2) Warmte vgl: $\partial_t u = \alpha^2 \partial_x^2 u$

$u(0,t) = T_0$ | $u(x,t) = u_p(x) + u_h(x,t)$

$u'(L,t) = 0$

$u(x,0) = f(x)$

① Scheiding der veranderlijken

$$T'X = \alpha^2 \cdot X''T$$

$$\Leftrightarrow \frac{T'}{T} = \alpha^2 \frac{X''}{X} \Rightarrow \begin{cases} T' = \alpha^2 \sigma T \\ X'' = \sigma X \end{cases}$$

Zoek eerst homogeen RVW'en

$X'' = \sigma X$, $X(0) = 0$, $X'(L) = 0$

Study all variants of σ:

1) $\sigma > 0$: $c_1 \cdot e^{\alpha x} + c_2 e^{-\alpha x}$, if $X(0) = 0$

$\Leftrightarrow$ $c_1 + c_2 = 0$, so $c_2 = -c_1$

$\Rightarrow$ $c_1(e^{\alpha x} - e^{-\alpha x}) = 0$ $\rightarrow$ c_1 is 0, trivial!

2) $\sigma = 0$: $X'' = 0$

$\Leftrightarrow$ $X(x) = c_1 x + c_2$, if $X(0) = 0$

$\Leftrightarrow$ $c_2 = 0$

if $X'(L) \Rightarrow$ $c_1 = 0$ $\rightarrow$ Trivial.

3) $\sigma < 0$: let $\sigma = -k^2$

$c_1 \cdot \cos(k \cdot x) + c_2 \cdot \sin(kx)$

if $X(0) \Rightarrow$ $c_1 = 0$ $\rightarrow$ $X(x) = c_2 \sin(kx)$

if $X'(L) \Rightarrow$ $k \cdot c_2 \cdot \cos(k \cdot L) = 0$

solve $\cos(kL) = 0$ $\rightarrow$ $\sigma = -\left(\frac{\pi}{2L} \cdot (2n+1)\right)^2$

$\Leftrightarrow$ $k = \frac{\pi(2n+1)}{2L}$

$\longrightarrow$ $X_n(x) = \sin\left(\frac{\pi(2n+1) \cdot x}{2L}\right)$

$$T' = \alpha^2 \cdot -\left(\frac{\pi}{2L} \cdot (2n+1)\right)^2 T$$

$$\rightarrow T_n(t) = e^{\frac{-\alpha^2 \pi^2 (2n+1)^2 t}{4L^2}}$$

$$u_h(x,t) = \sum_{n=0}^{\infty} a_n \cdot e^{\frac{-\alpha^2 \pi^2 (2n+1)^2 t}{4L^2}} \sin\left(\frac{\pi(2n+1) \cdot x}{2L}\right)$$

Nu zoeken we particuliere oplossingen

$\partial_x^2 u_p = 0$

$\Rightarrow$ $c_1 x + c_2 = 0$

if $u(0,t)$ $\rightarrow$ $\boxed{c_2 = T_0}$

if $u'(L,t)$ $\rightarrow$ trivial

So full solution is:

$$u(x,t) = T_0 + \sum_{n=0}^{\infty} a_n e^{\frac{-\alpha^2 \pi^2 (2n+1)^2 t}{4L^2}} \cdot \sin\left(\frac{\pi(2n+1)\,x}{2L}\right)$$

③ Coefficienten

1) $u(x,0) = f(x)$

$\Leftrightarrow$ $f(x) = T_0 + \sum_{n=0}^{\infty} a_n \sin\left(\frac{\pi(2n+1)\,x}{2L}\right)$

$\Leftrightarrow$ $a_n = \frac{2}{L} \cdot \int_0^L \sin\left(\frac{\pi(2n+1)\,x}{2L}\right)(f(x) - T_0)\,dx$