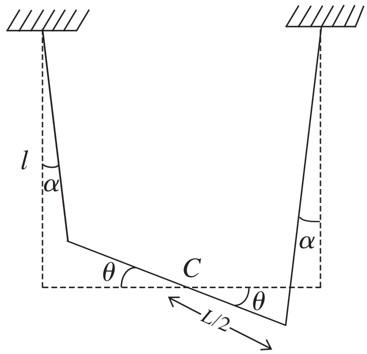
$$\frac{1}{2}\frac{mL^2}{12}\dot{\theta}^2 + mgl(1 - \cos\alpha') = \text{constant} \tag{1}$$

where L is the length of the rod, θ is the angular deviation of the rod from its equilibrium position, i.e., $\theta = 0$).

Differentiating Eq. (1) with respect to time

$$\frac{1}{2}\frac{mL^2}{12}2\dot{\theta}\,\ddot{\theta} + mgl\;\alpha'\dot{\alpha}' = 0 \quad (\text{for small } \alpha',\, \sin\alpha' \cong \alpha') \tag{2}$$

But from the figure



$$\left(\frac{L}{2}\right)\theta = 1\alpha' \quad \text{or} \quad \alpha' = \left(\frac{L}{2l}\right)\theta$$

So, $$\dot{\alpha}' = \left(\frac{L}{2l}\right)\dot{\theta}$$

Putting these values of α' and $d\alpha'/dt$ in Eq. (2), we get

$$\frac{d^2\theta}{dt^2} = -\frac{3g}{l}\theta$$

Thus, the sought time period

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{3g}} = 1.1 \text{ s}$$

(b) The sought oscillation energy

$$E = U_{\text{extreme}} = mgl\,(1 - \cos\alpha) = mgl\ 2\sin^2\frac{\alpha}{2}$$

$$\cong mgl\,2\frac{\alpha^2}{4} = \frac{mgl\alpha^2}{2} \quad (\text{for small angle } \alpha,\ \sin\alpha \cong \alpha)$$

$$= 0.05 \text{ J}$$

4.46 The kinetic energy of the disk is

$$\frac{1}{2}I\dot{\varphi}^2 = \frac{1}{2}\left(\frac{mR^2}{2}\right)\dot{\varphi}^2 = \frac{1}{4}mR^2\dot{\varphi}^2$$

The torsional potential energy is $1/2k\varphi^2$. Thus, the total energy is

$$E = \frac{1}{4}mR^2\dot{\varphi}^2 + \frac{1}{2}k\varphi^2 = \frac{1}{4}mR^2\dot{\varphi}_0^2 + \frac{1}{2}k\varphi_0^2$$

By definition of the amplitude $\varphi_m, \dot{\varphi} = 0$, when $\varphi = \varphi_m$. Thus, the total energy is

$$E = \frac{1}{2}k\varphi_m^2$$

Hence, $$\frac{1}{2}k\varphi_m^2 = \frac{1}{4}mR^2\dot{\varphi}_0^2 + \frac{1}{2}k\varphi_0^2$$

or $$\varphi_m = \varphi_0\sqrt{1 + \frac{mR^2}{2k}\frac{\dot{\varphi}_0^2}{\varphi_0^2}}$$

4.47 Moment of inertia of the rod equals $ml^2/3$ about its one end and perpendicular to its length. Thus, rotational kinetic energy of the rod is

$$\frac{1}{2}\left(\frac{ml^2}{3}\right)\dot{\theta}^2 = \left(\frac{ml^2}{6}\right)\dot{\theta}^2$$

when the rod is displaced by an angle θ its centre of gravity (C.G.) moves up by a distance

$$\frac{l}{2}(1-\cos\theta) \cong \frac{l\theta^2}{4} \quad \text{(for small } \theta\text{)}$$

Thus, the potential energy becomes

$$mg\frac{l\theta^2}{4}$$

Since the mechanical energy of oscillation of the rod is conserved

$$\frac{1}{2}\left(\frac{ml^2}{3}\right)\dot{\theta}^2 + \frac{1}{2}\left(\frac{mgl}{2}\right)\theta^2 = \text{constant}$$

On differentiating with respect to time and then simplifying, we get $\ddot{\theta} = -3g/2l\,\theta$ for small θ. We see that the angular frequency ω is $\sqrt{3g/2l}$.

We write the general solution of the angular oscillation as

$$\theta = A\cos\omega t + B\sin\omega t$$

But $\theta = \theta_0$ at $t = 0$, so $A = \theta_0$ and $B = \dot{\theta}_0/\omega$.

Thus, $$\theta = \theta_0\cos\omega t + \frac{\dot{\theta}_0}{\omega}\sin\omega t$$

Thus, the kinetic energy of the rod

$$T = \frac{ml^2}{6}\dot{\theta}^2 = [-\omega\theta_0\sin\omega t + \dot{\theta}_0\cos\omega t]^2\frac{ml^2}{6}$$

$$= \frac{ml^2}{6}[\dot{\theta}_0^2\cos^2\omega t + \omega^2\theta_0^2\sin^2\omega t - 2\,\omega\,\theta_0\dot{\theta}_0\sin\omega t\cos\omega t]$$

On averaging over one time period, the last term vanishes and $<\sin^2\omega t> = <\cos^2\omega t> = 1/2$. Thus,

$$<T> = \frac{1}{12}ml^2\,\dot{\theta}_0^2 + \frac{1}{8}mgl\,\theta_0^2 \quad \left(\text{where } \omega^2 = \frac{3g}{2l}\right)$$

4.48 Let l = distance between the C.G. (C) of the pendulum and its point of suspension O. Originally the pendulum is in inverted position and its C.G. is above O. When it falls to the normal (stable) position of equilibrium, its C.G. has fallen by a distance $2l$. In the equilibrium position, the total energy is equal to K.E. $= 1/2I\omega^2$ and we have from energy conservation

$$\frac{1}{2} I\omega^2 = mg2l \quad \text{or} \quad I = \frac{4mgl}{\omega^2}$$

Angular frequency of oscillation for a physical pendulum is given by $\omega_0^2 = mgl/I$. Thus,

$$T = 2\pi\sqrt{\frac{I}{mgl}} = 2\pi\sqrt{\frac{4mgl/\omega^2}{mgl}} = \frac{4\pi}{\omega}$$

4.49 Let moment of inertia of the pendulum about the axis be I, then using $N_Z = I\beta_z$ for the pendulum, we get

$$-mgx \sin\theta = I\dot{\theta} \quad \text{or} \quad \dot{\theta} = -\frac{mgx}{I}\theta \quad \text{(for small } \theta\text{)}$$

which is the required equation for S.H.M. So, the frequency of oscillation

$$\omega_1 = \sqrt{\frac{Mgx}{I}} \quad \text{or} \quad x = \frac{I}{Mg}\sqrt{\omega_1^2} \tag{1}$$

Now, when the mass m is attached to the pendulum at a distance l below the oscillating axis, we get

$$-Mgx \sin\theta' - mgl \sin\theta' = (I + ml^2)\frac{d^2\theta'}{dt^2}$$

or

$$-\frac{g(Mx + ml)}{(I + ml^2)}\theta' = \frac{d^2\theta'}{dt^2} \quad \text{(for small } \theta'\text{)}$$

which is again the equation of S.H.M. So, the new frequency

$$\omega_2 = \sqrt{\frac{g(Mx + ml)}{(I + ml^2)}} \tag{2}$$

Solving Eqs. (1) and (2), we get

$$\omega_2 = \sqrt{\frac{g[(I/g)\omega_1^2 + ml]}{(I + ml^2)}}$$

or

$$\omega_2^2 = \frac{I\omega_1^2 + mgl}{I + ml^2}$$

or

$$I(\omega_2^2 - \omega_1^2) = mgl - m\omega_2^2 l^2$$

Hence,

$$I = ml^2 \frac{(\omega_2^2 - g/l)}{\omega_1^2 - \omega_2^2} = 0.8 \text{ g·m}^2$$

4.50 When the two pendulums are joined rigidly and set to oscillate, each exerts torque on the other; these torques are equal and opposite. We write the law of motion for the two pendulums as

$$I_1\ddot{\theta} = -\,\omega_1^2 I_1\theta + G$$

$$I_2\ddot{\theta} = -\,\omega_2^2 I_2\theta - G$$

where $\pm G$ is the torque of mutual interactions. We consider the restoring forces on each pendulum in the absence of the other as $-\omega_1^2 I_1\theta$ and $-\omega_2^2 I_2\theta$, respectively. Then,

$$\ddot{\theta} = -\,\frac{I_1\omega_1^2 + I_2\omega_2^2}{I_1 + I_2}\,\theta = -\,\omega^2\theta$$

Hence,
$$\omega = \sqrt{\frac{I_1\omega_1^2 + I_2\omega_2^2}{I_1 + I_2}}$$

4.51 Let us locate the rod when it is at small angular position θ relative to its equilibrium position. If a is the sought distance, then from the conservation of mechanical energy of oscillation

$$mga\,(1 - \cos\theta) + \frac{1}{2}I_{OO'}(\dot{\theta})^2 = \text{constant}$$

Differentiating with respect to time, we get

$$mga\,\sin\theta\dot{\theta} + \frac{1}{2}I_{OO'}\,2\dot{\theta}\ddot{\theta} = 0$$

But,
$$I_{OO'} = \frac{ml^2}{12} + ma^2$$

Also, for small θ, $\sin\theta = \theta$, so

$$\ddot{\theta} = -\left[\frac{ga}{(l^2/12) + a^2}\right]\theta$$

Hence, the time period of one full oscillation becomes

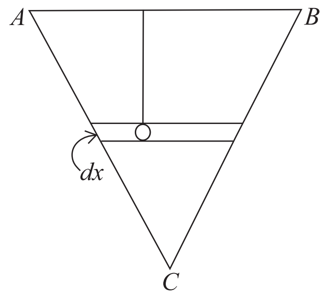
$$T = 2\pi\sqrt{\frac{(l^2/12) + a^2}{ag}} \quad \text{or} \quad T^2 = \frac{4\pi^2}{g}\left(\frac{l^2}{12a} + a\right)$$

For $T_{\min}$,
$$\frac{d}{da}\left(\frac{l^2}{12a} + a\right) = 0$$

So,
$$-\frac{l^2}{12a^2} + 1 = 0 \quad \text{or} \quad a = \frac{l}{2\sqrt{3}}$$

Hence,
$$T_{\min} = 2\pi\sqrt{\frac{l}{g\sqrt{3}}}$$

4.52 Consider the moment of inertia of the triangular plate about AB.



$$I = \iint x^2\, dm = \iint x^2\, \rho\, dx\, dy$$

$$= \int_0^h x^2\, \rho\, dx\, \frac{h-x}{h} \cdot \frac{2h}{\sqrt{3}} = \int_0^h x^2\, \frac{2\rho}{\sqrt{3}}\, (h-x)\, dx$$

$$= \frac{2\rho}{\sqrt{3}}\left(\frac{h^4}{3} - \frac{h^4}{4}\right) = \frac{\rho h^4}{6\sqrt{3}} = \frac{mh^2}{6}$$

on using the area of the triangle ABC, $\Delta = h^2/\sqrt{3}$ and $m = \rho\Delta$. Thus, kinetic energy

$$T = \frac{1}{2}\frac{mh^2}{6}\,\dot{\theta}^2$$

and potential energy

$$U = mg\frac{h}{3}(1 - \cos\theta) = \frac{1}{2}mgh\,\frac{\theta^2}{3}$$

Here θ is the angle that the instantaneous plane of the plate makes with the equilibrium position, which is vertical. (The plate rotates as a rigid body.)

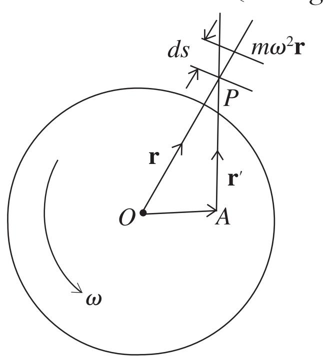
$$E = \frac{1}{2}\frac{mh^2}{6}\,\dot{\theta}^2 + \frac{1}{2}\frac{mgh}{3}\,\theta^2$$

Hence,
$$\omega^2 = \left(\frac{mgh/3}{mh^2/6}\right) = \frac{2g}{h}$$

So,
$$T = 2\pi\sqrt{\frac{h}{2g}} = \pi\sqrt{\frac{2h}{g}} \quad \text{and} \quad l_{\text{reduced}} = \frac{h}{2}$$

4.53 Let us consider the rotating frame, in which the disk is stationary. In this frame the rod is subjected to Coriolis and centrifugal forces, $\mathbf{F}_{\text{cor}}$ and $\mathbf{F}_{\text{cf}}$, where $\mathbf{F}_{\text{cor}} = \int 2\, dm\,(\mathbf{v}' \times \boldsymbol{\omega}_0)$ and $\mathbf{F}_{\text{cf}} = \int dm\, \omega_0^2 \mathbf{r}$, where $\mathbf{r}$ is the position of an elemental mass of the rod (see figure) with respect to point O (disk's centre) and

$$\mathbf{v}' = \frac{d\mathbf{r}'}{dt}$$



As
$$\mathbf{r} = \mathbf{OP} = \mathbf{OA} + \mathbf{AP}$$

so,
$$\frac{d\mathbf{r}}{dt} = \frac{d(\mathbf{AP})}{dt} = \mathbf{v}' \ \ (\text{as } \mathbf{OA} \text{ is constant})$$

Since the rod is vibrating transversely, $\mathbf{v}'$ is directed perpendicular to the length of the rod. Hence $2\, dm\,(\mathbf{v}' \times \boldsymbol{\omega})$ for each elemental mass of the rod is directed along **PA**.

Therefore, the net torque of Coriolis about A becomes zero. The net torque of centrifugal force about point A,

$$\boldsymbol{\tau}_{\mathrm{cf}(A)} = \int \mathbf{AP} \times dm\, \omega_0^2 \mathbf{r} = \int \mathbf{AP} \times \left(\frac{m}{l}\right) ds\, \omega_0^2\, (\mathbf{OA} + \mathbf{AP})$$

$$= \int \mathbf{AP} \times \left(\frac{m}{l} ds\right) \omega_0^2\, \mathbf{OA} = \int \frac{m}{l}\, ds\, \omega_0^2 s\, a \sin\theta\, (-\mathbf{k})$$

$$= \frac{m}{l}\, \omega_0^2\, a \sin\theta (-\mathbf{k}) \int_0^l s\, ds = m\omega_0^2\, a\, \frac{l}{2} \sin\theta (-\mathbf{k})$$

So, $$\tau_{\mathrm{cf}(Z)} = \boldsymbol{\tau}_{\mathrm{cf}(A)} \cdot \mathbf{k} = -\, m\omega_0^2\, a\, \frac{l}{2} \sin\theta$$

According to the equation of rotational dynamics, $\tau_{A(z)} = I_A \alpha_z$.

or $$-\, m\omega_0^2\, a \frac{l}{2} \sin\theta = \frac{ml^2}{3}\, \ddot{\theta}$$

or $$\ddot{\theta} = -\, \frac{3}{2}\, \frac{\omega_0^2 a}{l} \sin\theta$$

Thus, for small θ $$\ddot{\theta} \cong -\frac{3}{2}\, \frac{\omega_0^2 a}{l}\, \theta$$

This implies that the frequency of oscillation is

$$\omega_0 = \sqrt{\frac{3\omega^2 a}{2l}}$$

4.54 The physical system consists of a pulley and the block. Choosing an inertial frame, let us direct the x-axis as shown in the figure.

Initially the system is in equilibrium position. Now from the condition of translation equilibrium for the block

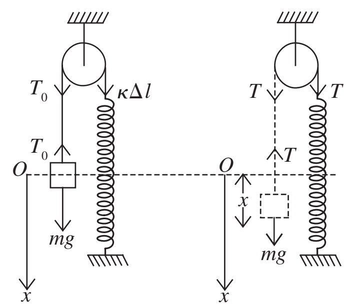
$$T_0 = mg \qquad (1)$$

Similarly for the rotational equilibrium of the pulley

$$\kappa \Delta l R = T_0 R$$

or $$T_0 = \kappa \Delta l \qquad (2)$$

From Eqs. (1) and (2) $$\Delta l = \frac{mg}{\kappa} \qquad (3)$$



Suppose the equilibrium of the system is disturbed in some way, in order to analyze its motion. At an arbitrary position shown in the figure, from Newton's second law of motion for the block

$$F_x = mw_x$$

$$mg - T = mw = m\ddot{x} \tag{4}$$

Similarly, for the pulley

$$N_Z = I\beta_Z$$

$$TR - \kappa\,(\Delta l + x)R = I\ddot{\theta} \tag{5}$$

$$w = \beta R \quad \text{or} \quad \ddot{x} = R\ddot{\theta} \tag{6}$$

From Eqs. (5) and (6) $$TR - \kappa\,(\Delta l + x)R = \frac{I}{R}\,\ddot{x} \tag{7}$$

Solving Eqs. (4) and (7) using the initial condition of the problem,

$$-\,\kappa R x = \left(mR + \frac{I}{R}\right)\ddot{x}$$

or $$\ddot{x} = -\left(\frac{\kappa}{m + (I/R^2)}\right)x$$

Hence, the sought time period

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m + I/R^2}{\kappa}} \quad \text{and} \quad \omega_0 = \sqrt{\frac{\kappa}{m + I/R^2}}$$

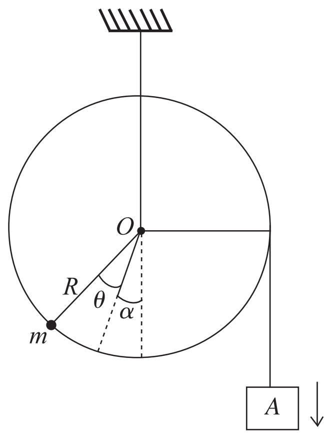
Note: *We may solve this problem by using the conservation of mechanical energy also.*

4.55 At the equilibrium position,

$$N_z = 0 \quad \text{(net torque about } O\text{)}$$

So, $$m_A gR - mg\,R\sin\alpha = 0 \quad \text{or} \quad m_A = m\sin\alpha \tag{1}$$

From the equation of rotational dynamics of a solid body about the stationary axis (say z-axis) of rotation, i.e., from $N_z = I\beta_Z$, when the pulley is rotated by the small angular displacement θ in clockwise sense relative to the equilibrium position (see figure), we get



$$m_A gR - mgR\sin(\alpha + \theta) = \left[\frac{MR^2}{2} + mR^2 + m_A R^2\right]\ddot{\theta}$$

Using Eq. (1)

$$mg\sin\alpha - mg\,(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$$

$$= \left\{\frac{MR + 2m\,(1 + \sin\alpha)\,R}{2}\right\}\ddot{\theta}$$

But for small θ, we may write $\cos\theta \cong 1$ and $\sin\theta \cong \theta$

Thus, we have

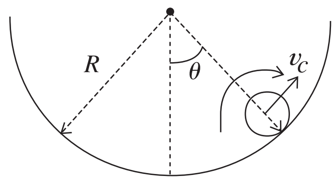
$$mg \sin \alpha - mg (\sin \alpha + \cos \alpha\theta) = \frac{\{MR + 2m(1 + \sin \alpha) R\}}{2} \ddot{\theta}$$

Hence, $$\ddot{\theta} = -\frac{2mg \cos \alpha}{[MR + 2m (1 + \sin \alpha) R]} \theta$$

Hence, the sought angular frequency,

$$\omega_0 = \sqrt{\frac{2mg \cos \alpha}{MR + 2mR (1 + \sin \alpha)}}$$

4.56 Let us locate a solid cylinder when it is displaced from its stable equilibrium position by the small angle θ during its oscillations (see figure). If v_0 is the instantaneous speed of the C.M. (C) of the solid cylinder which is in pure rolling, then its angular velocity about its own centre C is



$$\omega = \frac{v_C}{r} \tag{1}$$

Since C moves in a circle of radius $(R - r)$, the speed of C at the same moment can be written as

$$v_C = \dot{\theta} (R - r) \tag{2}$$

Thus, from Eqs. (1) and (2)

$$\omega = \dot{\theta} \frac{(R - r)}{r} \tag{3}$$

As the mechanical energy of oscillation of the solid cylinder is conserved, i.e.,

$$E = T + U = \text{constant}$$

So, $$\frac{1}{2} mv_C^2 + \frac{1}{2} I_C\omega^2 + mg (R - r) (1 - \cos \theta) = \text{constant}$$

(Here m is the mass of solid cylinder and I_C is the moment of inertia of the solid cylinder about an axis passing through its C.M. (C) and perpendicular to the plane of figure of solid cylinder.)

or $$\frac{1}{2} m\omega^2 r^2 + \frac{1}{2} \frac{mr^2}{2} \omega^2 + mg (R - r) (1 - \cos \theta) = \text{constant}$$

(using Eq. 1 and $I_C = mr^2/2$)

$$\frac{3}{4} r^2 (\dot{\theta})^2 \frac{(R - r)^2}{r^2} + g(R - r) (1 - \cos \theta) = \text{constant} \quad \text{(using Eq. 3)}$$

Differentiating with respect to time

$$\frac{3}{4}(R-r)\,2\dot{\theta}\,\ddot{\theta}+g\sin\theta\,\dot{\theta}=0$$

So, $$\ddot{\theta}=-\frac{2g}{3(R-r)}\theta \quad \text{(because for small } \theta,\ \sin\theta\cong\theta)$$

Thus, $$\omega_0=\sqrt{\frac{2g}{3(R-r)}}$$

Hence, the sought time period

$$T=\frac{2\pi}{\omega_0}=2\pi\sqrt{\frac{3(R-r)}{2g}}$$

4.57 Let κ_1 and κ_2 be the spring constants of left and right side springs. As the rolling of the solid cylinder is pure, its lowest point becomes the instantaneous centre of rotation. If θ is the small angular displacement of its uppermost point relative to its equilibrium position, the deformation of each spring becomes $2R\theta$. Since the mechanical energy of oscillation of the solid cylinder is conserved, $E=T+U=$ constant, i.e.,

$$\frac{1}{2}I_P\,(\dot{\theta})^2+\frac{1}{2}\kappa_1\,(2R\theta)^2+\frac{1}{2}\kappa_2\,(2R\theta)^2=\text{constant}$$

Differentiating with respect to time, we get

$$\frac{1}{2}I_P\,2\dot{\theta}\,\ddot{\theta}+\frac{1}{2}(\kappa_1+\kappa_2)\,4R^2\,2\,\theta\dot{\theta}=0$$

or $$\left(\frac{mR^2}{2}+mR^2\right)\ddot{\theta}+4R^2\kappa\theta=0$$

(because $I_P=I_C+mR^2=\dfrac{mR^2}{2}+mR^2$).

Hence, $$\ddot{\theta}=-\frac{8}{3}\frac{\kappa}{m}\theta$$

Thus, $$\omega_0=\frac{8\kappa}{3m}$$

and the sought time period,

$$T=\frac{2\pi}{\omega_0}=2\pi\sqrt{\frac{3m}{8\kappa}}=\pi\sqrt{\frac{3m}{2\kappa}}$$

4.58 In the C.M. frame (which is rigidly attached to the centre of mass of the two cubes) the cubes oscillate. We know that the kinetic energy of two-body system equals $1/2\ \mu v_{\text{rel}}^2$, where μ is the reduced mass and v_{rel} is the modulus of velocity of any one-body particle relative to other. From the conservation of mechanical energy of oscillation

$$\frac{1}{2}\kappa x^2 + \frac{1}{2}\mu\left\{\frac{d}{dt}(l_0 + x)\right\}^2 = \text{constant}$$

(Here l_0 is the natural length of the spring.)

Differentiating the above equation with respect to time, we get

$$\frac{1}{2}\kappa\, 2x\dot{x} + \frac{1}{2}\mu\, 2\dot{x}\ddot{x} = 0 \quad \left[\text{because } \frac{d(l_0 + x)}{dt} = \dot{x}\right]$$

Thus, $$\ddot{x} = -\frac{\kappa}{\mu}x \quad \left(\text{where } \mu = \frac{m_1 m_2}{m_1 + m_2}\right)$$

Hence the natural frequency of oscillation

$$\omega_0 = \sqrt{\frac{\kappa}{\mu}}$$

4.59 Suppose balls 1 and 2 are displaced by x_1 and x_2 respectively from their initial positions. Then the energy is

$$E = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}k(x_1 - x_2)^2 = \frac{1}{2}m_1 v_1^2$$

Also total momentum is

$$m_1\dot{x}_1 + m_2\dot{x}_2 = m_1 v_1$$

If $$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad \text{and} \quad x = x_1 - x_2$$

Then $$x_1 = X + \frac{m_2}{m_1 + m_2}x \quad \text{and} \quad x_2 = X - \frac{m_1}{m_1 + m_2}x$$

$$E = \frac{1}{2}(m_1 + m_2)\dot{X}^2 + \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{x}^2 + \frac{1}{2}kx^2$$

Now, from total momentum $$\dot{X} = \frac{m_1 v_1}{m_1 + m_2}$$

So, $$\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}m_1 v_1^2 - \frac{1}{2}\frac{m_1^2 v_1^2}{m_1 + m_2} = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}v_1^2$$

(a) From the above equation, we see

$$\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{3 \times 24}{2}} = 6 \text{ s}^{-1}, \text{ when } \mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{2}{3} \text{ kg}$$

(b) The energy of oscillation is

$$\frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} v_1^2 = \frac{1}{2} \times \frac{2}{3} \times (0.12)^2 = 48 \times 10^{-4} = 4.8 \text{ mJ}$$

We have $\qquad x = a \sin(\omega t + \alpha)$

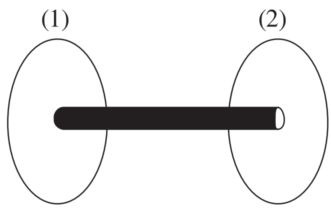
Initially, $x = 0$ at $t = 0$, so $\alpha = 0$. Then, $x = a \sin \omega t$. Also $x = v_1$ at $t = 0$. So, $\omega a = v_1$ and hence,

$$a = \frac{v_1}{\omega} = \frac{12}{6} = 2 \text{ cm}$$

4.60 Suppose the disk 1 rotates by angle θ_1 and the disk 2 by angle θ_2 in the opposite sense. Then total torsion of the rod $= \theta_1 + \theta_2$ and torsional potential energy is

$$\frac{1}{2} \kappa (\theta_1 + \theta_2)^2$$

The kinetic energy of the system (neglecting the moment of inertia of the rod) is



$$\frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2$$

So, total energy of the rod

$$E = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 + \frac{1}{2} \kappa (\theta_1 + \theta_2)^2$$

We can put the total angular momentum of the rod equal to zero since the frequency associated with the rigid rotation of the whole system must be zero (and is known).

Thus, $$I_1 \dot{\theta}_1 = I_2 \dot{\theta}_2 \quad \text{or} \quad \frac{\dot{\theta}_1}{1/I_1} = \frac{\dot{\theta}_2}{1/I_2} = \frac{\dot{\theta}_1 + \dot{\theta}_2}{1/I_1 + 1/I_2}$$

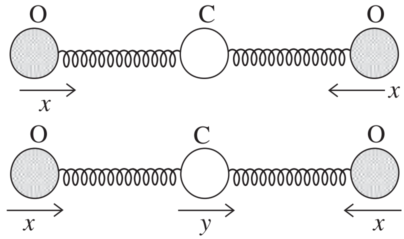
So, $$\dot{\theta}_1 = \frac{I_2}{I_1 + I_2} (\dot{\theta}_1 + \dot{\theta}_2) \quad \text{and} \quad \dot{\theta}_2 = \frac{I_1}{I_1 + I_2} (\dot{\theta}_1 + \dot{\theta}_2)$$

and $$E = \frac{1}{2} \frac{I_1 I_2}{I_1 + I_2} (\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2} \kappa (\theta_1 + \theta_2)^2$$

The angular oscillation frequency corresponding to this is

$$\omega^2 = \frac{\kappa}{I_1 I_2/(I_1 + I_2)} = \frac{\kappa}{I'} \quad \text{and} \quad T = 2\pi \sqrt{\frac{I'}{\kappa}} \quad \left(\text{where } I' = \frac{I_1 I_2}{I_1 + I_2}\right)$$

4.61 In the first mode, the carbon atom remains fixed and the oxygen atoms move in equal and opposite steps. Then total energy is



$$\frac{1}{2}\,2m_o\,\dot{x}^2 + \frac{1}{2}\,2\kappa x^2$$

where x is the displacement of one of the O atoms (say left one).

Thus, $$\omega_1^2 = \frac{\kappa}{m_o}$$

In the second mode, the oxygen atoms move in equal steps in the same direction but the carbon atom moves in such a way so as to keep the centre of mass fixed.

Thus, $$2m_o x + m_C y = 0 \quad \text{or} \quad y = -\frac{2m_o}{m_C}\,x$$

$$\text{K.E.} = \frac{1}{2}\,2m_o\dot{x} + \frac{1}{2}\,m_C\left(\frac{2m_o}{m_C}\,\dot{x}\right)^2 = \frac{1}{2}\,2m_o\dot{x}^2 + \frac{1}{2}\,2m_o\,\frac{2m_o}{m_C}\,\dot{x}^2 = \frac{1}{2}\,2m_o\left(1+\frac{2m_o}{m_C}\right)\dot{x}^2$$

$$\text{P.E.} = \frac{1}{2}\,\kappa\left(1+\frac{2m_o}{m_C}\right)^2 x^2 + \frac{1}{2}\,\kappa\left(1+\frac{2m_o}{m_C}\right)^2 x^2 = \frac{1}{2}\,2\kappa\left(1+\frac{2m_o}{m_C}\right)^2 x^2$$

Thus, $$\omega_2^2 = \frac{\kappa}{m_o}\left(1+\frac{2m_o}{m_C}\right) \quad \text{and} \quad \omega_2 = \omega_1\sqrt{1+\frac{2m_o}{m_C}}$$

Hence, $$\omega_2 = \omega_1\sqrt{1+\frac{32}{12}} = \omega_1\sqrt{\frac{11}{3}} \approx 1.91\,\omega_1$$

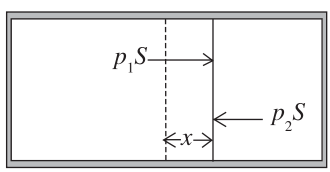
4.62 Let us displace the piston through small distance x towards right, then from $F_x = mw_x$

$$(p_1 - p_1)S = -\,m\ddot{x} \tag{1}$$

But, the process is adiabatic, so from pV^γ = constant

$$p_2 = \frac{p_0V_0^\gamma}{(V_0 - Sx)^\gamma} \quad \text{and} \quad p_1 = \frac{p_0V_0^\gamma}{(V_0 + Sx)^\gamma}$$

as the new volumes of the left and the right parts are now $(V_0 + Sx)$ and $(V_0 - Sx)$, respectively.



So, Eq. (1) becomes

$$\frac{p_0V_0^\gamma S}{m}\left\{\frac{1}{(V_0 - Sx)^\gamma} - \frac{1}{(V_0 + Sx)^\gamma}\right\} = -\,\ddot{x}$$

or $$\frac{p_0V_0^\gamma S}{m}\left\{\frac{(V_0 + Sx)^\gamma - (V_0 - Sx)^\gamma}{(V_0^2 - S^2x^2)^\gamma}\right\} = -\,\ddot{x}$$

or
$$\frac{p_0 V_0^{\gamma} S}{m}\left(\frac{\left(1+\frac{\gamma S\dot{x}}{V_0}\right)-\left(1-\frac{\gamma Sx}{V_0}\right)}{V_0^{\gamma}\left(1-\frac{\gamma S^2x^2}{V_0^2}\right)}\right)=-\ddot{x}$$

Neglecting the term $\frac{\gamma S^2 x^2}{V_0^2}$ in the denominator, as it is very small, we get

$$\ddot{x}=-\frac{2p_0 S^2\gamma x}{mV_0}$$

which is the equation for S.H.M. and hence the oscillating frequency will be

$$\omega_0 = S\sqrt{\frac{2p_0\gamma}{mV_0}}$$

4.63 In the absence of the charge, the oscillation period of the ball

$$T = 2\pi\sqrt{\frac{l}{g}}$$

when we impart the charge q to the ball, it will be influenced by the induced charges on the conducting plane. From the electric image method, the electric force on the ball by the plane equals $q^2/4\pi\varepsilon_0(2h)^2$ and is directed downward. Thus, in this case the effective acceleration of the ball will be

$$g' = g + \frac{q^2}{16\pi\varepsilon_0 mh^2}$$

and the corresponding time period

$$T' = 2\pi\sqrt{\frac{l}{g'}} = 2\pi\sqrt{\frac{l}{g+(q^2/16\pi\varepsilon_0 mh^2)}}$$

From the condition of the problem

$$T = \eta T'$$

So, $$T^2 = \eta^2 T'^2 \quad \text{or} \quad \frac{1}{g} = \eta^2\left(\frac{1}{g+(q^2/16\pi\varepsilon_0 mh^2)}\right)$$

Thus, on solving

$$q = 4h\sqrt{\pi\varepsilon_0 mg(\eta^2-1)} = 2\ \mu\text{C}$$

4.64 In a magnetic field of induction B the couple on the magnet is $-MB\sin\theta = -MB\,\theta$. Equating this to $I\ddot{\theta}$, we get

$$I\ddot{\theta} + MB\theta = 0$$

or $$\omega^2 = \frac{MB}{I} \quad \text{or} \quad T = 2\pi\sqrt{\frac{I}{MB}}$$

Given $$T_2 = \frac{T_1}{\eta}$$

or $$\sqrt{\frac{1}{B_2}} = \sqrt{\frac{1}{B_1}} \cdot \frac{1}{\eta} \quad \text{or} \quad \frac{1}{B_2} = \frac{1}{B_1} \cdot \frac{1}{\eta^2} \quad \text{or} \quad B_2 = \eta^2 B_1$$

The induction of the field increased $\eta^2 = 25$ times.

4.65 We have in the circuit at a certain instant of time (t), from Faraday's law of electromagnetic induction

$$L\frac{dI}{dt} = Bl\frac{dx}{dt} \quad \text{or} \quad L\,dI = Bl\,dx$$

As at $t = 0$, $x = 0$, so

$$LI = Blx \quad \text{or} \quad I = \frac{Bl}{L}x \tag{1}$$

For the rod from the second law of motion, $F_x = mw_x$, we have

$$-IlB = m\ddot{x}$$

Using Eq. (1), we get $$\ddot{x} = -\left(\frac{l^2B^2}{mL}\right)x = -\omega_0^2 x \tag{2}$$

where $$\omega_0 = \frac{lB}{\sqrt{mL}}$$

The solution of the above differential equation is of the form

$$x = a\sin(\omega_0 t + \alpha)$$

From the initial condition, at $t = 0$, $x = 0$, so $\alpha = 0$.

Hence, $$x = a\sin\omega_0 t \tag{3}$$

Differentiating with respect to time,

$$\dot{x} = a\omega_0\cos\omega_0 t$$

But from the initial condition of the problem at $t = 0$, $\dot{x} = v_0$.

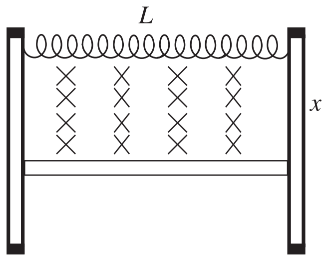
Thus, $$v_0 = a\omega_0 \quad \text{or} \quad a = \frac{v_0}{\omega_0} \tag{4}$$

Putting the value of a from Eq. (4) into Eq. (3), we obtain

$$x = \frac{v_0}{\omega_0}\sin\omega_0 t$$

4.66 As the connector moves, an emf is set up in the circuit and a current flows, since the emf is $\xi = -Bl\dot{x}$, we must have

$$-Bl\dot{x} + dI/dt = 0$$



So, $$I = \frac{Blx}{L}$$

provided x is measured from the initial position.

We then have,

$$m\ddot{x} = -\frac{Blx}{L} \cdot Bl + mg$$

Since by Lenz's law, the induced current will oppose downward sliding. Finally,

$$\ddot{x} + \frac{(Bl)^2}{mL} x = g$$

on putting $$\omega_0 = \frac{Bl}{\sqrt{mL}}$$

A solution of this equation is $x = g/\omega_0^2 + A \cos(\omega_0 t + \alpha)$.

But $x = 0$ and $\dot{x} = 0$ at $t = 0$.

This gives $$x = \frac{g}{\omega_0^2}(1 - \cos \omega_0 t)$$

4.67 We are given $x = a_0 e^{-\beta t} \sin \omega t$.

(a) The velocity of the point at $t = 0$ is obtained from

$$\mathbf{v}_0 = (\dot{x})_{t=0} = \omega a_0$$

The term "oscillation amplitude at the moment $t = 0$" is meaningless. Probably the implication is the amplitude for $t \ll 1/\beta$. Then $x \approx a_0 \sin \omega t$ and amplitude is a_0.

(b) When the displacement is an extremum $\dot{x} = (-\beta a_0 \sin \omega t + \omega a_0 \cos \omega t)\, e^{-\beta t} = 0$

Then, $$\tan \omega t = \frac{\omega}{\beta}$$

or $$t_n = \tan^{-1}\frac{\omega}{\beta} + n\pi \quad (\text{where } n = 0, 1, 2, \ldots)$$

4.68 Given $$\varphi = \varphi_0 e^{-\beta t} \cos \omega t$$

We have $$\dot{\varphi} = -\beta\varphi - \omega\varphi_0\, e^{-\beta t} \sin \omega t$$

$$\ddot{\varphi} = -\beta\dot{\varphi} + \beta\omega\, \varphi_0\, e^{-\beta t} \sin \omega t - \omega^2\, \varphi_0 e^{-\beta t} \cos \omega t$$

$$= \beta^2\varphi + 2\beta\omega\, \varphi_0\, e^{-\beta t} \sin \omega t - \omega^2\varphi$$

So,

(a) $(\dot{\varphi})_0 = -\beta\varphi_0, (\ddot{\varphi})_0 = (\beta^2 - \omega^2)\varphi_0$

(b) $\dot{\varphi} = -\varphi_0 e^{-\beta t}\,(\beta \cos \omega t + \omega \sin \omega t)$ becomes maximum (or minimum) when

$$\ddot{\varphi} = \varphi_0(\beta^2 - \omega^2)e^{-\beta t} \cos \omega t + 2\beta\omega\varphi_0\, e^{-Br} \sin \omega t = 0$$

or
$$\tan \omega t = \frac{\omega^2 - \beta^2}{2\beta\omega}$$

and
$$t_n = \frac{1}{\omega}\left[\tan^{-1} \frac{\omega^2 - \beta^2}{2\beta\omega} + n\pi\right] \quad (\text{where } n = 0, 1, 2, \ldots)$$

4.69 We write
$$x = a_0 e^{-\beta t} \cos(\omega t + \alpha)$$

(a) $x(0) = 0 \Rightarrow \alpha = \pm\pi/2 \Rightarrow x = \mp a_0 e^{-\beta t} \sin \omega t.$

Since a_0 is $+ve$, we must choose the upper sign if $\dot{x}(0) < 0$ and the lower sign if $\dot{x}(0) > 0$. Thus,

$$a_0 = \frac{|\dot{x}(0)|}{\omega} \quad \text{and} \quad \alpha = \begin{cases} +\pi/2 \text{ if } \dot{x}(0) < 0 \\ -\pi/2 \text{ if } \dot{x}(0) > 0 \end{cases}$$

(b) We write

$$x = \text{Re}\, Ae^{-\beta t + i\omega t} \quad \text{and} \quad A = a_0 e^{i\alpha}$$

Then,
$$\dot{x} = v_x = \text{Re}\,(-\beta + i\omega)Ae^{-\beta t + i\omega t}$$

From $v_x(0) = 0$, we get $\text{Re}\,(-\beta + i\omega)A = 0$.

This implies $A = \pm i(\beta \pm i\omega)B$ where B is real and positive. Also $x_0 = \text{Re}\, A = \mp\omega B$.

Thus, $B = |x_0|/\omega$ with $+ve$ sign in A if $x_0 < 0$ and $-ve$ sign in A if $x_0 > 0$.

So,
$$A = \pm i \frac{\beta + i\omega}{\omega} |x_0| = \left(\mp 1 + \pm \frac{i\beta}{\omega}\right) |x_0|$$

Finally
$$a_0 = \sqrt{1 + \left(\frac{\beta}{\omega}\right)^2}\, |x_0|$$

$$\tan \alpha = \frac{-\beta}{\omega} \quad \text{or} \quad \alpha = \tan^{-1}\left(\frac{-\beta}{\omega}\right)$$

α is in the fourth quadrant $(-\pi/2 < \alpha < 0)$ if $x_0 > 0$ and α is in the second quadrant $(\pi/2 < \alpha < \pi)$ if $x_0 < 0$.

4.70 We write $$x = a_0 e^{-\beta t} \cos(\omega t + \alpha)$$

Then, $$(\dot{x})_{t=0} = -\beta a_0 \cos\alpha - \omega a_0 \sin\alpha = 0$$

or $$\tan\alpha = -\frac{\beta}{\omega}$$

Also, $$(x)_{t=0} = a_0 \cos\alpha \, \frac{a_0}{\eta}$$

$$\sec^2\alpha = \eta^2 \quad \text{and} \quad \tan\alpha = -\sqrt{\eta^2 - 1}$$

Thus, $$\beta = \omega\sqrt{\eta^2 - 1} = 5 \text{ s}^{-1}$$

(We have taken the amplitude at $t = 0$ to be a_0.)

4.71 We write $$x = a_0 \, e^{-\beta t} \cos(\omega t + \alpha)$$

$$= \text{Re}\, A e^{-\beta t + i\omega t},\ A = a_0 e^{i\alpha}$$

$$\dot{x} = \text{Re}\, A(-\beta + i\omega)\, e^{-\beta t + i\omega t}$$

Velocity amplitude as a function of time is defined in the following manner.

Put $t = t_0 + \tau$, then

$$x = \text{Re}\, A e^{-\beta(t_0 + \tau)} \, e^{i\omega(t_0 + \tau)}$$

$$\approx \text{Re}\, A e^{-\beta t_0} \, e^{i\omega t_0 + i\omega\tau} \approx \text{Re}\, A e^{-\beta t_0} \, e^{i\omega t} \quad (\text{for } \tau \ll 1/\beta)$$

This means that the displacement amplitude around the time t_0 is $a_0 e^{-\beta t_0}$ and we can say that the displacement amplitude at time t is $a_0 e^{-\beta t}$. Similarly for the velocity amplitude

(a) Velocity amplitude at time t is

$$a_0 \sqrt{\beta^2 + \omega^2}\, e^{-\beta t}$$

Since $A(-\beta + i\omega) = a_0 e^{i\alpha}(-\beta + i\omega)$

$$= a_0 \sqrt{\beta^2 + \omega^2}\, e^{-i\gamma} \quad (\text{where } \gamma \text{ is another constant})$$

(b) $x(0) = 0 \Rightarrow \text{Re}\, A = 0$ or $A = \pm i a_0$ (where a_0 is real and positive)

Also, $$v_x = (0) = \dot{x}_0 = \text{Re} \pm i a_0 (-\beta + i\omega)$$

$$= \mp \omega a_0$$

Thus, $a_0 = |\dot{x}_0|/\omega$ and we take $-ve$ sign if $x_0 < 0$ and $+ve$ sign if $x_0 > 0$. Finally, the velocity amplitude is obtained as

$$|\dot{x}_0| \sqrt{1 + \left(\frac{\beta}{\omega}\right)^2}\; e^{-\beta t}$$

4.72 The first oscillation decays faster in time. But if one takes the natural time scale, the period T for each oscillation, the second oscillation attenuates faster during that period.

4.73 By definition of the logarithmic decrement ($\lambda = \beta\, 2\pi/\omega$), we get for the original decrement

$$\lambda_0 = \beta \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}} \quad \text{and finally} \quad \lambda = \frac{2\pi n\beta}{\sqrt{\omega_0^2 - n^2\beta^2}}$$

Now, $$\frac{\beta}{\sqrt{\omega_0^2 - \beta^2}} = \frac{\lambda_0}{2\pi} \quad \text{or} \quad \frac{\beta}{\omega_0} = \frac{\lambda_0/2\pi}{\sqrt{1 + (\lambda_0/2\pi)^2}}$$

So, $$\frac{\lambda/2\pi}{\sqrt{1 + (\lambda_0/2\pi)^2}} = \frac{n(\lambda_0/2\pi)}{\sqrt{1 + (\lambda_0/2\pi)^2}}$$

Hence, $$\frac{\lambda}{2\pi} = \frac{n\lambda_0/2\pi}{\sqrt{1 - (n^2 - 1)(\lambda_0/2\pi)^2}}$$

For critical damping $$\omega_0 = n_c\beta$$

So, $$\frac{1}{n_c} = \frac{\beta}{\omega_0} = \frac{\lambda_0/2\pi}{\sqrt{1 + (\lambda_0/2\pi)^2}} \quad \text{or} \quad n_c = \sqrt{1 + \left(\frac{2\pi}{\lambda_0}\right)^2}$$

4.74 The equation for the dead weight is

$$m\ddot{x} + 2\beta m\dot{x} + m\omega_0^2\, x = mg$$

So, $$\Delta x = \frac{g}{\omega_0^2} \quad \text{or} \quad \omega_0^2 = \frac{g}{\Delta x}$$

Now, $$\lambda = \frac{2\pi\beta}{\omega} = \frac{2\pi\beta}{\sqrt{\omega_0^2 - \beta^2}} \quad \text{or} \quad \frac{\omega_0}{\sqrt{\omega_0^2 - \beta^2}} = \sqrt{1 + \left(\frac{\lambda}{2\pi}\right)^2}$$

Thus, $$T = \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}} = \frac{2\pi}{\omega_0}\sqrt{1 + \left(\frac{\lambda}{2\pi}\right)^2}$$

$$= 2\pi\sqrt{\frac{\Delta x}{g}}\sqrt{1 + \left(\frac{\lambda}{2\pi}\right)^2} = \sqrt{\frac{\Delta x}{g}(4\pi^2 + \lambda^2)} = 0.70 \text{ s}$$

4.75 The displacement amplitude decreases η times every n oscillations. Thus,

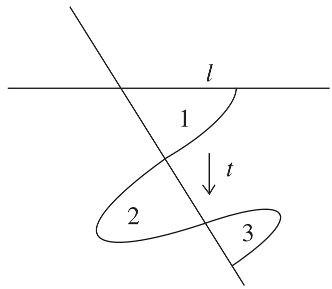
$$\frac{1}{\eta} = e^{-\beta\cdot 2\pi/\omega\cdot n}$$

or $$\frac{2\pi n\beta}{\omega} = \ln \eta \quad \text{or} \quad \frac{\beta}{\omega} = \frac{\ln \eta}{2\pi n}$$

So, $$Q = \frac{\omega}{2\beta} = \frac{\pi n}{\ln \eta} \approx 499$$

4.76 From $x = a_0 e^{-\beta t} \cos(\omega t + \alpha)$, we get using $(x)_{t=0} = l = a_0 \cos \alpha$

$$0 = (\dot{x})_{t=0} = -\beta a_0 \cos \alpha - \omega a_0 \sin \alpha$$



Then, $$\tan \alpha = -\frac{\beta}{\omega} \quad \text{or} \quad \cos \alpha = \frac{\omega}{\sqrt{\omega^2 + \beta^2}}$$

and $$x = \frac{l\sqrt{\omega^2 + \beta^2}}{\omega} e^{-\beta t} \cos\left(\omega t - \tan^{-1} \frac{\beta}{\omega}\right)$$

$$x = 0 \quad \text{at} \quad t = \frac{1}{\omega}\left(n\pi + \frac{\pi}{2} + \tan^{-1} \frac{\beta}{\omega}\right)$$

Total distance traveled in the first lap $= l$.

To get the maximum displacement in the second lap we note that

$$\dot{x} = \left[-\beta \cos\left(\omega t - \tan^{-1} \frac{\beta}{\omega}\right) - \omega \sin\left(\omega t - \tan^{-1} \frac{\beta}{\omega}\right)\right]$$

Now, $$x \frac{l\sqrt{\omega^2 + \beta^2}}{\omega} e^{-\beta t} = 0$$

when $\omega t = \pi, 2\pi, 3\pi, \ldots$.

Thus, $$\ddot{x}_{max} = -a_0 e^{-\pi\beta/\omega} \cos \alpha = -l e^{-\pi\beta/\omega} \quad (\text{for } t = \pi/\omega)$$

So, distance traversed in the second lap $= 2l e^{-\pi\beta/\omega}$.

Continuing total distance traversed $= l + 2l\, e^{-\pi\beta/\omega} + 2l e^{-2\pi\beta/\omega} + \ldots$

$$= l + \frac{2l e^{-\pi\beta/\omega}}{1 - e^{-\beta\pi/\omega}}$$

$$= l\, \frac{1 + e^{-\lambda/2}}{1 - e^{-\lambda/2}}$$

where $\lambda = 2\pi\beta/\omega$ is the logarithmic decrement. Substitution gives distance $= 2$ m.

4.77 For an undamped oscillator, the mechanical energy $E = 1/2\, m\dot{x}^2 + 1/2\, m\omega_0^2 x^2$ is conserved. For a damped oscillator

$$x = a_0\, e^{-\beta t} \cos(\omega t + \alpha), \quad \omega = \sqrt{\omega_0^2 - \beta^2}$$

and
$$E(t) = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\omega_0^2 x^2$$
$$= \frac{1}{2} ma_0^2\ e^{-2\beta r}\ \beta^2 \cos^2(\omega t + \alpha) + 2\beta\omega \cos(\omega t + \alpha)$$
$$\times \sin(\omega t + \alpha) + \omega^2\ \sin^2(\omega t + \alpha) + \frac{1}{2} ma_0^2\ \omega_0^2\ e^{-2\beta t}\ \cos^2(\omega t + \alpha)$$
$$= \frac{1}{2} ma_0^2\ \omega_0^2\ e^{-2\beta t} + \frac{1}{2} ma_0^2\ \beta^2\ e^{-2\beta t}\ \cos(2\omega t + 2\alpha)$$
$$+ \frac{1}{2} ma_0^2\ \beta\ \omega e^{-2\beta t}\ \sin(2\omega t + 2\alpha)$$

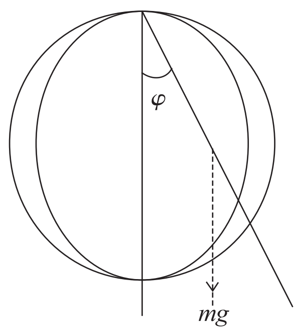
If $\beta \ll \omega$, then the average of the last two terms over many oscillations about the time t will vanish and
$$< E(t) > \cong \frac{1}{2} m\, a_0^2\ \omega_0^2\ e^{-2\beta t}$$

This is the relevant mechanical energy. In time τ this deceases by a factor $1/\eta$, so
$$e^{-2\beta\tau} = \frac{1}{\eta} \quad \text{or} \quad \tau = \frac{\ln \eta}{2\beta}$$
$$\beta = \frac{\ln \eta}{2\tau}$$

and
$$\lambda = \frac{2\pi\beta}{\sqrt{\omega_0^2 - \beta^2}} = \frac{2\pi}{\sqrt{(\omega_0/\beta)^2 - 1}} = \frac{2\pi}{\sqrt{[4g\tau^2/l(\ln \eta)^2] - 1}} \quad \left(\text{since } \omega_0^2 = \frac{g}{l}\right)$$

and
$$Q = \frac{\pi}{\lambda} = \frac{1}{2}\sqrt{\frac{4g\tau^2}{l(\ln \eta)^2} - 1} \cong 130$$

4.78 The restoring couple is
$$\Gamma = -mgR \sin \varphi \cong -mg\ R\varphi$$



The moment of inertia is
$$I = \frac{3mR^2}{2}$$

Thus, for undamped oscillations
$$\frac{3\,mR^2}{2}\ \ddot{\varphi} + mgR\,\varphi = 0$$

So, $$\omega_0^2 = \frac{2}{3}\frac{g}{R}$$

Also $$\lambda = \frac{2\pi\beta}{\omega} = \frac{2\pi\beta}{\sqrt{\omega_0^2 - \beta^2}}$$

Hence, $$\frac{\beta}{\sqrt{\omega_0^2 - \beta^2}} = \frac{\lambda}{2\pi} \quad \text{or} \quad \frac{\omega_0}{\sqrt{\omega_0 - \beta^2}} = \sqrt{1 + \left(\frac{\lambda}{2\pi}\right)^2}$$

Finally the period T of small oscillation comes to

$$T = \frac{2\pi}{\omega} = \frac{2\,\pi}{\omega_0} \times \frac{\omega_0}{\sqrt{\omega_0^2 - \beta^2}}\; 2\pi\sqrt{\frac{3R}{2g}\left(1 + \left(\frac{\lambda}{2\pi}\right)^2\right)}$$

$$= \sqrt{\frac{3R}{2g}(4\pi^2 + \lambda^2)} = 0.90 \text{ s}$$

4.79 Let us calculate the moment G_1 of all the resistive forces on the disk. When the disk rotates an element ($r\, dr\, d\theta$) with coordinates (r, θ) has velocity $r\dot{\varphi}$, where φ is the instantaneous angle of rotation from the equilibrium position and r is measured from the centre. Then,

$$G_1 = \int_0^{2\pi} d\theta \int_0^R dr \cdot r \cdot (F_1 \times r)$$

$$= \int_0^R \eta r\, \dot{\varphi} r^2 d \times 2\pi = \frac{\eta\pi R^4}{2}\,\dot{\varphi}$$

Also, moment of inertia $= mR^2/2$.

Thus, $$\frac{mR^2}{2}\,\ddot{\varphi} + \frac{\pi\eta R^4}{2}\,\dot{\varphi} + \alpha\varphi = 0$$

or $$\ddot{\varphi} + 2\frac{\pi\eta R^2}{2m}\,\dot{\varphi} + \frac{2\,\alpha}{mR^2}\,\varphi = 0$$

Hence, $$\omega_0^2 = \frac{2\alpha}{mR^2} \quad \text{and} \quad \beta = \frac{\pi\eta R^2}{2\,m}$$

and angular frequency $$\omega = \sqrt{\left(\frac{2\alpha}{mR^2}\right) - \left(\frac{\pi\eta R^2}{m}\right)^2}$$

(Normally by frequency we mean $\omega/2\pi$.)

4.80 From the law of viscosity, force per unit area = $\eta \, dv/dx$, so when the disk executes torsional oscillations the resistive couple on it is

$$\int_0^R \eta \cdot 2\pi r \cdot \frac{r\varphi}{h} \cdot r \cdot dr \times 2 = \frac{\eta \pi R^4}{h} \dot{\varphi} \qquad \text{(where } \varphi \text{ is torsion)}$$

(Here factor 2 is for the two sides of the disk; see the figure in the book.)

The equation of motion is

$$I\ddot{\varphi} + \frac{\eta \pi R^4}{h} \dot{\varphi} + c\varphi = 0$$

Comparing with $\qquad \ddot{\varphi} + 2\beta \dot{\varphi} + \omega_0^2 \varphi = 0$

we get, $\qquad \beta = \dfrac{\eta \pi R^4}{2hI}$

Now the logarithmic decrement λ is given by

$$\lambda = \beta T \quad \text{(where } T = \text{time period)}$$

Thus, $\qquad \eta = \dfrac{2\lambda hI}{\pi R^4 T}$

4.81 If φ = angle of deviation of the frame from its normal position, then emf

$$\varepsilon = Ba^2 \dot{\varphi}$$

is induced in the frame in the displaced position and a current $\varepsilon/R = B\,a^2\dot{\varphi}/R$ flows in it. A couple

$$\frac{B\,a^2\dot{\varphi}}{R} \cdot B \cdot a \cdot a = \frac{B^2a^4}{R} \dot{\varphi}$$

then acts on the frame in addition to any elastic restoring couple $c\varphi$. We write the equation of the frame as

$$I\ddot{\varphi} + \frac{B^2a^4}{R} \dot{\varphi} + c\varphi = 0$$

Thus, $\beta = B^2a^4/2IR$, where β is the damping coefficient.

Amplitude of oscillation dies out according to $e^{-\beta t}$, so time required for the oscillation to decrease e-fold is

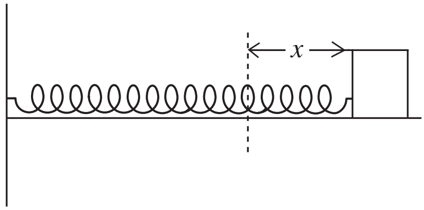
$$\frac{1}{\beta} = \frac{2\,IR}{B^2\,a^4}$$

4.82 We shall denote the stiffness constant by κ. Suppose the spring is stretched by x_0. The bar is then subjected to two horizontal forces: restoring force $-\kappa x$ and friction kmg opposing the motion of the bar. If

$$x_0 > \frac{kmg}{\kappa} = \Delta$$

the bar will come back. If $x_0 \le \Delta$, the bar will stay put. The equation of the bar when it is moving to the left is

$$m\ddot{x} = -\kappa x + kmg$$



This equation has the solution

$$x = \Delta + \left(x_0 - \Delta\right) \cos \sqrt{\frac{k}{m}}\, t$$

where we have used $x = x_0$, $\dot{x} = 0$ at $t = 0$. This solution is only valid till the bar comes to rest. This happens at

$$t_1 = \frac{\pi}{\sqrt{k/m}}$$

and at the time $x = x_1 = 2\Delta - x_0$. If $x_0 > 2\Delta$ the tendency of the bar will now be to move to the right. (If $\Delta < x_0 < 2\Delta$ the bar will stay put now.) Now the equation for rightward motion becomes

$$m\ddot{x} = -\kappa x - kmg$$

(The friction force has reversed.)

We notice that the bar will move to the right only if

$$\kappa\,(x_0 - 2\Delta) > kmg, \quad \text{i.e.,} \quad x_0 > 3\Delta$$

In this case, the solution is

$$x = -\Delta + (x_0 - 3\Delta) \cos \sqrt{\frac{k}{m}}\, t$$

Since $x = 2\Delta - x_0$ and $\dot{x} = 0$ at $t = t_1 = \dfrac{\pi}{\sqrt{k/m}}$ the bar will next come to rest at

$$t = t_2 = \frac{2\pi}{\sqrt{k/m}}$$

and at that instant $x = x_2 = x_0 - 4\Delta$. However, the bar will stay put unless $x_0 > 5\Delta$.

Thus,

(a) Time period of one full oscillation $= \dfrac{2\pi}{\sqrt{k/m}} = 0.28$ s.

(b) There is no oscillation if $0 < x_0 < \Delta$; one half oscillation if $\Delta < x_0 < 3\Delta$; two half oscillations if $3\Delta < x_0 < 5\Delta$, and so on

We can say that the number of full oscillations is one half of the integer n,

$$n = \left[\frac{x_0 - \Delta}{2\Delta}\right]$$

(where for any quantity a, $[a]$ = the smallest non-negative integer greater than a).

4.83 The equation of motion of the ball is

$$m(\ddot{x} + \omega_0^2 x) = F_0 \cos \omega t$$

This equation has the solution

$$x = A \cos(\omega_0 t + \alpha) + B \cos \omega t$$

where A and α are arbitrary and B is obtained by substitution in the above equation as

$$B = \frac{F_0/m}{\omega_0^2 - \omega^2}$$

The condition $x = 0,\ \dot{x} = 0$ at $t = 0$ gives

$$A \cos\alpha + \frac{F_0/m}{\omega_0^2 - \omega^2} = 0 \quad \text{and} \quad -\omega_0 A \sin \alpha = 0$$

This gives for $\alpha = 0$, $$A = -\frac{F_0/m}{\omega_0^2 - \omega^2} = \frac{F_0/m}{\omega^2 - \omega_0^2}$$

Finally, $$x = \frac{F_0/m}{\omega^2 - \omega_0^2}(\cos \omega_0 t - \cos \omega t)$$

4.84 We have to look for solutions of the equation

$$m\ddot{x} + kx = F \quad (\text{for } 0 < t < \tau)$$

$$m\ddot{x} + kx = 0 \quad (\text{for } t > \tau)$$

subjected to $x(0) = \dot{x}(0) = 0$, where F is constant.

The solution of this equation will be sought in the form

$$x = \frac{F}{k} + A\cos(\omega_0 t + \alpha) \quad (\text{for } 0 \le t \le \tau)$$

$$x = B\cos[\omega_0 (t - \tau) + \beta] \quad (\text{for } t > \tau)$$

A and α will be determined from the boundary condition at $t = 0$.

$$0 = \frac{F}{k} + A\cos\alpha$$

$$0 = -\omega_0 A \sin\alpha$$

Thus, $\alpha = 0$ and $A = -\dfrac{F}{k}$

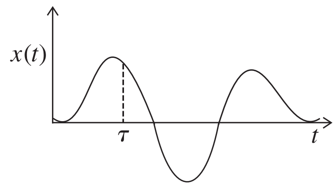
and $x = \dfrac{F}{k}(1 - \cos\omega_0 t)$ (for $0 \le t < \tau$)

B and β will be determined by the continuity of x and $\dot{x}$ at $t = \tau$. Thus,

$$\frac{F}{k}(1 - \cos\omega_0\tau) = B\cos\beta \quad \text{and} \quad \omega_0\frac{F}{k}\sin\omega_0\tau = -\omega_0 B\sin\beta$$

Thus, $B^2 = \left(\dfrac{F}{k}\right)^2(2 - 2\cos\omega_0\tau)$

or $B = 2\dfrac{F}{k}\left|\sin\dfrac{\omega_0\tau}{2}\right|$



4.85 For the spring, $mg = \kappa\Delta l$, where κ is its stiffness coefficient. Thus,

$$\omega_0^2 = \frac{\kappa}{m} = \frac{g}{\Delta l}$$

The equation of motion of the ball is

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

Here, $\lambda = \dfrac{2\pi\beta}{\sqrt{\omega_0^2 - \beta^2}}$ or $\dfrac{\beta}{\omega} = \dfrac{\lambda/2\pi}{\sqrt{1 + (\lambda/2\pi)^2}}$

To find the solution of the above equation we look for the solution of the auxiliary equation

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = \frac{F_0}{m}e^{i\omega t}$$

Clearly we can take Re $z = x$. Now we look for a particular integral for z of the form

$$z = Ae^{i\omega t}$$

Thus, substitution gives A and we get

$$z = \frac{(F_0/m)\,e^{i\omega t}}{\omega_0^2 - \omega^2 + 2i\beta\omega}$$

So, taking the real part

$$x = \frac{F_0}{m}\frac{[(\omega_0^2 - \omega^2)\cos\omega t + 2\beta\omega\sin\omega t]}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

or $$x = \frac{F_0}{m} \frac{\cos(\omega t - \varphi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \quad \text{and} \quad \varphi = \tan^{-1}\frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

The amplitude of this oscillation is maximum when the denominator is minimum. This happens when

$$\omega^4 - 2\omega_0^2\omega^2 + 4\beta^2\omega^2 + \omega_0^4 = (\omega^2 - \omega_0^2 + 2\beta^2) + 4\beta^2\omega_0^2 - 4\beta^4$$

is minimum, i.e., for $$\omega^2 = \omega_0^2 - 2\beta^2$$

Thus, $$\omega_{\text{res}}^2 = \omega_0^2\left(1 - \frac{2\beta^2}{\omega_0^2}\right)$$

$$= \frac{g}{\Delta l}\left[1 - \frac{2(\lambda/2\pi)^2}{1 + (\lambda/2\pi)^2}\right] = \sqrt{\frac{g}{\Delta l}\frac{1 - (\lambda/2\pi)^2}{1 + (\lambda/2\pi)^2}}$$

and $$a_{\text{res}} = \frac{F_0/m}{\sqrt{4\beta^2\omega_0^2 - 4\beta^4}} = \frac{F_0/m}{2\beta\sqrt{\omega_0^2 - \beta^2}} = \frac{F_0/m}{2\beta^2}\cdot\frac{\lambda}{2\pi}$$

$$= \frac{F_0}{2m\omega_0^2}\cdot\frac{1 + (\lambda/2\pi)^2}{\lambda/2\pi} = \frac{F_0\,\Delta l\,\lambda}{4\pi\, mg}\left(1 + \frac{4\pi^2}{\lambda^2}\right)$$

4.86 Since $$a = \frac{F_0/m}{\sqrt{(\omega^2 - \omega_0^2 + 2\beta^2)^2 + 4\beta^2(\omega_0^2 - \beta^2)}}$$

we must have $$\omega_1^2 - \omega_0^2 + 2\beta^2 = (\omega_2^2 - \omega_0^2 + 2\beta^2)$$

or $$\omega_0^2 - 2\beta^2 = \frac{\omega_1^2 + \omega_2^2}{2} = \omega_{\text{res}}^2$$

Therefore, $$\omega_{\text{res}} = \sqrt{\frac{\omega_1^2 + \omega_2^2}{2}}$$

$$= 5.1\times 10^2\ \text{s}^{-1}$$

4.87 We have $$x = \frac{F_0}{m}\frac{(\omega_0^2 - \omega^2)\cos\omega t + 2\beta\omega\sin\omega t}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

Then, $$\dot{x} = \frac{F_0\omega}{m}\frac{2\beta\omega\cos\omega t + (\omega^2 - \omega_0^2)\sin\omega t}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

Thus, the velocity amplitude is

$$v_0 = \frac{F_0\,\omega}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

$$= \frac{F_0}{m\sqrt{\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 + 4\beta^2}}$$

This is maximum when $\omega^2 = \omega_0^2 = \omega_{res}^2$

and then $v_{0\,res} = \dfrac{F_0}{2m\beta}$

Now at half maximum $\left(\dfrac{\omega_0^2}{\omega} - \omega\right)^2 = 12\,\beta^2$

or $\omega^2 \pm 2\sqrt{3}\,\beta\omega - \omega_0^2 = 0$

or $\omega = \beta\sqrt{3} + \sqrt{\omega_0^2 + 3\beta^2}$

where we have rejected a solution with negative sign before.

Using $\omega_1 = \sqrt{\omega_0^2 + 3\beta^2} + \beta\sqrt{3}$ and $\omega_2 = \sqrt{\omega_0^2 + 3\beta^2} - \beta\sqrt{3}$

we get,

(a) Velocity resonance frequency is $\omega_{res} = \omega_0\,\sqrt{\omega_1\omega_2}$

(b) Damping coefficient is $\beta = \dfrac{|\,\omega_1 - \omega_2\,|}{2\sqrt{3}}$

and damped oscillation frequency

$$\sqrt{\omega_0^2 - \beta^2} = \sqrt{\omega_1\omega_2 - \frac{(\omega_1 - \omega_2)^2}{12}}$$

4.88 In general for displacement amplitude

$$a = \frac{F_0}{m}\,\frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\,\beta^2\,\omega^2}}$$

$$= \frac{F_0}{m}\,\frac{1}{\sqrt{(\omega^2 - \omega_0^2 + 2\beta^2)^2 + 4\,\beta^2\,(\omega_0^2 - \beta^2)}}$$

Thus, $\eta = \dfrac{a_{res}}{a_{low}} = \dfrac{\omega_0^2}{\sqrt{4\beta^2\,(\omega_0^2 - \beta^2)}} = \dfrac{\omega_0^2}{2\,\beta\sqrt{\omega_0^2 - \beta^2}}$

But $\dfrac{\beta}{\omega_0} = \dfrac{\lambda/2\pi}{\sqrt{1 + (\lambda/2\pi)^2}}\,,\ \dfrac{\lambda}{2\pi} = \dfrac{\beta}{\sqrt{\omega_0^2 - \beta^2}}$

Hence, $$\eta = \frac{\omega_0^2}{2\,\beta^2} \cdot \frac{\lambda}{2\pi} = \frac{1}{2}\frac{1 + (\lambda/2\pi)^2}{\lambda/2\pi} = 2.1$$

4.89 The work done in one cycle is

$$A = \int F\,dx = \int_0^T Fv\;dt = \int_0^T F_0 \cos \omega t\,(-\omega a \sin(\omega t - \varphi))\;dt$$

$$= \int_0^T F_0 \omega a(-\cos \omega t \sin \omega t \cos \varphi + \cos^2 \omega t \sin \varphi)\;dt$$

$$= \frac{1}{2} F_0 \omega a \frac{T}{2} \sin \varphi = \pi a F_0 \sin \varphi$$

4.90 In the formula $$x = a \cos(\omega t - \varphi)$$

we have $$a = \frac{F_0}{m} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2 \omega^2}}$$

$$\tan \varphi = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

Thus, $$\beta = \frac{(\omega_0^2 - \omega^2)\tan \varphi}{2\omega}$$

Hence, $$\omega_0 = \sqrt{\frac{\kappa}{m}} = 20\ \text{s}^{-1}$$

(a) The quality factor

$$Q = \frac{\pi}{\beta T} = \frac{\sqrt{\omega_0^2 - \beta^2}}{2\beta} = \frac{1}{2}\sqrt{\frac{4\omega^2\omega_0^2}{(\omega_0^2 - \omega^2)^2 \tan^2 \varphi} - 1} = 2.2$$

(b) Work done is

$$A = \pi a F_0 \sin \varphi$$

$$= \pi\, m a^2 \sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}\,\sin \varphi = \pi m a^2 \times 2\beta\omega$$

$$= \pi\, m a^2\,(\omega_0^2 - \omega^2)\tan \varphi = 6\ \text{mJ}$$

4.91 Here as usual,

$$\tan \varphi = \frac{2\beta\omega}{\omega_0^2 - \omega^2} \quad \text{(where } \varphi \text{ is the phase lag of the displacement)}$$

$$x = a \cos(\omega t - \varphi)$$

and $$a = \frac{F_0}{m} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

(a) Mean power developed by the force over one oscillation period

$$<P> = \frac{\pi F_0 a \sin\varphi}{T} = \frac{1}{2} F_0\, a\omega \sin\varphi$$

$$= \frac{F_0^2}{m} \frac{\beta\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} = \frac{F_0^2\,\beta}{m} \frac{1}{(\omega_0^2/\omega - \omega)^2 + 4\beta^2}$$

(b) Mean power $<P>$ is maximum when $\omega = \omega_0$ (because the denominator is then minimum). So,

$$<P>_{\max} = \frac{F_0^2}{4m\beta}$$

4.92 Given $\beta = \omega_0/\eta$. Then from the previous problem

$$<P> = \frac{F_0^2\,\omega_0}{\eta m} \cdot \frac{1}{(\omega_0^2/\omega - \omega)^2 + 4\omega_0^2/\eta^2}$$

At displacement resonance $\omega = \sqrt{\omega_0^2 - 2\beta^2}$

$$<P>_{\text{res}} = \frac{F_0^2\omega_0}{\eta m} \frac{1}{4\beta^4/\omega_0^2 - 2\beta^2 + 4\omega_0^2/\eta^2} = \frac{F_0^2\,\omega_0}{\eta m} \frac{1}{\dfrac{4\omega_0^4/\eta^4}{\omega_0^2(1 - 2/\eta^2)} + 4\dfrac{\omega_0^2}{\eta^2}}$$

$$= \frac{F_0^2}{4\eta m\omega_0} \frac{\eta^2}{(1/\eta^2 - 2) + 1} = \frac{F_0^2\,\eta}{4m\omega_0} \frac{\eta^2 - 2}{\eta^2 - 1}$$

while $$<P>_{\max} = \frac{F_0^2\,\eta}{4m\,\omega_0}$$

Thus, $$\frac{<P>_{\max} - <P>_{\text{res}}}{<P>_{\max}} = \frac{100}{\eta^2 - 1}\ \%$$

4.93 The equation of the disk is

$$\ddot{\varphi} + 2\beta\dot{\varphi} + \omega_0^2\varphi = \frac{N_m \cos\omega t}{I}$$

Then, as before $$\varphi = \varphi_m \cos(\omega t - \alpha)$$

where $$\varphi_m = \frac{N_m}{I[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]^{1/2}}, \quad \tan\alpha = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

(a) Work performed by frictional forces is

$$-\int N_r \, d\varphi \quad \left(\text{where } N_r = -2I\beta\dot{\varphi} = -\int_0^T 2\,\beta I\,\dot{\varphi}^2\,dt = -2\pi\beta\,\omega I\,\varphi_m^2\right)$$

$$= -\pi I\varphi_m^2\left[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2\right]^{1/2}\sin\alpha = -\pi N_m\,\varphi_m \sin\alpha$$

(b) The quality factor

$$Q = \frac{\pi}{\lambda} = \frac{\pi}{\beta T} = \frac{\sqrt{\omega_0^2 - \beta^2}}{2\beta} = \frac{\omega\sqrt{\omega_0^2 - \beta^2}}{(\omega_0^2 - \omega^2)\tan\alpha}$$

$$= \frac{1}{2\tan\alpha}\left\{\frac{4\omega^2\,\omega_0^2}{(\omega_0^2 - \omega^2)^2} - \frac{4\,\beta^2\,\omega^2}{(\omega_0^2 - \omega^2)^2}\right\}^{1/2}$$

$$= \frac{1}{2\tan\alpha}\left\{\frac{4\,\omega^2\,\omega_0^2\,I^2\,\varphi_m^2}{N_m^2\,\cos^2\alpha} - \tan^2\alpha\right\}^{1/2} \qquad \left(\text{since } \omega_0^2 = \omega^2 + \frac{N_m}{I\varphi_m}\cos\alpha\right)$$

$$= \frac{1}{2\sin\alpha}\left\{\frac{4\,\omega^2\,\omega_0^2\,I^2\,\varphi_m^2}{N_m^2} - \sin^2\alpha\right\}^{1/2}$$

$$= \frac{1}{2\sin\alpha}\left\{\frac{4\,\omega^2\,I^2\,\varphi_m^2}{N_m^2}\left(\omega^2 + \frac{N_m\,\cos\alpha}{I\,\varphi_m}\right) + 1 - \cos^2\alpha\right\}^{1/2}$$

$$= \frac{1}{2\sin\alpha}\left\{\frac{4\,I^2\,\varphi_m^2}{N_m^2}\,\omega^4 + \frac{4\,I\,\varphi_m}{N_m}\,\omega^2\,\cos\alpha + \cos^2\alpha - 1\right\}^{1/2}$$

$$= \frac{1}{2\sin\alpha}\left\{\left(\frac{2I\,\varphi_m\omega^2}{N_m} + \cos\alpha\right)^2 - 1\right\}^{1/2}$$

4.2 Electric Oscillations

4.94 If the electrons (charge of each electron $= -e$) are shifted by a small distance x, a net $+ve$ charge density (per unit area) is induced on the surface. This will result in an electric field $E = n\,ex/\varepsilon_0$ in the direction of x and a restoring force on an electron of magnitude $-ne^2x/\varepsilon_0$.

Thus, $$m\ddot{x} = -\frac{ne^2x}{\varepsilon_0}$$

or $$\ddot{x} + \frac{ne^2}{m\varepsilon_0}x = 0$$

This gives $$\omega_p = \sqrt{\frac{ne^2}{m\varepsilon_0}}$$

$$= 1.645 \times 10^{16}\ \text{s}^{-1}$$

as the plasma frequency for the problem.

4.95 Since there are no sources of emf in the circuit,

$$\frac{q}{C} = -L\frac{dI}{dt}$$

where q = charge on the capacitor, $I = dq/dt$ current through the coil.

Then, $$\frac{d^2q}{dt^2} + \omega_0^2\, q = 0, \quad \omega_0^2 = \frac{1}{LC}$$

The solution of this equation is

$$q = q_m \cos(\omega_0 t + \alpha)$$

From the problem $$V_m = \frac{q_m}{C}$$

Then, $$I = -\omega_0 C V_m \sin(\omega_0 t + \alpha)$$

and $$V = V_m \cos(\omega_0 t - \alpha)$$

$$V^2 + \frac{I^2}{\omega_0^2\, C^2} = V_m^2$$

or $$V^2 + \frac{LI^2}{C} = V_m^2$$

By energy conservation

$$\frac{1}{2}LI^2 + \frac{q^2}{2C} = \text{constant}$$

When the potential difference across the capacitor takes its maximum value V_m, the current I must be zero.

Thus, $$\frac{1}{2}CV_m^2 = \text{constant}$$

Hence, once again $$\frac{LI^2}{C} + V^2 = V_m^2$$

4.96 After the switch is closed, the circuit satisfies

$$-L\frac{dI}{dt} = \frac{q}{C}$$

or $$\frac{d^2q}{dt^2} + \omega_0^2\, q = 0 \quad \Rightarrow \quad q = CV_m \cos \omega_0 t$$

where we have used the fact that when the switch is closed, we must have

$$V = \frac{q}{C} = V_m, \quad I = \frac{dq}{dt} = 0 \quad (\text{at } t = 0)$$

(a) Thus, $$I = \frac{dq}{dt} = -CV_m\, \omega_0 \sin \omega_0 t$$

$$= -V_m \sqrt{\frac{C}{L}} \sin \omega_0 t$$

(b) The electrical energy of the capacitor is

$$\frac{q^2}{2C} \cos^2 \omega_0 t$$

and of the inductor is

$$\frac{1}{2} LI^2 \sin^2 \omega_0 t$$

The two are equal when

$$\omega_0 t = \frac{\pi}{4}$$

At that instant, the emf of the self-inductance is

$$-L\frac{dI}{dt} = V_m \cos \omega_0 t = \frac{V_m}{\sqrt{2}}$$

4.97 The required work can be represented as an increment of the energy of the circuit

$$A = W' - W = \frac{q_m^2}{2}\left(\frac{1}{C'} - \frac{1}{C}\right) = W\left(\frac{C}{C'} - 1\right)$$

On the other hand, $\omega_0 \propto 1/\sqrt{C}$ and hence, $\eta = \omega_0'/\omega_0 = \sqrt{C/C'}$ and consequently $A = W\,(\eta^2 - 1)$.

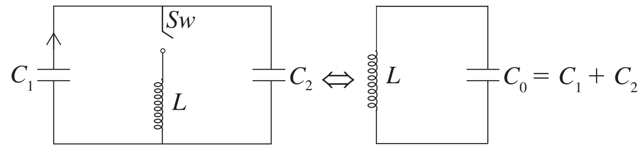
Alternate:

The required work can be represented as an increment of the energy of the circuit

$$A = W' - W = \frac{1}{2} L(\eta^2\, \omega_0^2\, q_m^2) - \frac{1}{2}\, L(\omega_0^2\, q_m^2) = \frac{1}{2} L\, \omega_0^2\, q_m^2\, (\eta^2 - 1)$$

taking into account $I_m = \omega_0\, q_m$.

4.98 Capacitors C_1 and C_2 are in parallel connection whose equivalent capacitance $C_0 = C_1 + C_2$. Equation of circuit becomes



$$-L\frac{dI}{dt} = \frac{q}{C_0}$$

or $$\ddot{q} + \omega_0^2 q = 0 \quad \left(\text{where } \omega_0^2 = \frac{1}{LC_0}\right)$$

Its general solution is

$$q = q_m \cos(\omega_0 t + \alpha)$$

At $t = 0$ charge is maximum, i.e.,

$$q = q_m = C_0 V = (C_1 + C_2)V$$

Using the initial condition, i.e., $\alpha = 0$

$$q = q_m \cos \omega_0 t$$

So, the frequency of oscillations

$$\omega_0 = \frac{1}{\sqrt{LC_0}} = \frac{1}{\sqrt{L(C_1 + C_2)}}$$

and period of oscillations $T = \dfrac{2\pi}{\omega_0} = 2\pi\sqrt{L(C_1 + C_2)} = 0.7 \text{ ms}$

Differentiating the equation $q = q_m \cos \omega_0 t$ with respect to time, we get

$$I = -\omega_0 q_m \sin \omega_0 t$$

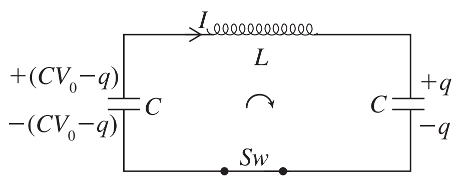
Thus, the peak value of current

$$I_m = \omega_0 q_m = \frac{1}{\sqrt{L(C_1 + C_2)}}(C_1 + C_2)V = V\sqrt{\frac{C_1 + C_2}{L}} = 8.05 \text{ A}$$

4.99 Assume that after closing the switch at $t = 0$, the right-hand side capacitor acquired charge q at $t = t$, equation of circuit will be

$$-L\frac{dI}{dt} = \frac{q}{C} - \frac{(CV_0 - q)}{C}$$

or $$\frac{d^2q}{dt^2} + \frac{2q}{LC} = \frac{V_0}{L} \qquad (1)$$



It is a form of differential equation

$$\ddot{x} + \omega_0^2 x = A \quad \text{(where } \omega_0 \text{ and } A \text{ are constant)}$$

General solution of such differential equation is

$$x = a \cos(\omega_0 t + \alpha) + \frac{A}{\omega^2}$$

Taking this into account, the general solution of Eq. (1) is

$$q = q_m \cos(\omega_0 t + \alpha) + \frac{CV_0}{2} \tag{2}$$

So,
$$I = -\omega_0 q_m \sin(\omega_0 t + \alpha) \tag{3}$$

Using the initial condition $I = 0$ and $q = 0$ at $t = 0$, in Eqs. (3) and (2), we get

$$\alpha = 0 \quad \text{and} \quad q_m = \frac{CV_0}{2}$$

Now Eq. (2) has the form

$$q = \frac{CV_0}{2}[1 - \cos \omega t]$$

So, voltage across the right-hand side capacitor

$$V = \frac{q}{C} = \frac{V_0}{2}[1 - \cos \omega t]$$

Thus, the charge on left-hand side capacitor

$$CV_0 - q = \frac{CV_0}{2}[1 + \cos \omega t]$$

and voltage across it is

$$\frac{V_0}{2}[1 + \cos \omega t]$$

Alternate:

Initially $q_1 = CV_0$ and $q_2 = 0$. After the switch is closed, charge flows and we get

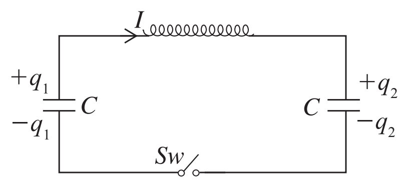
$$q_1 + q_2 = CV_0$$

$$\frac{q_1}{C} + L\frac{dI}{dt} - \frac{q_2}{C} = 0 \tag{1}$$

Also
$$I = \dot{q}_1 = -\dot{q}_2$$

Thus,
$$L\ddot{I} + \frac{2I}{C} = 0$$

Hence,
$$\ddot{I} + \omega_0^2 I = 0, \ \omega_0^2 = \frac{2}{LC}$$



The solution of this equation is subject to $I = 0$ at $t = 0$.

So, $$I = I_0 \sin \omega_0 t$$

Integrating, we get $$q_1 = A - \frac{I_0}{\omega_0} \cos \omega_0 t$$

and $$q_2 = B + \frac{I_0}{\omega_0} \cos \omega_0 t$$

Finally substituting in Eq. (1)

$$\frac{A - B}{C} - \frac{2I_0}{\omega_0 C} \cos \omega_0 t + LI_0\omega_0 \cos \omega_0 t = 0$$

Thus, $$A = B = \frac{CV_0}{2}$$

and $$\frac{CV_0}{2} + \frac{I_0}{\omega_0} = 0$$

So, $$q_1 = \frac{CV_0}{2}(1 + \cos \omega_0 t) \quad \text{and} \quad V_1 = \frac{q_1}{C} = \frac{V_0}{2}(1 + \cos \omega_0 t)$$

$$q_2 = \frac{CV_0}{2}(1 - \cos \omega_0 t) \quad \text{and} \quad V_2 = \frac{q_2}{C} = \frac{V_0}{2}(1 - \cos \omega_0 t)$$

4.100 The flux in the coil is

$$\Phi(t) = \begin{cases} \Phi \text{ (for } t<0) \\ 0 \text{ (for } t>0) \end{cases}$$

Due to sudden switching off of the external magnetic field at the moment $t = 0$, an induced current appears, but the capacitor still remains uncharged. In accordance with Ohm's law, we have

$$-\frac{d\Phi}{dt} - L\frac{dI}{dt} = 0$$

Hence, $$\Phi + LI = 0$$

This gives $\Phi = LI_0$, where I_0 is the initial current (immediately after switching off the field).

After switching off external magnetic field, the equation for LC circuit becomes

$$-\frac{q}{C} - L\frac{dI}{dt} = 0 \qquad (1)$$

Differentiation of this equation with respect to time gives

$$\frac{d^2 I}{dt^2} + \frac{I}{LC} = 0$$

Its general solution will be

$$I = I_m \cos(\omega_0 t + \alpha) \quad \left(\text{where } \omega_0^2 = \frac{1}{LC}\right)$$

Since at initial instant $t = 0$, the capacitor was uncharged, so $I_m = I_0$ (initial current) and $\alpha = 0$. As a result

$$I = I_0 \cos \omega_0 t$$

Finally, $$I = I_0 \cos \omega_0 t = \frac{\Phi}{L} \cos \omega_0 t = \frac{\Phi}{L} \cos\left(\frac{t}{\sqrt{LC}}\right)$$

4.101 Given $$V = V_m e^{-\beta t} \cos \omega t$$

(a) The phrase 'peak values' is not clear. The answer is obtained on taking

$$|\cos \omega t| = 1$$

i.e., $$t = \frac{\pi n}{\omega}$$

(b) For extremum $\dfrac{dV}{dt} = 0$

$$-\beta \cos \omega t - \omega \sin \omega t = 0 \quad \text{or} \quad \tan \omega t = \frac{-\beta}{\omega}$$

i.e., $$\omega t_n = n\pi + \tan^{-1}\left(\frac{-\beta}{\omega}\right)$$

Hence, $$t_n = \frac{1}{\omega}\left[n\pi + \tan^{-1}\left(\frac{-\beta}{\omega}\right)\right]$$

4.102 The equation of the circuit is

$$L\frac{d^2 Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

where Q = charge on the capacitor.

This has the solution

$$Q = Q_m e^{-\beta t} \sin(\omega t + \alpha)$$

where $$\beta = \frac{R}{2L}, \quad \omega = \sqrt{\omega_0^2 - \beta^2}, \quad \omega_0^2 = \frac{1}{LC}$$

Now, at $t = 0$ $\quad I = \frac{dQ}{dt} = 0$

so, $\quad Q_m \, e^{-\beta t}[-\beta \sin(\omega t + \alpha) + \omega \cos(\omega t + \alpha)] = 0$

Thus, $\quad \omega \cos\alpha = \beta \sin\alpha \quad \text{or} \quad \alpha = \tan^{-1}\frac{\omega}{\beta}$

Now, $\quad V_m = \frac{Q_m}{C} \quad \text{and} \quad V_0 = \frac{Q_m}{C}\sin\alpha \quad$ (where V_0 is P.D. at $t = 0$)

Therefore
$$\frac{V_0}{V_m} = \sin\alpha = \frac{\omega}{\sqrt{\omega^2 + \beta^2}} = \frac{\omega}{\omega_0} = \sqrt{1 - \frac{\beta^2}{\omega_0^2}} = \sqrt{1 - \frac{R^2 C}{4L}}$$

4.103 We write
$$-\frac{dQ}{dt} = I = I_m \, e^{-\beta t} \sin\omega t$$
$$= \text{gm } I_m \, e^{-\beta t + i\omega t} \quad \text{(where gm means imaginary part)}$$

Then,
$$Q = \text{gm } I_m \frac{e^{-\beta t + i\omega t}}{-\beta + i\omega}$$
$$= \text{gm } I_m \frac{e^{-\beta t + i\omega t}}{\beta - i\omega}$$
$$= \text{gm } I_m \frac{(\beta + i\omega)\, e^{-\beta t + i\omega t}}{\beta^2 + \omega^2}$$
$$= I_m \, e^{-\beta t} \frac{\beta \sin\omega t + \omega \cos\omega t}{\beta^2 + \omega^2}$$
$$= I_m \, e^{-\beta t} \frac{\sin(\omega t + \delta)}{\sqrt{\beta^2 + \omega^2}}, \quad \tan\delta = \frac{\omega}{\beta}$$

(An arbitrary constant of integration is assumed to be equal to zero.)

Thus,
$$V = \frac{Q}{C} = I_m \sqrt{\frac{L}{C}} \, e^{-\beta t} \sin(\omega t + \delta)$$
$$V(0) = I_m \sqrt{\frac{L}{C}} \sin\delta = I_m \sqrt{\frac{L}{C}} \frac{\omega}{\sqrt{\omega^2 + \beta^2}}$$
$$= I_m \sqrt{\frac{L}{C(1 + \beta^2/\omega^2)}}$$

4.104 We know that, $I = I_m\, e^{-\beta t} \sin \omega t$

$$\beta = \frac{R}{2L}, \quad \omega_0 = \sqrt{\frac{1}{LC}}, \quad \omega = \sqrt{\omega_0^2 - \beta^2}$$

$$I = -\dot{q} \quad \text{(where } q = \text{charge on the capacitor)}$$

Then, $$q = I_m\, e^{-\beta t} \frac{\sin(\omega t + \delta)}{\sqrt{\omega^2 + \beta^2}} \quad \text{and} \quad \tan \delta = \frac{\omega}{\beta}$$

Thus, $$W_{\text{mag}} = \frac{1}{2} L I_m^2\, e^{-2\beta t} \sin^2 \omega t$$

$$W_{\text{ele}} = \frac{I_m^2}{2C} \frac{e^{-2\beta t} \sin^2(\omega t + \delta)}{\omega^2 + \beta^2} = \frac{L I_m^2}{2} e^{-2\beta t} \sin^2(\omega t + \delta)$$

Current is maximum when

$$\frac{d}{dt} e^{-\beta t} \sin \omega t = 0$$

Thus, $$-\beta \sin \omega t + \omega \cos \omega t = 0$$

or $$\tan \omega t = \frac{\omega}{\beta} = \tan \delta$$

i.e., $$\omega t = n\pi + \delta$$

Hence, $$\frac{W_{\text{mag}}}{W_{\text{ele}}} = \frac{\sin^2(\omega t)}{\sin^2(\omega t + \delta)} = \frac{\sin^2 \delta}{\sin^2 2\delta} = \frac{1}{4 \cos^2 \delta}$$

$$= \frac{1}{4\beta^2/\omega_0^2} = \frac{\omega_0^2}{4\beta^2} = \frac{1}{LC} \times \frac{L^2}{R^2} = \frac{L}{CR^2} = 5$$

(W_{mag} is the magnetic energy of the inductance coil and W_{ele} is the electric energy of the capacitor.)

4.105 Clearly $L = L_1 + L_2, \quad R = R_1 + R_2$

4.106 We know, $$Q = \frac{\pi}{\beta T} \quad \text{or} \quad \beta = \frac{\pi}{QT}$$

Now, $$\beta t = \ln \eta$$

so, $$t = \frac{\ln \eta}{\pi} QT$$

$$= \frac{Q \ln \eta}{\pi \nu} = 0.5 \text{ s}$$

4.107 Current decreases e-fold in time

$$t = \frac{1}{\beta} = \frac{2L}{R}\ \mathrm{s} = \frac{2L}{RT}\ \text{oscillations}$$

$$= \frac{2L}{R}\frac{\omega}{2\pi}$$

$$= \frac{L}{\pi R}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \frac{1}{2\pi}\sqrt{\frac{4L}{R^2C} - 1} = 16\ \text{oscillations}$$

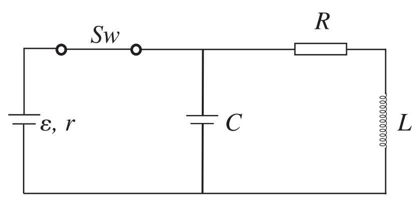
4.108 Since,
$$Q = \frac{\pi}{\beta T} = \frac{\omega}{2\beta}$$

Therefore,
$$\omega = 2\,\beta Q \quad \text{or} \quad \beta = \frac{\omega}{2Q}$$

Now,
$$\omega_0 = \omega\sqrt{1 + \frac{1}{4Q^2}} \quad \text{or} \quad \omega = \frac{\omega_0}{\sqrt{1 + 1/(4Q^2)}}$$

So,
$$\left|\frac{\omega_0 - \omega}{\omega_0}\right| \times 100\% \cong \frac{1}{8Q^2} \times 100\% = 0.5\%$$

4.109 At $t = 0$, current through the coil $= \dfrac{\varepsilon}{R + r}$



P.D. across the condenser $= \dfrac{\varepsilon}{R + r}$

(a) At $t = 0$, energy stored

$$W_0 = \frac{1}{2}L\left(\frac{\varepsilon}{R + r}\right)^2 + \frac{1}{2}C\left(\frac{\varepsilon R}{R + r}\right)^2 = \frac{1}{2}\varepsilon^2\frac{(L + CR^2)}{(R + r)^2} = 2.0\ \text{mJ}$$

(b) The current and the charge stored decrease as $e^{-tR/2L}$ so energy decreases as $e^{-tR/L}$.

Therefore,
$$W = W_0\, e^{-tR/L} = 0.10\ \text{mJ}$$

4.110 Since
$$Q = \frac{\pi}{\beta T} = \frac{\pi\nu}{\beta} = \frac{\omega}{2\beta} = \frac{\sqrt{\omega_0^2 - \beta^2}}{2\,\beta}$$

So,
$$\frac{\omega_0}{\beta} = \sqrt{1 + 4Q^2} \quad \text{or} \quad \beta = \frac{\omega_0}{\sqrt{1 + Q^2}}$$

Now,
$$W = W_0 e^{-2\beta t}$$

Thus, energy decreases η times in time

$$\frac{\ln \eta}{2\,\beta}$$

$$= \ln \eta \, \frac{\sqrt{1+4Q^2}}{2\omega_0} \approx \frac{Q \ln \eta}{2\pi v_0} \text{ s} = 1.033 \text{ ms}$$

4.111 In a leaky condenser

$$\frac{dq}{dt} = I - I' \quad \left(\text{where } I' = \frac{V}{R} = \text{leak curent}\right)$$

Now
$$V = \frac{q}{C} = -L\frac{dI}{dt} = -L\frac{d}{dt}\left(\frac{dq}{dt} + \frac{V}{R}\right)$$

$$= -L\frac{d^2q}{dt^2} - \frac{L}{RC}\frac{dq}{dt}$$

or
$$\ddot{q} + \frac{1}{RC}\frac{dq}{dt} + \frac{1}{LC}q = 0$$

Then
$$q = q_m\, e^{-\beta t} \sin(\omega t + \alpha)$$

(a) For damped oscillations,

$$\beta = \frac{1}{2RC}, \quad \omega_0^2 = \frac{1}{LC}, \quad \omega = \sqrt{\omega_0^2 - \beta^2}$$

Therefore,
$$\omega = \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}}$$

(b) The quality factor

$$Q = \frac{\omega}{2\beta} = RC\sqrt{\frac{1}{LC} - \frac{1}{4\,R^2\,C^2}}$$

$$= \frac{1}{2}\sqrt{\frac{4CR^2}{L} - 1}$$

4.112 Given
$$V = V_m\, e^{-\beta t} \sin \omega t, \quad \omega \cong \omega_0 \;\; \beta T \ll 1$$

$$\text{Power loss} = \frac{\text{Energy loss per cycle}}{T}$$

$$\cong \frac{1}{2}\, C V_m^2 \times 2\beta$$

(Energy decreases as $W_0\, e^{-\beta t}$ so loss per cycle is $W_0 \times 2\,\beta T$.)

Thus, $$<P> = \frac{1}{2}\, CV_m^2 \times \frac{R}{L}$$

or $$R = \frac{2<P>}{V_m^2}\,\frac{L}{C}$$

Hence, $$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \sqrt{\frac{C}{L}}\;\frac{V_m^2}{2<P>} = 100 \text{ (on substituting values)}$$

4.113 Energy is lost across the resistance and the mean power loss is

$$<P> = R<I^2> = \frac{1}{2}\, RI_m^2 = 20 \text{ mW}$$

This power should be fed to the circuit to maintain undamped oscillations.

4.114 We have $$<P> = \frac{RCV_m^2}{2L} \text{ (as in Problem 4.112)}$$

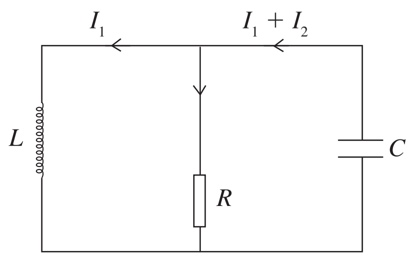
On substituting values, we get

$$<P> = 5 \text{ mW}$$

4.115 Given $$q = q_1 + q_2$$

$$I_1 = -\dot{q}_1, \quad I_2 = -\dot{q}_2$$

$$LI_1 = RI_2 = \frac{q}{C}$$



Thus, $$CL\,\ddot{q}_1 + (q_1 + q_2) = 0$$

$$RC\,\dot{q}_2 + q_1 + q_2 = 0$$

Putting $q_1 = A\,e^{i\omega t}$ and $q_2 = Be^{+i\omega t}$, we have

$$(1 - \omega^2 LC)\, A + B = 0$$

$$A + (1 + i\omega RC)\, B = 0$$

A solution exists only if

$$(1 - \omega^2 LC)\,(1 + i\omega RC) = 1$$

or $$i\omega RC - \omega^2 LC - i\omega^3\, LRC^2 = 0$$

or $$LRC^2\,\omega^2 - i\omega LC - RC = 0$$

and
$$\omega^2 - i\omega \frac{1}{RC} - \frac{1}{LC} = 0$$

So,
$$\omega = \frac{i}{2RC} \pm \sqrt{\frac{1}{LC} - \frac{1}{4R^2C^2}} \cong i\beta \pm \omega_0$$

Thus,
$$q_1 = (A_1 \cos \omega_0 t + A_2 \sin \omega_0 t)\, e^{-\beta t}$$

ω_0 is the oscillation frequency. Oscillations are possible only if $\omega_0^2 > 0$,

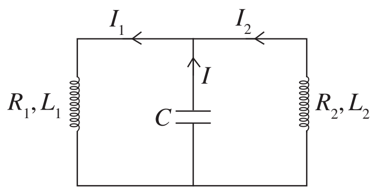
i.e.,
$$\frac{1}{4R^2} < \frac{C}{L}$$

4.116 We have

$$L_1\dot{I}_1 + R_1I_1 = L_2\dot{I}_2 + R_2I_2$$

$$= -\frac{\int I\,dt}{C}$$

$$I = I_1 + I_2$$



On differentiating we get the equations

$$L_1C\ddot{I}_1 + R_1C\dot{I}_1 + (I_1 + I_2) = 0$$

$$L_2C\ddot{I}_2 + R_2C\dot{I}_2 + (I_1 + I_2) = 0$$

On solving, we get $\quad I_1 = A_1\, e^{\alpha t}, \quad I_2 = A_2\, e^{\alpha t}$

Then,
$$(1 + \alpha^2 L_1C + \alpha R_1C)\, A_1 + A_2 = 0$$

$$A_1 + (1 + \alpha^2 L_2C + \alpha R_2C)\, A_2 = 0$$

This set of simultaneous equations has a non-trivial solution only if

$$(1 + \alpha^2 L_1C + \alpha R_1C)\,(1 + \alpha^2 L_2C + \alpha R_2C) = 1$$

or
$$\alpha^3 + \alpha^2 \frac{L_1R_2 + L_2R_1}{L_1L_2} + \alpha \frac{L_1 + L_2 + R_1R_2C}{L_1L_2C} + \frac{R_1 + R_2}{L_1L_2C} = 0$$

This cubic equation has one real root which we ignore and two complex conjugate roots. We require the condition that this pair of complex conjugate roots is identical with the roots of the equation

$$\alpha^2 LC + \alpha RC + 1 = 0$$

The general solution of this problem is not easy. So, we look for special cases.

If $R_1 = R_2 = 0$, then $R = 0$ and $L = L_1L_2/(L_1 + L_2)$.

If $L_1 = L_2 = 0$, then $L = 0$ and $R = R_1R_2/(R_1 + R_2)$.

These are the quoted solutions but they are misleading.

We shall give the solution for small R_1, R_2. Then we put $\alpha = -\beta + i\omega$, to get

$$(1 - \omega^2 L_1 C - 2i\beta\omega L_1 C - \beta R_1 C + i\omega R_1 C)(1 - \omega^2 L_2 C - 2i\beta\omega L_2 C - \beta R_2 C + i\omega R_2 C) = 1$$

when β is small, neglecting β^2 and βR_1, βR_2,we get

$$(1 - \omega^2 L_1 C)(1 - \omega^2 L_2 C) = 1 \quad \text{or} \quad \omega^2 = \frac{L_1 + L_2}{L_1 L_2 C}$$

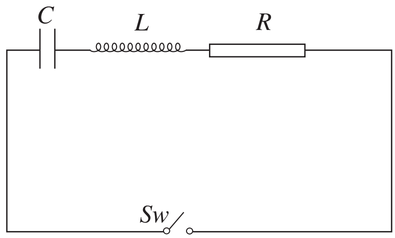
This is identical with $\omega^2 = \dfrac{1}{LC}$ if $L = \dfrac{L_1 L_2}{L_1 + L_2}$

Also $(2\beta L_1 - R_1)(1 - \omega^2 L_2 C) + (2\beta L_2 - R_2)(1 - \omega^2 L_1 C) = 0$

This gives $\beta = \dfrac{R}{2L} = \dfrac{R_1 L_2^2 + R_2 L_1^2}{2 L_1 L_2 (L_1 + L_2)} \Rightarrow R = \dfrac{R_1 L_2^2 + R_2 L_1^2}{(L_1 + L_2)^2}$

4.117 We have

$$0 = \frac{q}{C} + L\frac{dI}{dt} + RI, \quad I = +\frac{dq}{dt}$$



For the critical case, $R = 2\sqrt{\dfrac{L}{C}}$

Thus, $LC\ddot{q} + 2\sqrt{LC}\,\dot{q} + q = 0$

Let us look for a solution with $q \propto e^{-\beta t}$.

We know that, $\beta = \dfrac{1}{\sqrt{LC}}$

An independent solution is $t\,e^{-\beta t}$.

Thus, $q = (A + Bt)\,e^{-t/\sqrt{LC}}$

At $t = 0$, $q = CV_0$, thus $A = CV_0$. Also at $t = 0$, $\dot{q} = I = 0$.

So, $0 = B - A\dfrac{1}{\sqrt{LC}} \Rightarrow B = V_0\sqrt{\dfrac{C}{L}}$

Finally, $I = \dfrac{dq}{dt} = V_0\sqrt{\dfrac{C}{L}}\,e^{-t/\sqrt{LC}}$

$$= -\frac{1}{\sqrt{LC}}\left(CV_0 + V_0\sqrt{\frac{C}{L}}t\right)e^{-t/\sqrt{LC}}$$

$$= -\frac{V_0}{L} te^{-t/\sqrt{LC}}$$

The current has been defined to increase the charge. Hence, the minus sign.

The current is maximum when

$$\frac{dI}{dt} = -\frac{V_0}{L} e^{-t/\sqrt{LC}} \left(1 - \frac{t}{\sqrt{LC}}\right) = 0$$

This gives $t = \sqrt{LC}$ and the magnitude of the maximum current is

$$|I_{\max}| = \frac{V_0}{e}\sqrt{\frac{C}{L}}$$

4.118 The equation of the circuit is

$$L\frac{dI}{dt} + RI = V_m \cos \omega t$$

From the theory of differential equations

$$I = I_P + I_C$$

where I_P is a particular integral and I_C is the complementary function (solution of the differential equation with the R.H.S = 0).

Now, $\quad I_C = I_{\text{coil}}\, e^{-tR/L}$

and for I_P we write $\quad I_P = I_m \cos(\omega t - \varphi)$

Substituting, we get $\quad I_m = \dfrac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \quad$ and $\quad \varphi = \tan^{-1}\dfrac{\omega L}{R}$

Thus, $\quad I_m = \dfrac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t - \varphi) + I_{\text{coil}}\, e^{-tR/L}$

Now, in an inductive circuit $I = 0$ at $t = 0$ because a current cannot change suddenly.

Thus, $\quad I_{\text{coil}} = -\dfrac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos \varphi$

and so, $\quad I = \dfrac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \left[\cos(\omega t - \varphi) - \cos \varphi e^{-tR/L}\right]$

4.119 Here, the equation is

$$\frac{Q}{C} + R\frac{dQ}{dt} = V_m \cos \omega t \text{ (where } Q \text{ is charge on the capacitor)} \tag{1}$$

A solution subjected to $Q = 0$ at $t = 0$ is of the form (as in the previous problem)

$$Q = Q_m \left[\cos(\omega t - \bar{\varphi}) - \cos\bar{\varphi}\, e^{-t/RC}\right]$$

Substituting in Eq.(1), we get

$$\frac{Q_m}{C}\cos(\omega t - \bar{\varphi}) - \omega R Q_m \sin(\omega t - \bar{\varphi})$$

$$= V_m \cos\omega t$$

$$= V_m \left\{\cos\bar{\varphi}\cos(\omega t - \bar{\varphi}) - \sin\bar{\varphi}\sin(\omega t - \bar{\varphi})\right\}$$

So,
$$Q_m = C V_m \cos\bar{\varphi}$$

$$\omega R\, Q_m = V_m \sin\bar{\varphi}$$

This leads to

$$Q_m = \frac{C V_m}{\sqrt{1 + (\omega R C)^2}}, \quad \tan\bar{\varphi} = \omega R C$$

Hence,
$$I = \frac{dQ}{dt} = \frac{V_m}{\sqrt{R^2 + (1/\omega C)^2}}\left[-\sin(\omega t - \bar{\varphi}) + \frac{\cos^2\bar{\varphi}}{\sin\bar{\varphi}} e^{-t/RC}\right]$$

The solution given in the book satisfies $I = 0$ at $t = 0$. When $Q = 0$ at $t = 0$, this will not satisfy the equation at $t = 0$. Thus, $I \neq 0$. (Equation will be satisfied with $I = 0$ only if $Q \neq 0$ at $t = 0$.)

