Sensitivity Analysis, Duality Theory and Shadow Price

The concept of duality was the greatest discovery in the early development of linear programming. Corresponding to every linear programming problem, there is another linear programming problem. The given problem is called the ***"Primal"*** and the other is ***"Dual"***. Although the idea of duality is mostly mathematical, it has important interpretations. This can help managers answer questions about alternative courses of action and their effect on the values of the objective function.

So, what is duality?

When the primal problem is of the maximization type the dual is of the minimization type and vice versa. It is an interesting feature of the simplex method. We can use it to solve either the original problem the ***Primal*** or the ***Dual***. We can start either by solving the primal and then go to dual, or vice-versa but in the end, it gives us the resultant answer of both types.

Before we start to understand the importance of dual, the relationship between dual and primal etc.…; we need to know that all the linear programming problems must be in this format i.e.,

An LPP is said to be in standard form if

1. All constraints involve the sign in a problem of " <= " in the problem of Maximization.

2. All constraints involve the sign in a problem of " >= " in the problem of Minimization.

The relationships between the dual problem and the original problem (called the primal) prove to be extremely useful in a variety of ways,

- In duality, the objective constraints of the primal turn to be the RHS of the duality to talk about the least amount of profit that he or she needs to earn by employing the resources.
- Constraints in the primal are the resources that the producer is having when he wants to sell them off, they will be seeking a minimal value from it, so the objective function of the dual is going to be the minimization of the cost of the resources at which the producer has listed it out. This would eventually help any kind of organization to get an understanding towards the employment of factors of production i.e., land, labour, capital and organization.

The dual problem is constructed from the primal as follows:

- Each constraint in the primal problem will have a corresponding variable (dual variable y) in the dual problem.
- The elements of the right-hand side of the constraints in the primal are equal to the respective coefficient of the variables in the objective function in the dual [i.e., y1 will have a coefficient b1 in the objective function etc.].
- If the primal problem is maximization, the dual problem is minimization.
- The maximization problem has (< or =) type constraints and the minimization problem has (> or =) type constraints. If the constraints in the primal are mixed types, they are converted into constraints of the same type before formulating the dual.
- The coefficient matrix of the dual is the transpose of the coefficient matrix of the primal.
- The variables in both problems are non-negative.

Max Z, if the ***primal*** is

c1x1 + c2x2 + ---------- +cnxn

Subject to

a11x1 + a12x2 + --------+a1nxn <= b1

a21x1 + a22x2 + --------+a2nxn <= b2

----

am1x1 + am2x2 + -------+ amnxn <= bm

x1, x2, -----------, xn >= 0

Its ***dual*** is

Min C = b1y1 + b2y2 + ------ + bmym

Subject To

a11y1 + a21y2 + ------ + am1ym >= c1

a12y1 + a22y2 + ------ + am2ym >= c2

----

a1ny1 + a2ny2 + ------- + amnym >= cn

y1, y2, --------, ym >= 0

Formulation of Simplex, Dual and Shadow Prices

```r
library("lpSolve")

f.obj <- c(3,2)

f.con <- matrix(c(2,1,
                  2,3,
                  3,1), nrow=3, byrow=T)

f.dir <- c("<=",
           "<=",
           "<=")
```

```r
f.rhs <- c(18,42,24)

lp("max",f.obj,f.con,f.dir,f.rhs)

## Success: the objective function is 33

lp("max",f.obj,f.con,f.dir,f.rhs)$solution

## [1]  3 12

lp("max",f.obj,f.con,f.dir,f.rhs,compute.sens = T)$duals

## [1] 1.25 0.25 0.00 0.00 0.00
```

Here we were solving the primal problem of a business and got to see that the objective function is 33, x1 and x2 values are 3, 12. We can straightly get the dual values by employing the "$duals" function at the end of the lp command. While executing the $duals we get to see that the shadow price of the first resource is resulting in higher sensitivity, thus by investing more towards the first resource the company will surely be tasting the joy of success and profits.

Formulating the dual for the above primal problem

```r
f.obj1 <- c(18,42,24)

f.con1 <- matrix(c(2,2,3,
                  1,3,1),nrow=2,byrow=T)

f.dir1 <- c(">=",
           ">=")

f.rhs1 <- c(3,2)

lp("min", f.obj1,f.con1,f.dir1,f.rhs1)

## Success: the objective function is 33

lp("min",f.obj1,f.con1,f.dir1,f.rhs1)$solution

## [1] 1.25 0.25 0.00
```

Here we have constructed the dual problem from the above primal problem, the results are the same and it gives us the same notation to employ more of the first resource to experience benefits as the shadow price is more sensitive.

1. What is ***Sensitivity***?

Sensitivity is the responsiveness of the objective function to one unit change in one resource.

2. What does the ***Shadow Price*** inform the business world?

The shadow prices are the resultant values which we obtain from the dual problem, they give us an understanding of the sensitivity by employing one more unit of a resource. The greater the shadow price is the higher the sensitivity is and vice versa.

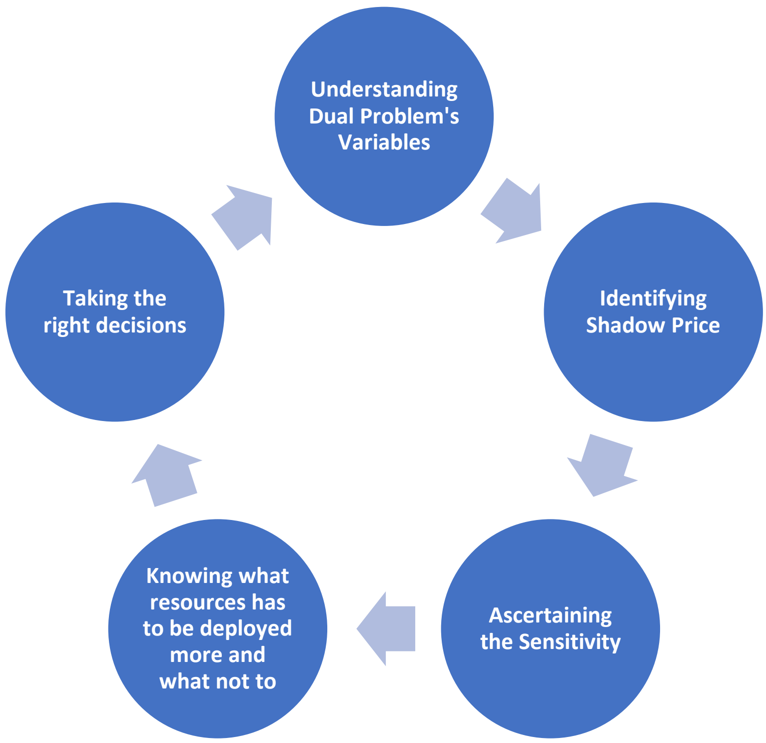
3. What is the relationship between ***resources*** and ***shadow price***?

They have an on-par relationship, by employing an additional amount of resources we get to see an increase or decrease in the value which is nothing else but sensitivity's response over the shadow price.

4. How does the business makes the decision on which resource to increase and which not?

As mentioned earlier the greater the shadow price is the larger the sensitivity is and it is always useful for a business to take informed decisions by using ***sensitivity, duality and shadow prices all together*** this way they will get to know which resource has to be deployed more and which to deploy less thereby saving the money of the firm. The decision-makers of a firm can therefore take impactful decisions by using Sensitivity, Duality and Shadow Prices all together.

Hence it's always useful for a firm to follow the below chain so that they can take value-added informed decisions promptly.



Reference:

1. https://www.mbaknol.com/management-science/duality-in-linear-programming/