7.3 oplossingsmethode 1D warmtevergelijking

Stap 1: zoek naar oplossing die zorgt voor scheiding van veranderlijken.

$$u(x,t) = X(x) \cdot T(t)$$

$$X(x) T'(t) = \alpha^2 X''(x) \cdot T(t)$$

→ deel door $x \cdot t$

$$\Rightarrow \frac{T'(t)}{T} = \alpha^2 \cdot \frac{X''(x)}{X(x)}$$

$$\boxed{X''(x) = \sigma \cdot X(x)} \quad \boxed{T'(t) = \alpha^2 \cdot \sigma T(t)}$$

$$u(x,t) = X(x) \cdot T(t)$$

Stap 2: bepaling normale modes

we nemen Dirichlet randvw: $X(0) = X(L) = 0$

$\sigma > 0$, $X(x) = c_1 e^{\sqrt{\sigma}\, x} + c_2 e^{-\sqrt{\sigma}\, x}$ (R.V $\Rightarrow c_1 = c_2 = 0$)

$\sigma = 0$, $X(x) = c_1 + c_2 x$ $\quad X(0) = 0 \Rightarrow c_1 = 0$, $X(L) = 0 \Rightarrow c_2 = 0$

$\sigma < 0$, $X(x) = c_1 \cos kx + c_2 \sin kx$ $\quad (k^2 = \sigma)$

$X(0) = 0 \Rightarrow c_1 = 0$

$X(L) = 0 \Rightarrow c_2 \sin kL = 0$

} triviale oplossingen

$kL = n \cdot \pi$

$$\boxed{X_n(x) = \sin\left(\frac{n\pi}{L}x\right) \text{ voor } n = 1, 2, \dots} \quad \left(\sigma = -\left(\frac{n\pi}{L}\right)^2\right)$$

↳

$$T'(t) = \alpha^2 \cdot \left(\frac{n\pi}{L}\right)^2 \cdot T_n(t) \Rightarrow T_n(t) = \exp\left(-\left(\frac{\alpha n\pi}{L}\right)^2\right) \cdot t$$

normale modes: $u_n(x,t) = \exp\left(-\left(\frac{\alpha n \pi}{L}\right)^2 \cdot t\right) \cdot \sin\left(\frac{n\pi x}{L}\right)$

$$u_h(x,t) = \underbrace{\sum_{n=1}^{+\infty} a_n \exp\left(-\left(\frac{\alpha}{L}\pi n\right)^2 t\right) \cdot \sin\left(\frac{n\pi x}{L}\right)}_{\text{evenwicht.}}$$

↳ homogeen.

evenwicht?

$\left(\frac{\alpha\pi}{L}\right)^2 t \gg 1$

$\Leftrightarrow t \gg \frac{L^2}{\alpha^2 \pi^2} = te$

$$\boxed{L \sim \sqrt{t}}$$

Nu niet homogeen:

$u(0,t) = T_1$, $u(L,t) = T_2$

$$\tilde{u}(x) = T_1 + (T_2 - T_1) \cdot \left(\frac{x}{L}\right)$$

$$u(x,t) = \underbrace{\tilde{u}(x)}_{\text{particuliere oplossing.}} + u_h(x,t)$$

Stap 3: Verwerken Beginvoorwaarden.

$$f(x) = u(x,0) = \tilde{u}(x) + \sum_{n=1}^{\infty} a_n \cdot \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) - \tilde{u}(x) = \sum_{n=1}^{\infty} a_n \cdot \sin\left(\frac{n\pi x}{L}\right)$$

$$\left\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{m\pi x}{L}\right)\right\rangle = \int_0^L dx \sin\left(\frac{n\pi x}{L}\right) \cdot \sin\left(\frac{m\pi x}{L}\right) = \frac{L}{2}\delta_{m,n}.$$

$$= \int_0^L \sin\left(\frac{m\pi x}{L}\right) \cdot \left(f(x) - \tilde{u}(x)\right) dx = \frac{L}{2} a_m$$

$$\Leftrightarrow a_m = \frac{2}{L}\int_0^L dx \cdot (\dots)$$