With the I obtained above

$$I(t = 0) = \frac{V_m}{R}$$

4.120 The current lags behind the voltage by the phase angle

$$\varphi = \tan^{-1}\frac{\omega L}{R}$$

Now,
$$L = \mu_0 n^2 \pi a^2 l \quad \text{(where } l \text{ is length of the solenoid)}$$

$$R = \frac{\rho\cdot 2\pi a n\cdot l}{\pi b^2} \quad \text{(where } 2b \text{ is diameter of the wire)}$$

But
$$2bn = 1 \quad \text{so,} \quad b = \frac{1}{2n}$$

Then,
$$\varphi = \tan^{-1}\frac{\mu_0 n^2 l \pi a^2\cdot 2\pi\nu}{\rho\cdot 2\pi a n l}\times\pi\frac{1}{4n^2}$$

$$= \tan^{-1}\frac{\mu_0 \pi^2 a\nu}{4\rho n}$$

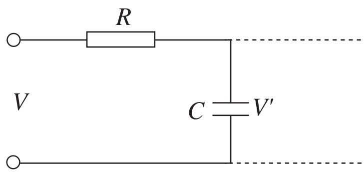
4.121 Here
$$V = V_m \cos \omega t$$
$$I = I_m \cos(\omega t + \varphi)$$

where
$$I_m = \frac{V_m}{\sqrt{R^2 + (1/\omega C)^2}}, \quad \tan\varphi = \frac{1}{\omega RC}$$

Now
$$R^2 + \frac{1}{(\omega C)^2} = \left(\frac{V_m}{I_m}\right)^2$$
$$\frac{1}{\omega RC} = \sqrt{\left(\frac{V_m}{RI_m}\right)^2 - 1}$$

Thus, the current is ahead of the voltage by
$$\varphi = \tan^{-1}\frac{1}{\omega RC} = \tan^{-1}\sqrt{\left(\frac{V_m}{RI_m}\right)^2 - 1} = 60°$$

4.122 (a) Here
$$V = IR + \frac{\int_0^t I dt}{C}$$



or
$$R\dot{I} + \frac{1}{C}I = \dot{V} = -\omega V_0 \sin\omega t$$

Ignoring transients, a solution has the form
$$I = I_0 \sin(\omega t - \alpha)$$
$$\omega R I_0 \cos(\omega t - \alpha) + \frac{I_0}{C}\sin(\omega t - \alpha) = -\omega V_0 \sin\omega t$$
$$= -\omega V_0 \left\{\sin(\omega t - \alpha)\cos\alpha + \cos(\omega t - \alpha)\sin\alpha\right\}$$

So,
$$RI_0 = -V_0 \sin\alpha$$
$$\frac{I_0}{\omega C} = -V_0 \cos\alpha, \quad \alpha = \tan^{-1}(\omega RC)$$
$$I_0 = \frac{V_0}{\sqrt{R^2 + (1/\omega C)^2}}$$

$$I = I_0 \sin(\omega t - \tan^{-1}\omega RC - \pi) = -I_0 \sin(\omega t - \tan^{-1}\omega RC)$$

Then, $$Q = \int_0^t I\,dt = Q_0 + \frac{I_0}{\omega}\cos(\omega t - \tan^{-1}\omega RC)$$

It satisfies $$V_0\left(1 + \cos\omega t\right) = R\frac{dQ}{dt} + \frac{Q}{C}$$

if $$V_0(1 + \cos\omega t) = -RI_0 \sin(\omega t - \tan^{-1}\omega RC) + \frac{Q_0}{C} + \frac{I_0}{\omega C}\cos(\omega t - \tan^{-1}\omega RC)$$

Thus, $$Q_0 = CV_0$$

and $$\frac{I_0}{\omega C} = \frac{V_0}{\sqrt{1 + (\omega RC)^2}}$$

$$RI_0 = \frac{V_0 \omega RC}{\sqrt{1 + (\omega RC)^2}}$$

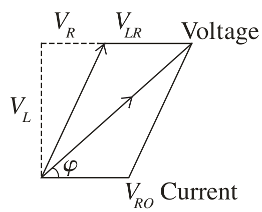
Hence, $$V' = \frac{Q}{C} = V_0 + \frac{V_0}{\sqrt{1 + (\omega RC)^2}}\cos(\omega t - \alpha)$$

(b) $$\frac{V_0}{\eta} = \frac{V_0}{\sqrt{1 + (\omega RC)^2}}$$

or $$\eta^2 - 1 = \omega^2 (RC)^2$$

or $$RC = \frac{\sqrt{\eta^2 - 1}}{\omega} = 22 \text{ ms}$$

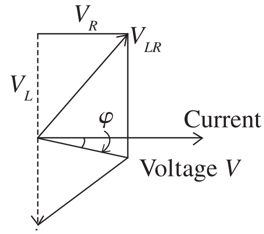
4.123 For Fig. (a) of the problem, the voltage vector diagram is



For Fig. (b) of the problem, we have

$$\tan\varphi = \frac{\omega L - 1/\omega C}{R} = -ve \quad \left(\text{as } \omega^2 < \frac{1}{LC}\right)$$

The voltage vector diagram is



4.124 (a) $$I_m = \frac{V_m}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \cong 4.48 \text{ A}$$

(b) $$\tan\varphi = \frac{\omega L - 1/\omega C}{R}, \quad \varphi \cong -60°$$

Current lags behind the voltage V by φ.

(c) $$V_C = \frac{I_m}{\omega C} = 0.65 \text{ kV}$$

$$V_L = I_m \sqrt{R^2 + \omega^2 L^2} = 0.5 \text{ kV}$$

4.125 (a) $$V_C = \frac{1}{\omega C}\frac{V_m}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$

$$= \frac{V_m}{\sqrt{(\omega RC)^2 + (\omega^2 LC - 1)^2}} = \frac{V_m}{\sqrt{(\omega^2/\omega_0^2 - 1)^2 + 4\beta^2\omega^2/\omega_0^4}}$$

$$= \frac{V_m}{\sqrt{(\omega^2/\omega_0^2 - 1 + 2\beta^2/\omega_0^2)^2 + 4\beta^2/\omega_0^2 - 4\beta^4/\omega_0^4}}$$

This is maximum when

$$\omega^2 = \omega_0^2 - 2\beta^2 = \frac{1}{LC} - \frac{R^2}{2L^2}$$

(b) $$V_L = I_m\,\omega L = V_m \frac{\omega L}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$

$$= \frac{V_m L}{\sqrt{R^2/\omega^2 + (L - 1/\omega^2 C)^2}} = \frac{V_m L}{\sqrt{L^2 - 1/\omega^2(2L/C - R^2) + 1/\omega^4 C^2}}$$

$$= \frac{V_m L}{\sqrt{(1/\omega^2 C - (L - CR^2/2))^2 + L^2 - (L - 1/2CR^2)^2}}$$

This is maximum when

$$\frac{1}{\omega^2 C} = L - \frac{1}{2}CR^2$$

or

$$\omega^2 = \frac{1}{LC - 1/2C^2R^2} = \frac{1}{1/\omega_0^2 - 2\beta^2/\omega_0^4}$$

$$= \frac{\omega_0^4}{\omega_0^2 - 2\beta^2} \quad \text{or} \quad \omega = \frac{\omega_0^2}{\sqrt{\omega_0^2 - 2\beta^2}}$$

4.126

$$V_L = I_m \sqrt{R^2 + \omega^2 L^2} = \frac{V_m \sqrt{R^2 + \omega^2 L^2}}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$

For a given ω, L, R, voltage amplitude is maximum when

$$\frac{1}{\omega C} = \omega L$$

or

$$C = \frac{1}{\omega^2 L} = 28.2\ \mu\text{F}$$

For calculated value of C,

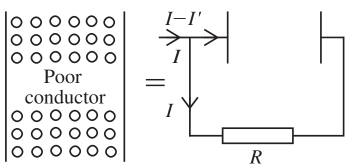
$$V_L = \frac{V\sqrt{R^2 + \omega^2 L^2}}{R} = V\sqrt{1 + \left(\frac{\omega L}{R}\right)^2} = 0.540\ \text{kV}$$

Also,

$$V_C = \frac{1}{\omega C}\frac{V_m}{R} = \frac{V_m \omega L}{R} = 0.509\ \text{kV}$$

4.127 We use the complex voltage $V = V_m\, e^{i\omega t}$. Then the voltage across the capacitor is

$$(I - I')\frac{1}{i\omega C}$$



and that across the resistance is RI' and both equal V.

Thus, $$I' = \frac{V_m}{R}\, e^{i\omega t} \quad \text{and} \quad I - I' = i\omega C V_m\, e^{i\omega t}$$

Hence,

$$I = \frac{V_m}{R}(1 + i\omega RC)e^{i\omega t}$$

The actual voltage is obtained by taking the real part.

Then,

$$I = \frac{V_m}{R}\sqrt{1 + (\omega RC)^2}\, \cos(\omega t + \varphi)$$

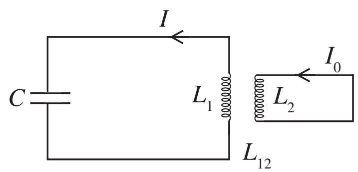
where $\tan\varphi = \omega RC$.

Note: *A condenser with poorly conducting material (dielectric of high resistance) between the plates is equivalent to an ideal condenser with a high resistance joined in parallel between its plates.*

4.128

$$L_1 \frac{dI_1}{dt} + \frac{\int I_1\,dt}{C} = -L_{12}\frac{dI_2}{dt}$$

$$L_2 \frac{dI_2}{dt} = -L_{12}\frac{dI_1}{dt}$$



From the second equation

$$L_2 I_2 = -L_{12} I_1$$

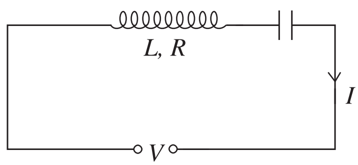
Then,
$$\left(L_1 - \frac{L_{12}^2}{L_2}\right)\ddot{I}_1 + \frac{I_1}{C} = 0$$

Thus, the current oscillates with frequency

$$\omega = \frac{1}{\sqrt{C(L_1 - L_{12}^2/L_2)}}$$

4.129 Given
$$V = V_m \cos\omega t$$

$$I = I_m \cos(\omega t - \varphi)$$



Here
$$I_m = \frac{V_m}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$

Then,
$$V_C = \frac{\int I\,dt}{C} = \frac{I_m \sin(\omega t - \varphi)}{\omega C}$$

$$= \frac{V_m}{\sqrt{(1-\omega^2 LC)^2 + (\omega RC)^2}}\sin(\omega t - \varphi)$$

At resonance, the voltage amplitude across the capacitor is

$$\frac{V_m}{RC\; 1/\sqrt{LC}} = \sqrt{\frac{L}{CR^2}}\,V_m = nV_m$$

So,
$$\frac{L}{CR^2} = n^2$$

Now,
$$Q = \sqrt{\frac{L}{CR^2} - \frac{1}{4}} = \sqrt{n^2 - \frac{1}{4}}$$

4.130 For maximum current amplitude

$$I_m = \frac{V_m}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$

$$L = \frac{1}{\omega^2 C} \quad \text{and then} \quad I_{m0} = \frac{V_m}{R}$$

Now,
$$\frac{I_{m0}}{\eta} = \frac{V_m}{\sqrt{R^2 + \dfrac{(n-1)^2}{\omega^2 C^2}}}$$

So,
$$\eta = \sqrt{1 + \frac{(n-1)^2}{(\omega RC)^2}}$$

or
$$\omega RC = \frac{n-1}{\sqrt{\eta^2 - 1}}$$

Now,
$$Q = \sqrt{\left(\frac{L}{CR^2}\right)^2 - \frac{1}{4}} = \sqrt{\left(\frac{1}{\omega RC}\right)^2 - \frac{1}{4}} = \sqrt{\frac{\eta^2 - 1}{(n-1)^2} - \frac{1}{4}}$$

4.131 At resonance $\quad \omega_0 L = (\omega_0 C)^{-1} \quad \text{or} \quad \omega_0 = \dfrac{1}{\sqrt{LC}}$

and
$$(I_m)_{\text{res}} = \frac{V_m}{R}$$

Now,
$$\frac{V_m}{nR} = \frac{V_m}{\sqrt{R^2 + \left(\omega_1 L - \dfrac{1}{\omega_1 C}\right)^2}} = \frac{V_m}{\sqrt{R^2 + \left(\omega_2 L - \dfrac{1}{\omega_2 C}\right)^2}}$$

Then,
$$\omega_1 L - \frac{1}{\omega_1 C} = \sqrt{n^2 - 1}\; R$$

$$\omega_2 L - \frac{1}{\omega_2 C} = +\sqrt{n^2 - 1}\; R \quad (\text{assuming } \omega_2 > \omega_1)$$

or
$$\omega_1 - \frac{\omega_0^2}{\omega_1} = -\omega_2 + \frac{\omega_0^2}{\omega_2} = -\sqrt{n^2 - 1}\,\frac{R}{L}$$

or
$$\omega_1 + \omega_2 = \frac{\omega_0^2}{\omega_1 \omega_2}(\omega_1 + \omega_2) \quad \Rightarrow \quad \omega_0 = \sqrt{\omega_1 \omega_2}$$

and
$$\omega_2 - \omega_1 = \sqrt{n^2 - 1}\,\frac{R}{L}$$

$$\beta = \frac{R}{2L} = \frac{\omega_2 - \omega_1}{2\sqrt{n^2 - 1}}$$

and $$Q = \sqrt{\frac{\omega_0^2}{4\beta^2} - \frac{1}{4}} = \sqrt{\frac{(n^2 - 1)\,\omega_1\omega_2}{(\omega_2 - \omega_1)^2} - \frac{1}{4}}$$

4.132 The quality factor

$$Q = \frac{\omega}{2\beta} \approx \frac{\omega_0}{2\beta} \quad \text{(for low damping)}$$

Now $$\frac{I_m}{\sqrt{2}} = \frac{RI_m}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$

(where I_m = current amplitude at resonance)

or $$\omega - \frac{\omega_0^2}{\omega} = \pm\frac{R}{L} = \pm 2\beta$$

Thus, $$\omega \cong \omega_0 \pm \beta$$

So, $$\Delta\omega = 2\beta \quad \text{and} \quad Q = \frac{\omega_0}{\Delta\omega}$$

4.133 At resonance $$\omega = \omega_0$$

$$I_m\,(\omega_0) = \frac{V_m}{R}$$

Then, $$I_m\,(\eta\,\omega_0) = \frac{V_m}{\sqrt{R^2 + (\eta\omega_0 L - 1/\eta\omega_0 C)^2}}$$

$$= \frac{V_m}{\sqrt{R^2 + (\eta - 1/\eta)^2 L/C}}$$

$$\frac{I_m(\omega_0)}{I_m(\eta\omega_0)} = \sqrt{1 + \left(Q^2 + \frac{l}{4}\right)\frac{(\eta^2 - 1)^2}{\eta^2}}$$

$$= 2.2 \ \text{(for } Q = 10)$$

and $$= 1.9 \ \text{(for } Q = 10)$$

4.134 The A.C. current must be

$$I = I_0\sqrt{2}\,\sin\omega t$$

Then D.C. component of the rectified current is

$$< I' > = \frac{1}{T}\int_0^{T/2} I_0\sqrt{2}\,\sin\omega t\;dt$$

$$= I_0\sqrt{2}\,\frac{1}{2\pi}\int_0^{\pi}\sin\theta\, d\theta$$

$$= \frac{I_0\sqrt{2}}{\pi}$$

Since the charge deposited must be the same

$$I_0 t_0 = \frac{I_0\sqrt{2}}{\pi}\,t \quad \text{or} \quad t = \frac{\pi t_0}{\sqrt{2}}$$

(The answer is incorrect in the answer sheet.)

4.135 (a) In this case $\qquad I(t) = I_1\,\dfrac{t}{T} \quad (\text{for } 0 \le t < T)$

$$I\,(t \pm T) = I(t)$$

Now, mean current

$$< I > = \frac{1}{T}\int_0^T I_1\,\frac{t}{T}\,dt = I_1\,\frac{T^2/2}{T^2} = \frac{I_1}{2}$$

Then, $\qquad I_1 = 2I_0 \;\text{ sin } < I > = I_0$

Now, mean square current

$$<I^2> = 4I_0^2\,\frac{1}{T}\int_0^T \frac{t^2}{T^2}\,dt = \frac{4I_0^2}{3}$$

So, $\qquad \text{effective current} = \dfrac{2I_0}{\sqrt{3}} \approx 1.51 I_0$

(b) In this case $\qquad I = I_1\,|\sin\omega t\,|$

and $\qquad I_0 = \dfrac{1}{T}\displaystyle\int_0^T I_1\,|\sin\omega t\,|\,dt$

$$= \frac{1}{2\pi}\,I_1\int_0^{2\pi}|\sin\theta\,|\,d\theta = \frac{I_1}{\pi}\int_0^{\pi}\sin\theta\,d\theta = \frac{2I_1}{\pi}$$

So, $\qquad I_1 = \dfrac{\pi I_0}{2}$

Then, mean square current

$$< I^2 > = \frac{\pi^2 I_0^2}{4T}\int_0^T \sin^2\,\omega t\;dt$$

$$= \frac{\pi^2 I_0^2}{4} \times \frac{1}{2\pi} \int_0^{2\pi} \sin^2\theta \, d\theta = \frac{\pi^2 I_0^2}{8}$$

So, effective current $= \dfrac{\pi I_0}{\sqrt{8}} \approx 1.11 I_0$

4.136 We have $$P_{\text{D.C.}} = \frac{V_0^2}{R}$$

and $$P_{\text{A.C.}} = \frac{V_0^2}{\sqrt{R^2 + \omega^2 L^2}} \cdot \frac{R}{\sqrt{R^2 + \omega^2 L^2}}$$

$$= \frac{V_0^2/R}{1 + (\omega L/R)^2} = \frac{P_{\text{D.C.}}}{\eta}$$

Thus, $$\frac{\omega L}{R} = \sqrt{\eta - 1}$$

or $$\omega = \frac{R}{L}\sqrt{\eta - 1}$$

$$\nu = \frac{R}{2\pi L}\sqrt{\eta - 1} = 2 \text{ kHz} \quad \text{(on substituting values)}$$

4.137 We have $$Z = \sqrt{R^2 + X_L^2} \quad \text{or} \quad R = \sqrt{Z^2 - X_L^2}$$

Then, $$\tan\varphi = \frac{X_L}{\sqrt{Z^2 - X_L^2}}$$

So, $$\cos\varphi = \frac{\sqrt{Z^2 - X_L^2}}{Z} = \sqrt{1 - \left(\frac{X_L}{Z}\right)^2}$$

Therefore, $$\varphi = \cos^{-1}\sqrt{1 - \left(\frac{X_L}{Z}\right)^2} = 37°$$

The current lags by φ behind the voltage.

Also $$P = VI\cos\varphi = \frac{V^2}{Z^2}\sqrt{Z^2 - X_L^2} = 0.16 \text{ kW}$$

4.138 We have $$P = \frac{V^2 (R + r)}{(R + r)^2 + \omega^2 L^2}$$

$$P = \frac{V^2}{R + r + \left(\frac{(\omega L)^2}{R + r}\right)} = \frac{V^2}{\left[\sqrt{R + r} - \frac{\omega L}{\sqrt{R + r}}\right]^2 + 2\omega L}$$

This is maximum when $R + r = \omega L$ or $R = \omega L - r$ and thus, $P_{max} = V^2/2\omega L$.

On substituting values, we get $R = 200\ \Omega$ and $P_{max} = 0.114$ kW.

4.139 We have
$$P = \frac{V^2 R}{R^2 + (X_L - X_C)^2}$$

Varying the capacitor does not change R, so if P increases n times
$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

must decrease $\sqrt{n}$ times.

Thus, $\cos\varphi = R/Z$ increases $\sqrt{n}$ times.

Therefore, percentage increase in $\cos\varphi = (\sqrt{n} - 1) \times 100\,\% = 30.4\,\%$

4.140 We have
$$P = \frac{V^2 R}{R^2 + (X_L - X_C)^2}$$

At resonance
$$X_L = X_C \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$$

Power generated will decrease n times when
$$(X_L - X_C)^2 = \left(\omega L - \frac{1}{\omega C}\right)^2 = (n - 1) R^2$$

or
$$\omega - \frac{\omega_0^2}{\omega} = \pm\sqrt{n - 1}\,\frac{R}{L} = \pm\sqrt{n - 1}\,2\beta$$

Thus,
$$\omega^2 \pm 2\sqrt{2n - 1}\,\beta\omega - \omega_0^2 = 0$$
$$(\omega \pm \sqrt{n - 1}\,\beta)^2 = \omega_0^2 + (n - 1)\beta^2$$

or
$$\frac{\omega}{\omega_0} = \sqrt{1 + (n - 1)\frac{\beta^2}{\omega_0^2}} \pm \sqrt{n - 1}\,\frac{\beta}{\omega_0}$$

(taking only the positive sign in the first term to ensure positive value for ω/ω_0).

Now, quality factor
$$Q = \frac{\omega}{2\beta} = \frac{1}{2}\sqrt{\left(\frac{\omega_0}{\beta}\right)^2 - 1}$$

$$\frac{\omega_0}{\beta} = \sqrt{1 + 4Q^2}$$

Thus,
$$\frac{\omega}{\omega_0} = \sqrt{1 + \frac{n-1}{(1+4Q^2)}} \pm \frac{\sqrt{n-1}}{\sqrt{1+4Q^2}}$$

For large Q

$$\left|\frac{\omega - \omega_0}{\omega_0}\right| = \frac{\sqrt{n-1}}{2Q} \times 100 = 0.5\%$$

4.141 We have
$$V_1 = \frac{VR}{\sqrt{(R+R_1)^2 + X_L^2}} \quad \text{and} \quad V_2 = \frac{V\sqrt{R_1^2 + X_L^2}}{\sqrt{(R+R_1)^2 + X_L^2}}$$

So,
$$(R+R_1)^2 + X_L^2 = \left(\frac{VR}{V_1}\right)^2 \quad \text{and} \quad R_1^2 + X_L^2 = \left(\frac{V_2 R}{V_1}\right)^2$$

Hence,
$$R^2 + 2RR_1 = \frac{R^2}{V_1^2}(V^2 - V_2^2)$$

or
$$R_1 = \frac{R}{2V_1^2}(V^2 - V_2^2 - V_1^2)$$

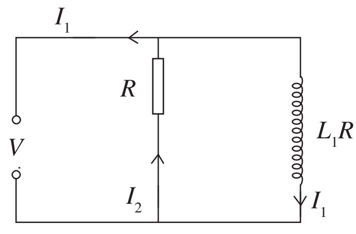
Heat power generated in the coil is

$$\frac{V^2 R_1}{\sqrt{(R_1 + R_2)^2 + X_L^2}} = \frac{V_1^2}{R^2} \times R_1 = \frac{V_1^2}{R^2} \times \frac{R^2}{2V_1^2}(V^2 - V_1^2 - V_2^2)$$

$$= \frac{V^2 - V_1^2 - V_2^2}{2R} = 30 \text{ W}$$

4.142 Here
$$I_2 = \frac{V}{R} \quad (V = \text{effective voltage})$$

$$I_1 = \frac{V}{\sqrt{R^2 + X_L^2}}$$



and
$$I = \frac{V\sqrt{(R+R_1)^2 + X_L^2}}{R\sqrt{R_1^2 + X_L^2}} \cong \frac{V}{R_{\text{eff}}}$$

R_{eff} is the impedance of the coil and the resistance in parallel.

Now,
$$\frac{I^2 - I_2^2}{I_2^2} = \frac{R^2 + 2RR_1}{R_1^2 + X_L^2} = \left(\frac{I_1}{I_2}\right)^2 + \frac{2RR_1}{R^2 + X_L^2}$$

$$\frac{I^2 - I_2^2 - I_1^2}{I_2^2} = \frac{2RR_1}{R^2 + X_L^2}$$

The mean power consumed in the coil is

$$I_1^2 R_1 = \frac{V^2 R_1}{R^2 + X_L^2} = I_2^2 R$$

$$= \frac{I^2 - I_1^2 - I_2^2}{2I_2^2} = \frac{1}{2} R\,(I^2 - I_1^2 - I_2^2) = 2.5 \text{ W}$$

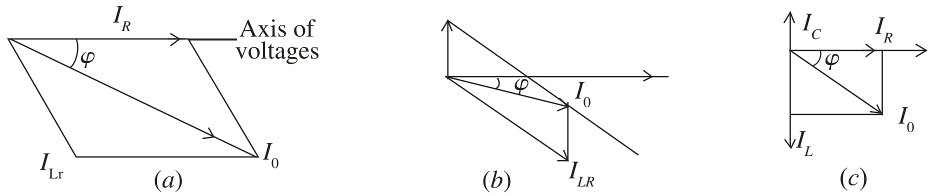
4.143 We have $\qquad \dfrac{1}{Z} = \dfrac{1}{R} + \dfrac{1}{1/i\,\omega C} = \dfrac{1}{R} + i\omega C = \dfrac{1 + i\omega RC}{R}$

$$|\,Z\,| = \frac{R}{\sqrt{1 + (\omega RC)^2}} = 40\ \Omega$$

4.144 For Fig. (a): The resistance, the voltage and the current are in phase. For the coil, the voltage is ahead of the current by less than 90°. The current is obtained by addition because the elements are in parallel.

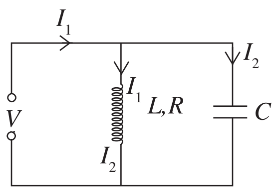
For Fig. (b): I_C is ahead of the voltage by 90°.

For Fig. (c): The coil has no resistance, so I_L is 90° behind the voltage.



4.145 When the coil and the condenser are in parallel, the equation is

$$L\frac{dI_1}{dt} + RI_1 = \frac{\int I_2\,dt}{C} = V_m \cos \omega t$$



And $\qquad I = I_1 + I_2$

Using complex voltages

$$I_1 = \frac{V_m e^{i\omega t}}{R + i\omega L} \quad \text{and} \quad I_2 = i\omega C V_m e^{i\omega t}$$

and $\qquad I = \left(\dfrac{1}{R + i\omega L} + i\omega C\right) V_m\ e^{i\omega t} = \left[\dfrac{R - i\omega L + i\omega C\,(R^2 + \omega^2 L^2)}{R^2 + \omega^2 L^2}\right] V_m\ e^{i\omega t}$

Thus, taking real parts

$$I = \frac{V_m}{|Z(\omega)|} \quad \cos(\omega t - \varphi)$$

when $$\frac{1}{|Z(\omega)|} = \frac{[R^2 + \{\omega C(R^2 + \omega^2 L^2) - \omega L\}^2]}{(R^2 + \omega^2 L^2)^{1/2}}$$

and $$\tan \varphi = \frac{\omega L - \omega C(R^2 + \omega^2 L^2)}{R}$$

(a) To get the frequency of resonance, first we must define resonance. One definition requires rapid change of phase with φ passing through zero at resonance. For the series circuit

$$I_m = \frac{V_m}{\{R^2 + (\omega L - 1/\omega C)^2\}^{1/2}} \quad \text{and} \quad \tan \varphi = \frac{\omega L - (1/\omega C)}{R}$$

Both definitions give $\omega^2 = 1/LC$ at resonance. In the present case, the two definitions do not agree (except when $R = 0$). The definition that has been adopted in the answer given in the book is the vanishing of phase. This requires

$$C(R^2 + \omega^2 L^2) = L$$

or $$\omega^2 = \frac{1}{LC} - \frac{R^2}{L^2} = \omega_{\text{res}}^2, \text{ so } \omega_{\text{res}} = 31.6 \times 10^3 \text{ rad/s}$$

Note that for small R, φ rapidly changes from $-\pi/2$ to $+\pi/2$ as ω passes through ω_{res} from $< \omega_{\text{res}} >$ to ω_{res}.

(b) At resonance $$I_m = \frac{V_m R}{L/C} = V_m \frac{CR}{L}$$

So effective value of total current

$$I = V\frac{CR}{L} = 3.1 \text{ mA}$$

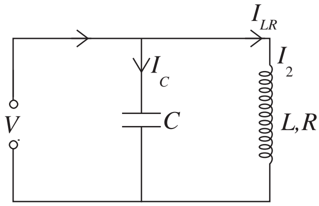
Similarly $$I_L = \frac{V}{\sqrt{L/C}} = V\sqrt{\frac{C}{L}} = 0.98 \text{ A}$$

And $$I_C = \omega CV = V\sqrt{\frac{C}{L} - \frac{R^2C^2}{L^2}} \cong 0.98 \text{ A}$$

Note: *The vanishing of phase (its passing through zero) is considered a more basic definition of resonance.*

4.146 We use the method of complex voltage

Here, $$V = V_0 e^{i\omega t}$$



Then, $$I_C = \frac{V_0 e^{i\omega t}}{1/i\omega C} = i\,\omega C V_0\, e^{i\omega t}$$

and $$I_{LR} = \frac{V_0 e^{i\omega t}}{R + i\omega L}$$

Then $$I = I_C + I_{LR} = V_0\,\frac{R - i\omega L + i\omega C\,(R^2 + \omega^2 L^2)}{R^2 + \omega^2 L^2}\,e^{i\omega t}$$

Then taking the real part

$$I = \frac{V_0\sqrt{R^2 + \{\omega C\,(R^2 + \omega^2 L^2) - \omega L\}^2}}{R^2 + \omega^2 L^2}\cos(\omega t - \varphi)$$

where $$\tan\varphi = \frac{-\,\omega L + \omega C\,(R^2 + \omega^2 L^2)}{R}$$

4.147 From the previous problem

$$Z = \frac{R^2 + \omega^2 L^2}{\sqrt{R^2 + \{\omega C\,(R^2 + \omega^2 L^2) - \omega L\}^2}}$$

$$= \frac{R^2 + \omega^2 L^2}{\sqrt{(R^2 + \omega^2 L^2)\,(1 - 2\omega^2 LC) + \omega^2 C^2\,(R^2 + \omega^2 L^2)^2}}$$

$$= \frac{\sqrt{R^2 + \omega^2 L^2}}{\sqrt{(1 - 2\omega^2 LC) + \omega^2 C^2\left(R^2 + \omega^2 L^2\right)}} = \frac{\sqrt{R^2 + \omega^2 L^2}}{\sqrt{(1 - \omega^2 LC)^2 + (\omega RC)^2}}$$

4.148 (a) We have

$$\varepsilon = -\frac{d\Phi}{dt} = \omega\Phi_0\,\sin\omega t = L\dot{I} + RI$$

Using $$I = I_m\,\sin(\omega t - \varphi)$$

we get $$\omega\Phi_0\,\sin\omega t = \omega\Phi_0\,\left\{\sin(\omega t - \varphi)\cos\varphi + \cos(\omega t - \varphi)\sin\varphi\right\}$$

$$= LI_m\,\omega\cos(\omega t - \varphi) + RI_m\,\sin(\omega t - \varphi)$$

$$RI_m = \omega\Phi_0\cos\varphi \quad \text{and} \quad LI_m = \Phi_0\sin\varphi$$

or $$I_m = \frac{\omega\Phi_0}{\sqrt{R^2 + \omega^2 L^2}} \quad \text{and} \quad \tan\varphi = \frac{\omega L}{R}$$

(b) Mean mechanical power required to maintain rotation = energy loss per unit time

$$= \frac{1}{T}\int_0^T RI^2\,dt = \frac{1}{2}RI_m^2 = \frac{1}{2}\,\frac{\omega^2\Phi_0^2\,R}{R^2 + \omega^2L^2}$$

4.149 We consider the force $\mathbf{F}_{12}$ that a circuit 1 exerts on another closed circuit 2:

$$\mathbf{F}_{12} = \oint I_2\,d\mathbf{l}_2 \times \mathbf{B}_{12}$$

Here $\mathbf{B}_{12}$ = magnetic field at the site of the current element $d\mathbf{l}_2$ due to the current I_1 flowing in 1 and is given by

$$\frac{\mu_0}{4\pi}\int \frac{I_1 d\,\mathbf{l}_1 \times \mathbf{r}_{12}}{r_{12}^3}$$

where $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ = vector from current element $d\mathbf{l}_1$ to the current element $d\mathbf{l}_2$.

Now,

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi}\iint I_1I_2\,\frac{d\mathbf{l}_2 \times (d\mathbf{l}_1 \times \mathbf{r}_{12})}{r_{12}^3} = \frac{\mu_0}{4\pi}\iint I_1I_2\,\frac{d\mathbf{l}_1\,(d\mathbf{l}_2\cdot\mathbf{r}_{12}) - (d\mathbf{l}_1\cdot d\mathbf{l}_2)\mathbf{r}_{12}}{r_{12}^3}$$

In the first term, we carry out the integration over $d\mathbf{l}_\tau$ first. Then,

$$\iint \frac{d\mathbf{l}_1\,(d\mathbf{l}_2\cdot\mathbf{r}_{12})}{r_{12}^3} = \int d\mathbf{l}_1 \oint \frac{d\mathbf{l}_2\cdot\mathbf{r}_{12}}{r_{12}^3} = -\int d\mathbf{l}_1 \oint d\mathbf{l}_2\cdot\nabla_2\frac{1}{r_{12}} = 0$$

because $\oint d\mathbf{l}_2\cdot\nabla_2\frac{1}{r_{12}} = \int d\mathbf{S}_2\ \text{curl}\left(\nabla\frac{1}{r_{12}}\right) = 0$

Thus, $\mathbf{F}_{12} = -\frac{\mu_0}{4\pi}\iint I_1I_2\,d\mathbf{l}_1\cdot d\mathbf{l}_2\,\frac{\mathbf{r}_{12}}{r_{12}^3}$

The integral involved will depend on the vector **a** that defines the separation of the (suitably chosen) centre of the coils. Let C_1 and C_2 be the centres of the two coils suitably defined.

Then, $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1 = \boldsymbol{\rho}_2 - \boldsymbol{\rho}_1 + \mathbf{a}$

where $\boldsymbol{\rho}_1$ ($\boldsymbol{\rho}_2$) is the distance of $d\mathbf{l}_1(d\mathbf{l}_2)$ from $C_1(C_2)$ and **a** stands for the vector $\mathbf{C}_1\mathbf{C}_2$.

Then, $\frac{\mathbf{r}_{12}}{r_{12}^3} = -\nabla_{\mathbf{a}}\frac{1}{r_{12}}$

and $\mathbf{F}_{12} = \nabla_{\mathbf{a}}\left[I_1I_2\,\frac{\mu_0}{4\pi}\iint\frac{d\,\mathbf{l}_1\cdot d\,\mathbf{l}_2}{r_{12}}\right]$

The bracket defines the mutual inductance L_{12}. Thus noting the definition of x

$$< F_x > = \frac{\partial L_{12}}{\partial x} < I_1I_2 >$$

where < > denotes time average.

Now $I_1 = I_0\cos\omega t$ = Real part of $I_0\,e^{i\omega t}$

The current in the coil 2 satisfies

$$RI_2 + L_2 \frac{dI_2}{dt} = -L_{12} \frac{dI_1}{dt}$$

or
$$I_2 = \frac{-i\omega L_{12}}{R + i\omega L_2} I_0 \, e^{i\omega t} \quad \text{(in the complex case)}$$

Taking the real part, we get

$$I_2 = \frac{\omega L_{12} I_0}{R^2 + \omega^2 L_2^2} (\omega L_2 \cos \omega t - R \sin \omega t) = -\frac{\omega L_{12}}{\sqrt{R^2 + \omega^2 L_\tau^2}} I_0 \cos (\omega t + \varphi)$$

(where $\tan \varphi = R/\omega L_2$).

Taking time average, we get

$$< F_x > = \frac{\partial L_{12}}{\partial x} I_0 \frac{\omega L_{12} I_0}{\sqrt{R^2 + \omega^2 L_2^2}} \cdot \frac{1}{2} \cos \varphi = \frac{\omega^2 L_2 L_{12} I_0^2}{2(R^2 + \omega^2 L_2^2)} \frac{\partial L_{12}}{\partial x}$$

The repulsive nature of the force is also consistent with Lenz's law, assuming of course, that L_{12} decreases with x.

4.3 Elastic Waves. Acoustics

4.150 Since the temperature varies linearly we can write the temperature as a function of x, which is the distance from the point A towards B, i.e.,

$$T = T_1 + \frac{T_2 - T_1}{l} \quad [\text{for } 0 < x < l]$$

Hence,
$$dT = \left(\frac{T_2 - T_1}{l}\right) dx \tag{1}$$

In order to travel an elemental distance of dx which is at a distance of x from A, the time taken will be

$$dt = \frac{dx}{\alpha \sqrt{T}} \tag{2}$$

From Eqs. (1) and (2), expressing dx in terms of dT, we get

$$dt = \frac{l}{\alpha \sqrt{T}} \left(\frac{dT}{T_2 - T_1}\right)$$

which on integration gives

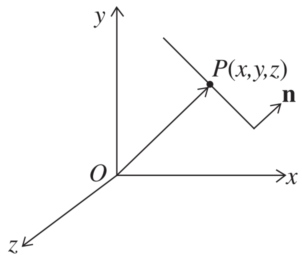
$$\int_0^t dt = \frac{l}{\alpha (T_2 - T_1)} \int_{T_1}^{T_2} \frac{dT}{\sqrt{T}}$$

or $$t = \frac{2l}{\alpha(T_2 - T_1)}\left(\sqrt{T_2} - \sqrt{T_1}\right)$$

Hence, the sought time

$$t = \frac{2l}{\alpha\left(\sqrt{T_1} + \sqrt{T_2}\right)}$$

4.151 Equation of plane wave is given by $\xi(r, t) = a \cos(\omega t - \mathbf{k} \cdot \mathbf{r})$, where $\mathbf{k} = \omega/v\ \mathbf{n}$ called the wave vector and $\mathbf{n}$ is the unit vector normal to the wave surface in the direction of the propagation of wave.



or $$\xi(x, y, z) = a \cos(\omega t - k_x x - k_y y - k_z z)$$

$$= a \cos(\omega t - kx \cos\alpha - ky \cos\beta - kz \cos\gamma)$$

Thus, $\xi(x_1, y_1, z_1, t) = a \cos(\omega t - kx_1 \cos\alpha - ky_1 \cos\beta - kz_1 \cos\gamma)$

and $\xi(x_2, y_2, z_2, t) = a \cos(\omega t - kx_2 \cos\alpha - ky_2 \cos\beta - kz_2 \cos\gamma)$

Hence, the sought wave phase difference

$$\varphi_2 - \varphi_1 = k[(x_1 - x_2)\cos\alpha + (y_1 - y_2)\cos\beta + (z_1 - z_2)\cos\gamma]$$

or $$\Delta\varphi = |\varphi_2 - \varphi_1| = k|[(x_1 - x_2)\cos\alpha + (y_1 - y_2)\cos\beta + (z_1 - z_2)\cos\gamma]|$$

$$= \frac{\omega}{v}|[(x_1 - x_2)\cos\alpha + (y_1 - y_2)\cos\beta + (z_1 - z_2)\cos\gamma]|$$

4.152 The phase of the oscillation can be written as

$$\Phi = \omega t - \mathbf{k} \cdot \mathbf{r}$$

When the wave moves along the x-axis

$$\Phi = \omega t - k_x x \quad (\text{on putting } k_y = k_z = 0)$$

Since the velocity associated with this wave is v_1

we have $$k_x = \frac{\omega}{v_1}$$

Similarly, $$k_y = \frac{\omega}{v_2} \quad \text{and} \quad k_z = \frac{\omega}{v_3}$$

Thus, $$\mathbf{k} = \frac{\omega}{v_1}\mathbf{e}_x + \frac{\omega}{v_2}\mathbf{e}_y + \frac{\omega}{v_3}\mathbf{e}_z = \omega\left(\frac{\mathbf{e}_x}{v_1} + \frac{\mathbf{e}_y}{v_2} + \frac{\mathbf{e}_z}{v_3}\right)$$

4.153 The wave equation propagating in the direction of +*ve* x-axis in medium K is given as

$$\xi = a\cos(\omega t - kx)$$

So, $$\xi = a\cos k(vt - x) \quad \left(\text{where } k = \frac{\omega}{v} \text{ and } v \text{ is the wave velocity}\right)$$

In the reference frame K', the wave velocity will be $(v - V)$ propagating in the direction of $+ve$ x-axis and x will be x'. Thus, the sought wave equation

$$\xi = a\cos k[(v - V)\,t - x']$$

or $$\xi = a\cos\left[\left(\omega - \frac{\omega}{v}V\right)t - kx'\right] = a\cos\left[\omega t\left(1 - \frac{V}{v}\right) - kx'\right]$$

4.154 This follows on actually putting

$$\xi = f(t + \alpha x)$$

in the wave equation $$\frac{\partial^2\xi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\xi}{\partial t^2}$$

(We have written the one-dimensional form of the wave equation.)

Then, $$\frac{1}{v^2}f''(t + \alpha x) = \alpha^2 f''(t + \alpha x)$$

The wave equation is satisfied if

$$\alpha = \pm\frac{1}{v'}$$

That is the physical meaning of the constant α.

4.155 The given wave equation

$$\xi = 60\cos(1800\,t - 5.3\,x)$$

is of the type $$\xi = a\cos(\omega t - kx)$$

where $a = 60 \times 10^{-6}$ m, $\omega = 1800\ \text{s}^{-1}$ and $k = 5.3\ \text{m}^{-1}$.

As $$k = \frac{2\pi}{\lambda} \quad \text{so,} \quad \lambda = \frac{2\pi}{k}$$

Also $$k = \frac{\omega}{v} \quad \text{so,} \quad v = \frac{\omega}{k} = 340\ \text{m/s}$$

(a) Sought ratio is $$\frac{a}{\lambda} = \frac{ak}{2\pi} = 5.1 \times 10^{-5}$$

(b) Since $$\xi = a \cos(\omega t - kx)$$

$$\frac{\partial \xi}{\partial t} = -a\omega \sin(\omega t - kx)$$

So, velocity oscillation amplitude

$$\left(\frac{\partial \xi}{\partial t}\right)_m \quad \text{or} \quad v_m = a\omega = 0.11 \text{ m/s} \tag{1}$$

and the sought ratio of velocity oscillation amplitude to the wave propagation velocity

$$\frac{v_m}{v} = \frac{0.11}{340} = 3.2 \times 10^{-4}$$

(c) Relative deformation

$$\frac{\partial \xi}{\partial x} = ak \sin(\omega t - kx)$$

So, relative deformation amplitude

$$\left(\frac{\partial \xi}{\partial x}\right)_m = ak = (60 \times 10^{-6} \times 5.3) \text{ m} = 3.2 \times 10^{-4} \text{ m} \tag{2}$$

From Eqs. (1) and (2)

$$\left(\frac{\partial \xi}{\partial x}\right)_m = ak = \frac{a\omega}{v} = \frac{1}{v}\left(\frac{\partial \xi}{\partial t}\right)_m$$

Thus, $$\left(\frac{\partial \xi}{\partial x}\right)_m = \frac{1}{v}\left(\frac{\partial \xi}{\partial t}\right)_m$$

where $v = 340$ m/s is the wave velocity.

4.156 (a) The given equation is

$$\xi = a \cos(\omega t - kx)$$

So at $t = 0$, $$\xi = a \cos kx$$

Now, $$\frac{d\xi}{dt} = -a\omega \sin(\omega t - kx)$$

and $$\frac{d\xi}{dt} = a\omega \sin kx \quad (\text{at } t = 0)$$

Also, $$\frac{d\xi}{dx} = +ak \sin(\omega t - kx)$$

and at $t = 0$,

$$\frac{d\xi}{dx} = -ak \sin kx$$

Hence, all the graphs are similar but have different amplitudes, as shown in the answer sheet of the problem book.

(b) At the points where $\xi = 0$, the velocity direction is positive, i.e., along $+ve$ x-axis in the case of longitudinal and $+ve$ y-axis in the case of transverse waves, where $d\xi/dt$ is positive and vice versa.

For sought plots see the answer sheet of the problem book.

4.157 In the given wave equation the particle's displacement amplitude $= ae^{-\gamma x}$.

Let there be two points x_1 and x_2, between which the displacement amplitude differs by $\eta = 1\%$.

So, $$ae^{-\gamma x_1} - ae^{-\gamma x_2} = \eta ae^{-\gamma x_1}$$

or $$e^{-\gamma x_1}(1-\eta) = e^{-\gamma x_2}$$

or $$\ln(1-\eta) - \gamma x_1 = -\gamma x_2$$

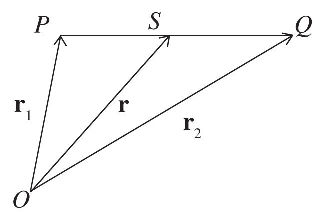
or $$x_2 - x_1 = -\frac{\ln(1-\eta)}{\gamma}$$

So, $$\text{path difference} = -\frac{\ln(1-\eta)}{\gamma}$$

and $$\text{phase difference} = \frac{2\pi}{\lambda} \times \text{path difference}$$

$$= -\frac{2\pi}{\lambda}\frac{\ln(1-\eta)}{\gamma} \cong \frac{2\pi\eta}{\lambda\gamma} = 0.3 \text{ rad}$$

4.158 Let S be the source whose position vector relative to the reference point O is $\mathbf{r}$. Since intensities are inversely proportional to the square of distances,



$$\frac{\text{Intensity at } P(I_1)}{\text{Intensity at } Q(I_2)} = \frac{d_2^2}{d_1^2}$$

where $d_1 = PS$ and $d_2 = QS$.

But intensity is also proportional to the square of amplitude.

So, $$\frac{a_1^2}{a_2^2} = \frac{d_2^2}{d_1^2} \quad \text{or} \quad a_1 d_1 = a_2 d_2 = k \text{(say)}$$

Thus, $$d_1 = \frac{k}{a_1} \quad \text{and} \quad d_2 = \frac{k}{a_2}$$

Let **n** be the unit vector along *PQ* directed from *P* to *Q*.

Then, $$\mathbf{PS} = d_1\mathbf{n} = \frac{k}{a_1}\mathbf{n}$$

and $$\mathbf{SQ} = d_2\mathbf{n} = \frac{k}{a_2}\mathbf{n}$$

From the triangle law of vector addition

$$\mathbf{OP} + \mathbf{PS} = \mathbf{OS} \quad \text{or} \quad \mathbf{r}_1 + \frac{k}{a_1}\mathbf{n} = \mathbf{r}$$

or $$a_1\mathbf{r}_1 + k\mathbf{n} = a_1\mathbf{r} \tag{1}$$

Similarly, $$\mathbf{r} + \frac{k}{a_2}\mathbf{n} = \mathbf{r}_2 \quad \text{or} \quad a_2\mathbf{r}_2 - k\mathbf{n} = a_2\mathbf{r} \tag{2}$$

Adding Eqs. (1) and (2), we get

$$a_1\mathbf{r}_1 + a_2\mathbf{r}_2 = (a_1 - a_2)\mathbf{r}$$

Hence, $$\mathbf{r} = \frac{a_1\mathbf{r}_1 + a_2\mathbf{r}_2}{a_1 + a_2}$$

4.159 (a) We know that the equation of a spherical wave in a homogenous absorbing medium of wave damping coefficient γ is

$$\xi = \frac{a_0' e^{-\gamma r}}{r}\cos(\omega t - kr)$$

Thus particle's displacement amplitude equals

$$\frac{a_0' e^{-\gamma r}}{r}$$

According to the conditions of the problem,

at $r = r_0$, $$a_0 = \frac{a_0' e^{-\gamma r_0}}{r_0} \tag{1}$$

and when $r = r$, $$\frac{a_0}{\eta} = \frac{a_0' e^{-\gamma r}}{r} \tag{2}$$

Thus, from Eqs. (1) and (2)

$$e^{\gamma(r - r_0)} = \eta\frac{r_0}{r}$$

or $$\gamma(r - r_0) = \ln(\eta r_0) - \ln r$$

or $$\gamma = \frac{\ln \eta + \ln r_0 - \ln r}{r - r_0}$$

$$= \frac{\ln 3 + \ln 5 - \ln 10}{5} = 0.08 \text{ m}^{-1}$$

(b) As $$\xi = \frac{a_0' e^{-\gamma r}}{r} \cos(\omega t - kr)$$

So, $$\frac{\partial \xi}{\partial t} = -\frac{a_0' e^{-\gamma r}}{r} \omega \sin(\omega t - kr)$$

$$\left(\frac{\partial \xi}{\partial t}\right)_n = \frac{a_0' e^{-\gamma r}}{r} \omega$$

But at point A, $$\frac{a_0' e^{-\gamma r}}{r} = \frac{a_0}{\eta}$$

So, $$\left(\frac{\partial \xi}{\partial t}\right)_m = \frac{a_0 \omega}{\eta} = \frac{a_0 2\pi\nu}{\eta}$$

$$= \frac{50 \times 10^{-6}}{3} \times 2 \times \frac{22}{7} \times 1.45 \times 10^3 = 15 \text{ cm/s}$$

4.160 (a) When the waves are transverse:

Equation of the resultant wave is

$$\xi = \xi_1 + \xi_2 = 2a \cos k\left(\frac{y - x}{2}\right) \cos\left\{\omega t - \frac{k(x + y)}{2}\right\}$$

$$= a' \cos\left\{\omega t - \frac{k(x + y)}{2}\right\} \quad \left[\text{where } a' = 2a \cos k\left(\frac{y - x}{2}\right)\right]$$

Now, the equation of wave pattern is

$$x + y = k \quad (k \text{ is a constant})$$

For sought plots see the answer sheet of the problem book.

For antinodes, i.e., maximum intensity

$$\cos \frac{k(y - x)}{2} = \pm 1 = \cos n\pi$$

or $$\pm(x - y) = \frac{2n\pi}{k} = n\lambda$$

or $$y = x \pm n\lambda \quad (\text{for } n = 0, 1, 2, \ldots)$$

Hence, the particles of the medium at the points lying on the solid straight lines $(y = x \pm n\lambda)$ oscillate with maximum amplitude. For nodes, i.e., minimum intensity,

$$\cos\frac{k(y-x)}{2} = 0$$

or $$\pm\frac{k(y-x)}{2} = (2n+1)\frac{\pi}{2}$$

or $$y = x \pm (2n+1)\frac{\lambda}{2}$$

and hence the particles at the points lying on dotted lines do not oscillate.

(b) When the waves are longitudinal:

For sought plots see the answer sheet of the problem book.

$$k(y-x) = \cos^{-1}\frac{\xi_1}{a}\cos^{-1}\frac{\xi_2}{a}$$

or $$\frac{\xi_1}{a} = \cos\left\{k(y-x) + \cos^{-1}\frac{\xi_2}{a}\right\}$$

$$= \frac{\xi_2}{a}\cos k(y-x) - \sin k(y-x)\sin\left(\cos^{-1}\frac{\xi_2}{a}\right)$$

$$= \frac{\xi_2}{a}\cos k(y-x) - \sin k(y-x)\sqrt{1-\frac{\xi_2^2}{a^2}} \qquad (1)$$

From Eq. (1), if $\sin k(y-x) = 0\sin(n\pi)$

$$\xi_1 = \xi_2\,(-1)^n$$

Thus, the particles of the medium at the points lying on the straight lines $(y = x \pm n\lambda/2)$ will oscillate along those lines (for even n) or at right angles to them (for odd n).

Also from Eq. (1),

$$\cos k(y-x) = 0 = \cos(2n-1)\frac{\pi}{2}$$

$$\frac{\xi_1^2}{a^2} = 1 - \frac{\xi_2^2}{a^2} \quad \text{(a circle)}$$

Thus, the particles at the points, where $y = x \pm (n \pm 1/4)\,\lambda$, will oscillate along circles. In general, all other particles will move along ellipses.

4.161 The displacement of oscillations is given by $\xi = a\cos(\omega t - kx)$. Without loss of generality, we confine ourselves to $x = 0$. Then the displacement maxima occurs at $\omega t = n\pi$. Now the energy density is given by

$$w = \rho a^2\omega^2\sin^2\omega t \qquad (\text{at } x = 0)$$

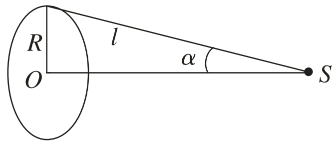
At $T/6$ time later than $t = 0$ (where $T = 2\pi/\omega$ is the time period),

$$w = \rho a^2\omega^2 \sin^2 \frac{\pi}{3} = \frac{3}{4} \rho a^2\omega^2 = w_0$$

Thus, $$< w > = \frac{1}{2}\rho a^2\omega^2 = \frac{2w_0}{3}$$

4.162 The power output of the source must be

$$4\pi l^2 I_0 = Q$$



The required flux of acoustic power is then

$$Q = \frac{\Omega}{4\pi}$$

where Ω is the solid angle subtended by the disk enclosed by the ring at *S*. This solid angle is

$$\Omega = 2\pi(1 - \cos \alpha) \tag{1}$$

So, flux

$$\Phi = 2\pi l^2 I_0 \left[1 - \frac{1}{\sqrt{1 + (R/L)^2}}\right]$$

Substitution gives $$\Phi = 2\pi \times 30 \left(1 - \frac{l}{\sqrt{1 + 1/4}}\right) \mu\text{W} = 20\ \mu\text{W}$$

Eq. (1) is a well-known result which is derived as follows. Let *SO* be the polar axis. Then the required solid angle is the area of that part of the surface of a sphere of such radius so that its co-latitude is $\leqslant \alpha$.

Thus, $$\Omega = \int_0^\alpha 2\pi \sin \theta\, d\theta = 2\pi\,(1 - \cos \alpha)$$

4.163 From the result of Problem 4.162, power flowing out through any one of the openings is

$$\frac{P}{2}\left(1 - \frac{h/2}{\sqrt{R^2 + (h/2)^2}}\right)$$

$$= \frac{P}{2}\left(1 - \frac{h}{\sqrt{4R^2 + h^2}}\right)$$

As total power output equals *P*, so the power reaching the lateral surface must be

$$P - 2\cdot \frac{P}{2}\left(1 - \frac{h}{\sqrt{4R^2 + h^2}}\right) = \frac{Ph}{\sqrt{4R^2 + h^2}} = 0.07\ \text{W}$$

4.164 We are given $$\xi = a \cos kx \cos \omega t$$

So, $$\frac{\partial \xi}{\partial x} = -ak \sin kx \cos \omega t \quad \text{and} \quad \frac{\partial \xi}{\partial t} = -a\omega \cos kx \sin \omega t$$

Thus, $$(\xi)_{t=0} = a \cos kx \quad \text{and} \quad (\xi)_{t=T/2} = -a \cos kx$$

Therefore, $$\left(\frac{\partial \xi}{\partial x}\right)_{t=0} = -ak \sin kx \quad \text{and} \quad \left(\frac{\partial \xi}{\partial x}\right)_{t=T/2} = ak \sin kx$$

(a) The graphs of (ξ) and $\left(\frac{\partial \xi}{\partial x}\right)$ are as shown in Fig. 35 of the answer sheet (p. 332).

(b) We can calculate the density as follows:

Take a parallelopiped of cross-section unity and length dx with its edges at x and $x + dx$. After the oscillation the edge at x goes to $x + \xi(x)$ and the edge at $x + dx$ goes to

$$x + dx + \xi\,(x + dx) = x + dx + \xi(x) + \frac{\partial \xi}{\partial x} dx$$

Thus, the volume of the element (originally dx) becomes

$$\left(1 + \frac{\partial \xi}{\partial x}\right) dx$$

and hence the density becomes

$$\rho = \frac{\rho_0}{1 + \partial \xi/\partial x}$$

On substituting, we get for the density $\rho(x)$, the curves shown in Fig. 35 referred to above.

(c) The velocity $v(x)$ at time $t = T/4$ is

$$\left(\frac{\partial \xi}{\partial t}\right)_{t=T/4} = -a\,\omega \cos kx$$

On plotting, we get the Fig. 36 given in the answer sheet (p. 332).

4.165 Given $$\xi = a \cos kx \cos \omega t$$

(a) The potential energy density (per unit volume) is the energy of longitudinal strain. This is

$$w_p = \left(\frac{1}{2} \text{ stress} \times \text{strain}\right) = \frac{1}{2} E \left(\frac{\partial \xi}{\partial x}\right)^2$$

(where $\partial \xi/\partial x$ is the longitudinal strain and E is the Young's modulus).

$$w_p = \frac{1}{2} E a^2 k^2 \sin^2 kx \cos^2 \omega t = \frac{1}{2} \rho a^2 \omega^2 \sin^2 kx \cos^2 \omega t$$

(b) The kinetic energy density is

$$w_k = \frac{1}{2}\rho\left(\frac{\partial \xi}{\partial t}\right)^2 = \frac{1}{2}\rho a^2 \omega^2 \cos^2 kx \sin^2 \omega t$$

On plotting we get Fig. 37 given in the answer sheet (p. 332).
For example, at $t = 0$

$$w = w_p + w_k = \frac{1}{2}\rho a^2 \omega^2 \sin^2 kx$$

and the displacement nodes are at $x = \pm \pi/2k$ so we do get the figure.

4.166 Let us denote the displacement of the elements of the string by

$$\xi = a \sin kx \cos \omega t$$

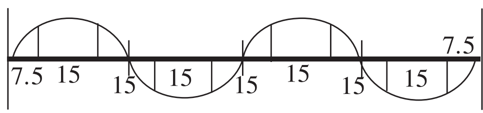
Since the string is 120 cm long we must have $k\,120 = n\pi$.
If x_1 is the distance at which the displacement amplitude first equals 3.5 mm, then

$$a \sin kx_1 = 3.5 = a \sin(kx_1 + 15k)$$

Then, $$kx_1 + 15k = \pi - kx_1 \quad \text{or} \quad kx_1 = \frac{\pi - 15k}{2}$$

Let us assume that the string has the form shown below in the figure.
It shows that

$$k \times 120 = 4\pi \quad \text{so,} \quad k = \frac{\pi}{30}\ \text{cm}^{-1}$$



Thus, we are dealing with the third overtone.

Also, $$kx_1 = \frac{\pi}{4} \quad \text{so,} \quad a = 3.5\sqrt{2}\ \text{mm} = 4.949\ \text{mm}$$

4.167 We have

$$n = \frac{1}{2l}\sqrt{T/m} = \frac{1}{2l}\sqrt{Tl/M} \quad \text{(where } M = \text{total mass of the wire)}$$

When the wire is stretched, total mass of the wire remains constant. For the first wire, the new length $= l + \eta_1 l$ and for the second wire, the length is $l + \eta_2 l$. Also $T_1 = \alpha(\eta_1 l)$, where α is a constant and $T_2 = \alpha(\eta_2 l)$. Substituting in the above formula

$$\nu_1 = \frac{1}{2(l + \eta_1 l)}\sqrt{\frac{(\alpha\,\eta_1 l)(l + \eta_1 l)}{M}}$$

$$\nu_2 = \frac{1}{2(l + \eta_2 l)}\sqrt{\frac{(\alpha\eta_2 l)(l + \eta_2 l)}{M}}$$

Therefore, $$\frac{\nu_2}{\nu_1} = \frac{1+\eta_1}{1+\eta_2}\sqrt{\frac{\eta_2}{\eta_1}\cdot\frac{1+\eta_2}{1+\eta_1}}$$

$$= \sqrt{\frac{\eta_2\,(1+\eta_1)}{\eta_1\,(1+\eta_2)}} = \sqrt{\frac{0.04\,(1+0.02)}{0.02\,(1+0.04)}} = 1.4$$

4.168 Let initial length and tension be l and T, respectively.

So, $$\nu_1 = \frac{1}{2l}\sqrt{\frac{T}{\rho_1}}$$

In accordance with the problem, the new length

$$l' = l - \frac{l \times 35}{100} = 0.65l$$

and new tension,

$$T' = T + \frac{T \times 70}{100} = 1.7T$$

Thus, the new frequency

$$\nu_2 = \frac{1}{2l'}\sqrt{\frac{T'}{\rho_1}} = \frac{1}{2 \times 0.65\,l}\sqrt{\frac{1.7\,T}{\rho_1}}$$

Hence, $$\frac{\nu_2}{\nu_1} = \frac{\sqrt{1.7}}{0.65} = \frac{1.3}{0.65} = 2$$

4.169 Obviously in this case the velocity of sound propagation will be

$$v = 2\nu\,(l_2 - l_1)$$

where l_2 and l_1 are consecutive lengths at which resonance occurs
In accordance with the problem,

$$(l_2 - l_1) = l$$

So, $$v = 2\nu l = 2 \times 2000 \times 8.5 \text{ cm/s} = 0.34 \text{ km/s}$$

4.170 (a) When the tube is closed at one end

$$\nu_n = \frac{v}{4l}\,(2n+1) \quad (\text{where } n = 0, 1, 2, \ldots)$$

$$= \frac{340}{4 \times 0.85}\,(2n+1) = 100\,(2n+1)$$

Thus for $n = 0, 1, 2, 3, 4, 5, 6, \ldots$, we get

$$\nu_1 = 100 \text{ Hz}, \nu_2 = 300 \text{ Hz}, \nu_3 = 500 \text{ Hz}, \nu_4 = 700 \text{ Hz},$$

$$\nu_5 = 900 \text{ Hz}, \nu_6 = 1100 \text{ Hz}, \nu_7 = 1300 \text{ Hz}$$

Since ν should be $> \nu_0 = 1250$ Hz, we need not go beyond ν_6.

Thus, 6 natural oscillations are possible.

(b) An organ pipe opened from both ends vibrates with all harmonics of the fundamental frequency. Now, the fundamental mode frequency is given as

$$\nu = \frac{v}{\lambda} \quad \text{(where } v \text{ is the velocity of sound)}$$

or $$\nu = \frac{v}{2l}$$

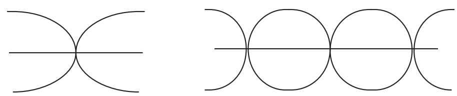
Here also, end correction has been neglected. So, the frequencies of higher modes of vibrations are given by

$$\nu = n\left(\frac{v}{2l}\right) \tag{1}$$

or $$\nu_1 = \frac{v}{2l}, \nu_2 = 2\left(\frac{v}{2l}\right), \nu_3 = 3\left(\frac{v}{2l}\right)$$

It may be checked by putting the values of n in the Eq. (1), that below 1285 Hz, there are a total of six possible natural oscillation frequencies of air column in the open pipe.

4.171 Since the copper rod is clamped at midpoint, it becomes a node and the two free ends will be antinodes. Thus, the fundamental node formed in the rod is as shown in Fig. (a).



(*a*) (*b*)

In this case, $$l = \frac{\lambda}{2}$$

So, $$\nu_0 = \frac{v}{2l} = \frac{1}{2l}\sqrt{\frac{E}{\rho}}$$

where E = Young's modulus and ρ is the density of copper.

Similarly, the second node or the first overtone in the rod is as shown above in Fig. (b).

Here, $$l = \frac{3\lambda}{2}$$

Hence, $$\nu_1 = \frac{3v}{2l} = \frac{3}{2l}\sqrt{\frac{E}{\rho}}$$

$$\nu = \frac{2n+1}{2l}\sqrt{\frac{E}{\rho}} \quad \text{(where } n = 0, 1, 2, \ldots)$$

Putting the given values of E and ρ in the general equation

$$\nu = 3.8(2n + 1) \text{ kHz}$$

Hence, $\nu_0 = 3.8$ kHz, $\nu_1 = (3.8 \times 3) = 11.4$ kHz, $\nu_2 = (3.8) \times 5 = 19$ kHz,

$\nu_3 = (3.8 \times 7) = 26.6$ kHz, $\nu_4 = (3.8 \times 9) = 34.2$ kHz,

$\nu_5 = (3.8 \times 11) = 41.8$ kHz, $\nu_6 = (3.8) \times 13 = 49.4$ kHz and

$\nu_7 = (3.8) \times 14 > 50$ kHz

Hence, the sought number of frequencies between 20 kHz to 50 kHz equals 4.

4.172 Let two waves, $\xi_1 = a \cos(\omega t - kx)$ and $\xi_2 = a \cos(\omega t + kx)$, superimpose and as a result, we have a standing wave (the resultant wave) in the string of the form $\xi = 2a \cos kx \cos \omega t$.

According to the problem $2a = a_m$. Hence, the standing wave excited in the string is

$$\xi = a_m \cos kx \cos \omega t$$

or
$$\frac{\partial \xi}{\partial t} = -\omega a_m \cos kx \sin \omega t$$

(a) So, the kinetic energy confined in the string element of length dx is given by

$$dT = \frac{1}{2}\left(\frac{m}{l}dx\right)\left(\frac{\partial \xi}{\partial t}\right)^2$$

or
$$dT = \frac{1}{2}\left(\frac{m}{l}dx\right) a_m^2 \omega^2 \cos^2 kx \sin^2 \omega t$$

or
$$dT = \frac{m a_m^2 \omega^2}{2l} \sin^2 \omega t \cos^2 \frac{2\pi}{\lambda} x dx$$

Hence, the kinetic energy confined in the string corresponding to the fundamental tone

$$T = \int dT = \frac{m a_m^2 \omega^2}{2l} \sin^2 \omega t \int_0^{\lambda/2} \cos^2 \frac{2\pi}{\lambda} x \, dx$$

(because for the fundamental tone, length of the string $l = \lambda/2$).

Integrating we get, $$T = \frac{1}{4} m a_m^2 \omega^2 \sin^2 \omega t$$

Hence, the sought maximum K.E. is

$$T_{\max} = \frac{1}{4} m a_m^2 \omega^2$$

(because for maximum K.E. $\sin^2 \omega t = 1$).

(b) Mean K.E. averaged over one oscillation period

$$< T > = \frac{\int T dt}{\int dt} = \frac{1}{4}\, m a_m^2 \omega^2 \, \frac{\int_0^{2\pi/\omega} \sin^2 \omega t \, dt}{\int_0^{2\pi/\omega} dt}$$

or $$< T > = \frac{1}{8}\, m a_m^2 \omega^2$$

4.173 We have a standing wave given by the equation

$$\xi = a \sin kx \cos \omega t$$

So, $$\frac{\partial \xi}{\partial t} = - a\omega \sin kx \sin \omega t$$

and $$\frac{\partial \xi}{\partial t} = ak \cos kx \cos \omega t$$

The K.E. confined in an element of length dx of the rod

$$dT = \frac{1}{2} (\rho S \, dx) \left(\frac{\partial \xi}{\partial t} \right)^2 = \frac{1}{2} \rho S a^2 \omega^2 \sin^2 \omega t \sin^2 kx \, dx$$

So the total K.E. confined in the rod

$$T = \int dT = \frac{1}{2} \rho S a^2 \omega^2 \sin^2 \omega t \int_0^{\lambda/2} \sin^2 \frac{2\pi}{\lambda} x \, dx$$

or $$T = \frac{\pi S a^2 \omega^2 \rho \sin^2 \omega t}{4k}$$

The P.E. in the above rod element

$$dU = \int \partial U = - \int_0^{\xi} F_{\xi} \, d\xi \quad \left(\text{where } F_{\xi} = (\rho S \, dx) \frac{\partial^2 \xi}{\partial t^2} \right)$$

or $$F_{\xi} = -(\rho S \, dx)\, \omega^2 \xi$$

So, $$dU = \omega^2 \rho S \, dx \int_0^{\xi} \xi \, d\xi$$

or $$dU = \frac{\rho \omega^2 S \xi^2}{2} dx = \frac{\rho \omega^2 S a^2 \cos^2 \omega t \sin^2 kx \, dx}{2}$$

Thus, the total P.E. stored in the rod

$$U = \int dU$$

or $$U = \rho\omega^2 Sa^2 \cos^2 \omega t \int_0^{\lambda/2} \sin^2 \frac{2\pi}{\lambda} x\, dx$$

$$= \frac{\pi\rho Sa^2\omega^2 \cos^2 \omega t}{4k}$$

To find the P.E. stored in the rod element we may adopt an easier way. We know that the potential energy density confined in a rod under elastic force

$$U_D = \frac{1}{2}(\text{stress} \times \text{strain}) = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}E\varepsilon^2$$

$$= \frac{1}{2}\rho v^2\varepsilon^2 = \frac{1}{2}\frac{\rho\omega^2}{k^2}\varepsilon^2$$

$$= \frac{1}{2}\frac{\rho\omega^2}{k^2}\left(\frac{\partial\xi}{\partial x}\right)^2 = \frac{1}{2}\rho a^2\omega^2 \cos^2 \omega t \cos^2 kx$$

Hence, the total P.E. stored in the rod will be

$$U = \int U_D\, dV = \int_0^{\lambda/2} \frac{1}{2}\rho a^2\omega^2 \cos^2 \omega t \cos^2 kx\ S\ dx$$

$$= \frac{\pi\rho Sa^2\, \omega^2 \cos^2 \omega t}{4k}$$

Hence, the sought mechanical energy confined in the rod between the two adjacent nodes

$$E = T + U = \frac{\pi\rho\omega^2 a^2 S}{4k}$$

4.174 Receiver R_1 registers the beating due to the sound waves reaching it directly from source and the other due to the reflection from the wall.

Frequency of sound reaching directly from S to R_1

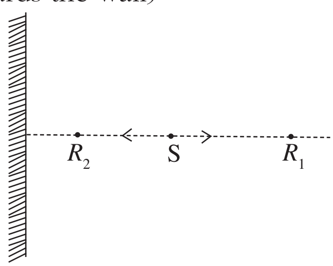
$$\nu_{S\to R_1} = \nu_0 \frac{v}{v-u} \quad \text{(when } S \text{ moves towards } R_1\text{)}$$

and $$\nu'_{S\to R_1} = \nu_0 \frac{v}{v+u} \quad \text{(when } S \text{ moves towards the wall)}$$

Now, frequency reaching R_1 after reflection from wall

$$\nu_{W\to R_1} = \nu_0 \frac{v}{v+u} \quad \text{(when } S \text{ moves towards } R_1\text{)}$$

and $$\nu'_{W\to R_1} = \nu_0 \frac{v}{v-u} \quad \text{(when } S \text{ moves towards the wall)}$$



Thus, the sought beat frequency

$$\Delta\nu = (\nu_{S \to R_1} - \nu_{W \to R_1})$$

$$= \nu_0 \frac{v}{v-u} - \nu_0 \frac{v}{v+u} = \frac{2\nu_0 v u}{v^2 - u^2} \cong \frac{2u\nu_0}{v} = 1\ \text{Hz}$$

4.175 Let the velocity of tuning fork be u. Thus, frequency reaching the observer due to the tuning fork that approaches the observer

$$\nu' = \nu_0 \frac{v}{v-u}$$

Frequency reaching the observer due to the tuning fork that recedes from the observer

$$\nu'' = \nu_0 \frac{v}{v+u}$$

So, beat frequency

$$\nu = \nu - \nu'' = \nu_0 v\left(\frac{1}{v-u} - \frac{1}{v+u}\right)$$

or $$\nu = \frac{2\nu_0 v u}{v^2 - u^2}$$

So, $$\nu u^2 + (2v\nu_0)u - v^2\nu = 0$$

Hence, $$u = \frac{-2v\nu_0 \pm \sqrt{4\nu_0^2 v^2 + 4\nu^2 v^2}}{2\nu}$$

Hence, the sought value of u, on simplifying and noting that $u > 0$, is

$$u = \frac{v\nu_0}{\nu}\left(\sqrt{1 + \left(\frac{\nu}{\nu_0}\right)^2} - 1\right) \approx 0.5\ \text{m/s}$$

4.176 Obviously, the maximum frequency will be heard when the source is moving with maximum velocity towards the receiver and minimum frequency will be heard when the source recedes with maximum velocity. As the source swings harmonically, its maximum velocity equals $a\omega$. Hence,

$$\nu_{\max} = \nu_0 \frac{v}{v - a\omega} \quad \text{and} \quad \nu_{\min} = \nu_0 \frac{v}{v + a\omega}$$

So, the frequency bandwidth

$$\Delta\nu = \nu_{\max} - \nu_{\min} = \nu_0 v\left(\frac{2a\omega}{v^2 - a^2\omega^2}\right)$$

or $$(\Delta\nu a^2)\omega^2 + (2\nu_0 va)\omega - \Delta\nu v^2 = 0$$

So, $$\omega = \frac{-2\nu_0 va \pm \sqrt{4\nu_0^2 v^2 a^2 + \Delta\nu^2 a^2 v^2}}{2\Delta\nu a^2}$$

On simplifying and taking $+ve$ sign as $\omega \to 0$ if $\Delta\nu \to 0$

$$\omega = \frac{v\nu_0}{\Delta\nu a}\left(\sqrt{1+\left(\frac{\Delta\nu}{\nu_0}\right)^2} - 1\right) = 34\ \text{s}^{-1}$$

4.177 It should be noted that the frequency emitted by the source at time t could not be received at the same moment by the receiver, because till that time the source will cover the distance $1/2wt^2$ and the sound wave will take further time $1/2\ wt^2/v$ to reach the receiver. Therefore, the frequency noted by the receiver at time t should be emitted by the source at the time $t_1 < t$. Therefore,

$$t_1 + \left(\frac{1/2wt_1^2}{v}\right) = t \tag{1}$$

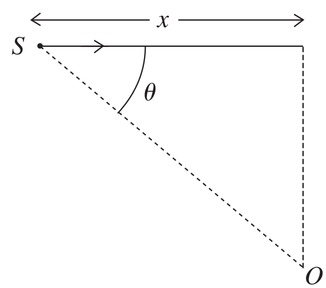
and the frequency noted by the receiver

$$\nu = \nu_0 \frac{v}{v + wt_1} \tag{2}$$

Solving Eqs. (1) and (2), we get

$$\nu = \frac{\nu_0}{\sqrt{1 + 2wt/v}} = 1.35\ \text{kHz}$$

4.178 (a) When the observer receives the sound, the source is the closest to him. It means that frequency is emitted by the source sometime before (see figure). Figure shows that the source approaches the stationary observer with velocity $v_s \cos\theta$.



Hence, the frequency noted by the observer

$$\nu = \nu_0\left(\frac{v}{v - v_s\cos\theta}\right)$$

$$= \nu_0\left(\frac{v}{v - \eta v\cos\theta}\right) = \frac{\nu_0}{1 - \eta\cos\theta} \tag{1}$$

But $$\frac{x}{v_s} = \frac{\sqrt{l^2 + x^2}}{v} \tag{2}$$

so, $$\frac{x}{\sqrt{l^2 + x^2}} = \frac{v_s}{v} = \eta$$

or $$\cos\theta = \eta$$

Hence, from Eqs. (1) and (2), the sought frequency

$$\nu = \frac{\nu_0}{1 - \eta^2} = 5 \text{ kHz}$$

(b) When the source is right in front of O, the sound emitted by it will not be Doppler shifted because $\theta = 90°$. This sound will be received at O at time $t = l/v$ after the source has passed it. The source will by then have moved ahead by a distance $v_s t = l\eta$. The distance between the source and observer at this time will be $l\sqrt{1 + \eta^2} = 0.32$ km.

4.179 Frequency of sound when it reaches the wall

$$\nu' = \nu \frac{v + u}{v}$$

The wall will reflect the sound with same frequency ν'. Thus, frequency noticed by a stationary observer after reflection from wall

$$\nu'' = \nu' \frac{v}{v - u}$$

(since wall behaves as a source of frequency ν').

Thus, $$\nu'' = \nu \frac{v + u}{v} \cdot \frac{v}{v - u} = \nu \frac{v + u}{v - u}$$

or $$\lambda'' = \lambda \frac{v - u}{v + u} \quad \text{or} \quad \frac{\lambda''}{\lambda} = \frac{v - u}{v + u}$$

So, $$1 - \frac{\lambda''}{\lambda} = 1 - \frac{v - u}{v + u} = \frac{2v}{v + u}$$

Hence, the sought percentage change in wavelength is

$$\frac{\lambda - \lambda''}{\lambda} \times 100 = \frac{2u}{v + u} \times 100\% = 2\% \text{ decrease}$$

4.180 Frequency of sound reaching the wall

$$\nu = \nu_0 \left(\frac{v - u}{v} \right) \tag{1}$$

For the observer, the wall becomes a source of frequency ν receding from it with velocity u.

Thus, the frequency reaching the observer

$$\nu' = \nu\left(\frac{v}{v+u}\right) = \nu_0\left(\frac{v-u}{v+u}\right) \quad \text{(using Eq. 1)}$$

Hence, the beat frequency registered by the receiver (observer)

$$\Delta\nu = \nu_0 - \nu' = \frac{2u\nu_0}{v+u} = 0.6 \text{ Hz}$$

4.181 Intensity of a spherical sound wave emitted from a point source in a homogenous absorbing medium of wave damping coefficient γ is given by

$$I = \frac{1}{2}\rho a^2 e^{-2\gamma r}\,\omega^2 v$$

So, intensity of sound at a distance r_1 from the source

$$\frac{I_1}{r_1^2} = \frac{1}{2}\frac{\rho a^2 e^{-2\gamma r_1}\,\omega^2 v}{r_1^2}$$

and intensity of sound at a distance r_2 from the source

$$\frac{I_2}{r_2^2} = \frac{1}{2}\frac{\rho a^2 e^{-2\gamma r_2}\,\omega^2 v}{r_2^2}$$

But according to the problem

$$\frac{1}{\eta}\frac{I_1}{r_1^2} = \frac{I_2}{r_2^2}$$

So, $$\frac{\eta r_1^2}{r_2^2} = e^{2\gamma(r_2 - r_1)} \quad \text{or} \quad \ln\frac{\eta r_1^2}{r_2^2} = 2\gamma(r_2 - r_1)$$

or $$\gamma = \frac{\ln(\eta r_1^2/r_2^2)}{2(r_2 - r_1)} = 6\times10^{-3}\ \text{m}^{-1}$$

4.182 (a) Loudness level in bels $= \log I/I_0$, where I_0 is the threshold of audibility.

So, loudness level in decibels

$$L = 10\log\frac{I}{I_0}$$

Thus, loudness level at $x = x_1$

$$L_{x_1} = 10\log\frac{I_{x_1}}{I_0}$$

Similarly, $$L_{x_2} = 10\log\frac{I_{x_2}}{I_0}$$

Thus, $$L_{x_2} - L_{x_1} = 10 \log \frac{I_{x_2}}{I_{x_1}}$$

or $$L_{x_2} = L_{x_1} + 10 \log \frac{1/2\rho a^2 \,\omega^2 v e^{-2\gamma x_2}}{1/2\rho a^2 \,\omega^2 v e^{-2\gamma x_1}}$$

$$= L_{x_1} + 10 \log e^{-2\gamma(x_2 - x_1)}$$

$$= L_{x_1} - 20\,\gamma(x_2 - x_1)\log e$$

Hence, $$L' = L - 20\gamma x \log e \quad (\text{since } x_2 - x_1 = x)$$

$$= 20 - 20 \times 0.23 \times 50 \times 0.4343$$

$$= 60 - 10 = 50 \text{ dB}$$

(b) The point at which the sound is not heard anymore, the loudness level should be zero. Thus,

$$0 = L - 20\gamma x \log e \quad \text{or} \quad x = \frac{L}{20\gamma \log e} = \frac{60}{20 \times 0.23 \times 0.4343} = 0.3 \text{ km}$$

4.183 (a) Since there is no damping, so

$$L_{r_0} = 10 \log \frac{I}{I_0} = 10 \log \frac{(1/2)\rho a^2 \omega^2 \, v/r_0^2}{(1/2)\rho a^2 \omega^2 v} = -20 \log r_0$$

Similarly, $$L_r = -20 \log r$$

So, $$L_r - L_{r_0} = 20 \log \left(\frac{r_0}{r}\right)$$

or $$L_r = L_{r_0} + 20 \log \left(\frac{r_0}{r}\right)$$

$$= 30 + 20 \times \log \frac{20}{10} = 36 \text{ dB}$$

(b) Let r be the sought distance at which the sound is not heard.

So, $$L_r = L_{r_0} + 20 \log \frac{r_0}{r} = 0 \quad \text{or} \quad L_{r_0} = 20 \log \frac{r}{r_0} \quad \text{or} \quad 30 = 20 \log \frac{r}{20}$$

So, $$\log_{10} \frac{r}{20} = \frac{3}{2} \quad \text{or} \quad 10^{(3/2)} = \frac{r}{20}$$

Thus, $$r = 200\sqrt{10} = 0.63 \text{ km}$$

Therefore, for $r > 0.63$ km no sound will be heard.

4.184 We treat the fork as a point source. In the absence of damping, the oscillation has the form

$$\frac{\text{constant}}{r}\cos(\omega t - kr)$$

Because of the damping of the fork, the amplitude of oscillation decreases exponentially with the retarded time (i.e., the time at which the wave started from the source). Thus, we write for the wave amplitude

$$\xi = \frac{\text{constant}}{r}\, e^{-\beta(t - r/v)}$$

This means that
$$\frac{e^{-\beta(t+\tau-r_A/v)}}{r_A} = \frac{e^{-\beta(t+\tau-r_B/v)}}{r_B}$$

Thus,
$$e^{-\beta\left(\tau + \frac{r_B - r_A}{v}\right)} = \frac{r_A}{r_B}$$

or
$$\beta = \frac{\ln r_B/r_A}{\tau + (r_B - r_A)/v} = 0.12\ \text{s}^{-1}$$

4.185 (a) Let us consider the motion of an element of the medium of thickness dx and unit area of cross-section. Let ξ = displacement of the particles of the medium at location x. Then, by the equation of motion

$$\rho\, dx\, \ddot{\xi} = -\,dp$$

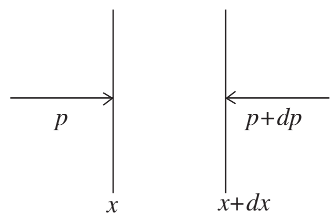
where dp is the pressure increment over the length dx.

Recalling the wave equation

$$\ddot{\xi} = v^2 \frac{\partial^2 \xi}{\partial x^2}$$

we can write the above equation as

$$\rho v^2 \frac{\partial^2 \xi}{\partial x^2}\, dx = -\,dp$$



Integrating this equation, we get

$$\Delta p = \text{surplus pressure} = -\,\rho v^2 \frac{\partial \xi}{\partial x} + \text{constant}$$

In the absence of a deformation (a wave), the surplus pressure is $\Delta p = 0$. So, constant = 0 and thus,

$$\Delta p = -\,\rho v^2 \frac{\partial \xi}{\partial x}$$

(b) We have found earlier that

$$w = w_k + w_p = \text{total energy density}$$

$$w_k = \frac{1}{2}\rho\left(\frac{\partial\xi}{\partial t}\right)^2 \quad \text{and} \quad w_p = \frac{1}{2}E\left(\frac{\partial\xi}{\partial x}\right)^2 = \frac{1}{2}\rho v^2\left(\frac{\partial\xi}{\partial x}\right)^2$$

It is easy to see that the space-time average of both densities is the same and the space-time average of total energy is then

$$< w > = \rho v^2\left(\frac{\partial\xi}{\partial x}\right)^2$$

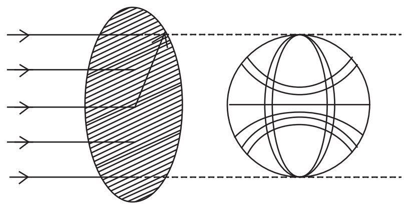
The intensity of the wave is

$$I = v < w > = \frac{(\Delta p)^2}{\rho v}$$

Using $$(\Delta p)^2 = \frac{1}{2}(\Delta p)_m^2$$

we get $$I = \frac{(\Delta p)_m^2}{2\rho v}$$

4.186 The intensity of the sound wave is



$$I = \frac{(\Delta p)_m^2}{2\rho v} = \frac{(\Delta p)_m^2}{2\rho\nu\lambda}$$

(here, ρ is the density of air and we have used $v = \nu\lambda$).

Thus, the mean energy flow reaching the ball is

$$< \Phi > = \int < \mathbf{j} > \cdot\, d\mathbf{S} = < \mathbf{j} > \cdot\, \mathbf{S}$$

(as mean energy flow density vector $< \mathbf{j} >$ is constant).

$$< \Phi > = -I\pi R^2$$

(Notice that $|< \mathbf{j} >|$ is nothing but intensity I, negative sign is due to energy inflow, and πR^2 being the effective area (area of cross-section) of the ball.)

Hence, $$< \Phi > = \pi R^2 I = \pi R^2\,\frac{(\Delta p)_m^2}{2\rho\nu\lambda}$$

$$= 10.9 \text{ mW (on substituting values)}$$

4.187 (a) We have $$\frac{P}{4\pi r^2} = \text{intensity} = \frac{(\Delta p)_m^2}{2\rho v}$$

or $$(\Delta p)_m = \sqrt{\frac{\rho v P}{2\pi r^2}}$$

$$= \sqrt{\frac{1.293 \text{ kg/m}^3 \times 340 \text{ m/s} \times 0.80 \text{ W}}{2\pi \times 1.5 \times 1.5 \text{ m}^2}}$$

$$= \sqrt{\frac{1.293 \times 340 \times 0.8}{2\pi \times 1.5 \times 1.5}} \left(\frac{\text{kg kg m}^2\text{s}^{-3} \text{ ms}^{-1}}{\text{m}^5}\right)^{1/2}$$

$$= 4.9877 \text{ (kg m}^{-1} \text{ s}^{-2}) = 5 \text{ Pa}$$

$$\frac{(\Delta p)_m}{p} = 5 \times 10^{-5}$$

(b) We have $$\Delta p = -\rho v^2 \frac{\partial \xi}{\partial x}$$

$$(\Delta p)_m = \rho v^2 k \xi_m = \rho v 2\pi \nu \xi_m$$

$$\xi_m = a = \frac{(\Delta p)_m}{2\pi \rho v \nu}$$

$$= \frac{5}{2\pi \times 1.293 \times 340 \times 600} = 3 \text{ μm}$$

$$\frac{\xi_m}{\lambda} = \frac{3 \times 10^{-6}}{340/600} = \frac{1800}{340} \times 10^{-6} = 5 \times 10^{-6}$$

4.188 Given that loudness level $L = 50$ dB $= 5$ bels.

Then the intensity at the relevant point (at a distance r from the source) is $I_0 \cdot 10^L$.

Had there been no damping, the intensity would have been $e^{2\gamma r} I_0 \cdot 10^L$.

Now this must equal the quantity

$$\frac{P}{4\pi r^2} \quad \text{(where } P = \text{sonic power of the source)}$$

Thus, $$\frac{P}{4\pi r^2} = e^{2\gamma r} I_0 \cdot 10^L$$

or $$P = 4\pi r^2 e^{2\gamma r} I_0 \cdot 10^L = 1.39 \text{ W}$$

4.4 Electomagnetic Waves. Radiation

4.189 The velocity of light in a medium of relative permittivity ε is $c/\sqrt{\varepsilon}$. Thus, the change in wavelength of light (from its value in vacuum to its value in the medium) is

$$\Delta\lambda = \frac{c/\sqrt{\varepsilon}}{\nu} - \frac{c}{\nu} = \frac{c}{\nu}\left(\frac{1}{\sqrt{\varepsilon}} - 1\right) = -50 \text{ m}$$

4.190 From the data of the problem, the relative permittivity of the medium varies as

$$\varepsilon(x) = \varepsilon_1\, e^{-(x/l)\ln \varepsilon_1/\varepsilon_2}$$

Hence, the local velocity of light

$$v(x) = \frac{c}{\sqrt{\varepsilon(x)}} = \frac{c}{\sqrt{\varepsilon_1}} \exp\left(\frac{x}{2l}\ln\frac{\varepsilon_1}{\varepsilon_2}\right)$$

Thus, the required time t

$$t = \int_0^t \frac{dx}{\nu(x)} = \frac{\sqrt{\varepsilon_1}}{c}\int_0^l \exp\left(-\frac{x}{2l}\ln\frac{\varepsilon_1}{\varepsilon_2}\right)dx$$

$$= \frac{\sqrt{\varepsilon_1}}{c}\,\frac{-\exp\left(\frac{1}{2}\ln\frac{\varepsilon_1}{\varepsilon_2}\right)+1}{\frac{1}{2l}\ln\frac{\varepsilon_1}{\varepsilon_2}} = \frac{2l}{c}\,\frac{\sqrt{\varepsilon_1}-\sqrt{\varepsilon_2}}{\ln\frac{\varepsilon_1}{\varepsilon_2}}$$

4.191 Conduction current density $\quad j_{\text{cond}} = \sigma\mathbf{E}$

Displacement current density of $\; j_{\text{dis}} = \frac{\partial \mathbf{D}}{\partial t} = \varepsilon\varepsilon_0\frac{\partial \mathbf{E}}{\partial t} = i\omega\varepsilon\varepsilon_0\,\mathbf{E}$

Ratio of amplitudes $\quad \frac{j_{\text{cond}}}{j_{\text{dis}}} = \frac{\sigma}{\omega\,\varepsilon\,\varepsilon_0} = 2$ (on substituting values)

4.192 We have $\quad \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu_0\frac{\partial \mathbf{H}}{\partial t}$

$$= \nabla\cos(\omega t - \mathbf{k}\cdot\mathbf{r})\times\mathbf{E}_m = \mathbf{k}\times\mathbf{E}_m\,\sin(\omega t - \mathbf{k}\cdot\mathbf{r})$$

At $\mathbf{r} = 0$ $\quad \frac{\partial \mathbf{H}}{\partial t} = \frac{\mathbf{k}\times\mathbf{E}_m}{\mu_0}\sin\omega t$

So integrating (ignoring a constant) and using $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$, we get

$$\mathbf{H} = \frac{\mathbf{k}\times\mathbf{E}_m}{\mu_0}\cos ckt\times\frac{1}{ck} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\,\frac{\mathbf{k}\times\mathbf{E}_m}{k}\cos ckt$$

4.193 As in the previous problem

$$\mathbf{H} = \frac{\mathbf{k}\times\mathbf{E}_m}{\mu_0\omega}\cos(\omega t - \mathbf{k}\cdot\mathbf{r})\,\frac{E_m}{\mu_0 c}\,\mathbf{e}_z\;\cos(kx - \omega t)$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}}E_m\mathbf{e}_z\;\cos(kx - \omega t)$$

Thus,

(a) At $t = 0$, $$\mathbf{H} = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_m \mathbf{e}_z \cos kx = -0.3\mathbf{e}_z$$

(b) At $t = t_0$, $$\mathbf{H} = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_m \mathbf{e}_z \cos(kx - \omega t_0) = 0.18\ \mathbf{e}_z$$

4.194 We have $$\xi_{\text{ind}} = \oint \mathbf{E} \cdot d\mathbf{l} = E_m l\,[\cos \omega t - \cos(\omega t - kl)]$$

$$= -2E_m l \sin \frac{\omega l}{2c} \sin\left(\omega t - \frac{\omega l}{2c}\right)$$

Putting the values $E_m = 50$ mV/m, $l = \frac{1}{2}$ m, we get

$$\frac{\omega l}{c} = \frac{2\pi v l}{c} = \frac{\pi \times 10^8}{3 \times 10^8} = \frac{\pi}{3}$$

so, $$\xi_{\text{ind}} = 50\text{ mV}\left(-\sin\frac{\pi}{6}\right)\sin\left(\omega t - \frac{\pi}{6}\right)$$

$$= -25 \sin\left(\omega t + \frac{\pi}{6} - \frac{\pi}{2}\right) = 25 \cos\left(\omega t - \frac{\pi}{3}\right)\text{ mV}$$

4.195 We have $$\mathbf{E} = \mathbf{j}E(t, x) \quad \text{and} \quad \mathbf{B} = \mathbf{k}B(t_1 x)$$

and $$\text{Curl } \mathbf{E} = \mathbf{k}\frac{\partial E}{\partial x} = -\frac{\partial \mathbf{B}}{\partial t} = -\mathbf{k}\frac{\partial B}{\partial t}$$

So, $$-\frac{\partial E}{\partial x} = \frac{\partial B}{\partial t}$$

Also, $$\text{Curl } \mathbf{B} = \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

and $$\text{Curl } \mathbf{B} = -\mathbf{j}\frac{\partial B}{\partial x} \quad \text{so,} \quad \frac{\partial B}{\partial x} = -\frac{1}{c^2}\frac{\partial E}{\partial t}$$

4.196 Given $$\mathbf{E} = \mathbf{E}_m \cos(\omega t - \mathbf{k} \cdot \mathbf{r})$$

Then, as before, $$\mathbf{H} = \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{\mathbf{k} \times \mathbf{E}_m}{k} \cos(\omega t - \mathbf{k} \cdot \mathbf{r})$$

So, $$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \sqrt{\frac{\varepsilon_0}{\mu_0}} \mathbf{E}_m \times (\mathbf{k} \times \mathbf{E}_m) \frac{1}{k} \cos^2(\omega t - \mathbf{k} \cdot \mathbf{r})$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}} \mathbf{E}_m^2 \frac{\mathbf{k}}{k} \cos^2(\omega t - \mathbf{k} \cdot \mathbf{r})$$

$$< \mathbf{S} > = \frac{1}{2} \sqrt{\frac{\varepsilon_0}{\mu_0}} \mathbf{E}_m^2 \frac{\mathbf{k}}{k}$$

4.197 We have $$E = E_m \cos(2\pi\nu t - kx)$$

(a) $$j_{\text{dis}} = \frac{\partial D}{\partial t} = -2\pi\varepsilon_0 \nu E_m \sin(\omega t - kx)$$

Thus, $$(j_{\text{dis}})_{\text{rms}} = < j_{\text{dis}}^2 >^{1/2}$$

$$= \sqrt{2}\pi\varepsilon_0 \nu E_m = 0.20 \text{ mA/m}^2$$

(b) As in Problem 4.196, we can write

$$< S_x > = \frac{1}{2} \sqrt{\frac{\varepsilon_0}{\mu_0}} E_m^2$$

Thus, $$< S_x > = 3.3 \ \mu\text{W/m}^2$$

4.198 For the Poynting vector, we can derive as in Problem 4.196

$$< S > = \frac{1}{2} \sqrt{\frac{\varepsilon\varepsilon_0}{\mu_0}} E_m^2 \text{ (along the direction of propagation)}$$

Hence, in time t (which is much longer than the time period T of the wave), the energy reaching the ball is

$$\pi R^2 \times \frac{1}{2} \sqrt{\frac{\varepsilon\varepsilon_0}{\mu_0}} E_m^2 \times t = 5 \text{ kJ}$$

4.199 Here $$\mathbf{E} = \mathbf{E}_m \cos kx \cos \omega t$$

From div $\mathbf{E} = 0$, we get $\mathbf{E}_{m_x} = 0$, so, $\mathbf{E}_m$ is in the y–z plane.

Also, $$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = -\nabla \cos kx \times \mathbf{E}_m \cos \omega t$$

$$= \mathbf{k} \times \mathbf{E}_m \sin kx \cos \omega t$$

So, $$\mathbf{B} = \frac{\mathbf{k} \times \mathbf{E}_m}{\omega} \sin kx \sin \omega t = \mathbf{B}_m \sin kx \sin \omega t$$

where $|\mathbf{B}_m| = E_m/c$ and $\mathbf{B}_m \perp \mathbf{E}_m$ in the y–z plane.

At $t = 0$, $\mathbf{B} = 0$, so $E = E_m \cos kx$.

At $t = \frac{T}{4}$, $\mathbf{E} = 0$, so $B = B_m \sin kx$.

4.200 Here

$$\mathbf{E} = \mathbf{E}_m \cos kx\, \omega t$$

$$\mathbf{H} = \frac{\mathbf{k} \times \mathbf{E}_m}{\mu_0 \omega} \sin kx \sin \omega t \quad \text{(exactly as in Problem 4.199)}$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \frac{\mathbf{E}_m \times (\mathbf{k} \times \mathbf{E}_m)}{\mu_0 \omega} \frac{1}{4} \sin 2\, kx \sin 2\omega t$$

Thus,

$$S_x = \frac{1}{4} \varepsilon_0 c E_m^2 \sin 2kx \sin 2\omega t \left(\text{as} \frac{1}{\mu_0 c} = \varepsilon_0 c \right)$$

$$< S_x > = 0$$

4.201 Inside the condenser, the peak electrical energy

$$W_{\text{ele}} = 1/2\, C V_m^2$$

$$= \frac{1}{2} V_m^2 \frac{\varepsilon_0 \pi R^2}{d}$$

where d = separation between the plates, πR^2 = area of each plate and $V = V_m \sin \omega t$, where V_m is the maximum voltage.

Changing electric field causes a displacement current

$$j_{\text{dis}} = \frac{\partial D}{\partial t} = \varepsilon_0 E_m \omega \cos \omega t$$

$$= \frac{\varepsilon_0 \omega V_m}{d} \cos \omega t$$

This gives rise to a magnetic field $B\ (r)$ (at a radial distance r from the centre of the plate)

$$B(r) \cdot 2\pi r = \mu_0 \pi r^2 j_{\text{dis}} = \mu_0 \pi r^2 \frac{\varepsilon_0 \omega V_m}{d} \cos \omega t$$

$$B = \frac{1}{2} \varepsilon_0 \mu_0 \omega \frac{r}{d} V_m \cos \omega t$$

Energy associated with this field is

$$\int \frac{B^2}{2\mu_0} dv$$

$$= \int \frac{B^2}{2\mu_0} (2\pi r\, dr) d$$

$$= \int \frac{B^2}{2\mu_0}(2\pi d) r\, dr$$

$$= \frac{\pi \varepsilon_0^2 \mu_0 \omega^2}{4d} V_m^2 \cos^2 \omega t \int_0^R r^3\, dr$$

$$= \frac{1}{16} \pi \varepsilon_0^2 \mu_0 \omega^2 \frac{\omega^2 R^4}{d} V_m^2 \cos^2 \omega t$$

Thus, the maximum magnetic energy

$$W_{\text{mag}} = \frac{\varepsilon_0^2 \mu_0}{16} (\omega R)^2 \frac{\pi R^2}{d} V_m^2$$

Hence, $$\frac{W_{\text{mag}}}{W_{\text{ele}}} = \frac{1}{8} \varepsilon_0 \mu_0 (\omega R)^2 = \frac{1}{8} \left(\frac{\omega R}{c} \right)^2 = 5 \times 10^{-15}$$

The approximation is valid only if $\omega R \ll c$.

4.202 Here $I = I_m \cos \omega t$, then the peak magnetic energy is

$$W_{\text{mag}} = \frac{1}{2} L I_m^2 = \frac{1}{2} \mu_0 n^2 I_m^2 \pi R^2 d$$

Changing magnetic field induces an electric field which by Faraday's law is

$$E \cdot 2\pi r = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{S} = \pi r^2 \mu_0 n I_m \omega \sin \omega t$$

$$E = \frac{1}{2} r \mu_0 n I_m \omega \sin \omega t$$

The associated peak electric energy is

$$W_{\text{ele}} = \int \frac{1}{2} \varepsilon_0 E^2 (2\pi r\, dr)\, d$$

$$= \frac{1}{8} \varepsilon_0 \mu_0^2 n^2 \omega^2 I_m^2 \sin^2 \omega t (2\pi d) \int_0^R r^3 dr$$

$$= \frac{1}{8} \varepsilon_0 \mu_0^2 n^2 \omega^2 I_m^2 \sin^2 \omega t \times \frac{\pi R^4}{2} d$$

Hence, $$\frac{W_{\text{ele}}}{W_{\text{mag}}} = \frac{1}{8} \varepsilon_0 \mu_0 (\omega R)^2 = \frac{1}{8} \left(\frac{\omega R}{c} \right)^2 = 5 \times 10^{-15}$$

Again, we expect the results to be valid if and only if

$$\left(\frac{\omega R}{c} \right) \ll 1$$

4.203 If the charge on the capacitor is Q, the rate of increase of the capacitor's energy is

$$\frac{d}{dt}\left(\frac{1}{2}\frac{Q^2}{C}\right) = \frac{Q\dot{Q}}{C} = \frac{d}{\varepsilon_0 \pi R^2} Q\dot{Q}$$

Now, electric field between the plates (inside it) is $E = Q/\pi R^2 \varepsilon_0$.

So, displacement current is

$$\frac{\partial D}{\partial t} = \frac{\dot{Q}}{\pi R^2}$$

This will lead to a magnetic field (circuital) inside the plates. At a radial distance r

$$2\pi r H_\theta(r) = \pi r^2 \frac{\dot{Q}}{\pi R^2} \quad \text{or} \quad H_\theta = \frac{\dot{Q} r}{2\pi R^2}$$

Hence, $$H_\theta(R) = \frac{\dot{Q}}{2\pi R} \text{ (at the edge)}$$

Thus, inward Poynting vector

$$S = \frac{\dot{Q}}{2\pi R} \times \frac{Q}{\pi R^2 \varepsilon_0}$$

Total flow is

$$2\pi R d \times S = \frac{Q\dot{Q}d}{\pi R^2 \varepsilon_0}$$

4.204 Suppose the radius of the conductor is R_0. Then the conduction current density is

$$j_c = \frac{I}{\pi R_0^2} = \sigma E \quad \text{or} \quad E = \frac{I}{\pi R_0^2 \sigma} = \frac{\rho I}{\pi R_0^2}$$

(where $\rho = \dfrac{1}{\sigma}$ is the resistivity).

Inside the conductor there is a magnetic field given by

$$H \cdot 2\pi R_0 = I \quad \text{or} \quad H = \frac{I}{2\pi R_0} \text{ (at the edge)}$$

Therefore, energy flowing in per second in a section of length l is

$$EH \times 2\pi R_0 l = \frac{\rho I^2 l}{\pi R_0^2}$$

But the resistance $$R = \frac{\rho l}{\pi R_0^2}$$

Thus, the energy flowing into the conductor $= I^2 R$.

4.205 Here
$$nev = \frac{I}{\pi R^2}$$

where R = radius of cross-section of the conductor and n = charge density (per unit volume).

Also, $$\frac{1}{2}mv^2 = eU \quad \text{or} \quad v = \sqrt{\frac{2eU}{m}}$$

Thus, the moving protons have a charge per unit length

$$ne\pi R^2 = I\sqrt{\frac{m}{2eU}}$$

This gives rise to an electric field at distance r given by

$$E = \frac{1}{\varepsilon_0}\frac{\sqrt{m/2eU}}{2\pi r}$$

The magnetic field is $$H = \frac{I}{2\pi r} \quad (\text{for } r > R)$$

Thus, $$S = \frac{I^2}{\varepsilon_0 4\pi^2 r^2}\sqrt{\frac{m}{2eU}} \quad (\text{radially outward from the axis})$$

This is the Poynting vector.

4.206 Within the solenoid, $B = \mu_0 nI$ and the rate of change of magnetic energy

$$\dot{W}_{\text{mag}} = \frac{d}{dt}\left(\frac{1}{2\mu_0 n^2 I^2 \pi R^2 l}\right) = \mu_0 n^2 \pi R^2 l I\dot{I}$$

where R = radius of cross-section of the solenoid and l = length.

Also, $H = B/\mu_0 = nI$ along the axis within the solenoid.

So, $$E_\theta 2\pi r = \pi n r^2 \dot{B} = \pi r^2 \mu_0 n\dot{I}$$

or $$E_\theta = \frac{1}{2}\mu_0 n\dot{I}r$$

So at the edge,

$$E_\theta(R) = \frac{1}{2}\mu_0 n\dot{I}R \quad (\text{circuital})$$

Then $$S_r = E_\theta H_z \quad (\text{radially inward})$$

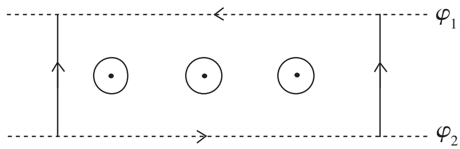
and $$\dot{W}_{\text{mag}} = \frac{1}{2}\mu_0 n^2 \dot{I} IR \times 2\pi Rl = \mu_0 n^2 \pi R^2 l I\dot{I} \quad (\text{as before})$$

4.207 Given $\varphi_2 > \varphi_1$.

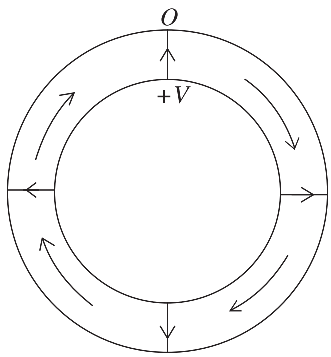
The electric field is as shown by the dashed lines (-------->---------).

The magnetic field is as shown ($\odot$) emerging out of the paper. $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is parallel to the wires and towards right.

Hence, the source must be on the left.



4.208 The electric field (——>) and the magnetic field (H —>) are as shown. The electric field by Gauss' theorem is



$$E_r = \frac{A}{r}$$

Integrating, we get
$$\varphi = A \ln \frac{r_2}{r}$$

So,
$$A = \frac{V}{\ln r_2/r_1} \quad (r_2 > r_1)$$

Then,
$$E = \frac{V}{r \ln r_2/r_1}$$

Magnetic field
$$H_\theta = \frac{I}{2\pi r}$$

The Poynting vector S is along the z-axis and non-zero between the two wires ($r_1 < r < r_2$). The total power flux is

$$\int_{r_1}^{r_2} \frac{IV}{2\pi r^2 \ln r_2/r_1} \cdot 2\pi r \, dr = IV$$

4.209 As in the previous problem

$$E_r = \frac{V_0 \cos \omega t}{r \ln r_2/r_1} \quad \text{and} \quad H_\theta = \frac{I_0 \cos(\omega t - \varphi)}{2\pi r}$$

Hence, time averaged power flux (along the z-axis) is

$$\frac{1}{2} V_0 I_0 \cos\varphi$$

(on using $< \cos \omega t \cos(\omega t - \varphi) > = \frac{1}{2}\cos\varphi$).

4.210 Let $\mathbf{n}$ be along the z-axis.

Then,
$$S_{1n} = E_{1x} H_{1y} - E_{1y} H_{1x}$$

and $$S_{2n} = E_{2x} H_{2y} - E_{2y} H_{2x}$$

Using the boundary condition $E_{1t} = E_{2t}$, $H_{1t} = H_{2t}$ at the boundary ($t = x$ or y), we see that

$$S_{1n} = S_{2n}$$

4.211 We know $$P \propto |\ddot{\mathbf{p}}|^2 \text{ when}$$

$$\mathbf{p} = \sum e_i \mathbf{r}_i = \sum \frac{e_i}{m_i} m_i \mathbf{r}_i = \frac{e}{m} \sum m_i \mathbf{r}_i$$

$\left(\text{if } \dfrac{e_i}{m_i} = \dfrac{e}{m} = \text{fixed} \right)$.

But $$\frac{d^2}{dt^2} \sum m_i \mathbf{r}_i = 0 \text{ (for a closed system)}$$

Hence, $P = 0$.

4.212 We have $$P = \frac{1}{4\pi\varepsilon_0} \frac{2(\ddot{\mathbf{p}})^2}{3c^3}$$

$$|\ddot{\mathbf{p}}|^2 = (e\omega^2 a)^2 \cos^2 \omega t$$

Thus, $$< P > = \frac{1}{4\pi\varepsilon_0} \frac{2}{3c^3} (e\omega^2 a)^2 \times \frac{1}{2}$$

$$= \frac{e^2 \omega^4 a^2}{12\pi\varepsilon_0 c^3} = 5.1 \times 10^{-1.5} \text{ W}$$

4.213 Here, $$\ddot{\mathbf{p}} = \frac{e}{m} \times \text{force} = \frac{e^2 q}{mR^2} \frac{1}{4\pi\varepsilon_0}$$

Thus, $$P = \frac{1}{(4\pi\varepsilon_0)^3} \left(\frac{e^2 q}{mR^2} \right)^2 \frac{2}{3c^3}$$

4.214 Most of the radiation occurs when the moving particle is closest to the stationary particle. In that region, we can write

$$R^2 = b^2 + v^2 t^2$$

and apply the previous problem's formula.

Thus, $$\Delta W \approx \frac{1}{(4\pi\varepsilon_0)^3} \frac{2}{3c^3} \int_{-\infty}^{\infty} \left(\frac{qe^2}{m} \right)^2 \frac{dt}{(b^2 + v^2 t^2)^2}$$

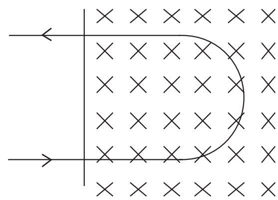
(the integral can be taken between $\pm\infty$ with little error).

Now,
$$\int_{-\infty}^{\infty}\frac{dt}{(b^2+v^2t^2)^2}=\frac{1}{v}\int_{-\infty}^{\infty}\frac{dx}{(b^2+x^2)^2}=\frac{\pi}{2vb^3}$$

Hence,
$$\Delta W\approx\frac{1}{(4\pi\varepsilon_0)^3}\frac{\pi q^2e^4}{3c^3m^2vb^3}$$

4.215 For the semicircular path on the right

$$\frac{mv^2}{R}=Bev\quad\text{or}\quad v=\frac{BeR}{m}$$



Thus, K.E.
$$T=\frac{1}{2}mv^2=\frac{B^2e^2R^2}{2m}$$

Power radiated
$$P=\frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\left(\frac{ev^2}{R}\right)^2$$

Hence, energy radiated
$$\Delta W=\frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\left(\frac{B^2e^3R}{m^2}\right)^2\cdot\frac{\pi R}{BeR}m=\frac{B^3e^5R^2}{6\varepsilon_0m^3c^3}$$

So,
$$\frac{\Delta W}{T}=\frac{Be^3}{3\varepsilon_0c^3m^2}=2.06\times10^{-18}$$

(neglecting the change in v due to radiation, the value is correct if $\Delta W/T\ll1$).

4.216 We have
$$R=\frac{mv}{eB}$$

Then,
$$P=\frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\left(\frac{ev^2}{R}\right)^2=\frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\left(\frac{e^2Bv}{m}\right)^2$$
$$=\frac{1}{3\pi\varepsilon_0c^3}\left(\frac{B^2e^4}{m^3}\right)T$$

This is the radiated power, so
$$\frac{dT}{dt}=\frac{-B^2e^4}{3\pi\varepsilon_0m^3c^3}T$$

Integrating, we get
$$T=T_0e^{-t/\tau}$$
$$\tau=\frac{3\pi\varepsilon_0m^3c^3}{B^2e^4}$$

τ is $(1836)^3 \approx 10^{10}$ times less for an electron than for a proton, so electrons radiate their energy much faster in a magnetic field.

