Practice Problems

1. MATRIX VECTOR COMPUTATIONS

Given the matrices A and B, and the vectors $\vec{x}$ and $\vec{y}$ as,

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 2 & 5 \end{pmatrix}, \; B = \begin{pmatrix} 3 & 2 & 0 \\ 4 & -1 & 0 \\ 2 & 1 & -1 \end{pmatrix}, \; x = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \; y = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

(a) Compute Ax and By.

(b) Compute $x^T A$ and $y^T B$. What do you notice when you compare Ax to $x^T A$ and why do you think we are getting this result ?

2. NORM EQUIVALENCES

We assume that we have a vector $\vec{x}$ with n elements $x_i, i = 1, ..., n$. In the lecture, we discussed three different kinds of vector norms,

- The 2-norm, also known as the Euclidean norm, and defined as,

$$\|x\|_2 = \Big(\sum_{i=1}^{n} x_i^2 \Big)^{1/2}$$

- Th 1-norm, given by,

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

- Th maximum norm, also known as the infinity norm, given by,

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|$$

We mentioned that the choice of an optimal norm among these three norms really depends on the application you are working on. In this problem, we show that these norms are equivalent, that is, that they are always within a constant factor of one another.

(a) Show that

$$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$$

(b) Show that

$$\|x\|_2 \le \|x\|_1 \le \sqrt{n}\|x\|_2$$

3. ORTHOGONAL MATRICES

We say that P is a permutation matrix if, given an arbitrary matrix A, PA permutes the rows of A and AP permutes the columns of A. An example of a 2×2 permutation matrix P is given by,

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

A permutation matrix can be thought of as the identity matrix I with permuted rows or columns.

- Show that any permutation matrix P is orthogonal.
- Extra Practice: Show that a rotation matrix is orthogonal.

4. PROJECTION MATRICES

Suppose that in two dimensions, we want to find the operator matrix P that takes any vector $\vec{x}$ and projects it orthogonally onto the x-axis. In other words, P takes a vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and makes it into $\begin{pmatrix} x_1 \\ 0 \end{pmatrix}$. The matrix P looks like $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and is a projection matrix. More formally, a square matrix P is a projection matrix if $P^2 = P$. We can see this as for given any vector $\vec{x}$, we would have $P^2x = Px$ which signifies that applying the matrix P to the vector Px keeps it the same.

- Show that a projection matrix P is not invertible.

5. DETERMINANTS

Given the matrices A and B,

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 2 & 5 \end{pmatrix}, \; B = \begin{pmatrix} 3 & 2 & 0 \\ 4 & -1 & 0 \\ 2 & 1 & -1 \end{pmatrix}, \; C = \begin{pmatrix} e^{234} & 2 & 0 \\ -4 & \frac{\sqrt{2}}{5} & 0 \\ 10 & 100 & 0 \end{pmatrix}$$

(a) Compute the determinant of A.
(b) Compute the determinant of B.
(c) Compute the determinant of C.

6. EIGENVALUES AND EIGENVECTORS

Given the matrices A and B,

$$A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 3 & 0 & 0 \\ 4 & -1 & 0 \\ 2 & 1 & -1 \end{pmatrix}$$

(a) Compute the eigenvalues and eigenvectors of A.

(b) Compute the eigenvalues and eigenvectors of B.

7. PROOF

Prove that the matrix A^{-1} exists if and only each $\vec{b}$ in $A\vec{x} = \vec{b}$ has a unique solution .