4.217 P is a fixed point at a distance l from the equilibrium position of the particle. Because $l \gg a$, to first order in a/l, the distance between P and the instantaneous position of the particle is still l. For the first case $y = 0$ so $t = T/4$.

The corresponding retarded time is

$$t' = \frac{T}{4} - \frac{l}{c}$$

Now, $$\ddot{y}(t') = -\omega^2 a \cos \omega\left(\frac{I}{4}\frac{l}{c}\right) = -\omega^2 a \sin \frac{\omega l}{c}$$

For the second case, $y = a$ at $t = 0$ so at the retarded time $t' = -\omega l/c$.

Thus, $$\ddot{y}(t') = -\omega^2 a \cos \frac{\omega l}{c}$$

The radiation fluxes in the two cases are proportional to $(\ddot{y}\,(t'))^2$, so

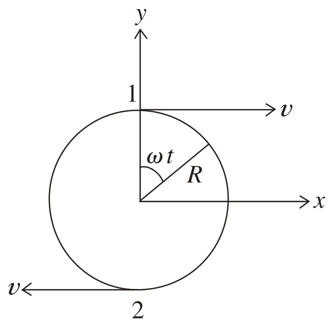
$$\frac{S_1}{S_2} = \tan^2 \frac{\omega l}{c} = 3.06 \text{ (on substituting values)}$$

Note: *The radiation received at P at time t depends on the acceleration of the charge at the retarded time.*

4.218 (a) Along the circle, $x = R \sin \omega t$, $y = R \cos \omega t$ where $\omega = v/R$. If t is the parameter in $x\,(t)$, $y\,(t)$ and t' is the observer time then

$$t' = t + \frac{l - x(t)}{c}$$

where we have neglected the effect of the y-coordinate which is of second order. The observed coordinates are



$$x'(t') = x\,(t) \quad \text{and} \quad y'\,(t') = y\,(t)$$

Then, $$\frac{dy'}{dt'} = \frac{dy}{dt'} = \frac{dt}{dt'}\frac{dy}{dt} = \frac{-\omega R \sin \omega t}{1-(\omega R/c)\cos \omega t} = \frac{\omega x}{1-\omega y/c} = \frac{vx/cR}{1-vy/cR}$$

and $$\frac{d^2y'}{dt'^2} = \frac{dt}{dt'}\frac{d}{dt}\left(\frac{-vx/R}{1-vy/cR}\right)$$

$$= \frac{1}{1-vy/cR}\left\{\frac{-v^2/R^2y}{1-vy/cR} + \frac{vx/R(v^2/cR^2x)}{(1-vy/cR)^2}\right\} = \frac{v^2/R(v/c - y/R)}{(1-vy/cR)^3}$$

This is the observed acceleration.

(b) Energy flow density of electromagnetic radiation S is proportional to the square of the y-projection of the observed acceleration of the particle (i.e., d^2y'/dt'^2).

Thus, $$\frac{S_1}{S_2} = \left[\frac{(v/c-1)}{(1-v/c)^3}\Big/\frac{(v/c+1)}{(1+v/c)^3}\right]^2 = \frac{(1+v/c)^4}{(1-v/c)^4}$$

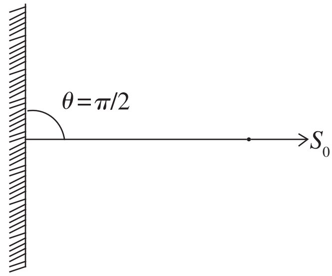
4.219 We know that $S_0\,(r) \propto 1/r^2$. At other angles $S\,(r, \theta) \propto \sin^2\theta$, thus

$S\,(r, \theta) = S_0\,(r)\sin^2\theta = S_0\sin^2\theta$

Average power radiated

$$P = S_0 \times 4\pi r^2 \times \frac{2}{3} = \frac{8\pi}{3}S_0 r^2$$

(Average of $\sin^2\theta$ over whole sphere is 2/3.)



4.220 From the previous problem

$$P_0 = \frac{8\pi S_0 r^2}{3}$$

or $$S_0 = \frac{3P_0}{8\pi r^2}$$

Thus, $$< w > = \frac{S_0}{c} = \frac{3P_0}{8\pi c r^2}$$

(Poynting flux vector is the energy contained in a box of unit cross-section and length c.)

4.221 The rotating dipole has moments

$$p_x = p\cos\omega t \quad \text{and} \quad p_y = p\sin\omega t$$

Thus, $$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\omega^4 p^2 = \frac{p^2\omega^4}{6\pi\varepsilon_0 c^3}$$

4.222 If the electric field of the wave is

$$\mathbf{E} = \mathbf{E}_0\cos\omega t$$

This induces a dipole moment whose second derivative is

$$\ddot{\mathbf{p}} = \frac{e^2\mathbf{E}_0}{m}\cos\omega t$$

Hence, radiated mean power

$$< P > = \frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\left(\frac{e^2E_0}{m}\right)^2 \times \frac{1}{2}$$

On the other hand, the mean incident Poynting flux is

$$< S_{\text{inc}} > = \sqrt{\frac{\varepsilon_0}{\mu_0}} \times \frac{1}{2}E_0^2$$

Thus,

$$\frac{P}{<S_{\text{inc}}>} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2}{3}(\varepsilon_0\mu_0)^{3/2}\left(\frac{e^2}{m}\right)^2 \times \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

$$= \frac{\mu_0^2}{6\pi}\left(\frac{e^2}{m}\right)^2$$

4.223 For the elastically bound electron

$$m\ddot{\mathbf{r}} + m\omega_0^2\mathbf{r} = e\mathbf{E}_0 \cos \omega t$$

This equation has a particular integral (i.e., neglecting the part which does not have the frequency of the impressed force), which gives

$$\mathbf{r} = \frac{e\mathbf{E}_0}{m}\frac{\cos \omega t}{\omega_0^2 - \omega^2}$$

So,

$$\ddot{\mathbf{p}} = -\frac{e^2\mathbf{E}_0\omega^2}{(\omega_0^2 - \omega^2)\,m}\cos \omega t$$

Hence, mean radiated power

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\left(\frac{e^2\omega^2}{m(\omega_0^2 - \omega^2)}\right)^2\frac{1}{2}E_0^2$$

The mean incident Poynting flux is

$$<S_{\text{inc}}> = \sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{1}{2}E_0^2$$

Thus,

$$\frac{P}{<S_{\text{inc}}>} = \frac{\mu_0^2}{6\pi}\left(\frac{e^2}{m}\right)^2\frac{\omega^4}{(\omega_0^2 - \omega^2)^2}$$

4.224 Let r = radius of the ball
R = distance between the ball and the Sun $(r \ll R)$
M_s = mass of the Sun
γ = gravitational constant

Then, $$\frac{\gamma M_s}{R^2}\frac{4\pi}{3}r^3\rho = \frac{P}{4\pi R^2}\pi r^2 \cdot \frac{1}{c}$$

(The factor $1/c$ converts the energy received on the right into momentum received. Then the right-hand side is the momentum received per unit time and must equal the negative of the impressed force for equilibrium.)

Thus, $$r = \frac{3P}{16\ \pi c\rho\gamma\ M_s} = 0.606\ \mu\text{m}$$

OPTICS

PART 5

5.1 Photometry and Geometrical Optics

5.1 (a) The relative spectral response $V(\lambda)$ shown in Fig. 5.11 of the book is so defined that $A/V(\lambda)$ is the energy flux of light of wavelength λ needed to produce a unit luminous flux at that wavelength. (A is the conversion factor defined in the book.)

At $\lambda = 0.51\ \mu\text{m}$, as per the figure

$$V(\lambda) = 0.50$$

and energy flux corresponding to a luminous flux of 1 lumen

$$= \frac{1.6}{0.50} = 3.2\ \text{mW}$$

At $\lambda = 0.64\ \mu\text{m}$, we have

$$V(\lambda) = 0.17$$

and energy flux corresponding to a luminous flux of 1 lumen

$$= \frac{1.6}{0.17} = 9.4\ \text{mW}$$

(b) Here, $$d\Phi_e(\lambda) = \frac{\Phi_e}{\lambda_2 - \lambda_1}$$

$d\lambda$ in the interval $\lambda_1 \le \lambda \le \lambda_2$, since energy is distributed uniformly.

Then, $$\Phi = \int_{\lambda_1}^{\lambda_2} V(\lambda)\, d\Phi_2 \frac{(\lambda)}{A} = \frac{\Phi_e}{A(\lambda_2 - \lambda_1)} \int_{\lambda_1}^{\lambda_2} V(\lambda)\, d\lambda$$

Since $V(\lambda)$ is assumed to vary linearly in the interval $\lambda_1 \le \lambda \le \lambda_2$, we have

$$\frac{1}{\lambda_1 - \lambda_2} \int_{\lambda_1}^{\lambda_2} V(\lambda)\, d\lambda = \frac{1}{2}\,(V(\lambda_1) + V(\lambda_2))$$

Thus,
$$\Phi = \frac{\Phi_e}{2A}(V(\lambda_1) + V(\lambda_2))$$

Using $V(0.58\ \mu\text{m}) = 0.85$ and $V(0.63\ \mu\text{m}) = 0.25$, we get

$$\Phi = \frac{\Phi_e}{2 \times 1.6} \times 1.1 = 1.55\ \text{lm}$$

5.2 We have
$$\Phi_e = \frac{\Phi A}{V(\lambda)}$$

But
$$\Phi_e = \underbrace{\frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}E_m^2}_{\text{mean energy flux vector}} \times \frac{4\pi r^2}{\text{area}}$$

or
$$E_m^2 = \frac{\Phi A}{2\pi r^2 V(\lambda)}\sqrt{\frac{\mu_0}{\varepsilon_0}}$$

For $\lambda = 0.59\ \mu\text{m}$ and $V(\lambda) = 0.74$, we have

$$E_m = 1.14\ \text{V/m}$$

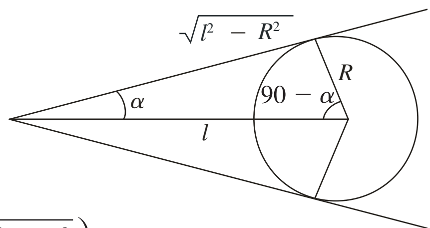
Also,
$$H_m = \sqrt{\frac{\varepsilon_0}{\mu_0}}E_m = 3.02\ \text{mA/m}$$

5.3 (a)
$$\text{Mean illuminance} = \frac{\text{Total luminous flux incident}}{\text{Total area illuminated}}$$

Now, to calculate the total luminous flux incident on the sphere, we note that the illuminance at the point of normal incidence is E_0. Thus, the incident flux is $E_0 \cdot \pi R^2$. So,

$$\text{mean illuminance} = \frac{\pi R^2 \cdot E_0}{2\pi R^2}$$

or
$$<E> = \frac{1}{2}E_0$$



(b) The sphere subtends a solid angle

$$2\pi(1 - \cos\alpha) = 2\pi\left(1 - \frac{\sqrt{l^2 - R^2}}{l}\right)$$

at the point source and therefore receives a total flux of

$$2\pi I\left(1-\frac{\sqrt{l^2-R^2}}{l}\right)$$

The area irradiated is

$$2\pi R^2\int_0^{90-\alpha}\sin\theta\,d\theta = 2\pi R^2(1-\sin\alpha) = 2\pi R^2\left(1-\frac{R}{l}\right)$$

Thus,
$$<E> = \frac{I}{R^2}\frac{1-\sqrt{1-(R/l)^2}}{1-R/l}$$

Substituting, we get

$$<E> = 50\text{ lx}$$

5.4 Luminance L is the light energy emitted per unit area of the emitting surface in a given direction per unit solid angle divided by $\cos\theta$. Luminosity M is simply energy emitted per unit area, so, we have

$$M = \int L\cos\theta\,d\Omega$$

where the integration must be in the forward hemisphere of the emitting surface (assuming light is being emitted in only one direction, say, outward direction of the surface). But

$$L = L_0\cos\theta$$

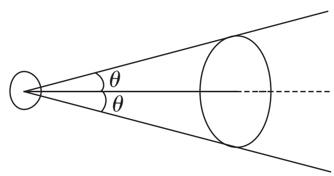
Thus,
$$M = \int L_0\cos^2\theta\,d\Omega = 2\pi\int_0^{\pi/2} L_0\cos^2\theta\sin\theta\,d\theta = \frac{2}{3}\pi L_0$$

5.5 (a) For a Lambert source, L = constant. The flux emitted into the cone is

$$\Phi = L\Delta S\int\cos\alpha\,d\Omega$$

$$= L\Delta S\int_0^{\theta} 2\pi\cos\alpha\sin\alpha\,d\alpha$$

$$= L\Delta S\pi(1-\cos^2\theta) = \pi L\Delta S\sin^2\theta$$



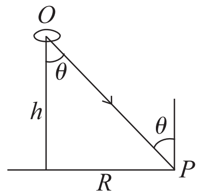
(b) The luminosity is obtained from the previous formula for $\theta = 90°$. So,

$$M = \frac{\Phi(\theta=90°)}{\Delta S} = \pi L$$

5.6 The equivalent luminous intensity in the direction OP is $LS\cos\theta$ and the illuminance at P is

$$\frac{LS\cos\theta}{(R^2+h^2)}\cos\theta = \frac{LSh^2}{(R^2+h^2)^2}$$

$$= \frac{LS}{(R^2/h+h)^2} = \frac{LS}{\left[(R/\sqrt{h}-\sqrt{h})^2+2R\right]^2}$$



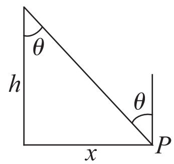
This is maximum when $R = h$ and the maximum illuminance is

$$\frac{LS}{4R^2} = \frac{1.6\times10^2}{4} = 40 \text{ lx}$$

5.7 The illuminance at P is

$$E_P = \frac{I(\theta)}{(x^2+h^2)}\cos\theta = \frac{I(\theta)\cos^3\theta}{h^2}$$

since this is constant at all x, we must have



$$I(\theta)\cos^3\theta = \text{constant} = I_0$$

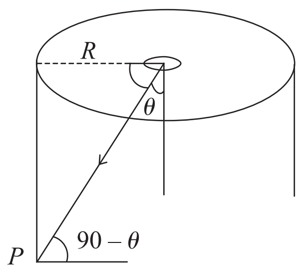
or
$$(\theta) = \frac{I_0}{\cos^3\theta}$$

The luminous flux reaching the table is

$$\Phi = \pi R^2 \times \frac{I_0}{h^2} = 314 \text{ lm}$$

5.8 The illuminated area acts as a Lambert source of luminosity $M = \pi L$ where $MS = \rho ES$ = total reflected light. Thus, the luminance

$$L = \frac{\rho E}{\pi}$$



The equivalent luminous intensity in the direction making an angle θ with the vertical is

$$LS\cos\theta = \frac{\rho ES}{\pi}\cos\theta$$

and the illuminance at point P is

$$\frac{\rho ES}{\pi}\frac{\cos\theta\sin\theta}{R^2\operatorname{cosec}^2\theta} = \frac{\rho ES}{\pi R^2}\cos\theta\sin^3\theta$$

This is maximum when

$$\frac{d}{d\theta}(\cos\theta\sin^3\theta) = -\sin^4\theta + 3\sin^2\theta\cos^2\theta = 0$$

or $$\tan^2\theta = 3 \Rightarrow \tan\theta = \sqrt{3}$$

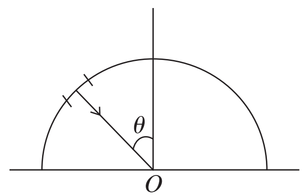
Then the maximum illuminance is

$$\frac{3\sqrt{3}}{16\pi}\frac{\rho ES}{R^2}$$

This illuminance is obtained at a distance $R\cot\theta = R/\sqrt{3}$ from the ceiling. Substitution gives the value 0.21 lx.

5.9 From the definition of luminance, the energy emitted in the radial direction by an element dS on the surface of the dome is

$$d\Phi = L\,dS\,d\Omega$$



Here L is constant. The solid $d\Omega$ is given by

$$d\Omega = \frac{dA\cos\theta}{R^2}$$

where dA is the area of an element on the plane illuminated by the radial light.

Then, $$d\Phi = \frac{L\,dS\,dA}{R^2}\cos\theta$$

The illuminance at O is then

$$E = \int\frac{d\Phi}{dA} = \int_0^{\pi/2}\frac{L}{R^2}2\pi R^2\sin\theta\,d\theta\cos\theta$$

$$= 2\pi L\int_0^{l}\sin\theta\,d(\sin\theta) = \pi L$$

5.10 Consider an element of area dS at point P. It emits light of flux

$$d\Phi = L\,dS\,d\Omega\cos\theta$$

$$= L\,dS\frac{dA}{h^2\sec^2\theta}\cos^2\theta$$

$$= \frac{L\,dS\,dA}{h^2}\cos^4\theta$$

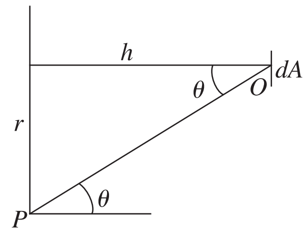
in the direction of the surface element dA at O.

The total illuminance at O is then

$$E = \int\frac{L\,dS}{h^2}\cos^4\theta$$

But $$dS = 2\pi r\,dr = 2\pi h \tan\theta\, d(h\tan\theta)$$
$$= 2\pi h^2 \sec^2\theta \tan\theta\, d\theta$$

Substitution gives $$E = 2\pi L \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = \pi L$$



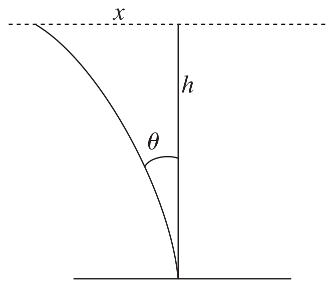
5.11 Consider an angular element of area

$$2\pi x\, dx = 2\pi h^2 \tan\theta \sec^2\theta\, d\theta$$

Light emitted from this ring is

$$d\Phi = L\, d\Omega(2\pi h^2 \tan\theta \sec^2\theta\, d\theta)\cos\theta$$

Now, $$d\Omega = \frac{dA\cos\theta}{h^2\sec^2\theta}$$



where dA = an element of area of the table just below the centre of the illuminant.

Then, the illuminance at the element dA will be

$$E_0 = \int_{\theta=0}^{\theta=\alpha} 2\pi L \sin\theta\cos\theta\, d\theta$$

where $$\sin\alpha = \frac{R}{\sqrt{h^2 + R^2}}$$

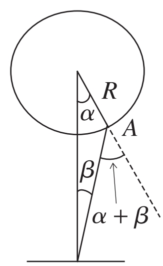
Finally, using luminosity $M = \pi L$, we get

$$E_0 = M\sin^2\alpha = M\frac{R^2}{h^2 + R^2}$$

or $$M = E_0\left(1 + \frac{h^2}{R^2}\right) = 700 \text{ lm/m}^2 = \left(1 \text{ lx} = 1\,\frac{\text{lm}}{\text{m}^2} \text{ dimensionally}\right)$$

5.12 The light emitted by an element of the illuminant towards the point O under consideration is (see figure)

$$d\Phi = L\, dS\, d\Omega \cos(\alpha + \beta)$$



The element dS has the area

$$dS = 2\pi R^2 \sin\alpha\, d\alpha$$

The distance

$$OA = \left[h^2 + R^2 - 2hR\cos\alpha\right]^{1/2}$$

We also have

$$\frac{OA}{\sin\alpha} = \frac{h}{\sin(\alpha+\beta)} = \frac{R}{\sin\beta}$$

From the figure, we have

$$\cos(\alpha+\beta) = \frac{h\cos\alpha - R}{OA}$$

$$\cos\beta = \frac{h - R\cos\alpha}{OA}$$

If we imagine a small area $d\Sigma$ at O, then

$$\frac{d\Sigma\cos\beta}{OA^2} = d\Omega$$

Hence, the illuminance at O is

$$\int \frac{d\Phi}{d\Sigma} = \int L2\pi R^2 \sin\alpha\, d\alpha \frac{(h\cos\alpha - R)(h - R\cos\alpha)}{(OA)^4}$$

The limit of α is $\alpha = 0°$ to that value for which $\alpha + \beta = 90°$, for then light is emitted tangentially. Thus,

$$\alpha_{\max} = \cos^{-1}\frac{R}{h}$$

So, $$E = \int_0^{\cos^{-1} R/h} L2\pi R^2 \sin\alpha\, d\alpha \frac{(h - R\cos\alpha)(h\cos\alpha - R)}{(h^2 + R^2 - 2hR\cos\alpha)^2}$$

We put $$y = h^2 + R^2 - 2hR\cos\alpha$$

So, $$dy = 2hR\sin\alpha\, d\alpha$$

Therefore, $$E = \int_{(h-R)^2}^{h^2-R^2} L2\pi R^2 \frac{dy}{2hR} \frac{\left(h - \dfrac{h^2 + R^2 - y}{2h}\right)\left(\dfrac{h^2 + R^2 - y}{2R} - R\right)}{y^2}$$

$$= \frac{L2\pi R^2}{8h^2R^2} \int_{(h-R)^2}^{h^2-R^2} \frac{(h^2 - R^2 + y)(h^2 - R^2 - y)}{y^2} dy$$

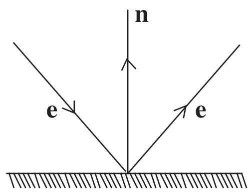
$$= \frac{\pi L}{4h^2} \int_{(h-R)^2}^{h^2-R^2} \left[\frac{(h^2 - R^2)^2}{y^2} - 1\right] dy = \frac{\pi L}{4h^2}\left[-\frac{(h^2 - R^2)^2}{y} - y\right]_{(h-R)^2}^{h^2-R^2}$$

$$= \frac{\pi L}{4h^2}\left[(h+R)^2 - (h^2 - R^2) - (h^2 - R^2) + (h-R)^2\right]$$

$$= \frac{\pi L}{4h^2}\left[2h^2 + 2R^2 - 2h^2 + 2R^2\right] = \frac{\pi L R^2}{h^2}$$

Substitution gives $E = 25.1$ lx.

5.13 We see in the figure that because of the law of reflection, the component of the incident unit vector $\mathbf{e}$ along $\mathbf{n}$ changes sign on reflection while the component parallel to the mirror remains unchanged.



So, $$\mathbf{e} = \mathbf{e}_{\parallel} + \mathbf{e}_{\perp}$$

where $$\mathbf{e}_{\perp} = \mathbf{n}(\mathbf{e}\cdot\mathbf{n})$$

and $$\mathbf{e}_{\parallel} = \mathbf{e} - \mathbf{n}(\mathbf{e}\cdot\mathbf{n})$$

We see that the reflected unit vector is

$$\mathbf{e}' = \mathbf{e}'_{\parallel} + \mathbf{e}'_{\perp} = \mathbf{e}_{\parallel} - \mathbf{e}_{\perp}$$

$$= [\mathbf{e} - \mathbf{n}(\mathbf{e}\cdot\mathbf{n})] - \mathbf{n}(\mathbf{e}\cdot\mathbf{n}) = \mathbf{e} - 2\mathbf{n}(\mathbf{e}\cdot\mathbf{n})$$

5.14 We choose the unit vectors perpendicular to the mirror as the x-, y-, z-axes in space. Then after reflection from the mirror with normal along x-axis, we have

$$\mathbf{e}' = \mathbf{e} - 2\mathbf{i}(\mathbf{i}\cdot\mathbf{e}) = -e_x\mathbf{i} + e_y\mathbf{j} + e_z\mathbf{k}$$

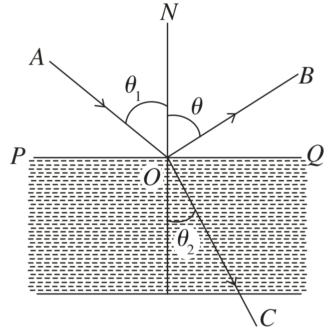
where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are the basic unit vectors. After a second reflection from the 2nd mirror, say, along y-axis

$$\mathbf{e}'' = \mathbf{e}' - 2\mathbf{j}(\mathbf{j}\cdot\mathbf{e}') = -e_x\mathbf{i} - e_y\mathbf{j} + e_z\mathbf{k}$$

Finally, after the third reflection

$$\mathbf{e}''' = -e_x\mathbf{i} - e_y\mathbf{j} - e_z\mathbf{k} = -\mathbf{e}$$

5.15 Let PQ be the surface of water and n be the refractive index (R.I.) of water. Let AO be the shaft of light with incident angle θ_1 and OB and OC be the reflected and refracted light rays at angles θ_1 and θ_2, respectively (see figure).



From the figure

$$\theta_2 = \pi/2 - \theta_1$$

From the law of refraction at the interface PQ

$$n = \frac{\sin\theta_1}{\sin\theta_2} = \frac{\sin\theta_1}{\sin(\pi/2 - \theta_1)}$$

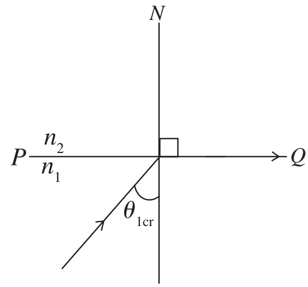
or $$n = \frac{\sin\theta_1}{\cos\theta_1} = \tan\theta_1$$

Hence, $\theta_1 = \tan^{-1} n$

So, $n = 53°$ (on substituting values)

5.16 Let two optical mediums of R.I., n_1 and n_2, respectively be such that $n_1 > n_2$. When angle of incidence is θ_{1cr} (see figure), from the law of reflection

$$n_1 \sin\theta_{1\,cr} = n_2 \qquad (1)$$



When the angle of incidence is θ_1, from the law of refraction at the interface of mediums 1 and 2,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

But in accordance with the problems

$$\theta_2 = \left(\frac{\pi}{2} - \theta_1\right) \qquad (2)$$

So, $n_1 \sin\theta_1 = n_2 \cos\theta_1$

Dividing Eq. (1) by Eq. (2)

$$\frac{\sin\theta_{1cr}}{\sin\theta_1} = \frac{1}{\cos\theta_1}$$

or $$\eta = \frac{1}{\cos\theta_1}$$

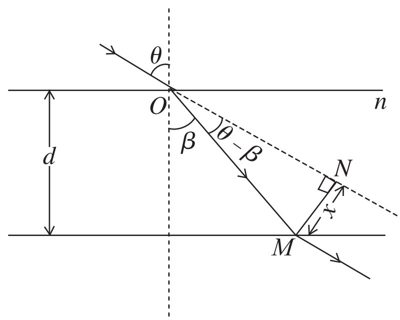
so, $$\cos\theta_1 = \frac{1}{\eta} \quad \text{and} \quad \sin\theta_1 = \frac{\sqrt{\eta^2 - 1}}{\eta} \qquad (3)$$

But, $$\frac{n_1}{n_2} = \frac{\cos\theta_1}{\sin\theta_1}$$

So, $$\frac{n_1}{n_2} = \frac{1}{\eta}\frac{\eta}{\sqrt{\eta^2 - 1}} \quad \text{(using Eq. 3)}$$

Thus, $$\frac{n_1}{n_2} = \frac{1}{\sqrt{\eta^2 - 1}} = 1.25$$

5.17 From the figure, the sought lateral shift



$$\begin{aligned} x &= OM \sin(\theta - \beta) \\ &= d \sec\beta \sin(\theta - \beta) \\ &= d \sec\beta (\sin\theta \cos\beta - \cos\theta \sin\beta) \\ &= d(\sin\theta - \cos\theta \tan\beta) \end{aligned}$$

But from the law of refraction

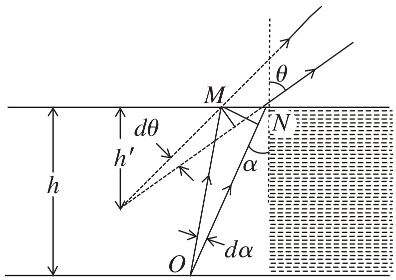
$$\sin\theta = n\sin\beta \quad \text{or} \quad \sin\beta = \frac{\sin\theta}{n}$$

So,
$$\cos\beta = \frac{\sqrt{n^2 - \sin^2\theta}}{n}$$

and
$$\tan\beta = \frac{\sin\theta}{\sqrt{n^2 - \sin^2\theta}}$$

Thus,
$$\begin{aligned} x &= d(\sin\theta - \cos\theta\,\tan\beta) \\ &= d\left(\sin\theta - \cos\theta\frac{\sin\theta}{\sqrt{n^2 - \sin^2\theta}}\right) \\ &= d\sin\theta\left[1 - \sqrt{\frac{1-\sin^2\theta}{n^2 - \sin^2\theta}}\right] = 3.1 \text{ cm} \end{aligned}$$

5.18 From the figure,



$$\sin d\alpha = \frac{MP}{OM} = \frac{MN\cos\alpha}{h\sec(\alpha + d\alpha)}$$

Since $d\alpha$ is very small, we have

$$d\alpha = \frac{MN\cos\alpha}{h\sec\alpha} = \frac{MN\cos^2\alpha}{h} \qquad (1)$$

Similarly,
$$d\theta = \frac{MN\cos^2\theta}{h'} \qquad (2)$$

From Eqs. (1) and (2)

$$\frac{d\alpha}{d\theta} = \frac{h'\cos^2\alpha}{h\cos^2\theta} \quad \text{or} \quad h' = \frac{h\cos^2}{\cos^2\alpha}\frac{d\alpha}{d\theta} \qquad (3)$$

From the law of refraction

$$n\sin\alpha = \sin\theta \qquad (A)$$

$$\sin\alpha = \frac{\sin\theta}{n} \quad \text{so,} \quad \cos\alpha = \sqrt{\frac{n^2 - \sin^2\theta}{n^2}} \qquad (B)$$

Differentiating Eq. (A)

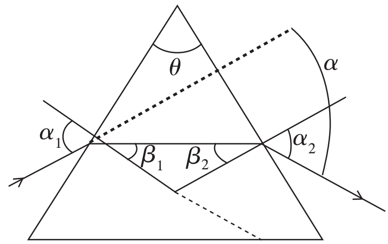
$$n\cos\alpha\,d\alpha = \cos\theta\,d\theta \quad \text{or} \quad \frac{d\alpha}{d\theta} = \frac{\cos\theta}{n\cos\alpha} \qquad (4)$$

Using Eq. (4) in Eq. (3), we get

$$h' = \frac{h\cos^3\theta}{n\cos^3\alpha} \tag{5}$$

Hence,
$$h' = \frac{h\cos^3\theta}{n\left(\dfrac{n^2-\sin^2\theta}{n^2}\right)^{3/2}} = \frac{n^2 h\cos^3\theta}{(n^2-\sin^2\theta)^{3/2}} \quad \text{(using Eq. B)}$$

5.19 The figure shows the passage of a monochromatic ray through the given prism, placed in air medium. From the figure, we have



$$\theta = \beta_1 + \beta_2 \tag{1}$$

and
$$\alpha = (\alpha_1 + \alpha_2) - (\beta_1 + \beta_2)$$
$$= (\alpha_1 + \alpha_2) - \theta \tag{2}$$

From Snell's law,

$$\sin\alpha_1 = n\sin\beta_1$$

or
$$\alpha_1 = n\beta_1 \quad \text{(for small angles)} \tag{3}$$

and
$$\sin\alpha_2 = n\sin\beta_2$$

or
$$\alpha_2 = n\beta_2 \quad \text{(for small angles)} \tag{4}$$

From Eqs. (2), (3) and (4), we get

$$\alpha = n(\beta_1 + \beta_2) - \theta$$

So,
$$\alpha = n(\theta) - \theta = (n-1)\theta \quad \text{(using Eq. 1)}$$

5.20 In general, for the passage of a monochromatic ray through a prism as shown in the figure of the solution of Problem 5.19.

$$\alpha = (\alpha_1 + \alpha_2) - \theta \tag{1}$$

Also, from Snell's law,

$$\sin\alpha_1 = n\sin\beta_1 \quad \text{or} \quad \alpha_1 = \sin^{-1}(n\sin\beta_1) \tag{2}$$

Similarly,

$$\alpha_2 = \sin^{-1}(n\sin\beta_2) = \sin^{-1}[n\sin(\theta-\beta_1)] \quad (\text{as } \theta = \beta_1 + \beta_2)$$

Using Eq. (2) in Eq. (1), we get

$$\alpha = [\sin^{-1}(n\sin\beta_1) + \sin^{-1}(n\sin(\theta-\beta_1))] - \theta$$

For α to be minimum,
$$\frac{d\alpha}{d\beta_1} = 0$$

or
$$\frac{n\cos\beta_1}{\sqrt{1-n^2\sin^2\beta_1}} - \frac{n\cos(\theta-\beta_1)}{\sqrt{1-n^2\sin^2(\theta-\beta_1)}} = 0$$

or
$$\frac{\cos^2\beta_1}{(1-n^2\sin^2\beta_1)} = \frac{\cos^2(\theta-\beta_1)}{1-n^2\sin^2(\theta-\beta_1)}$$

or
$$\cos^2\beta_1(1-n^2\sin^2(\theta-\beta_1)) = \cos^2(\theta-\beta_1)(1-n^2\sin^2\beta_1)$$

or
$$(1-\sin^2\beta_1)(1-n^2\sin^2(\theta-\beta_1)) = (1-\sin^2(\theta-\beta_1))(1-n^2\sin^2\beta_1)$$

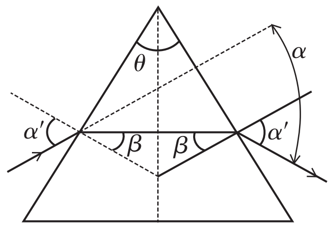
or
$$1-n^2\sin^2(\theta-\beta_1)-\sin^2\beta_1+\sin^2\beta_1 n^2\sin^2(\theta-\beta_1)$$
$$= 1-n^2\sin^2\beta_1-\sin^2(\theta-\beta_1)+\sin^2\beta_1 n^2\sin^2(\theta-\beta_1)$$

or
$$\sin^2(\theta-\beta_1)-n^2\sin^2(\theta-\beta_1) = \sin^2\beta_1(1-n^2)$$

or
$$\sin^2(\theta-\beta_1)(1-n^2) = \sin^2\beta_1(1-n^2)$$

or
$$\theta-\beta_1=\beta_1 \quad \text{or} \quad \beta_1 = \frac{\theta}{2}$$

But
$$\beta_1+\beta_2=\theta \quad \text{so,} \quad \beta_2=\frac{\theta}{2}=\beta_1$$



which is the case of symmetric passage of the ray.
In the case of symmetric passage of the ray

$$\alpha_1=\alpha_2=\alpha' \text{ (say)}$$

and
$$\beta_1=\beta_2=\beta=\frac{\theta}{2}$$

Thus, the total deviation
$$\alpha=(\alpha_1+\alpha_2)-\theta$$
$$\alpha=2\alpha'-\theta \quad \text{or} \quad \alpha'=\frac{\alpha+\theta}{2}$$

But from Snell's law,
$$\sin\alpha = n\sin\beta$$

so,
$$\sin\frac{\alpha+\theta}{2} = n\sin\frac{\theta}{2}$$

5.21 In this case, we have

$$\sin\frac{\alpha+\theta}{2} = n\sin\frac{\theta}{2} \quad \text{(see solution of Problem 5.20)}$$

In our problem $\alpha=\theta$.

So, $$\sin\theta = n\sin\left(\frac{\theta}{2}\right) \quad \text{or} \quad 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = n\sin\left(\frac{\theta}{2}\right)$$

Hence, $$\cos\left(\frac{\theta}{2}\right) = \frac{n}{2}$$

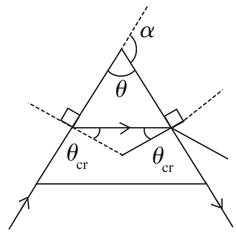
or $$\theta = 2\cos^{-1}\left(\frac{n}{2}\right) = 83° \quad (\text{where } n = 1.5)$$

5.22 In the case of minimum deviation

$$\sin\frac{\alpha+\theta}{2} = n\sin\frac{\theta}{2}$$

So, $$\alpha = 2\sin^{-1}\left\{n\sin\frac{\theta}{2}\right\} - \theta = 37° \quad (\text{for } n = 1.5)$$

Passage of ray for grazing incidence and grazing emergence is the condition for maximum deviation. From the figure



$$\alpha = \pi - \theta = \pi - 2\theta_{cr}$$

(where θ_{cr} is the critical angle).

So, $$\alpha = \pi - 2\sin\left(\frac{1}{n}\right) = 58°$$

for $n = 1.5$ = R.I. of glass.

5.23 The smallest deflection angle is given by the formula,

$$\delta = 2\alpha - \theta \tag{1}$$

where α is the angle of incidence at first surface and θ is the prism angle.

Also from Snell's law, $n_1 \sin\alpha = n_2 \sin(\theta/2)$, as the angle of refraction at first surface is equal to half the angle of prism for least deflection.

So, $$\sin\alpha = \frac{n_2}{n_1}\sin\left(\frac{\theta}{2}\right) = \frac{1.5}{1.33}\sin 30° = 0.5639$$

or $$\alpha = \sin^{-1}(0.5639) = 34.3259°$$

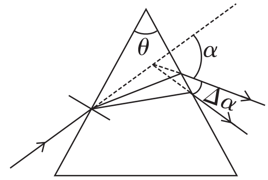
Substituting the values in Eq. (1), we get $\delta = 8.65°$.

5.24 From Cauchy's formula, and also experimentally, the R.I. of a medium depends upon the wavelength of the monochromatic ray, i.e., $n = f(\lambda)$. In the case of least deviation of a monochromatic ray passing through a prism, we have

$$n\sin\frac{\theta}{2} = \sin\frac{\alpha+\theta}{2} \tag{1}$$

The above equation tells us that we have $n = n(\alpha)$, so we may write

$$\Delta n = \frac{dn}{d\alpha}\Delta\alpha \qquad (2)$$



From Eq. (1)

$$dn \sin\frac{\theta}{2} = \frac{1}{2}\cos\frac{\alpha+\theta}{2}\, d\alpha$$

or
$$\frac{dn}{d\alpha} = \frac{\cos(\alpha+\theta)/2}{2\sin\theta/2} \qquad (3)$$

From Eqs. (2) and (3)

$$\Delta n = \frac{\cos(\alpha+\theta)/2}{\sin\theta/2}\Delta\alpha$$

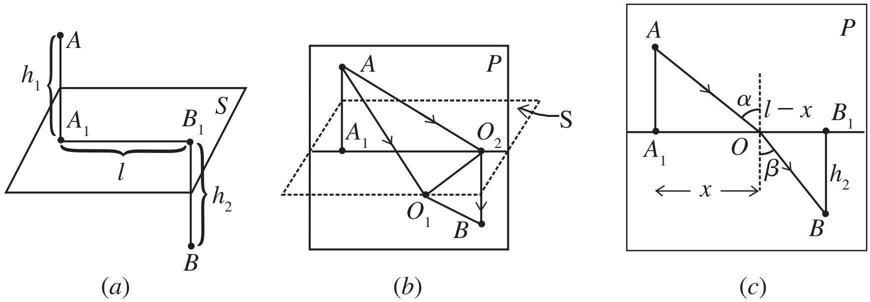
or
$$\Delta n = \frac{\sqrt{1-\sin^2(\alpha+\theta)/2}}{2\sin\theta/2}\Delta\alpha = \frac{\sqrt{1-n^2\sin^2\theta/2}}{2\sin\theta/2}\Delta\alpha \quad \text{(using Eq. 1)}$$

Thus,
$$\Delta\alpha = \frac{2\sin\theta/2}{\sqrt{1-n^2\sin^2\theta/2}}\Delta n = 0.44$$

5.25 Fermat's principle: The actual path of propagation of light (trajectory of a light ray) is the path which can be followed by light within the least time, in comparison with all other hypothetical paths between the same two points.

The above statement is the original wording of Fermat (a famous French scientist of 17th century). Deduction of the law of refraction from Fermat's principle:

Let the plane S be the interface between medium 1 and medium 2 [Fig. (a)] with the refractive indices $n_1 = c/v_1$ and $n_2 = c/v_2$. Assume, as usual, that $n_1 < n_2$. Two points are given — one above the plane S (point A), the other under plane S (point B). The various distances are: $AA_1 = h_1$, $BB_1 = h_2$, $A_1B_1 = $ l. We must find the path from A to B which can be covered by light faster than it can cover any other hypothetical path. Clearly, this path must consist of two straight lines, viz., AO in medium 1 and OB in medium 2. The point O in the plane S has to be found.



First of all, it follows from Fermat's principle that the point O must lie on the intersection of S and a plane P, which is perpendicular to S and passes through A and B.

Indeed, let us assume that this point does not lie in the plane P; let this be point O_1 in Fig. (b). Drop the perpendicular $O_1 O_2$ from O_1 onto P. Since $AO_2 < AO_1$ and $BO_2 < BO_1$, it is clear that the time required to traverse AO_2B is less than that needed to cover the path AO_1B. Thus, using Fermat's principle, we see that the first law of refraction is observed: the incident and the refracted rays lie in the same plane as the perpendicular to the interface at the point where the ray is refracted. This plane is the plane P in Fig. (b); it is called the plane of incidence.

Now let us consider light rays in the plane of incidence [Fig. (c)]. Designate A_1O as x and $OB_1 = l - x$. The time it takes a ray to travel from A to O and then from O to B is

$$T = \frac{AO}{v_1} + \frac{OB}{v_2} = \frac{\sqrt{h_2^2 + (l-x)^2}}{v_2} + \frac{\sqrt{h_1^2 + x^2}}{v_1} \tag{1}$$

The time depends on the value of x. According to Fermat's principle, the value of x must minimize the time T. At this value of x, the derivative dT/dx equals zero, i.e.,

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{h_1^2 + x^2}} - \frac{l-x}{v_2\sqrt{h_2^2 + (l-x)^2}} = 0 \tag{2}$$

Now,
$$\frac{x}{\sqrt{h_1^2 + x^2}} = \sin\alpha \quad \text{and} \quad \frac{l-x}{\sqrt{h_2^2 + (l-x)^2}} = \sin\beta$$

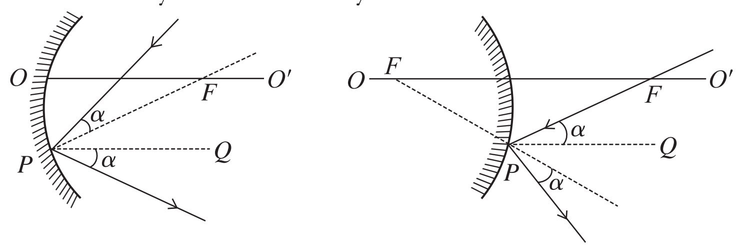
Consequently,

$$\frac{\sin\alpha}{v_1} - \frac{\sin\beta}{v_2} = 0 \quad \text{or} \quad \frac{\sin\alpha}{\sin\beta} = \frac{v_1}{v_2}$$

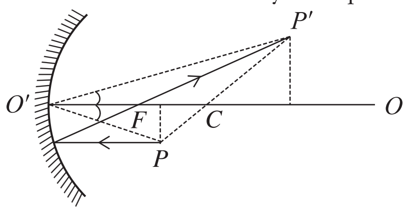
So,
$$\frac{\sin\alpha}{\sin\beta} = \frac{c/n_1}{c/n_2} = \frac{n_2}{n_1}$$

Note: *Fermat himself could not use Eq. 2, because mathematical analysis was developed later by Newton and Leibniz. To deduce the law of the refraction of light, Fermat used his own maximum and minimum method of calculus, which, in fact, corresponded to the subsequently developed method of finding the minimum (maximum) of a function by differentiating it and equating the derivative to zero.*

5.26 (a) Join F to P. Draw PQ parallel to OO'. Reflected ray makes an angle (say α) with PQ equal to that made by the incident ray with PF.

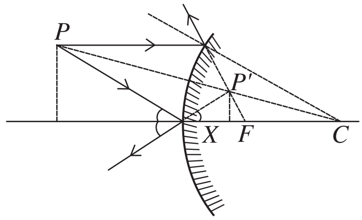


(b) In Fig. (a) of the problem book, P is the object and P' is its (real) image. Look for a point O' on the axis such that $O'P'$ and $O'P$ make equal angles with $O'O$. This determines the position of the mirror. Draw a ray from P parallel to the axis. This must on reflection pass through P'. The intersection of the reflected ray with principal axis determines the focus.



(a)

In Fig. (b) of the problem book, suppose P is the object and P' is the image. Then the mirror is convex because the image is virtual, erect and diminished. Look for a point X (between P and P') on the axis such that PX and $P'X'$ make equal angle with the axis.



(b)

5.27 (a) From the mirror formula,

$$\frac{1}{s'} + \frac{1}{s} = \frac{1}{f}$$

we get,
$$f = \frac{s's}{s' + s} \tag{1}$$

But from the problem

$$s - s' = l \frac{s'}{s} = \beta$$

From these two relations, we get

$$s = \frac{l}{1 - \beta} \quad \text{and} \quad s' = -\frac{l\beta}{1 + \beta}$$

Substituting it in the Eq. (1), we get

$$f = \frac{\beta(l/1 - \beta)^2}{l(1 - \beta/1 - \beta)} = \frac{l\beta}{(1 - \beta^2)} = 10 \text{ cm}$$

(b) Again we have,

$$\frac{1}{s'} + \frac{1}{s} = \frac{1}{f} \quad \text{or} \quad \frac{s}{s'} + 1 = \frac{s}{f}$$

or $$\frac{1}{\beta_1} = \frac{s}{f} - 1 = \frac{s-f}{f}$$

or $$\beta_1 = \frac{f}{s-f} \tag{2}$$

Now, it is clear from the above equation that for smaller β, s must be large, so the object is displaced away from the mirror in second position

i.e., $$\beta_2 = \frac{f}{s+l-f} \tag{3}$$

Eliminating s from Eqs. (2) and (3), we get

$$f = \frac{l\beta_1\beta_2}{(\beta_2 - \beta_1)} = -2.5 \text{ cm}$$

5.28 For a concave mirror

$$\frac{1}{s'} + \frac{1}{s} = \frac{1}{f} \quad \text{so} \quad s' = \frac{sf}{s-f}$$

(In coordinate convention, $s = -s$ is negative and $f = -|f|$ is also negative.)

If A is the area of the mirror (assumed small) and the object is on the principal axis, then the light incident on the mirror per second is I_0A/s^2. This follows from the definition of luminous intensity as light emitted per second per unit solid angle in a given direction and the fact that A/s^2 is the solid angle subtended by the mirror at the source. Of this a fraction ρ is reflected so if I is the luminous intensity of the image, then

$$\frac{IA}{s'^2} = \rho I_0 \frac{A}{s^2}$$

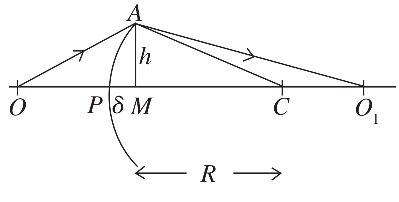
Hence, $$I = \rho I_0 \left(\frac{|f|}{|f| - s}\right)^2$$

(Because our convention makes f $-ve$ for a concave mirror, we have to write $|f|$.)

Substitution gives $I = 2.0 \times 10^3$ cd.

5.29 For O_1 to be the image, the optical paths of all rays OAO_1 must be equal upto terms of leading order in h. Thus,

$$nOA + n'AO_1 = \text{constant}$$



But, $OP = |s|$, $O_1P = |s'|$ and $PM = \delta$, so

$$OA = \sqrt{h^2 + (|s| + \delta^2)}$$

$$\cong |s| + \delta + \frac{h^2}{2|s|}$$

Similarly, $$O_1A = \sqrt{h^2 + (|s'| - \delta)} \cong |s'| - \delta + \frac{h^2}{2|s'|}$$

(Ignoring products $h^2\delta$.) Then,

$$n|s| + n'|s'| + n\delta - n'\delta + \frac{h^2}{2}\left(\frac{n_1}{|s|} + \frac{n'}{|s'|}\right) = \text{constant} \tag{1}$$

From the triangle AMC,

$$(R - \delta)^2 + h^2 = R^2$$

or $$h^2 = 2R\delta \quad (\text{where } CP = R, \text{ radius of curvature})$$

So, $$\delta = \frac{h^2}{2R} \tag{2}$$

Hence, $$n|s| + n'|s'| + \frac{h^2}{2}\left\{\frac{n - n'}{R} + \frac{n}{|s|} + \frac{n'}{|s'|}\right\} = \text{constant} \quad (\text{using Eq. 2 in Eq. 1})$$

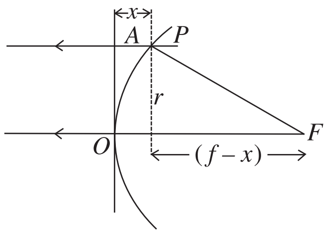
Since this must hold for all h, we have

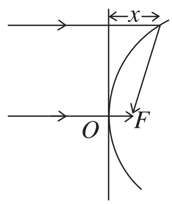
$$\frac{n'}{|s'|} + \frac{n}{|s|} = \frac{n' - n}{R}$$

From our sign convention, $s' > 0$, $s < 0$ so we get

$$\frac{n'}{s'} - \frac{n}{s} = \frac{n' - n}{R}$$

5.30 All rays focusing at a point must have traversed the same optical path. Thus,





$$x + n\sqrt{r^2 + (f - x)^2} = nf$$

or $$(nf - x)^2 = n^2r^2 + n^2(f - x)^2$$

or $$n^2r^2 = (nf - x)^2 - [n(f - x)]^2$$
$$= (nf - x + nf - nx)(nf - x - nf + nx)$$
$$= x(n - 1)(2nf - (n + 1)x)$$
$$= 2n(n - 1)fx - (n + 1)(n - 1)x^2$$

Thus, $\quad (n+1)(n-1)x^2 - 2n(n-1)fx + n^2r^2 = 0$

So, $$x = \frac{n(n-1)f \pm \sqrt{n^2(n-1)^2 f^2 - n^2r^2(n+1)(n-1)}}{(n+1)(n-1)}$$

$$= \frac{nf}{n+1}\left[1 \pm \sqrt{1 - \frac{n+1}{n-1}\frac{r^2}{f^2}}\right]$$

The ray must move forward so $x < f$, for $+ve$ sign $x > f$ for small r, so $-ve$ sign. (Also $x \to 0$ as $r \to 0$.) $x > f$ means ray is turning back in the direction of incidence (see figure).

Hence, $$x = \frac{nf}{n+1}\left[1 - \sqrt{1 - \frac{n+1}{n-1}\frac{r^2}{f^2}}\right]$$

For the maximum value of r,

$$\sqrt{1 - \frac{n+1}{n-1}\frac{r^2}{f^2}} = 0 \tag{1}$$

because the expression under the radical sign must be non-negative, which gives the maximum value of r.

Hence, $$r_{\max} = f\sqrt{(n-1)(n+1)} \quad \text{(using Eq. 1)}$$

5.31 Since the given lens has significant thickness, the thin lens formula cannot be used. For refraction at the front surface from the formula

$$\frac{n'}{s'} - \frac{n}{s} = \frac{n'-n}{R}$$

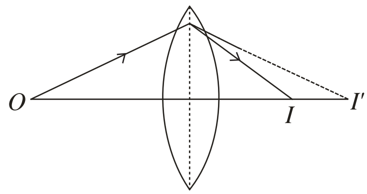
$$\frac{1.5}{s'} - \frac{1}{-20} = \frac{1.5-1}{5}$$

On simplifying, we get, $s' = 30$ cm.

Thus, the image I' produced by the front surface behaves as a virtual source for the rear surface at distance 25 cm from it, because the thickness of the lens is 5 cm. Again, from the refraction formula at curve surface

$$\frac{n'}{s'} - \frac{n}{s} = \frac{n'-n}{R}$$

$$\frac{1}{s'} - \frac{1.5}{25} = \frac{1-1.5}{-5}$$



On simplifying, $s' = +6.25$ cm.

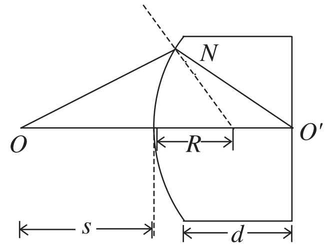
Thus, we get a real image I at a distance 6.25 cm beyond the rear surface (see figure).

5.32 (a) The formation of the image of a source S, placed at a distance u from the pole of the convex surface of plano-convex lens of thickness d is shown in the figure. On applying the formula for refraction through a spherical surface, we get

$$\frac{n}{s'} - \frac{1}{s} = \frac{(n-1)}{R} \quad (\text{here } n_2 = n \text{ and } n_1 = 1)$$

So,
$$\frac{n}{d} - \frac{1}{s} = \frac{(n-1)}{R} \quad \text{or} \quad \frac{1}{s} = \frac{n}{d} - \frac{(n-1)}{R}$$

or
$$\frac{s'}{s} = s'\left\{\frac{n}{d} - \frac{(n-1)}{R}\right\}$$



But in this case, optical path of the light corresponding to the distance v in the medium is v/n, so the magnification produced will be

$$\beta = \frac{s'}{ns} = \frac{s'}{n}\left\{\frac{n}{d} - \frac{(n-1)}{R}\right\} = \frac{d}{n}\left\{\frac{n}{d} - \frac{(n-1)}{R}\right\} = 1 - \frac{d(n-1)}{nR}$$

Substituting the values, we get magnification $\beta = -0.20$.

(b) If the transverse area of the object is A (assumed small), the area of the image is $\beta^2 A$. We shall assume that $\pi D^2/4 > A$. Then light falling on the lens is

$$LA\,\frac{\pi D^2/4}{s^2}$$

from the definition of luminance (see Eq. 5.1c) of the book. Here,

$$\cos\theta \approx 1 \text{ (if } D^2 \ll s^2) \quad \text{and} \quad d\Omega = \left(\frac{\pi D^2/4}{s^2}\right)$$

Then the illuminance of the image is

$$LA\,\frac{(\pi D^2/4)/s^2}{\beta^2 A} = \frac{Ln^2\pi D^2}{4d^2}$$

Substitution gives 42 lx.

5.33 (a) Optical power of a thin lens of R.I. n in a medium with R.I. n_0 is given by

$$\Phi = (n - n_0)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{1}$$

From Eq. (1), when the lens is placed in air

$$\Phi_0 = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{2}$$

Similarly, from Eq. (1), when the lens is placed in liquid

$$\Phi = (n - n_0)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{3}$$

Thus, from Eqs. (2) and (3)

$$\Phi = \frac{n - n_0}{n - 1}\Phi_0 = 2\,\text{D}$$

The second focal length is given by

$$f' = \frac{n'}{\Phi}$$

(where n' is the R.I. of the medium in which it is placed).

$$f' = \frac{n_0}{\Phi} = 85\ \text{cm}$$

(b) Optical power of a thin lens of R.I. n placed in medium of R.I. n_0 is given by

$$\Phi = (n - n_0)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{4}$$

For a biconvex lens placed in air medium from Eq. (4)

$$\Phi_0 = (n - 1)\left(\frac{1}{R} - \frac{1}{-R}\right) = \frac{2(n-1)}{R}$$

where R is the radius of each curve surface of the lens.

Optical power of a spherical refractive surface is given by

$$\Phi = \frac{n' - n}{R}$$

For the rear surface of the lens which divides air and glass medium

$$\Phi_a = \frac{n-1}{R} \quad \text{(here } n \text{ is the R.I. of glass)}$$

Similarly, for the front surface which divides water and glass medium

$$\Phi_l = \frac{n_0 - n}{-R} = \frac{n - n_0}{R} \tag{5}$$

Hence, the optical power of the given optical system

$$\Phi = \Phi_a + \Phi_l = \frac{n-1}{R} + \frac{n - n_0}{R} = \frac{2n - n_0 - 1}{R} \tag{6}$$

From Eqs. (4) and (6)

$$\frac{\Phi}{\Phi_0} = \frac{2n - n_0 - 1}{2(n-1)} \quad \text{so,} \quad \Phi = \frac{(2n - n_0 - 1)}{2(n-1)}\Phi_0$$

Focal length in air,

$$f = \frac{1}{\Phi} = 15 \text{ cm}$$

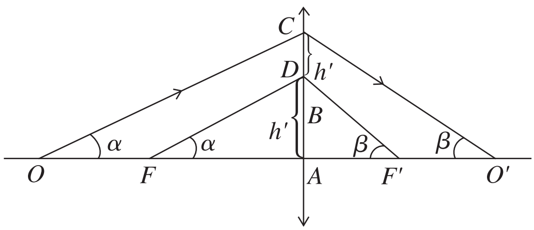
or $$\Phi = 6.7 \text{ D}$$

and focal length in water

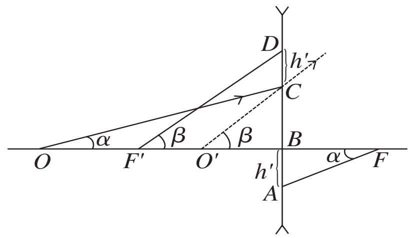
$$= \frac{n_0}{\Phi} = 20 \text{ cm} \quad \left(\text{for } n_0 = \frac{4}{3}\right)$$

5.34 (a) Clearly, the media on the sides are different. The front focus F is the position of the object (virtual or real) for which the image is formed at infinity. The rear focus F' is the position of the image (virtual or real) of the object at infinity.

(a) For Figs. 5.7 (a) and (b) of the problem book:



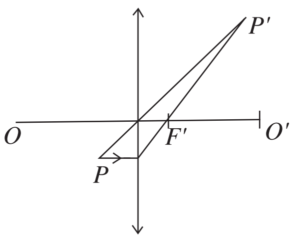
(*a*) Convex lens



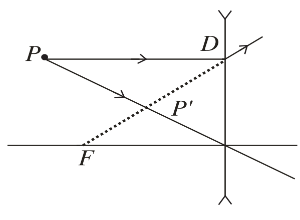
(*b*) Concave lens

This geometrical construction ensured that the second expression in Eq. (5.1g) is obeyed. In the figure above, FA is parallel to the incident ray. Join F' to D where $CD = AB$. Emergent ray is parallel to DF'.

(b) For Figs. 5.5 (a) and (b) of the problem book replacing mirror with lens:



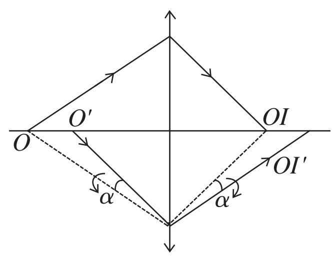
(*a*) Convex lens



(*b*) Concave lens

where P is the object.

(c) For Figs. 5.8 (a) and (b) of the problem book:



Clearly, it is important to note that when the rays (1) and (2) are not symmetric about the principal axis, the figure can be completed by reflection in the principal axis. Knowing one path we know the path of all rays connecting the two points. For a different object, we proceed as shown in the figure, we use the fact that a ray incident at a given height above the optic centre suffers a definite deviation.

The concave lens can be discussed similarly.

5.35 Since the image is formed on the screen, it is real, so for a converging lens, the object is on the incident side. Let s_1 and s_2 be the magnitudes of the object distance in the first and second cases, respectively. We have the lens formula

$$\frac{1}{s'} - \frac{1}{s} = \frac{1}{f} \tag{1}$$

In the first case, from Eq. (1)

$$\frac{1}{(+l)} - \frac{1}{(-s_1)} = \frac{1}{f} \quad \text{or} \quad s_1 = \frac{f(l)}{l-f} = 26.31 \text{ cm}$$

Similarly, from Eq. (1) in the second case

$$\frac{1}{(l-\Delta l)} - \frac{1}{(-s_2)} = \frac{1}{f} \quad \text{or} \quad s_2 = \frac{lf}{(l-\Delta l) - f} = 26.36 \text{ cm}$$

Thus, the sought distance

$$\Delta x = s_2 - s_1 = \frac{\Delta l f^2}{(l - f^2)} = 0.5 \text{ mm}$$

5.36 The distance between the object and the image is l. Let x = distance between the object and the lens. Then, since the image is real, we have as per convention,

$$s = -x, \quad s' = l - x$$

So,
$$\frac{1}{x} + \frac{1}{l-x} = \frac{1}{f}$$

or
$$x(l-x) = lf \quad \text{or} \quad x^2 - xl + lf = 0$$

Solving, we get the roots

$$x = \frac{1}{2}\left[l \pm \sqrt{l^2 - 4lf}\right]$$

(We must have $l > 4f$ for real roots.)

(a) If the distance between the two positions of the lens is Δl, then clearly

$$\Delta l = x_2 - x_1 = \text{difference between roots} = \sqrt{l^2 - 4lf}$$

So,
$$f = \frac{l^2 - \Delta l^2}{4l} = 20 \text{ cm}$$

(b) The two roots are conjugate in the sense that if one gives the object distance, the other gives the corresponding image distance (in both cases). Thus, the magnifications are

$$-\frac{l + \sqrt{l^2 - 4lf}}{l - \sqrt{l^2 - 4lf}} \text{ (enlarged)} \quad \text{and} \quad -\frac{l - \sqrt{l^2 - 4lf}}{l + \sqrt{l^2 - 4lf}} \text{ (diminished)}$$

The ratio of these magnifications being η, we have

$$\frac{l+\sqrt{l^2-4lf}}{l-\sqrt{l^2-4lf}}=\sqrt{\eta} \quad \text{or} \quad \frac{\sqrt{l^2-4lf}}{l}=\frac{\sqrt{\eta}-1}{\sqrt{\eta}+1}$$

or
$$1-\frac{4f}{l}=\left(\frac{\sqrt{\eta}-1}{\sqrt{\eta}+1}\right)^2=1-4\frac{\sqrt{\eta}}{(1+\sqrt{\eta})^2}$$

Hence,
$$f=l\frac{\sqrt{\eta}}{(1+\sqrt{\eta})^2}=20\text{ cm}$$

5.37 We know from the previous problem that the two magnifications are reciprocal of each other $(\beta'\beta''=1)$. If h is the size of the object, then

$$h'=\beta' h \quad \text{and} \quad h''=\beta'' h$$

Hence,
$$h=\sqrt{h'h''}$$
$$=3.0\text{ mm}$$

5.38 Refer to Problem 5.32 b. If A is the area of the object, then provided the angular diameter of the object at the lens is much smaller than other relevant angles like D/f, we calculate the light falling on the lens as $LA(\pi D^2/4s^2)$, where s^2 is the object distance squared. If β is the transverse magnification $(\beta = s'/s)$ then the area of the image is $\beta^2 A$.

Hence, the illuminance of the image (also taking account of the light lost in the lens)

$$E=(1-\alpha)LA\frac{\pi D^2}{4s^2}\frac{1}{\beta^2 A}=\frac{(1-\alpha)\pi D^2 L}{4f^2}$$

(since $s'=f$ for a distant object.)

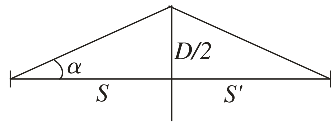
Substitution gives $E = 15$ lx

5.39 (a) If s = object distance, s' = average distance, L = luminance of the source, ΔS = area of the source assumed to be a plane surface held normal to the principal axis, then we find for the flux incident on the lens

$$\Delta\Phi=\int L\Delta S\cos\theta\,d\Omega$$
$$\approx L\Delta S\int_0^{\infty}\cos\theta\,2\pi\sin\theta\,d\theta$$
$$=L\Delta S\pi\sin^2\alpha\cong L\Delta S\frac{\pi D^2}{4s^2}$$

Here, we are assuming $D \ll s$, and ignoring the variation of L since α is small.

If L' is the luminance of the image and $\Delta S' = (S'/S)^2 \Delta S$ is the area of the image, then similarly



$$L'\Delta S' \frac{D^2}{4s'^2}\pi = L'\Delta S \frac{D^2}{4s^2}\pi = L\Delta S \frac{D^2}{4s^2}\pi$$

or $$L' = L \text{ (irrespective of } D)$$

(b) In this case, the image on the white screen is from a Lambert source. Then, if its luminance is L_0, its luminosity will be πL_0 and

$$\pi L_0 \frac{s'^2}{s^2}\Delta S = L\Delta S \frac{D^2}{4s^2}\pi$$

or $$L_0 \propto D^2$$

since s' depends on f and s but not on D.

5.40 Focal length of the converging lens, when it is submerged in water of R.I. n_0 (say)

$$\frac{1}{f_1} = \left(\frac{n_1}{n_0} - 1\right)\left(\frac{1}{R} - \frac{1}{-R}\right) = \frac{2(n_1 - n_0)}{n_0 R} \qquad (1)$$

Similarly, the focal length of diverging lens in water

$$\frac{1}{f_2} = \left(\frac{n_2}{n_0} - 1\right)\left(\frac{1}{-R} - \frac{1}{-R}\right) = \frac{-2(n_2 - n_0)}{n_0 R} \qquad (2)$$

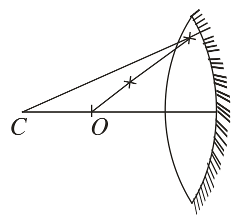
Now, when they are put together in the water, the focal length of the system, will be

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$

$$= \frac{2(n_1 - n_0)}{n_0 R} - \frac{2(n_2 - n_0)}{n_0 R} = \frac{2(n_1 - n_2)}{n_0 R}$$

or $$f = \frac{-n_0 R}{2(n_1 - n_2)} = 35 \text{ cm}$$

5.41 C is the centre of curvature of the silvered surface and O is the effective centre of the equivalent mirror in the sense that an object at O forms a coincident image. From the figure, using the formula for refraction at a spherical surface, we have

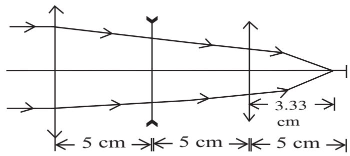


$$\frac{n}{-R} - \frac{1}{2f} = \frac{n-1}{R} \quad \text{or} \quad f = \frac{-R}{2(2n-1)}$$

(In our case f is $-ve$.)

Substitution gives $f = -10$ cm.

5.42 (a) Path of a ray, as it passes through the lens system is as shown in the figure. Focal length of all the three lenses



$$f = \frac{1}{10} \text{ m} = 10 \text{ cm} \quad \text{(neglecting their signs)}$$

Applying lens formula for the first lens, considering a ray coming from infinity

$$\frac{1}{s'} - \frac{1}{\infty} = \frac{1}{f} \quad \text{or} \quad s' = f = 10 \text{ cm}$$

and so the position of the image is 5 cm to the right of the second lens, when only the first one is present. The ray again gets refracted while passing through the second lens, so

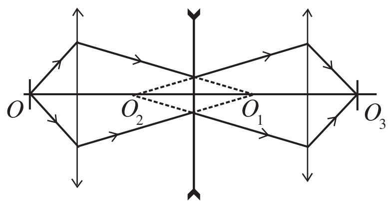
$$\frac{1}{s'} - \frac{1}{5} = \frac{1}{f} = \frac{1}{-10}$$

or $s' = 10$ cm, which is now 5 cm left to the third lens. So, for this lens

$$\frac{1}{s''} - \frac{1}{5} = \frac{1}{10} \quad \text{or} \quad \frac{1}{s''} = \frac{3}{10}$$

or $$s'' = 10/3 = 3.33 \text{ cm} \quad \text{(from the last lens)}$$

(b) This means that if the object is x cm to the left of the first lens on the axis OO' then the image is x on to the right of the third (last) lens. Call the lenses L_1, L_2, L_3 from the left and let O be the object, O_1 its image by the first lens, O_2 the image of O_1 by the second lens and O_3 the image of O_2 by the third lens.



O_1 and O_2 must be symmetrically located with respect to the lens L_2 and since this lens is concave, O_1 must be at a distance $2\,|\,f_2\,|$ to be the right of L_2 and O_2 must be $2\,|\,f_2\,|$ to be the left of L_2. One can check that this satisfies lens equation for the third lens L_3

$$s = -(2\,|\,f_2\,| + 5) = -25 \text{ cm}$$

$$s' = x, \quad f_3 = 10 \text{ cm}$$

Hence, $$\frac{1}{x} + \frac{1}{25} = \frac{1}{10} \quad \text{so,} \quad x = 16.67 \text{ cm}$$

We have written $|\,f_2\,|$ because f_2 is $-ve$ in our convention.

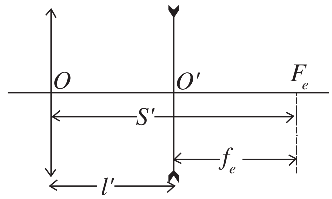
5.43 (a) Angular magnification for the Galilean telescope in normal adjustment is given as

$$\Gamma = \frac{f_o}{f_e}$$

or $$10 = \frac{f_o}{f_e} \quad \text{or} \quad f_o = 10 f_e \tag{1}$$

The length of the telescope in this case is given by

$$l = f_o - f_e = 45 \text{ cm}$$



So, using Eq. (1), we get

$$f_e = +5 \quad \text{and} \quad f_o = +50 \text{ cm}$$

(b) Using lens formula for the objective,

$$\frac{1}{s_0'} - \frac{1}{s_0} = \frac{1}{f_o} \quad \text{or} \quad s_0' = \frac{s_0 f_o}{s_0 + f_o} = 50.5 \text{ cm}$$

From the figure, it is clear that,

$$s_0' = l' + f_e \text{ (where } l' \text{ is the new tube length)}$$

or $$l' = v_0 - f_e = 50.5 - 5 = 45.5 \text{ cm}$$

So, the displacement of ocular is,

$$\Delta l = l' - l = 45.5 - 45 = 0.5 \text{ cm}$$

5.44 In the Keplerian telescope, in normal adjustment, the distance between the objective and eyepiece is $f_o + f_e$. The image of the mounting produced by the eyepiece is formed at a distance v to the right where

$$\frac{1}{s'} - \frac{1}{s} = \frac{1}{f_e}$$

But $$s = -(f_o + f_e)$$

so, $$\frac{1}{s'} = \frac{1}{f_e} - \frac{1}{f_o + f_e} = \frac{f_o}{f_e(f_o + f_e)}$$

The linear magnification produced by the eyepiece of the mounting is in magnitude,

$$|\beta| = \left|\frac{s'}{s}\right| = \frac{f_e}{f_o}$$

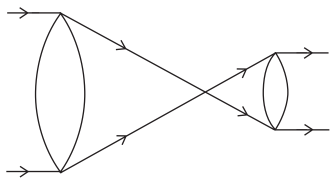
This equals d/D according to the problem so,

$$\Gamma = \frac{f_o}{f_e} = \frac{D}{d}$$

5.45 It is clear from the figure that a parallel beam of light, originally of intensity I_0 has, on emerging from the telescope, an intensity

$$I = I_0\left(\frac{f_o}{f_e}\right)^2$$

because it is concentrated over a section whose diameter is f_e/f_o of the diameter of the cross-section of the incident beam.



Thus, $$\eta = \left(\frac{f_o}{f_e}\right)^2$$

So, $$\Gamma = \frac{f_o}{f_e} = \sqrt{\eta}$$

Now, $$\Gamma = \frac{\tan\Psi'}{\tan\Psi} \cong \frac{\Psi'}{\Psi}$$

Hence, $\Psi = \Psi'/\sqrt{\eta} = 0.6'$, on substitution.

5.46 When a glass lens is immersed in water its focal length increases approximately four times. We check this as follows:

$$\frac{1}{f_a} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{f_w} = \left(\frac{n}{n_0} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$= \frac{(n/n_0) - 1}{n-1}\cdot\frac{1}{f_a} = \frac{n-n_0}{n_0(n-1)}\frac{1}{f_a}$$

Now, back to the problem, originally in air

$$\Gamma = \frac{f_o}{f_e} = 15 \quad \text{so,} \quad l = f_o + f_e = f_e(\Gamma + 1)$$

In water, $$f_e' = \frac{n_0(n-1)}{n-n_0}f_e$$

and the focal length of the replaced objective is given by the condition

$$f_o' + f_e' = l = (\Gamma + 1)f_e$$

or $$f_o' = (\Gamma + 1)f_e - f_e'$$

Hence, $$\Gamma' = \frac{f_o'}{f_e'} = (\Gamma + 1)\frac{n-n_0}{n_0(n-1)} - 1$$

Substitution gives $(n = 1.5, n_0 = 1.33)$, $\Gamma' = 3.09$.

5.47 If L is the luminance of the object, A is its area, s = distance of the object, then light falling on the object is

$$\frac{L\pi D^2}{4s^2}A$$

The area of the image formed by the telescope (assuming that the image coincides with the object) is $\Gamma^2 A$ and the area of the final image on the retina is

$$\left(\frac{f}{s}\right)^2 \Gamma^2 A$$

where f = focal length of the eye lens. Thus the illuminance of the image on the retina (when the object is observed through the telescope) is

$$\frac{L\pi D^2 A}{4s^2(f/s)^2\Gamma^2 A} = \frac{L\pi D^2}{4f^2\Gamma^2}$$

When the object is viewed directly, the illuminance is, similarly

$$\frac{L\pi d_0^2}{4f^2}$$

We want $$\frac{L\pi D^2}{4f^2\Gamma^2} \geq \frac{L\pi d_0^2}{4f^2}$$

So, $$\Gamma' \leq \frac{D}{d_0} = 20 \text{ (on substituting values)}$$

5.48 Obviously, $f_o = +1$ cm and $f_e = +5$ cm. Now, we know that magnification of a microscope is

$$\Gamma = \left(\frac{s_0'}{f_o} - 1\right)\frac{D}{f_e} \qquad \text{(for distinct vision)}$$

or $$50 = \left(\frac{s_0'}{1} - 1\right)\frac{25}{5} \quad \text{or} \quad s_0' = 11 \text{ cm}$$

Since distance between objective and ocular has increased by 2 cm, it will cause an increase in the tube length by 2 cm.

So, $$s_0'' = s_0' + 2 = 13 \text{ cm}$$

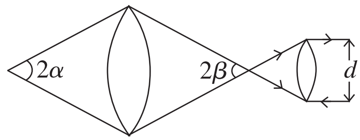
and hence, $$\Gamma' = \left(\frac{s_0''}{f_o} - 1\right)\frac{D}{f_e} = 60$$

5.49 It is implied in the problem that the final image of the object is at infinity (otherwise light coming out of the eyepiece will not have a definite diameter).

(a) We see that

$$s_0' 2\beta = |s_0| 2\alpha$$

or $$\beta = \frac{|s_0|}{s_0'}\alpha$$



Then, from the figure

$$d = 2f_e\beta = 2f_e\alpha \Big/ \frac{s_0'}{|s_0|}$$

But when the final image is at infinity, the magnification Γ in a microscope is given by

$$\Gamma = \frac{s_0'}{|s_0|}\cdot\frac{l}{f_e} \quad (l = \text{least distance of distinct vision})$$

So, $$d = \frac{2l\alpha}{T}$$

Since $d = d_0$ when $\Gamma = \Gamma_0$

$$d = \frac{2l\alpha}{d_0} = 15 \quad \text{(on substituting values)}$$

(b) If Γ is the magnification produced by the microscope, then the area of the image produced on the retina (when we observe an object through a microscope) is

$$\Gamma^2\left(\frac{f}{s}\right)^2 A$$

where s = distance of the image produced by the microscope from the eye lens, f = focal length of the eye lens and A = area of the object. If Φ = luminous flux reaching the objective from the object and $d \le d_0$ so that the entire flux is admitted into the eye), then illuminance of the final image on the retina is

$$\frac{\Phi}{\Gamma^2(f/s)^2 A}$$

But if $d \ge d_0^2$, then only a fraction $(d_0/d)^2$ of light is admitted into the eye and the illuminance becomes

$$\frac{\Phi}{A(f/s)^2\Gamma^2}\left(\frac{d_0}{d}\right)^2 = \frac{\Phi d_0^2}{A(f/s)^2(2l\alpha)^2}$$

independent of Γ. The condition for this is then

$$d \ge d_0 \quad \text{or} \quad \Gamma \le \Gamma_0 = 15$$

5.50 The primary and secondary focal lengths of a thick lens are given as

$$f = -(n/\Phi)\{1 - (d/n')\Phi_2\}$$

and $$f' = +(n''/\Phi)\{1 - (d/n')\Phi_1\}$$

where Φ is the lens power and n, n' and n'' are the refractive indices of first medium, lens material and the second medium beyond the lens, respectively. Φ_1 and Φ_2 are the powers of first and second spherical surface of the lens.

Here $n = 1$ for lens, $n' = n$ for air and $n'' = n_0$ for water.

So, $$\left.\begin{aligned} f &= -1/\Phi_1 \\ \text{and } f' &= +n_0/\Phi \end{aligned}\right\} \quad (\text{as } d \approx 0) \tag{1}$$

Now, power of a thin lens

$$\Phi = \Phi_1 + \Phi_2$$

where $$\Phi_1 = \frac{(n-1)}{R}$$

and $$\Phi_2 = \frac{(n_0 - n)}{-R}$$

So, $$\Phi = \frac{(2n - n_0 - 1)}{R} \tag{2}$$

From Eqs. (1) and (2), we get

$$f = \frac{-R}{(2n - n - 1)} = -11.2 \text{ cm}$$

and $$f' = \frac{n_0 R}{(2n - n_0 - 1)} = +14.9 \text{ cm}$$

Since the distance between the primary principal point and primary nodal point is given as

$$x = f'\left\{(n'' - n)/n''\right\}$$

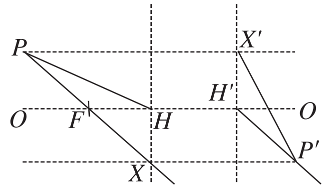
So, in this case

$$x = \frac{n_0}{\Phi} \times \frac{n_0 - 1}{n_0} = (n_0 - 1)/\Phi$$

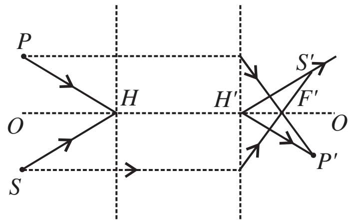
$$= \frac{n_0}{\Phi} - \frac{1}{\Phi} = f' + f = 3.7 \text{ cm}$$

5.51 See the answer sheet of the problem book.

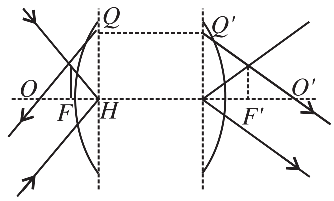
5.52 (a) Draw $P'X$ parallel to the axis OO' and let PF intersect it at X. That determines the principal point H. Since the medium on both sides of the system is the same, the principal point coincides with the nodal point. Draw a ray parallel to PH through P'. That determines H'. Draw a ray PS' parallel to the axis and join $P'X'$. That gives F'.



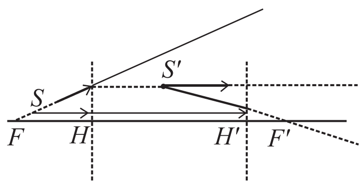
(b) We let H stand for the principal point (on the axis). Determine H' by drawing a ray $P'H'$ passing through P' and parallel to PH. One ray (conjugate to SH) can be obtained from this. To get the other ray one needs to know F or F'. This is easy because P and P' are known. Finally we get S'.



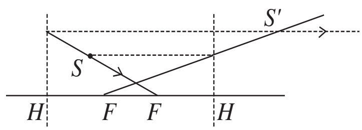
(c) From the incident ray we determine Q. A line parallel to OO' through Q determines Q and hence H'. H and H' are also the nodal points. A ray parallel to the incident ray through H will emerge parallel to itself through H'. That determines F'. Similarly, a ray parallel to the emergent ray through H determines F.



5.53 Here we do not assume that the media on the two sides of the system are the same.



(*a*)



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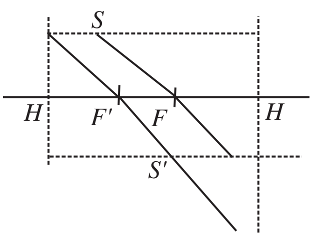

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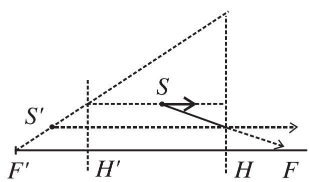


(*d*)

5.54 (a) Optical power of the system of combination of two lenses

$$\Phi = \Phi_1 + \Phi_2 - d\Phi_1\Phi_2$$

on substituting values, $\Phi = 4 \text{ D}$

or $$f = \frac{1}{\Phi} = 25 \text{ cm}$$

Now, the position of primary principal plane with respect to the vertex of converging lens is

$$X = \frac{d\Phi_2}{\Phi} = 10 \text{ cm}$$

Similarly, the distance of secondary principal plane with respect to the vertex of diverging lens is

$$X' = -\frac{d\Phi_1}{\Phi} = -10 \text{ cm} \quad \text{(i.e., 10 cm left to it)}$$

(b) The distance between the rear principal focal point F' and the vertex of converging lens is

$$l = d + \left(\frac{1}{\Phi}\right)(-d\Phi_1) = \frac{\Phi d}{\Phi} + \left(\frac{-d\Phi_1}{\Phi}\right)$$

and $$\frac{f}{l} = \left(\frac{1}{\Phi}\right)\Big/\frac{\Phi d}{\Phi} - \frac{d\Phi_1}{\Phi} \quad \left(\text{as } f = \frac{1}{\Phi}\right)$$

$$= \frac{1}{d\Phi} - d\Phi_1$$

$$= \frac{1}{d}(\Phi_1 + \Phi_2 - d\Phi_1\Phi_2) - d\Phi_1 = \frac{1}{d\Phi_2} - d^2\Phi_1\Phi_2$$

Now, if f/l is maximum for a certain value of d then l/f will be minimum for the same value of d. And for minimum l/f

$$d(l/f)/dd = \Phi_2 - 2d\Phi_1\Phi_2 = 0$$

or $$d = \Phi_2/2\Phi_1\Phi_2$$

or $$d = 1/2\Phi_1 = 5 \text{ cm}$$

So, the required maximum ratio of $f/l = 4/3$.

5.55 The optical power of first convex surface is

$$\Phi = \frac{P(n-1)}{R_1} = 5 \text{ D} \quad (\text{as } R_1 = 10 \text{ cm})$$

and the optical power of second concave surface is

$$\Phi_1 = \frac{(1-n)}{R_2} = -10 \text{ D}$$

So, the optical power of the system will be

$$\Phi = \Phi_1 + \Phi_2 - \frac{d}{n}\Phi_1\Phi_2 = -4 \text{ D}$$

Now, the distance of the primary principal plane from the vertex of convex surface is given as

$$x = \left(\frac{1}{\Phi}\right)\left(\frac{d}{n}\right)\Phi_2 \qquad (\text{here } n_1 = 1 \text{ and } n_2 = n)$$

$$= \frac{d\Phi_2}{\Phi n} = 5 \text{ cm}$$

and the distance of secondary principal plane from the vertex of second concave surface will be

$$x' = -\left(\frac{1}{\Phi}\right)\left(\frac{d}{n}\right)\Phi_1 = -\frac{d\Phi_1}{\Phi n} = 2.5 \text{ cm}$$

5.56 The optical power of the system of two thin lenses placed in air is given as

$$\Phi = \Phi_1 + \Phi_2 - d\Phi_1\Phi_2$$

or $$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2} \quad (\text{where } f \text{ is the equivalent focal length})$$

So, $$\frac{1}{f} = \frac{f_2 + f_1 - d}{f_1 f_2}$$

or $$f = \frac{f_1 f_2}{f_1 + f_2 - d}$$

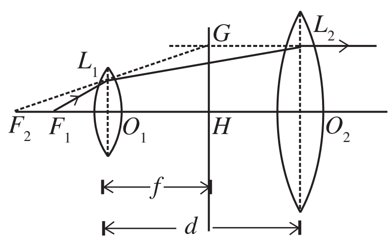
This equivalent focal length of the system of two lenses is measured from the primary principal plane. As is clear from the figure, the distance of the primary principal plane from the optical centre of the first is

$$O_1H = x = +(n/\Phi)(d/n')\Phi_1$$

$$= \frac{d\Phi_1}{\Phi} \quad (\text{as } n = n' = 1 \text{ for air})$$

$$= \frac{df}{f_1}$$

$$= \left(\frac{d}{f_1}\right)\left(\frac{f_1 f_2}{f_1 + f_2 - d}\right)$$

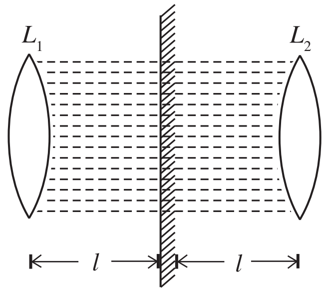


$$= \frac{df_2}{f_1 + f_2 - d}$$

If we place the equivalent lens at the primary principal plane of the lens system, it will provide the same transverse magnification as the system. So, the distance of equivalent lens from the vertex of the first lens is

$$x = \frac{df_2}{f_1 + f_2 - d}$$

5.57 The plane mirror forms the image of the lens and water, filled in the space between the two, behind the mirror, as shown in the figure. So, the whole optical system is equivalent to two similar lenses, separated by a distance $2l$ and thus, the power of this system



$$\Phi = \Phi_1 + \Phi_2 - \frac{d\Phi_1\Phi_2}{n_0}$$

where $\Phi_1 = \Phi_2 = \Phi_1'$ is the optical power of individual lens and n_0 is the R.I. of water.

Now, Φ' = optical power of first convex surface + optical power of second concave surface, so

$$\Phi' = \frac{(n-1)}{R} + \frac{n_0 - n}{R} \qquad \text{(since } n \text{ is the refractive index of glass)}$$

$$= \frac{(2n - n_0 - 1)}{R}$$

and so, the optical power of whole system will be

$$\Phi = 2\Phi' - \frac{2d\Phi'^2}{n_0} = 3.0 \text{ D (on substituting values)}$$

5.58 (a) A telescope in normal adjustment is a zero-power combination of lenses. Thus, we require

$$\Phi = 0 = \Phi_1 + \Phi_2 - \frac{d}{n}\Phi_1\Phi_2$$

But, Φ_1 = power of the convex surface $= \dfrac{n-1}{R_0 + \Delta R}$

Φ_2 = power of the concave surface $= -\dfrac{n-1}{R_0}$

Thus, $$0 = \frac{-(n-1)\Delta R}{R_0(R_0 + \Delta R)} + \frac{d}{n}\frac{(n-1)^2}{R_0(R_0 + \Delta R)}$$

So, $$d = \frac{n\Delta R}{n-1} = 4.5 \text{ cm} \quad \text{(on substituting values)}$$

(b) Here, $$\Phi = -1 = \frac{0.5}{0.1} - \frac{0.5}{0.075} + \frac{d}{1.5} \times \frac{0.5 \times 0.5}{0.1 \times 0.075}$$

$$= 5 - \frac{20}{3} + \frac{d \times 2}{3} \times \frac{5 \times 20}{3} = -\frac{5}{3} + \frac{200d}{9}$$

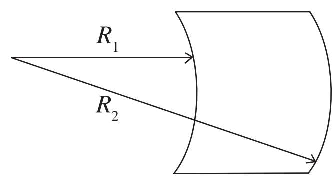
or $$\frac{200d}{9} = \frac{2}{3} \quad \text{or} \quad d = \left(\frac{3}{100}\right) \text{m} = 3 \text{ cm}$$

5.59 (a) The power of the lens is (as in the previous problem)

$$\Phi = \frac{n-1}{R} - \frac{n-1}{R} - \frac{d}{n}\left(\frac{n-1}{R}\right)\left(-\frac{n-1}{R}\right) = \frac{d(n-1)^2}{nR^2} > 0$$

The principal planes are located on the side of the convex surface at a distance d from each other, with the front principal plane being removed from the convex surface of the lens by a distance $R/(n-1)$.

(b) Here, $$\Phi = -\frac{n-1}{R_1} + \frac{n-1}{R_2} + \frac{R_2 - R_1}{n}\frac{(n-1)^2}{R_1 R_2}$$



$$= \frac{(n-1)(R_2 - R_1)}{R_2 R_1}\left[-1 + \frac{n-1}{n}\right]$$

$$= -\frac{n-1}{n}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) < 0$$

Both principal planes pass through the common centre of curvature of the surfaces of the lens.

5.60 Let the optical powers of the first and second surfaces of the ball of radius R_1 be Φ_1' and Φ_1'', respectively then

$$\Phi_1' = \frac{(n-1)}{R_1} \quad \text{and} \quad \Phi_1'' = (1-n)(-R_1) = \frac{(n-1)}{R_1}$$

This ball may be treated as a thick spherical lens of thickness $2R_1$. So, the optical power of this sphere is

$$\Phi = \Phi_1' - \frac{2R_1 \Phi_1' \Phi_1''}{n} = \frac{2(n-1)}{nR_1}$$

Similarly, the optical power of second ball

$$\Phi_2 = \frac{2(n-1)}{nR_2}$$

Let the distance between the centres of these balls be d. Then, the optical power of whole system

$$\begin{aligned}\Phi &= \Phi_1 + \Phi_2 - d\Phi_1\Phi_2 \\ &= \frac{2(n-1)}{nR_1} + \frac{2(n-1)}{nR_2} - \frac{4d(n-1)^2}{n^2R_1R_2} \\ &= \frac{2(n-1)}{nR_1R_2}\left[(R_1 + R_2) - \frac{2d(n-1)}{n}\right]\end{aligned}$$

Now, since this system serves as a telescope, the optical power of the system must be equal to zero. So,

$$(R_1 + R_2) = \frac{2d(n-1)}{n} \qquad \left(\text{as } \frac{2(n-1)}{nR_1R_2} \neq 0\right)$$

or
$$d = \frac{n(R_1 + R_2)}{2(n-1)} = 9 \text{ cm}$$

Since the diameter D of the objective is $2R_1$ and that of the eyepiece is $d = 2R_2$, so, the magnification

$$\Gamma = \frac{D}{d} = \frac{2R_1}{2R_2} = \frac{R_1}{R_2} = 5$$

5.61 Optical powers of the two surfaces of the lens are

$$\Phi_1 = \frac{(n-1)}{R} \quad \text{and} \quad \Phi_2 = \frac{(1-n)}{-R} = \frac{n-1}{R}$$

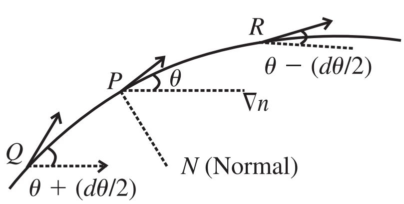
So, the power of the lens of thickness d will be

$$\Phi' = \Phi_1 + \Phi_2 - \frac{d\Phi_1\Phi_2}{n} = \frac{n-1}{R} + \frac{n-1}{R} - \frac{d(n-1)^2/R^2}{n^2} = \frac{n^2-1}{nR}$$

and optical power of the combination of these two thick lenses will be

$$\Phi = \Phi' + \Phi' = 2\Phi' = \frac{2(n^2-1)}{nR} = 37 \text{ D}$$

5.62 Consider a ray QPR in a medium of gradually varying refractive index n. At P, the gradient of n is a vector with the given direction which is nearly the same at neighboring points Q and R. The arc length QR is ds. We apply Snell's formula $n \sin\theta$ = constant, where θ is to be measured from the direction ∇n. The refractive indices at Q, R whose mid-point is P are



$$n \pm \frac{1}{2}|\nabla n|\, d\theta \cos\theta$$

So, $$\left(n - \frac{1}{2}|\nabla n|\, d\theta \cos\theta\right)\left(\sin\theta + \frac{1}{2}\cos\theta\, d\theta\right)$$

$$= \left(n + \frac{1}{2}|\nabla n|\, d\theta \cos\theta\right)\left(\sin\theta - \frac{1}{2}\cos\theta\, d\theta\right)$$

or $$n \cos\theta\, d\theta = |\nabla n|\, ds \cos\theta \sin\theta$$

(We have used $\sin\left(\theta \pm \frac{1}{2} d\theta\right) = \sin\theta \pm \frac{1}{2}\cos\theta\, d\theta$ here.)

Now, using the definition of the radius of curvature

$$\frac{1}{\rho} = \frac{d\theta}{ds}$$

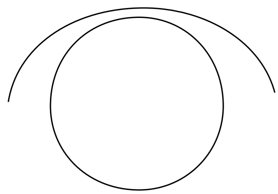
we get, $$\frac{1}{\rho} = \frac{1}{n}|\nabla n| \sin\theta$$

The quantity $|\nabla n|$ sin θ can be called $\delta n/\delta N$, i.e., the derivative of n along the normal N to the ray. Then,

$$\frac{1}{\rho} = \frac{\delta}{\delta N} \ln n$$

5.63 From the previous problem

$$\frac{1}{\rho} = \frac{1}{n}\hat{p}\cdot\nabla n \approx \hat{p}\cdot\nabla n \approx |\nabla n| = 3\times10^{-8}\ \text{m}^{-1}$$



(since $\hat{p} \parallel \nabla n$ both being vertical).

So, $$\rho = 3.3 \times 10^7 \text{ m}$$

For the ray of light to propagate all the way round the earth, we must have

$$\rho = R = 6400 \text{ km} = 6.4\times10^6 \text{ m}$$

Thus, $$|\nabla n| = 1.6\times10^{-7}\ \text{m}^{-1}$$

5.2 Interference of Light

5.64 (a) In this case the net vibration is given by

$$x = a_1 \cos\omega t + a_2 \cos(\omega t + \delta)$$

where δ is the phase difference between the two vibrations which varies rapidly and randomly in the interval (0, 2π). (This is what is meant by incoherence.)

Then, $$x = (a_1 + a_2 \cos\delta)\cos\omega t + a_2 \sin\delta \sin\omega t$$

The total energy will be taken to be proportional to the time average of the square of the displacement.

Thus, $$E = \langle (a_1 + a_2\cos\delta)^2 + a_2^2 \sin^2\delta \rangle = a_1^2 + a_2^2$$

as $\langle \cos\delta \rangle = 0$ and we have put $\langle \cos^2 \omega t \rangle = \langle \sin^2 \omega t \rangle = 1/2$ and this has been absorbed in the overall constant of proportionality. In the same units the energies of the two oscillations are a_1^2 and a_2^2, respectively, so the proposition is proved.

(b) Here $$\mathbf{r} = a_1 \cos\omega t\,\mathbf{i} + a_2 \cos(\omega t + \delta)\mathbf{j}$$

and the mean square displacement is

$$\alpha a_1^2 + a_2^2$$

If δ is fixed but arbitrary, then as in (a), we see that $E = E_1 + E_2$.

5.65 It is easier to do it analytically. Given that

$$\xi_1 = a\cos\omega t,\ \xi_2 = 2a\sin\omega t$$

and $$\xi_3 = \frac{3}{2}a\left(\cos\frac{\pi}{3}\cos\omega t - \sin\frac{\pi}{3}\sin\omega t\right)$$

Resultant vibration is

$$\xi = \frac{7a}{4}\cos\omega t + a\left(2 - \frac{3\sqrt{3}}{4}\right)\sin\omega t$$

This has an amplitude

$$\frac{a}{4}\sqrt{49 + (8 - 3\sqrt{3})^2} = 1.89a$$

5.66 We use the method of complex amplitudes. Then the amplitudes are

$$A_1 = a,\ A_2 = ae^{i\varphi}, \ldots, A_N = ae^{i(N-1)\varphi}$$

and the resultant complex amplitude is

$$A = A_1 + A_2 + \cdots + A_N = a(1 + e^{i\varphi} + e^{2i\varphi} + \cdots + e^{i(N-1)\varphi})$$

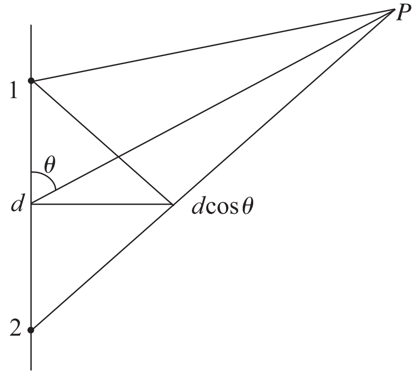
$$= a\frac{1 - e^{iN\varphi}}{1 - e^{i\varphi}}$$

The corresponding ordinary amplitude is

$$|A| = a\left|\frac{1 - e^{iN\varphi}}{1 - e^{i\varphi}}\right| = a\left[\frac{1 - e^{iN\varphi}}{1 - e^{i\varphi}} \times \frac{1 - e^{-iN\varphi}}{1 - e^{-i\varphi}}\right]^{1/2}$$

$$= a\left[\frac{2 - 2\cos N\varphi}{2 - 2\cos\varphi}\right]^{1/2} = a\frac{\sin N\varphi/2}{\sin\varphi/2}$$

5.67 (a) With dipole moment perpendicular to plane, there is no variation with θ of individual radiation amplitude. Then the intensity variation is due to interference only. In the direction given by angle θ, the phase difference is



$$\frac{2\pi}{\lambda}(d\cos\theta)+\varphi=2k\pi \quad \text{(for maxima)}$$

Thus, $$d\cos\theta=\left(k-\frac{\varphi}{2\pi}\right)\lambda$$

Here $k=0,\pm1,\pm2,\ldots$.

We have added φ to $2\pi/\lambda\ d\cos\theta$ because the extra path that the wave from 2 has to travel in going to P (as compared to 1) makes it lag more than it already has (due to φ).

(b) Maximum for $\theta=\pi$ gives

$$-d=\left(k-\frac{\varphi}{2\pi}\right)\lambda$$

Minimum for $\theta=0$ gives

$$d=\left(k'-\frac{\varphi}{2\pi}+\frac{1}{2}\right)\lambda$$

Adding we get $$\left(k+k'-\frac{\varphi}{\pi}+\frac{1}{2}\right)\lambda=0$$

This can be true only if $k'=-k,\ \varphi=\pi/2$, since $0<\varphi<\pi$.

Then, $$-d=\left(k-\frac{1}{4}\right)\lambda$$

Here $k=0,-1,-2,-3,\ldots$.

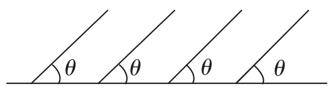
(Otherwise R.H.S. will become $+ve$.)

Putting $k=-\bar{k},\ \bar{k}=0,+1,+2,+3,\ldots$, we get

$$d=\left(\bar{k}+\frac{1}{4}\right)\lambda$$

5.68 If $\Delta\varphi$ is the phase difference between neighboring radiations, then for a maximum in the direction θ we must have

$$\frac{2\pi}{\lambda}d\cos\theta+\Delta\varphi=2\pi k$$



For scanning $$\theta=\omega t+\beta$$

Thus, $$\frac{d}{\lambda}\cos(\omega t + \beta) + \frac{\Delta\varphi}{2\pi} = k$$

or $$\Delta\varphi = 2\pi\left[k - \frac{d}{\lambda}\cos(\omega t + \beta)\right]$$

To get the answer same as the book, put $\beta = \alpha - (\pi/2)$.

5.69 From the general formula $$\Delta x = \frac{l\lambda}{d}$$

we find that $$\frac{\Delta x}{\eta} = \frac{l\lambda}{d + 2\Delta h}$$

Since d increases to $d + 2\Delta h$ when the source is moved away from the mirror plane by Δh. Thus,

$$\eta d = d + 2\Delta h \quad \text{or} \quad d = \frac{2\Delta h}{(\eta - 1)}$$

So, $$\lambda = \frac{2\Delta h\Delta x}{(\eta - 1)l} = 0.6\ \mu\text{m}$$

5.70 We can think of the two coherent plane waves as emitted from two coherent point sources very far away. Then

$$\Delta x = \frac{l\lambda}{d} = \frac{\lambda}{d/l}$$

But $$\frac{d}{l} = \Psi \quad (\text{if } \Psi \ll 1)$$

so, $$\Delta x = \frac{\lambda}{\Psi}$$

5.71 (a) Here, $$S'S'' = d = 2r\alpha$$

Then, $$\Delta x = \frac{(b + r)\lambda}{2\alpha r}$$

Putting $b = 1.3$ m, $r = 0.1$ m, $\lambda = 0.55\ \mu$m, $\alpha = 12' = \dfrac{1}{5 \times 57}$ radian, we get

$$\Delta x = 1.1 \text{ mm}$$

Number of possible maxima

$$\frac{2b\alpha}{\Delta x} + 1 \approx 8.3 + 1 = 9$$

(Here $2b\alpha$ is the length of the spot on the screen which gets light after reflection from both the mirrors. We add 1 above to take account of the fact that in a distance Δx there are two maxima.)

(b) When the slit moves by δl along the arc of radius r the incident ray on the mirror rotates by $\delta l/r$; this is also the rotation of the reflected ray. There is then a shift of the fringe of magnitude. Thus,

$$b\frac{\delta l}{r} = 13 \text{ mm} \quad \text{(on substituting values)}$$

(c) If the width of the slit is δ then we can imagine the slit to consist of two narrow slits with separation δ. The fringe pattern due to the wide slit is the superposition of the pattern due to these two narrow slits. The full pattern will not be sharp at all if the pattern due to the two narrow slits are $(1/2)\Delta x$ apart because then the maxima due to one will fill the minima due to the other. Thus,

$$\frac{b\delta_{\max}}{r} = \frac{1}{2}\Delta x = \frac{(b+r)\lambda}{4r\alpha}$$

or
$$\delta_{\max} = \left(1 + \frac{r}{b}\right)\frac{\lambda}{4\alpha} = 42.4 \text{ }\mu\text{m}$$

5.72 To get this case we must let $r \to \infty$ in the formula for Δx of the last example.

So,
$$\Delta x = \frac{(b+r)\lambda}{2\alpha r} \to \frac{\lambda}{2\alpha}$$

(A plane wave is like light emitted from a point source at ∞.)

Then,
$$\lambda = 2\alpha\Delta x = 0.64 \text{ }\mu\text{m}$$

5.73 (a) We show the upper half on the lens. The emergent light is at an angle $a/2f$ from the axis. Thus, the divergence angle of the two incident light beams is

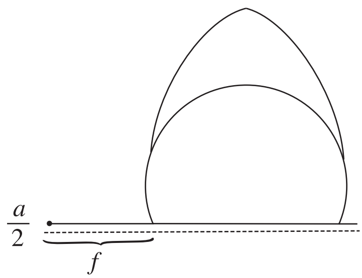
$$\Psi = \frac{a}{f}$$

When they interfere, the fringes produced have a width

$$\Delta x = \frac{\lambda}{\Psi} = \frac{f\lambda}{a} = 0.15 \text{ mm}$$

The patch on the screen illuminated by both rays of light has width $b\Psi$ and this contains

$$\frac{b\Psi}{\Delta x} = \frac{ba^2}{f^2\lambda} = 13 \text{ fringes}$$



(if we ignore 1 in comparison with $b\Psi/\Delta x$ (as in solution of Problem 5.71a)

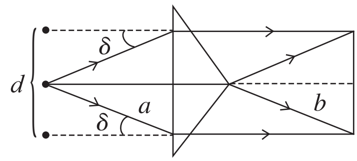
(b) We follow the logic of Problem 5.71c. From one edge of the slit to the other edge the distance is of magnitude δ (i.e., $a/2$ to $a/2 + \delta$). If we imagine the edge to shift by this distance, the angle $\Psi/2$ will increase by $\Delta\Psi/2 = \delta/2f$ and the light will shift $\pm\, b\cdot\delta/2f$. The fringe pattern will therefore shift by $\delta \cdot b/f$.

Equating this to $\Delta x/2 = f\lambda/2a$, we get

$$\delta_{max} = \frac{f^2\lambda}{2ab} = 37.5\ \mu m$$

5.74 We know

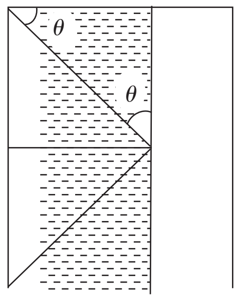
$$\Delta x = \frac{l\lambda}{d}$$



$$l = a + b$$
$$d = 2(n-1)\theta a$$
$$\delta = (n-1)\theta$$
$$d = 2\delta \cdot a$$
$$n = \text{R.I. of glass}$$

Thus,
$$\lambda = \frac{2(n-1)\theta a\,\Delta x}{a+b} = 0.64\ \mu m$$

5.75 It will be assumed that the space between the biprism and the glass plate filled with benzene constitutes complementary prisms as shown in the figure. Then the two prisms being oppositely placed, the net deviation produced by them is



$$\delta = (n-1)\theta - (n'-1)\theta = (n-n')\theta$$

Hence, as in the previous problem

$$d = 2a\delta = 2a\theta(n-n')$$

So,
$$\Delta x = \frac{(a+b)\lambda}{2a\theta(n-n')}$$

For plane incident wave, we let $a \to \infty$

So,
$$\Delta x = \frac{\lambda}{2\theta(n-n')} = 0.2\ \text{mm}$$

5.76 Extra phase difference introduced by the glass plate is

$$\frac{2\pi}{\lambda} = (n-1)h$$

This will cause a shift equal to $(n-1)h/\lambda$ fringe width,

i.e., by
$$(n-1)\frac{h}{\lambda} \times \frac{l\lambda}{d} = \frac{(n-1)hl}{d} = 2\ \text{mm}$$

The fringes move down if the lower slit is covered by the plate to compensate for the extra phase shift introduced by the plate.

5.77 No. of fringes shifted $$N = (n' - n)\frac{1}{\lambda} - N$$

So, $$n' = n + \frac{N\lambda}{l} = 1.000377$$

5.78 (a) Suppose the vector **E**, **E**$'$, **E**$''$ correspond to the incident, reflected and the transmitted waves respectively. Due to the continuity of the tangential component of the electric field across the interface, it follows that

$$E_\tau + E_\tau{}' = E_\tau{}'' \qquad (1)$$

where the subscript τ means tangential.

The energy flux density is

$$\mathbf{E} \times \mathbf{H} = \mathbf{S}$$

Since $$H\sqrt{\mu\mu_0} = E\sqrt{\varepsilon\varepsilon_0}$$

and $$H = E\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\frac{\varepsilon}{\mu}} = n\sqrt{\frac{\varepsilon_0}{\mu_0}}E$$

Now $S \sim nE^2$ and since the light is incident normally

$$n_1 E_\tau^2 = n_1 E_\tau^2 + n_2 E_\tau''^2 \qquad (2)$$

or $$n_1(E_\tau^2 - E_\tau'^2) = n_2 E_\tau''^2$$

So, $$n_1(E_\tau - E_\tau') = n_2 E_\tau'' \qquad (3)$$

or $$E_\tau'' = \frac{2n_1}{n_1 + n_2}E_\tau$$

Since E_τ'' and E_τ have the same sign, there is no phase change involved in this case.

(b) From Eqs. (1) and (3)

$$(n_2 + n_1)E_\tau{}' + (n_2 - 1)E_\tau = 0$$

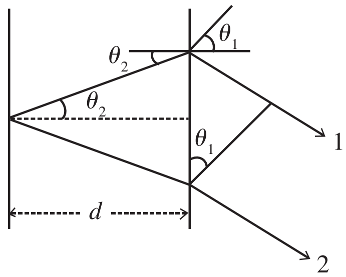
or $$E_\tau{}' = \frac{n_1 - n_2}{n_1 + n_2}E_\tau$$

If $n_2 > n_1$, then E_τ' and E_τ have opposite signs. Thus, the reflected wave has an abrupt change of phase by π if $n_2 < n_1$, i.e., on reflection from the interface between two media when light is incident from the rarer to denser medium.

5.79 Path difference between rays 1 and 2 is

$$2nd\sec\theta_2 - 2d\tan\theta_2\sin\theta_1$$

$$= 2d\frac{n - (\sin^2\theta_1/n)}{\sqrt{1 - (\sin^2\theta_1/n^2)}} = 2d\sqrt{n^2 - \sin^2\theta_1}$$



For bright fringes this must equal $(k + 1/2)\lambda$, where 1/2 comes from the phase change of π for ray 1.

Here $k = 0, 1, 2, \ldots$.

Thus, $4d\sqrt{n^2 - \sin^2\theta_1} = (2k + 1)\lambda$

or $$d = \frac{\lambda(1 + 2k)}{4\sqrt{n^2 - \sin^2\theta_1}} = 0.14(1 + 2k)\ \mu\text{m}$$

5.80 Given that

$$2d\sqrt{n^2 - \left(\frac{1}{4}\right)} = \left(k + \frac{1}{2}\right) \times 0.64\ \mu\text{m} \quad \text{(bright fringe)}$$

$$2d\sqrt{n^2 - \left(\frac{1}{4}\right)} = k' \times 0.40\ \mu\text{m} \quad \text{(dark fringe)}$$

(where k, k' are integers.)

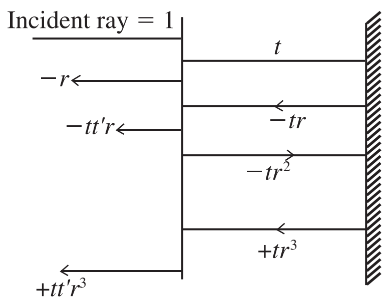
Then, $$64\left(k + \frac{1}{2}\right) = 40k' \quad \text{or} \quad 4(2k + 1) = 5k'$$

This means, for the smallest integer solutions, $k = 2$ and $k' = 4$

Hence, $$d = \frac{4 \times 0.40}{2\sqrt{n^2 - 1/4}} = 0.65\ \mu\text{m}$$

5.81 When the glass surface is coated with a material of R.I. $n' = \sqrt{n}$ (n = R.I. of glass) of appropriate thickness, reflection is zero because of interference between various multiplied reflected waves, as shown in the figure.

Let a wave of unit amplitude be normally incident from the left. The reflected amplitude is $-r$, where



$$r = \frac{\sqrt{n} - 1}{\sqrt{n} + 1}$$

Its phase is $-ve$ so we write the reflected wave as $-r$. The transmitted wave has amplitude t, where

$$t = \frac{2}{1 + \sqrt{n}}$$

This wave is reflected at the second face and has amplitude $-tr$, because

$$\frac{n - \sqrt{n}}{n + \sqrt{n}} = \frac{\sqrt{n} - 1}{\sqrt{n} + 1}$$

The emergent wave has amplitude $-tt'r$.

It is proved below that $-tt' = 1 - r^2$. There is also a reflected part of amplitude $trr' = -tr^2$, where r' is the reflection coefficient for a ray incident from the coating towards air. After reflection from the second face wave of amplitude

$$+tt'r^3 = +(1-r^2)r^3$$

emerges. Let δ be the phase of the wave after traversing the coating both ways.

Then, the complete reflected wave is

$$-r-(1-r^2)re^{i\delta}+(1-r^2)r^3e^{2i\delta}-(1-r^2)r^5e^{3i\delta}\ldots$$

$$=-r-(1-r^2)re^{i\delta}\frac{1}{1+r^2e^{i\delta}}$$

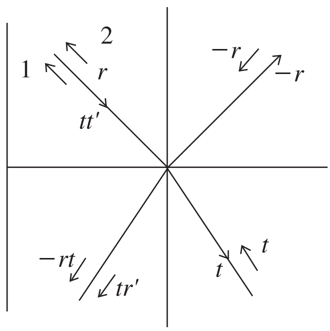
$$=-r[1+r^2e^{i\delta}+(1-r^2)e^{i\delta}]\frac{1}{1+r^2e^{i\delta}}$$

$$=-r\frac{1+e^{i\delta}}{1+r^2e^{i\delta}}$$

This vanishes if $\delta = (2k+1)\pi$. But

$$\delta=\frac{2\pi}{\lambda}2\sqrt{n}d$$

so,
$$d=\frac{\lambda}{4\sqrt{n}}(2k+1)$$



We now deduce that $tt' = 1 - r^2$ and $r' = +r$. This follows from the principle of reversibility of light path as shown in the figure.

$$tt'+r^2=1$$

$$-rt+r't=0$$

Therefore,
$$tt'=1-r^2$$

$$r'=+r$$

($-r$ is the reflection ratio for the wave entering a denser medium.)

5.82 We have the condition for maxima

$$2d\sqrt{n^2-\sin^2\theta_1}=\left(k+\frac{1}{2}\right)\lambda$$

This must hold for angle $\theta \pm \delta\theta/2$ with successive values of k. Thus,

$$2d\sqrt{n^2-\sin^2\left(\theta+\frac{\delta\theta}{2}\right)}=\left(k-\frac{1}{2}\right)\lambda$$

and $$2d\sqrt{n^2 - \sin^2\left(\theta - \frac{\delta\theta}{2}\right)} = \left(k + \frac{1}{2}\right)\lambda$$

Therefore, $$\lambda = 2d\left\{\sqrt{n^2 - \sin^2\theta + \delta\theta\sin\theta\cos\theta} - \sqrt{n^2 - \sin^2\theta - \delta\theta\sin\theta\cos\theta}\right\}$$

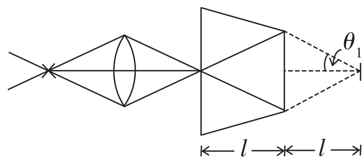
$$= 2d\frac{\delta\theta\sin\theta\cos\theta}{\sqrt{n^2 - \sin^2\theta}}$$

or $$d = \frac{\sqrt{n^2 - \sin^2\theta\lambda}}{\sin 2\theta\,\delta\theta} = 15.2\ \mu\text{m}$$

5.83 For small angles θ, we write for dark fringes

$$2d\sqrt{n^2 - \sin^2\theta} = 2d\left(n - \frac{\sin^2\theta}{2n}\right) = (k)\lambda$$

For the first dark fringe $\theta \cong 0$ and $2dn = (k_0)\lambda$.
For the i^{th} dark fringe



$$2d\left(n - \frac{\sin^2\theta_i}{2n}\right) = (k_0 - i + 1)\lambda$$

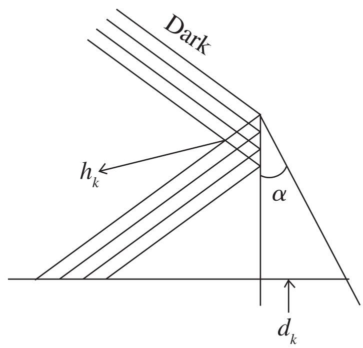
or $$\sin^2\theta_i = \frac{n\lambda}{d}(i-1) = \frac{r_i^2}{4l^2}$$

Then $$\frac{n\lambda}{d}(i-k) = \frac{r_i^2 - r_k^2}{4l^2}$$

so, $$\lambda = \frac{d(r_i^2 - r_k^2)}{4l^2 n(i-k)}$$

5.84 We have the usual equation for maxima

$$2h_k\alpha\sqrt{n^2 - \sin^2\theta_1} = \left(k + \frac{1}{2}\right)\lambda$$



Here, h_k = distance of the fringe from top and $h_k\,\alpha = d_k$ = thickness of the film.

Thus, on the screen placed at right angles to the reflected light

$$\Delta x = (h_k - h_{k-1})\cos\theta_1$$

$$= \frac{\lambda\cos\theta_1}{2\alpha\sqrt{n^2 - \sin^2\theta_1}}$$

5.85 (a) For normal incidence, we have, using the above formula,

$$\Delta x = \frac{\lambda}{2n\alpha}$$

so, $$\alpha = \frac{\lambda}{2n\Delta x} = 3 \text{ (on substituting values)}$$

(b) In a distance l on the wedge there are $N = l/\Delta x$ fringes. If the fringes disappear there, it must be due to the fact that the maxima due to the component of wavelength λ coincides with the minima due to the component of wavelength $\lambda + \Delta\lambda$. Thus,

$$N\lambda = \left(N - \frac{1}{2}\right)(\lambda + \Delta\lambda) \quad \text{or} \quad \Delta\lambda = \frac{\lambda}{2N}$$

so, $$\frac{\Delta\lambda}{\lambda} = \frac{1}{2N} = \frac{\Delta x}{2l} = \frac{0.21}{30} = 0.007$$

(The answer given in the answer sheet is off by a factor 2.)

5.86 We have $$r^2 = \frac{1}{2}k\lambda R$$

So, for k differing by 1 ($\Delta k = 1$), we have

$$2r\Delta r = \frac{1}{2}\Delta k\lambda R = \frac{1}{2}\lambda R$$

or $$\Delta r = \frac{\lambda R}{4r}$$

5.87 The path traversed in air film of the wave constituting the k^{th} ring is

$$\frac{r^2}{R} = \frac{1}{2}k\lambda$$

When the lens is moved a distance Δh the ring radius changes to r' and the path length becomes

$$\frac{r'^2}{R} + 2\Delta h = \frac{1}{2}k\lambda$$

Thus, $$r' = \sqrt{r^2 - 2R\Delta h} = 1.5 \text{ mm}$$

5.88 In this case, the path difference is $r^2 - r_0^2/R$ for $r > r_0$ and zero for $r \le r_0$. This must equal $(k - 1/2)\lambda$ (where $k = 6$ for the sixth bright ring).

Thus, $$r = \sqrt{r_0^2 + \left(k - \frac{1}{2}\right)\lambda R} = 3.8 \text{ mm}$$

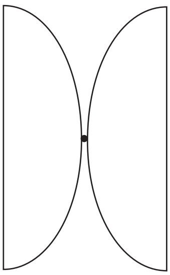
5.89 From the formula for Newton's rings, we derive for dark rings

$$\frac{d_1^2}{4} = k_1 R\lambda \quad \text{and} \quad \frac{d_2^2}{4} = k_2 R\lambda$$

So, $$\frac{d_2^2 - d_1^2}{4(k_2 - k_1)R} = \lambda = 0.5\ \mu\text{m (on substituting values)}$$

5.90 Path difference between waves reflected by the two convex surface is

$$r^2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$



Taking account of the phase change at the 2nd surface we write the condition of bright rings as

$$r^2\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{2k+1}{2}\lambda$$

In this case, $k = 4$ for the fifth bright ring.

Thus, $$\frac{1}{R_1} + \frac{1}{R_2} = \frac{9}{2}\lambda, \quad \frac{4}{d^2} = \frac{18\lambda}{d^2}$$

Now $$\frac{1}{f_1} = (n-1)\frac{1}{R_1}, \quad \frac{1}{f_2} = (n-1)\frac{1}{R_2}$$

So, $$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} = (n-1)\frac{18\lambda}{d^2} = \Phi = 2.40\,\text{D}$$

(Here n = R.I. of glass = 1.5.)

5.91 (a) Here $$\Phi = (n-1)\left(\frac{2}{R_1} - \frac{2}{R_2}\right)$$

so, $$\frac{1}{R_1} - \frac{1}{R_2} = \frac{\Phi}{2(n-1)}$$

As in the previous problem, for the dark rings, we have

$$r_k^2\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{\Phi}{2(n-1)}r_k^2 = k\lambda$$

As $k = 0$ is a dark spot; excluding it, we take $k = 10$ here.

Then, $$r = \sqrt{\frac{20\lambda(n-1)}{\Phi}} = 3.49\ \text{mm}$$

(b) Path difference in water film will be

$$n_0\bar{r}^2 - \left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

(where $\bar{r}$ = new radius of the ring.) Thus,

$$n_0\bar{r}^2 = r^2$$

or $$\bar{r} = \frac{r}{\sqrt{n_0}} = 3.03 \text{ mm}$$

(Here n_0 = R.I. of water = 1.33.)

5.92 The conditions for minima is

$$\frac{r^2}{R}n_2 = \left(k + \frac{1}{2}\right)\lambda$$

(Phase changes occur at both surfaces on reflection, hence, minima when path difference is half integer multiple of λ.)

In this case $k = 4$ for the fifth dark ring, counting from $k = 0$ for the first dark ring.

Thus, $$r = \sqrt{\frac{(2k-1)\lambda R}{2n_2}}, \quad k = 5$$

$$r = 1.17 \text{ mm (on substituting values)}$$

5.93 Sharpness of the fringe pattern is the worst when the maxima and minima intermingle

So, $$n_1\lambda_1 = \left(n_1 - \frac{1}{2}\right)\lambda_2$$

Using $\lambda_1 = \lambda, \lambda_2 = \lambda + \Delta\lambda$, we get

$$n_1\Delta\lambda = \frac{\lambda}{2}$$

or $$n_1 = \frac{\lambda}{2\Delta\lambda} = \frac{\lambda_1}{2(\lambda_2 - \lambda_1)} = 140$$

5.94 Interference pattern vanishes when the maxima due to one wavelength mingle with the minima due to the other. Thus,

$$2\Delta h = k\lambda_2 = (k+1)\lambda_1$$

(where Δh = displacement of the mirror between the sharpest patterns of rings).

Thus, $$k(\lambda_2 - \lambda_1) = \lambda_1$$

or $$k = \frac{\lambda_1}{\lambda_2 - \lambda_1}$$

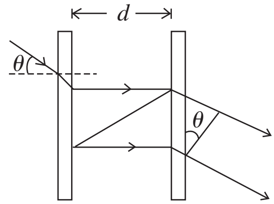
So, $$\Delta h = \frac{\lambda_1\lambda_2}{2(\lambda_2 - \lambda_1)} \cong \frac{\lambda^2}{2\Delta\lambda} \cong 0.29 \text{ mm}$$

5.95 (a) The path difference between rays 1 and 2 can be seen to be

$$\Delta = 2d\sec\theta - 2d\tan\theta\sin\theta$$
$$= 2d\cos\theta = k\lambda \text{ (for maxima)}$$

(Here k = half-integer.)

The order of interference decreases as θ increases, i.e., as the radius of the rings increases.



(b) Differentiating $$2d\cos\theta = k\lambda$$
$$-2d\sin\theta\,\delta\theta = \delta k\lambda$$
or $$2d\sin\theta\,\delta\theta = \lambda$$

(On putting $\delta k = -1$.)

Thus, $$\delta\theta = \frac{\lambda}{2d\sin\theta}$$

$\delta\theta$ decreases as θ increases.

5.96 (a) We have $$k_{max} = \frac{2d}{\lambda} \quad (\text{for } \theta = 0)$$

(b) We must have

$$2d\cos\theta = k\lambda = (k-1)(\lambda + \Delta\lambda)$$

Thus, $$\frac{1}{k} \cong \frac{\lambda}{2d} \quad \text{and} \quad \Delta\lambda = \frac{\lambda}{k} = \frac{\lambda^2}{2d} = 5 \text{ pm (on substituting values)}$$

5.3 Diffraction of Light

5.97 The radius of the periphery of the N^{th} Fresnel zone is

$$r_N = \sqrt{Nb\lambda}$$

Then, by conservation of energy

$$I_0\pi(\sqrt{Nb\lambda})^2 = \int_0^\infty 2\pi r\,dr\,I(r)$$

(Here r is the distance from the point P.)

Thus, $$I_0 = \frac{2}{Nb\lambda}\int_0^\infty r\,dr\,I(r)$$

5.98 By definition

$$r_k^2 = k\frac{ab\lambda}{a+b} \quad \text{(for the periphery of the } k^{\text{th}} \text{ zone)}$$

Then, $$ar_k^2 + br_k^2 = kab\lambda$$

So, $$b = \frac{ar_k^2}{ka\lambda - r_k^2} = \frac{ar^2}{3a\lambda - r^2} = 2\text{ m} \quad \text{(on substituting values)}$$

(It is given that $r = r_k$ for $k = 3$.)

5.99 Suppose maximum intensity is obtained when the aperture contains k zones. Then a minimum will be obtained for $k + 1$ zones. Another maximum will be obtained for $k + 2$ zones. Hence,

$$r_1^2 = k\lambda\frac{ab}{a+b}$$

and $$r_2^2 = (k+2)\lambda\frac{ab}{a+b}$$

Thus, $$\lambda = \frac{a+b}{2ab}(r_2^2 - r_1^2) = 0.598\ \mu\text{m} \quad \text{(on substituting values)}$$

5.100 (a) When the aperture is equal to the first Fresnel zone: The amplitude is A_1 and should be compared with the amplitude $A/2$ when the aperture is very wide. If I_0 is the intensity in the second case, the intensity in the first case will be $4I_0$.

When the aperture is equal to the internal half of the first zone: Suppose A_{in} and A_{out} are the amplitudes due to the two halves of the first Fresnel zone. Clearly, A_{in} and A_{out} differ in phase by $\pi/2$ because only half the Fresnel zone is involved. Also, in magnitude

$$|A_{\text{in}}| = |A_{\text{out}}|$$

Then, $$A_1^2 = 2|A_{\text{in}}|^2$$

so, $$|A_{\text{in}}|^2 = \frac{A_1^2}{2}$$

Hence, following the argument of the first case, $I_{\text{in}} = 2I_0$.

(b) The aperture was made equal to the first Fresnel zone and then half of it was closed along a diameter. In this case the amplitude of vibration is $A_1/2$. Thus, $I = I_0$.

5.101 (a) Suppose the disk does not obstruct light at all. Then,

$$A_{\text{disk}} + A_{\text{remainder}} = \frac{1}{2}A_{\text{disk}}$$

(because the disk covers the first Fresnel zone only).

So, $$A_{\text{remainder}} = \frac{1}{2} A_{\text{disk}}$$

Hence, the amplitude when half of the disk is removed along a diameter is

$$\frac{1}{2} A_{\text{disk}} + A_{\text{remainder}}$$
$$= \frac{1}{2} A_{\text{disk}} - \frac{1}{2} A_{\text{disk}} = 0$$

Hence, $I = 0$.

(b) In this case

$$A = \frac{1}{2} A_{\text{external}} + A_{\text{remainder}}$$
$$= \frac{1}{2} A_{\text{external}} - \frac{1}{2} A_{\text{disk}}$$

We write $$A_{\text{disk}} = A_{\text{in}} + iA_{\text{out}}$$

where A_{in} (A_{out}) stands for A_{internal} (A_{external}). The factor i takes account of the $\pi/2$ phase difference between two halves of the first Fresnel zone.

Thus, $$A = -\frac{1}{2} A_{\text{in}} \quad \text{and} \quad I = \frac{1}{4} A_{\text{in}}^2$$

On the other hand $$I_0 = \frac{1}{4}(A_{\text{in}}^2 + A_{\text{out}}^2) = \frac{1}{2} A_{\text{in}}^2$$

So, $$I = \frac{1}{2} I_0$$

5.102 When the screen is fully transparent, the amplitude of vibrations is $(1/2)A_1$ with intensity $I_0 = 1/4\, A_1^2$.

(a) In case of screen (1)

$$A = \frac{3}{4}\left(\frac{1}{2} A_1\right) \quad \text{and} \quad I = \frac{9}{16} I_0$$

In case of screen (2), half of the plane is blocked out so

$$A = \frac{1}{2}\left(\frac{1}{2} A_1\right) \quad \text{and} \quad I = \frac{1}{4} I_0$$

In case screen (3)

$$A = \frac{1}{4}\left(\frac{A_1}{2}\right) \quad \text{and} \quad I = \frac{1}{16} I_0$$

In case of screen (4)

$$A = \frac{1}{2}\left(\frac{1}{2} A_1\right) \quad \text{and} \quad I = \frac{I}{4} I_0, \quad \text{so} \quad I_4 = I_2$$

In general we get

$$I(\varphi) = I_0\left(1 - \left(\frac{\varphi}{2\pi}\right)\right)^2$$

where φ is the total angle blocked out by the screen.

(b) In case of screen (5)

$$A = \frac{3}{4}\left(\frac{1}{2}A_1\right) + \frac{1}{4}A_1$$

A_1 being the contribution of the first Fresnel zone.

Thus, $$A = \frac{5}{8}A_1 \quad \text{and} \quad I = \frac{25}{16}I_0$$

In case of screen (6)

$$A = \frac{1}{2}\left(\frac{1}{2}A_2\right) + \frac{1}{2}A_1 = \frac{3}{4}A_1 \quad \text{and} \quad I = \frac{9}{4}I_0$$

In case of screen (7)

$$A = \frac{1}{4}\left(\frac{1}{2}A_1\right) + \frac{3}{4}A_1 = \frac{7}{8}A_1 \quad \text{and} \quad I = \frac{49}{16}I_0$$

In case of screen (8)

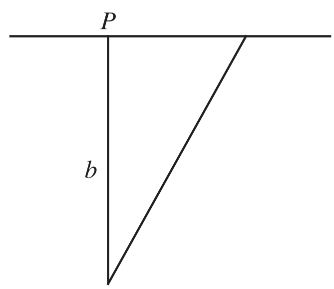
$$A = \frac{1}{2}\left(\frac{1}{2}A_1\right) + \frac{1}{2}A_1 = \frac{3}{4}A_1 \quad \text{and} \quad I = \frac{9}{4}I_0, \quad \text{so} \quad I_8 = I_6$$

In screens (5) to (8) the first term in the expression for the amplitude is the contribution of the plane part and the second term gives the expression for the Fresnel zone part. In general in screens (5) and (8)

$$I = I_0\left[1 + \left(\frac{\varphi}{2\pi}\right)\right]^2$$

where φ is the angle covered by the screen.

5.103 We would require the contribution to the amplitude of a wave at a point from half a Fresnel zone. For this, we proceed directly from the Fresnel-Huygen's principle. The complex amplitude is written as



$$E = \int K(\varphi)\,\frac{a_0}{r}\,e^{-ikr}\,dS$$

Here $K(\varphi)$ is a factor which depends on the angle φ between a normal **n** to the area dS and the direction from dS to the point P and r is the distance from the element dS to P.

We see that for the first Fresnel zone

$$E \cong \frac{a_0}{b} \int_0^{\sqrt{b\lambda}} e^{-ikb - ik\rho^2/2b} \, 2\pi\rho \, d\rho \quad (\text{where } K(\varphi) \approx 1)$$

$$\left(\text{using } r \cong b + \frac{\rho^2}{2b} (\text{for} \sqrt{\rho^2 + b^2})\right).$$

For the first Fresnel zone, $r = b + (\lambda/2)$ so, $r^2 \cong b^2 + b\lambda$ and $\rho^2 = b\lambda$.

Thus,

$$E \cong \frac{a_0}{b} e^{-ikb} 2\pi \int_0^{(b\lambda/2)} e^{-i\frac{kx}{b}} \, dx$$

$$= \frac{a_0}{b} 2\pi e^{-ikb} \frac{e^{-ik\lambda/2} - 1}{-ik/b}$$

$$= \frac{a_0}{k} 2\pi i e^{-ikb}(-2) = -\frac{4\pi}{k} i a_0 e^{-ikb} = A_1$$

For the next half-zone

$$E = \frac{a_0}{b} e^{-ikb} 2\pi \int_{b\lambda/2}^{3b\lambda/4} e^{-ikx/b} \, dx$$

$$= \frac{a_0}{k} 2\pi i e^{-ikb} (e^{-i(3k\lambda/4)} - e^{-ik\lambda/2})$$

$$= \frac{a_0}{k} 2\pi i e^{-ikb}(+1 + i) = -\frac{A_1(1 + i)}{2}$$

If we calculate the contribution of the full second Fresnel zone we will get $-A_1$. If we take account of the factors $K(\varphi)$ and $1/r$ which decrease monotonically, we expect the contribution to change to $-A_2$. Thus, we write for the contribution of the half zones in the second Fresnel zone as

$$-\frac{A_2(1 + i)}{2} \quad \text{and} \quad -\frac{A_2(1 - i)}{2}$$

The part lying in the recess has an extra phase difference

$$-\delta = -\frac{2\pi}{\lambda}(n - 1)h$$

Thus, the full amplitude is (note that the correct form is e^{-ikr})

$$\left(A_1 - \frac{A_2}{2}(1 + i)\right)e^{+i\delta} - \frac{A_2}{2}(1 - i) + A_3 + A_4 + \ldots$$

$$\approx \left(\frac{A_1}{2}(1 - i)\right)e^{+i\delta} - \frac{A_2}{2}(1 - i) + \frac{A_3}{2}$$

$$\approx \left(\frac{A_1}{2}(1-i)\right)e^{+i\delta} + i\frac{A_1}{2} \quad (\text{as } A_2 \approx A_3 \approx A_1)$$

and
$$A_3 - A_4 - A_5 \ldots = \frac{A_3}{2}$$

The corresponding intensity is

$$I = \frac{A_1^2}{4}\left[(1-i)e^{+i\delta} + \frac{i}{e}\right][(1+i)e^{-i\delta} - i]$$

$$= I_0[3 - 2\cos\delta + 2\sin\delta] = I_0\left[3 + 2\sqrt{2}\sin\left(\delta - \frac{\pi}{4}\right)\right]$$

(a) For maximum intensity

$$\sin\left(\delta - \frac{\pi}{4}\right) = +1$$

or
$$\delta - \frac{\pi}{4} = 2k\pi + \frac{\pi}{2} \quad (\text{where } k = 0, 1, 2, \ldots)$$

$$\delta = 2k\pi + \frac{3\pi}{4} = \frac{2\pi}{\lambda}(n-1)h$$

So,
$$h = \frac{\lambda}{n-1}\left(k + \frac{3}{8}\right)$$

(b) For minimum intensity

$$\sin\left(\delta - \frac{\pi}{4}\right) = 1$$

$$\delta - \frac{\pi}{4} = 2k\pi + \frac{3\pi}{2} \quad \text{or} \quad \delta = 2k\pi + \frac{7\pi}{4}$$

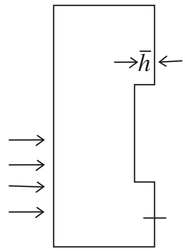
So,
$$h = \frac{\lambda}{n-1}\left(k + \frac{7\pi}{8}\right)$$

(c) For $I = I_0$, $\left.\begin{array}{r}\cos\delta = 0 \\ \sin\delta = -1\end{array}\right\}$ or $\left\{\begin{array}{l}\sin\delta = 0 \\ \cos\delta = +1\end{array}\right.$

Thus,
$$\delta = 2k\pi \quad \text{and} \quad h = \frac{k\lambda}{n-1}$$

or
$$\delta = 2k + \frac{3\pi}{2} \quad \text{and} \quad h = \frac{\lambda}{n-1}\left(k + \frac{3\pi}{4}\right)$$

5.104 The contribution to the wave amplitude of the inner half-zone is



$$\frac{2\pi a_0 e^{-ikb}}{b}\int_0^{\sqrt{b\lambda/2}} e^{-ik\rho^2/2b}\rho\,d\rho$$

$$= \frac{2\pi a_0 e^{-ikb}}{b}\int_0^{b\lambda/4} e^{-ikx/b}\,dx$$

$$= \frac{2\pi a_0 e^{-ikb}}{b}(e^{-ik\lambda/4}-1)\times\frac{1}{-ik/b}$$

$$= \frac{2\pi a_0 e^{-ikb}}{b}(-i-1) = +\frac{A_1}{2}(1+i)$$

With phase factor this becomes

$$\frac{A_1}{2}(1+i)e^{i\delta}\ \left(\text{where } \delta = \frac{2\pi}{\lambda}(n-1)h\right)$$

The contribution of the remaining aperture is $A_1/2(1-i)$ (so that the sum of the two parts when $\delta = 0$ is A_1).

Thus, the complete amplitude is

$$\frac{A_1}{2}(1+i)e^{i\delta} + \frac{A_1}{2}(1-i)$$

and the intensity is

$$I = I_0\,[(1+i)e^{i\delta} + (1-i)][(1-i)e^{-i\delta} + (1+i)]$$

$$= I_0[2+2+(1-i)^2e^{-i\delta} + (1+i)^2e^{i\delta}]$$

$$= I_0\,[4-2ie^{-i\delta} + 2ie^{i\delta}] = I_0\,(4-4\sin\delta)$$

Here $I_0 = A_1^2/4$ is the intensity of the incident light which is the same as the intensity due to an aperture of infinite extent (and no recess). Now, I is maximum when

$$\sin\delta = -1$$

or

$$\delta = 2k\pi + \frac{3\pi}{2}$$

So,

$$h = \frac{\lambda}{n-1}\left(k+\frac{3}{4}\right)$$

The maximum intensity $I_{max} = 8I_0$.

5.105 We follow the argument of Problem 5.103. We find that the contribution of the first Fresnel zone is

$$A_1 = -\frac{4\pi i}{k} a_0 e^{-ikb}$$

For the next half zone it is

$$-\frac{A_2}{2}(1+i)$$

$\Big($The contribution of the remaining part of the second Fresnel zone will be

$-\frac{A_2}{2}(1-i).\Big)$

If the disk has a thickness h, the extra phase difference suffered by the light wave in passing through the disk will be

$$\delta = \frac{2\pi}{\lambda}(n-1)h$$

Thus, the amplitude at P will be

$$E_P = \left(A_1 - \frac{A_2}{2}(1+i)\right)e^{-i\delta} - \frac{A_2}{2}(1-i) + A_3 - A_4 - A_5 + \ldots$$

$$= \left(\frac{A_1(1-i)}{2}\right)e^{-i\delta} + \frac{iA_1}{2} = \frac{A_1}{2}[(1-i)e^{-i\delta} + i]$$

The corresponding intensity will be

$$I = I_0(3 - 2\cos\delta - 2\sin\delta) = I_0\left(3 - 2\sqrt{2}\sin\left(\delta + \frac{\pi}{4}\right)\right)$$

The intensity will be a maximum when

$$\sin\left(\delta + \frac{\pi}{4}\right) = -1$$

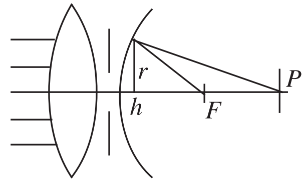
or
$$\delta + \frac{\pi}{4} = 2k\pi + \frac{3\pi}{2}$$

i.e.,
$$\delta = \left(k + \frac{5}{8}\right)\cdot 2\pi$$

So,
$$h = \frac{\lambda}{n-1}\left(k + \frac{5}{8}\right) \quad (\text{where } k = 0, 1, 2, \ldots)$$

Note: *It is not clear why $k = 2$ for $h_{\min}$. The normal choice will be $k = 0$. If we take $k = 0$, we get $h_{\min} = 0.59$ μm.*

5.106 Here the focal point acts as a virtual source of light. This means that we can take spherical waves converging towards F. Let us divide these waves into Fresnel zones just after they emerge from the stop. So,



$$r^2 = f^2 - (f - h)^2 = (b - m\lambda/2)^2 - (b - h)^2$$

(where r is the radius of the m^{th} Fresnel zone and h is the distance to the left of the foot of the perpendicular).

Thus, $$r^2 \approx 2fh = -bm\lambda + 2bh$$

So, $$h = bm\lambda/2\,(b - f)$$

and $$r^2 = fbm\,\lambda/(b - f)$$

The intensity maxima are observed when an odd number of Fresnel zones are exposed by the stop. Thus,

$$r_k = \sqrt{\frac{kbf\lambda}{b - f}}$$

$$= 0.90\sqrt{k}\text{ mm} \quad (\text{where } k = 1, 3, 5, \ldots)$$

5.107 For the radius of the periphery of the k^{th} zone, we have

$$r_k = \sqrt{k\lambda\frac{ab}{a + b}} = \sqrt{k\lambda b} \quad (\text{if } a = \infty)$$

If the aperture diameter is reduced η times, it will produce a similar diffraction pattern (reduced η times) if the radii of the Fresnel zones are also η times less. Thus,

$$r_k' = \frac{r_k}{\eta}$$

This requires $$b' = \frac{b}{\eta^2} = 1.0\text{ m}$$

5.108 (a) If a point source is placed before an opaque ball, the diffraction pattern consists of a bright spot inside a dark disk followed by fringes. The bright spot is on the line joining the point source and the centre of the ball. When the object is a finite source of transverse dimension y, every point of the source has its corresponding image on the line joining that point and the centre of the ball. Thus, the transverse dimension of the image is given by

$$y' = \frac{b}{a}y = 9\text{ mm}$$

(b) The minimum height of the irregularities covering the surface of the ball at random, at which the ball obstructs light is, according to the note at the end of the problem, comparable with the width of the Fresnel zone along which the edge of opaque screen passes.

So, $$h_{\min} \approx \Delta r$$

To find Δr, we note that

$$r^2 = \frac{k\lambda ab}{a+b}$$

or $$2r\Delta r = D\Delta r = \frac{\lambda ab}{a+b}\,\Delta k$$

where D = diameter of the disk (= diameter of the last Fresnel zone) and $\Delta k = 1$.

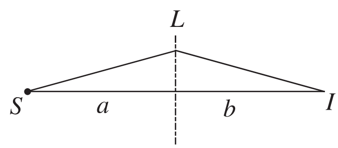
Thus, $$h_{\min} = \frac{\lambda ab}{D(a+b)} = 0.099 \text{ mm}$$

5.109 In a zone plate, an undarkened circular disk is followed by a number of alternately undarkened and darkened rings. For the proper case, these correspond to 1st, 2nd, 3rd , $\cdots$ Fresnel zones.

Let r_1 = radius of the central undarkened circle. Then for this to be the first Fresnel zone in the present case, we must have

$$SL + LI - SI = \lambda/2$$

Thus, if r_1 is the radius of the periphery of the first zone



$$\sqrt{a^2 + r_1^2} + \sqrt{b^2 + r_1^2} - (a+b) = \frac{\lambda}{2}$$

or $$\frac{r_1^2}{2}\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{\lambda}{2} \quad \text{or} \quad \frac{1}{a} + \frac{1}{b} = \frac{1}{r_1^2/\lambda}$$

It is clear that the plate is acting as a lens of focal length

$$f_1 = \frac{r_1^2}{\lambda} = \frac{ab}{a+b} = \frac{1.5}{2.5} \text{ m} = 0.6 \text{ m}$$

This is the principal focal length. Other maxima are obtained when

$$SL + LI - SI = 3\frac{\lambda}{2}, 5\frac{\lambda}{2}, \cdots$$

Thus, focal lengths are also

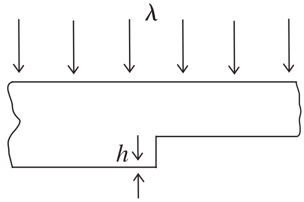
$$\frac{r_1^2}{3\lambda}, \frac{r_1^2}{5\lambda}, \cdots$$

5.110 Just below the ledge, the amplitude of the wave is given by

$$A = \frac{1}{2}(A_1 - A_2 + A_3 - A_4 + \ldots)e^{-i\delta} + \frac{1}{2}(A_1 - A_2 + A_3 - A_4 + \ldots)$$

Here the quantities in the brackets are the contributions of various Fresnel zones; the factor 1/2 is to take account of the division of the plate into two parts by the ledge; the phase factor δ is given by

$$\delta = \frac{2\pi}{\lambda} h(n-1)$$



and takes into account the extra length traversed by the waves on the left.

Using $$A_1 - A_2 + A_3 - A_4 + \ldots \approx \frac{A_1}{2}$$

we get $$A = \frac{A_1}{4}(1 + e^{-i\delta})$$

and the corresponding intensity is

$$I = I_0 \frac{1+\cos\delta}{2} \qquad \left(\text{where } I_0 \propto \left(\frac{A_1}{2}\right)^2\right)$$

(a) This is minimum when

$$\cos\delta = -1$$

So, $$\delta = (2k+1)\pi$$

and $$h = (2k+1)\frac{\lambda}{2(n-1)} \qquad (\text{where } k = 0, 1, 2, \ldots)$$

Using $n = 1.5$, $\lambda = 0.60$ μm, we get

$$h = 0.60(2k+1)\ \mu\text{m}$$

(b) Intensity will be twice as low as ($I = I_0/2$) when

$$\cos\delta = 0$$

or $$\delta = k\pi + \frac{\pi}{2} = (2k+1)\frac{\pi}{2}$$

Thus, in this case

$$h = 0.30(2k+1)\ \mu\text{m}$$

5.111 (a) From Cornu's spiral, the intensity of the first maximum is given as

$$I_{max} = 1.37 I_0$$

and the intensity of the first minimum is given by

$$I_{\min} = 0.78 I_0$$

So, the required ratio is

$$\frac{I_{\max}}{I_{\min}} = 1.76$$

(b) The value of the distance x is related to the parameter v in Fresnel's integral by

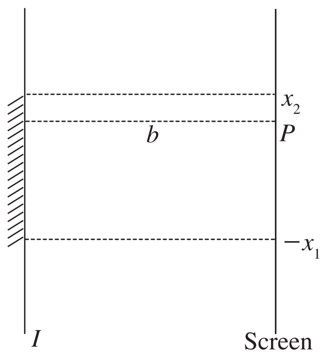
$$v = x\sqrt{\frac{2}{b\lambda}}$$

For the first two maxima, the distances x_1, x_2 are related to the parameters v_1, v_2 by

$$x_1 = \sqrt{\frac{b\lambda}{2}}\, v_1 \quad \text{and} \quad x_2 = \sqrt{\frac{b\lambda}{2}}\, v_2$$

Thus, $$(v_2 - v_1)\sqrt{\frac{b\lambda}{2}} = x_2 - x_1 = \Delta x$$

or $$\lambda = \frac{2}{b}\left(\frac{\Delta x}{v_2 - v_1}\right)^2$$



From Cornu's spiral, the positions of the maxima are

$$v_1 = 1.22,\ v_2 = 2.34,\ v_3 = 3.08, \text{ etc.}$$

Thus, $$\lambda = \frac{2}{b}\left(\frac{\Delta x}{1.12}\right)^2 \cong 0.7\ \mu\text{m}$$

5.112 We shall use the equation written down in Problem 5.103, the Fresnel-Huygen's formula.

Suppose we want to find the intensity at P which is such that the coordinates of the edges (x-coordinates) with respect to P are x_2 and $-x_1$. Then, the amplitude at P is

$$E = \int K(\varphi)\frac{a_0}{r} e^{-ikr}\, dS$$

We write $dS = dx\, dy$, y is to be integrated from $-\infty$ to $+\infty$. So,

$$r \approx b + \frac{x^2 + y^2}{2b} \tag{1}$$

where r is the distance of the element of surface on I from P. It is $\sqrt{b^2 + x^2 + y^2}$ and hence, approximately given by Eq. (1). We then get

$$E = A_0(b)\left[\int_{x_2}^{\infty} e^{-ikx^2/2b}\,dx + \int_{-\infty}^{-x_1} e^{-ikx^2/2b}\,dx\right]$$

$$= A_0'(b)\left[\int_{v_2}^{\infty} e^{-i\pi u^2/2}\,du + \int_{-\infty}^{-v_1} e^{-i\pi u^2/2}\,du\right]$$

where $$v_2 = \sqrt{\frac{2}{b\lambda}}x_2 \quad \text{and} \quad v_1 = \sqrt{\frac{2}{b\lambda}}x_1$$

The intensity is the square of the amplitude. In our case, at the centre

$$v_1 = v_2 = \sqrt{\frac{2}{b\lambda}}\cdot\frac{a}{2} = \sqrt{\frac{a^2}{2b\lambda}} = 0.64$$

(where a = width of the strip = 0.7 mm, b = 100 cm, λ = 0.60 μm).

At, say, the lower edge $v_1 = 0$, $v_2 = 1.28$.

Thus,

$$\frac{I_{\text{centre}}}{I_{\text{edge}}} = \left|\frac{\int_{0.64}^{\infty} e^{-i\pi u^2/2}\,du + \int_{-\infty}^{-0.64} e^{-i\pi u^2/2}\,du}{\int_{1.28}^{\infty} e^{-i\pi u^2/2}\,du + \int_{-\infty}^{0} e^{-i\pi u^2/2}\,du}\right| = 4\,\frac{\left(\frac{1}{2} - C(0.64)\right)^2 + \left(\frac{1}{2} - S(0.64)\right)^2}{\left(1 - C(1.28)\right)^2 + \left(1 - S(1.28)\right)^2}$$

where $$C(v) = \int_0^v \cos\frac{\pi u^2}{2}\,du$$

and $$S(v) = \int_0^v \sin\frac{\pi u^2}{2}\,du$$

Rough evaluation of the integrals using Cornu's spiral gives

$$\frac{I_{\text{centre}}}{I_{\text{edge}}} \sim 2.4$$

$$\left(\text{We have used } \int_0^{\infty} \cos\frac{\pi u^2}{2}\,du = \int_0^{\infty} \sin\frac{\pi u^2}{2}\,du = \frac{1}{2}.\right)$$

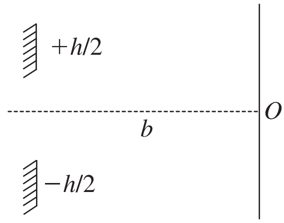
Thus, $$C(0.64) = 0.62, \quad S(0.64) = 0.15$$

and $$C(1.28) = 0.65, \quad S(1.28) = 0.67$$

5.113 If the aperture has width h then the parameters $(v, -v)$ associated with $(h/2, -h/2)$ are given by

$$v = \frac{h}{2}\sqrt{\frac{2}{b\lambda}} = \frac{h}{\sqrt{2b\lambda}}$$

The intensity of light at O on the screen is obtained as the square of the amplitude A of the wave at O which is



$$A \sim \text{constant} \int_{-v}^{v} e^{-i\pi u^2/2}\, du$$

Thus, $$I = 2I_0 \left\{ (C(v))^2 + (S(v))^2 \right\}$$

where $C(v)$ and $S(v)$ have been defined above and I_0 is the intensity at O due to an infinitely wide ($v = \infty$) aperture, for then

$$I = 2I_0 \left(\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \right) = 2I_0 \times \frac{1}{2} = I_0$$

By definition v corresponds to the first minimum of the intensity. This means

$$v = v_1 \approx 0.90$$

When we increase h to $h + \Delta h$, the corresponding second minimum of intensity

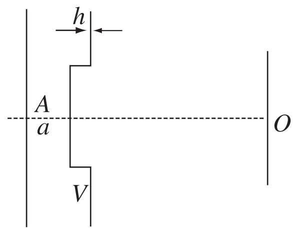
$$v_2 = \frac{h + \Delta h}{\sqrt{2b\lambda}}$$

From Cornu's spiral $v_2 \approx 2.75$.

Thus, $$\Delta h = \sqrt{2b\lambda}(v_2 - v_1) = 0.85\sqrt{2b\lambda}$$

or $$\lambda = \left(\frac{\Delta h}{0.85}\right)^2 \frac{1}{2b} = \left(\frac{0.70}{0.85}\right)^2 \frac{1}{2 \times 0.6} = 0.565\ \mu\text{m}$$

5.114 Let a = width of the recess and



$$v = \frac{a}{2}\sqrt{\frac{2}{b\lambda}} = \frac{a}{\sqrt{2b\lambda}}$$

$$= \frac{0.6}{\sqrt{2 \times 0.77 \times 0.65}} \approx 0.60$$

be the parameter along Cornu's spiral corresponding to the half-width of the recess. The amplitude of the diffracted wave is given by

$$A \sim \text{constant} \left[e^{i\delta} \int_{-v}^{v} e^{-i\pi u^2/2}\, du + \int_{v}^{\infty} e^{-i\pi u^2/2}\, du + \int_{-\infty}^{v} e^{-i\pi u^2/2}\, du \right]$$

where $\delta = 2\pi/\lambda(n-1)h$ is the extra phase due to the recess. (Actually an extra phase $e^{-i\delta}$ appears outside the recess. When we take it out and absorb it in the constant we get the expression written.)

Thus, $$A \sim \text{constant} \times \left[C(v) - iS(v)e^{i\delta} + \left(\frac{1}{2} - C(v)\right) - i\left(\frac{1}{2} - S(v)\right)\right]$$

From Cornu's spiral, the coordinates corresponding to the parameter $v = 0.60$ are $C(v) = 0.57$, $S(v) = 0.13$. So, the intensity at O is proportional to

$$\begin{aligned}&\left|[(0.57 - 0.13i)e^{i\delta} - 0.07 - 0.37i]\right|^2 \\ &= (0.57^2 + 0.13^2) + 0.07^2 + 0.37^2 \\ &\quad +(0.57 - 0.13i)(-0.07 + 0.37i)e^{i\delta} \\ &\quad +(0.57 + 0.13i)(-0.07 - 0.37i)e^{-i\delta}\end{aligned}$$

So,

$$0.57 \pm 0.13i = 0.585e^{\pm i\alpha}, \quad \alpha = 12.8°$$

$$-0.07 \pm 0.37i = 0.377e^{\pm i\beta}, \quad \beta = 100.7°$$

Thus, the cross-term is

$$2 \times 0.585 \times 0.377 \cos(\delta + 88°)$$

$$\approx 2 \times 0.585 \times 0.377 \cos\left(\delta + \frac{\pi}{2}\right)$$

For maximum intensity

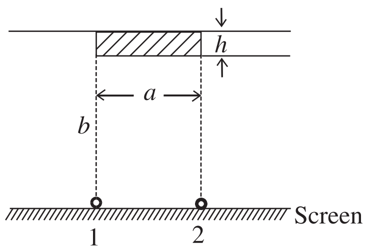
$$\delta + \frac{\pi}{2} = 2k'\pi \quad (\text{where } k' = 1, 2, 3, 4, \ldots)$$

$$= 2(k+1)\pi \quad (\text{where } k = 0, 1, 2, 3, \ldots)$$

or $$\delta = 2k\pi + \frac{3\pi}{2}$$

So, $$h = \frac{\lambda}{n-1}\left(k + \frac{3}{4}\right)$$

5.115 Using the method of Problem 5.103 we can write down the amplitudes at 1 and 2 as,



$$A_1 \sim \text{constant}\left[\int_{-\infty}^{0} e^{-i\pi u^2/2}\,du + e^{-i\delta}\int_{v}^{\infty} e^{-i\pi u^2/2}\,du\right]$$

$$A_2 \sim \text{constant}\left[\int_{-\infty}^{-v} e^{-i\pi u^2/2}\,du + e^{-i\delta}\int_{0}^{\infty} e^{-i\pi u^2/2}\,du\right]$$

where $v = a\sqrt{2/b\lambda}$

is the parameter of Cornu's spiral and constant factor is common to 1 and 2.

Using the usual notation

$$C = C(v) = \int_0^v \cos\frac{\pi u^2}{2}\,du$$

$$S = S(v) = \int_0^v \sin\frac{\pi u^2}{2}\,du$$

and the result $$\int_0^\infty \cos\frac{\pi u^2}{2}\,du = \int_0^\infty \sin\frac{\pi u^2}{2}\,du = \frac{1}{2}$$

We find the ratio of intensities as

$$\frac{I_2}{I_1} = \left|\frac{\left(\frac{1}{2} - C\right) - i\left(\frac{1}{2} - S\right) + e^{-i\delta}\,\frac{(1-i)}{2}}{\frac{1-i}{2} + e^{-i\delta}\left\{\left(\frac{1}{2} - C\right) - i\left(\frac{1}{2} - S\right)\right\}}\right|^2$$

(The constants in A_1 and A_2 must be the same by symmetry.)

In our case, $a = 0.30$ mm, $\lambda = 0.65\ \mu$m, $b = 1.1$ m .

So, $$v = 0.30 \times \sqrt{\frac{2}{1.1 \times 0.65}} = 0.50$$

Therefore, $$C(0.50) = 0.48 \quad \text{and} \quad S(0.50) = 0.06$$

Also, $$\frac{I_2}{I_1} = \left|\frac{0.02 - 0.44i + e^{-i\delta}\,\frac{(1-i)}{2}}{\frac{1-i}{2}e^{i\delta} + 0.02 - 0.44i}\right|^2 = \left|\frac{1 + (0.02 - 0.44i)\sqrt{2}e^{i\delta + i\pi/4}}{1 + (0.02 + 0.44i)\sqrt{2}e^{i\delta + i\pi/4}}\right|^2$$

But $0.02 - 0.44i = 0.44e^{-i\alpha}$ and $\alpha = 1.525$ radians$(\approx 87.4°)$.

So, $$\frac{I_2}{I_1} = \left|\frac{1 + 0.44 \times \sqrt{2} \times e^{i(\delta - 0.740)}}{1 + 0.44 \times \sqrt{2} \times e^{-i(\delta + 0.740)}}\right|^2 = \frac{1 + 2\,(0.44)^2 + 2\sqrt{2} \times 0.44\cos(\delta - 0.740)}{1 + 2\,(0.44)^2 + 2\sqrt{2} \times 0.44\cos(\delta + 0.740)}$$

I_2 is maximum when $\delta - 0.740 = 0$ (modulo 2π).

Thus, in that case

$$\frac{I_2}{I_1} = \frac{1.387 + 1.245}{1.387 + 1.245\cos(1.48)} = \frac{2.632}{1.5} \approx 1.9$$

5.116 We apply the formula of Problem 5.103 and calculate

$$\int_{\text{Aperture}} \frac{a_0}{r} e^{-ikr}\, dS = \int_{\text{Semicircle}} + \int_{\text{Slit}}$$

The contribution of the full first Fresnel zone has been evaluated in Problem 5.103. The contribution of the semicircle is one-half of it and is

$$-\frac{2\pi}{k} i a_0 e^{-ikb} = -i a_0 \lambda e^{-ikb}$$

The contribution of the slit is

$$\frac{a_0}{b} \int_0^{0.90\sqrt{b\lambda}} e^{-ikb} e^{-ikx^2/2b}\, dx \int_{-\infty}^{\infty} e^{-iky^2/2b}\, dy$$

Now,
$$\int_{-\infty}^{\infty} e^{-iky^2/2b}\, dy = \int_{-\infty}^{\infty} e^{-i\pi y^2/b\lambda}\, dy$$

or
$$\sqrt{\frac{b\lambda}{2}} \int_{-\infty}^{\infty} e^{-i\pi u^2/2}\, du = \sqrt{b\lambda}\, e^{-i\pi/4}$$

Thus, the contribution of the slit is

$$\frac{a_0}{b} \sqrt{b\lambda}\, e^{-ikb - i\pi/4} \int_0^{0.9 \times \sqrt{2}} e^{-i\pi u^2/2}\, du \sqrt{\frac{b\lambda}{2}}$$

$$= a_0 \lambda e^{-ikb - i\pi/4} \frac{1}{\sqrt{2}} \int_0^{1.27} e^{-i\pi u^2/2}\, du$$

Thus, the intensity at the observation point P on the screen is

$$a_0^2 \lambda^2 \left| -i + \frac{1-i}{2} (C(1.27) - iS\,(1.27)) \right|^2$$

$$= a_0^2 \lambda^2 \left| -i + \frac{(1-i)(0.67 - 0.65i)}{2} \right|^2$$

(On using $C\,(1.27) = 0.67$ and $S\,(1.27) = 0.65$.)

$$= a_0^2 \lambda^2 |-i + 0.01 - 0.66i|^2$$

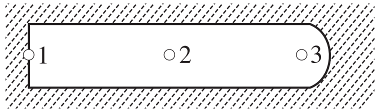
$$= a_0^2 \lambda^2 |\,0.01 - 1.66i|^2$$

$$= 2.76\, a_0^2 \lambda^2$$

Now $a_0^2 \lambda^2$ is the intensity due to half of first Fresnel zone and is therefore equal to I_0. (It can also be obtained by doing the x-integral over $-\infty$ to $+\infty$.)

Thus,
$$I = 2.76 I_0$$

5.117 From the statement of the problem, we know that the width of the slit = diameter of the first Fresnel zone $= 2\sqrt{b\lambda}$, where b is the distance of the observation point from the slit.



We calculate the amplitudes by evaluating the integral of Problem 5.103. We get

$$A_1 = \frac{a_0}{b} \int_{-\sqrt{b\lambda}}^{\sqrt{b\lambda}} e^{-ikb} e^{-ikx^2/2b}\, dx \int_0^{\infty} e^{-iky^2/2b}\, dy$$

$$= \frac{a_0}{b} e^{-ikb} \frac{b\lambda}{2} \int_{-\sqrt{2}}^{\sqrt{2}} e^{-i\pi u^2/2}\, du \times \int_0^{\infty} e^{-i\pi u^2/2}\, du$$

$$= \frac{a_0 \lambda}{2}(1 - i)e^{-ikb}(C(\sqrt{2}) - iS\,(\sqrt{2}))$$

$$A_2 = \frac{a_0}{b} \int_{-\sqrt{b\lambda}}^{\sqrt{b\lambda}} e^{-ikb} e^{-ikx^2/2b}\, dx \int_{-\infty}^{\infty} e^{-iky^2/2b}\, dy$$

$$= 2A_1$$

$$A_3 = -\, ia_0 \lambda e^{-ikb} + \frac{a_0 \lambda (1 - i)}{2}(C(\sqrt{2}) - iS(\sqrt{2}))e^{-ikb}$$

where the contribution of the first half Fresnel zone (in A_3, first term) has been obtained from the last problem.

Thus,
$$I_1 = a_0^2 \lambda^2 \left| \frac{(1 - i)(0.53 - 0.72i)}{2} \right|^2$$

(using $C(\sqrt{2}) = 0.53$ and $S(\sqrt{2}) = 0.72$.)

$$= a_0^2 \lambda^2 |-0.095 - 0.625i|^2 = 0.03996\, a_0^2 \lambda^2$$

$$I_2 = 4I_1$$

$$I_3 = a_0^2 \lambda^2 |-0.095 - 0.625i - i|^2$$

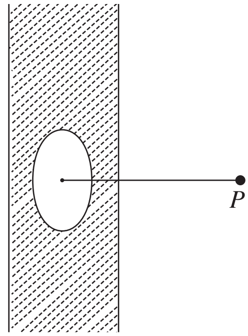
$$= a_0^2 \lambda^2 |-0.095 - 1.625i|^2$$

$$= 2.6496\, a_0^2 \lambda^2$$

So,
$$I_3 = 6.6 I_1$$

Thus,
$$I_1 : I_2 : I_3 \approx 1 : 4 : 7$$

5.118 The radius of the first-half Fresnel zone is $\sqrt{b\lambda/2}$ and the amplitude at P is obtained using Problem 5.103. So, we have



$$A = \frac{a_0}{b}\left[\int_{-\infty}^{-\eta\sqrt{b\lambda/2}} + \int_{\eta\sqrt{b\lambda/2}}^{\infty}\right] e^{-ikb - ikx^2/2b}\, dx$$

$$\int_{-\infty}^{\infty} e^{-iky^2/2b}\, dy + \frac{a_0}{b} e^{-ikb} \int_0^{\sqrt{b\lambda/2}} e^{-ik\rho^2/2b}\, 2\pi\rho d\rho$$

We use

$$\begin{aligned}
\int_{-\infty}^{-\eta\sqrt{b\lambda/2}} e^{-ikx^2/2b}\, dx &= \int_{\eta\sqrt{b\lambda/2}}^{\infty} e^{-ikx^2/2b}\, dx \\
&= \int_{\eta\sqrt{b\lambda/2}}^{\infty} e^{-i\pi x^2/b\lambda}\, dx \\
&= \int_{\eta}^{\infty} e^{-i\pi u^2/2} \sqrt{\frac{b\lambda}{2}} du \\
&= \sqrt{\frac{b\lambda}{2}} \left(\int_0^{\infty} - \int_0^{\eta}\right) e^{-i\pi u^2/2}\, du \\
&= \sqrt{\frac{b\lambda}{2}} \left(\left(\frac{1}{2} - C(\eta)\right) - i\left(\frac{1}{2} - S(\eta)\right)\right)
\end{aligned}$$

Thus,

$$A = a_0 \frac{\lambda}{2} \times 2 \times (1-i)e^{-ikb}\left[\left(\frac{1}{2} - C(\eta)\right) - i\left(\frac{1}{2} - S(\eta)\right)\right] + a_0\lambda(1-i)e^{-ikb}$$

where we have used

$$\int_0^{\sqrt{b\lambda/2}} e^{-ik\rho^2/2b}\, 2\pi\rho\, d\rho = \frac{2\pi ib}{k}(-1-i) = \frac{2\pi b}{k}(1-i) = \lambda b(1-i)$$

Thus, the intensity is

$$I = |A|^2 = a_0^2\lambda^2 \times 2\left[(3/2 - C(\eta))^2 + \left(\frac{1}{2} - S(\eta)\right)^2\right]$$

From Cornu's spiral,

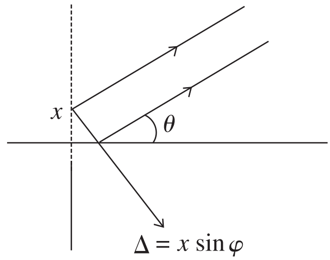
$$C(\eta) = C(1.07) = 0.76$$

and $$S(\eta) = S(1.07) = 0.50$$

$$I = a_0^2\lambda^2 \times 2 \times (0.74)^2 = 1.09a_0^2\lambda^2$$

As before $$I_0 = a_0^2\lambda^2 \quad \text{so,} \quad I \approx I_0$$

5.119 If a plane wave is incident normally from the left on a slit of width b and the diffracted wave is observed at a large distance, the resulting pattern is called Fraunhofer diffraction. The condition for this is $b^2 \ll l\lambda$, where l is the distance between the slit and the screen. In practice, light may be focused on the screen with the help of a lens (or a telescope).



Consider an element of the slit which is an infinite strip of width dx. We use the formula of Problem 5.103 with the following modifications:

The factor $1/r$ characteristic of spherical waves will be omitted. The factor $K(\varphi)$ will also be dropped if we confine ourselves to not too large φ. In the direction defined by the angle φ the extra path difference of the wave emitted from the element at x relative to the wave emitted from the centre is

$$\Delta = -x\sin\varphi$$

Thus, the amplitude of the wave is given by

$$\alpha \int_{-b/2}^{+b/2} e^{ik\sin\varphi}\, dx = \left(\frac{e^{i\frac{1}{2}kb\sin\varphi} - e^{-i\frac{1}{2}kb\sin\varphi}}{ik\sin\varphi} \right)$$

$$= \frac{\sin(\pi b/\lambda \sin\varphi)}{\pi b/\lambda \sin\varphi}$$

Thus, $$I = I_0 \frac{\sin^2\alpha}{\alpha^2}$$

where $\alpha = \pi b/\lambda \sin\varphi$ and I_0 is a constant.

Minima are observed for $\sin\alpha = 0$ but $\alpha \neq 0$. Thus, we find minima at angles given by

$$b\sin\varphi = k\lambda \quad (\text{where } k = \pm 1, \pm 2, \pm 3, \ldots)$$

5.120 Since $I(\alpha)$ is $+ve$ and vanishes for $b\sin\varphi = k\lambda$, i.e., for $\alpha = k\pi$, we expect maxima of $I(\alpha)$ between $\alpha = +\pi$ and $\alpha = +2\pi$, etc. We can get these values by

$$\frac{d}{d\alpha}(I(\alpha)) = I_0 2\frac{\sin\alpha}{\alpha}\frac{d}{d\alpha}\frac{\sin\alpha}{\alpha} = 0$$

or $$\frac{\alpha\cos\alpha - \sin\alpha}{\alpha^2} = 0$$

or $$\tan\alpha = \alpha$$

Solutions of this transcendental equation can be obtained graphically. The first three solutions are $\alpha_1 = 1.43\pi, \alpha_2 = 2.46\pi, \alpha_3 = 3.47\pi$, on the $+ve$ side.

Thus, $$b\sin\varphi_1 = 1.43\lambda$$

$$b\sin\varphi_2 = 2.46\lambda$$

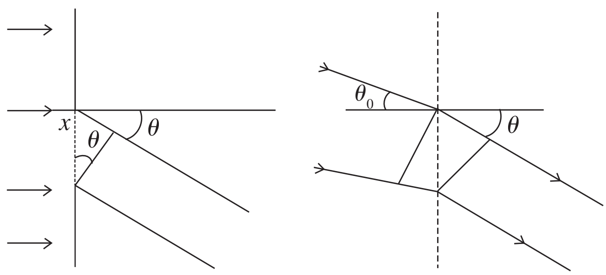
$$b\sin\varphi_3 = 3.47\lambda$$

(On the negative side the solutions are $-\alpha_1, -\alpha_2, -\alpha_3, \ldots$)

Asymptotically the solutions are

$$b\sin\varphi_m \simeq \left(m + \frac{1}{2}\right)\lambda$$

5.121 The relation $b\sin\theta = k\lambda$ for minima (when light is incident normally on the slit) has a simple interpretation: $b\sin\theta$ is the path difference between extreme rays emitted at angle θ. When light is incident at an angle θ_0, the path difference is



$$b(\sin\theta - \sin\theta_0)$$

and the condition of minima is

$$b(\sin\theta - \sin\theta_0) = k\lambda$$

For the first minima

$$b(\sin\theta - \sin\theta_0) = \pm\lambda \quad \text{or} \quad \sin\theta = \sin\theta_0 \pm \frac{\lambda}{b}$$

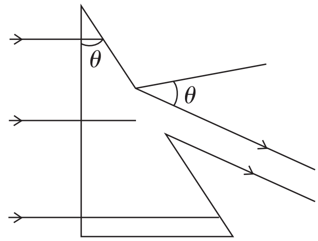
Putting in values $\theta_0 = 30°, \lambda = 0.50\ \mu\text{m}, b = 10\ \mu\text{m}$, we get

$$\sin\theta = \frac{1}{2} \pm \frac{1}{20} = 0.55 \quad \text{or} \quad 0.45$$

and $$\theta_{+1} = 33°\,20' \text{ and } \theta_{-1} = 26°\,44'$$

5.122 (a) This case is analogous to the previous one except that the incident wave moves in glass of R.I. n. Thus, the expression for the path difference for light diffracted at angle θ from the normal to the hypotenuse of the wedge is

$$b(\sin\theta - n\sin\Theta)$$



We write $\theta = \Theta + \Delta\theta$

Then, for the direction of Fraunhofer maximum

$$b(\sin(\Theta + \Delta\theta) - n\sin\Theta) = 0$$

or $$\Delta\theta = \sin^{-1}(n\sin\Theta) - \Theta$$

Using $\Theta = 15°$, $n = 1.5$, we get

$$\Delta\theta = 7.84°$$

(b) The width of the central maximum is obtained from

$$b(\sin\theta_1 - n\sin\Theta) = \pm\lambda \quad (\text{where } \lambda = 0.60\ \mu\text{m},\ b = 10\ \mu\text{m})$$

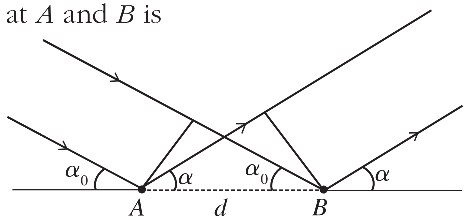
Thus, $$\theta_{+1} = \sin^{-1}\left(n\sin\Theta + \frac{\lambda}{b}\right) = 26.63°$$

and $$\theta_{-1} = \sin^{-1}\left(n\sin\Theta - \frac{\lambda}{b}\right) = 19.16°$$

Therefore, $$\delta\theta = \theta_{+1} - \theta_{-1} = 7.47°$$

5.123 The path difference between waves reflected at A and B is

$$d(\cos\alpha_0 - \cos\alpha)$$



and for maxima

$$d(\cos\alpha_0 - \cos\alpha) = k\lambda \ (k = 0, \pm 1, \pm 2, \ldots)$$

In our case, $k = 2$ and α_0, α are small in radians. Then,

$$2\lambda = d\left(\frac{\alpha^2 - \alpha_0^2}{2}\right)$$

or $$\lambda = \frac{(\alpha^2 - \alpha_0^2)d}{4}$$

$$= 0.61\ \mu\text{m}$$

(for $\alpha = 3\pi/180$, $\alpha_0 = \pi/180$ and $d = 10^{-3}$ m).

5.124 The general formula for diffraction from N slits is

$$I = I_0 \frac{\sin^2 \alpha}{\alpha^2} \frac{\sin^2 N\beta}{\sin^2 \beta}$$

where $$\alpha = \frac{\pi a \sin\theta}{\lambda}$$

$$\beta = \frac{\pi(a+b)\sin\theta}{\lambda}$$

and $N = 3$ in the two cases given here.

(a) In this case $a + b = 2a$

so, $\beta = 2\alpha$

and $$I = I_0 \frac{\sin^2 \alpha}{\alpha^2}(3 - 4\sin^2 2\alpha)^2$$

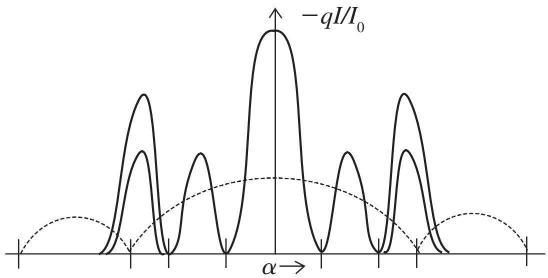
On plotting we get a curve that qualitatively looks like the given figure.

(b) In this case $a + b = 3a$

so, $\beta = 3a$

and $$I = I_0 \frac{\sin^2 \alpha}{\alpha^2}(3 - 4\sin^2 3\alpha)^2$$

This has 3 minima between the principal maxima.



5.125 From the diffraction formula

$$d \sin\theta = m\lambda$$

we have $$d \sin 45° = 2\lambda_1 = 2 \times 0.65\ \mu\text{m}$$

or $$d = 2\sqrt{2} \times 0.65\ \mu\text{m}$$

Then, for $\lambda_2 = 0.50$ in the third order

$$2\sqrt{2} \times 0.65 \sin\theta = 3 \times 0.50$$

$$\sin\theta = \frac{1.5}{1.3 \times \sqrt{2}} = 0.81602$$

This gives $$\theta = 54.68° \approx 55°$$

5.126 The diffraction formula is

$$d \sin\theta_0 = n_0\lambda$$

where $\theta_0 = 35°$ is the angle of diffraction corresponding to order n_0 (which is not yet known).

Thus, $$d = \frac{n_0\lambda}{\sin\theta_0} = n_0 \times 0.9327\ \mu\text{m}$$

on using $\lambda = 0.535\ \mu$m.

For the n^{th} order, we get

$$\sin\theta = \frac{n}{n_0}\sin\theta_0 = \frac{n}{n_0}(0.573576)$$

If $n_0 = 1$, then $n > n_0$ is at least 2 and $\sin\theta > 1$, so, $n = 1$ is the highest order of diffraction.

If $n_0 = 2$, then $n = 3, 4$, but $\sin\theta > 1$ for $n = 4$, thus, the highest order of diffraction is 3.

If $n_0 = 3$, then $n = 4, 5, 6$. For $n = 6, \sin\theta = 2 \times 0.57 > 1$, so not allowed, while for $n = 5, \sin\theta = 5/3 \times 0.573576 < 1$ is allowed. Thus, in this case the highest order of diffraction is 5 as given. Hence, $n_0 = 3$ and $d = 3 \times 0.9327 = 2.7981 \approx 2.8\ \mu\text{m}$.

5.127 Given that

$$d\sin\theta_1 = \lambda$$

and $$d\sin\theta_2 = d\sin(\theta_1 + \Delta\theta) = 2\lambda$$

Thus, $$\sin\theta_1\cos\Delta\theta + \cos\theta_1\sin\Delta\theta = 2\sin\theta_1$$

or $$\sin\theta_1(2 - \cos\Delta\theta) = \cos\theta_1\sin\Delta\theta$$

or $$\tan\theta_1 = \frac{\sin\Delta\theta}{2 - \cos\Delta\theta}$$

or $$\sin\theta_1 = \frac{\sin\Delta\theta}{\sqrt{\sin^2\Delta\theta + (2 - \cos\Delta\theta)^2}}$$

$$= \frac{\sin\Delta\theta}{\sqrt{5 - 4\cos\Delta\theta}}$$

Thus, $$\lambda = \frac{d\sin\Delta\theta}{\sqrt{5 - 4\cos\Delta\theta}}$$

Substitution gives $$\lambda \approx 0.534\ \mu\text{m}$$

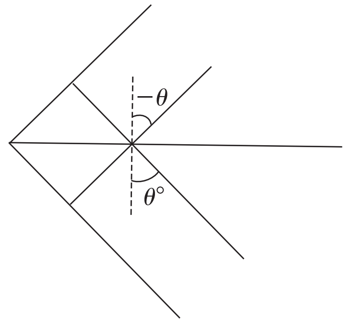
5.128 (a) Here the simple formula $d \sin\theta = m_1 \lambda$ holds.

Thus, $$1.5 \sin\theta = m \times 0.530 \quad \text{and} \quad \sin\theta = \frac{m \times 0.530}{1.5}$$

Highest permissible value of $m = 2$ because $\sin\theta > 1$ if $m = 3$.

Thus, $$\sin\theta = \frac{1.06}{1.50} \quad (\text{for } m = 2, \theta \cong 45°)$$

(b) Here $$d(\sin\theta_0 - \sin\theta) = n\lambda$$



Thus,
$$\sin\theta = \sin\theta_0 - \frac{n\lambda}{d}$$
$$= \sin 60° - n \times \frac{0.53}{1.5}$$
$$= 0.86602 - n \times 0.353333$$

For $n = 5$, $\sin\theta = -0.900645$.

For $n = 6$, $\sin\theta < -1$. Thus, the highest order is $n = 5$ and we get

$$\theta = \sin^{-1}(-0.900645) = -64°$$

5.129 For the lens

$$\frac{1}{f} = (n-1)\left(\frac{1}{R} - \frac{1}{\infty}\right) \quad \text{or} \quad f = \frac{R}{n-1}$$

For the grating

$$d \sin\theta_1 = \lambda \quad \text{or} \quad \sin\theta_1 = \frac{\lambda}{d}$$

$$\operatorname{cosec}\theta_1 = \frac{d}{\lambda}, \quad \cot\theta_1 = \sqrt{(d/\lambda)^2 - 1}$$

and $$\tan\theta_1 = \frac{1}{\sqrt{(d/\lambda)^2 - 1}}$$

Hence, the distance between the two symmetrically placed first order maxima is

$$2f \tan\theta_1 = \frac{2R}{(n-1)\sqrt{(d/\lambda)^2 - 1}}$$

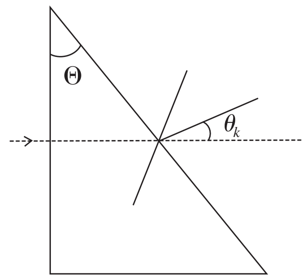
On putting $R = 20$, $n = 1.5$, $d = 6.0\ \mu\text{m}$, $\lambda = 0.60\ \mu\text{m}$, we get this distance as 8.04 cm.

5.130 The diffraction formula is easily obtained on taking account of the fact that the optical path in the glass wedge acquires a factor n (refractive index). We get

$$d(n\sin\Theta - \sin(\Theta\theta_k)) = k\lambda$$

Since $n > 0$, $\Theta - \theta_0 > \Theta$ so, θ_0 must be negative. We get, using $\Theta = 30°$

$$\frac{3}{2} \times \frac{1}{2} = \sin(30° - \theta_0) = \sin 48.6°$$



Thus, $\theta_0 = -18.6°$

Also, for $k = 1$

$$\frac{3}{4} - \sin(30° - \theta_{+1}) = \frac{\lambda}{d} = \frac{0.5}{2.0} = \frac{1}{4}$$

Thus, $\theta_{+1} = 0°$

We calculate θ_k for various values of k by the above formula. For $k = 6$

$$\sin(\theta_k - 30°) = \frac{3}{4} \Rightarrow \theta_k = 78.6°$$

For $k = 7$,

$$\sin(\theta_k - 30°) = 1 \Rightarrow \theta_k = 120°$$

This is inadmissible. Thus the highest order that can be observed is $k = 6$ corresponding to $\theta_k = 78.6°$.

(For $k = 7$ the diffracted ray will be grazing the wedge.)

5.131 The intensity of the central Fraunhofer maximum will be zero if the waves from successive grooves (not in the same plane) differ in phase by an odd multiple of π. Then, since the phase difference is

$$\delta = \frac{2\pi}{\lambda}(n-1)h$$

for the central ray, we have

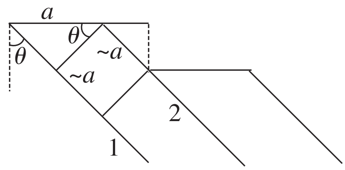
$$\frac{2\pi}{\lambda}(n-1)h = \left(k - \frac{1}{2}\right)2\pi \quad (\text{where } k = 1, 2, 3, \ldots)$$

or

$$h = \frac{\lambda}{n-1}\left(k - \frac{1}{2}\right)$$

The path difference between the rays 1 and 2 is approximately (neglecting terms of order θ^2).

$$a \sin\theta + a - na$$
$$= a\sin\theta - (n-1)a$$



Thus, for a maximum

$$a\sin\theta = \left(k' + \frac{1}{2}\right)\lambda = m\lambda$$

or $$a\sin\theta = \left(m + k' + \frac{1}{2}\right)\lambda$$

(where $k' = 0, 1, 2, \ldots$ and $m = 0, \pm 1, \pm 2, \ldots$).

The first maximum after the central minimum is obtained when $m + k' = 0$. We then get,

$$a\sin\theta_1 = \frac{1}{2}\lambda$$

5.132 When standing ultrasonic waves are sustained in the tank, they behave like a grating whose grating element is equal to wavelength of the ultrasonic, i.e.,

$$d = \frac{v}{\nu}$$

and v = velocity of ultrasonic. Thus, for maxima

$$\frac{v}{\nu}\sin\theta_m = m\lambda$$

On the other hand

$$f\tan\theta_m = m\Delta x$$

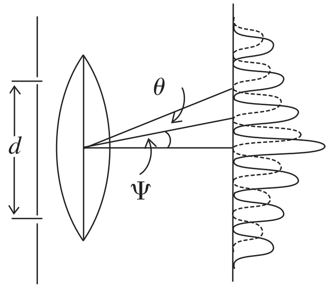
Assuming θ_m to be small (because $\lambda \ll \frac{v}{\nu}$), we get

$$\Delta x = \frac{f\tan\theta_m}{m} = \frac{f\tan\theta_m}{v/\nu\lambda\sin\theta_m} = \frac{\lambda\nu f}{v}$$

or $$v = \frac{\lambda\nu f}{\Delta x}$$

Putting the values $\lambda = 0.55\ \mu\text{m}$, $\nu = 4.7$ MHz, $f = 0.35$ m and $\Delta x = 0.60 \times 10^{-3}$ m, we get $v = 1.51$ km/s.

5.133 Each star produces its own diffraction pattern in the focal plane of the objective and these patterns superpose. The zeroeth order maxima of these patterns are separated by angle Ψ. As the distance d decreases the angle θ between the neighboring maxima in either diffraction pattern increases ($\sin\theta = \lambda/d$). When θ becomes equal to 2Ψ, the first deterioration of visibility occurs because the maxima of one system of fringes coincide with the minima of the other system. Thus, from the condition $\theta = 2\Psi$ and $\sin\theta = \lambda/d$, we get



$$\Psi = \frac{1}{2}\theta \approx \frac{\lambda}{2d} \quad \text{(radians)}$$

$$= 0.06'' \text{ (on substituting values)}$$

5.134 (a) For normal incidence, the maxima are given by

$$d\sin\theta = n\lambda$$

So, $$\sin\theta = n\frac{\lambda}{d} = n \times \frac{0.530}{1.500}$$

Clearly, $n \le 2$ as $\sin\theta > 1$ for $n = 3$. Thus, the highest order is $n = 2$. Then,

$$D = \frac{d\theta}{d\lambda} = \frac{k}{d\cos\theta} = \frac{k}{d}\frac{1}{\sqrt{1-(k\lambda/d)^2}}$$

Putting $k = 2$, $\lambda = 0.53$ μm, $d = 1.5$ μm $= 1500$ nm, we get

$$D = \frac{2}{1500}\frac{1}{\sqrt{1-\left(\frac{1.06}{1.5}\right)^2}} \times \frac{180}{\pi} \times 60 = 6.47 \text{ ang min/nm}$$

(b) We write the diffraction formula as

$$d(\sin\theta_0 + \sin\theta) = k\lambda$$

So, $$\sin\theta_0 + \sin\theta = k\frac{\lambda}{d}$$

Here $\theta_0 = 45°$ and $\sin\theta_0 = 0.707$, so

$$\sin\theta_0 + \sin\theta \le 1.707$$

Since $$\frac{\lambda}{d} = \frac{0.53}{1.5} = 0.353333$$

we see that $k \le 4$.

Thus, highest order corresponds to $k = 4$.

Now, as before $$D = \frac{d\theta}{d\lambda}$$

so, $$D = \frac{k}{d\cos\theta} = \frac{k/d}{\sqrt{1-(k\lambda/d - \sin\theta_0)^2}}$$

$$= 12.948 \text{ ang min/nm}$$

5.135 We have $$d \sin\theta = k\lambda$$

so, $$\frac{d\theta}{d\lambda} = D = \frac{k}{d\cos\theta} = \frac{\tan\theta}{\lambda}$$

5.136 For the second order principal maximum

$$d\sin\theta_2 = 2\lambda = k\lambda$$

or $$\frac{N\pi}{\lambda} d\sin\theta_2 = 2N\pi$$

Minima adjacent to this maximum occur at

$$\frac{N\pi}{\lambda} d\sin(\theta_2 \pm \Delta\theta) = (2N \pm 1)\pi$$

or $$d\cos\theta_2\, \Delta\theta = \frac{\lambda}{N}$$

Finally, angular width of the second principal maximum is

$$2\Delta\theta = \frac{2\lambda}{Nd\cos\theta_2}$$

$$= \frac{2\lambda}{Nd\sqrt{1-(k\lambda/d)^2}} = \frac{\tan\theta_2}{N}$$

$$= 11.019'' \text{ of arc (on substituting values)}$$

5.137 Using $$R = \frac{\lambda}{\delta\lambda} = kN = \frac{Nd\sin\theta}{\lambda}$$

$$= \frac{l\sin\theta}{\lambda}$$

or $$R \le \frac{l}{\lambda}$$

5.138 For the just resolved waves, the frequency difference

$$\delta\nu = \frac{c\delta\lambda}{\lambda} = \frac{c}{\lambda R} = \frac{c}{\lambda kN}$$

$$= \frac{c}{Nd\sin\theta} = \frac{1}{\delta t}$$

since $Nd\sin\theta$ is the path difference between waves emitted by the extremities of the grating.

5.139 It is given that $\delta\lambda = 0.050$ nm. So,

$$R = \frac{\lambda}{\delta\lambda} \simeq \frac{600}{0\cdot 05} = 12000 \text{ (nearly)}$$

On the other hand

$$d\sin\theta = k\lambda$$

Thus, $$\frac{l}{kN}\sin\theta = \lambda$$

where $l = 10^{-2}$ m is the width of the grating.

Hence, $$\sin\theta = 12000 \times \frac{\lambda}{l}$$

$$= 12000 \times 600 \times 10^{-7} = 0.72$$

or $$\theta = 46°$$

5.140 (a) We see that

$$N = 6.5 \times 10 \times 200 = 13000$$

Hence, to resolve lines with $\delta\lambda = 0.015$ nm and $\lambda \approx 670.8$ nm, we must have

$$R = \frac{670.8}{0.015} = 44720$$

Since $3N < R < 4N$, one must go to the fourth order to resolve the said components.

(b) We have $$d = \frac{1}{200}\text{ mm} = 5\ \mu\text{m}$$

So, $$\sin\theta = \frac{k\lambda}{d} = \frac{k \times 0.670}{5}$$

Since $\left|\sin\theta\right| \le 1,$ we must have

$$k \le 7.46$$

so,
$$k_{\max} = 7 \approx \frac{d}{\lambda}$$

Thus,
$$R_{\max} = k_{\max}N = 91000 \approx \frac{Nd}{\lambda} = \frac{l}{\lambda}$$

(where $l = 6.5$ cm is the grating width).

Finally,
$$\delta\lambda_{\min} = \frac{\lambda}{R_{\max}} = \frac{670}{91000} = 0.007 \text{ nm} = 7 \text{ pm} \approx \frac{\lambda^2}{l}$$

5.141 (a) Here,
$$R = \frac{\lambda}{\delta\lambda} = \frac{589.3}{0.6} = kN = 5N$$

so,
$$N = \frac{589.3}{3} = \frac{10^{-2}}{d}$$

$$d = \frac{3\times10^{-2}}{589.3} \text{ m} = 0.0509 \text{ mm}$$

(b) To resolve a doublet with $\lambda = 40.0$ nm and $\delta\lambda = 0.13$ nm in the third order, we must have

$$N = \frac{R}{3} = \frac{460}{3\times0.13} = 1179$$

This means that the grating is

$$Nd = 1179\times0.0509 = 60.03 \text{ mm wide} = 6 \text{ cm}$$

5.142 (a) From $d = \sin\theta = k\lambda,$ we get

$$\delta\theta = \frac{k\delta\lambda}{d\cos\theta}$$

On the other hand,
$$x = f\sin\theta$$

so,
$$\delta x = f\cos\theta\,\delta\theta = \frac{kf}{d}\delta\lambda$$

For $f = 0.80$ m, $\delta\lambda = 0.03$ nm and $d = 1/250$ mm, substituting we get

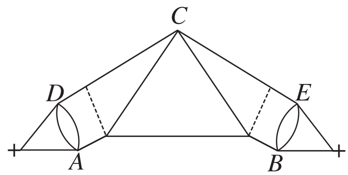
$$\delta x = \begin{cases} 6\,\mu\text{m} & \text{if } k=1 \\ 12\,\mu\text{m} & \text{if } k=2 \end{cases}$$

(b) Here $$N = 25 \times 250 = 6250$$

and $$\frac{\lambda}{\delta\lambda} = \frac{310.169}{0.03} = 10339 \cdot > N$$

and so to resolve we need $k = 2$ because $k = 1$ gives an R.P. of only 6250.

5.143 Suppose the incident light consists of two wavelengths λ and $\lambda + \delta\lambda$ which are just resolved by the prism. Then by Rayleigh's criterion, the maximum of the line of wavelength λ must coincide with the first minimum of the line of wavelength $\lambda + \delta\lambda$. Let us write both conditions in terms of the optical path differences for the extreme rays.



For the light of wavelength λ

$$bn - (DC + CE) = 0$$

For the light of wavelength $\lambda + \delta\lambda$

$$b(n + \delta n) - (DC + CE) = \lambda + \delta\lambda$$

because the path difference between extreme rays equals λ for the first minimum in a single slit diffraction (from the formula $a \sin\theta = \lambda$).

Hence, $$b\delta n \approx \lambda$$

and $$R = \frac{\lambda}{\delta\lambda} = b\left|\frac{\delta n}{\delta\lambda}\right| = b\left|\frac{dn}{\delta\lambda}\right|$$

5.144 (a) We know $$\frac{\lambda}{\delta\lambda} = R = b\left|\frac{dn}{d\lambda}\right| = 2Bb/\lambda^3$$

For $b = 5$ cm, $B = 0.01\ \mu\text{m}^2$, $\lambda_1 = 0.434\ \mu\text{m} = 5 \times 10^4\ \mu\text{m}$, we get

$$R_1 = 1.223 \times 10^4$$

For $\lambda_2 = 0.656\ \mu$m, we get

$$R_2 = 0.3542 \times 10^4$$

(b) To resolve the D-lines we require

$$R = \frac{5893}{6} = 982$$

Thus, $$982 = \frac{0.02 \times b}{(0.5893)^3}$$

$$b = \frac{982 \times (0.5893)^3}{0.02}\ \mu\text{m} = 1.005 \times 10^4\ \mu\text{m} = 1.005\text{ cm}$$

5.145 Given that
$$b\left|\frac{dn}{d\lambda}\right| = kN = 2 \times 10{,}000$$
or
$$b \times 0.10\ \mu\text{m}^{-1} = 2 \times 10^4$$
Thus,
$$b = 2 \times 10^5\ \mu\text{m} = 0.2\ \text{m} = 20\ \text{cm}$$

5.146 Resolving power of the objective is
$$\frac{D}{1.22\lambda} = \frac{5 \times 10^{-2}}{1.22 \times 0.55 \times 10^{-6}} = 7.45 \times 10^4$$
Let $(\Delta y)_{\text{min}}$ be the minimum distance between two points at a distance of 3.0 km which the telescope can resolve. Then,
$$\frac{(\Delta y)_{\text{min}}}{3 \times 10^3} = \frac{1.22\lambda}{D} = \frac{1}{7.45 \times 10^4}$$
or
$$(\Delta y)_{\text{min}} = \frac{3 \times 10^3}{7.45 \times 10^4} = 0.04026\ \text{m} = 4.03\ \text{cm}$$

5.147 The limit of resolution of a reflecting telescope is determined by diffraction from the mirror and obeys a formula similar to that from a refracting telescope. The limit of resolution is
$$\frac{1}{R} = \frac{1.22\lambda}{D} = \frac{(\Delta y)_{\text{min}}}{L}$$
(where L = distance between the Earth and the Moon = 384000 km).

Then, putting the values $\lambda = 0.55\ \mu$m, $D = 5.0$ m, we get
$$(\Delta y)_{\text{min}} = 51.6\ \text{m}$$

5.148 By definition, the magnification
$$\Gamma = \frac{\text{angle subtended by the image at the eye}}{\text{angle subtended by the object at the eye}} = \frac{\Psi'}{\Psi}$$
At the limit of resolution
$$\Psi = \frac{1.22\lambda}{D}$$
where D = diameter of the objective.

On the other hand, to be visible to the eye,
$$\Psi' \geq \frac{1.22\lambda}{d_0}$$

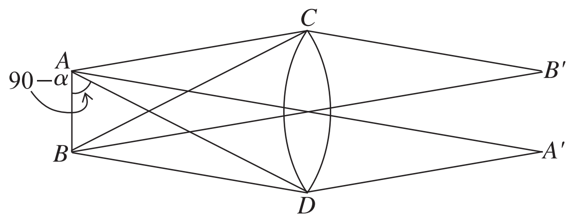
where d_0 = diameter of the pupil.

Thus, to avail of the resolution offered by the telescope we must have

$$\Gamma \geq \frac{1.22\lambda/d_0}{1.22\lambda/D} = \frac{D}{d_0}$$

Hence, $$\Gamma_{\min} = \frac{D}{d_0} = \frac{50 \text{ mm}}{4 \text{ mm}} = 12.5$$

5.149 Let A and B be two points in the field of a microscope which is represented by the lens CD. Let A' and B' be their image points which are at equal distances from the axis of the lens CD. Then all paths from A to A' are equal and the extreme difference of paths from A to B' is equal to



$$ADB' - ACB'$$
$$= AD + DB' - (AC + CB')$$
$$= AD + DB' - BD - DB' + BC + CB' - AC - CB'$$

(As $BD + DB' = BC + CB'$.)

$$= AD - BD + BC - AC$$
$$= 2AB \cos(90° - \alpha) = 2AB \sin\alpha$$

From the theory of diffraction by circular apertures, this distance must be equal to 1.22λ when B' coincides with the minimum of the diffraction due to A and A' with the minimum of the diffraction due to B. Thus

$$AB = \frac{1.22\lambda}{2\sin\alpha} = 0.61\frac{\lambda}{\sin\alpha}$$

Here 2α is the angle subtended by the objective of the microscope at the object.

Substituting values

$$AB = \frac{0.61 \times 0.55}{0.24} \text{ µm} = 1.40 \text{ µm}$$

5.150 Suppose $d_{\min}$ = minimum separation resolved by the microscope, Ψ = angle subtended at the eye by this object when the object is at the least distance of distinct vision l_0 (= 25 cm). The minimum angular separation resolved by the eye

$$\Psi' = \frac{1.22\lambda}{d_0}$$

From the previous problem

$$d_{\min} = \frac{0.61\lambda}{\sin\alpha}$$

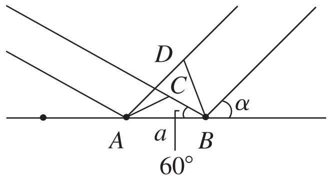
and
$$\Psi = \frac{d_{\min}}{l_0} = \frac{0.61\lambda}{l_0 \sin\alpha}$$

$$\Gamma = \text{magnifying power} = \frac{\text{angle subteneded at the eye by the image}}{\text{angle subtended at the eye by the object}}$$

When the object is at the least distance of distinct vision $\geqslant \frac{\Psi'}{\Psi}$.

Thus,
$$\Gamma_{\min} = 2\left(\frac{l_0}{d_0}\right)\sin\alpha = 2 \times \frac{25}{0.4} \times 0.24 = 30$$

5.151 Path difference is given by



$$BC - AD = a\,(\cos 60° - \cos\alpha)$$

For diffraction maxima

$$a\,(\cos 60° - \cos\alpha) = k\lambda$$

Since $\lambda = 2/5a$, we get

$$\cos\alpha = \frac{1}{2} - \frac{2}{5}k$$

If $k = -1$, $\quad \cos\alpha = \frac{1}{2} + \frac{2}{5} = 0.9$ and $\alpha = 26°$

If $k = 0$, $\quad \cos\alpha = \frac{1}{2} = 0.5$ and $\alpha = 60°$

If $k = 1$, $\quad \cos\alpha = \frac{1}{2} - \frac{2}{5} = 0.1$ and $\alpha = 84°$

If $k = 2$, $\quad \cos\alpha = \frac{1}{2} - \frac{4}{5} = -0.3$ and $\alpha = 107.5°$

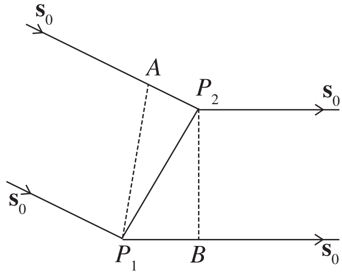
If $k = 3$, $\quad \cos\alpha = \frac{1}{2} - \frac{6}{5} = -0.7$ and $\alpha = 134.4°$

Other values of k are not allowed as they lead to $|\cos\alpha| > 1$.

5.152 We give here a simple derivation of the conditions for diffraction maxima, known as Laue equations. It is easy to see from the figure that the path difference between waves scattered by nearby scattering centres P_1 and P_2 is

$$P_2A - P_1B = \mathbf{r}\cdot\mathbf{s}_0 - \mathbf{r}\cdot\mathbf{s}$$

$$= \mathbf{r}\cdot(\mathbf{s}_0 - \mathbf{s}) = \mathbf{r}\cdot\mathbf{S}$$



$$a\cos\alpha = h\lambda$$

$$b\cos\beta = k\lambda$$

Here, cos α and cos β are the direction cosines of the ray with respect to the x- and y-axes of the two-dimensional crystal. So,

$$\cos\alpha = \frac{\Delta x}{\sqrt{(\Delta x)^2 + 4l^2}}$$

$$= \sin\left(\tan^{-1}\frac{\Delta x}{2l}\right) = 0.28735$$

Using $h = k = 2$, we get

$$a = \frac{40\times 2}{0.28735}\text{ pm} = 0.278\text{ nm}$$

Similarly, $$\cos\beta = \frac{\Delta y}{\sqrt{(\Delta y)^2 + 4l^2}} = \sin\left(\tan^{-1}\frac{\Delta y}{2l}\right) = 0.19612$$

Thus, $$b = \frac{80}{\cos\beta}\text{ pm} = 0.408\text{ nm}$$

5.153 Suppose α, β and γ are the angles between the direction of the diffraction maximum and the directions of the array along the periods a, b, and c, respectively (call them x-, y-, and z-axes). Then the value of these angles can be found from the following familiar conditions

$$a(1-\cos\alpha) = k_1\lambda$$

$$b\cos\beta = k_2\lambda \quad \text{and} \quad c\cos\gamma = k_3\lambda$$

where k_1, k_2, k_3 are integers.

(These formulas are, in effect, Laue equations, see any text book on modern physics.) Squaring and adding, we get, on using $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

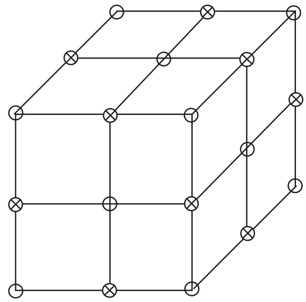
$$2 - 2\cos\alpha = \left[\left(\frac{k_1}{a}\right)^2 + \left(\frac{k_2}{b}\right)^2 + \left(\frac{k_3}{c}\right)^2\right]\lambda^2 = \frac{2k_1\lambda}{a}$$

Thus, $$\lambda = \frac{2k_1/a}{\left[(k_1/a)^2 + (k_2/a)^2 + (k_3/a)^2\right]}$$

Knowing a, b, c and the integers k_1, k_2, k_3, we can find α, β, γ as well as λ.

5.154 The unit cell of NaCl is shown in the figure. In an infinite crystal, there are four Na^+ and four Cl^- ions per unit cell. (Each ion on the middle of the edge is shared by four unit cells; each ion on the face centre by two unit cells, the ion in the middle of the cell by one cell only and finally each ion on the corner by eight unit cells.) Thus,

$$4\frac{M}{N_A} = \rho \cdot a^3$$



(where M = molecular weight of NaCl in grams = 58.5 g

N_A = Avogadro's number = 6.023×10^{23}).

Thus, $$\frac{1}{2}a = \sqrt{\frac{M}{2N_A\rho}} = 2.822 \text{ Å}$$

The natural facet of the crystal is one of the faces of the unit cell. The interplanar distance

$$d = \frac{1}{2}a = 2.822 \text{ Å}$$

Thus, $$2d\sin\alpha = 2\lambda$$

So, $$\lambda = d\sin\alpha = 2.822 \text{ Å} \times \frac{\sqrt{3}}{2} = 244 \text{ pm}$$

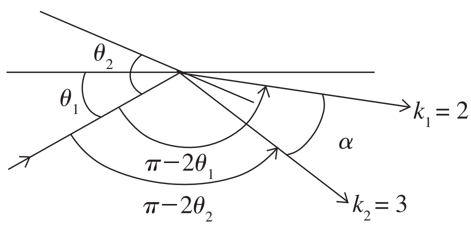
5.155 When the crystal is rotated, the incident monochromatic beam is diffracted from a given crystal plane of interplanar spacing d, whenever in the course of rotation the value of θ satisfies the Bragg equation.

We have the equations

$$2d\sin\theta_1 = k_1\lambda \quad \text{and} \quad 2d\sin\theta_2 = k_2\lambda$$

But, $$\pi - 2\theta_1 = \pi - 2\theta_2 + \alpha \quad \text{or} \quad 2\theta_1 = 2\theta_2 - \alpha$$

So, $$\theta_2 = \theta_1 + \frac{\alpha}{2}$$



Thus,

$$2d\left\{\sin\theta_1\cos\frac{\alpha}{2} + \cos\theta_1\sin\frac{\alpha}{2}\right\} = k_2\lambda$$

Hence, $$2d\sin\frac{\alpha}{2}\cos\theta_1 = \left(k_2 - k_1\cos\frac{\alpha}{2}\right)\lambda$$

Also $$2d \sin\frac{\alpha}{2} \sin\theta_1 = k_1\lambda \sin\frac{\alpha}{2}$$

Squaring and adding $$2d \sin\frac{\alpha}{2} = \left(k_1^2 + k_2^2 - 2k_1k_2 \cos\frac{\alpha}{2}\right)^{1/2} \lambda$$

Hence, $$d = \frac{\lambda}{2\sin\alpha/2}\left(k_1^2 + k_2^2 - 2k_1k_2 \cos\frac{\alpha}{2}\right)^{1/2}$$

Substituting $\alpha = 60°$, $k_1 = 2$, $k_2 = 3$, $\lambda = 174$ pm, we get

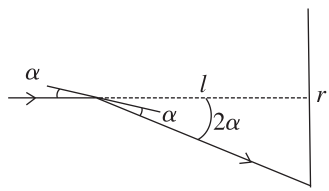
$$d = 281 \text{ pm} = 2.81 \text{ Å}$$

(Lattice parameters are typically in Å′s and not in fractions of a pm so the answer is not 0.281 pm as given in the book.)

5.156 In a polycrystalline specimen, microcrystals are oriented at various angles with respect to one another. The microcrystals which are oriented at certain special angles with respect to the incident beam produce diffraction maxima that appear as rings.

The radius of these rings are given by

$$r = l \tan 2\alpha$$



Bragg's law gives

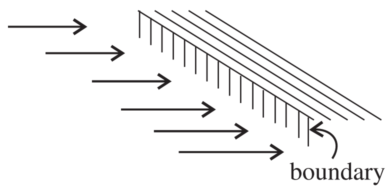
$$2d \sin\alpha = k\lambda$$

In this case $k = 2$, $d = 155$ pm, $\lambda = 17.8$ pm, so

$$\alpha = \sin^{-1}\frac{17.8}{155} = 6.6° \quad \text{and} \quad r = 3.52 \text{ cm}$$

5.4 Polarization of Light

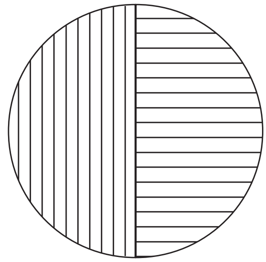
5.157 Natural light can be considered an incoherent mixture of two plane polarized light rays of intensity $I_0/2$ with mutually perpendicular planes of vibrations. The screen consisting of the two polaroid half-planes acts as an opaque half-screen for one or the other of these light waves. The resulting diffraction pattern has the alteration in intensity (in the illuminated region) characteristic of a straight edge on both sides of the boundary.



At the boundary the intensity due to either component is $(I_0/2)/4$ and the total intensity is $I_0/4$. (Recall that when light of intensity I_0 is incident on a straight edge, the illuminance in front of the edge is $I_0/4$.)

5.158 (a) Assume first that there is no polaroid and the amplitude due to the entire hole which extends over the first Fresnel zone is A_1. Then, we know, as usual, $I_0 = A_1^2/4$.

When the polaroid is introduced as shown above, each half transmits only the corresponding polarized light. If the full hole were covered by one polaroid, the amplitude transmitted will be $(A_1/\sqrt{2})$.



Therefore, the amplitude transmitted in the present case will be $A_1/2\sqrt{2}$ through either half. Since these transmitted waves are polarized in mutually perpendicular planes, the total intensity will be

$$\left(\frac{A_1}{2\sqrt{2}}\right)^2 + \left(\frac{A_1}{2\sqrt{2}}\right)^2 = \frac{A_1^2}{4} = I_0$$

(b) We interpret the problem to mean that the two polaroid pieces are separated along the circumference of the circle limiting the first half of the Fresnel zone. (This however is inconsistent with the polaroids being identical in shape; however no other interpretation makes sense.)

From Problem 5.103 and the previous problems we see that the amplitudes of the waves transmitted through the two parts is

$$\frac{A_1}{2\sqrt{2}}(1+i) \quad \text{and} \quad \frac{A_1}{2\sqrt{2}}(1-i)$$

and the intensity is

$$\left|\frac{A_1^2}{2\sqrt{2}}(1+i)\right|^2 + \left|\frac{A_1}{2\sqrt{2}}(1-i)\right|^2$$

$$= \frac{A_1^2}{2} = 2I_0$$

5.159 When the polarizer rotates with angular velocity ω, its instantaneous principal direction makes angle ωt from a reference direction which we choose to be along the direction of vibration of the plane polarized incident light. The transmitted flux at this instant is

$$\Phi_0 \cos^2 \omega t$$

and the total energy passing through the polarizer per revolution is

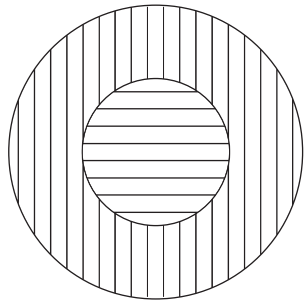
$$\int_0^T \Phi_0 \cos^2 \omega t \, dt \quad \left(\text{where } T = \frac{2\pi}{\omega}\right)$$

$$= \Phi_0 \frac{\pi}{\omega} = 0.6 \text{ mJ}$$

5.160 Let I_0 = intensity of the incident beam. Then the intensity of the beam transmitted through the first Nicol prism is

$$I_1 = \frac{1}{2} I_0$$

and through the second prism is



$$I_2 = \left(\frac{1}{2} I_0\right) \cos^2 \varphi$$

Then, through the N^{th} prism it will be

$$I_N = I_{N-1} \cos^2 \varphi$$

$$= \frac{1}{2} I_0 \cos^{2(N-1)} \varphi$$

Hence, fraction transmitted

$$\frac{I_N}{I_0} = \eta = \frac{1}{2} \cos^{(2N-1)} \varphi = 0.12 \quad (\text{for } N = 6)$$

and $\qquad \varphi = 30°$

5.161 When natural light is incident on the first polaroid, the fraction transmitted will be $1/2\tau$ (only the component polarized parallel to the principal direction of the polaroid will go).

The emergent light will be plane polarized and on passing through the second polaroid will be polarized in a different direction (corresponding to the principal direction of the second polaroid) and the intensity will have decreased further by $\tau \cos^2 \varphi$.

In the third polaroid the direction of polarization will again have to change by φ, thus only a fraction $\tau \cos^2 \varphi$ will go through.

Finally $\qquad I = I_0 \times \frac{1}{2} \tau^3 \cos^4 \varphi$

Thus, the intensity will have decreased. So,

$$\frac{I_0}{I} = \frac{2}{\tau^3 \cos^4 \varphi} = 60.2 \text{ times}$$

where $\tau = 0.81$, $\varphi = 60°$.

5.162 Suppose the partially polarized light consists of natural light of intensity I_1 and plane polarized light of intensity I_2 with direction of vibration parallel to, say, x-axis.

Then when a polaroid is used to transmit it, the light transmitted will have a maximum intensity $(1/2)I_1 + I_2$, when the principal direction of the polaroid is parallel to x-axis, and will have a minimum intensity $(1/2)I_1$ when the principal direction is perpendicular to x-axis.

Thus,
$$P = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{I_2}{I_1 + I_2}$$

So,
$$\frac{I_2}{I_1} = \frac{P}{1-P} = \frac{0.25}{0.75} = \frac{1}{3}$$

5.163 If, as above, I_1 is the intensity of natural component and I_2 is the intensity of plane polarized component, then

$$I_{\max} = \frac{1}{2}I_1 + I_2$$

and
$$I = \frac{I_{\max}}{\eta} = \frac{1}{2}I_1 + I_2\cos^2\varphi$$

So,
$$I_2 = I_{\max}\left(1 - \frac{1}{\eta}\right)\operatorname{cosec}^2\varphi$$

$$I_1 = 2I_{\max}\left[1 - \left(1 - \frac{1}{\eta}\right)\operatorname{cosec}^2\varphi\right] = \frac{2I_{\max}}{\sin^2\varphi}\left[\frac{1}{\eta} - \cos^2\varphi\right]$$

Then,
$$P = \frac{I_2}{I_1 + I_2} = \frac{1 - 1/\eta}{2(1/\eta - \cos^2\varphi) + 1 - 1/\eta} = \frac{\eta - 1}{1 - \eta\cos 2\varphi}$$

On putting $\eta = 3.0$, $\varphi = 60°$, we get

$$P = \frac{2}{1 + 3 \times 1/2} = \frac{4}{5} = 0.8$$

5.164 Let us represent the natural light as a sum of two mutually perpendicular components, both with intensity I_0. Suppose that each polarizer transmits a fraction α_1 of the light with oscillation plane parallel to the principal direction of the polarizer and a fraction α_2 with oscillation plane perpendicular to the principal of the polarizer. Then, the intensity of light transmitted through the two polarizers is

$$I_{\parallel} = \alpha_1^2 I_0 + \alpha_2^2 I_0$$

when their principal directions are parallel and

$$I_{\perp} = \alpha_1\alpha_2 I_0 + \alpha_2\alpha_1 I_0 = 2\alpha_1\alpha_2 I_0$$

when they are crossed. But,

$$\frac{I_{\perp}}{I_{\parallel}} = \frac{2\alpha_1\alpha_2}{\alpha_1^2 + \alpha_2^2} = \frac{1}{\eta}$$

So,
$$\frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2} = \sqrt{\frac{\eta - 1}{\eta + 1}}$$

(a) Now, the degree of polarization produced by either polarizer when used singly is

$$P_0 = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2}$$

(assuming, that $\alpha_1 > \alpha_2$).

Thus, $$P_0 = \sqrt{\frac{\eta - 1}{\eta + 1}} = \sqrt{\frac{9}{11}} = 0.905$$

(b) When both the polarizers are used with their principal directions parallel, the transmitted light, when analyzed, has the maximum intensity, $I_{max} = \alpha_1^2 I_0$ and the minimum intensity, $I_{min} = \alpha_2^2 I_0$.

So, $$P = \frac{\alpha_1^2 - \alpha_2^2}{\alpha_1^2 + \alpha_2^2} = \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2} \cdot \frac{(\alpha_1 + \alpha_2)^2}{\alpha_1^2 + \alpha_2^2}$$

$$= \sqrt{\frac{\eta - 1}{\eta + 1}}\left(1 + \frac{2\alpha_1\alpha_2}{\alpha_1^2 + \alpha_2^2}\right)$$

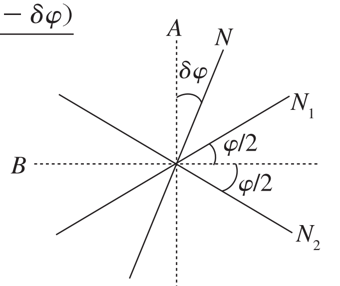
$$= \sqrt{\frac{\eta - 1}{\eta + 1}}\left(1 + \frac{1}{\eta}\right) = \frac{\sqrt{\eta^2 - 1}}{\eta} = \sqrt{1 - \frac{1}{\eta^2}} = 0.995$$

5.165 If the principal direction N of the Nicol prism is along A or B, the intensity of light transmitted is the same whether the light incident is one with oscillation plane N_1 or N_2. If N makes an angle $\delta\varphi$ with A as shown in the figure, then the fractional difference in intensity transmitted (when the light incident is N_1 or N_2) is

$$\left(\frac{\Delta I}{I}\right)_A = \frac{\cos^2(90° - \varphi/2 - \delta\varphi) - \cos^2(90° + \varphi/2 - \delta\varphi)}{\cos^2(90° - \varphi/2)}$$

$$= \frac{\sin^2(\varphi/2 + \delta\varphi) - \sin^2(\varphi/2 - \delta\varphi)}{\sin^2\varphi/2}$$

$$= \frac{2\sin\varphi/2 \cdot 2\cos\varphi/2\delta\varphi}{\sin^2\varphi/2} = 4\cot\frac{\varphi}{2}\delta\varphi$$



If N makes an angle $\delta\varphi$ ($\ll \varphi$) with B, then

$$\left(\frac{\Delta I}{I}\right)_B = \frac{\cos^2(\varphi/2 - \delta\varphi) - \cos^2(\varphi/2 + \delta\varphi)}{\cos^2\varphi/2} = \frac{2\cos\varphi/2 \cdot 2\sin\varphi/2\delta\varphi}{\cos^2\varphi/2} = 4\tan\frac{\varphi}{2}\delta\varphi$$

Thus, $$\eta = \frac{(\Delta I/I)_A}{(\Delta I/I)_B} = \cot^2\frac{\varphi}{2}$$

or $$\varphi = 2\tan^{-1}\frac{1}{\sqrt{\eta}}$$

This gives $\varphi = 11.4°$ for $\eta = 100$.

5.166 Fresnel equations read

$$I'_{\perp} = I_{\perp}\frac{\sin^2(\theta_1 - \theta_2)}{\sin^2(\theta_1 + \theta_2)} \quad \text{and} \quad I'_{\parallel} = I_{\parallel}\frac{\tan^2(\theta_1 - \theta_2)}{\tan^2(\theta_1 + \theta_2)}$$

At the boundary between vacuum and a dielectric $\theta_1 \neq \theta_2$, since by Snell's law $\sin\theta_1 = n\sin\theta_2$.

Thus $I'_{\perp}/I_{\perp}$ cannot be zero. However, if $\theta_1 + \theta_2 = 90°$, $I'_{\parallel} = 0$ and the reflected light is polarized in this case. The condition for this is

$$\sin\theta_1 = n\sin\theta_2 = n\sin(90° - \theta_1)$$

or
$$\tan\theta_1 = n\theta_2$$

is called Brewster's angle.

The angle between the reflected and refracted light is 90° in this case.

5.167 (a) From Fresnel's equations

$$I'_{\perp} = I_{\perp}\frac{\sin^2(\theta_1 - \theta_2)}{\sin^2(\theta_1 + \theta_2)}$$

and
$$I'_{\parallel} = 0 \text{ at Brewster's angle}$$

$$I'_{\perp} = I_{\perp}\sin^2(\theta_1 - \theta_2)$$

$$= \frac{1}{2}I(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)^2$$

Now,
$$\tan\theta_1 = n, \quad \sin\theta_1 = \frac{n}{\sqrt{n^2+1}}$$

$$\cos\theta_1 = \frac{1}{\sqrt{n^2+1}}, \quad \sin\theta_2 = \cos\theta_1$$

and
$$\cos\theta_2 = \sin\theta_1$$

Therefore,
$$I'_{\perp} = \frac{1}{2}I\left(\frac{n^2-1}{n^2+1}\right)^2$$

Thus reflection coefficient

$$\rho = \frac{I'_{\perp}}{I}$$

$$= \frac{1}{2}\left(\frac{n^2-1}{n^2+1}\right)^2 = 0.074$$

(on putting $n = 1.5$).

(b) For the refracted light

$$I''_{\perp} = I_{\perp} - I'_{\perp} = \frac{1}{2} I \left\{ 1 - \left(\frac{n^2 - 1}{n^2 + 1} \right)^2 \right\}$$

$$= \frac{1}{2} I \frac{4n^2}{(n^2 + 1)^2}$$

and $$I'_{\parallel} = \frac{1}{2} I$$

at the Brewster's angle.

Thus, the degree of polarization of the refracted light is

$$P = \frac{I''_{\parallel} - I''_{\perp}}{I''_{\parallel} + I''_{\perp}} = \frac{(n^2 + 1)^2 - 4n^2}{(n^2 + 1)^2 + 4n^2}$$

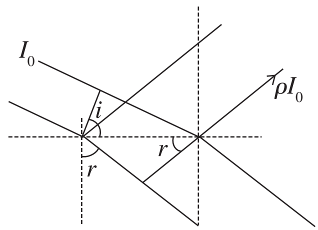
$$= \frac{(n^2 - 1)^2}{2(n^2 + 1)^2 - (n^2 - 1)^2} = \frac{\rho}{1 - \rho}$$

On putting $\rho = 0.074$, we get $P = 0.080$.

5.168 The energy transmitted is, by conservation of energy, the difference between incident energy and the reflected energy. However the intensity is affected by the change of the cross section of the beam by refraction. Let A_i, A_r, A_t be the cross sections of the incident, reflected and transmitted beams, respectively.

Then $$A_i = A_r$$

and $$A_t = A_i \frac{\cos r}{\cos i}$$



But at Brewster's angle $r = 90 - i$, so

$$A_t = A_i \tan i = nA_i$$

Thus, $$I_t = \frac{(1 - \rho)I_0}{n} = 0.721 I_0$$

5.169 The amplitude of the incident component whose oscillation vector is perpendicular to the plane of incidence is

$$A_{\perp} = A_0 \sin \varphi$$

and similarly $$A_{\parallel} = A_0 \cos \varphi$$

Then, $$I'_{\perp} = I_0 \frac{\sin^2(\theta_1 - \theta_2)}{\sin^2(\theta_1 + \theta_2)} \sin^2 \varphi$$

$$= I_0 \left[\frac{\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2}{\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2} \right]^2 \sin^2\varphi$$

$$= I_0 \left[\frac{n^2 - 1}{n^2 + 1} \right]^2 \sin^2\varphi$$

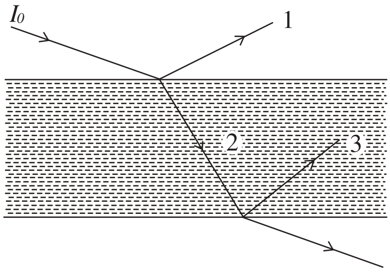
Putting $n = 1.33$ for water, we get $\rho = 0.0386$.

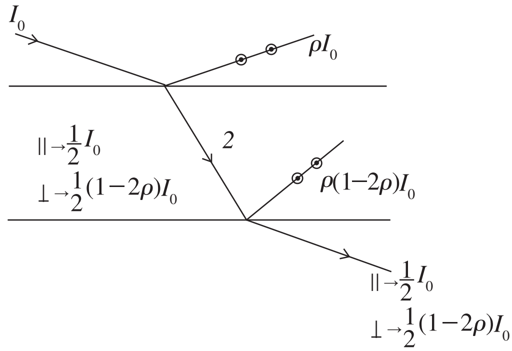
5.170 Since natural light is incident at the Brewster's angle, the reflected light 1 is completely polarized and $P_1 = 1$.

Similarly, the ray 2 is incident on glass air surface at Brewster's angle ($\tan^{-1} 1/n$) so 3 is also completely polarized. Thus, $P_3 = 1$.

Now as in Problem 5.167b

$$P_2 = \frac{\rho}{1 - \rho} = 0.087 \quad (\text{if } \rho = 0.080)$$





Finally, as shown in the figure

$$P_4 = \frac{1/2 - 1/2(1 - 2\rho)^2}{1/2 + 1/2(1 - 2\rho)^2} = \frac{2\rho(1 - \rho)}{1 - 2\rho(1 - \rho)} = 0.173$$

5.171 (a) In this case, from Fresnel's equations

$$I'_\perp = I_\perp \frac{\sin^2(\theta_1 - \theta_2)}{\sin^2(\theta_1 + \theta_2)}$$

we get $$I_1 = \left(\frac{n^2 - 1}{n^2 + 1} \right)^2 I_0 = \rho I_0 \ \ (\text{say})$$

Then $$I_2 = (1 - \rho) I_0, \ I_3 = \rho(1 - \rho) I_0$$

(Here ρ is invariant under the substitution $n \to 1/n$.)

Thus, $$I_4 = (1-\rho)^2 I_0 = \frac{16n^4}{(n^2+1)^4} I_0 = 0.726 I_0$$

(b) Suppose ρ' is the coefficient of reflection of the component of light whose electric vector oscillates at right angles to the incidence plane. From Fresnel's equations

$$\rho' = \left(\frac{n^2-1}{n^2+1}\right)^2$$

Then, in the transmitted beam we have a partially polarized beam which is a superposition of two (|| and $\perp$) components with intensities

$$\frac{1}{2} I_0 \quad \text{and} \quad \frac{1}{2} I_0 (1-\rho')^2$$

Thus, $$P = \frac{1-(1-\rho')^2}{1+(1-\rho')^2} = \frac{(n^2+1)^4 - 16n^4}{(n^2+1)^4 + 16n^4} = \frac{1-0.726}{1+0.726} \approx 0.158$$

5.172 (a) When natural light is incident on a glass plate at Brewster's angle, the transmitted light has

$$I''_{\parallel} = \frac{I_0}{2} \quad \text{and} \quad I''_{\perp} = \frac{16n^4}{(n^2+1)^4} \frac{I_0}{2} = \frac{\alpha^4 I_0}{2}$$

where I_0 is the incident intensity (see Problem 5.171a).
After passing through the second plate, we find

$$I''''_{\parallel} = \frac{1}{2} I_0 \quad \text{and} \quad I''''_{\perp} = (\alpha^4)^2 \frac{1}{2} I_0$$

Thus, after N plates $$I^{\text{trans}}_{\parallel} = \frac{1}{2} I_0$$

and $$I^{\text{trans}}_{\perp} = \alpha^{4N} \frac{1}{2} I_0$$

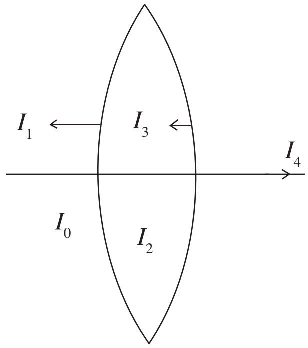
Hence, $$P = \frac{1-\alpha^{4N}}{1+\alpha^{4N}} \quad \left(\text{where } \alpha = \frac{2n}{1+n^2}\right)$$

(b) We have $\alpha^4 = 0.726$ for $n = 3/2$.

Thus, $P(N = 1) = 0.158$, $P(N = 2) = 0.310$ $P(N = 5) = 0.663$ and $P(N = 10) = 0.922$

5.173 (a) We decompose the natural light into two components with intensity $I_{\parallel} = I_0/2 = I_{\perp}$ where first component has its electric vector oscillating parallel to the plane of incidence and second component has the same perpendicular to it. By Fresnel's equations for normal incidence

$$\frac{I'_\perp}{I_\perp} = \lim_{\theta_1 \to 0} \frac{\sin^2(\theta_1 - \theta_2)}{\sin^2(\theta_1 + \theta_2)} = \lim_{\theta_1 \to 0} \left(\frac{\theta_1 - \theta_2}{\theta_1 + \theta_2}\right)^2 = \left(\frac{n-1}{n+1}\right)^2 = \rho$$



Similarly, $\quad \dfrac{I'_{||}}{I_{||}} = \rho = \left(\dfrac{n-1}{n+1}\right)^2$

Thus, $\quad \dfrac{I'}{I} = \rho = \left(\dfrac{0.5}{2.5}\right)^2 = \dfrac{1}{25} = 0.040$

(b) The reflected light at the first surface has the intensity

$$I_1 = \rho I_0$$

Then the transmitted light has the intensity

$$I_2 = (1 - \rho)I_0$$

At the second surface where light emerges from glass into air, the reflection coefficient is again ρ because ρ is invariant under the substitution $n \to 1/n$.

Thus, $\quad I_3 = \rho(1-\rho)I_0$ and $I_4 = (1-\rho)^2 I_0$

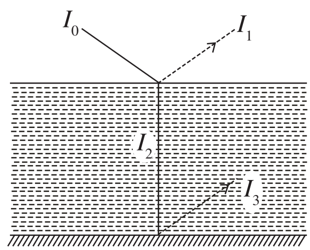
For N lenses the loss in luminous flux is then

$$\frac{\Delta\Phi}{\Phi} = 1 - (1-\rho)^{2N} = 0.335 \quad (\text{for } N = 5)$$

5.174 Suppose the incident light can be decomposed into waves with intensity $I_{||}$ and $I_\perp$ with oscillations of the electric vectors parallel and perpendicular to the plane of incidence.

For normal incidence, we have from Fresnel equations

$$I'_\perp = I_\perp \left(\frac{\theta_1 - \theta_2}{\theta_1 + \theta_2}\right)^2 \to I_\perp \left(\frac{n-1}{n+1}\right)^2$$



(where we have used $\sin\theta \approx \theta$ for small θ).

Similarly, $\quad I'_{||} = I_{||}\left(\dfrac{n'-1}{n'+1}\right)^2$

Then the refracted wave will be

$$I''_{||} = I_{||}\frac{4n'}{(n'+1)^2} \quad \text{and} \quad I''_\perp = I_\perp \frac{4n'}{(n'+1)^2}$$

At the interface with glass

$$I'''_\perp = I''_\perp \left(\frac{n'-n}{n'+n}\right)^2 \quad (\text{similarly for } I'''_{||})$$

we see that

$$\frac{I'_\perp}{I_\perp} = \frac{I'''_\perp}{I''_\perp} \quad (\text{if } n' = \sqrt{n}, \text{ similarly for } \| \text{ component})$$

This shows that the light reflected as a fraction of the incident light is the same on the two surfaces if $n' = \sqrt{n}$.

Note: *The statement of the problem given in the book is incorrect. Actual amplitudes are not equal; only the reflectance is equal.*

5.175 Here $\theta_1 = 45°$. So, we have

$$\sin\theta_2 = \frac{1}{\sqrt{2}} \times \frac{1}{n} = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3} = 0.4714$$

$$\theta_2 = \sin^{-1} 0.4714 = 28.1°$$

Hence,

$$I'_\perp = I_\perp \frac{\sin^2(\theta_1 - \theta_2)}{\sin^2(\theta_1 + \theta_2)}$$

$$= \frac{1}{2} I_0 \left(\frac{\sin 16.9°}{\sin 73.1°}\right)^2 = \frac{1}{2} I_0 \times 0.0923$$

Similarly,

$$I'_\| = \frac{1}{2} I_0 \left(\frac{\tan 16.9}{\tan 73.1}\right)^2 = \frac{1}{2} I_0 \times 0.0085$$

(a) Degree of polarization P of the reflected light

$$P = \frac{0.0838}{0.1008} = 0.831$$

(b) By conservation of energy

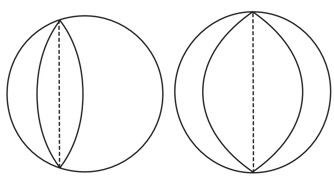
$$I''_\perp = \frac{1}{2} I_0 \times 0.9077$$

$$I''_\| = \frac{1}{2} I_0 \times 0.9915$$

Thus,

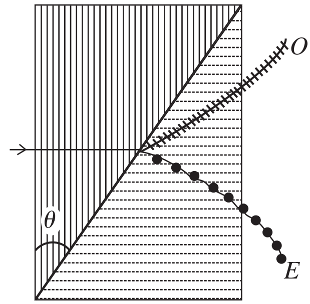
$$P = \frac{0.0838}{1.8982} = 0.044$$

5.176 The wave surface of uniaxial crystal consists of two sheets of which one is a sphere while the other is an ellipsoid of revolution. The optic axis is the line joining the points of contact.



To make the appropriate Huygen's construction we must draw the relevant section of the wave surface inside the crystal and determine the directions of the ordinary and extraordinary rays. The result is as shown in Fig. 42 (a, b and c) of the answer sheet.

5.177 In a uniaxial crystal, an unpolarized beam of light (or even a polarized one) splits up into *O* (for ordinary) and *E* (for extraordinary) light waves. The direction of vibration in the *O* and *E* waves are most easily specified in terms of the *O* and *E* principal planes. The principal plane of the ordinary wave is defined as the plane containing the *O* ray and the optic axis. Similarly, the principal plane of the *E* wave is the plane containing the *E* ray and the optic axis. In terms of these planes the following is true:



The *O* vibrations are perpendicular to the principal of the *O* ray while the *E* vibrations are in the principal plane of the *E* ray.

When we apply this definition to the Wollaston prism we find the following:

When unpolarized light enters from the left, the *O* and *E* waves travel in the same direction but with different speeds. The *O* ray on the left has its vibrations normal to the plane of the paper and it becomes *E* ray on crossing the diagonal boundary of the two prisms, similarly the *E* ray on the left becomes *O* ray on the right. In this case Snell's law is applicable only approximately. The two rays are incident on the boundary at an angle θ and in the right prism the ray which we have called *O* ray on the right emerges at

$$\sin^{-1}\left(\frac{n_E}{n_O}\sin\theta\right) = \sin^{-1}\left(\frac{1.658}{1.486}\times\frac{1}{2}\right) = 33.91°$$

where we have used

$$n_E = 1.1658,\ n_O = 1.486 \text{ and } \theta = 30°$$

Similarly, the *E* ray on the right emerges within the prism at

$$\sin^{-1}\left(\frac{n_O}{n_E}\sin\theta\right) = 26.62°$$

This means that the *O* ray is incident at the boundary between the prism and air at

$$33.91 - 30° = 3.91°$$

and will emerge into air with a deviation of

$$\sin^{-1} n_O \sin 3.91° = \sin^{-1}(1.658 \sin 3.91°) = 6.49°$$

The E ray will emerge with an opposite deviation of

$$\sin^{-1}(n_E \sin(30° - 26.62°))$$
$$= \sin^{-1}(1.486 \sin 3.38°) = 5.03°$$

Hence,
$$\delta \approx 6.49° + 5.03° = 11.52°$$

This result is accurate to first order in $(n_E - n_O)$ because Snell's law holds when $n_E = n_O$.

5.178 The wave is moving in the direction of z-axis.

(a) Here $E_x = E\cos(\omega t - kz)$ and $E_y = E\sin(\omega t - kz)$

or
$$\frac{E_x^2}{E^2} + \frac{E_y^2}{E^2} = 1$$

So the tip of the electric vector moves along a circle. For the right-handed coordinate system, this represents circular anticlockwise polarization when observed towards the incoming wave.

(b) Here, $E_x = E\cos(\omega t - kz)$ and $E_y = E\cos\left(\omega t - kz + \frac{\pi}{4}\right)$

or
$$\frac{E_y}{E} = \frac{1}{\sqrt{2}}\cos(\omega t - kz) - \frac{1}{\sqrt{2}}\sin(\omega t - kz)$$

or
$$\left(\frac{E_y}{E} - \frac{1}{\sqrt{2}}\frac{E_x}{E}\right)^2 = \frac{1}{2}\left(1 - \frac{E_x^2}{E^2}\right)$$

or
$$\frac{E_y^2}{E^2} + \frac{E_x^2}{E^2} - \sqrt{2}\,\frac{E_y E_x}{E^2} = \frac{1}{2}$$

This is clearly an ellipse. By comparing with the previous case (compare the phase of E_y in the two cases) we see this represents elliptical clockwise polarization when viewed towards the incoming wave.

We write the equations as

$$E_x + E_y = 2E\cos\left(\omega t - kz + \frac{\pi}{8}\right)\cos\frac{\pi}{8}$$

$$E_x - E_y = +\,2E\sin\left(\omega t - kz + \frac{\pi}{8}\right)\sin\frac{\pi}{8}$$

Thus,
$$\left(\frac{E_x + E_y}{2E\cos\pi/8}\right)^2 + \left(\frac{E_x - E_y}{2E\sin\pi/8}\right)^2 = 1$$

Since $\cos\pi/8 > \sin\pi/8$, the major axis is in the direction of the straight line $y = x$.

(c) Here, $E_x = E\cos(\omega t - kz)$

and $E_y = E\cos(\omega t - kz + \pi) = -E\cos(\omega t - kz)$

Thus, the tip of the electric vector traces the curve

$$E_y = -E_x$$

which is a straight line ($y = -x$). It corresponds to plane polarization.

5.179 For quartz

$$\left.\begin{aligned} n_E &= 1.553 \\ n_O &= 1.544 \end{aligned}\right\} \text{ for } \lambda = 589 \text{ nm}$$

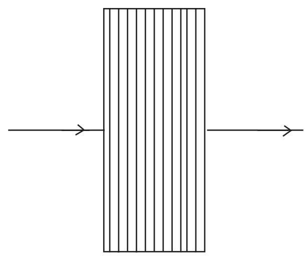
In a quartz plate cut parallel to its optic axis, plane polarized light incident normally from the left divides itself into O and E waves which move in the same direction with different speeds and as a result acquire a phase difference. This phase difference is

$$\delta = \frac{2\pi}{\lambda}(n_E - n_O)d$$

where d = thickness of the plate. In general, this makes the emergent light elliptically polarized.

(a) For emergent light to experience only rotation of polarization plane

$$\delta = (2k+1)\pi \quad (\text{where } k = 0, 1, 2, 3 \ldots)$$



For this
$$d = (2k+1)\frac{\lambda}{2(n_E - n_O)}$$
$$= (2k+1)\frac{0.589}{2 \times 0.009}\ \mu\text{m}$$
$$= (2k+1)\frac{0.589}{18}\ \text{mm}$$

The maximum value of $(2k + 1)$ for which this is less than 0.50 is obtained from

$$\frac{0.50 \times 18}{0.589} = 15.28$$

Then we must take

$$k = 7 \text{ and } d = 15 \times \frac{0.589}{18} = 0.4908 \text{ mm}$$

(b) For circular polarization

$$\delta = \frac{\pi}{2} \ (\text{modulo } \pi)$$

i.e.,
$$\delta = (4k+1)\frac{\pi}{2}$$

So, $$d = (4k+1)\frac{\lambda}{4(n_E - n_O)} = (4k+1)\frac{0.589}{36}$$

Now $$\frac{0.50 \times 36}{0.589} = 30.56$$

The nearest integer less than this which is of the form $4k + 1$ is 29 for $k = 7$. For this value of k, $d = 0.4749$ mm.

5.180 As in the previous problem, the quartz plate introduces a phase difference δ between the O and E components. When $\delta = \pi/2$ (modulo π), the resultant wave is circularly polarized. In this case intensity is independent of the rotation of the rear prism. Now,

$$\delta = \frac{2\pi}{\lambda}(n_E - n_O)d$$
$$= \frac{2\pi}{\lambda} 0.009 \times 0.5 \times 10^{-3} \text{ m}$$
$$= \frac{9\pi}{\lambda} \quad \text{(here } \lambda \text{ is in } \mu\text{m)}$$

For $\lambda = 0.50$ μm, $\delta = 18\pi$. The relevant values of δ have to be chosen in the form

$$\left(k + \frac{1}{2}\right)\pi$$

For $k = 17, 16, 15$, we get $\lambda = 0.5143$ μm, 0.5435 μm and 0.5806 μm.

These are the values of λ which lie between 0.50 μm and 0.60 μm.

5.181 As in the previous two problems, the quartz plate will introduce a phase difference δ. The light on passing through the plate will remain plane polarized only for $\delta = 2k\pi$ or $(2k + 1)\pi$. In the latter case the plane of polarization of the light incident on the plate will be rotated by 90°, so light passing through the analyzer (which was originally crossed) will be maximum. Thus, dark bands will be observed only for those λ for which $\delta = 2k\pi$.

Now $$\delta = \frac{2\pi}{\lambda}(n_E - n_O)d$$
$$= \frac{2\pi}{\lambda} \times 0.009 \times 1.5 \times 10^{-3} \text{ m}$$
$$= \frac{27\pi}{\lambda} \quad \text{(here } \lambda \text{ is in } \mu\text{m)}$$

For $\lambda = 0.55$, we get $\delta = 49.09\pi$.

Choosing $\delta = 48\pi, 46\pi, 42\pi$, we get $\lambda = 0.5625$ μm, 0.5870 μm, 0.6136 μm and 0.6429 μm. These are the only values of λ between 0.55 μm and 0.66 μm. Thus there are four bands.

5.182 Here,
$$\delta = \frac{2\pi}{\lambda} \times 0.009 \times 0.25 \text{ m}$$
$$= \frac{4.5\pi}{\lambda} \quad \text{(here } \lambda \text{ is in μm)}$$

We see that, for $\lambda = 428.6$ nm, $\delta = 10.5\pi$

for $\lambda = 529.4$ nm, $\delta = 8.5\pi$

for $\lambda = 692.3$ nm, $\delta = 6.5\pi$

These are the only values of λ for which the plate acts as a quarter wave plate.

5.183 Between crossed Nicols, a quartz plate, whose optic axis makes 45° angle with the principal directions of the Nicols, must introduce a phase difference of $(2k + 1)\pi$ so as to transmit the incident light (of that wavelength) with maximum intensity. In this case, the plane of polarization of the light emerging from the polarizer will be rotated by 90° and will go through the analyzer undiminished. Thus, we write for light of wavelength 643 nm
$$\delta = \frac{2\pi \times 0.009}{0.643 \times 10^{-6}} \times d(\text{mm}) \times 10^{-3}$$
$$= \frac{18\pi d}{0.643} = (2k + 1)\pi \tag{1}$$

To nearly block the light of wavelength 564 nm we require
$$\frac{18\pi d}{0.564} = (2k')\pi \tag{2}$$

We must have $2k' > 2k + 1$. For the smallest value of d, we take $2k' = 2k + 2$.

Thus, $$0.643(2k + 1) = 0.564 \times (2k + 2)$$

or $$0.079 \times 2k = 0.564 \times 2 - 0.643$$

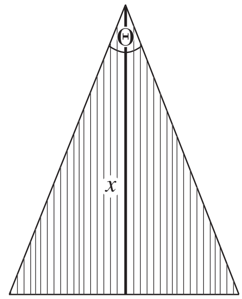
or $$2k = 6.139$$

This is not quite an integer but is close to it. This means that if we take $2k = 6$ Eq. (1) can be satisfied exactly while Eq. (2) will hold approximately.

Thus, $$d = \frac{7 \times 0.643}{18} = 0.250 \text{ mm}$$

5.184 If a ray traverses the wedge at a distance x below the joint, then the distance that the ray moves in the wedge is $2x \tan \Theta/2$ and this causes a phase difference

$$\delta = \frac{2\pi}{\lambda}(n_E - n_O)\, 2x \tan\frac{\Theta}{2}$$



between the E and O wave components of the ray. For a general x the resulting light is elliptically polarized and is not completely quenched by the analyzer polaroid. The condition for complete quenching is

$$\delta = 2k\pi - \text{dark fringe}$$

That for maximum brightness is

$$\delta = (2k+1)\pi - \text{bright fringe}$$

The fringe width is given by

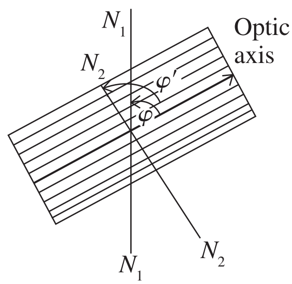
$$\Delta x = \frac{\lambda}{2(n_E - n_O)\tan \Theta/2}$$

Hence,
$$(n_E - n_O) = \frac{\lambda}{2\Delta x \tan \Theta/2}$$

Using $\tan\left(\frac{\Theta}{2}\right) = \tan 175° = 0.03055$ $\lambda = 0.55$ μm and $\Delta x = 1$ mm, we get

$$n_E - n_O = 9.001 \times 10^{-3}$$

5.185 Light emerging from the first polaroid is plane polarized with amplitude A where $A^2 = I_0/2$. N_1 is the principal direction of the polaroid and a vibration of amplitude can be resolved into two vibrations: E wave with vibration along the optic axis with amplitude $A \cos \varphi$ and the O wave with vibration perpendicular to the optic axis and having an amplitude $A \sin \varphi$. These acquire a phase difference δ on passing through the plate. The second polaroid transmits the components $A\cos\varphi\cos\varphi'$ and $A\sin\varphi\sin\varphi'$.



What emerges from the second polaroid is a set of two plane polarized waves in the same direction and same plane of polarization but with phase difference δ. They interfere and produce a wave of amplitude squared

$$R^2 = A^2[\cos^2\varphi\cos^2\varphi' + \sin^2\varphi\sin^2\varphi' + 2\cos\varphi\cos\varphi'\sin\varphi\sin\varphi'\cos\delta]$$

Using $\cos^2(\varphi - \varphi') = (\cos\varphi\cos\varphi' + \sin\varphi\sin\varphi')^2$

$$= \cos^2\varphi\cos^2\varphi' + \sin^2\varphi\sin^2\varphi' + 2\cos\varphi\cos\varphi' + 2\cos\varphi\cos\varphi'\sin\varphi\sin\varphi'$$

we easily find

$$R^2 = A^2\left[\cos^2(\varphi - \varphi') - \sin 2\varphi \sin 2\varphi' \sin^2 \frac{\delta}{2}\right]$$

Now $A^2 = I_0/2$ and $R^2 = I$, so the result is

$$I = \frac{1}{2} I_0 \left[\cos^2(\varphi - \varphi') - \sin 2\varphi \sin 2\varphi' \sin^2 \frac{\delta}{2}\right]$$

Special cases:

Crossed polaroids: Here $\varphi - \varphi' = 90°$ or $\varphi' = \varphi - 90°$ and $2\varphi' = 2\varphi - 180°$. Thus, in this case

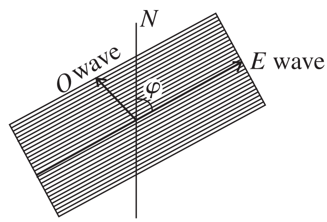
$$I = I_\perp = \frac{1}{2} I_0 \sin^2 2\varphi \sin^2 \frac{\delta}{2}$$

Parallel polaroids: Here $\varphi = \varphi'$. Thus,

$$I = I_\parallel = \frac{1}{2} I_0 \left(1 - \sin^2 2\varphi \sin^2 \frac{\delta}{2}\right)$$

With $\delta = \dfrac{2\pi}{\lambda}\Delta$, the conditions for the maximum and minimum are easily found to be those shown in the answer sheet.

5.186 Let the circularly polarized light be resolved into plane polarized components of amplitude A_0 with a phase difference $\pi/2$ between them.



On passing through the crystal, the phase difference becomes $\delta + \pi/2$ and the components of the E and O waves in the direction N are, respectively,

$$A_0 \cos \varphi \quad \text{and} \quad A_0 \sin \varphi$$

They interfere to produce a wave of amplitude squared

$$R^2 = A_0^2 \cos^2\varphi + A_0^2 \sin^2\varphi + 2A_0^2 \cos\varphi \sin\varphi \cos\left(\delta + \frac{\pi}{2}\right)$$

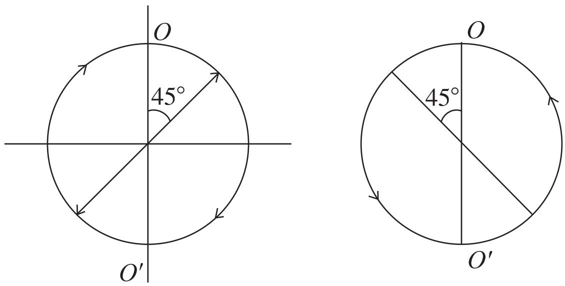
$$= A_0^2(1 + \sin 2\varphi \sin \delta)$$

Hence, $$I = I_0(1 + \sin 2\varphi \sin \delta)$$

Here I_0 is the intensity of the light transmitted by the polaroid when there is no crystal plate.

5.187 (a) The light with right circular polarization (viewed against the oncoming light, this means that the light vector is moving clockwise) becomes plane polarized on passing through a quarter wave plate. In this case the direction of oscillations of the electric vector of the electromagnetic wave forms an angle of $+45°$ with the axis of the crystal OO' (see Fig. a). In the case of left-hand circular polarizations, this angle will be $-45°$ (see Fig. b).

(b) If for any position of the plate the rotation of the polaroid (located behind the plate) does not bring about any variation in the intensity of the transmitted light, the incident light is unpolarized (i.e., natural). If the intensity of the transmitted light can drop to zero on rotating the analyzer polaroid for some position of the quarter wave plate, the incident light is circularly polarized. If it varies but does not drop to zero, it must be a mixture of natural and circularly polarized light.



(*a*) (*b*)

5.188 (a) The light from P is plane polarized with its electric vector vibrating at 45° with the plane of the paper. At first the sample S is absent. Light from P can be resolved into components vibrating in and perpendicular to the plane of the paper. The former is the E ray in the left half of the Babinet compensator (B.C.) and the latter is the O ray. In the right half the nomenclature it is the opposite. In the compensator the two components acquire a phase difference which depends on the relative position of the ray. If the ray is incident at a distance x above the central line through the compensator then the E ray acquires a phase

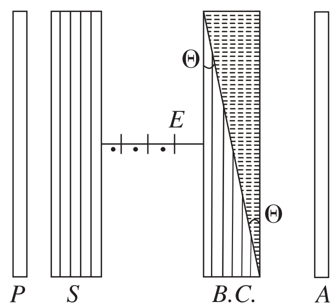
$$\frac{2\pi}{\lambda}[n_E(l-x)+n_O(l+x)]\tan\Theta$$

while the O ray acquires

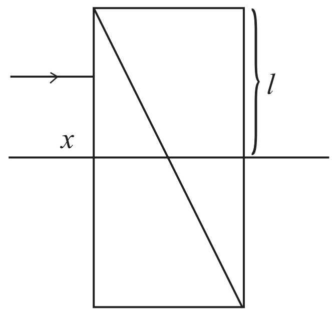
$$\frac{2\pi}{\lambda}[n_O(l-x)+n_E(l+x)]\tan\Theta$$

so the phase difference between the two rays is

$$\frac{2\pi}{\lambda}(n_O-n_E)2x\tan\Theta=\delta$$



We get dark fringes whenever $\delta = 2k\pi$ because then the emergent light is the same as that coming from the polarizer and is quenched by the analyzer. (If $\delta = (2k+1)\pi$, we get bright fringes because in this case, the plane of polarization of the emergent light has rotated by 90° and is therefore fully transmitted by the analyzer.)



It follows that the fringe width Δx is given by

$$\Delta x = \frac{\lambda}{2|n_E - n_O|\Theta}$$

(b) If the fringes are displaced upwards by δx, then the path difference introduced by the sample between the O and the E rays must be such so as to be exactly cancelled by the compensator. Thus,

$$\frac{2\pi}{\lambda}[d(n_O' - n_E') + (n_E - n_O)2\delta x \tan\Theta] = 0$$

or
$$d(n_O' - n_E') = -2(n_E - n_O)\delta x\,\Theta$$

using $\tan\Theta \approx \Theta$.

5.189 Light polarized along the x-direction (i.e., one whose electric vector has only an x-component) and propagating along the z-direction can be decomposed into left and right circularly polarized light in accordance with the formula

$$E_x = \frac{1}{2}(E_x + iE_y) + \frac{1}{2}(E_x - iE_y)$$

On passing through a distance l of an active medium these acquire the phases

$$\delta_R = \frac{2\pi}{\lambda} n_R l \quad \text{and} \quad \delta_L = \frac{2\pi}{\lambda} n_L l$$

So we get for the complex amplitude

$$\begin{aligned} E' &= \frac{1}{2}(E_x + iE_y)e^{i\delta_R} + \frac{1}{2}(E_x - iE_y)e^{i\delta_L} \\ &= e^{i\frac{\delta_R+\delta_L}{2}}\left[\frac{1}{2}(E_x + iE_y)e^{i\delta/2} + \frac{1}{2}(E_x - iE_y)e^{-i\delta/2}\right] \\ &= e^{i\frac{\delta_R+\delta_L}{2}}\left[E_x \cos\frac{\delta}{2} - E_y \sin\frac{\delta}{2}\right] \quad (\text{where } \delta = \delta_R - \delta_L) \end{aligned}$$

Apart from an overall phase $(\delta_R + \delta_L)/2$ (which is irrelevant) this represents a wave whose plane of polarization has rotated by

$$\frac{\delta}{2} = \frac{\pi}{\lambda}(\Delta n)l \quad (\text{where } \Delta n = |n_R - n_L|)$$

By definition this equals αl, so

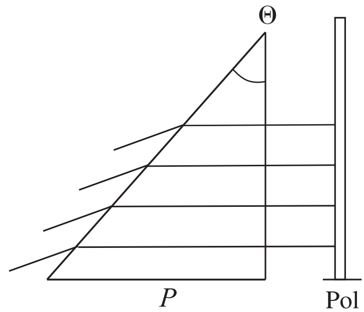
$$\Delta n = \frac{\alpha\lambda}{\pi}$$

$$= \frac{589.5 \times 10^{-6}\text{ mm} \times 21.72\text{ deg/mm}}{\pi} \times \frac{\pi}{180}\text{ (radians)}$$

$$= \frac{0.5895 \times 21.72}{180} \times 10^{-3}$$

$$= 0.71 \times 10^{-4}$$

5.190 Plane polarized light, on entering the wedge, decomposes into right and left circularly polarized light which travels with different speeds in prism(P) and the emergent light gets its plane of polarization rotated by an angle which depends on the distance travelled.



Given that Δx = fringe width, $\Delta x \tan\theta$ = difference in the path length traversed by two rays which form successive bright or dark fringes.

Then, $$\frac{2\pi}{\lambda}|n_R - n_L|\,\Delta x \tan\theta = 2\pi$$

Thus, $$\alpha = \frac{\pi\Delta n}{\lambda} = \frac{\pi}{\Delta x \tan\Theta}$$

$$= 20.8\text{ ang deg/mm}$$

Let x = distance on the polaroid Pol as measured from a maximum. Then, a ray that falls at this distance traverses an extra distance equal to

$$\pm x \tan\theta$$

and hence a rotation of

$$\pm \alpha x \tan\theta = \pm\frac{\pi x}{\Delta x}$$

By Malus′ law the intensity at this point will be

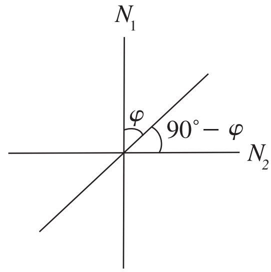
$$\sim \cos^2\left(\frac{\pi x}{\Delta x}\right)$$

5.191 If I_0 = intensity of natural light, then $(1/2)I_0$ = intensity of light emerging from the polarizer Nicol (N_1).

Suppose the quartz plate rotates this light by φ, then the analyzer (N_2) will transmit

$$\frac{1}{2} I_0 \cos^2(90 - \varphi)$$

$$= \frac{1}{2} I_0 \sin^2 \varphi$$



of this intensity.

Hence, $$\eta I_0 = \frac{1}{2} I_0 \sin^2 \varphi$$

or $$\varphi = \sin^{-1} \sqrt{2\eta}$$

But $$\varphi = \alpha d$$

so, $$d_{\min} = \frac{1}{\alpha} \sin^{-1} \sqrt{2\eta}$$

For minimum d we must take the principal value of inverse sine. Thus, using $\alpha = 17$ ang deg/mm we get $d_{\min} = 2.99$ mm.

5.192 For light of wavelength 436 nm

$$41.5^\circ \times d = k \times 180^\circ = 2k \times 90^\circ$$

(Light will be completely cut off when the quartz plate rotates the plane of polarization by a multiple of 180°.) Here d = thickness of quartz plate in mm.

For natural incident light, half the light will be transmitted when the quartz rotates light by an odd multiple of 90°. Thus,

$$31.1^\circ \times d = (2k' + 1) \times 90^\circ$$

Now, $$\frac{41.5^\circ}{31.1^\circ} = 1.3344 \approx \frac{4}{3}$$

Thus, $$k = 2 \text{ and } k' = 1$$

so, $$d = \frac{4 \times 90^\circ}{41.5^\circ} = 8.67 \text{ mm}$$

5.193 Two effects are involved here: rotation of plane of polarization by sugar solution and the effect of that rotation on the scattering of light in the transverse direction. The latter is shown in the given figure. It is easy to see from the figure that there will be no scattering of light in this transverse direction if the incident light has its electric vector parallel to the line of sight. In such a situation, we expect fringes to occur in the given experiment.

From the given data we see that in a distance of 50 cm, the rotation of plane of polarization must be 180°.

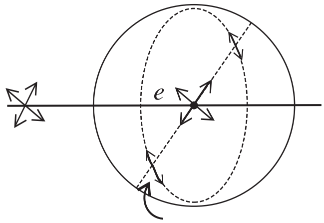
Thus, the specific rotation constant of sugar is

$$\frac{\text{rotation constant}}{\text{concentration}}$$

$$= \frac{180/50}{500\,\text{g/l}}\ \text{ang/deg/cm}$$

$$= \frac{180}{5.0\ \text{dm} \times (0.500\ \text{g/cm}^3}$$

$$= 72\ \text{ang deg/(dm} \cdot \text{g/cm}^3)$$



5.194 (a) In passing through the Kerr cell, the two perpendicular components of the electric field will acquire a phase difference. When this phase difference equals 90°, the emergent light will be circularly polarized because the two perpendicular components O and E have the same magnitude since it is given that the direction of electric field E in the capacitor forms an angle of 45° with the principal directions of the Nicols. In this case the intensity of light that emerges from this system will be independent of the rotation of the analyzer prism.

Now the phase difference introduced is given by

$$\delta = \frac{2\pi}{\lambda}(n_E - n_O)l$$

In the present case

$$\delta = \frac{\pi}{2} \quad \text{(for minimum electric field)}$$

So, $$n_E - n_O = \frac{\lambda}{4l}$$

Now $$n_E - n_O = B\lambda E^2$$

so, $$E_{\text{min}} = \sqrt{\frac{1}{4Bl}} = 10^5/\sqrt{88} = 10.66\ \text{kV/cm}$$

(b) If the applied electric field is

$$E = E_{\text{m}} \sin \omega t \quad (\text{where } \omega = 2\pi\nu)$$

then the Kerr cell introduces a time varying phase difference

$$\delta = 2\pi B\, l\, E_m^2 \sin^2 \omega \text{t}$$

$$= 2\pi \times 2.2 \times 10^{-10} \times 10 \times (50 \times 10^3)^2 \sin^2 \omega \text{t}$$

$$= 11\pi \sin^2 \omega t$$

In one half-cycle (i.e., in time $\pi/\omega = T/2 = 1/2\nu$) this reaches the value $2k\pi$ when

$$\sin^2 \omega t = 0, \frac{2}{11}, \frac{4}{11}, \frac{6}{11}, \frac{8}{11}, \frac{10}{11}, \frac{2}{11}, \frac{4}{11}, \frac{6}{11}, \frac{8}{11}, \frac{10}{11}$$

i.e., 11 times. On each of these occasions light will be interrupted. Thus, light will be interrupted

$$2\nu \times 11 = 2.2 \times 10^8 \text{ times per second}$$

[Light will be interrupted when the Kerr cell (placed between crossed Nicols) introduces a phase difference of $2k\pi$ and in no other case.]

5.195 From Problem 5.189, we know that

$$\Delta n = \frac{\alpha\lambda}{\pi}$$

where α is the rotation constant. Thus,

$$\Delta n = \frac{2\alpha}{2\pi/\lambda} = \frac{2\alpha c}{\omega}$$

On the other hand, $\alpha_{\text{mag}} = VH$

Thus, for the magnetic rotations

$$\Delta n = \frac{2cVH}{\omega}$$

5.196 Part of the rotation is due to Faraday effect and part of it is ordinary optical rotation. The latter does not change sign when magnetic field is reversed. Thus,

$$\varphi_1 = \alpha l + VlH$$

and $$\varphi_2 = \alpha l - VlH$$

Hence, $$2VlH = (\varphi_1 - \varphi_2)$$

or $$V = \frac{(\varphi_1 - \varphi_2)/2}{lH}$$

Putting the values, we get

$$V = \frac{510 \text{ ang min}}{2 \times 0.3 \times 56.5} \times 10^{-3} = 0.015 \text{ ang min/A}$$

5.197 As per the problem,

$$\varphi = \varphi_{\text{chem}} + \varphi_{\text{mag}}$$

We look against the transmitted beam and count the positive direction clockwise. The chemical part of the rotation is annulled by reversal of wave vector upon reflection.

Thus, $$\varphi_{\text{chem}} = \alpha l$$

since in effect there is a single transmission. On the other hand

$$\varphi_{\text{mag}} = -NHVl$$

To get the signs right, recall that dextrorotatory compounds rotate the plane of vibration in a clockwise direction on looking against the oncoming beam. The sense of rotation of light vibration in Faraday effect is defined in terms of the direction of the field; positive rotation being that of right handed screw advancing in the direction of the field. This is the opposite of the definition of φ_{chem} for the present case.

Finally,

$$\varphi = (\alpha - VNH)l$$

Note: *If plane polarized light is reflected back and forth through the same active medium in a magnetic field, the Faraday rotation increases with each traversal.*

5.198 There must be a Faraday rotation by 45° in the opposite direction so that light could pass through the second polaroid. Thus,

$$VlH_{\text{min}} = \pi/4$$

$$H_{\text{min}} = \frac{\pi/4}{Vl} = \frac{45 \times 60}{2.59 \times 0.26} \text{ A/m}$$

$$= 4.01 \text{ kA/m}$$

If the direction of magnetic field is changed then the sense of rotation will also change. Light will be completely quenched in the above case.

5.199 Let r = radius of the disk, then its moment of inertia about its axis $= 1/2mr^2$. In time t, the disk will acquire an angular momentum

$$t \cdot \pi r^2 \cdot \frac{I}{\omega}$$

when circularly polarized light of intensity I falls on it. By conservation of angular momentum, this must equal to

$$\frac{1}{2}mr^2 \cdot \omega_0$$

where ω_0 = final angular velocity. Equating

$$t = \frac{m\omega\omega_0}{2\pi I}$$

But $$\frac{\omega}{2\pi} = \nu = \frac{c}{\lambda} \quad \text{so,} \quad t = \frac{mc\omega_0}{I\lambda}$$

Substituting the values of the various quantities, we get

$$t = 11.9 \text{ h}$$

5.5 Dispersion and Absorption of Light

5.200 In a travelling plane electromagnetic wave, the intensity is simply the time averaged magnitude of the Poynting vector i.e.,

$$I = <|\mathbf{E} \times \mathbf{H}|> = <\sqrt{\frac{\varepsilon_0}{\mu_0}} E^2> = <c\varepsilon_0 E^2>$$

on using $$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \quad \text{and} \quad E\sqrt{\varepsilon_0} = H\sqrt{\mu_0}$$

(see Section 4.4 of the book).

Now, time averaged value of E^2 is $E_0^2/2$, so

$$I = \frac{1}{2} c\varepsilon_0 E_0^2 \quad \text{or} \quad E_0 = \sqrt{\frac{2I}{c\varepsilon_0}}$$

(a) Represent the electric field at any point by $E = E_0 \sin \omega t$. Then, for the electron we have the equation

$$m\ddot{x} = eE_0 \sin \omega t$$

So, $$x = -\frac{eE_0}{m\omega^2} \sin \omega t$$

The amplitude of the forced oscillation is

$$\frac{eE_0}{m\omega^2} = \frac{e}{m\omega^2}\sqrt{\frac{2I}{c\varepsilon_0}} = 5.1 \times 10^{-16} \text{ cm}$$

The velocity amplitude is clearly

$$\frac{eE_0}{m\omega} = 5.1 \times 10^{-16} \times 3.4 \times 10^{15} = 1.73 \text{ cm/s}$$

(b) For the electric force, amplitude of the electric force

$$F_{\text{ele}} = eE_0$$

For the magnetic force (which we have neglected above), it is

$$F_{\text{mag}} = (evB)$$

$$= (ev\mu_0 H)$$

$$= evE\sqrt{\varepsilon_0\mu_0} = ev\frac{E}{c}$$

Writing $$v = -v_0 \cos\omega t$$

where $$v_0 = \frac{eE_0}{m\omega}$$

We see that the magnetic force is (neglecting the sign)

$$F_{\text{mag}} = \frac{evE_0}{2c}\sin 2\omega t$$

Hence, the ratio of amplitudes of the two forces

$$\frac{F_{\text{mag}}}{F_{\text{ele}}} = \frac{v_0}{2c} = 2.9 \times 10^{-11}$$

This is negligible and justifies ignoring the magnetic field of the electromagnetic wave in calculating v_0.

5.201 (a) It turns out that the spatial dependence of the electric field as well as the magnetic field is negligible. Thus, for a typical electron

$$m\ddot{\mathbf{r}} = e\mathbf{E}_0 \sin\omega t$$

So, $$\mathbf{r} = -\frac{e\mathbf{E}_0}{m\omega^2}\sin\omega t \text{ (neglecting any non-sinusoidal part)}$$

The ions will be practically unaffected. Then,

$$\mathbf{P} = n_0 e\mathbf{r} = -\frac{n_0 e^2}{m\omega^2}\mathbf{E}_0$$

and $$\mathbf{D} = \varepsilon_0\mathbf{E}_0 + \mathbf{P} = \varepsilon_0\left(1 - \frac{n_0 e^2}{\varepsilon_0 m\omega^2}\right)\mathbf{E}_0$$

Hence, the permittivity

$$\varepsilon = 1 - \frac{n_0 e^2}{\varepsilon_0 m\omega^2}$$

(b) The phase velocity is given by

$$v = \frac{\omega}{K} = \frac{c}{\sqrt{\varepsilon}}$$

So, $$ck = \omega\sqrt{1 - \frac{\omega_P^2}{\omega^2}} \quad \left(\text{where } \omega_P^2 = \frac{n_0 e^2}{\varepsilon_0 m}\right)$$

or $$\omega^2 = c^2k^2 + \omega_P^2$$

Thus, $$v = c\sqrt{1 + \frac{\omega_P^2}{c^2k^2}} = c\sqrt{1 + \left(\frac{n_0 e^2}{4\pi^2 mc^2\varepsilon_0}\right)\lambda^2}$$

5.202 From the previous problem

$$n^2 = 1 - \frac{n_0 e^2}{\varepsilon_0 m\omega^2}$$

$$= 1 - \frac{n_0 e^2}{4\pi^2\varepsilon_0 m\nu^2}$$

Thus, $$n_0 = (4\pi^2\nu^2 m\varepsilon_0/e^2)(1 - n^2) = 2.36 \times 10^7 \text{ cm}^{-3}$$

5.203 For hard x-rays, the electrons in graphite will behave as if nearly free and the formula of previous problem can be applied. Thus,

$$n^2 = 1 - \frac{n_0 e^2}{\varepsilon_0 m\omega^2}$$

and $$n \approx 1 - \frac{n_0 e^2}{2\varepsilon_0 m\omega^2}$$

on taking square root and neglecting higher order terms.

So, $$n - 1 = -\frac{n_0 e^2}{2\varepsilon_0 m\omega^2} = -\frac{n_0 e^2\lambda^2}{8\pi^2\varepsilon_0 me^2}$$

We calculate n_0 as follows: There are $6 \times 6.023 \times 10^{23}$ electrons in 12 g of graphite of density 1.6 g/cm^3. Thus,

$$n_0 = \frac{6 \times 6.023 \times 10^{23}}{(12/1.6)} \text{ cm}^{-3}$$

Using the values of other constants and $\lambda = 50 \times 10^{-12}$ m, we get

$$n - 1 = -5.4 \times 10^{-7}$$

5.204 (a) The equation of the electron can (under the stated conditions) be written as

$$m\ddot{x} + \gamma\dot{x} + kx = eE_0 \cos\omega t$$

To solve this equation we shall find it convenient to use complex displacements. Consider the equation

$$m\ddot{z} + \gamma\dot{z} + kz = eE_0 e^{-i\omega t}$$

Its solution is

$$z = \frac{eE_0 e^{-i\omega t}}{-m\omega^2 - i\gamma\omega + k}$$

(We ignore transients.)

Now, $$\beta = \frac{\gamma}{2m} \quad \text{and} \quad \omega_0^2 = \frac{k}{m}$$

We find $$z = \frac{(eE_0/m)e^{-i\omega t}}{(\omega_0^2 - \omega^2 - 2i\beta\omega)}$$

Now $$x = \text{Real part of } z$$

$$= \frac{eE_0}{m} \cdot \frac{\cos(\omega t + \varphi)}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} = a\cos(\omega t + \varphi)$$

where $$\tan\varphi = \frac{2\beta\omega}{\omega^2 - \omega_0^2} \quad \text{and}$$

$$\sin\varphi = \frac{2\beta\omega}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2}}$$

(b) We calculate the power absorbed as

$$P = < F\dot{x} > = < eE_0 \cos\omega t\,(-\omega a \sin(\omega t + \varphi)) >$$

$$= eE_0 \cdot \frac{eE_0}{m}\frac{1}{2} \cdot \frac{2\beta\omega}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \cdot \omega$$

$$= \left(\frac{eE_0}{m}\right)^2 \frac{\beta m\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

This is clearly maximum when $\omega_0 = \omega$ because P can be written as

$$P = \left(\frac{eE_0}{m}\right)^2 \frac{\beta m}{[(\omega_0^2/\omega) - \omega]^2 + 4\beta^2}$$

and $$P_{\max} = \frac{m}{4\beta}\left(\frac{eE_0}{m}\right)^2 \quad (\text{for } \omega = \omega_0)$$

P can also be calculated from

$$P = <\gamma\dot{x}\cdot\dot{x}>$$

$$= \left(\frac{\gamma\omega^2 a^2}{2}\right) = \frac{\beta m\omega^2 (eE_0/m)^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

5.205 Let us write the solutions of the wave equation in the form

$$A = A_0 e^{i(\omega t - kx)}$$

where $k = 2\pi/\lambda$ and λ is the wavelength in the medium. If $n' = n + i\chi$, then

$$k = \frac{2\pi}{\lambda_0} n'$$

(where λ_0 is the wavelength in vacuum) and the equation becomes

$$A = A_0 e^{\chi' x} \exp[i(\omega t_1 - k' x)]$$

where $$\chi' = \frac{2\pi}{\lambda_0}\chi \quad \text{and} \quad k' = \frac{2\pi}{\lambda_0} n$$

In real form, $$A = A_0 e^{\chi' x}\cos(\omega t - k' x)$$

This represents a plane wave whose amplitude diminishes as it propagates to the right (provided $\chi' < 0$). When $n' = i\chi$, then similarly

$$A = A_0 e^{\chi' x}\cos\omega t$$

(on putting $n = 0$ in the above equation).

This represents a standing wave whose amplitude diminishes as one goes to the right (if $\chi' < 0$). The wavelength of the wave is infinite ($k' = 0$).

Waves of the former type are realized inside metals as well as inside dielectrics when there is total reflection (penetration of wave).

5.206 In the plasma radio, waves with wavelengths exceeding λ_0 are not propagated. We interpret this to mean that the permittivity becomes negative for such waves. Thus,

$$0 = 1 - \frac{n_0 e^2}{\varepsilon_0 m\omega^2} \quad \left(\text{if } \omega = \frac{2\pi c}{\lambda_0}\right)$$

Hence, $$\frac{n_0 e^2 \lambda_0^2}{4\pi^2 \varepsilon_0 m c^2} = 1$$

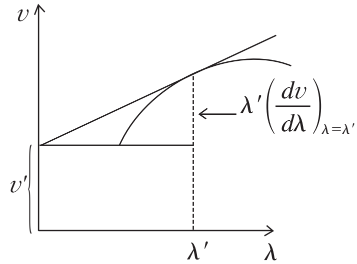
or $$n_0 = \frac{4\pi^2 \varepsilon_0 m c^2}{e^2 \lambda_0^2} = 1.984 \times 10^9 \text{ cm}^{-3}$$

5.207 By definition

$$u = \frac{d\omega}{dk}$$

$$= \frac{d}{dk}(vk) \quad (\text{as } \omega = vk)$$

$$= v + k\frac{dv}{dk}$$



Now, $$k = \frac{2\pi}{\lambda} \quad \text{so,} \quad dk = -\frac{2\pi}{\lambda^2}d\lambda$$

Thus, $$u = v - \lambda\frac{dv}{d\lambda}$$

Its interpretation is the following:

The slope of the $v - \lambda$ curve at $\lambda = \lambda'$ is

$$\left(\frac{dv}{d\lambda}\right)_{\lambda=\lambda'}$$

Thus, as is obvious from the figure

$$v' = v(\lambda') - \lambda'\left(\frac{dv}{d\lambda}\right)_{\lambda=\lambda'}$$

is the group velocity for $\lambda = \lambda'$.

5.208 (a) Given that $$v = a/\sqrt{\lambda} \quad (a \text{ is a constant})$$

Then, $$u = v - \lambda\frac{dv}{d\lambda}$$

$$= \frac{a}{\sqrt{\lambda}} - \lambda\left(-\frac{1}{2}a\lambda^{-3/2}\right) = \frac{3}{2}\cdot\frac{a}{\sqrt{\lambda}} = \frac{3}{2}v$$

(b) Given that $$v = bk = k \quad (b \text{ is a constant})$$

So, $$\omega = bk^2$$

and $$u = \frac{d\omega}{dk} = 2bk = 2v$$

(c) Given that $$v = \frac{c}{\omega^2} \quad \left(c \text{ is a constant} = \frac{\omega}{k}\right)$$

So, $$\omega^3 = ck \quad \text{or} \quad \omega = c^{1/3}k^{1/3}$$

Thus, $$u = \frac{d\omega}{dk} = c^{1/3}\,\frac{1}{3}\,k^{-2/3} = \frac{1}{3}\frac{\omega}{k} = \frac{1}{3}v$$

5.209 We have

$$uv = \frac{\omega}{k}\frac{d\omega}{dk} = c^2$$

Integrating we find

$$\omega^2 = A + c^2k^2 \quad (A \text{ is a constant})$$

So, $$k = \frac{\sqrt{\omega^2 - A}}{c}$$

and $$v = \frac{\omega}{k}\frac{c}{\sqrt{1 - A/\omega^2}}$$

Writing this as $c/\sqrt{\varepsilon(\omega)}$, we get

$$\varepsilon(\omega) = 1 - \frac{A}{\omega^2}$$

(A can be $+ve$ or $-ve$.)

5.210 The phase velocity of light in the vicinity of $\lambda = 534 \text{ nm} = \lambda_0$ is obtained as

$$v(\lambda_0) = \frac{c}{n(\lambda_0)} = \frac{3 \times 10^8}{1.640} = 1.829 \times 10^8 \text{ m/s}$$

To get the group velocity we need to calculate $(dn/d\lambda)_{\lambda = \lambda_0}$. We shall use linear interpolation in the two intervals. Thus,

$$\left(\frac{dn}{d\lambda}\right)_{\lambda = 521.5 \text{ nm}} = -\frac{0.007}{25} = -28 \times 10^{-5} \text{ nm}^{-1}$$

$$\left(\frac{dn}{d\lambda}\right)_{\lambda = 561.5 \text{ nm}} = -\frac{0.01}{55} = -18.2 \times 10^{-5} \text{ nm}^{-1}$$

The $(dn/d\lambda)$ values have been assigned to the mid-points of the two intervals. Interpolating again, we get

$$\left(\frac{dn}{d\lambda}\right)_{\lambda = 534\text{ nm}} = \left[-28 + \frac{9.8}{40} \times 12.5\right] 10^{-5}\ \text{nm}^{-1}$$

$$= -24.9 \times 10^{-5}\ \text{nm}^{-1}$$

Finally $$u = \frac{c}{n} - \lambda \frac{d}{d\lambda}\left(\frac{c}{n}\right) = \frac{c}{n}\left[1 + \frac{\lambda}{n}\left(\frac{dn}{d\lambda}\right)\right]$$

At $\lambda = 534$ nm

$$u = \frac{3 \times 10^8}{1.640}\left[1 - \frac{534}{1.640} \times 24.9 \times 10^{-5}\right]$$

$$= 1.59 \times 10^8\ \text{m/s}$$

5.211 We write

$$v = \frac{\omega}{k} = a + b\lambda$$

so, $$\omega = k(a + b\lambda) = 2\pi b + ak \qquad \left(\text{since } k = \frac{2\pi}{\lambda}\right)$$

Suppose a wave train at time $t = 0$ has the form

$$F(x, 0) = \int f(k) e^{ikx}\, dk$$

Then, at time t it will have the form

$$F(x, t) = \int f(k)^{ikx - i\omega t}\, dk$$

$$= \int f(k) e^{ikx - i(2\pi b + ak)t}\, dx$$

$$= \int f(k) e^{ik(x - at)} e^{-i2\pi bt}\, dk$$

At $t = 1/b = \tau$

$$F(x, \tau) = F(x - a\tau, 0)$$

So at time $t = \tau$, the wave train has regained its shape though it has advanced by $a\tau$.

5.212 On passing through the first (polarizer) Nicol, the intensity of light becomes $(1/2)I_0$ because one of the components has been cut off. On passing through the solution, the plane of polarization of the light will rotate by

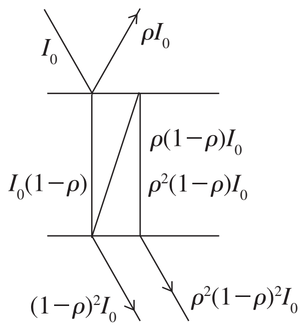
$$\varphi = VlH$$

and its intensity will also decrease by a factor $e^{-\chi l}$. The plane of vibration of the light wave will then make an angle $90° - \varphi$ with the principal direction of the analyzer Nicol. Thus by Malus′ law, the intensity of light coming out of the second Nicol will be

$$\frac{1}{2} I_0 e^{-\chi l} \cdot \cos^2(90° - \varphi)$$

$$= \frac{1}{2} I_0 e^{-\chi l} \sin^2\varphi$$

5.213 (a) The multiple reflections are shown in the figure below. Transmission gives a factor $(1 - \rho)$ while reflections give a factor of ρ. Thus the transmitted intensity, assuming incoherent light, is



$$(1-\rho)^2 I_0 + (1-\rho)^2\rho^2 I_0 + (1-\rho)^2\rho^4 I_0 + \dots$$

$$= (1-\rho)^2 I_0 (1 + \rho^2 + \rho^4 + \rho^6 + \dots)$$

$$= (1-\rho)^2 I_0 \times \frac{1}{1-\rho^2} = \left(I_0 \frac{(1-\rho)^2}{(1+\rho)^2} \right)$$

(b) When there is absorption, we pick up a factor $\sigma = e^{-\chi\delta d}$ in each traversal of the plate.

Thus, we get

$$(1-\rho)^2\sigma I_0 + (1-\rho)^2\sigma^3\rho^2 I_0 + (1-\rho)^2\sigma^5\rho^4 I_0 + \dots$$

$$= (1-\rho)^2\sigma I_0 (1 + \sigma^2\rho^2 + \sigma^4\rho^4 + \dots)$$

$$= I_0 \frac{\sigma(1-\rho)^2}{1-\sigma^2\rho^2}$$

5.214 We have
$$\tau_1 = e^{-\chi d_1}(1-\rho)^2$$

and
$$\tau_2 = e^{-\chi d_2}(1-\rho)^2$$

where ρ is the reflectivity (see previous problem) and multiple reflections have been ignored.

Thus,
$$\frac{\tau_1}{\tau_2} = e^{\chi(d_2 - d_1)}$$

or $$\chi = \frac{\ln(\tau_1/\tau_2)}{d_2 - d_1} = 0.35\ \text{cm}^{-1}$$

5.215 On each surface we pick up a factor $(1 - \rho)$ from reflection and a factor $e^{-\chi l}$ due to absorption in each plate.

Thus, $$\tau = (1-\rho)^{2N} e^{-\chi N l}$$

and $$\chi = \frac{1}{Nl} \ln \frac{(1-\rho)^{2N}}{\tau} = 0.034\ \text{cm}^{-1}$$

5.216 Apart from the factor $(1 - \rho)$ on each end face of the plate, we shall get a factor due to absorptions. This factor can be calculated by assuming the plate to consist of a large number of very thin slabs within each of which the absorption coefficient can be assumed to be constant. Thus, we shall get a product like

$$\ldots e^{-\chi(x)dx} e^{-\chi(x+dx)dx} e^{-\chi(x+2dx)dx} \ldots$$

This product is nothing but

$$e^{-\int_0^l \chi(x)dx}$$

Now $\chi(0) = \chi_1$, $\chi(l) = \chi_2$ and variation with x is linear, so

$$\chi(x) = \chi_1 + \frac{x}{l}(\chi_2 - \chi_1)$$

Thus, the factor becomes

$$e^{-\int_0^l \left[\chi_1 + \frac{x}{l}(\chi_2 - \chi_1)\right]dx} = e^{-1/2(\chi_2 + \chi_1)l}$$

Finally we collect the factors due to reflection at each surface to get the transmission coefficient as

$$(1-\rho)^2 e^{-(\chi_1 + \chi_2)\frac{l}{2}}$$

5.217 The spectral density of the incident beam (i.e., intensity of the components whose wavelength lies in the interval λ and $\lambda + d\lambda$) is

$$\frac{I_0}{\lambda_2 - \lambda_1} d\lambda \quad (\text{for } \lambda_1 \le \lambda \le \lambda_2)$$

The absorption factor for this component is

$$e^{-\left[\chi_1 + \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1}(\chi_2 - \chi_1)\right]l}$$

and the transmission factor due to reflection at the surface is $(1 - \rho)^2$. Thus, the intensity of the transmitted beam is

$$(1-\rho)^2 \frac{I_0}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} d\lambda \, e^{-l\left[\chi_1 + \frac{\lambda - \lambda_1}{\lambda_2 - \lambda_1}(\chi_2 - \chi_1)\right]}$$

$$= (1-\rho)^2 \frac{I_0}{\lambda_2 - \lambda_1} e^{-\chi_1 l} \left(\frac{1 - e^{-(\chi_2 - \chi_1)l}}{(\chi_2 - \chi_1)l}\right) \chi(\lambda_2 - \lambda_1)$$

$$= (1-\rho)^2 I_0 \frac{e^{-\chi_1 l} - e^{-\chi_2 l}}{(\chi_2 - \chi_1)l}$$

5.218 At the wavelength λ_0, the absorption coefficient vanishes and loss in transmission is entirely due to reflection. This factor is the same at all wavelengths and therefore cancels out in calculating the passband and we need not worry about it.

Now, $\quad T_0 = (\text{transmissivity at } \lambda = \lambda_0) = (1 - \rho)^2$

and $\quad T = (\text{transmissivity at } \lambda) = (1 - \rho)^2 \, e^{-\chi(\lambda)d}$

Thus,

$$\frac{T}{T_0} = e^{-\alpha d(1 - \lambda/\lambda_0)^2}$$

The edges of the passband are $\lambda_0 \pm \Delta\lambda/2$ and at the edge

$$\frac{T}{T_0} = e^{-\alpha d\left(\frac{\Delta\lambda}{2\lambda_0}\right)^2} = \eta$$

Thus,
$$\frac{\Delta\lambda}{2\lambda_0} = \sqrt{\frac{\ln 1/\eta}{\alpha d}}$$

or
$$\Delta\lambda = 2\lambda_0 \sqrt{\frac{1}{\alpha d}\left(\ln \frac{1}{\eta}\right)}$$

5.219 We have to derive the law for decrease of intensity in an absorbing medium taking into account the natural geometrical fall-off (inverse square law) as well as absorption.

Consider a thin spherical shell of thickness dx and internal radius x. Let $I(x)$ and $I(x + dx)$ be the intensities at the inner and outer surfaces of this shell.

Then, $\quad 4\pi x^2 I(x) e^{-\chi dx} = 4\pi (x + dx)^2 I(x + dx)$

Except for the factor $e^{-\chi dx}$ this is the usual equation. We rewrite this as

$$x^2 I(x) = I(x+dx)(x+dx)^2(1+\chi dx)$$

$$= \left(I + \frac{dI}{dx}dx\right)(x^2 + 2x\,dx)(1+\chi dx)$$

or
$$x^2\frac{dI}{dx} + \chi x^2 I + 2xI = 0$$

Hence,
$$\frac{d}{dx}(x^2 I) + \chi(x^2 I) = 0$$

So,
$$x^2 I = Ce^{-\chi x}$$

where C is a constant of integration.

In our case we apply this equation for $a \le x \le b$. For $x \le a$ the usual inverse square law gives

$$I(a) = \frac{\Phi}{4\pi a^2}$$

Hence,
$$C = \frac{\Phi}{4\pi}e^{\chi a}$$

and
$$I(b) = \frac{\Phi}{4\pi b^2}e^{-\chi(b-a)}$$

This does not take into account reflections. When we do that, we get

$$I(b) = \frac{\Phi}{4\pi b^2}(1-\rho)^2 e^{-\chi(b-a)}$$

5.220 The transmission factor is $e^{-\mu d}$ and so the intensity will decrease as

$$e^{\mu d} = e^{3.6\times 11.3\times 8.1} = 58.4 \text{ times}$$

(We have used $\mu = (\mu/\rho)\times\rho$ and used the known value of density of lead.)

5.221 We require
$$\mu_{Pb}d_{Pb} = \mu_{Al}d_{Al}$$

or
$$\left(\frac{\mu_{Pb}}{\rho_{Pb}}\right)\rho_{Pb}d_{Pb} = \left(\frac{\mu_{Al}}{\rho_{Al}}\right)\rho_{Al}d_{Al}$$

So,
$$72.0\times 11.3\times d_{Pb} = 3.48\times 2.7\times 2.6$$

or
$$d_{Pb} = 0.3 \text{ mm}$$

5.222 We require

$$\frac{1}{2} = e^{-\mu d}$$

or

$$d = \frac{\ln 2}{\mu} = \frac{\ln 2}{(\mu/\rho)\rho} = 0.80 \text{ cm}$$

5.223 We require N plates where

$$\left(\frac{1}{2}\right)^N = \frac{1}{50}$$

So,

$$N = \frac{\ln 50}{\ln 2} = 5.6$$

5.6 Optics of Moving Sources

5.224 In the Fizean experiment, light disappears when the wheel rotates to bring a tooth in the position formerly occupied by a gap in the time taken by light to go from the wheel to the mirror and back. Thus, distance travelled = $2l$. Suppose the m^{th} tooth after the gap has come in place of the latter. Then, time taken is

$$\frac{2(m-1)+1}{2zn_1} \text{ s} \quad \text{(in the first case)}$$

and

$$\frac{2m+1}{2zn_2} \text{ s} \quad \text{(in the second case)}$$

Then

$$\frac{2l}{c} = \frac{1}{z(n_2 - n_1)}$$

Hence,

$$c = 2lz\,(n_2 - n_1) = 3.024 \times 10^8 \text{ m/s}$$

5.225 When $v \ll c$, time dilation effect of relativity can be neglected (i.e., $t' \approx t$) and we can use time in the reference frame fixed to the observer. Suppose the source emits short pulses with intervals T_0. Then in the reference frame fixed to the receiver the distance between two successive pulses is $\lambda = cT_0 - v_r T_0$ when measured along the observation line.

Here $v_r = v\cos\theta$ is the projection of the source velocity on the observation line. The frequency of the pulses received by the observer is

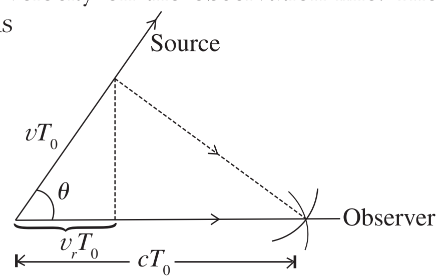
$$\nu = \frac{c}{\lambda} = \frac{\nu_0}{1 - v_r/c} \approx \nu_0\left(1 + \frac{v_r}{c}\right)$$

(The formula is accurate to first order only.)

Thus,

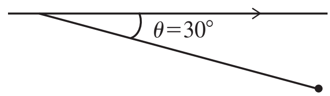
$$\frac{\nu - \nu_0}{\nu_0} = \frac{v_r}{c} = \frac{v\cos\theta}{c}$$

The frequency increases when the source is moving towards the observer.



5.226 From the previous problem

$$\frac{\Delta\nu}{\nu} = \frac{v}{c}\cos\theta$$



But $\nu\lambda = c$ gives on differentiation

$$\frac{\Delta\nu}{\nu} = -\frac{\Delta\lambda}{\lambda}$$

So, $$\Delta\lambda = -\lambda\sqrt{\frac{v^2}{c^2}}\cos\theta = -\lambda\sqrt{\frac{2T}{mc^2}}\cos\theta$$

On using $T = \frac{1}{2}mv^2$, m = mass of He^+ ion and $mc^2 = 4 \times 938$ MeV and putting other values, we get

$$\Delta\lambda = -26 \text{ nm}$$

5.227 One end of the solar disk is moving towards us while the other end is moving away from us. The angle θ between the direction in which the edges of the disk are moving and the line of observation is small ($\cos\theta \approx 1$). Thus,

$$\frac{\Delta\lambda}{\lambda} = \frac{2\omega R}{c}$$

where $\omega = 2\pi/T$ is the angular velocity of the Sun. Thus,

$$\omega = \frac{c\Delta\lambda}{2R\lambda}$$

So, $$T = \frac{4\pi R\lambda}{c\Delta\lambda}$$

Putting the values ($R = 6.95 \times 10^8$ m), we get

$$T = 24.85 \text{ days}$$

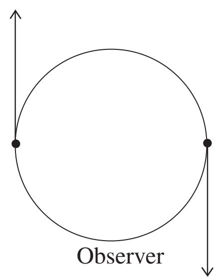
5.228 Maximum splitting of the spectral lines will occur when both of the stars are moving along the direction of line of observation as shown in the figure.

We then have the equations

$$\left(\frac{\Delta\lambda}{\lambda}\right)_m = \frac{2v}{c}$$

$$\frac{mv^2}{R} = \frac{\gamma m^2}{4R^2}$$

$$\tau = \frac{\pi R}{v}$$



From these we get

$$d = 2R = \left(\frac{\Delta\lambda}{\lambda}\right)_m \frac{c\tau}{\pi} = 2.97 \times 10^7 \text{ km}$$

and

$$m = \left(\frac{\Delta\lambda}{\lambda}\right)_m^3 \frac{c^3\tau}{2\pi\gamma} = 2.9 \times 10^{29} \text{ kg}$$

5.229 We define the frame S (the lab frame) by the condition of the problem. In this frame the mirror is moving with velocity V (along say x-axis) towards left and light of frequency ω_0 is approaching it from the left. We introduce the frame S' whose axes are parallel to those of S but which is moving with velocity V along x-axis towards left (so that the mirror is at rest in S'). In S', the frequency of the incident light is

$$\omega_1 = \omega_0 \left(\frac{1 + V/c}{1 - V/c}\right)^{1/2}$$

In S', the reflected light still has frequency ω_1 but it is now moving towards left. When we transform back to S, this reflected light has the frequency

$$\omega = \omega_1 \left(\frac{1 + V/c}{1 - V/c}\right)^{1/2} = \omega_0 \left(\frac{1 + V/c}{1 - V/c}\right)$$

In the non-relativistic limit

$$\omega \approx \omega_0 \left(1 + \frac{2V}{c}\right)$$

5.230 From the previous problem, the beat frequency is clearly

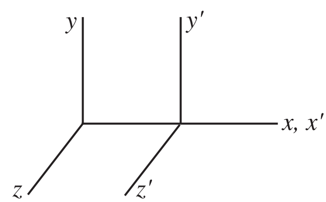
$$\Delta\nu = \nu_0 \frac{2v}{c} = \frac{2v}{(c/\nu_0)} = \frac{2v}{\lambda_0}$$

Hence,

$$v = \frac{1}{2}\lambda\Delta\nu = \frac{10^3}{2} \times 50 \text{ cm/s} = 900 \text{ km/h}$$

5.231 From the invariance of phase under Lorentz transformations, we get

$$\omega t - kx = \omega' t' - k'x'$$



Here $\omega = ck$. The primed coordinates refer to the frame S' which is moving to the right with velocity v.

Then,

$$x' = \gamma\,(x - vt)$$

and

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

where $\gamma = 1/(\sqrt{1 - v^2/c^2})$.

Substituting and equating the coefficients of t and x, we get

$$\omega = \gamma\omega' + \gamma k' v = \omega' \frac{1 + v/c}{\sqrt{1 - v^2/c^2}}$$

and
$$k = \gamma \frac{\omega' v}{c^2} + \gamma k' = k' \frac{1 + v/c}{\sqrt{1 - v^2/c^2}}$$

5.232 From the previous problem using $k = 2\pi/\lambda$, we get

$$\lambda' = \lambda\sqrt{\frac{1 + v/c}{1 - v/c}}$$

Thus,
$$\frac{1 + v/c}{1 - v/c} = \frac{\lambda'^2}{\lambda^2}$$

or
$$\frac{v}{c} = \frac{\lambda'^2 - \lambda^2}{\lambda'^2 + \lambda^2} = \frac{564^2 - 434^2}{564^2 + 434^2} = 0.256$$

5.233 As in the previous problem

$$\frac{v}{c} = \frac{\lambda^2 - \lambda'^2}{\lambda^2 + \lambda'^2}$$

So,
$$v = c\frac{(\lambda/\lambda')^2 - 1}{(\lambda/\lambda')^2 + 1} = 7.1 \times 10^4 \text{ km/s}$$

5.234 We go to the frame in which the observer is at rest. In this frame, the velocity of the source of light is, by relativistic velocity addition formula

$$v = \frac{\frac{3}{4}c - \frac{1}{2}c}{1 - \left[\frac{3}{4}c \cdot \frac{1}{2} \cdot \frac{c}{c^2}\right]} = \frac{2c}{5}$$

When this source emits light of proper frequency ω_0, the frequency recorded by observer will be

$$\omega = \omega_0\sqrt{\frac{1 - v/c}{1 + v/c}} = \sqrt{\frac{3}{7}}\,\omega_0$$

Note that $\omega < \omega_0$ as the source is moving away from the observer (red shift).

5.235 In transverse Doppler effect

$$\omega = \omega_0\sqrt{1-\beta^2} \approx \omega_0\left(1 - \frac{1}{2}\beta^2\right)$$

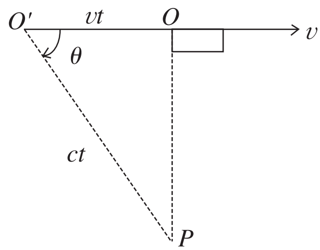
So, $$\lambda = \frac{c}{\omega} = \frac{c}{\omega_0}\left(1+\frac{1}{2}\beta^2\right) = \lambda_0\left(1+\frac{1}{2}\beta^2\right)$$

Hence, $$\Delta\lambda = \frac{1}{2}\beta^2\lambda$$

Using $$\beta^2 = \frac{v^2}{c^2} = \frac{2T}{mc^2} \quad \text{(where } T = \text{K.E. of H atoms)}$$

we get, $$\Delta\lambda = \frac{T}{mc^2}\lambda = \frac{1}{938}\times 656.3 \text{ nm} = 0.70 \text{ nm}$$

5.236 (a) If light is received by the observer at P at the moment when the source is at O, it must have been emitted by the source when it was at O'' and travelled along $O'P$. Then if $O'P = ct$, $O'O = vt$ and $\cos\theta = v/c = \beta$.



In the frame of the observer, the frequency of the light is ω while its wave vector is

$$\frac{\omega}{c}(\cos\theta, \sin\theta, 0)$$

We can calculate the value of ω by relating it to proper frequency ω_0. The relation is

$$\omega_0 = \frac{\omega}{\sqrt{1-\beta^2}}(1-\beta\cos\theta)$$

To derive the formula in this form it is easiest to note that

$$\frac{\omega}{\sqrt{1-v^2/c^2}} - \frac{\mathbf{k}\cdot\mathbf{v}}{\sqrt{1-v^2/c^2}}$$

is an invariant which takes the value ω_0 in the rest frame of the source.

Thus, $$\omega = \frac{\omega_0\sqrt{1-\beta^2}}{1-\beta^2} = \frac{\omega_0}{\sqrt{1-\beta^2}} = 5\times 10^{10}\ \text{s}^{-1}$$

(b) For the light to be received at the instant observer sees the source at O, light must be emitted when the observer is at O at $\theta = 90°$ or $\cos\theta = 0$.

Then, as before

$$\omega_0 = \frac{\omega}{\sqrt{1-\beta^2}} \quad \text{or} \quad \omega = \omega_0\sqrt{1-\beta^2} = 1.8\times 10^{10}\ \text{s}^{-1}$$

In this case the observer will receive light along OP and he will "see" that the source is at O even though the source will have moved ahead at the instant the light is received.

5.237 An electron moving in front of a metal mirror sees an image charge of equal and opposite type. The two together constitute a dipole. Let us look at the problem in the rest frame of the electron. In this frame the grating period is Lorentz contracted to

$$d' = d\sqrt{1 - v^2/c^2}$$

Because the metal has etchings, the dipole moment of electron–image pair is periodically disturbed with a period d'/v. The corresponding frequency is v/d', which is also the proper frequency of radiation emitted. Due to Doppler effect, the frequency observed at an angle θ is

$$\nu = \nu' \frac{\sqrt{1 - (v/c)^2}}{1 - v/c \cos\theta} = \frac{v/d}{1 - v/c \cos\theta}$$

The corresponding wavelength is

$$\lambda = \frac{c}{\nu} = d\left(\frac{c}{v} - \cos\theta\right)$$

Putting $c \approx v$, $\theta = 45°$, $d = 2\mu\text{m}$, we get

$$\lambda = 0.586\ \mu\text{m}$$

5.238 (a) Let v_x be the projection of velocity vector of the radiating atom in the observer's direction. The number of atoms with projections falling within the interval v_x and $v_x + dv_x$ is

$$n(v_x)dv_x \sim \exp(-mv_x^2/2kT)dv_x$$

The frequency of light emitted by the atoms moving with velocity v_x is

$$\omega = \omega_0\left(1 + \frac{v_x}{c}\right)$$

From the expressions the frequency distribution of atoms can be found: $n(\omega)d\omega = n(v_x)dv_x$. Now using

$$v_x = c\frac{\omega - \omega_0}{\omega_0}$$

we get
$$n(\omega)d\omega \sim \exp\left(-\frac{mc^2}{2kT}\left(1 - \frac{\omega}{\omega_0}\right)^2\right)\frac{cd\omega}{\omega_0}$$

Now the spectral radiation density $I_\omega \propto n_\omega$.

Hence, $$I_\omega = I_0 e^{-a\left(1-\frac{\omega}{\omega_0}\right)^2} \quad \left(\text{where } a = \frac{mc^2}{2kT}\right)$$

(The constant of proportionality is fixed by I_0.)

(b) On putting $\omega = \omega_0 \pm 1/2\,\Delta\omega$ and using $I_\omega = I_0/2$ in the above relation, we get

$$\frac{1}{2} = e^{-a\left(\frac{\Delta\omega}{2\omega_0}\right)^2}$$

So, $$a\left(\frac{\Delta\omega}{2\omega_0}\right)^2 = \ln 2$$

Hence, $$\frac{\Delta\omega}{2\omega_0} = \sqrt{(2\ln 2)\frac{kT}{mc^2}}$$

and $$\frac{\Delta\omega}{\omega} = 2\sqrt{(2\ln 2)\frac{kT}{mc^2}}$$

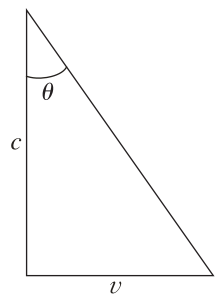
5.239 In vacuum, inertial frames are all equivalent; the velocity of light is c in any frame. This equivalence of inertial frames does not hold in material media and here the frame in which the medium is at rest is singled out. It is in this frame that the velocity of light is c/n where n is the refractive index of light for that medium.

The velocity of light in the frame in which the medium is moving is given by the law of addition of velocities as

$$\frac{c/n + V}{1 + c/n \cdot V/c^2} = \frac{c/n + V}{1 + V/cn}\left(\frac{c}{n} + V\right)\left(1 - \frac{V}{cn} + \cdots\right)$$
$$= \frac{c}{n} + V - \frac{V}{n^2} + \cdots$$
$$\approx \frac{c}{n} + V\left(1 - \frac{1}{n^2}\right)$$

This is the velocity of light in the medium in a frame in which the medium is moving with velocity $V \ll c$.

5.240 Although speed of light is the same in all inertial frames of reference according to the principles of relativity, the direction of a light ray can appear different in different frames. This phenomenon is called aberration and to first order in v/c, can be calculated by the elementary law of addition of velocities applied to light waves.



The angle of aberration is $\tan^{-1} v/c$ and in the present case it equals $1/2\,\delta\theta$ on either side. Thus, on equating

$$\frac{v}{c} = \tan\frac{1}{2}\delta\theta \approx \frac{1}{2}\delta\theta \quad (\delta\theta \text{ in radians})$$

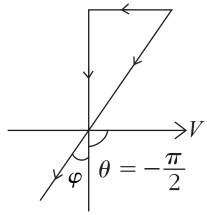
or $$v = \frac{c}{2}\delta\theta = \frac{3\times 10^8}{2} \times \frac{41}{3600} \times \frac{\pi}{180}$$

$$= \frac{3\times 4.1\times \pi}{3.6\times 3.6} \times 10^4 \text{ m/s} = 29.8 \text{ km/s}$$

5.241 We consider the invariance of the phase of a wave moving in the x–y plane. We write

$$\omega' t' - k'_x \chi' - k'_y y' = \omega t - k_x \chi - k_y y$$

From Lorentz transformations, L.H.S. is



$$\omega'\gamma\left(t - \frac{Vx}{c^2}\right) - k'_x(x - Vt)\gamma - k'_y y$$

So equating $$\omega = \gamma\,(\omega' + Vk'_x)$$

$$k_x = \gamma\left(k'_x + \frac{V\omega'}{c^2}\right)$$

and $$k_y = k'_y$$

on inverting $$\omega' = \gamma(\omega - Vk_x)$$

$$k'_x = \gamma\left(k_x - \frac{V\omega}{c^2}\right)$$

$$k'_y = k_y$$

So, $$k'_x = k' \cos\theta',\ k_x = k\cos\theta$$

and $$k'_y = k' \sin\theta',\ k_y = k\sin\theta$$

We get on using $ck' = \omega'$ and $ck = \omega$

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}$$

where $\beta = V/c$ and the primed frame is moving with velocity V in the x-direction with respect to the unprimed frame. For small $\beta \ll 1$, the situation is as given.

We see that $\cos\theta' \approx -\beta$ if $\theta = -\pi/2$. Then,

$$\theta' = -\left(\frac{\pi}{2} + \sin^{-1}\beta\right)$$

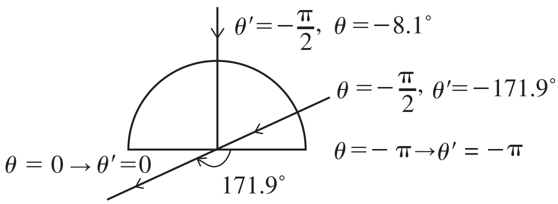
This is exactly what we get from elementary non-relativistic law of addition of velocities.

5.242 The statement of the problem is not quite properly worked out and is in fact misleading. The correct situation is described below. We consider, for simplicity, stars in the x–z plane. Then the previous formula is applicable, and we have

$$\cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta} = \frac{\cos\theta - 0.99}{1 - 0.99\cos\theta}$$

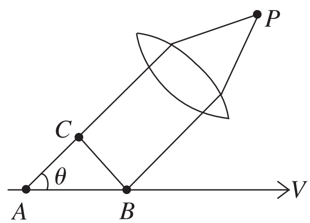
The distribution of θ' is given in the figure.

The light that appears to come from the forward quadrant in the frame K ($\theta = -\pi$ to $\theta = -\pi/2$) is compressed into an angle of magnitude 8.1° in the forward direction while the remaining stars are spread out.



The three-dimensional distribution can also be found out from the three-dimensional generalization of the formula in the previous problems.

5.243 The field induced by a charged particle moving with velocity V excites the atoms of the medium turning them into sources of light waves. Let us consider two arbitrary points A and B along the path of the particle. The light waves emitted from these points when the particle passes them reach the point P simultaneously and reinforce each other provided they are in phase, which is the case in general if the time taken by the light wave to propagate from the point A to the point C is equal to that taken by the particle to fly over the distance AB. Hence, we obtain



$$\cos\theta = \frac{v}{V}$$

where $v = c/n$ is the phase velocity of light. It is evident that the radiation is possible only if $V > v$, i.e., when the velocity of the particle exceeds the phase velocity of light in the medium.

5.244 We must have

$$V \geq \frac{c}{n} = \frac{3}{1.6} \times 10^8 \text{ m/s} \quad \text{or} \quad \frac{V}{c} \geq \frac{1}{1.6}$$

For electrons this means a K.E. greater than

$$T_e \geq m_e c^2 \left[\frac{n}{\sqrt{n^2 - 1}} - 1\right] = \frac{m_e c^2}{\sqrt{1 - (1/1.6)^2}} - m_e c^2$$

$$= 0.511\left[\frac{1}{\sqrt{1-(1/1.6)^2}} - 1\right] \quad (\text{using } m_e c^2 = 0.511 \text{ MeV})$$

$$= 0.144 \text{ MeV}$$

For protons with $m_p c^2 = 938$ MeV, we have

$$T_p \geq 938\left[\frac{1}{\sqrt{1-(1/1.6)^2}} - 1\right] = 264 \text{ MeV} = 0.264 \text{ GeV}$$

Also, $$T_{\min} = 29.6 \text{ MeV} = mc^2\left[\frac{1}{\sqrt{1-(1/1.6)^2}} - 1\right]$$

Then $mc^2 = 105.3$ MeV. This is very nearly the mass of muons.

5.245 From $\cos\theta = v/V$, we get

$$V = v \sec\theta$$

So, $$\frac{V}{c} = \frac{v}{c}\sec\theta = \frac{\sec\theta}{n} = \frac{\sec 30^\circ}{1.5} = \frac{2/\sqrt{3}}{3/2} = \frac{4}{3\sqrt{3}}$$

Thus, for electrons

$$T_e = 0.511\left[\frac{1}{\sqrt{1-16/27}} - 1\right] = 0.511\left[\sqrt{\frac{27}{11}} - 1\right] = 0.289 \text{ MeV}$$

Generally $$T = mc^2\left[\frac{1}{\sqrt{1-1/n^2\cos^2\theta}} - 1\right]$$

5.7 Thermal Radiation. Quantum Nature of Light

5.246 (a) The most probable radiation frequency ω_{pr} is the frequency for which

$$\frac{d}{d\omega}u_\omega = 3\omega^2 F\left(\frac{\omega}{T}\right) + \frac{\omega^2}{T}F'(\omega/T) = 0$$

The maximum frequency is the root other than $\omega = 0$ of this equation. It is

$$\omega = -\frac{3TF(\omega/T)}{F'(\omega/T)}$$

or $$\omega_{pr} = x_0\, T$$

where x_0 is the solution of the transcendental equation

$$3F(x_0) + x_0 F'(x_0) = 0$$

(b) The maximum spectral density is the density corresponding to most probable frequency.

$$(u_\omega)_{\max} = x_0^3 F(x_0) T^3 \propto T^3$$

where x_0 is defined above.

(c) The radiosity is

$$M_e = \frac{c}{4}\int_0^\infty \omega^3 F\left(\frac{\omega}{T}\right) d\omega = T^4 \left[\frac{c}{4}\int_0^\infty x^3 F(x) dx\right] \propto T^4$$

5.247 For the first black body

$$(\lambda_m)_1 = \frac{b}{T_1}$$

Then, $$(\lambda_m)_2 = \frac{b}{T_1} + \Delta\lambda = \frac{b}{T_2}$$

Hence, $$T_2 = \frac{b}{b/T + \Delta\lambda} = \frac{bT_1}{b + T_1\Delta\lambda} = 1.747 \text{ kK}$$

5.248 From the radiosity we get the temperature of the black body. It is

$$T = \left(\frac{M_e}{\sigma}\right)^{1/4} = \left(\frac{3.0 \times 10^4}{5.67 \times 10^{-8}}\right)^{1/4} = 852.9 \text{ K}$$

Hence, the wavelength corresponding to the maximum emissive capacity of the body is

$$\frac{b}{T} = \frac{0.29}{852.9} \text{ cm} = 3.4 \times 10^{-4} \text{ cm} = 3.4 \ \mu\text{m}$$

(Note that $3.0 \text{ W/cm}^2 = 3.0 \times 10^4 \text{ W/m}^2$.)

5.249 The black body temperature of the Sun may be taken as

$$T = \frac{0.29}{0.48 \times 10^{-4}} = 6042 \text{ K}$$

Thus, the radiosity is

$$M = 5.67 \times 10^{-8} (6042)^4 = 0.755 \times 10^8 \text{ W/m}^2$$

Energy lost by Sun is

$$4\pi(6.95)^2 \times 10^{16} \times 0.7555 \times 10^8 = 4.5855 \times 10^{26} \text{ W}$$

This corresponds to a mass loss of

$$\frac{4.5855 \times 10^{26}}{9 \times 10^{16}} \text{ kg/s} = 5.1 \times 10^{9} \text{ kg/s}$$

The Sun loses 1% of its mass in

$$\frac{1.97 \times 10^{30} \times 10^{-2}}{5.1 \times 10^{9}} \text{ s} \approx 1.22 \times 10^{11} \text{ years}$$

2.250 For an ideal gas, $p = nkT$, where n = number density of the particles and $k = R/N_A$ is Boltzmann's constant. In a fully ionized hydrogen plasma, both H ions (protons) and electrons contribute to pressure but since the mass of electrons is quite small ($\approx m_p/1836$), only protons contribute to mass density. Thus,

$$n = \frac{2\rho}{m_{\mathrm{H}}}$$

and
$$P = \frac{2\rho R}{N_A m_{\mathrm{H}}} T$$

(where $m_{\mathrm{H}} \approx m_p$ is the proton or hydrogen mass).

Equating this to thermal radiation pressure, we get

$$\frac{2\rho R}{N_A m_{\mathrm{H}}} T = \frac{u}{3} = \frac{M_e}{3} \times \frac{4}{c} = \frac{4\sigma T^4}{3c}$$

Then
$$T^3 = \frac{3c\rho R}{2\sigma N_A m_{\mathrm{H}}} = \frac{3c\rho R}{\sigma M}$$

(where $M = 2\,N_A m_{\mathrm{H}}$ = molecular weight of hydrogen $= 2 \times 10^{-3}$ kg).

Thus,
$$T = \left(\frac{3c\rho R}{\sigma M}\right)^{1/3} \approx 1.89 \times 10^{7} \text{ K}$$

5.251 In time dt after the instant t when the temperature of the ball is T, it loses $\pi d^2 \sigma T^4 dt$ Joules of energy. As a result its temperature falls by $-dT$ and

$$\pi d^2 \sigma T^4 dt = -\frac{\pi}{6} d^3 \rho C dT$$

(where ρ = density of copper and C = its specific heat).

Thus,
$$dt = \frac{C\rho d}{6\sigma} \frac{dT}{T^4}$$

or
$$t_0 = \frac{C\rho d}{6\sigma} = \int_{T_0}^{T_0/\eta} -\frac{dT}{T^4}$$

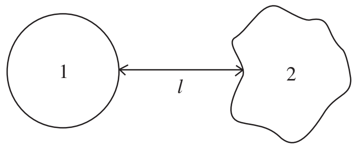
$$= \frac{C\rho d}{18\sigma T_0^3}(\eta^3 - 1) = 2.94 \text{ h}$$

5.252 Taking account of cosine law of emission, we write for the energy radiated per second by the hole in cavity 1 as

$$dI(\Omega) = A\cos\theta\, d\Omega$$

where A is a constant and $d\Omega$ is an element of solid angle around some direction defined by the symbol Ω. Integrating over the whole forward hemisphere, we get

$$I = A\int_0^{\pi/2} \cos\theta\, 2\pi \sin\theta\, d\theta = \pi A$$



We find A by equating this to the quantity

$$\sigma T_1^4 \cdot \frac{\pi d^2}{4}$$

(where σ is Stefan-Boltzmann constant and d is the diameter of the hole).

Then $$A = \frac{1}{4}\sigma d^2 T_1^4$$

Now energy reaching cavity 2 from cavity 1 is

$$= \frac{1}{4}\sigma d^2 T_1^4 \cdot \Delta\Omega \quad (\text{where } \cos\theta \approx 1)$$

where $\Delta\Omega = (\pi d^2/4)/l^2$ is the solid angle subtended by the hole of cavity 2 at cavity 1. (We are assuming $d \ll l$ so $\Delta\Omega$ = area of hole/(distance)^2). This must be equal to $\sigma T_2^4 \pi d^2/4$ which is the energy emitted by cavity 2. Thus equating, we get

$$\frac{1}{4}\sigma d^2 T_1^4 \frac{\pi d^2}{4l^2} = \sigma T_2^4 \frac{\pi d^2}{4}$$

or $$T_2 = T_1\sqrt{\frac{d}{2l}}$$

Substituting, we get T_2 = 0.380 kK = 380 K.

5.253 (a) The total internal energy of the cavity is

$$U = \frac{4\sigma}{c}T^4 V$$

Hence, $$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{16\sigma}{c}T^3 V$$

$$= \frac{16 \times 5.67 \times 10^{-8}}{3 \times 10^8} \times 10^9 \times 10^{-3} \text{ J/K}$$

$$= \frac{1.6 \times 5.67}{3} \text{ nJ/K} = 3.024 \text{ nJ/K}$$

(b) From the first law

$$TdS = dU + pdV$$

$$= VdU + UdV + \frac{U}{3} dV \quad \left(\text{as } p = \frac{U}{3}\right)$$

$$= VdU + \frac{4U}{3} dV$$

$$= \frac{16\sigma}{c} VT^3 dT + \frac{16\sigma}{3c} T^4 dV$$

So, $$dS = \frac{16\sigma}{c} VT^2 dT + \frac{16\sigma}{3c} T^3 dV$$

$$= d\left(\frac{16\sigma}{3c} VT^3\right)$$

Hence, $$S = \frac{16\sigma}{3c} VT^3 = \frac{1}{3} C_V = 1.008 \text{ nJ/K}$$

5.254 We are given

$$u(\omega, T) = A\omega^3 \exp(-a\omega/T)$$

(a) Then $$\frac{du}{d\omega} = \left(\frac{3}{\omega} - \frac{a}{T}\right) u = 0$$

So, $$\omega_{\text{pr}} = \frac{3T}{a} = \frac{6000}{7.64} \times 10^{12} \text{ s}^{-1} = 7.853 \times 10^{14} \text{ s}^{-1}$$

(b) We determine the spectral distribution in wavelength

$$-\tilde{u}(\lambda, T)\, d\lambda = u(\omega, T)\, d\omega$$

But $$\omega = \frac{2\pi c}{\lambda} \quad \text{or} \quad \lambda = \frac{2\pi c}{\omega} = \frac{C'}{\omega}$$

So, $$d\lambda = -\frac{C'}{\omega^2} d\omega \quad \text{and} \quad d\omega = -\frac{C'}{\lambda^2} d\lambda$$

(We have put a minus sign before $d\lambda$ to subserve just this fact that $d\lambda$ is $-ve$ where $d\omega$ is $+ve$.)

Then, $$\tilde{u}(\lambda, T) = \frac{C'}{\lambda^2} u\left(\frac{C'}{\lambda}, T\right) = \frac{C'^4 A}{\lambda^5} \exp\left(-\frac{aC'}{\lambda^2 T}\right)$$

This is maximum when

$$\frac{\partial \tilde{u}}{\partial \lambda} = 6 = \tilde{u}\left[\frac{-5}{\lambda} + \frac{aC'}{\lambda^2 T}\right]$$

or $$\lambda_{pr} = \frac{aC'}{5T} = \frac{2\pi ca}{5T} = 1.44\ \mu\text{m}$$

5.255 From Planck's formula

$$u_\omega = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{\hbar\omega/kT} - 1}$$

(a) In a range $\hbar\omega \ll kT$ (long wavelength or high temperature)

$$u_\omega \to \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\hbar\omega/kT}$$

$$= \frac{\omega^2}{\pi^2 c^3} kT \quad \text{(using } e^x \equiv 1 + x \text{ for small } x\text{)}$$

(b) In the range $\hbar\omega \gg T$ (high frequency or low temperature)

$$\frac{\hbar\omega}{kT} \gg 1 \quad \text{so,} \quad e^{\frac{\hbar\omega}{kT}} \gg 1$$

and $$u_\omega \equiv \frac{\hbar\omega^3}{\pi^2 c^3} e^{-\hbar\omega/kT}$$

5.256 We write

$$u_\omega d\omega = \tilde{u}_\nu d\nu \quad \text{(where } \omega = 2\pi\nu\text{)}$$

Then $$\tilde{u}_\nu = \frac{2\pi\hbar(2\pi\nu)^3}{\pi^2 c^3} \frac{1}{e^{2\pi\hbar\nu/kT} - 1} = \frac{16\pi^2\hbar\nu^3}{c^3} \frac{1}{e^{2\pi\hbar\nu/kT} - 1}$$

Also $$-\tilde{u}(\lambda, T) d\lambda = u_\omega d\omega \quad \left(\text{where } \lambda = \frac{2\pi c}{\omega}\right)$$

$$d\omega = -\frac{2\pi c}{\lambda^2} d\lambda$$

$$\tilde{u}(\lambda, T) = \frac{2\pi c}{\lambda^2} u\left(\frac{2\pi c}{\lambda}, T\right)$$

$$= \frac{2\pi c}{\lambda^2}\left(\frac{2\pi c}{\lambda}\right)^3 \frac{\hbar}{\pi^2 c^3} \frac{1}{e^{2\pi\hbar c/\lambda kT} - 1} = \frac{16\pi^2 c\hbar}{\lambda^5} \frac{1}{e^{2\pi\hbar c/\lambda kT} - 1}$$

5.257 We write the required power in terms of the radiosity by considering only the energy radiated in the given range. Then, from the previous problem we have

$$\Delta P = \frac{c}{4}\tilde{u}(\lambda_m, T)\Delta\lambda$$

$$= \frac{4\pi^2 c^2 \hbar}{\lambda_m^5}\frac{\Delta\lambda}{e^{2\pi c\hbar/k\lambda_m T} - 1}$$

But $\lambda_m T = b$, so

$$\Delta P = \frac{4\pi^2 c^2 \hbar T^5}{b^5}\frac{1}{e^{2\pi c\hbar/kb} - 1}\Delta\lambda$$

Using the given data, we get

$$\frac{2\pi c\hbar}{kb} = \frac{2\pi \times 3 \times 10^8 \times 1.05 \times 10^{-34}}{1.38 \times 10^{-23} \times 2.9 \times 10^{-3}} = 4.9643$$

$$\frac{1}{e^{2\pi c\hbar/kb} - 1} = 7.03 \times 10^{-3}$$

and $$\Delta P = 0.312 \text{ W/cm}^2$$

5.258 (a) From the curve of the function $y(x)$, we see that $y = 0.5$ when $x = 1.41$.

Thus, $$\lambda = 1.41 \times \frac{0.29}{3700} \text{ cm} = 1.105 \ \mu\text{m}$$

(b) At 5000 K,

$$\lambda = \frac{0.29}{0.5} \times 10^{-6} \text{ m} = 0.58 \ \mu\text{m}$$

So the visible range (0.40 to 0.70) μm corresponds to a range (0.69 to 1.31) of x. From the curve

$$y(0.69) = 0.07$$

and $$y(1.31) = 0.44$$

So the required fraction is 0.37.

(c) The values of x corresponding to 0.76 are

$$x_1 = \frac{0.76}{0.29/0.3} = 0.786 \text{ at } 3000 \text{ K}$$

and $$x_2 = \frac{0.76}{0.29/0.5} = 1.31 \text{ at } 5000 \text{ K}$$

The requisite fraction is then

$$\underbrace{\frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^4}_{\text{ratio of total power}} \times \underbrace{\frac{1-y_2}{1-y_1}}_{\text{ratio of the fraction of required wavelengths in the radiated power}}$$

$$= \left(\frac{5}{3}\right)^4 \times \frac{1-0.44}{1-0.12} = 4.91$$

5.259 We use the formula $\varepsilon = \hbar\omega$.

Then, the number of photons in the spectral interval (ω, $\omega + d\omega$) is

$$n(\omega)d\omega = \frac{u(\omega, T)d\omega}{\hbar\omega} = \frac{\omega^2}{\pi^2 c^3}\frac{1}{e^{\hbar\omega/kT}-1}d\omega$$

Using $n(\omega)d\omega = -\tilde{n}(\lambda)d\lambda$, we get

$$d\lambda\,\tilde{n}(\lambda) = n\left(\frac{2\pi c}{\lambda}\right)\frac{2\pi c}{\lambda^2}d\lambda$$

$$= \frac{(2\pi c)^3}{\pi^2 c^3 \lambda^4}\frac{1}{e^{2\pi\hbar c/k\lambda T}-1}d\lambda$$

$$= \frac{8\pi}{\lambda^4}\frac{1}{e^{2\pi\hbar c/k\lambda T}-1}$$

5.260 (a) The mean density of the flow of photons at a distance r is

$$<j> = \frac{P/4\pi r^2}{2\pi\hbar c/\lambda} = \frac{P\lambda}{8\pi^2\hbar c r^2}\ \mathrm{m^{-2}s^{-1}}$$

$$= \frac{10 \times 0.589 \times 10^{-6}}{8\pi^2 \times 1.054 \times 10^{-34} \times 10^8 \times 4}\ \mathrm{m^{-2}s^{-1}}$$

$$= \frac{10 \times 0.589}{8\pi^2 \times 1.054 \times 12} \times 10^{16}\ \mathrm{cm^{-2}s^{-1}}$$

$$= 5.9 \times 10^{13}\ \mathrm{cm^{-2}\,s^{-1}}$$

(b) If $n(r)$ is the mean concentration (number per unit volume) of photons at a distance r from the source, then, since all photons are moving outwards with a velocity c, there is an outward flux of cn which is balanced by the flux from the source. In steady state, the two are equal, so

$$n(r) = \frac{<j>}{c} = \frac{P\lambda}{8\pi^2 \hbar c^2 r^2} = n$$

So,
$$r = \frac{1}{2\pi c}\frac{P\lambda}{2\hbar n}$$

$$= \frac{1}{6\pi \times 10^8}\sqrt{\frac{10 \times 0.589 \times 10^{-6}}{2 \times 1.054 \times 10^{-34} \times 10^2 \times 10^6}}$$

$$= \frac{10^2}{6\pi}\sqrt{\frac{5.89}{2.108}} = 8.87 \text{ m}$$

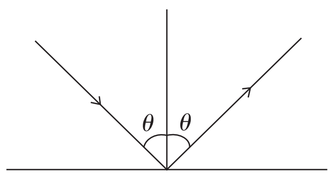
5.261 The statement made in the question is not always correct. However, it is correct in certain cases, for example, when light is incident on a perfect reflector or perfect absorber. Consider the former case. If light is incident at an angle θ and reflected at the angle θ, then momentum transferred by each photon is

$$2\frac{h\nu}{c}\cos\theta$$

If there are $n(\nu)\, d\nu$ photons in frequency interval $(\nu,\ \nu + d\nu)$, then total momentum transferred is

$$\int_0^\infty 2n(\nu)\frac{h\nu}{c}\cos\theta d\nu$$

$$= \frac{2\Phi_e}{c}\cos\theta$$



Thus,
$$\frac{dp}{dt} = \frac{\Phi_e}{c}$$

5.262 Momentum of a photon

$$p = \frac{h}{\lambda} = \frac{h\nu}{c} = \frac{E}{c}$$

So, the change in momentum per unit time will be

$$2\rho\frac{E}{c\tau} + (1-\rho)\frac{E}{c\tau} = (1+\rho)\frac{E}{c\tau} \tag{1}$$

The first term in Eq. (1) is the momentum transferred on reflection and the second on absorption.

The mean pressure $<p>$ is related to the force F exerted by the beam by the expression

$$<p> \times \frac{\pi d^2}{4} = F$$

The force F equals momentum transferred per second. This is (assuming that photons that are not reflected are absorbed)

$$2\rho \frac{E}{c\tau} + (1-\rho)\frac{E}{c\tau} = (1+\rho)\frac{E}{c\tau}$$

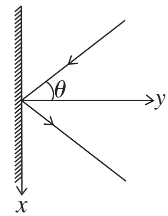
The first term is the momentum transferred on reflection (see Problem 5.261); the second on absorption. Hence,

$$<p> = \frac{4(1+\rho)E}{\pi d^2 c\tau}$$

$$= 48.3 \text{ atm (on substituting values)}$$

5.263 The momentum transferred to the plate is

$$\underbrace{\frac{E}{c}(1-\rho)\{\sin\theta\mathbf{i} - \cos\theta\mathbf{j}\}}_{\begin{pmatrix}\text{momentum}\\ \text{transferred}\\ \text{on absorption}\end{pmatrix}} + \underbrace{\frac{E}{C}\rho\{-2\cos\theta\mathbf{j}\}}_{\begin{pmatrix}\text{momentum}\\ \text{transferred}\\ \text{on reflection}\end{pmatrix}}$$



$$= \frac{E}{c}(1-\rho)\sin\theta\mathbf{j} - \frac{E}{c}(1+\rho)\cos\theta\mathbf{j}$$

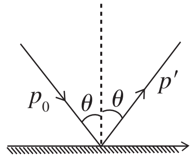
Its magnitude is

$$\frac{E}{c}\sqrt{(1-\rho)^2\sin^2\theta + (1+\rho)^2\cos^2\theta} = \frac{E}{c}\sqrt{1+\rho^2+2\rho\cos 2\theta}$$

Substitution gives 35 nN s as the answer.

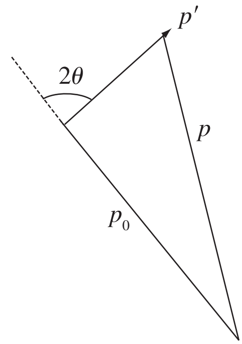
Alternate:

The Fig. (a) shows incident and reflected beams.



(a)

Having drawn the triangle of momenta in Fig. (b), where p_0, p' and p are the momenta of the incident beam, reflected beam, and the momentum transferred to the plate, respectively, and taking into account $p = E/c$, $p' = \rho\, E/c$, we obtain



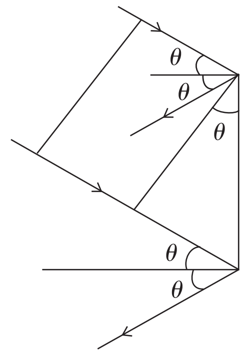
(b)

$$p = \frac{E}{c}\sqrt{1 + \rho^2 + 2\rho\cos 2\theta} = 35 \text{ nN s}$$

(on substituting values).

5.264 Suppose the mirror has a surface area A. The incident beam then has a cross-section of $A \cos \theta$ and the incident energy is $IA \cos \theta$. Then, from the previous problem, the momentum transferred per second (= force) is

$$-\frac{IA\cos\theta}{c}(1+\rho)\cos\theta\mathbf{j} + \frac{IA\cos\theta}{c}(1-\rho)\sin\theta\mathbf{i}$$



The normal pressure is then

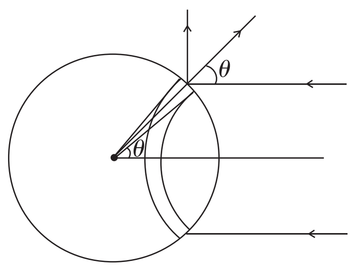
$$p = \frac{I}{c}(1+\rho)\cos^2\theta$$

(Here $\mathbf{j}$ is the unit vector perpendicular to the plane mirror.)

Putting in the values, we get

$$p = \frac{0.20 \times 10^4}{3 \times 10^8} \times 1.8 \times \frac{1}{2} = 0.6 \text{ nN/cm}^2$$

5.265 We consider a strip defined by the angular range $(\theta, \theta + d\theta)$. From the previous problem, the normal pressure exerted on this strip is



$$\frac{2I}{c}\cos^2\theta$$

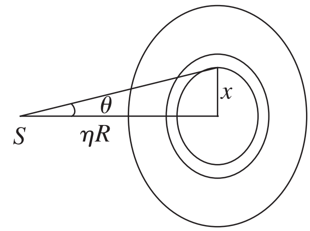
This pressure gives rise to a force whose resultant, by symmetry, is in the direction of the incident light.

Thus, $$F = \frac{2I}{c}\int_0^{\pi/2} \cos^2\theta\cos 2\pi R^2 \sin\theta d\theta = \pi R^2 \frac{I}{c}$$

Putting in the values, we get

$$F = \pi \times 25 \times 10^{-4}\, \frac{0.70 \times 10^4}{3 \times 10^8} \text{ N} = 0.183\ \mu\text{N}$$

5.266 Consider a ring of radius x on the plate. The normal pressure on this ring is, by Problem 5.264,



$$\frac{2}{c}\frac{P}{4\pi(x^2+\eta^2R^2)}\cos^2\theta$$

$$=\frac{P}{2\pi c}\frac{\eta^2R^2}{(x^2+\eta^2R^2)^2}$$

The total force is then

$$\int_0^R \frac{P}{2\pi c}\frac{\eta^2R^2}{(x^2+\eta^2R^2)^2}2\pi x dx$$

$$=\frac{P\eta^2R^2}{2c}\int_{\eta^2R^2}^{R^2(1+\eta^2)}\frac{dy}{y^2}$$

$$=\frac{P\eta^2R^2}{2c}\left[\frac{1}{\eta^2R^2}-\frac{1}{R^2(1+\eta^2)}\right]=\frac{P}{2c(1+\eta^2)}$$

5.267 (a) In the reference frame fixed to the mirror, the frequency of the photon is, by the Doppler shift formula

$$\bar{\omega}=\omega\sqrt{\frac{1+\beta}{1-\beta}}\left(=\omega\frac{\sqrt{1-\beta^2}}{1-\beta}\right)\text{ (see Eq. 5.6b of the book)}$$

In this frame, momentum imparted to the mirror is

$$\frac{2\hbar\bar{\omega}}{c}=\frac{2\hbar\omega}{c}\sqrt{\frac{1+\beta}{1-\beta}}$$

(b) In the K frame, the incident particle carries a momentum of $\hbar\omega/c$ and returns with momentum

$$\frac{\hbar\omega}{c}\frac{1+\beta}{1-\beta}\quad\text{(see Problem 5.229)}$$

The momentum imparted to the mirror, then, has the magnitude

$$\frac{\hbar\omega}{c}\left[\frac{1+\beta}{1-\beta}+1\right]=\frac{2\hbar\omega}{c}\frac{1}{1-\beta}$$

Here $\beta=V/c$.

5.268 When light falls on a small mirror and is reflected by it, the mirror recoils. The energy of recoil is obtained from the incident beam photon and the frequency of reflected photons is less than the frequency of the incident photons. This shift of frequency can however be neglected in calculating quantities related to recoil (to the first approximation).

Thus, the momentum acquired by the mirror as a result of the laser pulse is

$$\left|\mathbf{p}_f - \mathbf{p}_i\right| = \frac{2E}{c}$$

On assuming $\mathbf{p}_i = 0$, we get

$$\left|\mathbf{p}_f\right| = \frac{2E}{c}$$

Hence the K.E. of the mirror is

$$\frac{p_f^2}{2m} = \frac{2E^2}{mc^2}$$

Suppose the mirror is deflected by an angle θ. Then by conservation of energy,

$$\text{Final P.E.} = mgl(1 - \cos\theta) = \text{Initial K.E.} = \frac{2E^2}{mc^2}$$

or
$$mgl2\sin^2\frac{\theta}{2} = \frac{2E^2}{mc^2}$$

or
$$\sin\frac{\theta}{2} = \left(\frac{E}{mc}\right)\frac{1}{\sqrt{gl}}$$

Using the given data,

$$\sin\frac{\theta}{2} = \frac{13}{10^{-5} \times 3 \times 10^8\sqrt{9.8 \times 0.1}} = 4.377 \times 10^{-3}$$

or
$$\theta = 0.502°$$

5.269 We shall only consider stars which are not too compact so that the gravitational field at their surface is weak

$$\frac{\gamma M}{c^2 R} \ll 1$$

We shall also explain the problem by making clear the meaning of the (slightly changed) notation.

Suppose the photon is emitted by some atom whose total relativistic energies (including the rest mass) are E_1 and E_2 with $E_1 < E_2$. These energies are defined in the absence of gravitational field and we have

$$\omega_0 = \frac{E_2 - E_1}{\hbar}$$

as the frequency at infinity of the photon that is emitted in $2 \to 1$ transition. On the surface of the star, the energies become

$$E_2' = E_2 - \frac{E_2}{c^2} \cdot \frac{\gamma M}{R} = E_2\left(1 - \frac{\gamma M}{c^2 R}\right)$$

$$E_1' = E_1\left(1 - \frac{\gamma M}{c^2 R}\right)$$

Thus, from $\hbar\omega = E_2' = E_1'$, we get

$$\omega = \omega_0\left(1 - \frac{\gamma M}{c^2 R}\right)$$

Here ω is the frequency of the photon emitted in the transition $2 \to 1$ when the atom is on the surface of the star. It shows that the frequency of spectral lines emitted by atoms on the surface of some star is less than the frequency of lines emitted by atoms here on Earth (where the gravitational effect is quite small).

Finally, $$\frac{\Delta\omega}{\omega_0} \approx -\frac{\gamma M}{c^2 R}$$

(The answer given in the book is incorrect in general though it agrees with the above result for $\frac{\gamma M}{c^2 R} \ll 1$.)

5.270 The general formula is

$$\frac{2\pi\hbar c}{\lambda} = eV$$

Thus, $$\lambda = \frac{2\pi\hbar c}{eV}$$

Now, $$\Delta\lambda = \frac{2\pi\hbar c}{eV}\left(1 - \frac{1}{\eta}\right)$$

Hence, $$V = \frac{2\pi\hbar c}{e\Delta\lambda}\left(\frac{\eta - 1}{\eta}\right) = 15.9 \text{ kV}$$

5.271 We have, as in the previous problem

$$\frac{2\pi\hbar c}{\lambda} = eV$$

On the other hand, from Bragg's law

$$2d \sin \alpha = k\lambda = \lambda$$

since $k = 1$ when α takes its smallest value.

Thus, $$V = \frac{\pi\hbar c}{ed \sin \alpha} = 30.974 \text{ k}V \approx 31 \text{ kV}$$

5.272 The wavelength of X-rays is the least when all the K.E. of the electrons approaching the anticathode is converted into the energy of X-rays. But the K.E. of electron is

$$T_m = mc^2\left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right]$$

(mc^2 = rest mass energy of electrons = 0.511 MeV).

Thus, $$\frac{2\pi\hbar c}{\lambda} = T_m$$

or $$\lambda = \frac{2\pi\hbar c}{T_m} = \frac{2\pi\hbar}{mc}\left[\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right]^{-1} = \frac{2\pi\hbar}{mc(\gamma - 1)}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = 2.70 \text{ pm}$$

5.273 The work function of zinc is

$$A_{\text{Zn}} = 3.74 \text{ eV} = 3.74 \times 1.602 \times 10^{-19} \text{ J}$$

The threshold wavelength for photoelectric effect is given by

$$\frac{2\pi\hbar c}{\lambda_0} = A_{\text{Zn}}$$

or $$\lambda_0 = \frac{2\pi\hbar c}{A} = 331.6 \text{ nm}$$

The maximum velocity of photoelectrons liberated by light of wavelength λ is given by

$$\frac{1}{2}mv_{\text{max}}^2 = 2\pi\hbar c\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)$$

So, $$v_{max} = \sqrt{\frac{4\pi\hbar c}{m}\left(\frac{1}{\lambda} - \frac{1}{\lambda_0}\right)} = 6.55 \times 10^5 \text{ m/s}$$

5.274 From the last equation of the previous problem, we find

$$\eta = \frac{(v_1)_{max}}{(v_2)_{max}} = \sqrt{\frac{1/\lambda_1 - 1/\lambda_0}{1/\lambda_2 - 1/\lambda_0}}$$

Thus, $$\eta^2\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_0}\right) = \frac{1}{\lambda_1} - \frac{1}{\lambda_0}$$

or $$\frac{1}{\lambda_0}(\eta^2 - 1) = \frac{\eta^2}{\lambda_2} - \frac{1}{\lambda_1}$$

and $$\frac{1}{\lambda_0} = \frac{(\eta^2/\lambda_2 - 1/\lambda_1)}{(\eta^2 - 1)}$$

So, $$A = \frac{2\pi\hbar c}{\lambda_0} = \frac{2\pi\hbar c}{\lambda_2}\frac{\eta^2 - \lambda_2/\lambda_1}{\eta^2 - 1} = 1.88 \text{ eV}$$

5.275 When light of sufficiently short wavelength falls on the ball, photoelectrons are ejected and the copper ball gains positive charge. The charged ball tends to resist further emission of electrons by attracting them. When the copper ball has enough charge, even the most energetic electrons are unable to leave it. We can calculate this final maximum potential of the copper ball. It is obviously equal in magnitude (in volt) to the maximum K.E. of electrons (in volt) initially emitted.

Hence, $$\varphi_{max} = \frac{2\pi\hbar c}{\lambda e} - A_{Cu}$$
$$= 8.86 - 4.47 = 4.39 \text{ V}$$

(Here A_{Cu} is the work function of copper.)

5.276 We are given

$$E = a\,(1 + \cos\omega t)\cos\omega_0 t$$
$$= a\cos\omega_0 t + \frac{a}{2}[\cos(\omega_0 - \omega)t + \cos(\omega_0 + \omega)t]$$

It is obvious that light has three frequencies and the maximum frequency is $\omega_0 + \omega$. Then, the maximum K.E. of photoelectrons ejected is

$$\hbar(\omega + \omega_0) - A_{Li} = 0.37 \text{ eV}$$

on substituting values and using $A_{Li} = 2.39$ eV.

5.277 Suppose N photons fall on the photocell per second. Then the power incident is

$$N\frac{2\pi\hbar c}{\lambda}$$

This will give rise to a photocurrent of

$$N\frac{2\pi\hbar c}{\lambda}\cdot J$$

which means that number of electrons emitted is

$$N\frac{2\pi\hbar c}{e\lambda}\cdot J$$

Thus, the number of photoelectrons produced by each photon is

$$w=\frac{2\pi\hbar cJ}{e\lambda}=0.0198\approx 0.02 \text{ (on substituting values)}$$

5.278 A simple application of Einstein's equation

$$\frac{1}{2}mv_{\max}^2=h\nu-h\nu_0=\frac{2\pi\hbar c}{\lambda}-A_{\mathrm{Cs}}$$

gives an incorrect result in this case because the photoelectrons emitted by the cesium electrode are retarded by the small electric field that exists between the cesium electrode and the copper electrode even in the absence of external emf. This small electric field is caused by the contact potential difference whose magnitude equals the difference of work functions

$$\frac{1}{e}(A_{\mathrm{Cu}}-A_{\mathrm{Cs}})\ \mathrm{V}$$

Its physical origin is explained below.

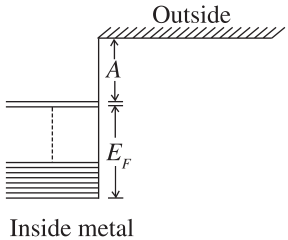
The maximum velocity of the photoelectrons reaching the copper electrode is then

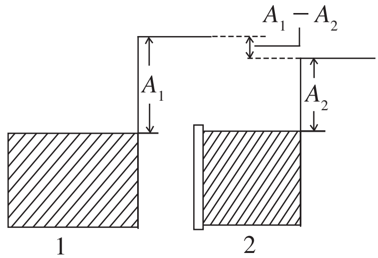
$$\frac{1}{2}mv_{\max}^2=\frac{1}{2}mv_0^2-(A_{\mathrm{Cu}}-A_{\mathrm{Cs}})=\frac{2\pi\hbar c}{\lambda}-A_{\mathrm{Cu}}$$

Here v_0 is the maximum velocity of the photoelectrons immediately after emission. Putting the values and using $A_{\mathrm{Cu}}=4.47$ eV, $\lambda=0.22$ μm, we get

$$v_{\max}=6.41\times 10^5 \text{ m/s}$$

The origin of contact potential difference is the following. Inside the metals free electrons can be thought of as Fermi gas which occupy energy levels up to a maximum called the Fermi energy E_F. The work function A measures the depth of the Fermi level.





When metals 1 and 2 are in contact, electrons flow from one to the other till their Fermi levels are the same. This requires the appearance of contact potential difference of $A_1 - A_2$ between the two metals externally.

5.279 The maximum K.E. of the photoelectrons emitted by the Zn cathode is

$$E_{\max} = \frac{2\pi\hbar c}{\lambda} - A_{\text{Zn}}$$

On calculating this comes out to be 0.993 eV ≈ 1.0 eV.

Since an external decelerating voltage of 1.5 V is required to cancel this current, we infer that a contact potential difference of $1.5 - 1.0 = 0.5$ V exists in the circuit whose polarity is opposite to the decelerating voltage.

5.280 The unit of $\hbar$ is Joule-seconds. Since mc^2 is the rest energy, $\hbar/mc^2$ has the dimension of time and multiplying by c we get a quantity

$$\bar{\lambda}_c = \frac{\hbar}{mc}$$

whose dimension is length. This quantity is called reduced Compton wavelength. (The name Compton wavelength is traditionally reserved for $2\pi\hbar/mc$)

5.281 We consider the collision in the rest frame of the initial electron. Then the reaction is

$$\gamma + e\,(\text{rest}) \rightarrow e\,(\text{moving})$$

Energy momentum conservation gives

$$\hbar\omega + m_0 c^2 = \frac{m_0 c^2}{\sqrt{1-\beta^2}}$$

or

$$\frac{\hbar\omega}{c} = \frac{m_0 c\beta}{\sqrt{1-\beta^2}}$$

(where ω is the angular frequency of the photon).

Eliminating $\hbar\omega$, we get

$$m_0c^2 = m_0c^2 \frac{1-\beta}{\sqrt{1-\beta^2}} = m_0c^2 \sqrt{\frac{1-\beta}{1+\beta}}$$

This gives $\beta = 0$ which implies $\hbar\omega = 0$.

But a zero energy photon means no photon.

5.282 (a) Compton scattering is the scattering of light by free electrons (free electrons are the electrons whose binding is much smaller than the typical energy transfer to the electrons). For this reason the increase in wavelength $\Delta\lambda$ is independent of the nature of the scattered substance.

(b) This is because the effective number of free electrons increases in both cases. With increasing angle of scattering, the energy transferred to electrons increases. With diminishing atomic number of the substance, the binding energy of the electrons decreases.

(c) The presence of a non-displaced component in the scattered radiation is due to scattering from strongly bound (inner) electrons as well as nuclei. For scattering by these, the atom essentially recoils as a whole, there is very little energy transfer.

5.283 Let λ_0 = wavelength of the incident radiation. Then wavelength of the radiation scattered at $\theta_1 = 60°$ is

$$\lambda_1 = \lambda_0 + 2\pi\bar{\lambda}_c\,(1-\cos\theta_1) \quad \left(\text{where } \bar{\lambda}_c = \frac{\hbar}{mc}\right)$$

Similarly $$\lambda_2 = \lambda_0 + 2\pi\bar{\lambda}_c\,(1-\cos\theta_2)$$

From the data $\theta_1 = 60°$, $\theta_2 = 120°$ and $\lambda_2 = \eta\lambda_1$, we get

$$(\eta-1)\lambda_0 = 2\pi\bar{\lambda}_c\,[1-\cos\theta_2 - \eta(1-\cos\theta_1)]$$

$$= 2\pi\bar{\lambda}_c\,[1-\eta+\eta\cos\theta_1 - \cos\theta_2]$$

Hence, $$\lambda_0 = 2\pi\bar{\lambda}_c\left[\frac{\eta\cos\theta_1 - \cos\theta_2}{\eta-1} - 1\right]$$

$$= 4\pi\bar{\lambda}_c\left[\frac{\sin^2\theta_2/2 - \eta\sin^2\theta_1/2}{\eta-1}\right]$$

$$= 1.21 \text{ pm (on substituting values)}$$

Note: *The expression λ_0 given in the book contains misprints.*

5.284 The wavelength of the photon has increased by a fraction η so its final wavelength is

$$\lambda_f = (2 + \eta)\,\lambda_i$$

and its energy is

$$\frac{\hbar\omega}{1+\eta}$$

The K.E. of the Compton electron is the energy lost by the photon and is given by

$$T = \hbar\omega\left(1 - \frac{1}{1+\eta}\right) = \hbar\omega\frac{\eta}{1+\eta}$$

$$= 0.20 \text{ MeV}$$

5.285 (a) From the Compton formula

$$\lambda' = 2\pi\bar{\lambda}_c\,(1 - \cos 90°) + \lambda$$

Thus, $$\omega' = \frac{2\pi c}{\lambda'} = \frac{2\pi c}{\lambda + 2\pi\bar{\lambda}_c} \quad \left(\text{where } 2\pi\bar{\lambda}_c = \frac{\hbar}{mc}\right)$$

Substituting the values, we get $\omega' = 2.24 \times 10^{20}$ rad/s.

(b) The K.E. of the scattered electron (in the frame in which the initial electron was stationary) is simply

$$T = \hbar\omega - \hbar\omega'$$

$$= \frac{2\pi\hbar c}{\lambda} - \frac{2\pi\hbar c}{\lambda + 2\pi\bar{\lambda}_c}$$

$$= \frac{4\pi^2\hbar c\bar{\lambda}_c}{\lambda(\lambda + 2\pi\bar{\lambda}_c)} = \frac{2\pi\hbar c/\lambda}{1 + \lambda/2\pi\bar{\lambda}_c} = 59.5 \text{ keV}$$

5.286 The wavelength of the incident photon is

$$\lambda_0 = \frac{2\pi c}{\omega}$$

Then, the wavelength of the final photon is

$$\frac{2\pi c}{\omega} + 2\pi\bar{\lambda}_c\,(1 - \cos\theta\,)$$

and the energy of the final photon is

$$\hbar\omega' = \frac{2\pi\hbar c}{2\pi c/\omega + 2\pi\bar{\lambda}_c(1-\cos\theta)} = \frac{\hbar\omega}{1 + \hbar\omega/mc^2(1-\cos\theta)}$$

$$= \frac{\hbar\omega}{1 + 2(\hbar\omega/mc^2)\sin^2(\theta/2)} = 144.2 \text{ keV}$$

5.287 We use the equation

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar}{p}$$

Then, from the Compton formula

$$\frac{2\pi\hbar}{p'} = \frac{2\pi\hbar}{p} + 2\pi\frac{\hbar}{mc}(1 - \cos\theta)$$

So,
$$\frac{1}{p'} = \frac{1}{p} + \frac{1}{mc}\cdot 2\sin^2\frac{\theta}{2}$$

Hence,
$$\sin^2\frac{\theta}{2} = \frac{mc}{2}\left(\frac{1}{p'} - \frac{1}{p}\right)$$

$$= \frac{mc(p - p')}{2pp'}$$

or
$$\sin\frac{\theta}{2} = \sqrt{\frac{mc(p - p')}{2pp'}}$$

Substituting the values, we get

$$\sin\frac{\theta}{2} = \sqrt{\frac{mc^2(cp - cp')}{2cp\cdot cp'}} = \sqrt{\frac{0.511(1.02 - 0.255)}{2\times 1.02\times 0.255}}$$

or
$$\theta = 120.2°$$

5.288 From the Compton formula

$$\lambda = \lambda_0 + \frac{2\pi\hbar}{mc}(1 - \cos\theta)$$

From conservation of energy

$$\frac{2\pi\hbar c}{\lambda_0} = \frac{2\pi\hbar c}{\lambda} + T$$

$$= \frac{2\pi\hbar c}{\lambda_0 + 2\pi\hbar/mc\,(1 - \cos\theta)} + T$$

or
$$\frac{4\pi\hbar}{mc}\sin^2\frac{\theta}{2} = \frac{T}{2\pi\hbar c}\lambda_0\left(\lambda_0 + \frac{4\pi\hbar}{mc}\sin^2\frac{\theta}{2}\right)$$

or $$\frac{2\sin^2\theta/2}{mc^2} = \frac{T}{\hbar\omega_0}\left(\frac{1}{\hbar\omega_0} + \frac{2}{mc^2}\sin^2\frac{\theta}{2}\right) \quad (\text{using } \hbar\omega_0 = 2\pi\hbar c/\lambda_0)$$

Hence, $$\left(\frac{1}{\hbar\omega_0}\right)^2 + 2\frac{1}{\hbar\omega_0}\frac{\sin^2\theta/2}{mc^2} - \frac{2\sin^2\theta/2}{mc^2 T} = 0$$

$$\left(\frac{1}{\hbar\omega_0} + \frac{\sin^2\theta/2}{mc^2}\right)^2 = \frac{2\sin^2\theta/2}{mc^2 T} + \left(\frac{\sin^2\theta/2}{mc^2}\right)^2$$

$$\frac{1}{\hbar\omega_0} = \frac{\sin^2\theta/2}{mc^2}\left[\sqrt{1 + \frac{2mc^2}{T\sin^2\theta/2}} - 1\right]$$

or $$\hbar\omega_0 = \frac{mc^2/\sin^2\theta/2}{\sqrt{1 + \dfrac{2mc^2}{T\sin^2\theta/2}}} - 1$$

$$= \frac{T}{2}\left[\sqrt{1 + \frac{2mc^2}{T\sin^2\theta/2}} + 1\right]$$

$$\hbar\omega_0 = 0.677 \text{ MeV} \ \ (\text{on substituting values})$$

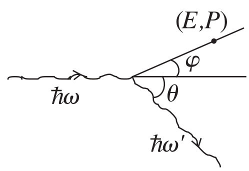
5.289 We see from the previous problem that the electron gains the maximum K.E. when the photon is scattered backwards, i.e., $\theta = 180°$, then

$$\omega_0 = \frac{mc^2/\hbar}{\sqrt{1 + 2mc^2/T_{max}} - 1}$$

Hence, $$\lambda_0 = \frac{2\pi c}{\omega_0} = \frac{2\pi\hbar}{mc}\left[\sqrt{1 + \frac{2mc^2}{T_{max}}} - 1\right]$$

Substituting the values, we get $\lambda_0 = 3.695$ pm.

5.290 Refer to the figure. Energy momentum conservation gives



$$\frac{\hbar\omega'}{c} - \frac{\hbar\omega'}{c}\cos\theta = p\cos\varphi$$

$$\frac{\hbar\omega'}{c}\sin\theta = p\sin\varphi$$

$$\hbar\omega + mc^2 = \hbar\omega' + E$$

where $E^2 = c^2p^2 + m^2c^4$. We see

$$\tan\varphi = \frac{\omega'\sin\theta}{\omega - \omega'\cos\theta} = \frac{1/\lambda'\sin\theta}{1/\lambda - 1/\lambda'\cos\theta}$$

$$= \frac{\lambda\sin\theta}{\lambda' - \lambda\cos\theta} = \frac{\sin\theta}{\Delta\lambda/\lambda + 2\sin^2\theta/2}$$

where $\Delta\lambda = \lambda' - \lambda = 2\pi\lambda_c(1-\cos\theta) = 4\pi\lambda_c \sin^2\dfrac{\theta}{2}$

Hence, $$\tan\varphi = \frac{2\sin\theta/2\,\cos\theta/2}{\Delta\lambda/\lambda + \Delta\lambda/2\pi\lambda_c}$$

But $$\sin\theta = 2\sqrt{\frac{\Delta\lambda}{4\pi\bar{\lambda}_c}}\sqrt{1 - \frac{\Delta\lambda}{4\pi\bar{\lambda}_c}} = \frac{\Delta\lambda}{2\pi\bar{\lambda}_c}\sqrt{\frac{4\pi\bar{\lambda}_c}{\Delta\lambda} - 1}$$

Thus, $$\tan\varphi = \frac{\sqrt{(4\pi\hbar/mc\Delta\lambda) - 1}}{1 + (2\pi\hbar/mc\lambda)} = \frac{\sqrt{(4\pi\hbar/mc\Delta\lambda) - 1}}{1 + (\pi\hbar/mc^2)}$$

or $$\varphi = 31.3^\circ \text{ (on substituting values)}$$

5.291 By head-on collision we understand that the electron moves on in the direction of the incident photon after the collision and the photon is scattered backwards. Then,

$$\hbar\omega = \eta mc^2$$

$$\hbar\omega' = \sigma mc^2$$

$$(E, p) = (\varepsilon mc^2, \mu mc) \text{ of the electron.}$$

Then by energy momentum conservation (cancelling factors of mc^2 and mc) we have

$$1 + \eta = \sigma + \varepsilon$$

$$\eta = \mu - \sigma$$

$$\varepsilon^2 = 1 + \mu^2$$

On eliminating σ and ε, we get

$$1 + \eta = -\eta + \mu + \sqrt{\mu^2 + 1}$$

or $$(1 + 2\eta - \mu) = \sqrt{\mu^2 + 1}$$

Squaring $$(1 + 2\eta)^2 - 2\mu\,(1 + 2\eta) = 1$$

or $$4\eta + 4\eta^2 = 2\mu\,(1 + 2\eta)$$

or $$\mu = \frac{2\eta(1+\eta)}{1+2\eta}$$

Thus, the momentum of the Compton electron is

$$p = \mu mc = \frac{2\eta(1+\eta)mc}{1+2\eta}$$

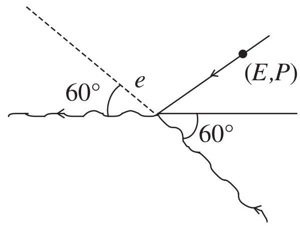
Now, in a magnetic field

$$p = Be\rho$$

Thus, $$\rho = \frac{2\eta(1+\eta)}{(1+2\eta)mc/Be}$$

Substituting values, we get $\rho = 3.412$ cm.

5.292 This is the inverse of usual Compton scattering. When we write down the energy–momentum conservation equations for this process we find that they are the same for the inverse process as they are for the usual process. It follows that the formula for Compton shift is applicable except that the energy (frequency) of the photon is increased on scattering and the wavelength is shifted downward. With this understanding, we write



$$\begin{aligned}\Delta\lambda &= 2\pi\frac{\hbar}{mc}(1-\cos\theta)\\ &= 4\pi\left(\frac{\hbar}{mc}\right)\sin^2\frac{\theta}{2}\\ &= 1.21 \text{ pm (on substituting values)}\end{aligned}$$

ATOMIC AND NUCLEAR PHYSICS

PART 6

For this chapter in the book the formulas are given in the CGS units. Since most students are familiar only with MKS units, we shall do the problems in MKS units. However, wherever needed, we shall also write the formulas in the Gaussian units.

6.1 Scattering of Particles. Rutherford–Bohr Atom

6.1 The Thomson model consists of a uniformly charged nucleus in which the electrons are at rest at certain equilibrium points (the plum in the pudding model). For the hydrogen nucleus the charge on the nucleus is $+e$ while the charge on the electron is $-e$. The electron by symmetry must be at the centre of the nuclear charge where the potential (from Problem 3.38a) is

$$\varphi_0 = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{3e}{2R}$$

where R is the radius of the nuclear charge distribution. The potential energy of the electron is $-e\varphi_0$ and since the electron is at rest, this is also the total energy. To ionize such an electron will require an energy of $E = e\varphi_0$. From this we find

$$R = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{3e^2}{2E}$$

(In Gaussian system the factor $1/4\pi\varepsilon_0$ is absent.)

On putting the values, we get $R = 0.16$ nm.

Light is emitted when the electron vibrates. If we displace the electron slightly inside the nucleus by giving it a push r in some radial direction and an energy δE of oscillation, then since the potential at a distance r in the nucleus is

$$\varphi(r) = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{e}{R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right)$$

the total energy of the nucleus becomes

$$\frac{1}{2}m\dot{r}^2 - \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{e^2}{R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right) = -e\varphi_0 + \delta E$$

or
$$\delta E = \frac{1}{2} m\dot{r}^2 + \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{e^2}{2R^3} r^2$$

This is the energy of a harmonic oscillator whose frequency is

$$\omega^2 = \left(\frac{1}{4\pi\varepsilon_0}\right)\frac{e^2}{mR^3}$$

The vibrating electron emits radiation of frequency ω whose wavelength is

$$\lambda = \frac{2\pi c}{\omega} = \frac{2\pi c}{e}\sqrt{mR^3}(4\pi\varepsilon_0)^{1/2}$$

(In Gaussian units the factor $(4\pi\varepsilon_0)^{1/2}$ is absent.)

On putting the values, we get $\lambda = 0.237$ μm.

6.2 Equation (6.1a) of the book can be written in MKS units as

$$\tan\frac{\theta}{2} = \frac{(q_1 q_2/4\pi\varepsilon_0)}{2bT}$$

Thus,
$$b = \left(\frac{q_1 q_2}{4\pi\varepsilon_0}\right)\frac{\cot\theta/2}{2T}$$

(In Gaussian units the factor $1/4\pi\varepsilon_0$ is absent.)

For α particle, $q_1 = 2e$, for gold, $q_2 = 79e$.

Substituting, we get $b = 0.731$ pm.

6.3 (a) In the case of Pb, we shall ignore the recoil of the nucleus, both because Pb is quite heavy ($A_{Pb} = 208 = 52 \times A_{He}$), and because Pb is not free. Then for a head-on collision, at the distance of closest approach, the K.E. of the α-particle must become zero (because α-particle will turn back at this point). Then,

$$\frac{2Ze^2}{(4\pi\varepsilon_0)r_{\min}} = T$$

(In Gaussian units the factor $4\pi\varepsilon_0$ is absent.)

So,
$$r_{\min} = 0.591 \text{ pm} \quad \text{(on substituting values)}$$

(b) Here we have to take into account that part of the energy is spent in the recoil of Li nucleus. Suppose x_1 = coordinate of the α-particle from some arbitrary point on the line joining it to the Li nucleus, x_2 = coordinate of the Li nucleus with respect to the same point. Then, we have the energy momentum equations

$$\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{2\times 3e^2}{(4\pi\varepsilon_0)\,|\,x_1 - x_2\,|} = T$$

$$m_1\dot{x}_1 + m_2\dot{x}_2 = \sqrt{2m_1T}$$

Here m_1 = mass of He^{++} nucleus, m_2 = mass of Li nucleus. Eliminating $\dot{x}_2$, we get

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2m_2}\left(\sqrt{2m_1T} - m_1\dot{x}_1\right)^2 + \frac{6e^2}{(4\pi\varepsilon_0)(x_1 - x_2)}$$

We complete the square on the right-hand side and rewrite the above equation as

$$\frac{m_2}{m_1 + m_2}T = \frac{1}{2m_2}\left[\sqrt{m_1(m_1 + m_2)}\dot{x}_1 - \sqrt{\frac{m_1}{m_1 + m_2}}\sqrt{2m_1T}\right]^2$$
$$+ \frac{6e^2}{(4\pi\varepsilon_0)\,|\,x_1 - x_2\,|}$$

For the least distance of approach, the second term on the right must be greatest which implies that the first term must vanish.

Thus,
$$|x_1 - x_2|_{\min} = \frac{6e^2}{(4\pi\varepsilon_0)T}\left(1 + \frac{m_1}{m_2}\right)$$

Using $m_1/m_2 = 4/7$ and other values, we get $|\,x_1 - x_2\,|_{\min} = 0.034$ pm.

(In Gaussian units the factor $4\pi\varepsilon_0$ is absent.)

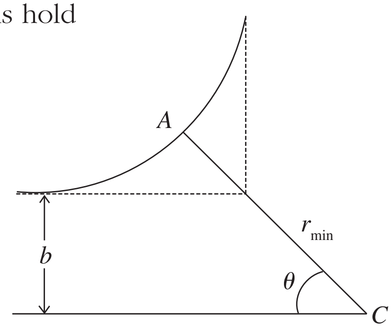
6.4 We shall ignore the recoil of Hg nucleus.

(a) Let A be the point of closest approach to the centre C, so $AC = r_{\min}$. At A the motion is instantaneously circular because the radial velocity vanishes. Then if v_0 is the speed of the particle at A, the following equations hold

$$\Gamma = \frac{1}{2}mv_0^2 + \frac{Z_1Z_2\,e^2}{(4\pi\varepsilon_0)r_{\min}} \qquad (1)$$

$$mv_0r_{\min} = \sqrt{2mT}\,b \qquad (2)$$

$$\frac{mv_0^2}{\rho_{\min}} = \frac{Z_1Z_2\,e^2}{(4\pi\varepsilon_0)r_{\min}^2} \qquad (3)$$



(This is Newton's law. Here ρ is the radius of curvature of the path at A and ρ is minimum at A by symmetry). Finally, we have Eq. (6.1a) of the book in the form

$$b = \frac{Z_1Z_2\,e^2}{(4\pi\varepsilon_0)2T}\cot\frac{\theta}{2} \qquad (4)$$

From Eqs. (2) and (3)

$$\frac{2Tb^2}{\rho_{\min}} = \frac{Z_1Z_2\,e^2}{(4\pi\varepsilon_0)}$$

or $$\rho_{\min} = \frac{Z_1 Z_2 e^2}{(4\pi\varepsilon_0)2T}\cot^2\frac{\theta}{2}$$

With $z_1 = 2$, $z_2 = 80$, we get $\rho_{\min} = 0.231$ pm.

(b) From Eqs. (2) and (4), we get

$$r_{\min} = \frac{Z_1 Z_2 e^2}{(4\pi\varepsilon_0)\sqrt{2mT}}\frac{\cot\theta/2}{v_0}$$

Substituting in Eq. (1)

$$T = \frac{1}{2}mv_0^2 + \sqrt{2mT}v_0\tan\frac{\theta}{2}$$

Solving for v_0 we get

$$v_0 = \sqrt{\frac{2T}{m}}\left(\sec\frac{\theta}{2} - \tan\frac{\theta}{2}\right)$$

Then $$r_{\min} = \frac{Z_1 Z_2 e^2}{(4\pi\varepsilon_0)2T}\frac{\cot\theta/2}{\sec\theta/2 - \tan\theta/2}$$

$$= \frac{Z_1 Z_2 e^2}{(4\pi\varepsilon_0)2T}\cot\frac{\theta}{2}\left(\sec\frac{\theta}{2} + \tan\frac{\theta}{2}\right)$$

$$= \frac{Z_1 Z_2 e^2}{(4\pi\varepsilon_0)2T}\left(1 + \operatorname{cosec}\frac{\theta}{2}\right) = 0.557 \text{ pm}$$

6.5 By momentum conservation

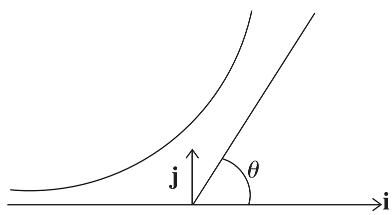
$$\mathbf{p} + \mathbf{p}_i = \mathbf{p}' + \mathbf{p}_f$$

where $\mathbf{p}$ and $\mathbf{p}'$ are the momenta of proton and $\mathbf{p}_i$ and $\mathbf{p}_f$ are the momenta of gold nucleus, before and after the scattering.

Thus, the momentum transferred to the gold nucleus is clearly

$$\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = \mathbf{p} - \mathbf{p}'$$

Although the momentum transferred to the gold nucleus is not small, the energy associated with this recoil is quite small and its effect on the motion of the proton can be neglected to a first approximation. Then,



$$\Delta\mathbf{p} = \sqrt{2mT}(1 - \cos\theta)\mathbf{i} + \sqrt{2mT}\sin\theta\,\mathbf{j}$$

Here $\mathbf{i}$ is the unit vector in the direction of the incident proton and $\mathbf{j}$ is normal to it on the side on which it is scattered. Thus,

$$|\Delta \mathbf{p}| \approx 2\sqrt{2mT}\sin\frac{\theta}{2}$$

Using $$\tan\frac{\theta}{2} = \frac{Ze^2}{(4\pi\varepsilon_0)2bT} \quad \text{(for the proton)}$$

we get, $$|\Delta \mathbf{p}| \approx 2\sqrt{2mT\Big/\left[1+\left(\frac{2bT(4\pi\varepsilon_0)}{Ze^2}\right)^2\right]}$$

6.6 The proton moving by the electron first accelerates and then decelerates and it is not easy to calculate the energy lost by the proton, so energy conservation does not do the trick. Instead we must directly calculate the momentum acquired by the electron. By symmetry that momentum is along **OA** and its magnitude is

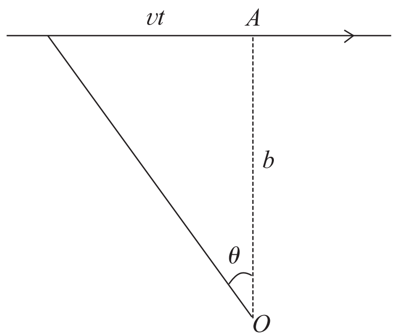
$$p_d = \int F_\perp \, dt$$

where $F_\perp$ is the component of the force on electron along **OA**. Thus,

$$p_d = \int_{-\infty}^{\infty} \frac{e^2}{4\pi\varepsilon_0} \frac{b}{\sqrt{b^2 + v^2t^2}} \cdot \frac{1}{b^2 + v^2t^2} dt$$

$$= \frac{e^2 b}{4\pi\varepsilon_0 v} \cdot \int_{-\infty}^{\infty} \frac{dx}{(b^2 + x^2)^{3/2}} \quad (\text{where } x = vt)$$

Evaluate the integral by substituting $x = b\tan\theta$.



Then, $$p_e = \frac{2e^2}{(4\pi\varepsilon_0)vb}$$

and $$T_e = \frac{p_e^2}{2m_e} = \frac{m_p e^4}{(4\pi\varepsilon_0)^2 T b^2 m_e}$$

(In Gaussian units there is no factor $(4\pi\varepsilon_0)^2$.)

On substituting the values we get $T_e = 3.82$ eV.

6.7 In the region where potential is non-zero, the K.E. of the particle is, by energy conservation, $T + U_0$ and the momentum of the particle has the magnitude $\sqrt{2m(T+U_0)}$. On the boundary the force is radial, so the tangential component of the momentum does not change.

So, $$\sqrt{2mT}\sin\alpha = \sqrt{2m(T+U_0)}\sin\varphi$$

or $$\sin\varphi = \sqrt{\frac{T}{T+U_0}}\sin\alpha = \frac{\sin\alpha}{n}$$

where $$n = \sqrt{1 + \frac{U_0}{T}}$$

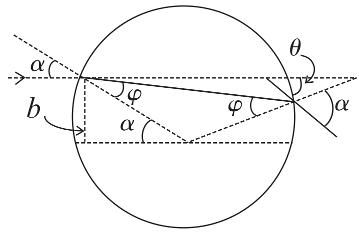
We also have $\theta = 2(\alpha - \varphi)$. Therefore,

$$\sin\frac{\theta}{2} = \sin(\alpha - \varphi) = \sin\alpha\cos\varphi - \cos\alpha\sin\varphi$$

$$= \sin\alpha\left(\cos\varphi - \frac{\cos\alpha}{n}\right)$$

or $$\frac{n\sin\theta/2}{\sin\alpha} = \sqrt{n^2 - \sin^2\alpha} - \cos\alpha$$

or $$\left(\frac{n\sin\theta/2}{\sin\alpha} + \cos\alpha\right)^2 = n^2 - \sin^2\alpha$$



$$n^2\sin^2\frac{\theta}{2}\cot^2\alpha + 2n\sin\frac{\theta}{2}\cot\alpha + 1 = n^2\cos^2\frac{\theta}{2}$$

or $$\cot\alpha = \frac{n\cos\theta/2 - 1}{n\sin\theta/2}$$

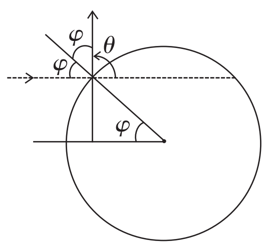
Hence, $$\sin\alpha = \frac{n\sin\theta/2}{\sqrt{1 + n^2 - 2n\cos\theta/2}}$$

Finally, the impact parameter is

$$b = R\sin\alpha = \frac{nR\sin\theta/2}{\sqrt{1 + n^2 - 2n\cos\theta/2}}$$

6.8 It is implied that the ball is too heavy to recoil.

(a) The trajectory of the particle is symmetrical about the radius vector through the point of impact. It is clear from the diagram that



$$\theta = \pi - 2\varphi \quad \text{or} \quad \varphi = \frac{\pi}{2} - \frac{\theta}{2}$$

Also, $$b = (R + r)\sin\varphi = (R + r)\cos\frac{\theta}{2}$$

(b) With b defined above, the fraction of particles scattered between θ and $\theta + d\theta$ (or the probability of the same) is

$$dP = \frac{|2\pi b\,db|}{\pi(R + r)^2} = \frac{1}{2}\sin\theta\,d\theta$$

(c) The probability of particle to be deflected after collision is

$$P = \int_0^{\pi/2} \frac{1}{2} \sin\theta \, d\theta = \frac{1}{2} \int_{-1}^{0} d(-\cos\theta) = \frac{1}{2}$$

6.9 From the Eq. (6.1b) of the book

$$\frac{dN}{N} = n \left(\frac{Ze^2}{(4\pi\varepsilon_0) 2T} \right)^2 \frac{d\Omega}{\sin^4 \theta/2}$$

(We have put $q_1 = 2e$, $q_2 = Ze$ here. Also n = number of Pt nuclei in the foil per unit area.)

$$= \underbrace{(At\,\rho)}_{\text{mass of the foil}} \cdot \underbrace{\frac{N_A}{A_{\text{Pt}}}}_{\text{number of nuclei per unit mass}} \cdot \frac{1}{A} = \frac{N_A \rho t}{A_{\text{Pt}}}$$

Using the values $A_{\text{Pt}} = 195$, $\rho = 21.5 \times 10^3$ kg/m^3, $N_A = 6.023 \times 10^{26}$/kmol, we get $n = 6.641 \times 10^{22}$ per m^2.

Also, $$d\Omega = \frac{dS_n}{r^2} = 10^{-2} \text{ sr}$$

Therefore, $$\frac{dN}{N} = 3.36 \times 10^{-5} \text{ (on substituting values)}$$

6.10 A scattered flux density of J (particles per unit area per second) equals $J/(1/r^2) = r^2 J$ particles scattered per unit time per steradian in the given direction. Let n = concentration of the gold nuclei in the foil. Then, $n = N_A \rho / A_{\text{Au}}$ and the number of Au nuclei per unit area of the foil is nd where d = thickness of the foil. Then from Eq. (6.1b) of the book (note that $n \to nd$ here)

$$r^2 J = dN = ndI \left(\frac{Ze^2}{(4\pi\varepsilon_0) 2T} \right)^2 \operatorname{cosec}^4 \frac{\theta}{2}$$

(Here I is the number of α-particles falling on the foil per second.)

Hence, $$d = \frac{4T^2 r^2 J}{nI(Ze^2/4\pi\varepsilon_0)^2} \sin^4 \frac{\theta}{2}$$

Using $Z = 79$, $A_{\text{Au}} = 197$, $\rho = 19.3 \times 10^3$ kg/m^3, $N_A = 6.023 \times 10^{26}$/kmol and other data from the problem, we get $d = 1.47$ μm.

6.11 From the Eq. (6.1b) of the book, we find

$$\frac{dN_{\mathrm{Pt}}}{dN_{\mathrm{Ag}}} = \frac{n_{\mathrm{Pt}}}{n_{\mathrm{Ag}}} \cdot \frac{Z_{\mathrm{Pt}}^2}{Z_{\mathrm{Ag}}^2} = \eta$$

But since the foils have the same mass thickness ($= \rho d$), we have

$$\frac{n_{\mathrm{Pt}}}{n_{\mathrm{Ag}}} = \frac{A_{\mathrm{Ag}}}{A_{\mathrm{Pt}}} \quad \text{(See the Problem 6.9)}$$

Hence,
$$Z_{\mathrm{Pt}} = Z_{\mathrm{Ag}} \cdot \sqrt{\frac{\eta A_{\mathrm{Pt}}}{A_{\mathrm{Ag}}}}$$

Substituting $Z_{\mathrm{Ag}} = 47$, $A_{\mathrm{Ag}} = 108$, $A_{\mathrm{Pt}} = 195$ and $\eta = 1.52$, we get $Z_{\mathrm{Pt}} = 77.86 \approx 78$.

6.12 (a) From Eq. (6.1b) of the book, we get

$$dN = I_0 \tau \frac{\rho d N_A}{A_{\mathrm{Au}}} \left(\frac{Ze^2}{(4\pi\varepsilon_0)2T} \right)^2 \frac{2\pi \sin\theta \, d\theta}{\sin^4 \theta/2}$$

(We have used $d\Omega = 2\pi \sin\theta \, d\theta$ and $N = I_0 I$.)

From the data

$$d\theta = 2° = \frac{2}{57.3} \text{ radians}$$

Also, using $Z_{\mathrm{Au}} = 79$, $A_{\mathrm{Au}} = 197$, we get $dN = 1.63 \times 10^6$.

(b) This number is

$$N(\theta_0) = I_0 \tau \left(\frac{\rho \, d N_A}{A_{\mathrm{Au}}} \right) \left(\frac{Ze^2}{(4\pi\varepsilon_0)2T} \right)^2 4\pi \int_{\theta_0}^{\pi} \frac{\cos(\theta/2) d\theta}{\sin^3 \theta/2}$$

The integral is

$$2 \int_{\sin\theta_0/2}^{1} \frac{dx}{x^2} = \frac{1}{2} \left[\frac{-1}{2x^2} \right]_{\sin\theta_0/2}^{1} = \cot^2 \frac{\theta_0}{2}$$

Thus,
$$N(\theta_0) = \pi n d \left(\frac{Ze^2}{(4\pi\varepsilon_0)T} \right)^2 I_0 \tau \cot^2 \frac{\theta_0}{2}$$

where n is the concentration of nuclei in the foil ($n = \rho N_1 / A_{\mathrm{Au}}$).
Substitution gives $N(\theta_0) = 2.02 \times 10^7$.

6.13 The requisite probability can be written easily by analogy with part (b) of the previous problem. It is

$$P = \frac{N(\pi/2)}{I_0 \tau} = nd\left(\frac{Ze^2}{(4\pi\varepsilon_0)2mv^2}\right)^2 4\pi \int_{\pi/2}^{\pi} \frac{\cos\theta/2\, d\theta}{\sin^3\theta/2}$$

The integral is unity. Thus,

$$P = \pi nd\left(\frac{Ze^2}{(4\pi\varepsilon_0)mv^2}\right)^2$$

On substitution, we get

$$n = \frac{\rho_{\text{Ag}} N_A}{A_{\text{Ag}}} = \frac{10.5 \times 10^3 \times 6.023 \times 10^{23}}{108} \quad \text{and} \quad P = 0.006$$

6.14 Because of the cosec^4 $\theta/2$ dependence of the scattering, the number of particles (or fractions) scattered through $\theta < \theta_0$ cannot be calculated directly. But we can write this fraction as

$$P(\theta_0) = 1 - Q(\theta_0)$$

where $Q(\theta_0)$ is the fraction of particles scattered through $\theta \geq \theta_0$. This fraction has been calculated before and is (see the solution of Problem 6.12b)

$$Q(\theta_0) = \pi n\left(\frac{Ze^2}{(4\pi\varepsilon_0)T}\right)^2 \cot^2\frac{\theta_0}{2}$$

where n is number of nuclei/cm^2. Using the data, we get

$$Q = 0.4$$

Thus, $\quad P(\theta_0) = 0.6$

6.15 The relevant fraction can be written as (see Problem 6.12b). (Note that the projectiles are protons.)

$$\frac{\Delta N}{N} = \left(\frac{e^2}{(4\pi\varepsilon_0)2T}\right)^2 \pi\cot^2\frac{\theta_0}{2}\cdot(n_1 Z_1^2 + n_2 Z_2^2)$$

Here $n_1(n_2)$ is the number of Zn(Cu) nuclei per cm^2 of the foil and $Z_1(Z_2)$ is the atomic number of Zn(Cu).

Now $\quad n_1 = \dfrac{\rho d N_A}{M_1} = 0.7 \quad \text{and} \quad n_2 = \dfrac{\rho d N_A}{M_2} = 0.3$

(Here M_1, M_2 are the mass numbers of Zn and Cu.)

Then, substituting the values $Z_1 = 30$, $Z_2 = 29$, $M_1 = 65.4$, $M_2 = 63.5$, we get

$$\frac{\Delta N}{N} = 1.43 \times 10^{-3}$$

6.16 From the Rutherford scattering formula

$$\frac{d\sigma}{d\Omega} = \left(\frac{Ze^2}{(4\pi\varepsilon_0)2T}\right)^2 \frac{1}{\sin^4 \theta/2}$$

or

$$d\sigma = \left(\frac{Ze^2}{(4\pi\varepsilon_0)2T}\right)^2 \frac{2\pi \sin\theta \, d\theta}{\sin^4 \theta/2}$$

$$= \left(\frac{Ze^2}{(4\pi\varepsilon_0)T}\right)^2 \pi \frac{\cos\theta/2 \, d\theta}{\sin^3 \theta/2}$$

Then integrating from $\theta = \theta_0$ to $\theta = \pi$, we get the required cross-section

$$\Delta\sigma = \left(\frac{Ze^2}{(4\pi\varepsilon_0)T}\right)^2 \pi \int_{\theta_0}^{\pi} \frac{\cos\theta/2 \, d\theta}{\sin^3 \theta/2}$$

$$= \left(\frac{Ze^2}{(4\pi\varepsilon_0)2T}\right)^2 \cot^2 \frac{\theta_0}{2}$$

For uranium nucleus, $Z = 92$ and on putting the values we get, $\Delta\sigma = 737$ b $= 0.737$ kb (where 1b is 1 barn $= 10^{-28}$ m^2).

6.17 (a) From the previous problem

$$\Delta\sigma = \left(\frac{Ze^2}{(4\pi\varepsilon_0)2T}\right)^2 \pi \cot^2 \frac{\theta_0}{2}$$

or

$$T = \frac{Ze^2}{4\pi\varepsilon_0} \cot \frac{\theta_0}{2} \sqrt{\frac{\pi}{\Delta\sigma}}$$

Substituting the values with $Z = 79$, $\theta_0 = 90°$, we get $T = 0.903$ MeV.

(b) The differential scattering cross-section is

$$\frac{d\sigma}{d\Omega} = C \operatorname{cosec}^4 \frac{\theta}{2}$$

where

$$\Delta\sigma(\theta > \theta_0) = 4\pi C \cot^2 \frac{\theta_0}{2}$$

Thus, from the given data

$$C = \frac{500}{4\pi} b = 39.79 \text{ b/sr}$$

So,

$$\frac{d\sigma}{d\Omega}(\theta = 60°) = 39.79 \times 16 \text{ b/sr} = 0.637 \text{ kb/sr}$$

6.18 The formula in MKS units is

$$\frac{dE}{dt} = \frac{\mu_0 e^2}{6\pi c}\mathbf{w}^2$$

For an electron performing (linear) harmonic vibrations, **w** is in some definite direction with

$$w_x = -\omega^2 x \text{ (say)}$$

Thus, $$\frac{dE}{dt} = -\frac{\mu_0 e^2 \omega^4}{6\pi c} x^2$$

If the radiation loss is small (i.e., if ω is not too large), then the motion of the electron is always close to simple harmonic with slowly decreasing amplitude. Then, we can write

$$E = \frac{1}{2} m\omega^2 a^2$$

and $$x = a \cos \omega t$$

and averaging the above equation, ignoring variation of a in any cycle, we get the equation (on using $< x^2 > = 1/2\ a^2$)

$$\frac{dE}{dt} = \frac{\mu_0 e^2 \omega^4}{6\pi c} \cdot \frac{1}{2} a^2 = -\frac{\mu_0 e^2 \omega^2}{6\pi mc} E$$

since $E = 1/2\ m\omega^2 a^2$ for a harmonic oscillator.

This equation integrates to

$$E = E_0 e^{-t/T}$$

where $T = 6\pi mc/e^2\omega^2\mu_0$.

It is then seen that energy decreases η times in time t_0 given by

$$t_0 = T \ln \eta = \frac{6\pi mc}{e^2 \omega^2 \mu_0} \ln \eta = 14.7 \text{ ns}$$

6.19 Moving around the nucleus, the electron radiates and its energy decreases. This means that the electron gets nearer to the nucleus. By the statement of the problem, we can assume that the electron is always moving in a circular orbit and the radial acceleration by Newton's law is

$$w = \frac{e^2}{(4\pi\varepsilon_0) m r^2} \quad \text{(directed inwards)}$$

Thus, $$\frac{dE}{dt} = \frac{\mu_0 e^6}{6\pi c} \frac{1}{(4\pi\varepsilon_0)^2 m^2 r^4}$$

On the other hand, in a circular orbit

$$E = -\frac{e^2}{(4\pi\varepsilon_0)2r}$$

So, $$\frac{e^2}{(4\pi\varepsilon_0)2r^2}\frac{dr}{dt} = -\frac{\mu_0 e^6}{(4\pi\varepsilon_0)^2 6\pi c m^2 r^4}$$

or $$\frac{dr}{dt} = -\frac{\mu_0 e^4}{(4\pi\varepsilon_0)3\pi c m^2 r^2}$$

On integrating $$r^3 = r_0^3 - \frac{\mu_0 e^4}{4\pi^2\varepsilon_0 c m^2}t$$

and so the radius falls to zero in time

$$t_0 = \frac{4\pi^2\varepsilon_0 c m^2 r_0^3}{\mu_0 e^4}\text{ s} = 13.1\text{ ps}$$

6.20 In a circular orbit, we have the following formula

$$\frac{mv^2}{r} = \frac{Ze^2}{(4\pi\varepsilon_0)r^2}$$

$$mvr = n\hbar$$

Then, $$v = \frac{Ze^2}{(4\pi\varepsilon_0)n\hbar}$$

and $$r = \frac{n^2}{Z}\frac{\hbar(4\pi\varepsilon_0)}{me^2}$$

The energy is

$$E_n = \frac{1}{2}mv^2 - \frac{Ze^2}{(4\pi\varepsilon_0)r}$$

$$= \left(\frac{Ze^2}{4\pi\varepsilon_0}\right)^2 \frac{m}{2\hbar^2 n^2} - \left(\frac{Ze^2}{4\pi\varepsilon_0}\right)\frac{m}{\hbar n^2} = m\left(\frac{Ze^2}{4\pi\varepsilon_0}\right)^2 \Big/ 2\hbar^2 n^2$$

and the circular frequency of this orbit is

$$\omega_n = \frac{v}{r} = \left(\frac{Ze^2}{4\pi\varepsilon_0}\right)^2 \frac{m}{\hbar^3 n^3}$$

On the other hand, the frequency ω of the light emitted when the electron makes a transition $n + 1 \rightarrow n$ is

$$\omega = \left(\frac{Ze^2}{4\pi\varepsilon_0}\right)^2 \frac{m}{2\hbar^2}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$$

Thus, the inequality $\omega_n > \omega > \omega_{n+1}$ will result if

$$\frac{1}{n^3} > \frac{1}{2}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) > \frac{1}{(n+1)^3}$$

Multiplying by $n^2\,(n+1)^2$, we have to prove

$$\frac{(n+1)^2}{n} > \frac{1}{2}(2n+1) > \frac{n^2}{n+1}$$

This can be written as

$$n + 2 + \frac{1}{n} > n + \frac{1}{2} > n + 1 - 2 + \frac{1}{n+1}$$

This is obvious because

$$-1 + \frac{1}{n+1} < -\frac{1}{2} \quad \text{(since } n \geq 1\text{)}$$

For large n $$\frac{\omega_n}{\omega_{n+1}} = \left(\frac{n+1}{n}\right)^3 = 1 + \frac{3}{n}$$

So, $$\frac{\omega_n}{\omega_{n+1}} \to 1 \text{ and we may say } \frac{\omega}{\omega_n} \to 1$$

6.21 We have the following equations (we ignore reduced mass effects)

$$\frac{mv^2}{r} = kr$$

and $$mvr = n\hbar$$

So, $$mv = \sqrt{mk}\,r$$

Therefore $$r = \sqrt{\frac{n\hbar}{\sqrt{mk}}}$$

and $$v = \frac{\sqrt{n\hbar\sqrt{mk}}}{m}$$

The energy levels are

$$E_n = \frac{1}{2}mv^2 + \frac{1}{2}kr^2$$

$$= \frac{1}{2}\frac{n\hbar\sqrt{mk}}{m} + \frac{1}{2}k\frac{n\hbar}{\sqrt{mk}}$$

$$= n\hbar\sqrt{\frac{k}{m}}$$

6.22 The basic equations have been derived in the Problem 6.20. We rewrite them here and determine the required values.

(a) We have
$$r_1 = \frac{\hbar^2}{m(Ze^2/4\pi\varepsilon_0)}$$

Thus, $r_1 = 52.8$ pm for H atom and $r_1 = 26.4$ pm for He^+ ion, on using $Z = 1$ for H and $Z = 2$ for He^+.

Also,
$$v_1 = \frac{Ze^2}{(4\pi\varepsilon_0)\hbar}$$

Thus, $v_1 = 2.191 \times 10^6$ m/s for H atom and $v_1 = 4.382 \times 10^6$ m/s for He^+ ion.

(b) we have
$$T = \frac{1}{2}mv_1^2 = \frac{m(Ze^2)^2}{(4\pi\varepsilon_0)^2\, 2\hbar^2}$$

On substituting values, $T = 13.65$ eV for H atom and $T = 54.6$ eV for He^+ ion.

In both the cases, binding energy $E_b = T$ because $E_b = -E$ and $E \approx -T$.

(Recall that for coulomb force $V = -2T$.)

(c) The ionization potential φ_i is given by
$$e\varphi_i = E_b$$

So $\varphi_i = 13.65$ V for H atom and $\varphi_i = 54.6$ V for He^+ ion, on substituting values.

The energy levels are $E_n = -\dfrac{13.65}{n^2}$ eV (for H atom)

and $E_n = -\dfrac{54.6}{n^2}$ eV (for He^+ ion)

Thus
$$\varphi_1 = 13.65\left(1 - \frac{1}{4}\right) = 10.23 \text{ V (for H atom)}$$

and
$$\varphi_1 = 4 \times 10.23 = 40.9 \text{ V (for He}^+ \text{ ion)}$$

The wavelength of the resonance line ($n' = 2 \to n = 1$) is given by
$$\frac{2\pi\hbar c}{\lambda} = -\frac{13.6}{4} + \frac{13.6}{1} = 10.23 \text{ eV for H atom}$$

So
$$\lambda = 121.2 \text{ nm (for H atom)}$$

and
$$\lambda = \frac{121.2}{4} = 30.3 \text{ nm (for He}^+ \text{ ion)}$$

6.23 This has been calculated before in Problem 6.20. It is
$$\omega = \frac{m(Ze^2/4\pi\varepsilon_0)^2}{\hbar^3 n^3} = 2.08 \times 10^{16} \text{ rad/s}$$

6.24 An electron moving in a circle with a time period T constitutes a current

$$I = \frac{e}{T}$$

and forms a current loop of area πr^2. This is equivalent to magnetic moment,

$$\mu = I\pi r^2 = \frac{e\pi r^2}{T} = \frac{evr}{2}$$

where $v = 2\pi r/T$. Thus, for the n^{th} orbit

$$\mu_n = \frac{emvr}{2m} = \frac{ne\hbar}{2m}$$

(In Gaussian units $\mu_n = ne\hbar/2mc$.)

We see that
$$\mu_n = \frac{e}{2m} M_n$$

where $M_n = n\hbar = mvr$ is the angular momentum.

Thus,
$$\frac{\mu_n}{M_n} = \frac{e}{2m}$$

$$\mu_1 = \frac{e\hbar}{2m} = \mu_B = 9.27 \times 10^{-24} \text{ A m}^2$$

(In CGS units $\mu_1 = \mu_B = 9.27 \times 10^{-21}$ erg/G.)

6.25 The revolving electron is equivalent to a circular current.

So,
$$I = \frac{e}{T} = \frac{e}{2\pi r/v} = \frac{ev}{2\pi r}$$

The magnetic induction

$$B = \frac{\mu_0 I}{2r} = \frac{\mu_0 ev}{4\pi r^2} = \frac{\mu_0}{4\pi} \cdot e \cdot \frac{e^2}{(4\pi\varepsilon_0)\hbar} \cdot \left[\frac{me^2}{\hbar^2(4\pi\varepsilon_0)}\right]^2$$

$$= \frac{\mu_0 m^2 e^7}{256\pi^4 \varepsilon_0^3 \hbar^5}$$

Substitution gives, $B = 12.56$ T at the centre.

(In Gaussian units, $B = m^2e^7/c\hbar^5 = 125.6$ kG.)

6.26 From the general formula for the transition $n_2 \rightarrow n_1$

$$\hbar\omega = E_H\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) \text{ (where } E_{\text{H}} = 13.65 \text{ eV)}$$

(1) For Lyman series: $n_1 = 1$; $n_2 = 2, 3, \ldots$. Thus,

$$\hbar\omega \geq \frac{3}{4} E_{\mathrm{H}} = 10.238 \text{ eV}$$

This corresponds to

$$\lambda = \frac{2\pi c\hbar}{\hbar\omega} = 0.121\ \mu\text{m}$$

and Lyman lines have $\lambda \leq 0.121\ \mu$m with the series limit at 0.0909 μm.

(2) For Balmer series: $n_2 = 2$; $n_3 = 3, 4, \ldots$. Thus,

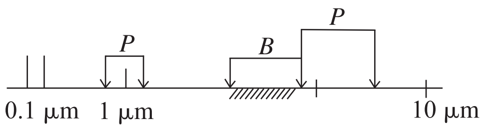
$$\hbar\omega \geq E_{\mathrm{H}}\left(\frac{1}{4} - \frac{1}{9}\right) = \frac{5}{36} E_{\mathrm{H}} = 1.876 \text{ eV}$$

This corresponds to $\lambda = 0.65\ \mu$m and so, Balmer series has $\lambda \leq 0.65\ \mu$m with the series limit at $\lambda = 0.363\ \mu$m.

(3) For Paschen series: $n_2 = 3$; $n_1 = 4, 5, \ldots$. Thus,

$$\hbar\omega \geq E_{\mathrm{H}}\left(\frac{1}{9} - \frac{1}{16}\right) = \frac{7}{144} E_{\mathrm{H}} = 0.6635 \text{ eV}$$

This corresponds to $\lambda = 1.869\ \mu$m with the series limit at $\lambda = 0.818\ \mu$m.



6.27 The Balmer line of wavelength 486.1 nm is due to the transition $4 \to 2$ while the Balmer line of wavelength 410.2 nm is due to the transition $6 \to 2$. The line whose wave number corresponds to the difference in wave numbers of these two lines is due to the transition $6 \to 4$. That line belongs to the Brackett series. The wavelength of this line is

$$\lambda = \frac{1}{1/\lambda_2 - 1/\lambda_1} = \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2} = 2.627\ \mu\text{m}$$

6.28 The energies are

$$E_{\mathrm{H}}\left(\frac{1}{4} - \frac{1}{9}\right) = \frac{5}{36} E_{\mathrm{H}}$$

$$E_{\mathrm{H}}\left(\frac{1}{4} - \frac{1}{16}\right) = \frac{3}{16} E_{\mathrm{H}}$$

and
$$E_H\left(\frac{1}{4}-\frac{1}{25}\right)=\frac{21}{100}E_H$$

They correspond to wavelengths 654.2 nm, 484.6 nm and 433 nm, respectively.

The n^{th} line of the Balmer series has the energy

$$E_H\left(\frac{1}{4}-\frac{1}{(n+2)^2}\right)$$

For $n = 19$, we get the wavelength 366.7450 nm and for $n = 20$, we get the wavelength 366.4470 nm.

To resolve these lines we require a resolving power of

$$R \approx \frac{\lambda}{\delta\lambda}=\frac{366.6}{0.298}=1.23\times 10^3$$

6.29 For the Balmer series

$$\hbar\omega_n = \hbar R\left(\frac{1}{4}-\frac{2}{n^2}\right)\text{(for } n \geq 3)$$

where $\hbar R = E_H = 13.65$ eV.

Thus,
$$\frac{2\pi\hbar c}{\lambda_n}=\hbar R\left(\frac{1}{4}-\frac{1}{n^2}\right)$$

or
$$\frac{2\pi\hbar c}{\lambda_{n+1}}-\frac{2\pi\hbar c}{\lambda_n}=\hbar R\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)$$

$$=\hbar R\left(\frac{2n+1}{n^2(n+1)^2}\right)\approx\frac{2R}{n^3}\quad\text{(for } n \gg 1)$$

Thus,
$$\frac{2\pi\hbar c}{\lambda_n^2}\delta\lambda\approx\frac{2R\hbar}{n^3}$$

or
$$\frac{\lambda_n}{\delta\lambda}\approx\frac{\pi\hbar c n^3}{\lambda_n R\hbar}=\frac{\pi c n^3}{\lambda_n R}$$

On the other hand, for just resolution in a diffraction grating

$$\frac{\lambda}{\delta\lambda}=kN=k\frac{l}{d}=\frac{l}{\lambda d}k\lambda=\frac{l}{\lambda d}d\sin\theta=\frac{l}{\lambda}\sin\theta$$

Hence,
$$\sin\theta=\frac{\pi c n^3}{lR}$$

$$\theta\approx 59.4^\circ\quad\text{(on substituting values)}$$

6.30 If all wavelengths are four times shorter but otherwise similar to the hydrogen atom spectrum, then the energy levels of the given atom must be four times greater. This means

$$E_n = -\frac{4E_H}{n^2}$$

compared to $E_n = -E_H/n^2$ for hydrogen atom. Therefore, the spectrum is that of He^+ ion ($Z = 2$).

6.31 Because of cascading, all possible transitions are seen. Thus, we look for the number of ways in which we can select upper and lower levels. The number of ways we can do this is

$$\frac{1}{2}n(n-1)$$

where the factor 1/2 takes account of the fact that the photon emission always arises from upper $\rightarrow$ lower transition.

6.32 These are the Lyman lines

$$\hbar\omega = E_H\left(\frac{1}{1} - \frac{1}{n^2}\right)$$

Here $n = 2, 3, 4, \ldots.$

For $n = 2$ we get $\lambda = 121.1$ nm;

for $n = 3$ we get $\lambda = 102.2$ nm;

for $n = 4$ we get $\lambda = 96.9$ nm;

for $n = 5$ we get $\lambda = 94.64$ nm and

for $n = 6$ we get $\lambda = 93.45$ nm.

Thus, at the level of accuracy of our calculation, there are four lines 121.1 nm, 102.2 nm, 96.9 nm and 94.64 nm falling.

6.33 If the wavelengths are λ_1, λ_2, then the total energy of the excited state must be

$$E_n = E_1 + \frac{2\pi c\hbar}{\lambda_1} + \frac{2\pi c\hbar}{\lambda_2}$$

But $E_1 = -4E_H$ and $E_n = -4E_H/n^2$, where we are ignoring reduced mass effects.

Then,
$$4E_H = \frac{4E_H}{n^2} + \frac{2\pi c\hbar}{\lambda_1} + \frac{2\pi c\hbar}{\lambda_2}$$

Substituting the values we get $n^2 = 23$, which we take to mean $n = 5$. The result is sensitive to the values of the various quantities and small difference gets multiplied because difference of two large quantities is involved:

$$n^2 = \frac{E_H}{E_H - \frac{\pi c\hbar}{2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)}$$

6.34 For the longest wavelength (first) line of the Balmer series, we have on using the generalized Balmer formula

$$\omega = Z^2 R\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$$

the result $$\lambda_{1,\text{ Lyman}} = \frac{2\pi c}{Z^2 R(1 - 1/4)} = \frac{8\pi c}{3Z^2 R}$$

Then, $$\Delta\lambda = \lambda_{1,\text{ Balmer}} - \lambda_{1,\text{ Lyman}} = \frac{176\pi c}{15Z^2 R}$$

So, $$R = \frac{176\pi c}{15Z^2 \Delta\lambda} = 2.07 \times 10^{16}\ \text{s}^{-1}$$

6.35 From the formula of the previous problem

$$\Delta\lambda = \frac{176\pi c}{15Z^2 R}$$

or $$Z = \sqrt{\frac{176\pi c}{15R\Delta\lambda}}$$

Substitution of $\Delta\lambda = 59.3$ nm and R from the previous problem gives $Z = 3$. This identifies the ions as Li^+.

6.36 We start from the generalized Balmer formula

$$\omega = RZ^2\left(\frac{1}{n^2} - \frac{1}{m^2}\right)$$

Here $m = n + 1, n + 2, \ldots, \infty$

The interval between extreme lines of this series (series n) is

$$\Delta\omega = RZ^2\left(\frac{1}{n^2} - \frac{1}{(\infty)^2}\right) - RZ^2\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = \frac{RZ^2}{(n+1)^2}$$

Hence, $$n = Z\sqrt{\frac{R}{\Delta\omega}} - 1$$

Then the angular frequency of the first line of this series (series n) is

$$\omega_1 = RZ^2\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = \Delta\omega\left[\left(\frac{n+1}{n}\right)^2 - 1\right]$$

$$= \Delta\omega\left[\left\{\frac{Z\sqrt{R/\Delta\omega}}{Z\sqrt{R/\Delta\omega} - 1}\right\}^2 - 1\right]$$

$$= \Delta\omega \; \frac{2Z\sqrt{R/\Delta\omega} - 1}{\left(Z\sqrt{R/\Delta\omega} - 1\right)^2}$$

Then the wavelength will be

$$\lambda_1 = \frac{2\pi c}{\omega_1} = \frac{2\pi c}{\Delta\omega}\,\frac{\left(Z\sqrt{R/\Delta\omega} - 1\right)^2}{2Z\sqrt{R/\Delta\omega} - 1}$$

Substitution (with the value of R from Problem 6.34 which is also the correct value determined directly) gives $\lambda_1 = 0.468$ μm.

6.37 For the third line of Balmer series

$$\omega = RZ^2\left(\frac{1}{2^2} - \frac{1}{5^2}\right) = \frac{21}{100}RZ^2$$

Hence, $$\lambda = \frac{2\pi c}{\omega} = \frac{200\pi c}{21RZ^2}$$

or $$Z = \sqrt{\frac{200\pi c}{21R\lambda}}$$

Substitution gives $Z = 2$. Hence, the binding energy (B.E.) of the electron in the ground state of this ion is $E_b = 4E_H = 4 \times 13.65 = 54.6$ eV. Therefore, ion is He^+.

6.38 To remove one electron requires 24.6 eV. The ion that is left is He^+ which in its ground state has a binding energy of $4E_H = 4\hbar R$. The complete binding energy of both electrons is then

$$E = E_0 + 4\hbar R$$

$$= 79.1 \text{ eV} \quad \text{(on substituting values)}$$

6.39 By conservation of energy

$$\frac{1}{2}mv^2 = \frac{2\pi\hbar c}{\lambda} - E_b$$

where $E_b = 4\hbar R$ is the binding energy of the electron in the ground state of He^+. (Recoil of He^{2+} nucleus is negligible). Then,

$$v = \sqrt{\frac{2}{m}\left(\frac{2\pi\hbar c}{\lambda} - E_b\right)}$$

$$= 2.25 \times 10^6 \text{ m/s} \quad \text{(on substituting values)}$$

6.40 Photon can be emitted in H–H collision only if one H is excited to an $n = 2$ state, which then de-excites to $n = 1$ state by emitting a photon. Let v_1 and v_2 be the velocities of the two hydrogen atoms after the collision and M their masses. Then, from energy momentum conservation

$$Mv_1 + Mv_2 = \sqrt{2MT}$$

(in the frame of the stationary H atom)

and
$$\frac{1}{2}Mv_1^2 + \frac{1}{2}Mv_2^2 + \frac{3}{4}\hbar R = T$$

where $\frac{3}{4}\hbar R = \hbar R\left(1 - \frac{1}{4}\right)$ is excitation energy of $n = 2$ state from ground state.

Eliminating v_2, we get

$$\frac{1}{2}M\left\{v_1^2 + \left(\sqrt{\frac{2T}{M}} - v_1\right)^2\right\} + \frac{3}{4}\hbar R = T$$

or
$$\frac{1}{2}M\left\{2v_1^2 - 2\sqrt{\frac{2T}{M}}v_1 + \frac{2T}{M}\right\} + \frac{3}{4}\hbar R = T$$

$$M\left\{\left(v_1 - \frac{1}{2}\sqrt{\frac{2T}{M}}\right)^2\right\} + \frac{1}{2}T + \frac{3}{4}\hbar R = T$$

or
$$M\left\{\left(v_1 - \frac{1}{2}\sqrt{\frac{2T}{M}}\right)^2\right\} + \frac{3}{4}\hbar R = \frac{1}{2}T$$

For minimum T, the square on the left should vanish. Thus, $T = \frac{3}{2}\hbar R = 20.4$ eV.

6.41 In the rest frame of the original excited nucleus we have the equations

$$O = \mathbf{p}_\gamma + \mathbf{p}_H$$

and
$$\frac{3}{4}\hbar R = c\left|\mathbf{p}_\gamma\right| + \frac{p_H^2}{2M}$$

(where $3/4\hbar R$ is the energy available in $n = 2 \to 1$ transition corresponding to the first Lyman line).

Then
$$p_H^2 + 2Mcp_H - \frac{3\hbar RM}{2} = 0$$

or
$$(p_H + Mc)^2 = M^2c^2 + \frac{3}{2}\hbar RM$$

$$p_H = Mc + \sqrt{M^2c^2 + \frac{3}{2}\hbar RM}$$

$$= -Mc + Mc\left(1 + \frac{3\hbar R}{2Mc^2}\right)^{1/2} \approx \frac{3\hbar R}{4c}$$

(We could have written this directly by noting that $p_H^2/2M \ll cp_\gamma$). Then,

$$v_H = \frac{3\hbar R}{4Mc} = 3.3 \text{ m/s}$$

6.42 We have
$$\varepsilon = \frac{3}{4}\hbar R \quad \text{and} \quad \varepsilon' \approx \frac{3}{4}\hbar R - \frac{1}{2M}\left(\frac{3\hbar R}{4c}\right)^2$$

Then,
$$\frac{\varepsilon - \varepsilon'}{\varepsilon} = \frac{3\hbar R}{8Mc^2}$$

$$= \frac{v_H}{2c} = 5.5 \times 10^{-9} \times 100 = 0.55 \times 10^{-6}\ \%$$

6.43 We neglect recoil effects. The energy of the first Lyman line photon emitted by He^+ is

$$4\hbar R\left(1 - \frac{1}{4}\right) = 3\hbar R$$

The velocity v of the photoelectron that this photon liberates is given by

$$3\hbar R = \frac{1}{2}mv^2 + \hbar R$$

(where $\hbar R$ on the right is the binding energy of the $n = 1$ electron in H atom).

Thus,
$$v = \sqrt{\frac{4\hbar R}{m}} = 2\sqrt{\frac{\hbar R}{m}} = 3.1 \times 10^6 \text{ m/s}$$

Here, m is the mass of the electron.

6.44 Since $\Delta\lambda$ (= 0.20 nm) $\ll \lambda$ (= 121 nm) of the first Lyman line of H atom, we need not worry about v^2/c^2 effects. Then,

$$\omega' = \frac{\omega}{1-\beta\cos\theta} \quad \left(\text{where } \beta = \frac{v}{c}\right)$$

Hence, $$1-\beta\cos\theta = \frac{\omega}{\omega'} = \frac{\lambda'}{\lambda}$$

or $$\beta\cos\theta = 1-\frac{\lambda'}{\lambda} = \frac{\Delta\lambda}{\lambda}$$

But $$\omega = \frac{3}{4}R \quad \text{so} \quad \lambda = \frac{2\pi c}{\omega} = \frac{8\pi c}{3R}$$

Hence, $$v = c\beta = \frac{3R\Delta\lambda}{8\pi\cos\theta}$$

Substitution $(\cos\theta = 1/\sqrt{2})$ gives $v = 7.0 \times 10^5$ m/s.

6.45 (a) If we measure energy from the bottom of the well, then $V(x) = 0$ inside the walls. Then the quantization condition reads

$$\oint p\,dx = 2lp = 2\pi n\hbar$$

or $$p = \frac{\pi n\hbar}{l}$$

Hence, $$E_n = \frac{p^2}{2m} = \frac{\pi^2 n^2 \hbar}{2ml^2}$$

(Here $\oint p\,dx = 2lp$ because we have to consider the integral from $-l/2$ to $l/2$ and then back to $-l/2$.)

(b) Here, $$\oint p\,dx = 2\pi rp = 2\pi n\hbar$$

or $$p = \frac{n\hbar}{r}$$

Hence, $$E_n = \frac{n^2\hbar^2}{2mr^2}$$

(c) By energy conservation

$$\frac{p^2}{2m} + \frac{1}{2}\alpha x^2 = E$$

so, $$p = \sqrt{2mE - m\alpha x^2}$$

Then, $$\oint p\,dx = \oint \sqrt{2mE - m\alpha x^2}\,dx$$

$$= 2\sqrt{m\alpha} \int_{-\sqrt{2E/\alpha}}^{\sqrt{2E/\alpha}} \sqrt{\frac{2E}{\alpha} - x^2}\, dx$$

The integral is

$$\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = a^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta = \frac{a^2}{2} \int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\, d\theta$$

$$= a^2 \frac{\pi}{2}$$

Thus, $$\oint p\, dx = \pi\sqrt{m\alpha} \cdot \frac{2E}{\alpha} = E \cdot 2\pi \cdot \sqrt{\frac{m}{\alpha}} = 2\pi n\hbar$$

Hence, $$E_n = n\hbar\sqrt{\frac{\alpha}{m}}$$

(d) It is required to find the energy levels of the circular orbit for the potential

$$U(r) = -\frac{\alpha}{r}$$

In a circular orbit, the particle only has tangential velocity and the quantization condition reads $\oint p\, dx = mv \cdot 2\pi r = 2\pi n\hbar$

so, $$mvr = M = n\hbar$$

The energy of the particle is

$$E = \frac{n^2\hbar^2}{2mr^2} - \frac{\alpha}{r}$$

Equilibrium requires that the energy as a function of r be minimum. Thus,

$$\frac{n^2\hbar^2}{mr^3} = \frac{\alpha}{r^2} \quad \text{or} \quad r = \frac{n^2\hbar^2}{m\alpha}$$

Hence, $$E_n = -\frac{m\alpha^2}{2n^2\hbar^2}$$

6.46 The total energy of the H atom in an arbitrary frame is

$$E = \frac{1}{2} m\mathbf{V}_1^2 + \frac{1}{2} M\mathbf{V}_2^2 - \frac{e^2}{(4\pi\varepsilon_0)|\mathbf{r}_1 - \mathbf{r}_2|}$$

Here $\mathbf{V}_1 = \dot{\mathbf{r}}_1$, $\mathbf{V}_2 = \dot{\mathbf{r}}_2$, and $\mathbf{r}_1$ and $\mathbf{r}_2$ are the coordinates of the electron and protons.

We define $$\mathbf{R} = \frac{m\mathbf{r}_1 + M\mathbf{r}_2}{M + m}$$

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

Then,
$$\mathbf{V} = \frac{m\mathbf{V}_1 + M\mathbf{V}_2}{m + M}$$

$$\mathbf{v} = \mathbf{V}_1 - \mathbf{V}_2$$

or
$$\mathbf{V}_1 = \mathbf{V} + \frac{M}{m + M}\mathbf{v}$$

and
$$\mathbf{V}_2 = \mathbf{V} - \frac{m}{m + M}\mathbf{v}$$

We get
$$E = \frac{1}{2}(m + M)V^2 + \frac{1}{2}\frac{mM}{m + M}v^2 - \frac{e^2}{4\pi\varepsilon_0 r}$$

In the frame $\mathbf{V} = 0$, this reduces to the energy of a particle of mass

$$\mu = \frac{mM}{m + M}$$

where μ is called the reduced mass.

Then,
$$E_b = \frac{\mu e^4}{2\hbar^2} \quad \text{and} \quad R = \frac{\mu e^4}{2\hbar^3}$$

Since,
$$\mu = \frac{m}{1 + \dfrac{m}{M}} \approx m\left(1 - \frac{m}{M}\right)$$

these values differ by m/M ($\cong$ 0.54%) from the values obtained without considering nuclear motion ($M = 1837m$).

6.47 The difference between the binding energies is

$$\Delta E_b = E_b(\mathrm{D}) - E_b(\mathrm{H})$$
$$= \frac{m}{1 + m/M}\frac{e^4}{2\hbar^2} + \frac{m}{1 + m/2M}\frac{e^4}{2\hbar^2}$$
$$= \frac{me^4}{2\hbar^2}\left(\frac{m}{2m}\right)$$

Substitution gives $\Delta E_b = 3.7$ meV.

For the first line of the Lyman series

$$\frac{2\pi\hbar c}{\lambda} = \hbar R\left(\frac{1}{1} - \frac{1}{4}\right) = \frac{3}{4}\hbar R$$

or
$$\lambda = \frac{8\pi c}{3R} = \frac{8\pi\hbar c}{3E_b}$$

Hence,
$$\lambda_H - \lambda_D = \frac{8\pi\hbar c}{3}\left(\frac{1}{E_b(H)} - \frac{1}{E_b(D)}\right)$$
$$= \frac{8\pi\hbar c}{3}\cdot\left(\frac{me^4}{2\hbar^2}\right)^{-1}\left(1 + \frac{m}{M} - 1 - \frac{m}{2M}\right)$$
$$= \frac{8\pi\hbar c}{3\left(\dfrac{me^4}{2\hbar^2}\right)}\cdot\frac{m}{2M}$$
$$= \frac{m}{2M}\times\lambda_1$$

(where λ_1 is the wavelength of the first line of Lyman series without considering nuclear motion).

Using $\lambda_1 = 121$ nm (see Problem 6.21), we get $\Delta\lambda = 33$ pm.

6.48 (a) In the mesonic system, the reduced mass of the system is related to the masses of the meson (m_μ) and proton (m_p) by

$$\mu = \frac{m_\mu m_p}{m_\mu + m_p} = 186.04\ m_e$$

Then, separation between the particles in the ground state is

$$\frac{\hbar^2}{\mu e^2} = \frac{1}{186}\frac{\hbar^2}{me^2}$$
$$= 0.284 \text{ pm}$$

$$E_b(\text{meson}) = \frac{\mu e^4}{2\hbar^2} = 186\times 13.65 \text{ eV}$$
$$= 2.54 \text{ keV}$$

$$\lambda_1(\text{meson}) = \frac{8\pi\hbar c}{3E_b(\text{meson})} = \frac{\lambda_1(H)}{186} = 0.65 \text{ nm}$$

(On using λ_1 (H) = 121 nm.)

(b) In the positronium

$$\mu = \frac{m_e^2}{2m_e} = \frac{m_e}{2}$$

Thus, separation between the particles in the ground state is

$$2\frac{\hbar^2}{m_e e^2} = 105.8 \text{ pm}$$

$$E_b(\text{positronium}) = \frac{m_e}{2} \cdot \frac{e^4}{2\hbar^2} = \frac{1}{2} E_b(\text{H}) = 6.8 \text{ eV}$$

$$\lambda_1 \text{ (positronium)} = 2\lambda_1(\text{H}) = 0.243 \text{ nm}$$

6.2 Wave Properties of Particles. Schrodinger Equation

6.49 The kinetic energy is non-relativistic in all three cases. Now

$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{\sqrt{2mT}}$$

Using $T = 1.602 \times 10^{-17}$ J, we get $\lambda_e = 122.6$ pm and $\lambda_p = 2.86$ pm, $\lambda_U = \lambda_p/\sqrt{238} =$ 0.185 pm.

(Here we have used a mass number of 238 for the U nucleus).

6.50 From

$$\lambda = \frac{2\pi\hbar}{p} = \frac{2\pi\hbar}{\sqrt{2mT}}$$

we find

$$T = \frac{4\pi^2\hbar^2}{2m\lambda^2} = \frac{2\pi^2\hbar^2}{m\lambda^2}$$

Thus,

$$T_2 - T_1 = \frac{2\pi^2\hbar^2}{m}\left(\frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2}\right)$$

Substitution gives $\Delta T = 451$ eV $= 0.451$ keV.

6.51 We shall use $M_d \approx 2M_n$. The centre of mass (C.M.) is moving with velocity

$$V = \frac{\sqrt{2M_n T}}{3M_n} = \sqrt{\frac{2T}{9M_n}}$$

with respect to the laboratory frame. In the C.M. frame, the velocity of neutron is

$$v_n' = v_n - V = \sqrt{\frac{2T}{M_n}} - \sqrt{\frac{2T}{9M_n}} = \sqrt{\frac{2T}{M_n}} \cdot \frac{2}{3}$$

and

$$\lambda_n' = \frac{2\pi\hbar}{M_n v_n'} = \frac{3\pi\hbar}{\sqrt{2M_n T}}$$

Substitution gives $\lambda'_n = 8.6$ pm.

Since the momenta are equal in the C.M. frame, the de Broglie wavelengths will also be equal. If we do not assume $M_d \approx 2M_n$, we shall get

$$\lambda'_n = \frac{2\pi\hbar(1 + M_n/M_d)}{\sqrt{2M_n T}}$$

6.52 If $\mathbf{p}_1$, $\mathbf{p}_2$ are the momenta of the two particles then their momenta in the C.M. frame will be $\pm\,(\mathbf{p}_1 - \mathbf{p}_2)/2$ as the particles are identical. Hence, their de Broglie wavelength will be

$$\lambdabar = \frac{2\pi\hbar}{\frac{1}{2}|\mathbf{p}_1 - \mathbf{p}_2|} = \frac{4\pi\hbar}{\sqrt{p_1^2 + p_2^2}} \quad \text{(because } \mathbf{p}_1 \perp \mathbf{p}_2\text{)}$$

$$= \frac{2}{\sqrt{\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}}} = \frac{2\lambda_1\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}}.$$

6.53 In thermodynamic equilibrium, Maxwell's velocity distribution law holds, so

$$dN(v) = \Phi(v)dv = Av^2 e^{-mv^2/2kT}\,dv$$

$\Phi\,(v)$ is maximum when

$$\Phi'(v) = \Phi(v)\left[\frac{2}{v} - \frac{mv}{kT}\right] = 0$$

This defines the most probable velocity,

$$v_{\text{pr}} = \sqrt{\frac{2kT}{m}}$$

The de Broglie wavelength of H molecules with the most probable velocity is

$$\lambda = \frac{2\pi\hbar}{mv_{\text{pr}}} = \frac{2\pi\hbar}{\sqrt{2mkT}}$$

Substituting the appropriate values $m = m_{H_2} = 2m_H$, $T = 300$ K, we get $\lambda = 128$ pm.

6.54 To find the most probable de Broglie wavelength of a gas in thermodynamic equilibrium, we determine the distribution in λ corresponding to Maxwellian velocity distribution. It is given by

$$\Psi\,(\lambda)\,d\lambda = -\Phi\,(v)\,dv$$

(where $-ve$ sign takes into account that λ decreases as v increases).

Now, $$\lambda = \frac{2\pi\hbar}{mv} \quad \text{or} \quad v = \frac{2\pi\hbar}{m\lambda}$$

$$dv = -\frac{2\pi\hbar}{m\lambda^2}\,d\lambda$$

Thus, $$\Psi(\lambda) = +\,Av^2 e^{-mv^2/2kT}\left(-\frac{dv}{d\lambda}\right)$$

$$= A\left(\frac{2\pi\hbar}{m\lambda}\right)^2\left(\frac{2\pi\hbar}{m\lambda^2}\right)\exp\left[\frac{-m}{2kT}\cdot\left(\frac{2\pi\hbar}{m\lambda}\right)^2\right]$$

$$= \text{constant} \times \lambda^{-4} e^{-a/\lambda^2}$$

where $$a = \frac{2\pi^2\hbar^2}{mkT}$$

This is maximum when

$$\Psi'(\lambda) = 0 = \Psi(\lambda)\left[\frac{-4}{\lambda} + \frac{2a}{\lambda^3}\right]$$

or $$\lambda_{\text{pr}} = \sqrt{\frac{a}{2}} = \frac{\pi\hbar}{\sqrt{mkT}}$$

Using the result of the previous problem, it is

$$\lambda_{\text{pr}} = \frac{126}{\sqrt{2}} = 89.1 \text{ pm}$$

6.55 For a relativistic particle

$$\text{Total energy} = T + mc^2 = \sqrt{c^2 p^2 + m^2 c^4}$$

Squaring, rearranging and taking positive root

$$\sqrt{T(T + 2mc^2)} = cp$$

Hence, $$\lambda = \frac{2\pi\hbar c}{\sqrt{T(T + 2mc^2)}}$$

$$= \frac{2\pi\hbar}{\sqrt{2mT(1 + T/2mc^2)}}$$

If we use the non-relativistic formula,

$$\lambda_{\text{NR}} = \frac{2\pi\hbar}{\sqrt{2mT}}$$

so, $$\frac{\Delta\lambda}{\lambda} = \frac{\lambda_{\text{NR}} - \lambda}{\lambda_{\text{NR}}} \approx \frac{T}{4mc^2}$$

Thus, $$T \le \frac{4mc^2\Delta\lambda}{\lambda} \quad \text{(if the error is less than } \Delta\lambda)$$

Here we have used that if $T/2\ mc^2 \ll 1$, we can write $(1 + T/2mc^2)^{-1/2} \approx 1 - T/4mc^2$.

For an electron, the error is not more than 1% if

$$T \le 4 \times 0.511 \times 0.01 \text{ MeV}$$

i.e., $$T \le 20.4 \text{ keV}$$

For a proton, the error is not more than 1% if

$$T \le 4 \times 938 \times 0.01 \text{ MeV}$$

i.e., $$T \le 37.5 \text{ MeV}$$

6.56 The de Broglie wavelength is

$$\lambda_{dB} = \frac{2\pi\hbar/m_0 v}{\sqrt{1 - v^2/c^2}} = \frac{2\pi\hbar}{m_0 v}\sqrt{1 - v^2/c^2}$$

and the Compton wavelength is

$$\lambda_{comp} = \frac{2\pi\hbar}{m_0 c}$$

The two are equal if $\beta = \sqrt{1 - \beta^2}$, where $\beta = \dfrac{v}{c}$ or $\beta = \dfrac{1}{\sqrt{2}}$

The corresponding K.E. is

$$T = \frac{m_0 c^2}{\sqrt{1 - \beta^2}} - m_0 c^2 = \left(\sqrt{2} - 1\right) m_0 c^2 = 0.21 \text{ MeV}$$

Here m_0 is the rest mass of the particle (which here is an electron).

6.57 For relativistic electrons, the formula for the short wavelength limit of X-rays will be

$$\frac{2\pi\hbar c}{\lambda_{sh}} = m_0 c^2 \left(\frac{1}{\sqrt{1 - \beta^2}} - 1 \right) = c\sqrt{p^2 + m^2c^2} - mc^2$$

or $$\left(\frac{2\pi\hbar}{\lambda_{sh}} + mc \right)^2 = p^2 + m^2c^2$$

or $$\left(\frac{2\pi\hbar}{\lambda_{sh}} \right)\left(\frac{2\pi\hbar}{\lambda_{sh}} + 2mc \right) = p^2$$

or $$p = \frac{2\pi\hbar}{\lambda_{sh}} \sqrt{1 + \frac{mc\lambda_{sh}}{\pi\hbar}}$$

Hence, $$\lambda_{dB} = \lambda_{sh} \Big/ \sqrt{1 + \frac{mc\lambda_{sh}}{\pi\hbar}} = 3.29 \text{ pm}$$

6.58 The first minimum in Fraunhofer diffraction is given by $b \sin\theta = \lambda$ (where b is the width of the slit).

Here, $$\sin\theta = \frac{\Delta x/2}{\sqrt{l^2 + (\Delta x/2)^2}} \approx \frac{\Delta x}{2l}$$

Thus, $$\lambda = \frac{b\Delta x}{2l} = \frac{2\pi\hbar}{mv}$$

so, $$v = \frac{4\pi\hbar l}{mb\Delta x} = 2.02 \times 10^6 \text{ m/s}$$

6.59 From the Young's slit formula

$$\Delta x = \frac{l\lambda}{d} = \frac{l}{d} \cdot \frac{2\pi\hbar}{\sqrt{2meV}}$$
$$= 4.90 \ \mu\text{m} \quad \text{(on substituting values)}$$

6.60 From Bragg's law, for the first case

$$2d\sin\theta = n_0\lambda = n_0 \frac{2\pi\hbar}{\sqrt{2meV_0}}$$

where n_0 is an unknown integer. For the next higher voltage

$$2d\sin\theta = (n_0 + 1)\frac{2\pi\hbar}{\sqrt{2me\eta V_0}}$$

Thus, $$n_0 = \frac{n_0 + 1}{\sqrt{\eta}}$$

or $$n_0\left(1 - \frac{1}{\sqrt{\eta}}\right) = \frac{1}{\sqrt{\eta}} \quad \text{or} \quad n_0 = \frac{1}{\sqrt{\eta} - 1}$$

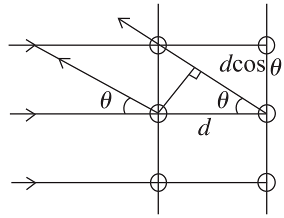
Going back we get

$$V_0 = \frac{\pi^2\hbar^2}{2med^2\sin^2\theta} \frac{1}{\left(\sqrt{\eta} - 1\right)^2} = 0.150 \text{ keV}$$

Note: *In the Bragg's formula, θ is the glancing angle and not the angle of incidence. We have obtained the correct result by taking θ to be the glancing angle. If θ is the angle of incidence, then the glancing angle will be $90° - \theta$. Then final answer will be smaller by a factor $\tan^2\theta = 1/3$.*

6.61 Path difference is

$$d + d\cos\theta = 2d\cos^2\frac{\theta}{2}$$



Thus, for reflection maximum of the k^{th} order

$$2d\cos^2\frac{\theta}{2} = k\lambda = k\frac{2\pi\hbar}{\sqrt{2mT}}$$

Hence, $$d = \frac{k\pi\hbar}{\sqrt{2mT}}\sec^2\frac{\theta}{2}$$

Substitution with $k = 4$, gives $d = 0.232$ nm.

6.62 See the analogous Problem 5.156 with X-rays. The glancing angle is obtained from

$$\tan 2\theta = \frac{D}{2l}$$

where D = diameter of the ring, l = distance from the foil to the screen.

Then for the third-order Bragg reflection,

$$2d\sin\theta = k\lambda = k\frac{2\pi\hbar}{\sqrt{2mT}} \quad (\text{where } k = 3)$$

Thus, $$d = \frac{\pi\hbar k}{\sqrt{2mT}\sin\theta} = 0.232 \text{ nm}$$

6.63 Inside the metal, there is a negative potential energy of $-\,eV_i$. (This potential energy prevents electrons from leaking out and can be measured in photoelectric effect, etc.) An electron whose K.E. is eV outside the metal will find its K.E. increased to $e\,(V + V_i)$ in the metal. Then,

(a) de Broglie wavelength in the metal

$$\lambda_m = \frac{2\pi\hbar}{\sqrt{2me(V + V_i)}}$$

Also, de Broglie wavelength in vacuum

$$\lambda_0 = \frac{2\pi\hbar}{\sqrt{2mVe}}$$

Hence, refractive index $$n = \frac{\lambda_0}{\lambda_m} = \sqrt{1 + \frac{V_i}{V}}$$

Substituting, we get $$n = \sqrt{1 + \frac{1}{10}} \approx 1.05$$

(b) We have $$n - 1 = \sqrt{1 + \frac{V_i}{V}} - 1 \le \eta$$

then, $$1 + \frac{V_i}{V} \le (1 + \eta)^2$$

or $$V_i \le \eta\,(2 + \eta)\,V$$

or $$\frac{V}{V_i} \ge \frac{1}{\eta(2 + \eta)}$$

For $\eta = 1\% = 0.01$, we get

$$\frac{V}{V_i} \ge 50$$

6.64 The energy inside the well is entirely kinetic if energy is measured from the value inside.

We require $$l = \frac{n\lambda}{2} = n\frac{\pi\hbar}{\sqrt{2mE}}$$

or $$E_n = \frac{n^2\pi^2\hbar^2}{2ml^2} \quad (\text{for } n = 1, 2, \ldots)$$

6.65 The Bohr condition is

$$\oint p\,dx = \oint \frac{2\pi\hbar}{\lambda}\,dx = 2\pi n\hbar$$

For the case when λ is constant (for example in circular orbits), this means

$$2\pi r = n\lambda$$

(Here r is the radius of the circular orbit.)

6.66 From the uncertainty principle (Eq. 6.2b of the book)

$$\Delta x \Delta p_x \ge \hbar$$

Thus, $$\Delta p_x = m\Delta v_x \ge \frac{\hbar}{\Delta x}$$

or $$\Delta v_x \ge \frac{\hbar}{m\Delta x}$$

If $\Delta x = 1\ \mu\text{m} = 10^{-6}$ m, the uncertainty in velocity is:

For an electron $$\Delta v_x = 1.16 \times 10^4 \text{ cm/s}$$

For a proton $\qquad \Delta v_x = 6.3$ cm/s

For a ball $\qquad \Delta v_x = 1 \times 10^{-20}$ cm/s

6.67 As in the previous problem

$$\Delta v \gtrsim \frac{\hbar}{ml} = 1.16 \times 10^6 \text{ m/s}$$

The actual velocity v_1 has been expressed in Problem 6.21. It is $v_1 = 2.21 \times 10^6$ m/s. Thus, $\Delta v \sim v_1$ (they are of the same order of magnitude).

6.68 Given that

$$\Delta x = \frac{\lambda}{2\pi} = \frac{2\pi\hbar}{p} \cdot \frac{1}{2\pi}$$

$$= \frac{\hbar}{p} = \frac{\hbar}{mv}$$

Then,

$$\Delta v \gtrsim \frac{\hbar}{m\Delta x} = v$$

Thus, Δv is of the same order as v.

6.69 We know that initial uncertainty is $\Delta v \gtrsim \hbar/ml$.

With this uncertainty the wave train will spread out to a distance ηl in time t_0, then

$$t_0 \approx \frac{\eta l}{\hbar/ml} \approx \frac{\eta m l^2}{\hbar} \text{ s}$$

$$= 8.6 \times 10^{-16} \text{ s} \sim 10^{-15} \text{ s}$$

6.70 Clearly

$$\Delta x \le l \quad \text{so,} \quad \Delta p_x \ge \frac{\hbar}{l}$$

Now $p_x \ge \Delta p_x$ and so,

$$T = \frac{p_x^2}{2m} \ge \frac{\hbar^2}{2ml^2}$$

Thus,

$$T_{\min} = \frac{\hbar^2}{2ml^2} \simeq 0.95 \text{ eV}$$

6.71 The momentum of the electron is $\Delta p_x = \sqrt{2mT}$. Uncertainty in its momentum is

$$\Delta p_x \ge \frac{\hbar}{\Delta x} = \frac{\hbar}{l}$$

Hence, relative uncertainty

$$\frac{\Delta p_x}{p_x} = \frac{\hbar}{l\sqrt{2mT}} = \sqrt{\frac{\hbar^2/2ml^2}{T}} = \frac{\Delta v}{v}$$

Substitution gives $$\frac{\Delta v}{v} = 9.75 \times 10^{-5} \approx 10^{-4}$$

6.72 By uncertainty principle, the uncertainty in momentum

$$\Delta p \geq \frac{\hbar}{l}$$

For the ground state, we expect $\Delta p \sim p$, so

$$E \sim \frac{\hbar^2}{2ml^2}$$

The force exerted on the wall can be obtained as

$$F = -\frac{\partial U}{\partial l} = \frac{\hbar^2}{ml^3}$$

6.73 We write $$p \sim \Delta p \sim \frac{\hbar}{\Delta x} \sim \frac{\hbar}{x}$$

i.e., all four quantities are of the same order of magnitude. Then,

$$E \approx \frac{\hbar^2}{2mx^2} + \frac{1}{2}kx^2 = \frac{1}{2m}\left(\frac{\hbar}{x} - \sqrt{mk}x\right)^2 + \hbar\sqrt{\frac{k}{m}}$$

Thus, we get an equilibrium situation (E = minimum) when

$$x = x_0 = \sqrt{\frac{\hbar}{\sqrt{mk}}}$$

and then $$E = E_0 \sim \hbar\sqrt{\frac{k}{m}} = \hbar\omega$$

Quantum mechanics gives

$$E_0 = \frac{\hbar\omega}{2}$$

6.74 We write $$r \sim \Delta r, \quad p \sim \Delta p \sim \frac{\hbar}{\Delta r}$$

Then, $$E = \frac{\hbar^2}{2mr^2} - \frac{e^2}{r}$$

$$= \frac{1}{2m}\left(\frac{\hbar}{r} - \frac{me^2}{\hbar}\right)^2 - \frac{me^4}{2\hbar^2}$$

Hence, $$r_{\text{eff}} = \frac{\hbar^2}{me^2} = 53 \text{ pm (for the equilibrium state)}$$

and then $$E = -\frac{me^4}{2\hbar^2} = -13.6 \text{ eV}$$

6.75 Suppose the width of the slit (its extension along the y-axis) is δ. Then each electron has an uncertainty $\Delta y \sim \delta$. This translates to an uncertainty $\Delta p_y \sim \hbar/\delta$. We must therefore have

$$p_y \geq \frac{\hbar}{\delta}$$

For the image, broadening has two sources. We write

$$\Delta(\delta) = \delta + \Delta'(\delta)$$

where Δ' is the width caused by the spreading of electrons due to their transverse momentum.

We have $$\Delta' = v_y \frac{l}{v_x} \simeq p_y \frac{l}{p} = \frac{l\hbar}{mv\delta}$$

Thus, $$\Delta(\delta) = \delta + \frac{l\hbar}{mv\delta}$$

For large δ, $\Delta(\delta) \sim \delta$ and quantum effect is unimportant. For small δ, quantum effects are large. But $\Delta(\delta)$ is minimum when

$$\delta = \sqrt{\frac{l\hbar}{mv}}$$

as we see by completing the square.

Substitution gives $\delta = 1.025 \times 10^{-5}$ m ≈ 0.01 mm.

6.76 The Schrodinger equation in one dimension for a free particle is

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}$$

We write $$\Psi(x, t) = \varphi(x)\,\chi(t)$$

Then, $$\frac{i\hbar}{\chi}\frac{d\chi}{dt} = -\frac{\hbar^2}{2m}\frac{1}{\varphi}\frac{d^2\varphi}{dx^2} = E \text{ (say)}$$

Then, $$\chi(t) \sim \exp\left(-\frac{iEt}{\hbar}\right)$$

$$\varphi(x) \sim \exp\left(i\frac{\sqrt{2mE}}{\hbar}x\right)$$

E must be real and positive if $\varphi(x)$ is to be bound everywhere. Then,

$$\psi(x, t) = \text{constant} \times \exp\left(\frac{i}{\hbar}\left(\sqrt{2mE}x - Et\right)\right)$$

This particular solution describes plane waves.

6.77 We look for the solution of Schrodinger equation with

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \quad (\text{for } 0 \le x \le l) \tag{1}$$

The boundary condition of impenetrable walls means

$\psi(x) = 0$ for $x = 0$ and $x = l$ (as $\psi(x) = 0$ for $x < 0$ and $x > l$).

The solution of Eq. (1) is

$$\psi(x) = A\sin\frac{\sqrt{2mE}}{\hbar}x + B\cos\frac{\sqrt{2mE}}{\hbar}x$$

Then $$\psi(0) = 0 \Rightarrow B = 0$$

$$\psi(l) = 0 \Rightarrow A\sin\frac{\sqrt{2mE}}{\hbar}l = 0$$

As $A \neq 0$, so $$\frac{\sqrt{2mE}}{\hbar}l = n\pi$$

Hence, $$E_n = \frac{n^2\pi^2\hbar^2}{2ml^2} \quad (\text{for } n = 1, 2, 3, \ldots)$$

Thus, the ground state wave function is

$$\psi(x) = A\sin\frac{\pi x}{l}$$

We evaluate A by normalization

$$1 = A^2\int_0^l \sin^2\frac{\pi x}{l}dx = A^2\frac{l}{\pi}\int_0^{\pi}\sin^2\theta\, d\theta = A^2\frac{l}{\pi}\cdot\frac{\pi}{2}$$

Thus, $$A = \sqrt{\frac{2}{l}}$$

Finally, the probability P of the particle lying in $l/2 \le x \le 2l/3$ is

$$P = P\left(\frac{l}{3} \le x \le \frac{2l}{3}\right) = \frac{2}{l}\int_{l/3}^{2l/3} \sin^2 \frac{\pi x}{l}\, dx$$

$$= \frac{2}{\pi}\int_{\pi/3}^{2\pi/3} \sin^2\theta\, d\theta = \frac{1}{\pi}\int_{\pi/3}^{2\pi/3} (1-\cos 2\theta)\, d\theta$$

$$= \frac{1}{\pi}\left[\theta - \frac{1}{2}\sin 2\theta\right]_{\pi/3}^{2\pi/3} = \frac{1}{\pi}\left(\frac{2\pi}{3} - \frac{\pi}{3} - \frac{1}{2}\sin\frac{4\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3}\right)$$

$$= \frac{1}{\pi}\left(\frac{\pi}{3} + \frac{1}{2}\frac{\sqrt{3}}{2} + \frac{1}{2}\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{3} + \frac{\sqrt{3}}{2\pi} = 0.609$$

6.78 Here

$$-\frac{l}{2} \le x \le \frac{l}{2}$$

So, we have

$$\psi(x) = B\cos\frac{\sqrt{2mE}x}{\hbar} + A\sin\frac{\sqrt{2mE}x}{\hbar}$$

Then the boundary condition

$$\psi\left(\pm\frac{l}{2}\right) = 0$$

gives

$$B\cos\frac{\sqrt{2mE}l}{2\hbar} \pm A\sin\frac{\sqrt{2mE}l}{2\hbar} = 0$$

There are two cases:

(1) $A = 0$,

$$\sqrt{\frac{2mE}{2\hbar}}l = n\pi + \frac{\pi}{2}$$

or

$$\sqrt{2mE} = (2n+1)\frac{\pi\hbar}{l}$$

and

$$E_n = (2n+1)^2\frac{\pi^2\hbar^2}{2ml^2}$$

$$\psi_n^e(x) = \sqrt{\frac{2}{l}}\cos(2n+1)\frac{\pi x}{l} \quad (\text{for } n = 0, 1, 2, 3, \ldots)$$

This solution is even under $x \to -x$.

(2) $B = 0$,

$$\frac{\sqrt{2mE}}{2\hbar} l = n\pi \quad (\text{for } n = 1, 2, \ldots)$$

$$E_n = (2n\pi)^2 \frac{\hbar^2}{2ml^2}$$

$$\psi_n^o = \sqrt{\frac{2}{l}} \sin \frac{2n\pi x}{l} \quad (\text{for } n = 1, 2, \ldots)$$

This solution is odd.

6.79 The wave function is given in Problem 6.77. We see that

$$\int_0^l \psi_n(x)\psi_{n'}(x)dx = \frac{2}{l}\int_0^l \sin\frac{n\pi x}{l}\sin\frac{n'\pi x}{l}dx$$

$$= \frac{1}{l}\int_0^l \left[\cos(n-n')\frac{\pi x}{l} - \cos(n+n')\frac{\pi x}{l}\right]dx$$

$$= \frac{1}{l}\left[\frac{\sin(n-n')\pi x/l}{(n-n')\pi/l} - \frac{\sin(n+n')\pi x/l}{(n+n')\pi x/l}\right]_0^l e$$

If $n \neq n'$, this expression is zero as n and n' are integers.

6.80 We have found that

$$E_n = \frac{n^2\pi^2\hbar^2}{2ml^2}$$

Let $N(E)$ = number of states upto E. This number is n. The number of states up to $E + dE$ is $N(E + dE) = N(E) + dN(E)$. Then, $dN(E) = 1$ and

$$\frac{dN(E)}{dE} = \frac{1}{\Delta E}$$

(where ΔE = difference in energies between the n^{th} and $(n+1)^{\text{th}}$ levels).

Now,
$$\Delta E = \frac{(n+1)^2 - n^2}{2ml^2}\pi^2\hbar^2$$

$$= \frac{2n+1}{2ml^2}\pi^2\hbar^2$$

$$\cong \frac{\pi^2 \hbar^2}{2ml^2} 2n \text{ (neglecting } 1 \ll n)$$

$$= \frac{\pi^2 \hbar^2}{2ml^2} \times \sqrt{\frac{2ml^2}{\pi^2 \hbar^2}} \sqrt{E} \times 2$$

$$= \frac{\pi \hbar}{l} \sqrt{\frac{2}{m}} \sqrt{E}$$

Thus,
$$\frac{dN(E)}{dE} = \frac{l}{\pi \hbar} \sqrt{\frac{m}{2E}}$$

For the given case this gives

$$\frac{dN(E)}{dE} = 0.816 \times 10^7 \text{ levels per eV}$$

6.81 (a) Here the Schrodinger equation is

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = E\psi$$

we take the origin at one of the corners of the rectangle where the particle can lie. Then the wave function must vanish for

$$x = 0 \quad \text{or} \quad x = l_1$$

or
$$y = 0 \quad \text{or} \quad y = l_2$$

We look for a solution in the form

$$\psi = A \sin k_1 x \sin k_2 y$$

cosines are not permitted by the boundary conditions.

Then,
$$k_1 = \frac{n_1 \pi}{l_1} \quad \text{and} \quad k_2 = n_2 \frac{\pi}{l_2}$$

and
$$E = \frac{k_1^2 + k_2^2}{2m} \hbar^2 = \frac{\pi^2 \hbar^2}{2m}\left(\frac{n_1^2}{l_1^2} + \frac{n_2^2}{l_2^2}\right)$$

Here n_1, n_2 are non-zero integers.

(b) If $l_1 = l_2 = l$, then

$$\frac{E}{\hbar^2/ml^2} = \frac{n_1^2 + n_2^2}{2} \pi^2$$

1st level:
$$n_1 = n_2 = 1 \rightarrow \pi^2 = 9.87$$

2nd level: $\left.\begin{matrix} n_1 = 1, n_2 = 2 \\ n_1 = 2, n_2 = 1 \end{matrix}\right\} \rightarrow \frac{5}{2}\pi^2 = 24.7$

3rd level: $n_1 = n_2 = 2 \rightarrow 4\pi^2 = 39.5$

4th level: $\left.\begin{matrix} n_1 = 1, n_2 = 3 \\ n_1 = 3, n_2 = 1 \end{matrix}\right\} \rightarrow 5\pi^2 = 49.3$

6.82 The wave function for the ground state is

$$\psi_{11}(x, y) = A \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$$

We find A by normalization

$$1 = A^2 \int_0^a dx \int_0^b dy \sin^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{b} = A^2 \frac{ab}{4}$$

Thus, $A = \frac{2}{\sqrt{ab}}$

Then, the requisite probability

$$P = \int_0^{a/3} dx \int_0^b dy \frac{4}{ab} \sin^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{b}$$

$$= \frac{2}{a} \int_0^{a/3} dx \sin^2 \frac{\pi x}{a} \quad \text{(on doing the } y \text{ integral)}$$

$$= \frac{1}{a} \int_0^{a/3} d\left(1 - \cos \frac{2\pi x}{a}\right) = \frac{1}{a}\left(\frac{a}{3} - \frac{\sin 2\pi/3}{2\pi/a}\right)$$

$$= \frac{1}{3} - \frac{\sqrt{3}}{4\pi} = 0.196 = 19.6\%$$

6.83 We proceed exactly as in Problem 6.81. The wave function is chosen in the form

$$\psi(x, y, z) = A \sin k_1 x \sin k_2 y \sin k_3 z$$

(The origin is at one corner of the box and the axes of coordinates are along the edges.) The boundary condition is that $\psi = 0$ for $x = 0, x = a, y = 0, y = a, z = 0, z = a$

This gives $k_1 = \frac{n_1 \pi}{a}, \quad k_2 = \frac{n_2 \pi}{a}, \quad k_3 = \frac{n_3 \pi}{a}$

The energy eigenvalues are

$$E(n_1, n_2, n_3) = \frac{\pi^2\hbar^2}{2ma^2}(n_1^2 + n_2^2 + n_3^2)$$

(a) The first level is (1, 1, 1). The second has (1, 1, 2), (1, 2, 1) and (2, 1, 1). The third level is (1, 2, 2) or (2, 1, 2) or (2, 2, 1) with energy as

$$E_3 = \frac{9\pi^2\hbar^2}{2ma^2}$$

The fourth energy level is (1, 1, 3) or (1, 3, 1) or (3, 1, 1) with energy

$$E_4 = \frac{11\pi^2\hbar^2}{2ma^2}$$

(b) Thus, $$\Delta E = E_4 - E_3 = \frac{\hbar^2\pi^2}{ma^2}$$

(c) The fifth level is (2, 2, 2). The sixth level is (1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1) with energy

$$E_6 = \frac{7\hbar^2\pi^2}{ma^2}$$

and its degree of degeneracy is 6.

6.84 We can safely assume that the discontinuity occurs at the point $x = 0$. Now the Schrodinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\psi(x) = E\psi(x)$$

We integrate this equation around $x = 0$, i.e., from $x = -\varepsilon_1$ to $x = +\varepsilon_2$ where ε_1, ε_2 are small positive numbers. Then,

$$-\frac{\hbar^2}{2m}\int_{-\varepsilon_1}^{+\varepsilon_2}\frac{d^2\psi}{dx^2}dx = \int_{-\varepsilon_1}^{+\varepsilon_2}\left[E - U(x)\psi(x)dx\right]$$

or $$\left(\frac{d\psi}{dx}\right)_{+\varepsilon_2} - \left(\frac{d\psi}{dx}\right)_{-\varepsilon_1} = -\frac{2m}{\hbar^2}\int_{-\varepsilon_1}^{\varepsilon_2}\left[E - U(x)\right]dx\,\psi(x)$$

Since, the potential and the energy E are finite and $\psi(x)$ is bound by assumption, the integral on the right exists and approaches 0 as ε_1, $\varepsilon_2 \to 0$.

Thus, $$\left(\frac{d\psi}{dx}\right)_{+\varepsilon_2} = \left(\frac{d\psi}{dx}\right)_{-\varepsilon_1} \quad (\text{as } \varepsilon_1,\ \varepsilon_2 \to 0)$$

So $(d\psi/dx)$ is continuous at $x = 0$ (the point where $U(x)$ has a finite jump discontinuity).

6.85 (a) Starting from Schrodinger equation in the regions I and II

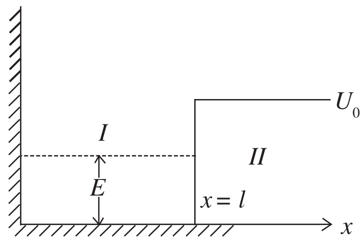
$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi = 0 \quad (\text{for } x \text{ in } I) \tag{1}$$

$$\frac{d^2\psi}{dx^2} - \frac{2mE(U_0 - E)}{\hbar^2}\psi = 0 \quad (\text{for } x \text{ in } II) \tag{2}$$

where $U_0 > E > 0$, we easily derive the solutions in I and II, as

$$\psi_I(x) = A \sin kx + B \cos kx \tag{3}$$

$$\psi_{II}(x) = Ce^{\alpha x} + De^{-\alpha x} \tag{4}$$



where $\quad k^2 = \dfrac{2mE}{\hbar^2},\ \alpha^2 = \dfrac{2m(U_0 - E)}{\hbar^2}$

The boundary condition is that

$$\psi(0) = 0 \tag{5}$$

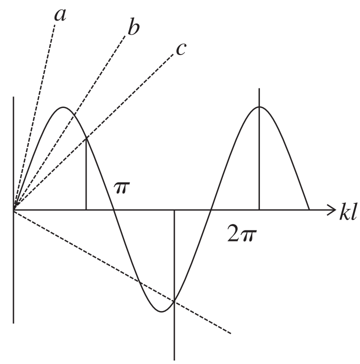
So ψ and $\left(\dfrac{d\psi}{dx}\right)$ are continuous at $x = l$, and ψ must vanish at $x = +\infty$.

Then, $\quad \psi_I = A \sin kx$

and $\quad \psi_{II} = De^{-\alpha x}$

so, $\quad A \sin kl = De^{-\alpha l}$

$$kA \cos kl = -\alpha\, De^{-\alpha l}$$



From this, we get

$$\tan kl = -\frac{k}{\alpha}$$

or

$$\sin kl = \frac{\pm kl}{\sqrt{k^2l^2 + \alpha^2l^2}}$$

$$= \frac{\pm kl}{\sqrt{2mU_0l^2/\hbar^2}}$$

$$= \pm kl\sqrt{\frac{\hbar^2}{2mU_0l^2}} \tag{6}$$

Plotting the left and right sides of this equation we can find the points at which the straight lines cross the sine curve. The roots of the equation corresponding to the eigenvalues of energy E_i and found from the intersection points $(kl)_i$, for which $\tan (kl)_i < 0$ (i.e., second and fourth and other even quadrants). It is seen that bound states do not always exist. For the first bound state to appear (refer to the line b above), we need

$$(kl)_{1,\,\min} = \frac{\pi}{2}$$

(b) Substituting, we get $$(l^2 U_0)_{1,\,\min} = \frac{\pi^2 h^2}{8m}$$

as the condition for the appearance of the first bound state. The second bound state will appear when kl is in the fourth quadrant. The magnitude of the slope of the straight line must then be less than $1/(3\pi/2)$ corresponding to

$$kl_{2,\,\min} = \frac{3\pi}{2} = (3)\frac{\pi}{2}$$

$$= (2 \times 2 - 1)\frac{\pi}{2}$$

For n bound states, it is easy to convince oneself that the slope of the appropriate straight line (upper or lower) must be less than $2/(2n - 1)\pi$ corresponding to

$$(kl)_{n,\,\min} = (2n - 1)\frac{\pi}{2}$$

Then, $$(l^2 U_0)_{n,\,\min} = \frac{(2n-1)^2 \pi^2 \hbar^2}{8m}$$

Do not forget to note that for large n both $+ve$ and $-ve$ signs in the Eq. (6) contribute to solutions.

6.86 Given $$U_0 l^2 = \left(\frac{3}{4}\pi\right)^2 \frac{\hbar^2}{m}$$

and $$El^2 = \left(\frac{3}{4}\pi\right)^2 \frac{\hbar^2}{2m}$$

or $$kl = \frac{3}{4}\pi$$

It is easy to check that the condition of the bound state is satisfied. Also,

$$\alpha l = \sqrt{\frac{2m}{\hbar^2}(U_0 - E)l^2} = \sqrt{\frac{mU_0}{\hbar^2}\, l^2} = \frac{3}{4}\pi$$

Then, from the previous problem

$$D = Ae^{\alpha l} \sin kl = A\frac{e^{3\pi/4}}{\sqrt{2}}$$

By normalization

$$I = A^2\left[\int_0^l \sin^2 kx\,dx + \int_l^\infty \frac{e^{3\pi/2}}{2} e^{-\frac{3\pi}{2}\frac{x}{l}}\,dx\right]$$

$$= A^2\left[\frac{l}{2}\int_0^l (1-\cos 2kx)\,dx + l\int_0^\infty \frac{1}{2} e^{-\frac{3\pi}{2}y}\,dy\right]$$

$$= A^2\left[\frac{1}{2}\left(-\frac{\sin 2kl}{2k}\right) + \frac{1}{2}\cdot\frac{l/3\pi}{2}\right] = A^2 l\left[\frac{1}{2}\left(1+\frac{1/3\pi}{2}\right) + \frac{1}{2}\frac{(1/3\pi)}{2}\right]$$

$$= A^2 l\left[\frac{1}{2} + \frac{2}{3\pi}\right] = A^2\,\frac{l}{2}\left(1+\frac{4}{3\pi}\right) \quad \text{or} \quad A = \sqrt{\frac{2}{l}}\left(1+\frac{4}{3\pi}\right)^{-1/2}$$

The probability of the particle to be located in the region $x > l$ is

$$P = \int_l^\infty \psi^2\,dx = \frac{2}{l}\left(1+\frac{4}{3\pi}\right)^{-1}\int_l^\infty \frac{e^{3\pi/2}}{2} e^{-\frac{3\pi}{2}\frac{x}{l}}\,dx$$

$$= \left(1+\frac{4}{3\pi}\right)^{-1}\int_l^\infty e^{3\pi/2}\, e^{-\frac{3\pi}{2}y}\,dy = \frac{2}{3\pi}\times\frac{3\pi}{3\pi+4} = 14.9\,\%$$

6.87 The Schrodinger equation is

$$\nabla^2\psi + \frac{2m}{\hbar^2}\left[E - U(r)\right]\psi = 0$$

When ψ depends on r only,

$$\nabla^2\psi = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right)$$

If we put $$\psi = \frac{\chi(r)}{r},\quad \frac{d\psi}{dr} = \frac{\chi'}{r} - \frac{\chi}{r^2}$$

and $$\nabla^2\psi = \frac{\chi''}{r}$$

we get $$\frac{d^2\chi}{dr^2} + \frac{2m}{\hbar^2}\left[E - U(r)\right]\chi = 0$$

Thus, the solution is $$\chi = A\sin kr \quad (\text{for } r < r_0)$$

$$k^2 = \frac{2mE}{\hbar^2} \quad \text{and} \quad \chi = 0 \quad (\text{for } r > r_0)$$

(For $r < r_0$ we have rejected a term $B \cos kr$ as it does not vanish at $r = 0$.) Continuity of the wave function at $r = r_0$ requires

$$kr_0 = n\pi$$

Hence,
$$E_n = \frac{n^2\pi^2\hbar^2}{2mr_0^2}$$

6.88 (a) The normalized wave functions are obtained from the normalization of

$$1 = \int |\psi|^2\, dV = \int |\psi|^2\, 4\pi r^2\, dr$$

$$= \int_0^{r_0} A^2 4\pi \chi^2\, dr = 4\pi A^2 \int_0^{r_0} \sin^2 \frac{n\pi r}{r_0}\, dr$$

$$= 4\pi A^2 \frac{r_0}{n\pi} \int_0^{n\pi} \sin^2 r\, dr = 4\pi A^2 \frac{r_0}{n\pi} \cdot \frac{n\pi}{2} = r_0 \cdot 2\pi A^2$$

Hence,
$$A = \frac{1}{\sqrt{2\pi r_0}} \quad \text{and} \quad \psi = \frac{1}{\sqrt{2\pi \cdot r_0}} \frac{\sin(n\pi r/r_0)}{r}$$

(b) The radial probability distribution function is

$$P_n(r) = 4\pi r^2 (\psi)^2 = \frac{2}{r_0} \sin^2 \frac{n\pi r}{r_0}$$

For the ground state, $n = 1$

so,
$$P_1(r) = \frac{2}{r_0} \sin^2 \frac{\pi r}{r_0}$$

By inspection this is maximum for $r = r_0/2$. Thus, $r_{pr} = r_0/2$.

Thus, the probability for the particle to be found in the region $r < r_{pr}$ is clearly 50% as one can immediately see from a graph of $\sin^2 x$.

6.89 If we take
$$\psi = \frac{\chi(r)}{r}$$

the equation for $\chi(r)$ has the form

$$\chi'' + \frac{2m}{\hbar^2}[E - U(r)]\chi(r) = 0$$

which can be written as $\quad \chi'' + k^2\chi = 0 \quad (\text{for } 0 \le r < r_0)$

and $\quad \chi'' - \alpha^2\chi = 0 \quad (\text{for } r_0 < r < \infty)$

where $$k^2 = \frac{2mE}{\hbar^2} \quad \text{and} \quad \alpha^2 = \frac{2m(U_0 - E)}{\hbar^2}$$

The boundary condition is $\chi(0) = 0$ and χ, χ' are continuous at $r = r_0$. These are exactly same as in the one-dimensional problem in Problem 6.85.

We therefore omit further details.

6.90 The Schrodinger equation is

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}kx^2\right)\psi = 0$$

We are given $$\psi = Ae^{-\alpha x^2/2}$$

Then, $$\psi' = -\alpha x A e^{-\alpha x^2/2}$$

$$\psi'' = -\alpha x A e^{-\alpha x^2/2} + \alpha^2 x^2 A e^{-\alpha x^2/2}$$

On substituting, we find the following equation must hold

$$\left[(\alpha^2 x^2 - \alpha) + \frac{2m}{\hbar^2}\left(E - \frac{1}{2}kx^2\right)\right]\psi = 0$$

since $\psi \neq 0$, the bracket must vanish. This means that the coefficient of x^2 as well the term independent of x must vanish. We get

$$\alpha^2 = \frac{mk}{\hbar^2} \quad \text{and} \quad \alpha = \frac{2mE}{\hbar^2}$$

Putting $k/m = \omega^2$ gives us

$$\alpha = \frac{m\omega}{2\hbar} \quad \text{and} \quad E = \frac{\hbar\omega}{2}$$

6.91 The Schrodinger equation for the problem in Gaussian units is

$$\nabla^2\psi + \frac{2m}{\hbar^2}\left[E + \frac{e^2}{r}\right]\psi = 0$$

In MKS units we should read $(e^2/4\pi\varepsilon_0)$ for e^2. So, we put

$$\psi = \frac{\chi(r)}{r}$$

Then $$\chi'' + \frac{2m}{\hbar^2}\left[E + \frac{e^2}{r}\right]\chi = 0 \qquad (1)$$

We are given that $\chi = r\psi = Ar(1 + ar)\, e^{-\alpha r}$

so, $\chi' = A(1 + 2ar)\, e^{-\alpha r} - \alpha Ar(1 + ar)\, e^{-\alpha r}$

and $\chi'' = \alpha^2 Ar(1 + ar)\, e^{-\alpha r} - 2\alpha A(1 + 2ar)\, e^{-\alpha r} + 2aAe^{-\alpha r}$

Substitution in Eq. (1) gives the condition

$$\alpha^2(r + ar^2) - 2\alpha(1 + 2ar) + 2a + \frac{2m}{\hbar^2}(Er + e^2) \times (1 + ar) = 0$$

Equating the coefficients of r^2, r, and constant term to zero, we get

$$2a - 2\alpha + \frac{2me^2}{\hbar^2} = 0 \tag{2}$$

$$a\alpha^2 + \frac{2m}{\hbar^2} Ea = 0 \tag{3}$$

$$\alpha^2 - 4a\alpha + \frac{2m}{\hbar^2}(E + e^2 a) = 0 \tag{4}$$

From Eq. (3), either $a = 0$ or $E = -\dfrac{\hbar^2 \alpha^2}{2m}$

In the first case $\alpha = \dfrac{me^2}{\hbar^2}, \quad E = -\dfrac{\hbar^2}{2m}\alpha^2 = -\dfrac{me^4}{2\hbar^2}$

This state is the ground state.

In the second case $a = \alpha - \dfrac{me^2}{\hbar^2}, \quad \alpha = \dfrac{1}{2}\dfrac{me^2}{\hbar^2}$

and $E = -\dfrac{me^4}{8\hbar^2}$ and $a = -\dfrac{1}{2}\dfrac{me^2}{\hbar^2}$

This state has $n = 2(2s)$.

6.92 We first find A by normalization

$$1 = \int_0^\infty 4\pi A^2 e^{-2r/r_1} r^2\, dr = \frac{\pi A^2}{2} r_1^3 \int_0^\infty e^{-x} x^2\, dx = \pi A^2 r_1^3$$

since the integral has the value 2.

Thus, $A^2 = \dfrac{1}{\pi r_1^3}$ or $A = \dfrac{1}{\sqrt{r_1^3 \pi}}$

(a) The most probable distance r_{pr} is that value of r for which

$$P(r) = 4\pi r^2 \left|\psi(r)\right|^2 = \frac{4}{r_1^3} r^2 e^{-2r/r_1}$$

is maximum. This requires

$$P'(r) = \frac{4}{r_1^3}\left[2r - \frac{2r^2}{r_1}\right]e^{-2r/r_1} = 0$$

or $$r = r_1 = r_{\text{pr}}$$

(b) The coulomb force is given by $-e^2/r^2$, then mean value of its modulus is

$$< F >= \int_0^\infty 4\pi r^2 \frac{1}{\pi r_1^3}\, e^{-2r/r_1}\, \frac{e^2}{r^2}\, dr$$

$$= \int_0^\infty \frac{4e^2}{r_1^3} e^{-2r/r_1}\, dr = \frac{2e^2}{r_1^2}\int_0^\infty e^{-x}\, dx = \frac{2e^2}{r_1^2}$$

In MKS units we should read $(e^2/4\pi\varepsilon_0)$ for e^2.

(c) $$< U >= \int_0^\infty 4\pi r^2 \frac{1}{\pi r_1^3}\, e^{-2r/r_3}\, \frac{-e^2}{r}\, dr = -\frac{e^2}{r_1}\int_0^\infty xe^{-x}\, dx = -\frac{e^2}{r_1}$$

In MKS units it shall read as $(e^2/4\pi\varepsilon_0)$ for e^2.

6.93 We find A by normalization as above. We get

$$A = \frac{1}{\sqrt{\pi r_1^3}}$$

Then, the electronic charge density is

$$\rho = -e|\psi|^2 = -e\frac{e^{-2r/r_1}}{\pi r_1^3} \equiv \rho(\mathbf{r})$$

The potential $\psi(\mathbf{r})$ due to this charge density is

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|}\, d^3r'$$

So at the origin $$\varphi(0) = \frac{1}{4\pi\varepsilon_0}\int_0^\infty \frac{\rho(r')}{r'} 4\pi r'^2\, dr' = \frac{-e}{4\pi\varepsilon_0}\int_0^\infty \frac{4r'}{r_1^3} e^{-2r'/r_1}\, dr'$$

$$= -\frac{e}{4\pi\varepsilon_0 r_1}\int_0^\infty xe^{-x}\, dx = -\frac{e}{(4\pi\varepsilon_0)r_1}$$

6.94 (a) We start from the Schrodinger equation

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}\left[E - U(x)\right]\psi = 0$$

which we write as $\psi_I'' + k^2\psi_I = 0 \quad (\text{for } x < 0)$

$$k^2 = \frac{2mE}{\hbar^2}$$

and $\psi_{II}'' + \alpha^2\psi_{II} = 0 \quad (\text{for } x > 0)$

$$\alpha^2 = \frac{2m}{\hbar^2}(E - U_0) > 0$$

It is convenient to look for solutions in the form

$$\psi_I = e^{ikx} + Re^{-ikx} \quad (\text{for } x < 0)$$

and $\psi_{II} = Ae^{i\alpha x} + Be^{-i\alpha x} \quad (\text{for } x > 0)$

In region I ($x < 0$), the amplitude of e^{ikx} is written as unity by convention. In region II we expect only a transmitted wave to the right, then $B = 0$. So,

$$\psi_{II} = Ae^{i\alpha x} \quad (\text{for } x > 0)$$

The boundary conditions follow from the continuity of ψ and $d\psi/dx$ at $x = 0$.

So, $1 + R = A$

$$ik(1 - R) = i\alpha A$$

Then $\dfrac{1 - R}{1 + R} = \dfrac{\alpha}{k} \quad \text{or} \quad R = \dfrac{k - \alpha}{k + \alpha}$

The reflection coefficient is the absolute square of R.

Thus, $|R|^2 = \left|\dfrac{k - \alpha}{k + \alpha}\right|^2$

(b) In this case, $E < U_0$, $\alpha^2 = -\beta^2 < 0$. Then ψ_I is unchanged in form but

$$\psi_{II} = Ae^{-\beta x} + Be^{+\beta x}$$

(we must have $B = 0$, since, otherwise $\psi(x)$ will become unbound as $x \to \infty$).

Finally $\psi_{II} = Ae^{-\beta x}$

Inside the barrier, the particle then has a probability density equal to

$$|\psi_{II}|^2 = |A|^2 e^{-2\beta x}$$

This decreases to $1/e$ of its value in

$$x_{\text{eff}} = \frac{1}{2\beta} = \frac{\hbar}{2\sqrt{2m(U_0 - E)}}$$

6.95 The formula is

$$D \approx \exp\left[-\frac{2}{\hbar}\int_{x_1}^{x_2}\sqrt{2m(V(x)-E)}\,dx\right]$$

Here $V(x_2) = V(x_1) = E$ and $V(x) > E$ in the region $x_2 > x > x_1$.

(a) For the problem, the integral is trivial

$$D \approx \exp\left[-\frac{2l}{\hbar}\sqrt{2m(U_0 - E)}\right]$$

(b) We can without loss of generality take $x = 0$ at the point the potential begins to climb.

Then, $$U(x) = \begin{cases} 0 & (\text{for } x < 0) \\ U_0\dfrac{x}{l} & (\text{for } 0 < x < l) \\ 0 & (\text{for } x > l) \end{cases}$$

Then, $$D \approx \exp\left[-\frac{2}{\hbar}\int_{lE/U_0}^{l}\sqrt{2m\left(U_0\frac{x}{l} - E\right)}dx\right]$$

$$= \exp\left[-\frac{2}{\hbar}\sqrt{\frac{2mU_0}{l}}\int_{x_0}^{l}\sqrt{x - x_0}\,dx\right] \quad (\text{where } x_0 = lE/U_0)$$

$$= \exp\left[-\frac{2}{\hbar}\sqrt{\frac{2mU_0}{l}}\,\frac{2}{3}(x - x_0)^{3/2}\Big|_{x_0}^{l}\right]$$

$$= \exp\left[-\frac{4}{3\hbar}\sqrt{\frac{2mU_0}{l}}\left(l - l\frac{E}{U_0}\right)^{3/2}\right]$$

$$= \exp\left[-\frac{4l}{3\hbar U_0}(U_0 - E)^{3/2}\sqrt{2m}\right].$$

6.96 The potential is

$$U(x) = U_0\left(1 - \frac{x^2}{l^2}\right)$$

The turning points are $\dfrac{E}{U_0} = 1 - \dfrac{x^2}{l^2}$ or $x = \pm l\sqrt{1 - \dfrac{E}{U_0}}$

Then $$D \approx \exp\left[-\frac{4}{\hbar}\int_0^{l\sqrt{1-(E/U_0)}} \sqrt{2m\left\{U_0\left(1-\frac{x^2}{l^2}\right)-E\right\}}\,dx\right]$$

$$= \exp\left[-\frac{4}{\hbar}\int_0^{l\sqrt{1-(E/U_0)}} \sqrt{2mU_0}\sqrt{1-\frac{E}{U_0}-\frac{x^2}{l^2}}\,dx\right]$$

$$= \exp\left[-\frac{4l}{\hbar}\sqrt{2mV_0}\int_0^{x_0}\sqrt{x_0^2-x^2}\,dx\right] \quad (\text{where } x_0 = \sqrt{1-E/V_0})$$

The integral is

$$\int_0^{x_0}\sqrt{x_0^2-x^2}\,dx = x_0^2\int_0^{\pi/2}\cos^2\theta\,d\theta = \frac{\pi}{4}x_0^2$$

Thus, $$D \approx \exp\left[-\frac{\pi l}{\hbar}\sqrt{2mU_0}\left(1-\frac{E}{U_0}\right)\right]$$

$$= \exp\left[-\frac{\pi l}{\hbar}\sqrt{\frac{2m}{U_0}}(U_0-E)\right]$$

6.3 Properties of Atoms. Spectra

6.97 From the Rydberg formula, we find

$$E_n = -\frac{\hbar R}{(n+\alpha_l)^2}$$

We use $\hbar R = 13.6$ eV. Then, for $n = 2$ state

$$5.39 = -\frac{13.6}{(2+\alpha_0)^2} \quad (\text{for } l = 0\ (S) \text{ state})$$

Thus, $$\alpha_0 \cong -0.41$$

For P state

$$3.54 = -\frac{13.6}{(2+\alpha_1)^2}$$

$$\alpha_1 \approx -0.039$$

6.98 The energy of the $3P$ state must be $-(E_0 - e\varphi)$, where $-E_0$ is the energy of the $3S$ state.

Then,
$$E_0 - e\varphi_1 = \frac{\hbar R}{(3 + \alpha_1)^2}$$

So,
$$\alpha_1 = \sqrt{\frac{\hbar R}{E_0 - e\varphi_1}} - 3 = -0.885$$

6.99 For the first line of the sharp series ($3S \to 2P$) in a Li atom

$$\frac{2\pi\hbar c}{\lambda_1} = -\frac{\hbar R}{(3 + \alpha_0)^2} + \frac{\hbar R}{(2 + \alpha_1)^2}$$

For the short wave, cut-off wavelength of the same series

$$\frac{2\pi\hbar c}{\lambda_2} = \frac{\hbar R}{(2 + \alpha_1)^2}$$

From these two equations, we get on subtraction

$$3 + \alpha_0 = \sqrt{\hbar R \Big/ \frac{2\pi\hbar c(\lambda_1 - \lambda_2)}{\lambda_1\lambda_2}}$$

$$= \sqrt{\frac{R\lambda_1\lambda_2}{2\pi c\Delta\lambda}} \quad (\text{where } \Delta\lambda = \lambda_1 - \lambda_2)$$

Thus, in the ground state, the binding energy of the electron is

$$E_b = \frac{\hbar R}{(2 + \alpha_0)^2} = \frac{\hbar R}{\left(\sqrt{\frac{R\lambda_1\lambda_2}{2\pi c\Delta\lambda}} - 1\right)^2} = 5.32 \text{ eV}$$

6.100 The energy of the $3S$ state is

$$E(3S) = -\frac{\hbar R}{(3 - 0.41)^2} = -2.03 \text{ eV}$$

The energy of a $2S$ state is

$$E(2S) = -\frac{\hbar R}{(2 - 0.41)^2} = -5.39 \text{ eV}$$

The energy of a $2P$ state is

$$E(2P) = -\frac{\hbar R}{(2 - 0.04)^2} = -3.55 \text{ eV}$$

We see that $E(2S) < E(2P) < E(3S)$. The transitions are $3S \to 2P$ and $2P \to 2S$. Direct $3S \to 2S$ transition is forbidden by selection rules. The wavelengths are determined by

$$E_2 - E_1 = \Delta E = \frac{2\pi\hbar c}{\lambda}$$

Substitution gives $\lambda = 0.816\ \mu\text{m}$ (for $3S \to 2P$)

and $\lambda = 0.674\ \mu\text{m}$ (for $2P \to 2S$)

6.101 The splitting of the Na lines is due to the fine structure splitting of $3P$ lines. (The $3S$ state is nearly single except for possible hyperfine effects.) The splitting of the $3P$ level then equals the energy difference

$$\Delta E = \frac{2\pi\hbar c}{\lambda_1} - \frac{2\pi\hbar c}{\lambda_2} = \frac{2\pi\hbar c(\lambda_2 - \lambda_1)}{\lambda_1\lambda_2} \approx \frac{2\pi\hbar c\Delta\lambda}{\lambda^2}$$

Here, $\Delta\lambda$ = wavelength difference and λ = average wavelength.

Substitution gives $\Delta E = 2.0$ meV.

6.102 The sharp series arise from the transitions $nS \to mP$. The S lines are unsplit so the splitting is entirely due to the P level. The frequency difference between sequent lines is $\Delta E/\hbar$ and is the same for all lines of the sharp series and is given by

$$\frac{1}{\hbar}\left(\frac{2\pi\hbar c}{\lambda_1} - \frac{2\pi\hbar c}{\lambda_2}\right) = \frac{2\pi c\Delta\lambda}{\lambda_1\lambda_2}$$

Substitution gives 1.645×10^{14} rad/s.

6.103 We shall ignore hyperfine interactions. The state with principal quantum number $n = 3$ has orbital angular momentum quantum number $l = 0, 1, 2$. The levels with these terms are $3S$, $3P$, $3D$. The total angular momentum is obtained by combining spin and angular momenta. For a single electron this leads to

$$J = \frac{1}{2} \quad (\text{if } L = 0)$$

$$J = L - \frac{1}{2} \quad \text{and} \quad L + \frac{1}{2} \quad (\text{if } L \neq 0)$$

We then get the final designations $3S_{1/2}$, $3P_{1/2}$, $3P_{3/2}$, $3D_{3/2}$, $3D_{5/2}$.

6.104 The rule is that if $\mathbf{J} = \mathbf{L} + \mathbf{S}$ then $\mathbf{J}$ takes the values $|L - S|$ to $L + S$. Thus,

(a) The values are 1, 2, 3, 4, 5.

(b) The values are 0, 1, 2, 3, 4, 5, 6.

(c) The values are $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$.

6.105 For the state $4P$, $L = 1$, $S = 3/2$ (since $2S + 1 = 4$). For the state $5D$, $L = 2$, $S = 2$. The possible values of J are

$$J = \begin{cases} \frac{5}{2}, \frac{3}{2}, \frac{1}{2} & \text{for } 4P \\ 4, 3, 2, 1, 0 & \text{for } 5D \end{cases}$$

The value of the magnitude of angular momentum is $\hbar\sqrt{J(J+1)}$. Substitution gives the values

For $4P$ $\qquad \hbar\sqrt{\frac{1}{2}\cdot\frac{3}{2}} = \frac{\hbar\sqrt{3}}{2}, \hbar\sqrt{\frac{3}{2}\cdot\frac{5}{2}} = \frac{\hbar\sqrt{15}}{2}$

and $\qquad \hbar\sqrt{\frac{5}{2}\cdot\frac{7}{2}} = \frac{\hbar\sqrt{35}}{2}$

for $5D$ $\qquad 0, \hbar\sqrt{2}, \hbar\sqrt{6}, \hbar\sqrt{12}, \hbar\sqrt{20}$

6.106 (a) For the Na atoms, the valence electron has principal quantum number $n = 4$, and the possible values of orbital angular momentum are $l = 0, 1, 2, 3$ so $I_{\max} = 3$. The state is ^2F, maximum value of J is 7/2. Thus, the state with maximum angular momentum will be $^2F_{7/2}$. For this state

$$M_{\max} = \hbar\sqrt{\frac{7}{2}\cdot\frac{9}{2}} = \frac{\hbar\sqrt{63}}{2}$$

(b) For the atom with electronic configuration $1s^2\ 2p^3d$, there are two unequivalent valence electrons. The total orbital angular moments will be 1, 2, 3 so we pick $L = 3$. The total spin angular momentum will be $S = 0, 1$ so we pick up $S = 1$. Finally J will be 2, 3, 4 so we pick up 4. Thus, maximum angular momentum state is $^3F_4$. For this state

$$M_{\max} = \hbar\sqrt{4 \times 5} = 2\hbar\sqrt{5}$$

6.107 For the F state $L = 3$, for the D state $L = 2$. Now if the state has spin S, the possible angular momenta are $|L - S|$ to $L + S$. The number of J angular momentum values is $2S + 1$ if $L \geq S$ and $2L + 1$ if $L < S$. Since the number of states is 5, we must have $S \geq L = 2$ for D state while $S \leq 3$ and $2S + 1 = 5$ imply $S = 2$ for F state. Thus for the F state, total spin angular momentum

$$M_S = \hbar\sqrt{2 \times 3} = \hbar\sqrt{6}$$

while for D state

$$M_S \geq \hbar\sqrt{6}$$

6.108 Multiplicity is $2S + 1$ so $S = 1$. Total angular momentum is

$$\hbar\sqrt{J(J+1)}$$

So, $J = 4$. Then, L must equal 3, 4, 5 in order that $J = 4$ may be included in $|L - S|$ to $L + S$.

6.109 (a) Here $J = 2$, $L = 2$. Then $S = 0, 1, 2, 3, 4$ and the multiplicities $(2S - 1)$ are 1, 3, 5, 7, 9.

(b) Here $J = 3/2$, $L = 1$. Then $S = 5/2, 3/2, 1/2$ and the multiplicities are 6, 4, 2.

(c) Here $J = 1$, $L = 3$. Then $S = 2, 3, 4$ and the multiplicities are 5, 7, 9.

6.110 The total angular momentum is greatest when L, S are both greatest and add to give J. Now, for a triplet of S, P, D electrons, we have

Maximum spin $S = 3/2$ corresponding to

$$M_S = \hbar\sqrt{\frac{3}{2}\cdot\frac{5}{2}} = \frac{\hbar\sqrt{15}}{2}$$

Maximum orbital angular momentum $L = 3$ corresponding to

$$M_L = \hbar\sqrt{3 \times 4} = \hbar\sqrt{12}$$

Maximum total angular momentum $J = 9/2$ corresponding to

$$M = \frac{\hbar}{2}\sqrt{99}$$

In vector model

$$\mathbf{L} = \mathbf{J} - \mathbf{S}$$

or if magnitude is squared

$$L(L+1)\hbar^2 = J(J+1)\hbar^2 + S(S+1)\hbar^2 - 2\mathbf{J}\cdot\mathbf{S}$$

Thus,
$$\cos(\angle\,\mathbf{J}, \mathbf{S}) = \frac{J(J+1) + S(S+1) - L(L+1)}{2\sqrt{J(J+1)}\sqrt{S(S+1)}}$$

Substitution gives $\angle\,(\mathbf{J}, \mathbf{S}) = 31.1°$.

6.111 Total angular momentum $\hbar\sqrt{6}$ means $J = 2$. It is given that $S = 1$. This means that $L = 1, 2,$ or 3. From vector model relation

$$L(L+1)\hbar^2 = 6\hbar^2 + 2\hbar^2 - 2\hbar^2\sqrt{6}\sqrt{2}\cos 73.2°$$

$$= 5.998\hbar^2 \approx 6\hbar^2$$

Thus, $L = 2$ and the spectral symbol of the state is $^3D_2$.

6.112 In a system containing a p electron and a d electron, $S = 0, 1$ and $L = 1, 2, 3$.

For $S = 0$, we have the terms $^1P_1$, $^1D_2$, $^1F_3$.

For $S = 1$, we have the terms $^3P_0$, $^3P_1$, $^3P_2$, $^3D_1$, $^3D_2$, $^3D_3$, $^3F_2$, $^3F_3$, $^3F_4$.

6.113 The atom has $S_1 = 1/2$, $L_1 = 1$, $J_1 = 3/2$. The electron has $s_2 = 1/2$, $l_2 = 2$, so the total angular momentum quantum number must be

$$j_2 = \frac{3}{2} \text{ or } \frac{5}{2}$$

In L–S coupling we get $S = 0, 1$; $L = 1, 2, 3$ and the terms that can be formed are the same as written in the problem above. The possible values of angular momentum are consistent with the addition $J_1 = 3/2$ to $j_2 = 3/2$ or $5/2$. Using $j_2 = 3/2$ gives us $J = 0, 1, 2, 3$ and using $j_2 = 5/2$ gives $J = 1, 2, 3, 4$. All these values are reached in the problem above.

6.114 Selection rules are

$$\Delta S = 0$$

$$\Delta L = \pm 1$$

$$\Delta J = 0, \pm 1 \quad (0 \to 0 \text{ is not allowed})$$

Thus,

$$^2D_{3/2} \to {}^2P_{1/2} \text{ is allowed}$$

$$^3P_1 \to {}^2S_{1/2} \text{ is not allowed}$$

$$^3F_3 \to {}^2S_{1/2} \text{ is not allowed}$$

$$^3F_3 \to {}^3P_2 \text{ is not allowed} \quad (\text{since } \Delta L = 2)$$

$$^4F_{7/2} \to {}^4D_{5/2} \text{ is allowed}$$

6.115 For a $3D$ state of an Li atom, $S = 1/2$ because there is only one electron and $L = 2$. The total degeneracy is

$$g = (2L + 1)\,(2S + 1) = 5 \times 2 = 10$$

The states are $^2D_{3/2}$ and $^2D_{5/2}$ and we find that

$$g = \left(2 \times \frac{3}{2} + 1\right) + \left(2 \times \frac{5}{2} + 1\right) = 4 + 6$$

6.116 The states with greatest possible total angular momentum are

For a ^2P state $\qquad J = \dfrac{1}{2} + 1 = \dfrac{3}{2}$, i.e., $^2P_{3/2}$

Its degeneracy is $2 \times (3/2) + 1 = 4$.

For a ^3D state $\qquad J = 1 + 2 = 3$, i.e., $^3D_3$

Its degeneracy is $2 \times 3 + 1 = 7$.

For a ^4F state $\qquad J = \dfrac{3}{2} + 3 = \dfrac{9}{2}$, i.e., $^4F_{9/2}$

Its degeneracy is $2 \times (9/2) + 1 = 10$.

6.117 The degeneracy is $2J + 1 = 7$. So we must have $J = 3$. From $L = 3S$, we see that S must be an integer since L is integer and S can be either integer or half integer. If $S = 0$ then $L = 0$ but this is consistent with $J = 3$. For $S \geq 2$, $L \geq 6$ and so $J = 3$. Thus, the state is $^3F_3$.

6.118 If the order of filling is K, L, M shells, then electrons occupy $4s^2$, $3d^{10}$ followed by $4p^3$. Hence, electronic configuration of the element will be $1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 3d^{10} 4p^3$. (There must be three $4p$ electrons.)

The number of electrons is $Z = 33$ and the element is As. (The $3d$ subshell must be filled before $4p$ gets filled.)

6.119 (a) When the partially filled shell contains three p electrons, the total spin S must equal 1/2 or 3/2. The state $S = 3/2$ has maximum spin and is totally symmetric under exchange of spin labels. By Pauli's exclusion principle this implies that the angular part of the wave function must be totally antisymmetric. Since the angular part of the wave function of a p electron is vector $\mathbf{r}$, the total wave function of three p electrons is the totally antisymmetric combination of $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$. The only possible combination is

$$\mathbf{r}_1 \cdot (\mathbf{r}_2 \times \mathbf{r}_3) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

This combination is scalar and hence, $L = 0$. The spectral term of the ground state is then

$$^4S_{3/2} \quad (\text{since } J = 3/2)$$

(b) We can think of four P electrons as consisting of a full p shell with two p holes. The state of maximum spin S is then $S = 1$. By Pauli's principle the orbital angular momentum part must be antisymmetric and can only have the form $\mathbf{r}_1 \times \mathbf{r}_2$ where $\mathbf{r}_1$, $\mathbf{r}_2$ are the coordinates of holes. The result is harder to understand if we do not use the concept of holes. Four p electrons can have $S = 0, 1, 2$ but the $S = 2$ state is totally symmetric. The corresponding angular wave function must be totally antisymmetric. But this is impossible; there is no quantity which is antisymmetric in four vectors. Thus, the maximum allowed S is $S = 1$. We can construct such a state by coupling the spins of electrons 1 and 2 to $S = 1$ and of electrons 3 and 4 to $S = 1$ and then coupling the resultant spin states to $S = 1$. Such a state is symmetric under the exchange of spins of (1, 2) and (3, 4) but antisymmetric under the simultaneous exchange of (1, 2) and (3, 4). The conjugate angular wave function must be antisymmetric under the exchange of (1, 2) and under the exchange of (3, 4) by Pauli's principle. It must also be antisymmetric under the simultaneous exchange of (1, 2) and (3, 4). (This is because two exchanges of electrons are involved.) The required angular wave function then has the form

$$(\mathbf{r}_1 \times \mathbf{r}_2) \times (\mathbf{r}_3 \times \mathbf{r}_4)$$

and is a vector and hence, $L = 1$. Thus, using also the fact that the shell is more than half-filled, we find the spectral term $^3P_2$ where ($J = L + S$).

6.120 (a) The maximum spin angular momentum of three electrons can be $S = 3/2$. This state is totally symmetric and hence, the conjugate angular wave function must be antisymmetric. By Pauli's exclusion principle the totally antisymmetric state must have different magnetic quantum numbers. It is easy to see that for d electrons, the maximum value of the magnetic quantum number for orbital angular momentum $|M_{L-S}| = 3$ (from $2 + 1 + 0$). Higher values violate Pauli's principle. Thus, the state of highest orbital angular momentum consistent with Pauli's principle is $L = 3$.

The state of the atom is then ^4F_J where $J = L - S$ by Hund's rule. Thus, the state is

$$^4F_{3/2}$$

The magnitude of the angular momentum is

$$\hbar\sqrt{\frac{3}{2}\cdot\frac{5}{2}} = \frac{\hbar}{2}\sqrt{15}$$

(b) Seven d electrons mean three holes. Then $S = 3/2$ and $L = 3$ as before. But $J = L + S = 9/2$ by Hund's rule for more than half-filled shell. Thus, the state is

$$^4F_{9/2}$$

Total angular momentum has the magnitude

$$\hbar\sqrt{\frac{9}{2}\cdot\frac{11}{2}} = \frac{3\hbar}{2}\sqrt{11}$$

6.121 (a) For $^3F_2$: The maximum value of spin is $S = 1$ here. This means there are 2 electrons. $L = 3$ so s and p electrons are ruled out. Thus, the simplest possibility is of d electrons. This is the correct choice for if we were considering f electrons, the maximum value of L allowed by Pauli's principle will be $L = 5$ (maximum value of the magnitude of magnetic quantum number will be $3 + 2 = 5$).

Thus, the atom has two d electrons in the unfilled shell.

(b) For $^2P_{3/2}$: Here $L = 1$, $S = 1/2$ and $J = 3/2$. Since $J = L + S$, Hund's rule implies that shell is more than half-filled. This means one electron less than a filled shell. On the basis of the whole picture it is easy to see that we have p electrons. Thus, the atom has five p electrons.

(c) For $^6S_{5/2}$: Here $S = 5/2$, $L = 0$. We either have five electrons or five holes. The angular part is antisymmetric. For five d electrons, the maximum value of the quantum number consistent with Pauli's exclusion principle is $(2 + 1 + 0 - 1 - 2) = 0$ so $L = 0$. For f or g electrons $L > 0$ whether the shell has five electrons or five holes. Thus, the atom has five d electrons.

6.122 (a) If $S = 1$ is the maximum spin then there must be two electrons. (If there are two holes then the shell will be more than half-filled.) This means that there are 6 electrons in the completely filled shell so it is a p shell. By Pauli's principle the only antisymmetric combination of two electrons has $L = 1$. Also, $J = L - S$ as the shell is less than half-filled. Thus, the term is $^3P_0$.

(b) $S = 3/2$ means either 3 electrons or 3 holes. As the shell is more than half-filled the former possibility is ruled out. Thus, we must have seven D electrons. Then as in Problem 6.120, the ground state is $^4F_{9/2}$.

6.123 With three electrons $S = 3/2$ and the spin part is totally symmetric. It is given that the basic term has $L = 3$, so this is the state of highest orbital angular momentum. This is not possible with p electron so we must have d electrons for which $L = 3$ for three electrons. For the f, g electrons $L > 3$. Thus, we have three d electrons. Then as in Problem 6.120, the ground state is $^4F_{3/2}$.

6.124 We have five d electrons in the only unfilled shell. Then $S = 5/2$. Maximum value of L consistent with Pauli's exclusion principle is $L = 0$. Then $J = 5/2$.

So by Lande's formula

$$g = 1 + \frac{(5/2)\,(7/2) + (5/2)\,(7/2) - 0}{2\times(5/2)\,(7/2)} = 2$$

Thus, $$\mu = g\sqrt{J(J+1)}\,\mu_B$$

$$= 2\frac{\sqrt{35}}{2}\mu_B = \sqrt{35}\,\mu_B$$

The ground state is $^6S_{5/2}$.

6.125 By Boltzmann's formula

$$\frac{N_2}{N_1} = \frac{g_2}{g_1}e^{-\Delta E/kT}$$

Here ΔE = energy difference between $n = 1$ and $n = 2$ states.

$$\Delta E = 13.6\left(1 - \frac{1}{4}\right) = 10.22 \text{ eV}$$

Also, $g_1 = 2$ and $g_2 = 8$ (counting $2S$ and $2P$ states)

Thus, $\frac{N_2}{N_1} = 4e^{-10.22 \times 1.602 \times 10^{-19}/1.38 \times 10^{-23} \times 3000} = 2.7 \times 10^{-17}$

Explicitly $\eta = \frac{N_2}{N_1} = n^2 e^{-\Delta E_n/kT}, \quad \Delta E_n = \hbar R\left(1 - \frac{1}{n^2}\right)$

for the n^{th} excited state because the degeneracy of the state with principal quantum number n is $2n^2$.

6.126 We have $\frac{N}{N_0} = \frac{g_P}{g_S}e^{-\hbar\omega/kT} = \frac{g_P}{g_S}e^{-2\pi\hbar c/\lambda kT}$

Here g_P = degeneracy of the $3P$ state = 6, g_S = degeneracy of the $3S$ state = 2, λ = wavelength of the $3P \to 3S$ line and $2\pi\hbar c/\lambda$ = energy difference between $3P$ and $3S$ levels.

Thus, $\frac{N}{N_0} = 1.13 \times 10^{-4}$ (on substituting values)

6.127 Let τ = mean lifetime of the excited atoms. Then the number of excited atoms will decrease with time as $e^{-t/\tau}$. In time t the atom travels a distance vt, so $t = l/v$. Thus, the number of excited atoms in a beam that has traversed a distance l has decreased by $e^{-l/v\tau}$

The intensity of the line is proportional to the number of excited atoms in the beam. Thus,

$$e^{-l/v\tau} = \frac{1}{\eta}$$

or $$\tau = \frac{l}{v \ln \eta} = 1.29 \times 10^{-6} \text{ s}$$

6.128 As a result of lighting by the mercury lamp, a number of atoms are pumped to the excited state. In equilibrium the number of such atoms is N. Since the mean lifetime of the atom is τ, the number of atoms decaying per unit time is N/τ. Since a photon of energy $2\pi\hbar c/\lambda$ results from each decay, the total radiated power will be

$$P = \frac{2\pi\hbar c}{\lambda} \cdot \frac{N}{\tau}$$

Thus,

$$N = \frac{P\tau}{2\pi\hbar c/\lambda} = \frac{P\tau\lambda}{2\pi\hbar} = 6.7 \times 10^9$$

6.129 The number of excited atoms per unit volume of the gas in $2P$ state is

$$N = n\frac{g_P}{g_S} e^{-2\pi\hbar c/\lambda kT}$$

Here g_P = degeneracy of the $2P$ state = 6, g_S = degeneracy of the $2S$ state = 2 and λ = wavelength of the resonant line $2P \to 2S$. The rate of decay of these atoms is N/τ per second per unit volume. Since each such atom emits light of wavelength λ, we must have

$$\frac{1}{\tau}\frac{2\pi\hbar c}{\lambda} n \frac{g_P}{g_S} e^{-2\pi\hbar c/\lambda kT} = P$$

Thus, $$\tau = \frac{1}{P}\frac{2\pi\hbar c}{\lambda} n \frac{g_P}{g_S} e^{-2\pi\hbar c/\lambda kT}$$

$$= 65.4 \times 10^{-9} \text{ s} = 65.4 \text{ ns}$$

6.130 (a) We know that

$$P_{21}^{\text{sp}} = A_{21}$$

$$P_{21}^{\text{ind}} = B_{21}u_\omega$$

$$= \frac{\pi^2 c^3}{\hbar\omega^3} A_{21} \cdot \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{e^{-\hbar\omega/kT} - 1} = \frac{A_{21}}{e^{-\hbar\omega/kT} - 1}$$

Thus, $$\frac{P_{21}^{\text{ind}}}{P_{21}^{\text{sp}}} = \frac{1}{e^{-\hbar\omega/kT} - 1}$$

For the transition $2P \to 1S$, $\hbar\omega = (3/4)\hbar R$.

Substitution gives 7×10^{-34}.

(b) The two rates become equal when $e^{-\hbar\omega/kT} = 2$

or $$T = \frac{\hbar\omega}{k \ln 1/2} = 1.71 \times 10^5 \text{ K}$$

6.131 Because of the resonant nature of the process, we can ignore non-resonant processes. We also ignore spontaneous emission since it does not contribute to the absorption coefficient and is a small term if the beam is intense enough.

Suppose I is the intensity of the beam at some point. The decrease in the value of this intensity on passing through the layer of the substance of thickness dx is equal to

$$-dI = \chi I\, dx = (N_1 B_{12} - N_2 B_{21})\left(\frac{I}{c}\right)\hbar\omega\, dx$$

Here, N_1 = number of atoms in lower level. N_2 = number of atoms in the upper level per unit volume. B_{12}, B_{21} are Einstein coefficients and I/c is energy density in the beam, c is velocity of light.

A factor $\hbar\omega$ arises because each transition result in a loss or gain of energy $\hbar\omega$.

Hence, $$\chi = \frac{\hbar\omega}{c} N_1 B_{12}\left(1 - \frac{N_2 B_{21}}{N_1 B_{12}}\right)$$

But $g_1 B_{12} = g_2 B_{21}$, so

$$\chi = \frac{\hbar\omega}{c} N_1 B_{12}\left(1 - \frac{g_1}{g_2}\frac{N_2}{N_1}\right)$$

By Boltzmann's factor $$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-\hbar\omega/kT}$$

When $\hbar\omega \gg kT$, we can put $N_1 \equiv N_0$ as the total number or atoms per unit volume.

Then, $$\chi = \chi_0(1 - e^{-\hbar\omega/kT})$$

where $\chi_0 = (\hbar\omega/c)N_0 B_{12}$ is the absorption coefficient for $T \to 0$.

6.132 A short-lived state of mean lifetime τ has an uncertainty in energy of $\Delta E \sim \hbar/\tau$ which is transmitted to the photon it emits as natural broadening. Then,

$$\Delta\omega_{\text{nat}} = \frac{1}{\tau} \quad \text{so,} \quad \Delta\lambda_{\text{nat}} = \frac{\lambda^2}{2\pi c\tau}$$

The Doppler broadening on the other hand arises from the thermal motion of radiating atoms. The effect is non-relativistic and the maximum broadening can be written as

$$\frac{\Delta\lambda_{\text{Dopp}}}{\lambda} = 2\beta = \frac{2v_{\text{pr}}}{c}$$

Thus, $$\frac{\Delta\lambda_{\text{Dopp}}}{\Delta\lambda_{\text{nat}}} = \frac{4\pi v_{\text{pr}}\tau}{\lambda}$$

Now, $$v_{\text{pr}} = \sqrt{\frac{2RT}{M}} = 157 \text{ m/s}$$

Therefore, $$\frac{\Delta\lambda_{\text{Dopp}}}{\Delta\lambda_{\text{nat}}} \approx 1.2 \times 10^3$$

Note: *Our formula is an order of magnitude estimate.*

6.133 From Moseley's law

$$\omega_{K_\alpha} = \frac{3}{4} R(Z-1)^2$$

or $$\lambda_{K_\alpha} = \frac{4}{3R} \frac{1}{(Z-1)^2}$$

Thus $$\frac{\lambda_{K_\alpha}(\text{Cu})}{\lambda_{K_\alpha}(\text{Fe})} = \left(\frac{Z_{\text{Fe}}-1}{Z_{\text{cu}}-1}\right)^2 = \left(\frac{25}{28}\right)^2$$

Substitution gives λ_{K_α} (Cu) = 153.9 pm.

6.134 (a) From Moseley's law

$$\omega_{K_\alpha} = \frac{3}{4} R(Z-\sigma)^2$$

or $$\lambda_{K_\alpha} = \frac{2\pi c}{\omega K_\alpha} = \frac{8\pi c}{3R} \frac{1}{(Z-\sigma)^2}$$

We shall take $\sigma = 1$. Then, for aluminium ($Z = 13$)

$$\lambda_{K_\alpha}(\text{Al}) = 843.2 \text{ pm}$$

and for cobalt ($Z = 27$)

$$\lambda_{K_\alpha}(\text{Co}) = 179.6 \text{ pm}$$

(b) This difference is nearly equal to the energy of the K_α line which by Moseley's law is equal to ($Z = 23$ for vanadium)

$$\Delta E = \hbar\omega_{K_\alpha} = \frac{3}{4} \times 13.62 \times 22 \times 22 = 4.94 \text{ keV}$$

6.135 We calculate the Z values corresponding to the given wavelengths using Moseley's law. (See Problem 6.134.)

Substitution gives $Z = 23$ corresponding to $\lambda = 250$ pm

and $Z = 27$ corresponding to $\lambda = 179$ pm

There are thus three elements in a row between those whose wavelengths of K_α lines are equal to 250 pm and 179 pm.

6.136 From Moseley's law

$$\lambda_{K_\alpha}(\text{Ni}) = \frac{8\pi c}{3R} \frac{1}{(Z-1)^2}$$

where $Z = 28$ for Ni. Substitution gives $\lambda_{K_\alpha}(\text{Ni}) = 166.5$ pm.

Now, the short wavelength cut-off of the continuous spectrum must be more energetic (smaller wavelength) otherwise K_α lines will not emerge. Then, since

$$\Delta\lambda = \lambda_{K_\alpha} - \lambda_0 = 84 \text{ pm}$$

we get $\lambda_0 = 82.5$ pm

This corresponds to a voltage of

$$V = \frac{2\pi\hbar c}{e\lambda_0}$$

Substitution gives $V = 15.0$ kV.

6.137 Since the short wavelength cut-off of the continuous spectrum is $\lambda = 0.50$ nm, the voltage applied must be

$$V = \frac{2\pi\hbar c}{e\lambda_0} = 2.48 \text{ kV}$$

Since this is greater than the excitation potential of the K series of the characteristic spectrum (which is only 1.56 kV) the latter will be observed.

6.138 Suppose λ_0 = wavelength of the characteristic X-ray line. Then using the formula for short wavelength limit of continuous radiation, we get

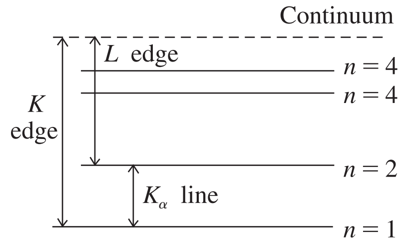
$$\frac{\lambda_0 - (2\pi\hbar c/eV_1)}{\lambda_0 - (2\pi\hbar c/eV_2)} = \frac{1}{n}$$

Hence, $$\lambda_0 = \frac{2\pi\hbar c}{eV_1}\frac{(n - V_1/V_2)}{n-1}$$

Using also Moseley's law, we get

$$Z = 1 + \sqrt{\frac{8\pi c}{3R\lambda}} = 1 + 2\sqrt{\frac{n-1}{3\hbar R}\frac{eV_1}{n - V_1/V_2}} = 29$$

6.139 The difference in frequencies of the K and L absorption edges is equal, according to the Bohr picture, to the frequency of the K_α line (see figure). Thus, by Moseley's formula



$$\Delta\omega = \frac{3}{4}R(Z-1)^2$$

or $$Z = 1 + \sqrt{\frac{4\Delta\omega}{3R}} = 22$$

Therefore, the metal is titanium.

6.140 From the figure in Problem 6.139, we see that the binding energy E_b of a K electron is the sum of the energy of a K_α line and the energy corresponding to the L edge of absorption spectrum. So,

$$E_b = \frac{2\pi\hbar c}{\lambda_L} + \frac{3}{4}\hbar R(Z-1)^2$$

For vanadium, $Z = 23$ and the energy of K_α line of vanadium has been calculated in Problem 6.134b. Using

$$\frac{2\pi\hbar c}{\lambda_L} = 0.51 \text{ keV for } \lambda_L = 2.4 \text{ nm}$$

we get $$E_b = 5.46 \text{ keV}$$

6.141 By Moseley's law

$$\hbar\omega = \frac{2\pi\hbar c}{\lambda} E_K - E_L = \frac{3}{4}\hbar R(Z-1)^2$$

where E_K is the energy of the K electron and E_L is the energy of the L electron. Also the energy of the line corresponding to the short wavelength cut-off of the K series is

$$E_K = \frac{2\pi\hbar c}{\lambda - \Delta\lambda} = \frac{2\pi\hbar c}{(2\pi c/\omega) - \Delta\lambda}$$

$$= \frac{\hbar}{(1/\omega) - \Delta\lambda/2\pi c} = \frac{\hbar\omega}{1 - \omega\Delta\lambda/2\pi c}$$

Hence, $$E_L = \frac{\hbar\omega}{1 - (\omega\Delta\lambda/2\pi c)} - \hbar\omega = \frac{\hbar\omega}{(2\pi c/\omega\Delta\lambda) - 1}$$

Substitution gives, for titanium ($Z = 22$),

$$\omega = 6.85 \times 10^{18}\ \text{s}^{-1}$$

and hence $$E_L = 0.47\ \text{keV}$$

6.142 The energy of the K_α radiation of Zn is

$$\hbar\omega = \frac{3}{4}\hbar R(Z-1)^2$$

where Z = atomic number of zinc = 30. The binding energy of the K electrons in iron is obtained from the wavelength of K absorption edge as $E_K = 2\pi\hbar c/\lambda_K$. Hence, by Einstein equation,

$$T = \frac{3}{4}\hbar R(Z-1)^2 - \frac{2\pi\hbar c}{\lambda_K}$$

$$T = 1.463\ \text{keV (on substituting values)}$$

This corresponds to a velocity of the photoelectrons of $v = 2.27 \times 10^6$ m/s.

6.143 From the Lande's formula

$$g = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}$$

(a) For S state, $L = 0$. This implies $J = S$. Then, if $S \neq 0$, $g = 2$.
(For singlet states, g is not defined if $L = 0$.)

(b) For singlet states, $J = L$. Then, for $S = 0$, we have

$$g = 1 + \frac{J(J+1) - L(L+1)}{2J(J+1)} = 1$$

6.144 (a) For the $^6F_{1/2}$ state

$$S = \frac{5}{2},\ L = 3,\ J = \frac{1}{2}$$

So, $$g = 1 + \frac{3/4 + 35/4 - 12}{2 \times 3/4} = 1 + \frac{38 - 48}{6} = -\frac{2}{3}$$

(b) For the $^4D_{1/2}$ state

$$S = \frac{3}{2},\ L = 2,\ J = \frac{1}{2}$$

So,
$$g = 1 + \frac{3/4 + 15/4 - 6}{2 \times 3/4} = 1 + \frac{18 - 24}{6} = 0$$

(c) For the $^5F_2$ state

$$S = 2,\ L = 3,\ J = 2$$

So,
$$g = 1 + \frac{6 + 6 - 12}{2 \times 6} = 1$$

(d) For the $^5P_1$ state

$$S = 2,\ L = 1,\ J = 1$$

So,
$$g = 1 + \frac{2 + 6 - 2}{2 \times 2} = \frac{5}{2}$$

(e) For the $^3P_0$ state with $J = 0$, $L = S$, the g factor is indeterminate.

6.145 (a) For the ^1F state

$$S = 0,\ L = 3 = J$$

So,
$$g = 1 + \frac{3 \times 4 - 3 \times 4}{2 \times 3 \times 4} = 1$$

Hence,
$$\mu = \sqrt{3 \times 4}\,\mu_B = \sqrt{12}\,\mu_B$$

(b) For the $^2D_{3/2}$ state

$$S = \frac{1}{2},\ L = 2,\ J = \frac{3}{2}$$

So,
$$g = 1 + \frac{15/4 + 3/4 - 6}{2 \times 15/4} = 1 + \frac{18 - 24}{30} = \frac{4}{5}$$

Hence,
$$\mu = \frac{4}{5}\sqrt{\frac{15}{4}}\,\mu_B = \frac{2}{5}\sqrt{15}\,\mu_B = 2\sqrt{\frac{3}{5}}\,\mu_B$$

(c) We have
$$\frac{4}{3} = 1 + \frac{J(J+1) + 2 - 6}{2J(J+1)}$$

or
$$\frac{2}{3}J(J+1) = J(J+1) - 4$$

or $$J(J+1) = 12 \Rightarrow J = 3$$

Hence, $$\mu = \frac{4}{3}\sqrt{12}\,\mu_B = \frac{8}{\sqrt{3}}\,\mu_B$$

6.146 The expression for the projection of the magnetic moment is

$$\mu_Z = g m_J \mu_B$$

where m_J is the projection of **J** on the z-axis. Maximum value of the m_J is J.

Thus, $$gJ = 4$$

Since $J = 2$, we get $g = 2$. Now

$$2 = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}$$

$$= 1 + \frac{6 + S(S+1) - 6}{2 \times 6} \quad (\text{as } L = 2)$$

$$= 1 + \frac{S(S+1)}{12}$$

Hence, $$S(S+1) = 12 \quad \text{or} \quad S = 3$$

Thus, $$M_S = \hbar\sqrt{3 \times 4} = 2\sqrt{3}\,\hbar$$

6.147 The angle between the angular momentum vector and the field direction is the least when the angular momentum projection is maximum, i.e., $J\hbar$.

Thus, $$J\hbar = \sqrt{J(J+1)}\,\hbar\cos 30^\circ$$

or $$\sqrt{\frac{J}{J+1}} = \frac{\sqrt{3}}{2}$$

Hence, $$J = 3$$

Then $$g = 1 + \frac{3 \times 4 + 1 \times 2 - 2 \times 3}{2 \times 3 \times 4} = 1 + \frac{8}{24} = \frac{4}{3}$$

and $$\mu = \frac{4}{3}\sqrt{3 \times 4}\,\mu_B = \frac{8}{\sqrt{3}}\,\mu_B$$

6.148 For a state with $n = 3$, $L = 2$. Thus, the state with maximum angular momentum is ${}^2D_{5/2}$.

Then, $$g = 1 + \frac{5/2 \times 7/2 + 1/2 \times 3/2 - 2 \times 3}{2 \times 5/2 \times 7/2}$$

$$= 1 + \frac{35 + 3 - 24}{70} = 1 + \frac{1}{5} = \frac{6}{5}$$

Hence, $$\mu = \frac{6}{5}\sqrt{\frac{5}{2} \times \frac{7}{2}}\,\mu_B = 3\sqrt{\frac{7}{5}}\,\mu_B$$

6.149 To get the greatest possible angular momentum we must have $S = S_{max} = 1$, $L = L_{max} = 1 + 2 = 3$ and $J = L + S = 4$.

Then $$g = 1 + \frac{4 \times 5 + 1 \times 2 - 3 \times 4}{2 \times 4 \times 5} = 1 + \frac{10}{40} = \frac{5}{4}$$

and $$\mu = \frac{5}{4}\sqrt{4 \times 5}\,\mu_B = \frac{5\sqrt{5}}{2}\,\mu_B$$

6.150 Since $\mu = 0$, we must have either $J = 0$ or $g = 0$. But $J = 0$ is incompatible with $L = 2$ and $S = 3/2$. Hence, $g = 0$. So, we have

$$0 = 1 + \frac{J(J+1) + 3/2 \times 5/2 - 2 \times 3}{2J(J+1)}$$

or $$-3J(J+1) = \frac{15}{4} - 6 = -\frac{9}{4}$$

Hence, $$J = \frac{1}{2}$$

Thus, $$M = \hbar\sqrt{\frac{1}{2} \times \frac{3}{2}} = \frac{\hbar\sqrt{3}}{2}$$

6.151 From the relation

$$M = \hbar\sqrt{J+1} = \sqrt{2}\hbar$$

we find $J = 1$. From the zero value of the magnetic moment, we find $g = 0$

or $$1 + \frac{1 \times 2L(L+1) + 2 \times 3}{2 \times 1 \times 2} = 0$$

$$1 + \frac{-L(L+1) + 8}{4} = 0$$

or $$12 = L(L+1)$$

Hence, $L = 3$. The state is $^5F_1$.

6.152 If **M** is the total angular momentum vector of the atom, then there is a magnetic moment

$$\boldsymbol{\mu}_m = \frac{g\mu_B \mathbf{M}}{\hbar}$$

associated with it. Here, g is the Lande's factor. In a magnetic field of induction **B**, the energy

$$H' = \frac{g\mu_B \mathbf{M}\cdot\mathbf{B}}{\hbar}$$

is associated with it. This interaction term corresponds with a precession of the angular momentum vector because it leads to an equation of motion of the angular momentum vector of the form

$$\frac{d\mathbf{M}}{dt} = \boldsymbol{\Omega}\times M \quad \left(\text{where } \boldsymbol{\Omega} = \frac{g\mu_B\mathbf{B}}{\hbar}\right)$$

Using Gaussian unit expression of μ_B, we get $\mu_B = 0.927\times 10^{-20}$ erg/G, $B = 10^3$ G, $\hbar = 1.054\times 10^{-27}$ erg s and for the $^2P_{3/2}$ state

$$g = 1 + \frac{3/2\times 5/2 + 1/2\times 3/2 - 1\times 2}{2\times 3/2\times 5/2} = 1 + \frac{1}{3} = \frac{4}{3}$$

and $$\Omega = 1.17\times 10^{10}\text{ rad/s}$$

The same formula is valid in MKS units also. But $\mu_B = 0.927\times 10^{-23}$ A m^2, $B = 10^{-1}$ T and $\hbar = 1.054\times 10^{-34}$ J s. The answer is the same.

6.153 The force on an atom with magnetic moment $\boldsymbol{\mu}$ in a magnetic field of induction **B** is given by

$$\mathbf{F} = (\boldsymbol{\mu}\cdot\boldsymbol{\nabla})\,\mathbf{B}$$

In the present case, the maximum force arises when $\boldsymbol{\mu}$ is along the axis or close to it.

Then $$F_Z = (\mu_Z)_{\max}\frac{\partial B}{\partial Z}$$

Here, $(\mu_Z)_{\max} = g\mu_B J$. The Lande's factor g is for the $^2P_{1/2}$.

Then, $$g = 1 + \frac{1/2\times 3/2 + 1/2\times 3/2 - 1\times 2}{2\times 1/2\times 3/2} = 1 - \frac{1/2}{3/2} = \frac{2}{3}$$

and $$J = \frac{1}{2} \quad \text{so,} \quad (\mu_Z)_{\max} = \frac{1}{3}\mu_B$$

The magnetic field is given by

$$B_Z = \frac{\mu_0}{4\pi}\cdot\frac{2I\pi r^2}{(r^2 + Z^2)^{3/2}}$$

or $$\frac{\partial B_Z}{\partial Z} = -\frac{\mu_0}{4\pi} 6I\pi r^2 \frac{Z}{(r^2 + Z^2)^{5/2}}$$

Thus, $$\left(\frac{\partial B_Z}{\partial Z}\right)_{Z=r} = \frac{\mu_0}{4\pi} \frac{3I\pi}{\sqrt{8} r^2}$$

Thus, the maximum force is

$$F = \frac{1}{3} \mu_B \frac{\mu_0}{4\pi} \frac{3\pi}{\sqrt{8}} \frac{I}{r^2}$$

Substitution gives (using data in MKS units)

$$F = 4.1 \times 10^{-27} \text{ N}$$

6.154 The magnetic field at a distance r from a long current carrying wire is mostly tangential and given by

$$B_\varphi = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0}{4\pi} \frac{2I}{r}$$

The force on a magnetic dipole of moment $\boldsymbol{\mu}$ due to this magnetic field is also tangential and has a magnitude $(\boldsymbol{\mu} \cdot \nabla) B_\varphi$.This force is non-vanishing only when the component of $\boldsymbol{\mu}$ along $\mathbf{r}$ is non-zero. Then

$$F = \mu_r \frac{\partial}{\partial r} B_\varphi = -\mu_r \frac{\mu_0}{4\pi} \frac{2I}{r^2}$$

Now, the maximum value of $\mu_r = \pm \mu_B$. Thus, the force is

$$F_{\max} = \mu_B \frac{\mu_0}{4\pi} \frac{2I}{r^2} = 2.97 \times 10^{-26} \text{ N}$$

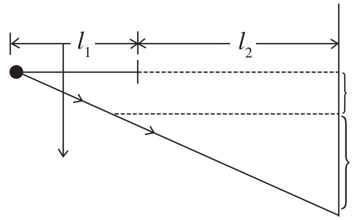
6.155 In the homogenous magnetic field, the atom experiences force

$$F = gJ\mu_B \frac{\partial B}{\partial Z}$$

Depending on the sign of J, this can be either upward or downward. Suppose the latter is true. The atom then traverses first along a parabola inside the field and, once outside, in a straight line. The total distance between extreme lines on the screen will be

$$\delta = 2gJ\mu_B \frac{\partial B}{\partial Z} \left\{ \frac{1}{2} \left(\frac{l_1}{v} \right)^2 + \frac{l_1}{v} \cdot \frac{l_2}{v} \right\} \Big/ m_V$$

Here m_V is the mass of the vanadium atom. (The first term is the displacement within the field and the second term is the displacement due to the transverse velocity acquired in the magnetic field.)



Thus, using $$\frac{1}{2} m_V v^2 = T$$

we get $$\frac{\partial B}{\partial Z} = \frac{2T\delta}{g\mu_B J l_1 (l_1 + 2l_2)}$$

For vanadium atom in the ground state $^4F_{3/2}$

$$g = 1 + \frac{(3\times 5)/4 + (3\times 5)/4 - 3\times 4}{2\times(3\times 5)/4}$$

$$= 1 + \frac{30-48}{30} = 1 - \frac{18}{30} = \frac{2}{5}$$

Using $J = 3/2$, and other data, on substituting, we get

$$\frac{\partial B}{\partial Z} = 1.45\times 10^{13}\ \text{G/cm}$$

This value differs from the answer given in the book by almost a factor of 10^9. For neutral atoms in Stern-Gerlach experiments, the value $T = 22$ MeV is much too large. A more appropriate value will be $T = 2$ meV, i.e., 10^9 times smaller. Then one gets the right answer.

6.156 (a) The term $3P_0$ does not split in weak magnetic field as it has zero total angular momentum.

(b) The term $^2F_{5/2}$ will split into $2\times 5/2 + 1 = 6$ sublevels. The shift in each sublevel is given by

$$\Delta E = -g\mu_B M_Z B$$

where $M_J = -J(J-1), \ldots, J$ and g is the Lande's factor

So, $$g = 1 + \frac{(5\times 7)/4 + (1\times 3)/4 - 3\times 4}{2\times(5\times 7)/4}$$

$$= 1 + \frac{38-48}{70} = \frac{6}{7}$$

(c) For the $^4D_{1/2}$ term

$$g = 1 + \frac{(1\times 3)/4 + (3\times 5)/4 - 2\times 3}{2\times(1\times 3)/4}$$

$$= 1 + \frac{3+15-24}{6} = 1 - 1 = 0$$

Thus, the energy differences vanish and the level does not split.

6.157 (a) For the $^1D_2$ term

$$g = 1 + \frac{2\times 3 + 0 - 2\times 3}{2\times 2\times 3} = 1$$

and $\Delta E = -\mu_B M_J B$

where $M_J = -2, -1, 0, +1, +2$.

Thus, the splitting is

$$\Delta E = 4\mu_B B$$

Substitution gives $\Delta E = 57.9\ \mu\text{eV}$.

(b) For the $^3F_4$ term

$$g = 1 + \frac{4\times 5 + 1\times 2 - 3\times 4}{2\times 4\times 5}$$

$$= 1 + \frac{10}{40} = \frac{5}{4}$$

and $\Delta E = -\frac{5}{4}\mu_B B M_J$

where $M_J = -4$ to $+4$.

Thus, $\Delta E = \frac{5}{4}\mu_B B \times 8 = 10\mu_B B (= 2gJ\mu_B)$

Substitution gives $\Delta E = 144.7\ \mu\text{eV}$.

6.158 (a) The term $^1P_1$ splits into 3 lines with $M_Z = \pm 1, 0$ in accordance with the formula

$$\Delta E = -g\mu_B B M_Z$$

where $g = 1 + \frac{1\times 2 + 0 - 1\times 2}{2\times 1\times 2} = 1$

The term $^1S_0$ does not split in weak magnetic field. Thus, the transitions between $^1P_1$ and $^1S_0$ will result in 3 lines, i.e., a normal Zeeman triplet.

(b) The term $^2D_{5/2}$ will split of into 6 lines in accordance with the formula

$$\Delta E = -g\mu_B B M_Z$$

$$M_Z = \pm\frac{5}{2}, \pm\frac{3}{2}, \pm\frac{1}{2}$$

and $g = 1 + \frac{5\times 7 + 1\times 3 - 4\times 2\times 3}{2\times 5\times 7} = \frac{6}{5}$

The term $^2P_{3/2}$ will split into 4 lines in accordance with the above formula with

$$M_Z = \pm\frac{3}{2}, \pm\frac{1}{2} \quad \text{and} \quad g = 1 + \frac{3\times 5 + 1\times 3 - 4\times 1\times 2}{2\times 3\times 5} = \frac{4}{5}$$

It is seen that the Zeeman splitting is anomalous as g factors are different.

(c) The term $^3D_1$ splits into 3 lines ($g = 5/2$). The term $^3P_0$ does not split. Thus, the Zeeman spectrum is normal.

(d) For the $5I_5$ term

$$g = 1 + \frac{5\times 6 + 2\times 3 - 6\times 7}{2\times 5\times 6}$$

$$= 1 + \frac{36-42}{60} = 1 - \frac{1}{10} = \frac{9}{10}$$

For the $^5H_4$ term

$$g = 1 + \frac{4\times 5 + 2\times 3 - 5\times 6}{2\times 4\times 5} = 1 + \frac{26-30}{40} = \frac{9}{10}$$

We see that the splitting in the 2 lines given by $\Delta E = -g\mu_B BM_Z$ is the same though the number of lines is different (11 and 9). It is then easy to see that only the lines with these energies occur: $\hbar\omega_0$, $\hbar\omega_0 \pm g\mu_B B$. The Zeeman pattern is normal.

6.159 For a singlet term $S = 0$, $L = J$, $g = 1$. Then the total splitting is $\Delta E = 2J\mu_B B$. Substitution gives $J = 3$ [$= \Delta E/(2\mu_B B)$]. The term is $^1F_3$.

6.160 Since the spectral line is caused by transition between singlet terms, the Zeeman effect will be normal (since $g = 1$ for both terms). The energy difference between extreme components of the line will be $2\mu_B B$. This must equal

$$-\Delta\left(\frac{2\pi\hbar c}{\lambda}\right) = \frac{2\pi\hbar c\Delta\lambda}{\lambda^2}$$

Thus,

$$\Delta\lambda = \frac{\mu_B B\lambda^2}{\pi\hbar c} = 35 \text{ pm}$$

6.161 From the previous problem, if the components are λ, $\lambda \pm \Delta\lambda$, then

$$\frac{\lambda}{\Delta\lambda} = \frac{2\pi\hbar c}{\mu_B B\lambda}$$

For resolution

$$\frac{\lambda}{\Delta\lambda} \le R = \frac{\lambda}{\delta\lambda} \quad \text{(of the instrument)}$$

Thus,
$$\frac{2\pi\hbar c}{\mu_B B\lambda} \le R \quad \text{or} \quad B \ge \frac{2\pi\hbar c}{\mu_B \lambda R}$$

Hence, the minimum magnetic induction is

$$B_{\min} = \frac{2\pi\hbar c}{\mu_B \lambda R} = 4\text{ kG} = 0.4\,\text{T}$$

6.162 The $^3P_0$ term does not split. The $^3D_1$ term splits into 3 lines corresponding to the shift.

$$\Delta E = -g\mu_B B M_Z$$

with $M_Z = \pm 1, 0$. The interval between neighboring components is then given by

$$\hbar\Delta\omega = g\mu_B B$$

Hence,
$$B = \frac{\hbar\Delta\omega}{g\mu_B}$$

Now, for the $^3D_1$ term

$$g = 1 + \frac{1\times 2 + 1\times 2 - 2\times 3}{2\times 1\times 2} = 1 + \frac{4-6}{4} = \frac{1}{2}$$

Substitution gives $B = 3$ kG $= 0.3$ T

6.163 (a) For the $^2P_{3/2}$ term

$$g = 1 + \frac{3/2\times 5/2 + 1/2\times 3/2 - 1\times 2}{2\times 3/2\times 5/2} = 1 + \frac{10}{30} = \frac{4}{3}$$

and the energy of the $^2P_{3/2}$ sublevels will be

$$E(M_Z) = E_0 - \frac{4}{3}\mu_B B M_Z$$

where $M_Z = \pm 3/2, \pm 1/2$. Thus, between neighboring sublevels

$$\delta E(^2P_{3/2}) = \frac{4}{3}\mu_B B$$

For the $^2P_{1/2}$ term

$$g = 1 + \frac{1/2\times 3/2 + 1/2\times 3/2 - 1\times 2}{2\times 1/2\times 3/2}$$

$$= 1 + \frac{6-8}{6} = 1 - \frac{1}{3} = \frac{2}{3}$$

and the separation between the two sublevels into which the $^2P_{1/2}$ term will split is

$$\Delta E(^2P_{1/2}) = \frac{2}{3}\mu_B B$$

The ratio of the two splittings is 2 : 1.

(b) The interval between neighboring Zeeman sublevels of the $^2P_{3/2}$ term is $(4/3)\mu_B B$. The energy separation between D_1 and D_2 lines is

$$\frac{2\pi\hbar c}{\lambda^2}\Delta\lambda$$

(this is the natural separation of the ^2P term).

Thus, $$\frac{4}{3}\mu_B B = \frac{2\pi\hbar c\Delta\lambda}{\lambda^2\eta}$$

or $$B = \frac{3\pi\hbar c\Delta\lambda}{2\mu_B\lambda^2\eta}$$

$B = 5.46$ kG (on substituting values)

6.164 For the $^2P_{3/2}$ level, $g = 4/3$ (see Problem 6.163) and the energies of sublevels are

$$E' = E'_0 - \frac{4}{3}\mu_B B M'_Z$$

where $M'_Z = \pm 3/2, \pm 1/2$ for the four sublevels.

For the $^2S_{1/2}$ level, $g = 2$ (since $L = 0$) and

$$E = E_0 - 2\mu_0 B M_Z \quad (\text{where } M_Z = \pm 1/2)$$

Permitted transitions must have $\Delta M_Z = 0, \pm 1$.

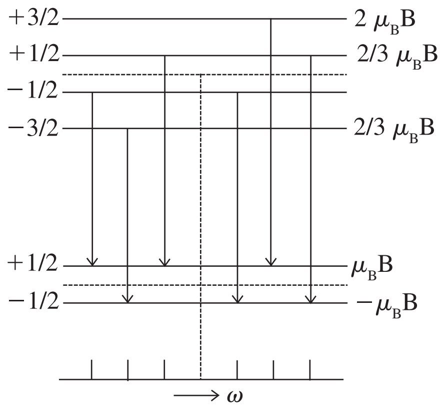
Thus, only the following transitions occur

$$\left.\begin{array}{r} \frac{3}{2} \to \frac{1}{2} \\ -\frac{3}{2} \to -\frac{1}{2} \end{array}\right\} \Delta\omega = \pm\mu_B B/\hbar = 3.96 \times 10^{10} \text{ rad/s}$$

$$\left.\begin{array}{r} \frac{1}{2} \to \frac{1}{2} \\ -\frac{1}{2} \to -\frac{1}{2} \end{array}\right\} \Delta\omega = \pm\frac{1}{3}\mu_B B/\hbar = 1.32 \times 10^{10} \text{ rad/s}$$

$$\left.\begin{array}{l}\frac{1}{2} \to \frac{-1}{2} \\ -\frac{1}{2} \to \frac{1}{2}\end{array}\right\} \Delta\omega = \pm\frac{5}{3}\frac{\mu_B B}{\hbar} = 6.6 \times 10^{10} \text{ rad/s}$$

These six lines are shown in the figure.



6.165 The difference arises because of different selection rules in the two cases. In direction 1 the line is emitted perpendicular to the field. The selection rules are then

$$\Delta M_Z = 0, \pm 1$$

In direction 2, the light is emitted along the direction of the field. Then, the selection rules are

$$\Delta M_Z = \pm 1$$

$$\Delta M_Z = 0 \text{ is forbidden.}$$

(a) The transition ${}^2P_{3/2} \to {}^2S_{1/2}$ has been considered in the Problem 6.164. In direction 1 we get all the six lines shown in that problem.

In direction 2 the line corresponding to $\frac{1}{2} \to \frac{1}{2}$ and $-\frac{1}{2} \to -\frac{1}{2}$ is forbidden. Then we get four lines.

(b) In the transition ${}^3P_2 \to {}^3S_1$, for the ${}^3P_2$ level,

$$g = 1 + \frac{2 \times 3 + 1 \times 2 - 1 \times 2}{2 \times 2 \times 3} = \frac{3}{2}$$

So the energies of the sublevels are

$$E'(M'_Z) = E'_0 - \frac{3}{2}\mu_B BM'_Z \quad (\text{where } M'_Z = \pm 2, \pm 1, 0)$$

For the $^3S_1$ lines, $g = 2$ and the energies of the sublevels are

$$E(M_Z) = E_0 - 2\mu_B BM_Z \quad (\text{where } M_Z = \pm 1, 0)$$

The lines are

$$\Delta M_Z = +1 \ \text{ for } -2 \to -1, -1 \to 0 \text{ and } 0 \to 1$$

$$\Delta M_Z = 0 \ \text{ for } -1 \to -1, 0 \to 0 \text{ and } 1 \to 1$$

$$\Delta M_Z = 1 \ \text{ for } 2 \to 1, 1 \to 0 \text{ and } 0 \to -1$$

All energy differences are unequal because the two g values are unequal. There are then nine lines if viewed along direction 1 and six lines if viewed along direction 2.

6.166 For the two levels

$$E'_0 = E_0 - g'\mu'_B M'_Z B$$

$$E_0 = E'_0 - g\mu_B M_Z B$$

and hence, the shift of the component is the value of

$$\Delta\omega = \frac{\mu_B B}{\hbar}[g'M'_Z - gM_Z]$$

subject to the selection rule $\Delta M_Z = 0, \pm 1$.

For the $^3D_3$ state $$g' = 1 + \frac{3\times 4 + 1\times 2 - 2\times 3}{2\times 3\times 4} = 1 + \frac{8}{24} = \frac{4}{3}$$

For the $^3P_2$ state $$g = 1 + \frac{2\times 3 + 1\times 2 - 1\times 2}{2\times 2\times 3} = \frac{3}{2}$$

Thus, $$\Delta\omega = \frac{\mu_B B}{\hbar}\left|\frac{4}{3}M'_Z - \frac{3}{2}M_Z\right|$$

For the different transitions we have the following table

Transition	$M'_Z g' - M_Z g$	Transition	$M'_Z g' - M_Z g$
$3 \to 2$	$\mu_B B$	$1 \to 1$	$-1/6\ \mu_B B$
$2 \to 2$	$-1/3\ \mu_B B$	$1 \to 0$	$4/3\ \mu_B B$
$2 \to 1$	$7/6\ \mu_B B$	$0 \to 1$	$-3/2\ \mu_B B$
$1 \to 2$	$-5/3\ \mu_B B$	$0 \to 0$	0

Transition	$M'_Z g' - M_Z g$	Transition	$M'_Z g' - M_Z g$
$0 \to -1$	$3/2\ \mu_B B$	$-2 \to -1$	$-7/6\ \mu_B B$
$-1 \to 0$	$-4/3\ \mu_B B$	$-2 \to -2$	$1/3\ \mu_B B$
$-1 \to -1$	$1/6\ \mu_B B$	$-3 \to -2$	$-\mu_B B$
$-1 \to -2$	$5/3\ \mu_B B$		

There are 15 lines in all.

The lines farthest out are $1 \to 2$ and $-1 \to -2$. The splitting between them is the total splitting. It is given by

$$\Delta\omega = \frac{10}{3}\frac{\mu_B B}{\hbar}$$

Substitution gives $\Delta\omega = 7.8 \times 10^{10}$ rad/s.

6.4 Molecules and Crystals

6.167 In the first excited rotational level, $J = 1$.

So,
$$E_J = 1 \times 2\frac{\hbar^2}{2I} = \frac{1}{2}I\omega^2 \quad \text{(classically)}$$

Thus,
$$\omega = \sqrt{2}\frac{\hbar}{I}$$

Now,
$$I = \Sigma m_i r_i^2 = \frac{m}{2}\frac{d^2}{4} + \frac{m}{2}\frac{d^2}{4} = m\frac{d^2}{4}$$

where m is the mass of the molecule and r_i is the distance of the atom from the axis.

Thus,
$$\omega = \frac{4\sqrt{2}\hbar}{md^2} = 1.56 \times 10^{11}\ \text{rad/s}$$

6.168 The axis of rotation passes through the centre of mass (C.M.) of the HCl molecule. The distance of the two atoms from the C.M. is

$$d_{\text{H}} = \frac{m_{\text{Cl}}}{m_{\text{HCl}}}d, \quad d_{\text{Cl}} = \frac{m_{\text{H}}}{m_{\text{HCl}}}d$$

Thus, moment of inertia about the axis

$$I = \frac{4}{2}m_{\text{H}}d_{\text{H}}^2 + m_{\text{Cl}}d_{\text{Cl}}^2 = \frac{m_{\text{H}}m_{\text{Cl}}}{m_{\text{H}} + m_{\text{Cl}}}d^2$$

The energy difference between two neighboring levels whose quantum numbers are J and $J-1$ is

$$\frac{\hbar^2}{2I}\cdot 2J = \frac{J\hbar^2}{I} = 7.86 \text{ meV}$$

Hence, $J = 3$ and the levels have quantum numbers 2 and 3.

6.169 The angular momentum is

$$\sqrt{2IE} = M$$

Now, $$I = \frac{md^2}{4} = 1.9584 \times 10^{-39} \text{ g cm}^2$$

(where m is the mass of O_2 molecule).

So, $$M = 3.68 \times 10^{-27} \text{ erg s} = 3.49\hbar$$

(This corresponds to $J = 3$.)

6.170 From $$E_J = \frac{\hbar^2}{2I} J(J+1)$$

and the selection rule $\Delta J = 1$ or $J \to J-1$ for a pure rotational spectrum, we get

$$\omega(J, J-1) = \frac{\hbar J}{I}$$

Thus, transition lines are equispaced in frequency $\Delta\omega = \hbar/I$.

In the case of CH

$$I = \frac{\hbar}{\Delta\omega} = 1.93 \times 10^{-40} \text{ g cm}^2$$

Also, $$I = \frac{m_c m_H}{m_c + m_H} d^2$$

so, $$d = 1.117 \times 10^{-8} \text{ cm} = 111.7 \text{ pm}$$

6.171 If the vibrational frequency is ω_0, the excitation energy of the first vibrational level will be $\hbar\omega_0$. Thus, if there are J rotational levels contained in the band between the ground state and the first vibrational excitation, then

$$\hbar\omega_0 = \frac{J(J+1)\hbar^2}{2I}$$

where, as stated in the problem, we have ignored any coupling between the two. For HF molecule

$$I = \frac{m_H m_F}{m_H + m_F} d^2 = 1.336 \times 10^{-4} \text{ g cm}^2$$

Then
$$J(J+1) = \frac{2I\omega_0}{\hbar} = 197.4$$

For $J = 14$, $J(J+1) = 210$. For $J = 13$, $J(J+1) = 182$. Thus, 13 levels lie between the ground state and the first vibrational excitation.

6.172 We proceed as above. On calculating, we get

$$\frac{2I\omega_0}{\hbar} \approx 1118$$

Now this must equal

$$J(J+1) \approx \left(J + \frac{1}{2}\right)^2$$

Taking the square root we get, $J \approx 33$.

6.173 From the formula

$$J(J+1)\frac{\hbar^2}{2I} = E$$

we get
$$J(J+1) = \frac{2IE}{\hbar^2}$$

or
$$\left(J + \frac{1}{2}\right)^2 - \frac{1}{4} = \frac{2IE}{\hbar^2}$$

Hence,
$$J = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2IE}{\hbar^2}}$$

So,
$$J + 1 = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2I}{\hbar^2}(E + \Delta E)}$$

we find
$$1 = \sqrt{\frac{1}{4} + \frac{2I}{\hbar^2}E + \frac{2I}{\hbar^2}\Delta E} - \sqrt{\frac{1}{4} + \frac{2IE}{\hbar^2}}$$

$$= \sqrt{\frac{1}{4} + \frac{2I}{\hbar^2}E}\left[\left(1 + \frac{\Delta E}{E + \hbar^2/8I}\right)^{1/2} - 1\right]$$

$$= \sqrt{\frac{1}{4} + \frac{2I}{\hbar^2}E}\, \frac{\Delta E}{2(E + \hbar^2/8I)}$$

$$= \sqrt{\frac{2I}{\hbar^2}}\, \frac{\Delta E}{2\sqrt{E + \hbar^2/8I}}$$

The quantity dN/dE is equal to $1/\Delta E$. For large E it is

$$\frac{dN}{dE} = \sqrt{\frac{I}{2\hbar^2 E}}$$

For an iodine molecule

$$I = \frac{m_1 d^2}{2} = 7.57 \times 10^{-38} \text{ g cm}^2$$

Thus, for $J = 10$

$$\frac{dN}{dE} = \sqrt{\frac{I}{2\hbar^2 \cdot \frac{\hbar^2}{2I} J(J+1)}} = \frac{1}{\sqrt{J(J+1)\hbar^2}}$$

$$= 1.04 \times 10^4 \text{ levels per eV} \quad \text{(on substituting values)}$$

6.174 For the first rotational level

$$E_{\text{rot}} = 2\frac{\hbar^2}{2I} = \frac{\hbar^2}{I}$$

and for the first vibrational level

$$E_{\text{vib}} = \hbar\omega$$

Thus,
$$\xi = \frac{E_{\text{vib}}}{E_{\text{rot}}} = \frac{I\omega}{\hbar}$$

Here ω = frequency of vibration.

Now,
$$I = \mu d^2 = \frac{m_1 m_2}{m_1 + m_2} d^2$$

(a) For H_2 molecule, $I = 4.58 \times 10^{-41}$ g cm^2 and $\xi = 36$.

(b) For HI molecule, $I = 4.247 \times 10^{-40}$ g cm^2 and $\xi = 175$.

(c) For I_2 molecule, $I = 7.57 \times 10^{-38}$ g cm^2 and $\xi = 2872$.

6.175 The energy of the molecule in the first rotational level will be $\hbar^2/I$. The ratio of the number of molecules at the first excited vibrational level to the number of molecules at the first excited rotational level is

$$\frac{e^{-\hbar\omega/kT}}{(2J+1)e^{-\frac{\hbar^2 J(J+1)}{2IkT}}}$$

$$= \frac{1}{3}e^{\hbar\omega/kT} \times e^{-\hbar^2/IkT} = \frac{1}{3}e^{-\hbar(\omega - 2B)/kT}$$

where $B = \hbar/2I$.

For the hydrogen molecule

$$I = \frac{1}{2}m_{\mathrm{H}}d^2 = 4.58 \times 10^{-41} \text{ g cm}^2$$

Substitution gives ratio as 3.04×10^{-4}.

6.176 By definition

$$<E> = \frac{\sum E_v e^{E_v/kT}}{\sum e^{-E_v/kT}} = \frac{\frac{\partial}{\partial\beta}\sum_{v=0}^{\infty} e^{-\beta E_v}}{\sum_{v=0}^{\infty} e^{-\beta E_v}}$$

$$= -\frac{\partial}{\partial\beta}\ln\sum_{v=0}^{\infty} e^{-\beta(v+1/2)\hbar\omega} \quad (\text{where } \beta = 1/kT)$$

$$= -\frac{\partial}{\partial\beta}\ln e^{-(1/2)\beta\hbar\omega}\frac{1}{1-e^{-\beta\hbar\omega}}$$

$$= -\frac{\partial}{\partial\beta}\left[-\frac{1}{2}\hbar\omega\beta - \ln(1-e^{-\beta\hbar\omega})\right]$$

$$= \frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}$$

Thus for one gram mole of diatomic gas

$$C_{V_{\mathrm{vib}}} = N\frac{\partial <E>}{\partial T} = \frac{R\left(\frac{\hbar\omega}{kT}\right)^2 e^{\hbar\omega/kT}}{(e^{\hbar\omega/kT} - 1)^2}$$

where $R = Nk$ is the gas constant.

In the present case

$$\frac{\hbar\omega}{kT} = 2.7088$$

and $$C_{V_{\mathrm{vib}}} = 0.56\,R$$

6.177 In the rotation vibration band, the main transition is due to change in vibrational quantum number $\nu \to \nu - 1$. Along with this the rotational quantum number may change. The "Zeroth line" $0 \to 0$ is forbidden in this case so the neighboring lines arise due to $1 \to 0$ or $0 \to 1$ in the rotational quantum number.

Now, $$E = E_\nu + \frac{\hbar^2}{2I} J(J+1)$$

Thus, $$\hbar\omega = \hbar\omega_0 + \frac{\hbar^2}{2I}(\pm 2)$$

Hence, $$\Delta\omega = \frac{2\hbar}{I} = \frac{2\hbar}{\mu d^2}$$

so, $$d = \sqrt{\frac{2\hbar}{\mu\Delta\omega}}$$

Substitution gives $d = 0.128$ nm.

6.178 If λ_R = wavelength of the red satellite and λ_V = wavelength of the violet satellite, then

$$\frac{2\pi\hbar c}{\lambda_R} = \frac{2\pi\hbar c}{\lambda_0} - \hbar\omega$$

and $$\frac{2\pi\hbar c}{\lambda_V} = \frac{2\pi\hbar c}{\lambda_0} + \hbar\omega$$

Substitution gives $\lambda_R = 424.3$ nm and $\lambda_V = 386.8$ nm.

The two formulas can be combined to give

$$\lambda = \frac{2\pi c}{2\pi c/\lambda_0 \pm \omega} = \frac{\lambda_0}{1 \pm \lambda_0\omega/2\pi c}$$

6.179 As in the previous problem

$$\omega = \pi c\left(\frac{1}{\lambda_V} - \frac{1}{\lambda_R}\right)$$

$$= \frac{\pi c(\lambda_R - \lambda_V)}{\lambda_R\lambda_V} = 1.368 \times 10^{14} \text{ rad/s}$$

The force constant x is defined by $x = \mu\omega^2$, where μ = reduced mass of the S_2 molecule.

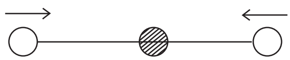
Substitution gives $x = 4.96$ N/cm.

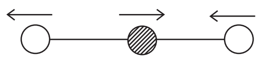
6.180 The violet satellite arises from the transition $1 \to 0$ in the vibrational state of the scattering molecule while the red satellite arises from the transition $0 \to 1$. The intensities of these two transitions are in the ratio of initial populations of the two states, i.e., in the ratio $e^{-\hbar\omega/kT}$.

Thus,
$$\frac{I_V}{I_R} = e^{-\hbar\omega/kT} = 0.067$$

If the temperature is doubled, the ratio increases to 0.259, an increase of 3.9 times.

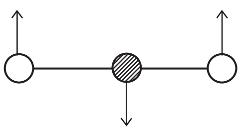
6.181 (a) In the case of CO_2(O−C−O): The molecule has 9 degrees of freedom, 3 for each atom. This means that it can have up to 9 frequencies. Three degrees of freedom correspond to rigid translation, the frequency associated with this is zero as the potential energy of the system cannot change under rigid translation. The potential energy will not change under rotations about axes passing through the C-atom and perpendicular to the O−C−O line. Thus, there can be atmost four non-zero frequencies. We must look for modes different from the above.

ω_1 

ω_2 

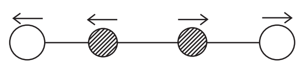
These are the only collinear modes.

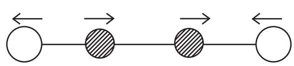
A third mode is doubly degenerate:

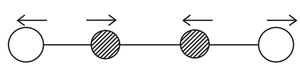
ω_3 

(vibration in and perpendicular to the plane of paper).

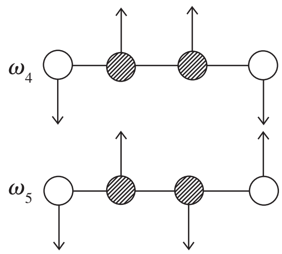
(b) In case of C_2H_2 (H − C − C − H): There are $4 \times 3 - 3 - 2 = 7$ different vibrations. There are three collinear modes.

ω_1 

ω_2 

ω_3 

Two other doubly degenerate frequencies are



together with their counterparts in the plane perpendicular to the paper.

6.182 Suppose the string is stretched along the x-axis from $x = 0$ to $x = l$ with the end points fixed. Suppose $y\,(x,\, t)$ is the transverse displacement of the element at x at time t. Then $y\,(x,\, t)$ obeys

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

We look for a stationary wave solution of this equation. It is

$$y(x,\, t) = A \sin \frac{\omega}{v} x \sin(\omega t + \delta)$$

where A and δ are constants. In this form $y = 0$ at $x = 0$. The other condition $y = 0$ at $x = l$ implies

$$\frac{\omega l}{v} = N\pi \quad (\text{where } N > 0)$$

or
$$N = \frac{l}{\pi v}\,\omega$$

N is the number of modes of frequency and is $\leq \omega$.

Thus,
$$dN = \frac{l}{\pi v}\,d\omega$$

6.183 Let $\xi\,(x,\, y,\, t)$ be the displacement of the element at $(x,\, y)$ at time t. Then it obeys the equation

$$\frac{\partial^2 \xi}{\partial t^2} = v^2 \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right)$$

where $\xi = 0$ at $x = 0$, $x = l$, $y = 0$ and $y = l$.

We look for a solution in the form

$$\xi = A \sin k_1 x \sin k_2 y \sin(\omega t + \delta)$$

Then, $$\omega^2 = v^2(k_1^2 + k_2^2)$$

and $$k_1 = \frac{n\pi}{l}, k_2 = \frac{m\pi}{l}$$

We write this as

$$n^2 + m^2 = \left(\frac{l\omega}{\pi v}\right)^2$$

Here n, $m < 0$. Each pair (n, m) determines a mode. The total number of modes whose frequency is $\le \omega$ is the area of the quadrant of a circle of radius $l\omega/\pi v$, i.e.,

$$N = \frac{\pi}{4}\left(\frac{l\omega}{\pi v}\right)^2$$

Then, $$dN = \frac{l^2}{2\pi v^2}\omega\, d\omega = \frac{S}{2\pi v^2}\omega\, d\omega$$

where $S = l^2$ is the area of the membrane.

6.184 For transverse vibration of a three-dimensional continuum in the form of a cube (say), we have the equation

$$\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = v^2\nabla^2\boldsymbol{\xi}\ (\text{div}\,\boldsymbol{\xi} = 0)$$

Here $\boldsymbol{\xi} = \boldsymbol{\xi}(x, y, z, t)$. We look for solution in the form

$$\boldsymbol{\xi} = \mathbf{A}\sin k_1 x, \sin k_2 y, \sin k_3 z, \sin(\omega t + \delta)$$

This requires $$\omega^2 = v^2(k_1^2 + k_2^2 + k_2^3)$$

From the boundary condition that $\xi = 0$ for $x = 0$, $x = l$, $y = 0$, $y = l$, $z = 0$, $z = l$, we get

$$k_1 = \frac{n_1\pi}{l}, k_2 = \frac{n_2\pi}{l}, k_3 = \frac{n_3\pi}{l}$$

where n_1, n_2, n_3 are non-zero positive integers.

We then get $$n_1^2 + n_2^2 + n_3^2 = \left(\frac{l\omega}{\pi v}\right)^2$$

Each triplet (n_1, n_2, n_3) determines a possible mode and the number of such modes, whose frequency $\le \omega$, is the volume of the all positive octant of a sphere of radius $l\omega/\pi v$. Considering also the fact that the subsidiary condition div $\xi = 0$ implies two independent values of $\mathbf{A}$ for each choice of the wave vector (k_1, k_2, k_3).

We find

$$N(\omega) = \frac{1}{8}\frac{4\pi}{3}\left(\frac{l\omega}{\pi v}\right)^3 2 = \frac{V\omega^3}{3\pi^2 v^3}$$

Thus,
$$dN = \frac{V\omega^2}{\pi^2 v^3}\, d\omega$$

6.185 To determine the Debye temperature we cut off the high-frequency modes in such a way so as to get the total number of modes correctly.

(a) In a linear crystal with $n_0 l$ atoms, the number of modes of transverse vibrations in any given plane cannot exceed $n_0 l$. Then

$$n_0 l = \frac{l}{\pi v}\int_0^{\omega_0} d\omega = \frac{l}{\pi v}\,\omega_0$$

The cut-off frequency ω_0 is related to the Debye temperature Θ by

$$\hbar\omega_0 = k\Theta$$

Thus,
$$\Theta = \left(\frac{\hbar}{k}\right)\pi n_0 v$$

(b) In a square lattice, the number of modes of transverse oscillations cannot exceed $n_0 S$.

Thus,
$$n_0 S = \frac{S}{2\pi v^2}\int_0^{\omega_0} \omega\, d\omega = \frac{S}{4\pi v^2}\,\omega_0^2$$

or
$$\Theta = \frac{\hbar}{k}\,\omega_0 = \left(\frac{\hbar}{k}\right)\left(\sqrt{4\pi n_0}\right)v$$

(c) In a cubic crystal, the maximum number of transverse waves must be $2n_0 V$ (two for each atom). Thus, we have

$$n_0 V = \frac{V}{\pi^2 v^3}\int_0^{\omega_0} \omega^2\, d\omega = \frac{V\omega_0^3}{3\pi^2 v^3}$$

Thus,
$$\Theta = \left(\frac{\hbar}{k}\right)v\,(6\pi^2 n_0)^{1/3}$$

6.186 We proceed as in the previous problem. The total number of modes must be $3n_0 v$ (total transverse and one longitudinal per atom). On the other hand the number of transverse modes per unit frequency interval is given by

$$dN^{\perp} = \frac{V\omega^2}{\pi^2 v_{\perp}^3}\, d\omega$$

while the number of longitudinal nodes per unit frequency interval is given by

$$dN^{\parallel} = \frac{V\omega^2}{2\pi^2 v_{\parallel}^3} d\omega$$

The total number per unit frequency interval is

$$dN = \frac{V\omega^2}{2\pi^2}\left(\frac{2}{v_{\perp}^3} + \frac{1}{v_{\parallel}^3}\right) d\omega$$

If the high frequency cut-off is at $\omega_0 = k\Theta/\hbar$, the total number of modes will be

$$3n_0 V = \frac{V}{6\pi^2}\left(\frac{2}{v_{\perp}^3} + \frac{1}{v_{\parallel}^3}\right)\left(\frac{k\Theta}{\hbar}\right)^3$$

Here n_0 is the number of iron atoms per unit volume.

Thus, $$\Theta = \frac{\hbar}{k}\left[\frac{18\pi^2 n_0}{2/v_{\perp}^3 + 1/v_{\parallel}^3}\right]^{1/3}$$

For iron $$n_0 = \frac{N_A}{M/\rho} = \frac{\rho N_A}{M}$$

(where ρ = density, M = atomic weight of iron, N_A = Avogadro number).

Thus, $$n_0 = 8.389 \times 10^{22}\ \text{cm}^{-3}$$

Substituting the data, we get $\Theta = 469.1$ K.

6.187 We apply the same formula but assume $v_{||} \approx v_{\perp}$. Then

$$\Theta = \frac{\hbar}{k} v(6\pi^2 n_0)^{1/3}$$

or $$v = \frac{k\Theta}{\hbar(6\pi^2 n_0)^{1/3}}$$

For aluminum $$n_0 = \frac{\rho N_A}{M} = 6.023\times 10^{22}\ \text{cm}^{-3}$$

Thus, $$v = 3.39\ \text{km/s}$$

The tabulated values are $v_{||} = 6.3$ km/s and $v_{\perp} = 3.1$ km/s.

6.188 In the Debye approximation, the number of modes per unit frequency interval is given by

$$dN = \frac{l}{\pi v} d\omega \quad \left(\text{for } 0 \leq \omega \leq \frac{k\Theta}{\hbar}\right)$$

But $$\frac{k\Theta}{\hbar} = \pi n_0 v$$

Thus, $$dN = \frac{l}{\pi v} d\omega \quad (\text{for } 0 \le \omega \le \pi n_0 v)$$

The energy per mode is

$$< E >= \frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}$$

Then the total interval energy of the chain is

$$U = \frac{l}{\pi v}\int_0^{\pi n_0 v} \frac{1}{2}\hbar\omega\, d\omega + \frac{l}{\pi v}\int_0^{\pi n_0 v} \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1} d\omega$$

$$= \frac{l\hbar}{4\pi v}(\pi n_0 v)^2 + \frac{l}{\pi v\hbar}(kT)^2 \int_0^{\Theta/T} \frac{xdx}{e^x - 1}$$

$$= ln_0 k \frac{\hbar}{k}(\pi n_0 v)\frac{1}{4} + ln_0 k \frac{T^2}{(\pi n_0 v\hbar/k)} \int_0^{\Theta/T} \frac{xdx}{e^x - 1}$$

We put $ln_0\, k = R$ for 1 mole of the chain.

Then $$U = R\Theta\left\{\frac{1}{4} + \left(\frac{T}{\Theta}\right)^2 \int_0^{\Theta/T} \frac{xdx}{e^x - 1}\right\}$$

Hence the molar heat capacity is by differentiation

$$C_V = \left(\frac{\partial U}{\partial T}\right)_\theta = R\left[2\left(\frac{T}{\Theta}\right)\int_0^{\Theta/T} \frac{xdx}{e^x - 1} - \frac{\Theta/T}{e^{\theta/T} - 1}\right]$$

when $T \gg \Theta$, $C_V \approx R$.

6.189 (a) If the chain has N atoms, we can assume atom number 0 and $N + 1$ are fixed. Then the displacement of the n^{th} atom has the form

$$u_n = A\left(\sin\frac{m\pi}{L}\cdot n\alpha\right)\sin\omega t$$

Putting $k = m\pi/L$, the allowed frequencies then have the form

$$\omega = \omega_{\max}\sin\frac{ka}{2}$$

In our form only $+ve$ k values are allowed. The number of modes in a wave number range dk is

$$dN = \frac{L dk}{\pi} = \frac{L}{\pi}\frac{dk}{d\omega}d\omega$$

But $$d\omega = \frac{a}{2}\omega_{\max}\cos\frac{ka}{2}dk$$

Hence, $$\frac{d\omega}{dk} = \frac{a}{2}\sqrt{\omega_{\max}^2 - \omega^2}$$

So, $$dN = \frac{2L}{\pi a}\frac{d\omega}{\sqrt{\omega_{\max}^2 - \omega^2}}$$

(b) The total number of modes is

$$N = \int_0^{\omega_{\max}} \frac{2L}{\pi a}\frac{d\omega}{\sqrt{\omega_{\max}^2 - \omega^2}} = \frac{2L}{\pi a}\frac{\pi}{2} = \frac{L}{a}$$

which is the number of atoms in the chain.

6.190 Molar zero point energy is

$$\frac{9}{8}R\Theta$$

The zero point energy per gram of copper is

$$\frac{9R\Theta}{8M_{Cu}}$$

where M_{Cu} is the atomic weight of copper.

Substitution gives 48.6 J/g.

6.191 (a) By Dulong and Petit's law, the classical heat capacity is $3R = 24.94$ J/mol-K.

Thus, $$\frac{C}{C_{Cl}} = 0.6014$$

From the graph we see that this value of C/C_{Cl} corresponds to

$$\frac{T}{\Theta} = 0.29$$

Hence, $$\Theta = \frac{65}{0.29} = 224 \text{ K}$$

(b) The molar heat capacity of aluminum = 22.4 J/mol-K corresponds to dimensionless heat capacity

$$\frac{22.4}{3 \times 8.314} = 0.898$$

From the graph this corresponds to $T/\Theta = 0.65$. This gives

$$\Theta = \frac{250}{0.65} \approx 385\,\text{K}$$

Then 80 K corresponds to $T/\Theta = 0.208$.

The corresponding value of C/C_{Cl} is 0.42. Hence, $C = 10.5$ J/mol-K.

(c) We calculate Θ from the data that $C/C_{\text{Cl}} = 0.75$ at $T = 125\,\text{K}$. The x-coordinate corresponding to 0.75 is 0.40.

Hence, $$\Theta = \frac{125}{0.4} = 3125\,\text{K}$$

Now $$k\Theta = \hbar\omega_{\max}$$

So, $$\omega_{\max} = 4.09 \times 10^{13}\ \text{rad/s}$$

6.192 We use the Eq. (6.4d) of the book

$$U = 9R\Theta\left[\frac{1}{8} + \left(\frac{T}{\Theta}\right)^4 \int_0^{\Theta/T} \frac{x^3 dx}{e^x - 1}\right]$$

$$= 9R\Theta\left[\frac{1}{8} + \left\{\int_0^{\infty} \frac{x^3 dx}{e^x - 1}\right\}\left(\frac{T}{\Theta}\right)^4 - \left(\frac{T}{\Theta}\right)^4 \int_{\Theta/T}^{\infty} \frac{x^3 dx}{e^x - 1}\right]$$

In the limit $T \ll \Theta$, the third term in the bracket is exponentially small along with its derivatives. Then we can drop the last term, to get

$$U = \text{constant} + \frac{9R}{\Theta^3} T^4 \int_0^{\infty} \frac{x^3 dx}{e^x - 1}$$

Thus, $$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_\Theta = 36R\left(\frac{T}{\Theta}\right)^3 \int_0^{\infty} \frac{x^3 dx}{e^x - 1}$$

Now from the appendix 9 of the book

$$\int_0^{\infty} \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$$

Thus, $$C_V = \frac{12\pi^4}{5}\left(\frac{T}{\Theta}\right)^3$$

If we call the 3rd term in the bracket above $-U_3$, then

$$U_3 = \left(\frac{T}{\Theta}\right)^4 \int_{\Theta/T}^{\infty} \frac{x^3}{2\sin h(x/2)} \cdot e^{-x/2}\, dx$$

The maximum value of $x^3/(2\sin hx/2)$ is a finite $+ve$ quantity C_0 for $0 \le x < \infty$.

Thus, $$U_3 \le 2C_0 \left(\frac{T}{\Theta}\right)^2 e^{-\Theta/2T}$$

We see that U_3 is exponentially small as $T \to 0$ and so is dU_3/dT.

6.193 At low temperatures $C \propto T^3$. This is also a test of the "lowness" of the temperature. We see that

$$\left(\frac{C_1}{C_2}\right)^{1/3} = 1.4982 \approx 1.5 = \frac{T_1}{T_2} = \frac{30}{20}$$

Thus, T^3 law is obeyed and T_1, T_2 can be regarded low.

6.194 The total zero point energy of 1 mole of the solid is $(9/8)R\Theta$. Dividing this by the number of modes $3N$ we get the average zero point energy per mode as $(3/8)\,k\Theta$.

6.195 In the Debye model

$$dN_\omega = A\omega^2 \quad (\text{for } 0 \le \omega \le \omega_m)$$

Then, $$3N = \int_0^{\omega_m} dN_\omega = \frac{A\omega_m^3}{3} \quad (\text{total number of modes is } 3N)$$

Thus, $$A = \frac{9N}{\omega_m^3}$$

We get $$U = \frac{9N}{\omega_m^3}\int_0^{\omega_m} \frac{\omega^2 \cdot \hbar\omega}{e^{\hbar\omega/kT} - 1}\, d\omega \quad (\text{ignoring zero point energy})$$

$$= 9N\hbar\omega_m \int_0^1 \frac{x^3 dx}{e^{\hbar\omega_m x/kT} - 1} \quad \left(\text{using } x = \frac{\omega}{\omega_m}\right)$$

$$= 9R\Theta \int_0^1 \frac{x^3 dx}{e^{x\Theta/T} - 1} \quad (\text{using } \Theta = \hbar\omega_m/k)$$

Thus, $$\frac{1}{9R\Theta}\frac{dU(x)}{dx} = \frac{x^3}{e^{x\Theta/T} - 1} \quad (\text{for } 0 \le x \le 1)$$

For $T = \dfrac{\Theta}{2}$, this expression is $\qquad \dfrac{x^3}{e^{2x} - 1}$

For $T = \dfrac{\Theta}{4}$, the expression is $\qquad \dfrac{x^3}{e^{4x} - 1}$

On plotting, we get the figures given in the answer sheet.

9.196 The maximum energy of the photon is $\hbar\omega_m = k\Theta = 28.4$ meV (if $\Theta = 330$ K).

To get the corresponding value of the maximum momentum we must know the dispersion relation $\omega = \omega(\mathbf{k})$. For small $\mathbf{k}$ we know $\omega = v|\mathbf{k}|$ where v is velocity of sound in the crystal. For an order of magnitude estimate, we continue to use this result for high $|\mathbf{k}|$. Then we estimate v from the values of the modulus of elasticity and density

$$v \sim \sqrt{\frac{E}{\rho}}$$

Using $E \sim 100$ GPa, $\rho = 8.9 \times 10^3$ kg/m^3, we get

$$v \sim 3 \times 10^3 \text{ m/s}$$

Hence, $$\hbar|\mathbf{k}|_{\max} \sim \frac{\hbar\omega_m}{v} \sim 1.5 \times 10^{-19} \text{ g cm/s}$$

6.197 (a) From the formula

$$dn = \frac{\sqrt{2}\, m^{3/2}}{\pi^2\hbar^3} E^{1/2}\, dE$$

The maximum value $E_{\max}$ is determined in terms of n by

$$n = \frac{\sqrt{2}\, m^{3/2}}{\pi^2\hbar^3} \int_0^{E_{\max}} E^{1/2}\, dE$$

$$= \frac{\sqrt{2}\, m^{3/2}}{\pi^2\hbar^3} \frac{2}{3} E_{\max}^{3/2}$$

or $$E_{\max}^{3/2} = \left(\frac{\hbar^2}{2m}\right)^{3/2} (3\pi^2 n)$$

$$E_{\max} = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$$

(b) Mean kinetic energy

$$<E> = \frac{\int_0^{E_{\max}} E\,dn}{\int_0^{E_{\max}} dn}$$

$$= \frac{\int_0^{E_{\max}} E^{3/2}\,dE}{\int_0^{E_{\max}} E^{1/2}\,dE}$$

$$\frac{(2/5)E_{\max}^{5/2}}{(2/3)E_{\max}^{3/2}} = \frac{3}{5}E_{\max}$$

6.198 The fraction is

$$\eta = \frac{\int_{1/2E_{\max}}^{E_{\max}} E^{1/2}\,dE}{\int_0^{E_{\max}} E^{1/2}\,dE}$$

$$= 1 - 2^{-3/2} = 0.646 = 64.6\%$$

6.199 We calculate the concentration n of electrons in the Na metal from

$$E_{\max} = E_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$$

Using Fermi level, $E_F = 3.07$ eV, we get $n = 2.447 \times 10^{22}$ cm^{-3}.

From this we get the number of electrons per Na atom as

$$\frac{n}{\rho} \cdot \frac{M}{N_A}$$

where ρ = density of Na, M = molar weight in grams of Na, N_A = Avogadro number. Substituting values, we get 0.963 electrons per Na atom.

6.200 The mean K.E. of electrons in a Fermi gas is $(3/5)E_F$. This must equal $(3/2)kT$. Thus,

$$T = \frac{2E_F}{5k}$$

We calculate E_F first.

For Cu $$n = \frac{N_A}{M/\rho} = \frac{\rho N_A}{M} = 8.442 \times 10^{22}\ \text{cm}^{-3}$$

Then $E_F = 7.01$ eV and $T = 3.25 \times 10^4$ K.

6.201 We write the expression for the number of electrons as

$$dN = \frac{V\sqrt{2}m^{3/2}}{\pi^2\hbar^3} E^{1/2}\, dE$$

Hence, if ΔE is the spacing between neighboring levels near the Fermi level, we must have

$$2 = \frac{V\sqrt{2}m^{3/2}}{\pi^2\hbar^3} E_F^{1/2}\, \Delta E$$

(Number 2 is used on the L.H.S. is to take care of both spins of electrons.)

Thus, $$\Delta E = \frac{\sqrt{2}\pi^2\hbar^3}{Vm^{3/2}E_F^{1/2}}$$

But $$E_F^{1/2} = \frac{\hbar}{\sqrt{2}m^{1/2}}(3\pi^2 n)^{1/3}$$

So, $$\Delta E = \frac{2\pi^2\hbar^2}{mV(3\pi^2 n)^{1/3}}$$

Substituting the data, we get $\Delta E = 1.79 \times 10^{-22}$ eV.

6.202 (a) From $$dn(E) = \frac{\sqrt{2}m^{3/2}}{\pi^2\hbar^3} E^{1/2}\, dE$$

On using $E = (1/2)mv^2$ and $mv^2 dn(E) = dn(v)$, we get

$$dn(v) = \frac{\sqrt{2}m^{3/2}}{\pi^2\hbar^3}\frac{1}{\sqrt{2}} m^{1/2}\, v\, mv\, dv = \frac{m^3}{\pi^2\hbar^3} v^2 dv$$

This holds for $0 < v < v_F$, where $(1/2)mv_F^2 = E_F$ and $dn(v) = 0$ for $v > v_F$.

(b) Mean velocity is

$$<v> = \frac{\int_0^{v_F} v^3 dv}{\int_0^{v_F} v^2 dv} = \frac{3}{4} v_F$$

Thus, $$\frac{<v>}{v_F} = \frac{3}{4}$$

6.203 Using the formula of the previous section

$$dn(v) = \frac{m^3}{\pi^2 \hbar^3} v^2 dv$$

We put $$mv = \frac{2\pi\hbar}{\lambda} \quad \text{(where } \lambda = \text{de Broglie wavelength)}$$

Then $$mdv = -\frac{2\pi\hbar}{\lambda^2} d\lambda$$

Taking account of the fact that λ decreases when v increases, we write

$$dn(\lambda) = -dn(v) = \frac{(2\pi)^3 d\lambda}{\pi^3 \lambda^4} = \frac{8\pi}{\lambda^4} d\lambda$$

6.204 From the kinetic theory of gases we know

$$p = \frac{2}{3}\frac{U}{V}$$

Here U is the total internal energy of the gas. This result is applicable to Fermi gas also. Now at $T = 0$, $U = U_0 = N < E > = nV < E >$.

So, $$p = \frac{2}{3} n <E>$$

$$= \frac{2}{3} n \times \frac{3}{5} E_F = \frac{2}{5} nE_F$$

$$= \frac{\hbar^2}{5m} (3\pi^2)^{2/3} n^{5/3}$$

Substituting the values, we get $p = 4.92 \times 10^4$ atm.

6.205 From Richardson's equation

$$I = aT^2 e^{-A/kT}$$

where A is the work function in eV. When T increases by ΔT, I increases to $(1 + \eta)I$.

Then $$1 + \eta = \left(\frac{T + \Delta T}{T}\right)^2 e^{-\frac{A}{kT}\left(\frac{T}{T+\Delta T} - 1\right)} = \left(1 + \frac{\Delta T}{T}\right)^2 e^{+\frac{A}{kT}\cdot\frac{\Delta T}{T+\Delta T}}$$

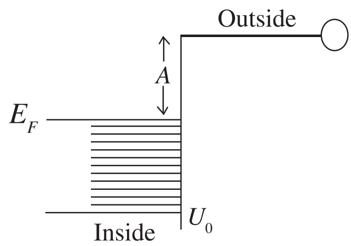
Expanding and neglecting higher powers of $\Delta T/T$, we get

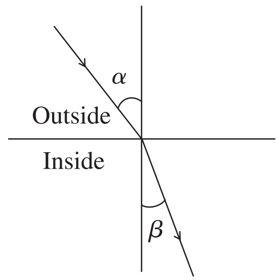
$$\eta = 2\frac{\Delta T}{T} + \frac{A}{kT^2}\Delta T$$

Thus, $$A = kT\left(\frac{\eta T}{\Delta T} - 2\right)$$

Substituting we get, $A = 4.48$ eV.

6.206





The potential energy inside the metal is $-U_0$ for the electron and is related to the work function A by

$$U_0 = E_F + A \tag{1}$$

If T is the K.E. of electrons outside the metal, their K.E. inside the metal will be $(E + U_0)$. On entering the metal, electron cannot experience any tangential force so the tangential component of momentum is unchanged. Then

$$\sqrt{2mT}\,\sin\alpha = \sqrt{2m(T + U_0)}\,\sin\beta$$

Hence, $$\frac{\sin\alpha}{\sin\beta} = \sqrt{1 + \frac{U_0}{T}} = n \quad \text{(by definition of refractive index)} \tag{2}$$

In sodium with one free electron per Na atom, $n = 2.54 \times 10^{22}\ \text{cm}^{-3}$, $E_F = 3.15$ eV and $A = 2.27$ eV (from table).

Using these values, we get

$$U_0 = 5.42 \text{ eV} \quad \text{(using Eq. 1)}$$

and $$n = 1.02 \quad \text{(using Eq. 2)}$$

6.207 In a pure (intrinsic) semiconductor, the conductivity is related to the temperature very closely by the following formula

$$\sigma = \sigma_0 e^{-\Delta\varepsilon/2kT}$$

where $\Delta\varepsilon$ is the energy gap between the top of valence band and the bottom of conduction band; it is also the minimum energy required for the formation of electron–hole pair. The conductivity increases with temperature and we have

$$\eta = e^{+\frac{\Delta\varepsilon}{2k}\left(\frac{1}{T_1} - \frac{1}{T_2}\right)}$$

or
$$\ln \eta = \frac{\Delta\varepsilon}{2k} \frac{T_2 - T_1}{T_1 T_2}$$

Hence,
$$\Delta\varepsilon = \frac{2kT_1T_2}{T_2 - T_1} \ln \eta$$

Substitution gives $\Delta\varepsilon = 0.333 \text{ eV} = E_{\min}$.

6.208 The photoelectric threshold determines the band gap $\Delta\varepsilon$ by

$$\Delta\varepsilon = \frac{2\pi\hbar c}{\lambda_{\text{th}}}$$

On the other hand, the temperature coefficient of resistance is defined by

$$\alpha = \frac{1}{\rho}\frac{d\rho}{dT} = \frac{d}{dT}\ln \rho = -\frac{d}{dT}\ln \sigma$$

where ρ is the resistivity and σ is the conductivity. But

$$\ln \sigma = \ln \sigma_0 - \frac{\Delta\varepsilon}{2kT}$$

Then,
$$\alpha = -\frac{\Delta\varepsilon}{2kT^2} = -\frac{\pi\hbar c}{kT^2\lambda_{\text{th}}} = -0.047 \text{ K}^{-1}$$

6.209 At high temperatures (small values of $1/T$) most of the conductivity is intrinsic, i.e., it is due to the transition of electrons from the upper levels of the valence band into the lower levels of conduction bands.

For this we can apply the formula

$$\sigma = \sigma_0 \exp\left(-\frac{E_g}{2kT}\right)$$

or
$$\ln \sigma = \ln \sigma_0 - \frac{E_g}{2kT}$$

From this we get the band gap

$$E_g = -2k\frac{\Delta \ln \sigma}{\Delta(1/T)}$$

The slope must be calculated at small $1/T$. Evaluation gives the slope

$$-\frac{\Delta \ln \sigma}{\Delta(1/T)} = 7000\,\mathrm{K}$$

Hence, $$E_g = 1.21 \text{ eV}$$

At low temperatures (high values of $1/T$) the conductance is mostly due to impurities. If E_0 is the ionization energy of donor levels then we can write the approximate formula (valid at low temperature)

$$\sigma' = \sigma_0' \exp\left(-\frac{E_0}{2kT}\right)$$

So, $$E_0 = -2kT\,\frac{\Delta \ln \sigma'}{\Delta(1/T)}$$

The slope must be calculated at low temperature. Evaluation gives the slope

$$-\frac{\Delta \ln \sigma'}{\Delta(1/T)} = \frac{1}{3} \times 1000 \text{ K}$$

Then, $$E_0 \sim 0.057 \text{ eV}$$

6.210 We write the conductivity of the sample as $\sigma = \sigma_i + \sigma_\gamma$, where σ_i = intrinsic conductivity and σ_γ is the photo conductivity. At $t = 0$, assuming saturation, we have

$$\frac{1}{\rho_1} = \frac{1}{\rho} + \sigma_{\gamma 0} \quad \text{or} \quad \sigma_{\gamma 0} = \frac{1}{\rho_1} - \frac{1}{\rho}$$

At time t after light source is switched off, we have, because of recombination of electron and holes in the sample

$$\sigma = \sigma_i + \sigma_{\gamma 0}\, e^{-t/\tau}$$

where τ = mean lifetime of electrons and holes.

Thus, $$\frac{1}{\rho_2} = \frac{1}{\rho} + \left(\frac{1}{\rho_1} - \frac{1}{\rho}\right) e^{-t/\tau}$$

$$\frac{1}{\rho_2} - \frac{1}{\rho} = \left(\frac{1}{\rho_1} - \frac{1}{\rho}\right) e^{-t/\tau}$$

or $$e^{t/\tau} = \frac{\dfrac{1}{\rho_1} - \dfrac{1}{\rho}}{\dfrac{1}{\rho_2} - \dfrac{1}{\rho}} = \frac{\rho_2(\rho - \rho_1)}{\rho_1(\rho - \rho_2)}$$

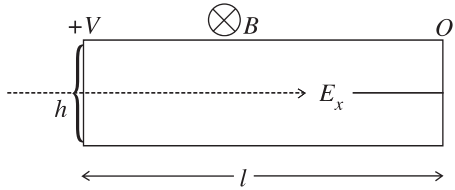
Hence, $$\tau = \frac{t}{\ln\left\{\frac{\rho_2(\rho - \rho_1)}{\rho_1(\rho - \rho_2)}\right\}}$$

Substitution gives $\tau = 9.87$ ms ~ 0.01 s.

6.211 We shall ignore minority carriers.

Drifting holes experience a sideways force in the magnetic field and react by setting up a Hall electric field E_y to counterbalance it. Thus

$$v_x B = E_y = \frac{V_H}{h}$$



If the concentration of carriers is n, then

$$j_x = nev_x$$

Hence, $$n = \frac{j_x}{ev_x} = \frac{j_x/eV_H}{hB} = \frac{j_x hB}{eV_H}$$

Also, using $$j_x = \sigma E_x = \frac{E_x}{\rho} = \frac{V}{\rho l}$$

we get $$n = \frac{VhB}{e\rho l V_H}$$

Substituting the data (note that in MKS unit $B = 5.0$ kG $= 0.5$ T, $\rho = 2.5 \times 10^{-2}$ ohm-m), we get

$$n = 4.99 \times 10^{21}\ \text{m}^{-3}$$
$$= 4.99 \times 10^{15}\ \text{cm}^{-3}$$

Also, the mobility is $$u_0 = \frac{v_x}{E_x} = \frac{V_H}{\hbar B} \times \frac{l}{V} = \frac{V_H l}{hBV}$$

Substitution gives $u_0 = 0.05$ m^2/V-s.

6.212 If an electric field E_x is present in a sample containing equal amounts of both electrons and holes, the two drift in opposite directions.

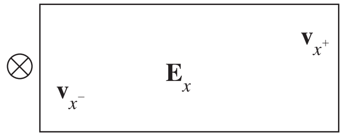
In the presence of a magnetic field $B_Z = B$, they set up Hall voltages in opposite directions.

The net Hall electric field is given by

$$E_y = (v_x^+ - v_x^-)B$$

$$= (u_0^+ - u_0^-)E_x B$$

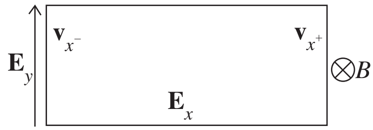
But $$\frac{E_y}{E_x} = \frac{1}{\eta}$$



Hence, $$|u_0^+ - u_0^-| = \frac{1}{\eta B}$$

Substitution gives $|u_0^+ - u_0^-| = 0.2$ m²/V-s.

6.213 When the sample contains unequal number of carriers of both types whose mobilities are different, static equilibrium (i.e., no transverse movement of either electron or holes) is impossible in a magnetic field. The transverse electric field acts differently on electrons and holes. If the E_y that is set up is as shown in the figure, the net Lorentz force per unit charge (effective transverse electric field) on electrons is



$$E_y - v_x^- B$$

and on holes is

$$E_y + v_x^+ B$$

(We are assuming that $B = B_Z$.) There is then a transverse drift of electrons and holes and the net transverse current must vanish in equilibrium. Using mobility

$$u_0^- N_e e(E_y - u_0^- E_x B) + N_h e u_0^+ (E_y + u_0^+ E_x B) = 0$$

or $$E_y = \frac{N_e u_0^{-2} - N_h u_0^{+2}}{N_e u_0^- + N_h u_0^+} E_x B$$

On the other hand

$$j_x = (N_e u_0^- + N_h u_0^+) e E_x$$

Thus, the Hall coefficient is

$$R_H = \frac{E_y}{j_x B} = \frac{1}{e} \frac{N_e u_0^{-2} - N_h u_0^{+2}}{(N_e u_0^- + N_h u_0^+)^2}$$

We see that $R_H = 0$ when

$$\frac{N_e}{N_h} = \left(\frac{u_0^+}{u_0^-}\right)^2 = \frac{1}{\eta^2} = \frac{1}{4}$$

Thus, $$\eta^2 = 4$$

6.5 Radioactivity

6.214 (a) The probability of survival (i.e., not decaying) in time t is $e^{-\lambda t}$. Hence, the probability of decay is $1 - e^{-\lambda t}$.

(b) The probability that the particle decays in time dt around time t is the difference

$$e^{-\lambda t} - e^{-\lambda(t + dt)} = e^{-\lambda t}[1 - e^{-e\lambda dt}] = \lambda e^{-\lambda t} dt$$

Therefore, the mean lifetime is

$$\tau = \frac{\int_0^\infty t\lambda e^{-\lambda t}\, dt}{\int_0^\infty \lambda e^{-\lambda t}\, dt} = \frac{\frac{1}{\lambda}\int_0^\infty xe^{-x}\, dx}{\int_0^\infty e^{-x} dx} = \frac{1}{\lambda}$$

6.215 We calculate λ first

$$\lambda = \frac{\ln 2}{T_{1/2}} = 9.722 \times 10^{-3} \text{ per day}$$

Hence, fraction decaying in a month $= 1 - e^{-\lambda t} = 0.253$.

6.216 Here

$$N_0 = \frac{1\mu g}{24 g} \times 6.023 \times 10^{23} = 2.51 \times 10^{16}$$

Also

$$\lambda = \frac{\ln 2}{T_{1/2}} = 0.04621 \text{ h}^{-1}$$

So the number of β rays emitted in one hour is

$$N_0(1 - e^{-\lambda}) = 1.13 \times 10^{15}$$

6.217 If N_0 is the number of radionuclei present initially, then

$$N_1 = N_0(1 - e^{-t_1/\tau})$$

and

$$\eta N_1 = N_0(1 - e^{-t_2/\tau})$$

where $\eta = 2.66$ and $t_2 = 3t_1$. Then,

$$\eta = \frac{1 - e^{-t_2/\tau}}{1 - e^{t_1/\tau}}$$

or

$$\eta - \eta e^{-t_1/\tau} = 1 - e^{-t_2/\tau}$$

Substituting the values

$$1.66 = 2.66 e^{-2/\tau} - e^{-6/\tau}$$

Putting $e^{-2/\tau} = x$, we get

$$x^3 - 2.66x + 1.66 = 0$$

$$(x^2 - 1)\,x - 1.66(x - 1) = 0$$

or $$(x - 1)\,(x^2 + x - 1.66) = 0$$

Now $x \neq 1$, so $$x^2 + x - 1.66 = 0$$

or $$x = \frac{-1 \pm \sqrt{1 + 4 \times 1.66}}{2}$$

Negative root has to be rejected because $x > 0$.

Thus, $$x = 0.882$$

This gives $$\tau = \frac{-2}{\ln 0.882} = 15.9 \text{ s}$$

6.218 If the half-life is T days, then

$$(2)^{-7/T} = \frac{1}{2.5}$$

Hence, $$\frac{7}{T} = \frac{\ln 2.5}{\ln 2}$$

or $$T = \frac{7 \ln 2}{\ln 2.5} = 5.30 \text{ days}$$

6.219 The activity is proportional to the number of parent nuclei (assuming that the daughter is not radioactive). In half of its half-life period, the number of parent nuclei decreases by a factor

$$(2)^{-1/2} = \frac{1}{\sqrt{2}}$$

So activity decreases to $650/\sqrt{2} = 460$ particles per minute.

6.220 If the decay constant (per hour) is λ, then the activity after one hour will decrease by a factor $e^{-\lambda}$. Hence,

$$0.96 = e^{-\lambda}$$

or $$\lambda = 1.11 \times 10^{-5} \text{ s}^{-1} = 0.0408 \text{ h}^{-1}$$

The mean lifetime is 24.5 h.

6.221 Here $$N_0 = \frac{1}{238} \times 6.023 \times 10^{23}$$

$$= 2.531 \times 10^{21}$$

The activity is $$A = 1.24 \times 10^4 \text{ dis/s}$$

Then $$\lambda = \frac{A}{N_0} = 4.90 \times 10^{-18}\ \text{s}^{-1}$$

Hence, the half-life is $$T = \frac{\ln 2}{\lambda} = 4.49 \times 10^9 \text{ years}$$

6.222 In old wooden atoms, the number of C^{14} nuclei steadily decreases because of radioactive decay. (In live trees biological processes keep replenishing C^{14} nuclei, thus maintaining a balance. This balance starts getting disrupted as soon as the tree is felled.)

If T is the half-life of C^{14}, then

$$e^{-t \times \frac{\ln 2}{T}} = \frac{3}{5}$$

Hence, $$t = T\frac{\ln 5/3}{\ln 2}$$

$$= 4105 \text{ years} \approx 4.1 \times 10^3 \text{ years}$$

6.223 The problem indicates that in the time since the ore was formed, U^{238} nuclei that have remained undecayed is

$$\frac{\eta}{1+\eta}$$

Thus, $$\frac{\eta}{1+\eta} = e^{-t \times \frac{\ln 2}{T}}$$

or $$t = T\frac{\ln\left(\dfrac{1+\eta}{\eta}\right)}{\ln 2}$$

Substituting $T = 4.5 \times 10^9$ years, $\eta = 2.8$ we get, $t = 1.98 \times 10^9$ years.

6.224 The specific activity of Na^{24} is

$$\lambda\frac{N_A}{M} = \frac{N_A \ln 2}{MT} = 3.22 \times 10^{17} \text{ dis / (g-s)}$$

Here M = molar weight of $Na^{24} = 24$ g, N_A is Avogadro number and T is the half-life of Na^{24}. Similarly, the specific activity of U^{235} is

$$\frac{6.023 \times 10^{23} \times \ln 2}{235 \times 10^{8} \times 365 \times 86400} = 0.793 \times 10^{5} \text{ dis/(g-s)}$$

6.225 Let V = volume of blood in the body of the human being. Then the total activity of the blood is $A'V$. Assuming all this activity is due to the injected Na^{24} and taking account of the decay of this radionuclide, we get

$$VA' = Ae^{-\lambda t}$$

Now $$\lambda = \frac{\ln 2}{15} \text{ h}^{-1} \quad (\text{where } t = 5 \text{ h})$$

Thus, $$V = \frac{A}{A'} e^{-\ln 2/3} = \frac{2.0 \times 10^{3}}{(16/60)} e^{-\ln 2/3} \text{ cm}^{3} = 5.95 \text{ l}$$

6.226 We see that the specific activity of the sample, i.e., activity of M grams of Co^{58} in the sample is

$$\frac{1}{M + M'}$$

Here M and M' are the masses of Co^{58} and Co^{59} in the sample. Now activity of M grams of Co^{58} is

$$\frac{M}{58} \times 6.023 \times 10^{23} \times \frac{\ln 2}{71.3 \times 86400} \text{ dis/s}$$

$$= 1.168 \times 10^{15} M$$

Thus, from the problem

$$1.168 \times 10^{15} \frac{M}{M + M'} = 2.2 \times 10^{12}$$

or $$\frac{M}{M + M'} = 1.88 \times 10^{-3} \times 100 = 0.188\%$$

6.227 Suppose N_1, N_2 are the initial numbers of component nuclei whose decay constants are λ_1, λ_2 (per hour). Then the activity at any instant is

$$A = \lambda_1 N_1 e^{-\lambda_1 t} + \lambda_2 N_2 e^{-\lambda_2 t}$$

The activity so defined is in units dis/h. We assume that data ln A given is of its natural logarithm. The daughter nuclei are assumed non-radioactive.

We see from the data that at large t, the change ln A per hour of elapsed time is constant and equal to -0.07.

Thus, $$\lambda_2 = 0.07 \text{ h}^{-1}$$

We can then see that the best-fit data is obtained by

$$A(t) = 51.1\ e^{-0.66t} + 10.0\ e^{-0.07t}$$

[To get the best-fit we calculate $A(t)\ e^{0.07t}$. We see that it reaches the constant value 10.0 at $t = 7, 10, 14, 20$ h very nearly. This fixes the second term. The first term is then obtained by subtracting out the constant value 10.0 from each value of $A(t)e^{0.07t}$ in the data for small t.]

Thus, we get $\qquad \lambda_1 = 0.66\ \text{h}^{-1}$

$$\left.\begin{aligned} T_1 &= 1.05\ \text{h} \\ T_2 &= 9.9\ \text{h} \end{aligned}\right\}\text{half-lives}$$

Ratio of radioactive nuclei is given by

$$\frac{N_1}{N_2} = \frac{51.1}{10.0} \times \frac{\lambda_2}{\lambda_1} = 0.54$$

The answer given in the book is incorrect.

6.228 Production of the nucleus is governed by the equation

$$\frac{dN}{dt} = \underset{\substack{\uparrow \\ \text{supply}}}{g} - \underset{\text{decay}}{\lambda N} \tag{1}$$

We see that N will approach a constant value g/λ. This can also be proved directly. So, we multiply Eq. (1) by $e^{\lambda t}$ and write

$$\frac{dN}{dt} e^{\lambda t} + \lambda e^{\lambda t} N = g e^{\lambda t}$$

Then, $$\frac{d}{dt}(N e^{\lambda t}) = g e^{\lambda t}$$

or $$N e^{\lambda t} = \frac{g}{\lambda} e^{\lambda t} + \text{constant}$$

At $t = 0$ when the production is started, $N = 0$.

So, $$0 = \frac{g}{\lambda} + \text{constant}$$

Hence, $$N = \frac{g}{\lambda}(1 - e^{-\lambda t})$$

Now the activity is

$$A = \lambda N = g\,(1 - e^{-\lambda t})$$

From the problem $$\frac{1}{2.7} = 1 - e^{\lambda t}$$

This gives $$\lambda t = 0.463$$

so, $$t = \frac{0.463}{\lambda} = \frac{0.463 \times T}{0.693} = 9.5 \text{ days}$$

Algebraically $$t = -\frac{T}{\ln 2}\ln\left(1 - \frac{A}{g}\right)$$

6.229 (a) Suppose N_1 and N_2 are the number of two radionuclides A_1, A_2 at time t. Then

$$\frac{dN_1}{dt} = -\lambda_1 N_1 \tag{1}$$

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2 \tag{2}$$

From Eq. (1) $$N_1 = (N_1)_0 e^{\lambda_1 t}$$

where $(N_1)_0$ is the initial number of nuclides A_1 at time $t = 0$.

From Eq. (2)

$$\left(\frac{dN_2}{dt} + \lambda_2 N_2\right)e^{\lambda_2 t} = \lambda_1 (N_1)_0 e^{-(\lambda_1 - \lambda_2)t}$$

or $$(N_2 e^{\lambda_2 t}) = \text{constant} - \frac{\lambda_1 (N_1)_0}{\lambda_1 - \lambda_2} e^{-(\lambda_1 - \lambda_2)t}$$

Since $N_2 = 0$ at $t = 0$,

$$\text{constant} = \frac{\lambda_1 (N_1)_0}{\lambda_1 - \lambda_2}$$

Thus, $$N_2 = \frac{\lambda_1 (N_1)_0}{\lambda_1 - \lambda_2}(e^{-\lambda_1 t} - e^{-\lambda_2 t})$$

(b) The activity of radionuclide A_2 is $\lambda_2 N_2$. This is maximum when N_2 is maximum. That happens when $dN_2/dt = 0$.

This requires

$$\lambda_2 e^{-\lambda_2 t} = \lambda_1 e^{-\lambda_1 t}$$

or $$t = \frac{\ln(\lambda_1/\lambda_2)}{\lambda_1 - \lambda_2}$$

6.230 (a) This case can be obtained from the previous one on putting

$$\lambda_2 = \lambda_1 - \varepsilon$$

where ε is very small and letting $\varepsilon \to 0$ at the end. Then

$$N_2 = \frac{\lambda_1 (N_1)_0}{\varepsilon}(e^{\varepsilon t} - 1)e^{-\lambda_1 t} = \lambda_1 t e^{-\lambda_1 t}(N_1)_0$$

On dropping the subscript 1 as the two values are equal, we get

$$N_2 = (N_1)_0 \lambda t e^{-\lambda t}$$

(b) This is maximum when

$$\frac{dN_2}{dt} = 0 \quad \text{or} \quad t = \frac{1}{\lambda}$$

6.231 Here we have the equations

$$\frac{dN_1}{dt} = -\lambda_1 N_1$$

$$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2$$

and $$\frac{dN_3}{dt} = \lambda_2 N_2$$

From Problem 6.229 $$N_1 = (N_1)_0 e^{-\lambda_1 t}$$

and $$N_2 = \frac{\lambda_1 (N_1)_0}{\lambda_1 - \lambda_2}(e^{-\lambda_2 t} - e^{-\lambda_1 t})$$

Then $$\frac{dN_3}{dt} = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2}(N_1)_0 (e^{-\lambda_2 t} - e^{-\lambda_1 t})$$

or $$N_3 = \text{constant} - \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2}\left(\frac{e^{-\lambda_2 t}}{\lambda_2} - \frac{e^{-\lambda_1 t}}{\lambda_1}\right)(N_1)_0$$

Since $N_3 = 0$ initially,

$$\text{constant} = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_1}(N_1)_0\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)$$

So, $$N_3 = \frac{\lambda_1 \lambda_2 (N_1)_0}{\lambda_1 - \lambda_2}\left[\frac{1}{\lambda_2}(1 - e^{-\lambda_2 t}) - \frac{1}{\lambda_1}(1 - e^{-\lambda_1 t})\right]$$

$$= (N_1)_0\left[1 + \frac{\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t}}{\lambda_2 - \lambda_1}\right]$$

6.232 We have the chain

$$\begin{array}{ccccc} Bi^{210} & \xrightarrow[\lambda_1]{\beta} & Po^{210} & \xrightarrow[\lambda_2]{\alpha} & Pb^{206} \\ A_1 & & A_2 & & A_3 \end{array}$$

of the previous problem. Initially

$$(N_1)_0 = \frac{10^{-3}}{210} \times 6.023 \times 10^{23} = 2.87 \times 10^{18}$$

A month after preparation

$$N_1 = 4.54 \times 10^{16}$$

$$N_2 = 2.52 \times 10^{18}$$

using the results of the previous problem.

Then $$A_\beta = \lambda_1 N_1 = 0.725 \times 10^{11} \text{ dis/s}$$

$$A_\alpha = \lambda_2 N_2 = 1.46 \times 10^{11} \text{ dis/s}$$

6.233 (a) For Ra^{226}, $Z = 88$ and $A = 226$.

After 5α and 4β (electron) emissions,

$$A = 226 - 20 = 206$$

$$Z = 88 + 4 - 5 \times 2 = 82$$

The product is ${}_{82}Pb^{206}$.

(b) We require $$-\Delta Z = 10 = 2n - m$$

$$-\Delta A = 32 = n \times 4$$

Here n = number of α emissions and m = number of β emissions.

Thus, $n = 8$ and $m = 6$.

6.234 The momentum of the α-particle is $\sqrt{2M_\alpha T}$. This is also the recoil momentum of the daughter nucleus in the opposite direction. The recoil velocity of the daughter nucleus is

$$\frac{\sqrt{2M_\alpha T}}{M_d}$$

$$= \frac{2}{196}\sqrt{\frac{2T}{M_p}} = 3.39 \times 10^5 \text{ m/s}$$

The energy of the daughter nucleus is $(M_\alpha / M_d)T$ and this represents a fraction of total energy given by

$$\frac{M_\alpha/M_d}{1+M_\alpha/M_d} = \frac{M_\alpha}{M_\alpha + M_d} = \frac{4}{200} = \frac{1}{50} = 0.02$$

(Here M_d is the mass of the daughter nucleus.)

6.235 The number of nuclei initially present is

$$\frac{10^{-3}}{210} \times 6.023 \times 10^{23} = 2.87 \times 10^{18}$$

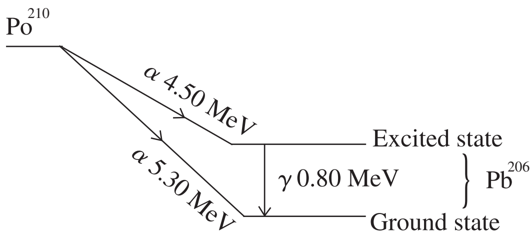
In the mean lifetime of these nuclei, the number decaying is the fraction

$$1 - \frac{1}{e} = 0.632$$

Thus, the energy released is

$$2.87 \times 10^{18} \times 0.632 \times 5.3 \times 1.602 \times 10^{-13}\ \text{J} = 1.54\ \text{MJ}$$

6.236 We neglect all recoil effects. Then the figure below, gives the energy of the γ-ray quanta emitted, which is 0.80 MeV.



6.237 (a) For an α-particle with initial K.E. 7.0 MeV, the initial velocity is

$$v_0 = \sqrt{\frac{2T}{M_\alpha}}$$

$$= \sqrt{\frac{2 \times 7 \times 1.602 \times 10^{-6}}{4 \times 1.672 \times 10^{-24}}}$$

$$= 1.83 \times 10^9\ \text{cm/s}$$

Thus, $$R = 6.02\ \text{cm}$$

(b) Over the whole path, the number of ion pairs is

$$\frac{7 \times 10^6}{34} = 2.06 \times 10^5$$

Over the first half of the path, we write the formula for the mean path as $R \propto E^{3/2}$ where E is the initial energy. Thus, if the energy of the α-particle after traversing the first half of the path is E_1, then,

$$R_0 E_1^{3/2} = \frac{1}{2} R_0 E_0^{3/2} \quad \text{or} \quad E_1 = 2^{-2/3} E_0$$

Hence, number of ion pairs formed in the first half of the path length is

$$\frac{E_0 - E_1}{34 \text{ eV}} = (1 - 2^{-2/3}) \times 2.06 \times 10^5 = 0.76 \times 10^5$$

6.238 In β^- decay

$$_Z X^A \to {}_{Z+1} Y^A + e^- + Q$$

$$Q = (M_X - M_Y - m_e)\, c^2$$

$$= [(M_X + Zm_e) - (M_Y + Zm_e + m_e)]c^2$$

$$= (M_p - M_d)\, c^2$$

Since M_p, M_d are the masses of the parent and daughter atoms, the binding energy of the electrons is ignored.

In K capture,

$$e_K^- + {}_Z X^A \to {}_{Z-1} Y^A + Q$$

$$Q = (M_X - M_Y)\, c^2 + m_e c^2$$

$$= (M_X c^2 + Zm_e c^2) - (M_Y c^2 + (Z-1) m_e c^2)$$

$$= c^2 (M_p - M_d)$$

In β^+ decay

$$_Z X^A \to {}_{Z-1} Y^A + e^+ + Q$$

Then

$$Q = (M_X - M_Y - m_e)\, c^2$$

$$= [M_X + Zm_e]c^2 - [M_Y + (Z-1)\, m_e]c^2 - 2m_e c^2$$

$$= (M_p - M_d - 2m_e)c^2$$

6.239 The reaction is

$$\text{Be}^{10} \to \text{B}^{10} + e^- + \overline{\nu}_e$$

For maximum K.E. of electrons we can assume the energy of neutrino ($\overline{\nu}_e$) to be zero. The atomic masses are

$$\text{Be}^{10} = 10.016711 \text{ amu}$$

$$B^{10} = 10.016114 \text{ amu}$$

So the K.E. of electrons is (see previous problem)

$$0.597 \times 10^{-6} \text{ amu} \times c^2 = 0.56 \text{ MeV}$$

The momentum of electrons with this K.E. is $0.941 \text{ MeV}/c$ and the recoil energy of the daughter nucleus is

$$\frac{(0.941)^2}{2 \times M_d c^2} = \frac{(0.941)^2}{2 \times 10 \times 938} \text{MeV} = 47.2 \text{ eV}$$

6.240 The masses are

$$Na^{24} = 24 - 0.00903 \text{ amu} \quad \text{and} \quad Mg^{24} = 24 - 0.01496 \text{ amu}$$

The reaction is

$$Na^{24} \rightarrow Mg^{24} + e^- + \bar{\nu}_e$$

The maximum K.E. of electrons is

$$0.00593 \times 931 \text{ MeV} = 5.52 \text{ MeV}$$

Average K.E. according to the problem is then $5.52/3 = 1.84$ MeV.

The initial number of Na^{24} is

$$\frac{10^{-3} \times 6.023 \times 10^{23}}{24} = 2.51 \times 10^{19}$$

The fraction decaying in a day is

$$1 - (2)^{-24/15} = 0.67$$

Hence the heat produced in a day is

$$0.67 \times 2.51 \times 10^{19} \times 1.84 \times 1.602 \times 10^{-13} \text{ J} = 4.95 \text{ MJ}$$

6.241 We assume that the parent nucleus is at rest. Then since the daughter nucleus does not recoil, we have

$$\mathbf{p} = -\mathbf{p}_\nu$$

i.e., momenta of positron and neutrino (ν) are equal and opposite. On the other hand, total energy released (including rest energy) is

$$\sqrt{c^2 p^2 + m_e^2 c^4} + cp = Q$$

(Here we have used the fact that energy of the neutrino is $c|\mathbf{p}_\nu| = cp$.)

Now, $$Q = [(\text{Mass of } C^{11} \text{ nucleus}) - (\text{Mass of } B^{11} \text{ nucleus})]c^2$$

$$= [\text{Mass of C}^{11}\text{ atom} - \text{Mass of B}^{11}\text{ atom} - m_e]c^2$$

$$= (0.00213 \times 931 - 0.511)\text{ MeV} = 1.47\text{ MeV}$$

Then $$c^2p^2 + (0.511)^2 = (1.47 - cp)^2$$

$$= (1.47)^2 - 2.94\, cp + c^2p^2$$

Thus, $$cp = 0.646\text{ MeV} = \text{energy of neutrino}$$

Also K.E. of electron is

$$1.47 - 0.646 - 0.511 = 0.313\text{ MeV}$$

6.242 The K.E. of the positron is maximum when the energy of neutrino is zero. Since the recoil energy of the nucleus is quite small, it can be calculated by successive approximation. The reaction is

$$\text{N}^{13} \rightarrow \text{C}^{13} + e^+ + \nu_e$$

The maximum energy available to the positron (including its rest energy) is

$$c^2\,(\text{Mass of N}^{13}\text{ nucleus} - \text{Mass of C}^{13}\text{ nucleus})$$

$$= c^2\,(\text{Mass of N}^{13}\text{ atom} - \text{Mass of C}^{13}\text{ atom} - m_e)$$

$$= 0.00239\, c^2 - m_ec^2$$

$$= (0.00239 \times 931 - 0.511)\text{ MeV}$$

$$= 1.71\text{ MeV}$$

The momentum corresponding to this energy is 1.636 MeV/c. The recoil energy of the nucleus is then

$$E = \frac{p^2}{2M} = \frac{(1.636)^2}{2 \times 13 \times 931} = 111\text{ eV} = 0.111\text{ keV}$$

(on using $Mc^2 = 13 \times 931$ MeV).

6.243 The process is

$$e_K^- + \text{Be}^7 \rightarrow \text{Li}^7 + \nu$$

The energy available in the process is

$$Q = c^2\,(\text{Mass of Be}^7\text{ atom} - \text{Mass of Li}^7\text{ atom})$$

$$= 0.00092 \times 931\text{ MeV} = 0.86\text{ MeV}$$

The momentum of a K electron is negligible. So in the rest frame of the Be^7 atom, most of the energy is taken by the neutrino whose momentum is very nearly

0.86 MeV/c. The momentum of the recoiling nucleus is equal and opposite. The velocity of recoil is

$$\frac{0.86 \text{ MeV/c}}{M_{\text{Li}}} = c \times \frac{0.86}{7 \times 931} = 3.96 \times 10^6 \text{ cm/s}$$

6.244 In internal conversion, the total energy is used to knock out K electrons. The K.E. of these electrons is the difference of energy available and binding energy of K electrons, i.e.,

$$(87 - 26) = 61 \text{ keV}$$

The total energy including rest mass of electrons is 0.511 + 0.061 = 0.572 MeV.

The momentum corresponding to this total energy is

$$\sqrt{(0.572)^2 - (0.511)^2}\,/c = 0.257 \text{ MeV}/c$$

The velocity is then $\qquad \dfrac{c^2 p}{E} = c \times \dfrac{0.257}{0.572} = 0.449\, c$

6.245 With recoil neglected, the γ-particle will have 129 keV energy. To a first approximation, its momentum will be 129 keV/c and the energy of recoil will be

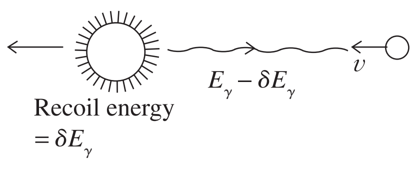
$$\frac{(0.129)^2}{2 \times 191 \times 931} \text{ MeV} = 4.18 \times 10^{-8} \text{ MeV}$$

In the next approximation we therefore write

$$E_\gamma \approx 0.129 - 8.2 \times 10^{-8} \text{ MeV}$$

So, $\qquad \dfrac{\delta E_\gamma}{E_\gamma} = 3.63 \times 10^{-7}$

6.246 For maximum (resonant) absorption, the absorbing nucleus must be moving with enough speed to cancel the momentum of the oncoming photon and have just the right amount of energy (ε = 129 keV) available for transition to the excited state.



Since $\delta E_\gamma \approx \varepsilon^2/2Mc^2$ and momentum of photon is ε/c, these conditions can be satisfied if the velocity of the nucleus is

$$\frac{\varepsilon}{Mc} = c\frac{\varepsilon}{Mc^2} = 218 \text{ m/s} = 0.218 \text{ km/s}$$

6.247 Because of the gravitational shift, the frequency of the γ-ray at the location of the absorber is increased by

$$\frac{\delta\omega}{\omega} = \frac{gh}{c^2}$$

For this to be compensated by the Doppler shift (assuming that resonant absorption is possible in the absence of gravitational field), we must have

$$\frac{gh}{c^2} = \frac{v}{c} \quad \text{or} \quad v = \frac{gh}{c} = 0.65\ \mu\text{m/s}$$

6.248 The natural lifetime is

$$\Gamma = \frac{\hbar}{\tau} = 4.7 \times 10^{-10}\ \text{eV}$$

Thus, the condition $\delta E_\gamma \geq \Gamma$ implies

$$\frac{gh}{c^2} \geq \frac{\Gamma}{\varepsilon} = \frac{\hbar}{\tau\varepsilon}$$

or

$$h \geq \frac{c^2\hbar}{\tau\varepsilon g} = 4.64\ \text{m}$$

(Here h is height of the place, not Planck's constant.)

6.6 Nuclear Reactions

6.249 Initial momentum of the α-particle is $\sqrt{2mT_\alpha}\,\mathbf{i}$ (where $\mathbf{i}$ is a unit vector in the incident direction). Final momenta are, respectively, $\mathbf{p}_\alpha$ and $\mathbf{p}_{\text{Li}}$. Conservation of momentum reads

$$\mathbf{p}_\alpha + \mathbf{p}_{\text{Li}} = \sqrt{2mT_\alpha}\,\mathbf{i}$$

Squaring,

$$p_\alpha^2 + p_{\text{Li}}^2 \cdot 2p_\alpha p_{\text{Li}} \cos\Theta = 2mT_\alpha \qquad (1)$$

(where Θ is the angle between $\mathbf{p}_\alpha$ and $\mathbf{p}_{\text{Li}}$).

Also, by energy conservation, we have

$$\frac{p_\alpha^2}{2m} + \frac{p_{\text{Li}}^2}{2M} = T_\alpha$$

(Here m and M are, respectively, the masses of α-particle and Li^6.)

So,

$$p_\alpha^2 + \frac{m}{M} p_{\text{Li}}^2 = 2mT_\alpha \qquad (2)$$

Subtracting Eq. (2) from Eq. (1) we see that

$$p_{\text{Li}}\left[\left(1-\frac{m}{M}\right)p_{\text{Li}}+2p_\alpha\cos\Theta\right]=0$$

Thus, if $p_{\text{Li}} \neq 0$

$$p_\alpha=-\frac{1}{2}\left(1-\frac{m}{M}\right)p_{\text{Li}}\sec\Theta$$

Since p_α, p_{Li} are both positive numbers (being magnitudes of vectors), we must have

$$-1 \le \cos\Theta < 0 \quad (\text{if } m < M)$$

This being understood, we write

$$\frac{p_{\text{Li}}^2}{2M}\left[1+\frac{M}{4m}\left(1-\frac{m}{M}\right)^2\sec^2\Theta\right]=T_\alpha$$

Hence the recoil energy of the Li nucleus is

$$\frac{p_{\text{Li}}^2}{2M}=\frac{T_\alpha}{1+\dfrac{(M-m)^2}{4mM}\sec^2\Theta}$$

As we pointed out above, $\Theta \neq 60°$. If we take $\Theta = 120°$, we get recoil energy of Li = 6 MeV.

6.250 (a) In a head-on collision

$$\sqrt{2mT}=p_d+p_n$$

$$T=\frac{p_d^2}{2M}+\frac{p_n^2}{2m}$$

(where p_d and p_n are the momenta of deuteron and neutron after the collision). Squaring, we get

$$p_d^2+p_n^2+2p_dp_n=2mT$$

$$p_n^2+\frac{m}{M}p_d^2=2mT$$

Since $p_d \neq 0$ in head-on collisions

$$p_n=-\frac{1}{2}\left(1-\frac{m}{M}\right)p_d$$

Going back to energy conservation

$$\frac{p_d^2}{2M}\left[1+\frac{M}{4m}\left(1-\frac{m}{M}\right)^2\right]=T$$

So,
$$\frac{p_d^2}{2M}=\frac{4mM}{(m+M)^2}T$$

This is the energy lost by neutron. So the fraction of energy lost is

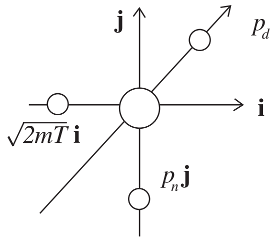
$$\eta=\frac{4mM}{(m+M)^2}=\frac{8}{9}$$

(b) In this case neutron is scattered by 90°. Then we have from the figure

$$\mathbf{p}_d = p_n\mathbf{j}+\sqrt{2mT}\,\mathbf{i}$$

Then, by energy conservation

$$\frac{p_n^2+2mT}{2M}+\frac{p_n^2}{2m}=T$$



or
$$\frac{p_n^2}{2m}\left(1+\frac{m}{M}\right)=T\left(1-\frac{m}{M}\right)$$

or
$$\frac{p_n^2}{2m}=\frac{M-m}{M+m}\cdot T$$

The energy lost by neutron is then

$$T-\frac{p_n^2}{2m}=\frac{2m}{M+m}T$$

or fraction of energy lost is

$$\eta=\frac{2m}{M+m}=\frac{2}{3}$$

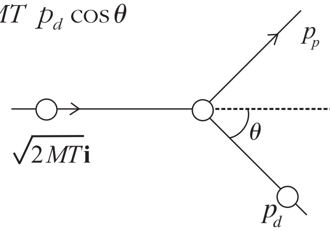
6.251 From conservation of momentum

$$\sqrt{2MT}\,\mathbf{i}=\mathbf{p}_d+\mathbf{p}_p$$

or
$$p_p^2=2MT+p_d^2-2\sqrt{2MT}\,p_d\cos\theta$$

From energy conservation

$$T=\frac{p_d^2}{2M}+\frac{p_p^2}{2m}$$



(where M = mass of deuteron, m = mass of proton).

So,
$$p_p^2=2mT-\frac{m}{M}p_d^2$$

Hence, $$p_d^2\left(1+\frac{m}{M}\right)-2\sqrt{2MT}\,p_d\cos\theta+2(M-m)T=0$$

For real roots $$4(2MT)\cos^2\theta-4\times 2(M-m)T\left(1+\frac{m}{M}\right)\geq 0$$

or $$\cos^2\theta\geq\left(1-\frac{m^2}{M^2}\right)$$

Hence, $$\sin^2\theta\leq\frac{m^2}{M^2}$$

i.e., $$\theta\leq\sin^{-1}\frac{m}{M}$$

For deuteron–proton scattering, $\theta_{max}=30°$.

6.252 This problem has a misprint. Actually the radius R of a nucleus is given by

$$R=1.3\sqrt[3]{A}\text{ fm}$$

(where 1 fm = 10^{-15} m).

Then the number of nucleons per unit volume is

$$\frac{A}{4\pi/3R^3}=\frac{3}{4\pi}\times(1.3)^{-3}\times 10^{+39}\text{ cm}^{-3}=1.09\times 10^{38}\text{ cm}^{-3}$$

The corresponding mass density is

$(1.09\times 10^{38}\times\text{mass of a nucleon})\text{ cm}^{-3}=1.82\times 10^{11}\text{ kg/cm}^3$

6.253 (a) The particle x must carry two nucleons and a unit of positive charge. The reaction is

$$B^{10}\,(d,\alpha)\,Be^8$$

(b) The particle x must contain a proton in addition to the constituents of O^{17}. Thus the reaction is

$$O^{17}\,(d,n)\,F^{18}$$

(c) The particle x must carry nucleon number 4 and two units of $+ve$ charge. Thus the particle must be $x=\alpha$ and the reaction is

$$Na^{23}\,(p,\alpha)\,Ne^{20}$$

(d) The particle x must carry mass number 37 and have one unit less of positive charge. Thus $x=Cl^{37}$ and the reaction is

$$Cl^{37}\,(p,n)\,Ar^{37}$$

6.254 From the basic formula

$$E_b = Zm_{\mathrm{H}} + (A - Z)\, m_n - M$$

We define $\Delta_{\mathrm{H}} = m_{\mathrm{H}} - 1 \text{ amu}$

$$\Delta_n = m_n - 1 \text{ amu}$$

$$\Delta = M - A \text{ amu}$$

Then clearly $E_b = Z\Delta_{\mathrm{H}} + (A - Z)\, \Delta_n - \Delta$

6.255 The mass number of the given nucleus must be

$$\frac{27}{(3/2)^3} = 8$$

Thus, the nucleus is Be^8. Then the binding energy is

$$E_b = 4 \times 0.00867 + 4 \times 0.000783 - 0.00531$$

$$= 0.06069 \text{ amu} = 56.5 \text{ MeV}$$

(on using 1 amu = 931 MeV).

6.256 (a) Total binding energy of the O^{16} nucleus is

$$E_b = 8 \times 0.00867 + 8 \times 0.00783 + 0.00509$$

$$= 0.13709 \text{ amu} = 127.6 \text{ MeV}$$

So, B.E. per nucleon is 7.98 MeV.

(b) B.E. of neutron in B^{11} nucleus is

$$\text{B.E. of } \mathrm{B}^{11} - \text{B.E. of } \mathrm{B}^{10}$$

(Since on removing a neutron from B^{11} we get B^{10}.)

$$= \Delta_n - \Delta_{\mathrm{B}^{11}} + \Delta_{\mathrm{B}^{10}}$$

$$= 0.00867 - 0.00930 + 0.01294$$

$$= 0.01231 \text{ amu} = 11.46 \text{ MeV}$$

B.E. of an α-particle in B^{11} is

$$= \text{B.E. of } \mathrm{B}^{11} - \text{B.E. of } \mathrm{Li}^7 - \text{B.E. of } \alpha$$

(Since on removing an α from B^{11} we get Li^7.)

$$= \Delta_{\mathrm{B}^{11}} + \Delta_{\mathrm{Li}^7} + \Delta_\alpha$$

$$= -0.00930 + 0.01601 + 0.00260$$

$$= 0.00931 \text{ amu} = 8.67 \text{ MeV}$$

(c) This energy is

[B.E. of O^{16} + 4 (B.E. of α particles)]

$$= -\Delta_{O^{16}} + 4\Delta_\alpha$$

$$= 4 \times 0.00260 + 0.00509$$

$$= 0.01549 \text{ amu} = 14.42 \text{ MeV}$$

6.257 B.E. of a neutron in B^{11} − B.E. of a proton in B^{11}

$$= (\Delta_n - \Delta_{B^{11}} + \Delta_{Be^{10}}) - (\Delta_p - \Delta_{B^{11}} + \Delta_{Be^{10}})$$

$$= \Delta_n - \Delta_p + \Delta_{B^{10}} - \Delta_{Be^{10}}$$

$$= 0.00867 - 0.00783 + 0.01294 - 0.01354$$

$$= 0.00024 \text{ amu} = 0.223 \text{ MeV}$$

The difference in binding energy is essentially due to the coulomb repulsion between the proton and the residual nucleus Be^{10} which together constitute B^{11}.

6.258 Required energy is simply the difference in total binding energies, i.e.,

$$\text{B.E. of } Ne^{20} - 2\,(\text{B.E. of } He^4) - \text{B.E. of } C^{12}$$

$$= 20\varepsilon_{Ne} - 8\varepsilon_{He} - 12\varepsilon_C$$

(Here ε is binding energy per unit nucleon.)

Substitution gives 11.88 MeV.

6.259 (a) We have for Li^8

$$41.3 \text{ MeV} = 0.044361 \text{ amu} = 3\Delta_H + 5\Delta_n - \Delta$$

Hence, $\Delta = 3 \times 0.00783 + 5 \times 0.00867 - 0.09436 = 0.022480 \text{ amu}$

Hence, the mass of Li^8 is 8.0225 amu.

(b) We have for C^{10}

$$10 \times 6.04 = 60.4 \text{ MeV} = 0.06488 \text{ amu} = 6\Delta_H + 4\Delta_n - \Delta$$

Hence, $\Delta = 6 \times 0.00783 + 4 \times 0.00867 - 0.06488 = 0.01678 \text{ amu}$

Hence, the mass of C^{10} is 10.01678 amu.

6.260 Suppose M_1, M_2, M_3, M_4 are the rest masses of the nuclei A_1, A_2, A_3 and A_4 participating in the reaction, respectively, then

$$A_1 + A_2 \rightarrow A_3 + A_4 + Q$$

Here Q is the energy released. Then, by conservation of energy

$$Q = c^2 (M_1 + M_2 - M_3 - M_4)$$

Now, $$M_1 c^2 = c^2 (Z_1 m_H + (A_1 - Z)\, m_n) - E_1, \text{ etc}$$

and $$Z_1 + Z_2 = Z_3 + Z_4 \text{ (conservation of charge)}$$

$$A_1 + A_2 = A_3 + A_4 \text{ (conservation of heavy particles)}$$

Hence, $$Q = (E_3 + E_4) - (E_1 + E_2)$$

6.261 (a) The energy liberated in the fission of 1 kg of U^{235} is

$$\frac{1000}{235} \times 6.023 \times 10^{23} \times 200 \text{ MeV} = 8.21 \times 10^{10} \text{ kJ}$$

The mass of coal with equivalent calorific value is

$$\frac{8.21 \times 10^{10}}{30000} = 2.74 \times 10^6 \text{ kg}$$

(b) The required mass is

$$\frac{30 \times 10^9 \times 4.1 \times 10^3}{200 \times 1.602 \times 10^{-13} \times 6.023 \times 10^{23}} \times \frac{235}{1000} = 1.49 \text{ kg}$$

6.262 The reaction is (in effect)

$$H^2 + H^2 \rightarrow He^4 + Q$$

Then
$$\begin{aligned} Q &= 2\Delta_{H^2} - \Delta_{He^4} \\ &= 0.02820 - 0.00260 \\ &= 0.02560 \text{ amu} = 23.8 \text{ MeV} \end{aligned}$$

Hence, the energy released in 1 g of He^4 is

$$\frac{6.023 \times 10^{23}}{4} \times 23.8 \times 16.02 \times 10^{-13} \text{ J} = 5.75 \times 10^8 \text{ kJ}$$

This energy can be derived from

$$\frac{5.75 \times 10^8}{30000} = 1.9 \times 10^4 \text{ kg of coal}$$

6.263 The energy released in the reaction $Li^6 + H^2 \to 2He^4$ is

$$\begin{aligned}&\Delta_{Li^6} + \Delta_{H^2} - 2\Delta_{He^4}\\ &= 0.01513 + 0.01410 - 2 \times 0.00260 \text{ amu}\\ &= 0.02403 \text{ amu} = 22.37 \text{ MeV}\end{aligned}$$

(This result for change in B.E. is correct because the contribution of Δ_n and Δ_H cancels out by conservation law for protons and neutrons.)

Energy per nucleon is then

$$\frac{22.37}{8} = 2.796 \text{ MeV/nucleon}$$

This should be compared with the value 200/235 = 0.85 MeV/nucleon.

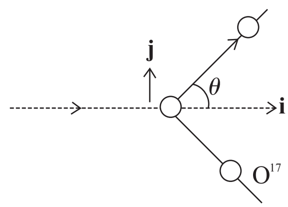
6.264 The energy of reaction $Li^7 + p \to 2He^4$ is

$$\begin{aligned}&2 \times \text{B.E. of } He^4 - \text{B.E. of } Li^7\\ &= 8\varepsilon_\alpha - 7\varepsilon_{Li}\\ &= 8 \times 7.06 - 7 \times 5.60 = 17.3 \text{ MeV}\end{aligned}$$

6.265 The reaction is $N^{14}(\alpha, p)O^{17}$.

It is given that (in the laboratory frame where N^{14} is at rest)

$$T_\alpha = 4.0 \text{ MeV}$$



The momentum of incident α particle is

$$\sqrt{2m_\alpha T_\alpha}\,\mathbf{i} = \sqrt{2\eta_\alpha m_O T_\alpha}\,\mathbf{i}$$

The momentum of outgoing proton is

$$\sqrt{2m_p T_p}(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) = \sqrt{2\eta_p m_O T_P}(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$$

where $\eta_p = m_p/m_O$, $\eta_\alpha = m_\alpha/m_O$, and m_0 is the mass of O^{17}.

Then, the momentum of O^{17} is

$$\left(\sqrt{2\eta_\alpha m_0 T_\alpha} - \sqrt{2\eta_p m_0 T_p}\cos\theta\right)\mathbf{i} - \sqrt{2m_O\eta_p T_p}\sin\theta\,\mathbf{j}$$

By energy conservation (conservation of energy including rest mass energy and kinetic energy)

$$M_{N^{14}}c^2 + M_\alpha c^2 + T_\alpha$$

$$= M_p c^2 + T_p + M_{O^{17}} c^2$$
$$+ \left[\left(\sqrt{\eta_\alpha T_\alpha} - \sqrt{\eta_p T_p} \cos\theta \right)^2 \right.$$
$$\left. + \eta_p T_p \sin^2\theta + \eta_p T_p \sin^2\theta \right]$$

Hence, by definition, energy of reaction is

$$Q = M_{N^{14}} c^2 + M_\alpha c^2 - M_p c^2 - M_{O^{17}} c^2$$
$$= T_p + \eta_\alpha T_\alpha + \eta_p T_p - 2\sqrt{\eta_p \eta_\alpha T_p T_\alpha} \times \cos\theta - T_\alpha$$
$$= (1 + \eta_p) T_p + T_\alpha (1 - \eta_\alpha) - 2\sqrt{\eta_p \eta_\alpha T_\alpha T_p} \cos\theta$$
$$= 1.19 \text{ MeV (on substituting values)}$$

6.266 (a) The reaction is Li^7 (p, n) Be^7 and the energy of the reaction is

$$Q = (M_{Be^7} + M_{Li^7}) c^2 + (M_p - M_n) c^2$$
$$= (\Delta_{Li^7} - \Delta_{Be^7}) c^2 + \Delta_p - \Delta_n$$
$$= [0.01601 + 0.00783 - 0.01693 - 0.00867] \text{ amu} \times c^2$$
$$= -1.64 \text{ MeV}$$

(b) The reaction is Be^9 (n, γ) Be^{10}. Mass of γ is taken zero. The energy of the reaction is

$$Q = (M_{Be^9} + M_n - M_{Be^{10}}) c^2$$
$$= (\Delta_{Be^9} + \Delta_n - \Delta_{Be^{10}}) c^2$$
$$= (0.01219 + 0.00867 - 0.01354) \text{ amu} \times c^2$$
$$= 6.81 \text{ MeV}$$

(c) The reaction is Li^7 $(\alpha, n)B^{10}$. The energy of the reaction is

$$Q = (\Delta_{Li^7} + \Delta_\alpha - \Delta_n - \Delta_{B^{10}}) c^2$$
$$= (0.01601 + 0.00260 - 0.00867 - 0.01294) \text{ amu} \times c^2$$
$$= -2.79 \text{ MeV}$$

(d) The reaction is O^{16} (d, α) N^{14}. The energy of the reaction is

$$Q = (\Delta_{O^{16}} + \Delta_d - \Delta_\alpha - \Delta_{N^{14}}) c^2$$
$$= (-0.00509 + 0.01410 - 0.00260 - 0.00307) \text{ amu} \times c^2$$
$$= 3.11 \text{ MeV}$$

6.267 The reaction is B^{10} (n, α) Li^7. The energy of the reaction is

$$\begin{aligned} Q &= (\Delta_{B^{10}} + \Delta_n - \Delta_\alpha - \Delta_{Li^7})c^2 \\ &= (0.01294 + 0.00867 - 0.00260 - 0.01601) \text{ amu} \times c^2 \\ &= 2.79 \text{ MeV} \end{aligned}$$

Since the incident neutron is very slow and B^{10} is stationary, the final total momentum must also be zero. So the reaction products must emerge in opposite directions. If their speeds are, respectively, v_α and v_{Li}, then

$$4v_\alpha = 7v_{Li}$$

and
$$\frac{1}{2}(4v_\alpha^2 + 7v_{Li}^2) \times 1.672 \times 10^{-24} = 2.79 \times 1.602 \times 10^{-6}$$

So,
$$\frac{1}{2} \times 4v_\alpha^2\left(1 + \frac{4}{7}\right) = 2.70 \times 10^{18} \text{ cm}^2/\text{s}^2$$

or
$$v_\alpha = 9.27 \times 10^6 \text{ m/s}$$

Then,
$$v_{Li} = 5.3 \times 10^6 \text{ m/s}$$

6.268 The Q value of the reaction Li^7 (p, n) Be^7 was calculated in Problem 6.266a. It is -1.64 MeV. We have by conservation of momentum and energy

$$p_p = p_{Be}$$

(since initial Li and final neutron are both at rest).

$$\frac{p_p^2}{2m_p} = \frac{p_{Be}^2}{2m_{Be}} + 1.64$$

Then,
$$\frac{p_p^2}{2m_p}\left(1 - \frac{m_p}{m_{Be}}\right) = 1.64$$

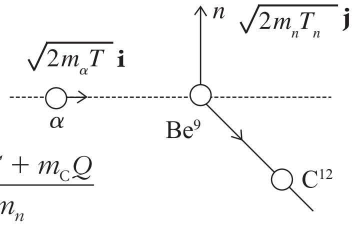
Hence,
$$T_p = \frac{p_p^2}{2m_p} = \frac{7}{6} \times 1.64 \text{ MeV} = 1.91 \text{ MeV}$$

6.269 It is understood that Be^9 is initially at rest. The moment of the outgoing neutron is $\sqrt{2m_nT_n}\,\mathbf{j}$. The momentum of C^{12} is

$$\sqrt{2m_\alpha T}\,\mathbf{i} - \sqrt{2m_nT_n}\,\mathbf{j}$$

Then by energy conservation

$$T + Q = T_n + \frac{2m_\alpha T + 2m_n T_n}{2m_C}$$



(Here m_C is the mass of C^{12}.)

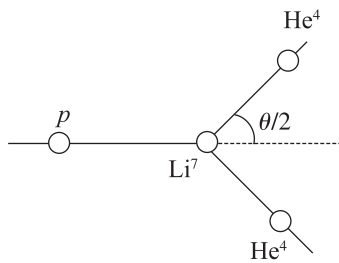
Thus,
$$T_n = \frac{m_C(T+Q) - m_\alpha T}{m_C + m_n} = \frac{(m_C - m_\alpha)T + m_C Q}{m_C + m_n}$$
$$= \frac{Q + (1 - m_\alpha/m_C)T}{1 + m_n/m_C} = 8.52 \text{ MeV}$$

6.270 The Q value of the reaction Li^7 (p, α) He^4 is

$$Q = (\Delta_{Li^7} + \Delta_H - 2\Delta_{He^4})\,c^2$$
$$= (0.01601 + 0.00783 - 0.00520) \text{ amu} \times c^2$$
$$= 0.01864 \text{ amu} \times c^2 = 17.35 \text{ MeV}$$

Since the direction of each He^4 nuclei is symmetrical, their momenta must also be equal. Let T be the K.E. of each He^4. Then

$$p_p = 2\sqrt{2m_{He}T}\cos\frac{\theta}{2}$$



(Here p_p is the momentum of proton.)

Also
$$\frac{p_p^2}{2m_p} + Q = 2T = T_p + Q$$

Hence,
$$T_p + Q = 2\frac{p_p^2 \sec^2\theta/2}{8m_{He}}$$
$$= T_p \frac{m_p}{2m_{He}}\sec^2\frac{\theta}{2}$$

Hence,
$$\cos\frac{\theta}{2} = \sqrt{\frac{m_p}{2m_{He}}\frac{T_p}{T_p + Q}}$$

Substitution gives $\qquad \theta = 170.53°$

Also $\qquad T = \frac{1}{2}(T_p + Q) = 9.18 \text{ MeV}$

6.271 Energy required is minimum when the reaction products all move in the direction of the incident particle with the same velocity (so that the combination is at rest in the C.M. frame). We then have

$$\sqrt{2mT_{\text{th}}} = (m + M)v$$

(Total mass is constant in the non-relativistic limit.)

$$T_{\text{th}} - |Q| = \frac{1}{2}(m + M)v^2 = \frac{mT_{\text{th}}}{m + M}$$

or
$$T_{\text{th}} \frac{M}{m + M} = |Q|$$

Hence,
$$T_{\text{th}} = \left(1 + \frac{m}{M}\right)|Q|$$

6.272 The result of the previous problem applies and we find that energy required to split a deuteron is

$$T \geq \left(1 + \frac{M_p}{M_d}\right)E_b = 3.3 \text{ MeV}$$

6.273 Since the reaction Li^7 (p, n) Be^7 ($Q = -1.65$ MeV) is initiated, the incident proton energy must be

$$T \geq \left(1 + \frac{M_p}{M_{\text{Li}}}\right) \times 1.65 = 1.89 \text{ MeV}$$

Since the reaction Be^9 (p, n) B^9 ($Q = -1.85$ MeV) is not initiated,

$$T \leq \left(1 + \frac{M_p}{M_{\text{Be}}}\right) \times 1.85 = 2.06 \text{ MeV}$$

Thus, $1.89 \text{ MeV} \leq T_p \leq 2.06$ MeV.

6.274 We have
$$4.0 = \left(1 + \frac{m_n}{M_{B^{11}}}\right)|Q|$$

or
$$Q = -\frac{11}{12} \times 4 = -3.67 \text{ MeV}$$

6.275 The Q value of the reaction Li^7 (p, n) Be^7 was calculated in Problem 6.266a. It is -1.64 MeV. Hence, the threshold K.E. of protons for initiating this reaction is

$$T_{\text{th}} = \left(1 + \frac{m_p}{m_{\text{Li}}}\right)|Q|$$

$$= \frac{8}{7} \times 1.64 = 1.87 \text{ MeV}$$

For the reaction Li^7 (p, d) Li^6, we find

$$Q = (\Delta_{Li^7} + \Delta_p - \Delta_d - \Delta_{Li^6})c^2$$

$$= (0.01601 + 0.00783 - 0.01410 - 0.01513) \text{ amu} \times c^2$$

$$= -5.02 \text{ MeV}$$

The threshold proton energy for initiating this reaction is

$$T_{th} = \left(1 + \frac{m_p}{m_{Li^7}}\right) \times |Q| = 5.73 \text{ MeV}$$

6.276 The Q value of the reaction Li^7 (α, n) B^{10} was calculated in Problem 6.266c. It is $Q = 2.79$ MeV. Then the threshold energy of α-particle is

$$T_{th} = \left(1 + \frac{m_\alpha}{m_{Li}}\right)|Q|$$

$$= \left(1 + \frac{4}{7}\right)2.79 = 4.38 \text{ MeV}$$

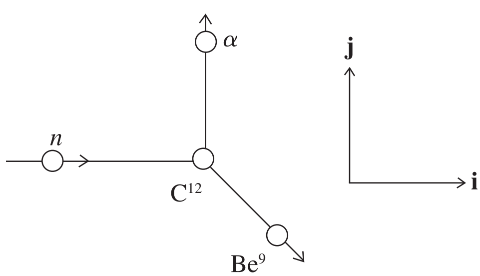
The velocity of B^{10} in this case is simply the velocity of centre of mass

$$v = \frac{\sqrt{2m_\alpha T_{th}}}{m_\alpha + m_{Li}} = \frac{1}{1 + m_{Li}/m_\alpha}\sqrt{\frac{2T_{th}}{m_\alpha}}$$

This is because both B^{10} and n are at rest in the C.M. frame at threshold. Substituting the values of various quantities, we get

$$v = 5.27 \times 10^6 \text{ m/s}$$

6.277 The momentum of incident neutron $\sqrt{2m_n T}\,\mathbf{i}$ that of α particle is $\sqrt{2m_\alpha T_\alpha}\,\mathbf{j}$ and of Be^9 is $-\sqrt{2m_\alpha T_\alpha}\,\mathbf{j} + \sqrt{2m_n T}\,\mathbf{i}$.



By conservation of energy

$$T = T_\alpha + \frac{m_\alpha T_\alpha + m_n T}{M} + |Q|$$

(Here M is the mass of Be^9.)

Thus, $$T_\alpha = \left[T\left(1 - \frac{m_n}{M}\right) - |Q|\right]\frac{M}{M + m_\alpha}$$

Using $$T_{\text{th}} = \left(1 + \frac{m_n}{M'}\right)|Q|$$

we get $$T_\alpha = \frac{M}{M + m_\alpha}\left[\left(1 - \frac{m_n}{M}\right)T - \frac{T_{\text{th}}}{1 + m_n/M'}\right]$$

(Here M' is the mass of C^{12} nucleus.)

or $$T_\alpha = \frac{1}{M + m_\alpha}\left[(M - m_n)T - \frac{MM'}{M' + m_n}T_{\text{th}}\right]$$

$$= 2.21 \text{ MeV (on substituting values)}$$

6.278 The formula of Problem 6.271 does not apply here because the photon is always relativistic. At threshold, the energy of the photon E_γ implies a momentum E_γ/c. The velocity of centre of mass with respect to the rest frame of initial H^2 is

$$\frac{E_\gamma}{(m_n + m_p)c}$$

Since both n and p are at rest in C.M. frame at threshold, we write

$$E_\gamma = \frac{E_\gamma^2}{2(m_n + m_p)c^2} + E_b$$

by conservation of energy. Since the first term is a small correction, we have

$$E_\gamma \cong E_b + \frac{E_b^2}{2(m_n + m_p)c^2}$$

Thus, $$\frac{\delta E}{E_b} = \frac{E_b}{2(m_n + m_p)c^2}$$

$$= \frac{2.2}{2 \times 2 \times 938} = 5.9 \times 10^{-4} \times 100 \sim 0.06\%$$

6.279 The reaction is

$$p + d \rightarrow He^3$$

Excitation energy of He^3 is just the energy available in centre of mass. The velocity of the centre of mass is

$$\frac{\sqrt{2m_pT_p}}{m_p + m_d} = \frac{1}{3}\sqrt{\frac{2T_p}{m_p}}$$

In the C.M. frame, the K.E. available is ($m_d \approx 2\, m_p$). So,

$$\frac{1}{2}m_p\left(\frac{2}{3}\sqrt{\frac{2T_p}{m_p}}\right)^2 + \frac{1}{2}2m_p\left(\frac{1}{3}\sqrt{\frac{2T}{m_p}}\right)^2 = \frac{2T}{3}$$

The total energy available is then

$$Q + \frac{2T}{3}$$

Now, $$Q = c^2(\Delta_n + \Delta_d - \Delta_{\mathrm{He}^3})$$

$$= c^2 \times (0.00783 + 0.01410 - 0.01603)\ \text{amu}$$

$$= 5.49\ \text{MeV}$$

Thus, $$E_{\text{exc}} = 6.49\ \text{MeV}$$

6.280 The reaction is

$$d + \mathrm{C}^{13} \to \mathrm{N}^{15*} \to n + \mathrm{N}^{14}$$

Maxima of yields determine the energy levels of N^{15*}. As in the previous problem, the excitation energy is

$$E_{\text{exc}} = Q + E_K$$

where E_K = available K.E. This is found as in the previous problem. The velocity of the centre of mass is

$$\frac{\sqrt{2m_dT_i}}{m_d + m_{\mathrm{C}}} = \frac{m_d}{m_d + m_{\mathrm{C}}}\sqrt{\frac{2T_i}{m_d}}$$

So, $$E_K = \frac{1}{2}m_d\left(1 - \frac{m_d}{m_d + m_{\mathrm{C}}}\right)^2\frac{2T_i}{m_d} + \frac{1}{2}m_{\mathrm{C}}\left(\frac{m_d}{m_d + m_{\mathrm{C}}}\right)^2\frac{2T_i}{m_d} = \frac{m_{\mathrm{C}}}{m_d + m_{\mathrm{C}}}T_i$$

Q is the Q value for the ground state of N^{15}. We have

$$Q = c^2 \times (\Delta_d + \Delta_{\mathrm{C}^{13}} - \Delta_{\mathrm{N}^{15}})$$

$$= c^2 \times (0.01410 + 0.00335 - 0.00011)\ \text{amu}$$

$$= 16.14\ \text{MeV}$$

The excitation energies then are 16.66 MeV, 16.92 MeV, 17.49 MeV and 17.70 MeV.

6.281 We have the relation

$$\frac{1}{\eta} = e^{-n\sigma d}$$

Here $1/\eta$ = attenuation factor, n = number of Cd nuclei per unit volume, σ = effective cross-section and d = thickness of the plate.

Now, $$n = \frac{\rho N_A}{M}$$

(Here ρ = density, M = molar weight of Cd, N_A = Avogadro number.)

Thus, $$\sigma = \frac{M}{\rho N_A d} \ln \eta = 2.53 \text{ kb}$$

6.282 Here $$\frac{1}{\eta} = e^{-(n_2\sigma_2 + n_1\sigma_1)d}$$

where 1 refers to oxygen and 2 to deutrium nuclei.

Using $n_2 = 2n$, $n_1 = n$ = concentration of oxygen nuclei in heavy water, we get

$$\frac{1}{\eta} = e^{-(2\sigma_2 + \sigma_1)nd}$$

Now, using the data for heavy water

$$n = \frac{1.1 \times 6.023 \times 10^{23}}{20} = 3.313 \times 10^{22} \text{ cm}^{-3}$$

Thus, substituting the values

$$\eta = 20.4 = \frac{I_0}{I}$$

6.283 In traversing a distance d, the fraction which is either scattered or absorbed is clearly

$$1 - e^{-n(\sigma_s + \sigma_a)d}$$

by the usual definition of the attenuation factor. Of this, the fraction scattered is (by definition of scattering and absorption cross-section)

$$w = \left\{1 - e^{-n(\sigma_s + \sigma_a)d}\right\} \frac{\sigma_s}{\sigma_s + \sigma_a}$$

In iron $$n = \frac{\rho \times N_A}{M} = 8.39 \times 10^{22} \text{ cm}^{-3}$$

Substitution gives $w = 0.352$.

6.284 (a) Assuming of course, that each reaction produces a radionuclide of the same type, the decay constant λ of the radionuclide is k/w. Hence,

$$T = \frac{\ln 2}{\lambda} = \frac{w}{k}\ln 2$$

(b) Number of bombarding particles is It/e (where e = charge on proton).

Then the number of Be^7 produced is

$$\frac{It}{e}w$$

If λ = decay constant of $Be^7 = (\ln 2)/T$, then the activity is

$$A = \frac{It}{e}w\cdot\frac{\ln 2}{T}$$

Hence,
$$w = \frac{eAT}{It\ln 2} = 1.98\times 10^{-3}$$

6.285 (a) Suppose N_0 = number of Au^{197} nuclei in the foil. Then the number of Au^{197} nuclei transformed in time t is $N_0 \cdot J \cdot \sigma \cdot t$. For this to equal ηN_0, we must have

$$t = \eta/(J\cdot\sigma)$$
$$= 323 \text{ years (on substituting values)}$$

(b) Rate of formation of the Au^{198} nuclei is $N_0 \cdot J \cdot \sigma$ per second and rate of decay is λn, where n is the number of Au^{198} at any instant.

Thus,
$$\frac{dn}{dt} = N_0\cdot J\cdot\sigma - \lambda n$$

The maximum number of Au^{198} nuclei is clearly

$$n_{\max} = \frac{N_0\cdot J\cdot\sigma}{\lambda} = \frac{N_0\cdot J\cdot\sigma\cdot T}{\ln 2}$$

because if n is smaller, $dn/dt > 0$ and n will increase further and if n is large, $dn/dt < 0$ and n will decrease. (Actually $n_{\max}$ is approached steadily as $t \to \infty$.)

Substituting $N_0 = 3.057 \times 10^{19}$ and other values, we get $n_{\max} = 1.01 \times 10^{13}$.

6.286 Rate of formation of the radionuclide is $n \cdot J \cdot \sigma$ per unit area per second and rate of decay is λN.

Thus, $$\frac{dN}{dt} = n \cdot J \cdot \sigma - \lambda N$$

Then $$\left(\frac{dN}{dt} + \lambda N\right) e^{\lambda t} = n \cdot J \cdot \sigma \cdot e^{\lambda t}$$

or $$\frac{d}{dt}(Ne^{\lambda t}) = n \cdot J \cdot \sigma \cdot e^{\lambda t}$$

Hence, $$Ne^{\lambda t} = \text{constant} + \frac{n \cdot J \cdot \sigma}{\lambda} e^{\lambda t}$$

The number of radionuclide at $t = 0$ when the process starts is zero. So,

$$\text{constant} = -\frac{n \cdot J \cdot \sigma}{\lambda}$$

Then, $$N = \frac{n \cdot J \cdot \sigma}{\lambda}(1 - e^{-\lambda t})$$

6.287 We apply the formula of the previous problem except that here we have N = number of radionuclide and n = number of host nuclei originally.

Here $$n = \frac{0.2}{197} \times 6.023 \times 10^{23} = 6.115 \times 10^{20}$$

Then, after time t

$$N = \frac{n \cdot J \cdot \sigma \cdot T}{\ln 2}(1 - e^{t \ln 2/T})$$

(where T = half-life of the radionuclide).

After the source of neutrons is cut-off, the activity after time τ will be

$$A = \frac{n \cdot J \cdot \sigma \cdot T}{\ln 2}(1 - e^{-t \ln 2/T})\, e^{-\tau \ln 2/T} \times \ln 2/T = n \cdot J \cdot \sigma (1 - e^{-t \ln 2/T})\, e^{-\tau \ln 2/T}$$

Thus, $$J = Ae^{\tau \ln 2/T} / n\sigma (1 - e^{-t \ln 2/T}) = 5.92 \times 10^{9} \text{ particle/cm}^2\text{-s}$$

6·288 Number of nuclei in the first generation = Number of nuclei initially = N_0.

N_0 in the second generation = $N_0 \times$ multiplication factor (k) = $N_0 \cdot k$.

N_0 in the third generation = $N_0 \cdot k \cdot k = N_0 k^2$.

N_0 in the n^{th} generation = $N_0 k^{n-1}$.

Substitution gives 1.25×10^5 neutrons.

6.289 Number of fissions per unit time is clearly P/E. Hence, number of neutrons produced per unit time is $\nu P/E$. Substitution gives 7.80×10^{18} neutrons/s.

6.290 (a) This number is k^{n-1} where n = number of generations in time $t = t/\tau$

Substitution gives number of neutrons generated per unit time as 388.

(b) We write
$$k^{n-1} = e^{\left(\frac{T}{\tau}-1\right)\ln k}$$
or
$$\frac{T}{\tau} - 1 = \frac{1}{\ln k}$$
$$T = \tau\left(1 + \frac{1}{\ln k}\right) = 10.15 \text{ s}$$

6.7 Elementary Particles

6.291 The formula is
$$T = \sqrt{c^2p^2 + m_0^2c^4} - m_0c^2$$
Thus,
$$T = 5.3 \text{ MeV for } p = 0.10\frac{\text{GeV}}{c} = 5.3 \times 10^{-3} \text{ GeV}$$
$$T = 0.433 \text{ GeV for } p = 1.0\frac{\text{GeV}}{c}$$
$$T = 9.106 \text{ GeV for } p = 10\frac{\text{GeV}}{c}$$
(Here we have used $m_0c^2 = 0.938$ GeV.)

6.292 Energy of pions is $(1 + \eta)\, m_0c^2$, so
$$(1+\eta)m_0c^2 = \frac{m_0c^2}{\sqrt{1-\beta^2}}$$
Hence,
$$\frac{1}{\sqrt{1-\beta^2}} = 1 + \eta$$
or
$$\beta = \frac{\sqrt{\eta(2+\eta)}}{1+\eta}$$
Here $\beta = v/c$ of pion. Hence, time dilation factor is $1 + \eta$ and the distance traversed by the pion in its lifetime will be

$$\frac{c\beta\tau_0}{\sqrt{1-\beta^2}} = c\tau_0\sqrt{\eta(2+\eta)}$$

$$= 15.0 \text{ m (on substituting values)}$$

Note: *The factor* $1/\sqrt{1-\beta^2}$ *can be looked at as a time dilation effect in the laboratory frame or as length contraction factor brought to the other side in the proper frame of the pion.*

6.293 From the previous problem

$$l = c\tau_0\sqrt{\eta(\eta+2)}$$

(where $\eta = T/(m_\pi c^2)$ and m_π is the rest mass of pions).

Substitution gives
$$\tau_0 = \frac{l}{c\sqrt{\eta(2+\eta)}}$$

$$= \frac{lm_\pi c}{\sqrt{T(T+2m_\pi c^2)}}$$

$$= 26.3 \text{ ns}$$

(where we have used $\eta = 100/139.6 = 0.716$).

6.294 Here
$$\eta = \frac{T}{mc^2} = 1$$

so the lifetime of the pion in the laboratory frame is

$$\eta = (1+\eta)\,\tau_0 = 2\tau_0$$

The law of radioactive decay implies that the flux will decrease by the factor

$$\frac{J}{J_0} = e^{-t/\tau} = e^{-l/v\tau} = e^{-l/c\tau_0\sqrt{\eta(2+\eta)}}$$

$$= \exp\left(-\frac{mcl}{\tau_0\sqrt{T(T+2mc^2)}}\right) = 0.221$$

6.295 Energy momentum conservation implies

$$0 = \mathbf{p}_\mu + \mathbf{p}_\nu$$

or $$m_\pi c^2 = E_\mu + E_\nu \quad \text{or} \quad m_\pi c^2 - E_\nu = E_\mu$$

But $$E_\nu = c\ |\mathbf{p}_\nu| = c\ |\mathbf{p}_\mu|$$

Thus, $$m_\pi^2 c^4 - 2m_\pi c^2 \cdot c|\mathbf{p}_\mu| + c^2 p_\mu^2 = E_\mu^2 = c^2 p_\mu^2 + m_\mu^2 c^4$$

Hence, $$c|\mathbf{p}_\mu| = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} \cdot c^2$$

So, $$T_\mu = \sqrt{c^2 p_\mu^2 + m_\mu^2 c^4} - m_\mu c^2 = \left[\sqrt{\frac{(m_\pi^2 - m_\mu^2)^2}{4m_\pi^2} + m_\mu^2}\right] \cdot c^2 - m_\mu c^2$$

$$= \frac{m_\pi^2 + m_\mu^2}{2m_\pi} c^2 - m_\mu c^2 = \frac{(m_\pi - m_\mu^2)}{2m_\pi} c^2$$

Substituting $m_\pi c^2 = 139.6$ MeV and $m_\mu c^2 = 105.7$ MeV, we get

$$T_\mu = 4.12 \text{ MeV}$$

Also, $$E_\nu = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} c^2 = 29.8 \text{ MeV}$$

6.296 We have $$0 = \mathbf{p}_n + \mathbf{p}_\pi \tag{1}$$

$$m_\Sigma c^2 = E_n + E_\pi$$

or $$(m_\Sigma c^2 - E_n)^2 = E_\pi^2$$

or $$m_\Sigma^2 c^4 - 2m_\Sigma c^2 E_n = E_\pi^2 - E_n^2 = c^4 m_\pi^2 - c^4 m_n^2$$

(because Eq. (1) implies $E_\pi^2 - E_n^2 = m_\pi^2 c^4 - m_\pi^2 c^4$)

Hence, $$E_n = \frac{m_\Sigma^2 + m_n^2 - m_\pi^2}{2m_\Sigma} c^2$$

and $$T_n = \left(\frac{m_\Sigma^2 + m_n^2 - m_\pi^2}{2m_\Sigma} - m_n\right) c^2 = \frac{(m_\Sigma - m_n)^2 - m_\pi^2}{2m_\Sigma} c^2$$

Substitution gives $T_n = 19.55$ MeV.

6.297 The reaction is

$$\mu^+ \to e^+ + \nu_e + \overline{\nu}_\mu$$

The neutrinos are massless. The positron will carry the largest momentum if both neutrinos (ν_e and $\overline{\nu}_\mu$) move in the same direction in the rest frame of the nuon. Then, the final product is effectively a two-body system and we get from Problem 6.295

$$(T_{e^+})_{\max} = \frac{(m_\mu - m_e)^2}{2m_\mu} c^2$$

Substitution gives $(T_{e^+})_{\max} = 52.35$ MeV.

6.298 By conservation of energy and momentum

$$Mc^2 = E_p + E_\pi$$

$$0 = \mathbf{p}_p + \mathbf{p}_\pi$$

Then
$$m_\pi^2 c^4 = E_\pi^2 - \mathbf{p}_\pi^2 c^2$$
$$= (Mc^2 - E_p)^2 - c^2 \mathbf{p}_p^2$$
$$= M^2c^4 - 2Mc^2 E_p + m_p^2 c^4$$

This is a quadratic equation in M, in the form

$$M^2 - 2\frac{E_p}{c^2} M + m_p^2 - m_\pi^2 = 0$$

or using $E_p = m_p c^2 + T$ and solving

$$\left(M - \frac{E_p}{c^2}\right)^2 = \frac{E_p^2}{c^4} - m_p^2 + m_\pi^2$$

Hence,
$$M = \frac{E_p}{c^2} + \sqrt{\frac{E_p^2}{c^4} - m_p^2 + m_\pi^2}$$

taking the positive sign. Thus,

$$M = m_p + \frac{T}{c^2} + \sqrt{m_\pi^2 + \frac{T}{c^2}\left(2m_p + \frac{T}{c^2}\right)}$$

Substitution gives $M = 1115.4$ MeV/c^2

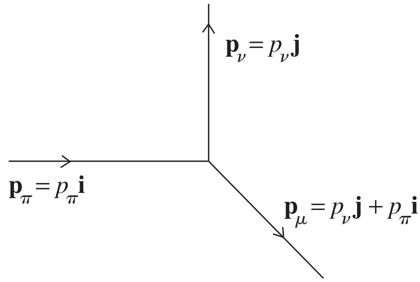
From the table of masses we identify the particle as a Λ-particle.

6.299 From the figure, by conservation of energy

$$\sqrt{m_\pi^2 c^4 + c^2 p_\pi^2} = cp_\nu + \sqrt{m_\mu^2 c^4 + p_\pi^2 c^2 + c^2 p_\nu^2}$$

or
$$\left(\sqrt{m_\pi^2 c^4 + c^2 p_\pi^2} - cp_\nu\right)^2 = m_\mu^2 c^4 + c^2 p_\pi^2 + c^2 p_\nu^2$$

or
$$m_\pi^2 c^4 - 2cp_\nu \sqrt{m_\pi^2 c^4 + c^2 p_\pi^2} = m_\mu^2 c^4$$



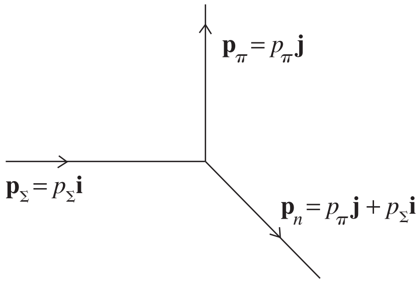
Hence, the energy of the neutrino is

$$E_\nu = cp_\nu = \frac{m_\pi^2 c^4 - m_\mu^2 c^4}{2(m_\pi c^2 + T)}$$

(on writing $\sqrt{m_\pi^2 c^4 + c^2 p_\pi^2} = m_\pi c^2 + T$)

Substitution gives $E_\nu = 21.93$ MeV.

6.300 By energy conservation



$$\sqrt{m_\Sigma^2 c^4 + c^2 p_\Sigma^2} = \sqrt{m_\pi^2 c^4 + c^2 p_\pi^2} + \sqrt{m_n^2 c^4 + c^2 p_\pi^2 + c^2 p_\Sigma^2}$$

or $$\left(\sqrt{m_\Sigma^2 c^4 + c^2 p_\Sigma^2} - \sqrt{m_\pi^2 c^4 + c^2 p_\pi^2}\right)^2 = m_n^2 c^4 + c^2 p_\pi^2 + c^2 p_\Sigma^2$$

or $$m_\Sigma^2 c^4 + c^2 p_\Sigma^2 + m_\pi^2 c^4 + c^2 p_\pi^2 - 2\sqrt{m_\pi^2 c^4 + c^2 p_\Sigma^2}\sqrt{m_\pi^2 c^2 + c^2 p_\pi^2}$$
$$= m_n^2 c^2 + c^2 p_\pi^2 + c^2 p_\Sigma^2$$

or using the K.E. of Σ and π

$$m_n^2 = m_\Sigma^2 + m_\pi^2 - 2\left(m_\Sigma + \frac{T_\Sigma}{c^2}\right)\left(m_\pi + \frac{T_\pi}{c^2}\right)$$

and $$m_n = \sqrt{m_\Sigma^2 + m_\pi^2 - 2\left(m_\Sigma + \frac{T_\Sigma}{c^2}\right)\left(m_\pi + \frac{T_\pi}{c^2}\right)} = 0.949 \text{ GeV}/c^2$$

6.301 Here, by conservation of momentum

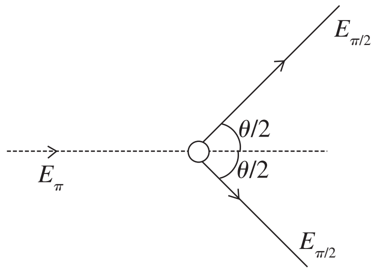
$$p_\pi = 2 \times \frac{E_\pi}{2c} \times \cos\frac{\theta}{2}$$

or $$cp_\pi = E_\pi \cos\frac{\theta}{2}$$

Thus, $$E_\pi^2 \cos^2\frac{\theta}{2} = E_n^2 - m_\pi^2 c^4$$

or $$E_\pi = \frac{m_\pi c^2}{\sin\theta/2}$$

and $$T_\pi = m_\pi c^2\left(\operatorname{cosec}\frac{\theta}{2} - 1\right)$$



Substitution gives

$$T_\pi = m_\pi c^2 = 135 \text{ MeV (for } \theta = 60°)$$

Also $$E_\gamma = \frac{m_\pi c^2 + T_\pi}{2} = \frac{m_\pi c^2}{2}\operatorname{cosec}\frac{\theta}{2}$$

$$= m_\pi c^2 \text{ (since } \theta = 60°)$$

6.302 With particle masses standing for the names of the particles, the reaction is

$$m + M \rightarrow m_1 + m_2 + \ldots$$

On R.H.S. let the energy momenta be $(E_1, c\mathbf{p}_1)$, $(E_2, c\mathbf{p}_2)$, etc. On the L.H.S. the energy momentum of the particle m is $(E, c\mathbf{p})$ and that of the other particle is $(Mc^2, \mathbf{O})$, where, of course, the usual relations, $E^2 - c^2\mathbf{p}^2 = m^2c^4$, etc., hold.

From the conservation of energy momentum, we see that

$$(E + Mc^2)^2 - c^2\mathbf{p}^2 = (\Sigma E_i)^2 - (\Sigma c\mathbf{p}_i)^2$$

The L.H.S. is

$$m^2c^4 + M^2c^4 + 2Mc^2E$$

We evaluate the R.H.S. in the frame where $\Sigma\mathbf{p}_i = 0$ (C.M. frame of the decay product).

Then, R.H.S. $= (\Sigma E_i)^2 \geq (\Sigma m_i c^2)^2$ because all energies are $+ve$. Therefore, we have the result

$$E \geq \frac{(\Sigma m_i)^2 - m^2 - M^2}{2M} c^2$$

or since $E = mc^2 + T$, we see that $T \geq T_{\text{th}}$, where

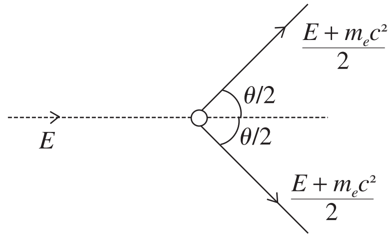
$$T_{\text{th}} = \frac{(\Sigma m_i)^2 - (m + M)^2}{2M} c^2$$

6.303 By momentum conservation

$$\sqrt{E^2 - m_e^2 c^4} = 2\frac{E + m_e c^2}{2}\cos\frac{\theta}{2}$$

or $$\cos\frac{\theta}{2} = \sqrt{\frac{E - m_e c^2}{E + m_e c^2}} = \sqrt{\frac{T}{T + 2m_e c^2}}$$

Substitution gives $\theta = 98.8°$.



6.304 The formula of Problem 6.302 gives

$$E_{\text{th}} = \frac{(\Sigma m_i)^2 - M^2}{2M} c^2$$

when the projectile is a photon.

(a) For $\gamma + e^- \to e^- + e^- + e^+$

$$E_{\text{th}} = \frac{9m_e^2 - m_e^2}{2m_e} c^2 = 4m_e c^2 = 2.04 \text{ MeV}$$

(b) For $\gamma + p \to p + \pi^+ + \pi^-$

$$E_{\text{th}} = \frac{(M_p + 2m_\pi)^2 - M_p^2}{2M_p} c^2$$

$$= \frac{4m_\pi M_p + 4m_\pi^2}{2M_p} c^2$$

$$= 2\left(m_\pi + \frac{m_\pi^2}{2M_p}\right) c^2$$

$$= 320.8 \text{ MeV}$$

6.305 (a) For $p + p \to p + p + p + \overline{p}$

$$T \geq T_{\text{th}} = \frac{16m_p^2 - 4m_p^2}{2m_p} c^2 = 6m_p c^2 = 5.63 \text{ GeV}$$

(b) For $p + p \to p + p + \pi^0$

$$T \geq T_{\text{th}} = \frac{(2m_p + m_{\pi^0})^2 - 4m_p^2}{2m_p} c^2$$

$$= \left(2m_\pi + \frac{m_{\pi^0}^2}{2m_p}\right) c^2 = 0.280 \text{ GeV}$$

6.306 (a) Here

$$T_{\text{th}} = \frac{(m_K + m_\Sigma)^2 - (m_\pi + m_p)^2}{2m_p} c^2$$

$$= 0.904 \text{ GeV (on substituting values)}$$

(b) Here

$$T_{\text{th}} = \frac{(m_{K^+} + m_{\Lambda^0})^2 - (m_{\pi^0} + m_p)^2}{2m_p} c^2$$

$$= 0.77 \text{ GeV (on substituting values)}$$

6.307 From the Gell-Mann Nishijima formula

$$Q = T_Z + \frac{Y}{2}$$

We get

$$0 = \frac{1}{2} + \frac{Y}{2} \quad \text{or} \quad Y = -1$$

Also,

$$Y = B + S \Rightarrow S = -2$$

Thus, the particle is Ξ^0 (neutral Xi).

6.308 (1) The process $n \to p + e^- + \nu_e$ cannot occur as there are two more leptons (e^-, ν_e) on the right compared to on the left.

(2) The process $\pi^+ \to \mu^+ + e^- + e^+$ is forbidden because this corresponds to a change of lepton number by 1, (0 on the left, -1 on the right).

(3) The process $\pi^- \to \mu^- + \nu_\mu$ is forbidden because μ^-, ν_μ being both leptons, $\Delta L = 2$ here.

The following processes (4), (5), (6) are allowed (except that one must distinguish between muon neutrinos and electron neutrinos). The correct names would be

(4) $p + e^- \to n + \nu_e$

(5) $\mu^+ \to e^+ + \nu_e + \tilde{\nu}_\mu$

(6) $K^- \to \mu^- + \tilde{\nu}_\mu$

6.309 (1) $\pi^- + p \to \Sigma^- + K^+$

0 0 -1 1

So $\Delta S = 0$ and process is allowed.

(2) $\pi^- + p \to \Sigma^+ + K^-$

0 0 -1 1

So $\Delta S = -2$ and process is forbidden.

(3) $\pi^- + p \to K^- + K^+ + n$

$0 \quad 0 \to -1 \quad 1 \quad 0$

So $\Delta S = 0$ and process is allowed.

(4) $n + p \to \Lambda^0 + \Sigma^+$

$0 \quad 0 \quad -1 \quad -1$

So $\Delta S = -2$ and process is forbidden.

(5) $\pi^- + n \to \Xi^- + K^+ + K^-$

$0 \quad 0 \to -2 \quad 1 \quad -1$

So $\Delta S = -2$ and process is forbidden.

(6) $K^- + p \to \Omega^- + K^+ + K^0$

$-1 \quad 0 \quad -3 \quad +1 \quad +1$

So $\Delta S = 0$ and process is allowed.

6.310 (1) $\Sigma^- \to \Lambda^0 + \pi^-$ is forbidden by energy conservation. The mass difference will be

$$M_{\Sigma^-} - \mathrm{M}_{\Lambda^0} = 82\frac{\mathrm{MeV}}{c^2} < m_\pi^-$$

[The process (1) $\to$ (2) + (3) will be allowed only if $m_1 > m_2 + m_3$.]

(2) $\pi^- + p \to K^+ + K^-$ is disallowed by conservation of baryon number.

(3) $K^- + n \to \Omega^- + K^+ + K^0$ is forbidden by conservation of charge.

(4) $n + p \to \Sigma^+ + \Lambda^0$ is forbidden by strangeness conservation.

(5) $\pi^- \to \mu^- + e^- + e^+$ is forbidden by conservation of muon number (or lepton number).

(6) $\mu^- \to e^- + \nu_e + \tilde{\nu}_\mu$ is forbidden by the separate conservation of muon number as well as lepton number.

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