

> restart : with(LinearAlgebra) :

First we need to define the Potential

$$\begin{aligned} > V := -\cos(3 \cdot (x - y)) \cdot \cos(z + y) \cdot \cos(5 \cdot (x + y)) \\ & \quad V := -\cos(3x - 3y) \cos(z + y) \cos(5x + 5y) \end{aligned} \quad (1)$$

> #Now we define the hessian, which will be a 3x3 matrix (3 variables)

$$> H := \text{Matrix}([\text{diff}(V, x, x), \text{diff}(V, x, y), \text{diff}(V, x, z)], [\text{diff}(V, x, y), \text{diff}(V, y, y), \text{diff}(V, y, z)], [\text{diff}(V, x, z), \text{diff}(V, z, y), \text{diff}(V, z, z)])$$

$$\begin{aligned} H := & [[34 \cos(3x - 3y) \cos(z + y) \cos(5x + 5y) - 30 \sin(3x - 3y) \cos(z + y) \sin(5x + 5y), 16 \cos(3x - 3y) \cos(z + y) \cos(5x + 5y) - 3 \sin(3x - 3y) \sin(z + y) \cos(5x + 5y) - 5 \cos(3x - 3y) \sin(z + y) \sin(5x + 5y), \\ & -3 \sin(3x - 3y) \sin(z + y) \cos(5x + 5y) - 5 \cos(3x - 3y) \sin(z + y) \sin(5x + 5y)], \\ & [16 \cos(3x - 3y) \cos(z + y) \cos(5x + 5y) - 3 \sin(3x - 3y) \sin(z + y) \cos(5x + 5y) - 5 \cos(3x - 3y) \sin(z + y) \sin(5x + 5y), 35 \cos(3x - 3y) \cos(z + y) \cos(5x + 5y) + 6 \sin(3x - 3y) \sin(z + y) \cos(5x + 5y) \\ & + 30 \sin(3x - 3y) \cos(z + y) \sin(5x + 5y) - 10 \cos(3x - 3y) \sin(z + y) \sin(5x + 5y), 3 \sin(3x - 3y) \sin(z + y) \cos(5x + 5y) + \cos(3x - 3y) \cos(z + y) \cos(5x + 5y) - 5 \cos(3x - 3y) \sin(z + y) \sin(5x + 5y)], \\ & [-3 \sin(3x - 3y) \sin(z + y) \cos(5x + 5y) - 5 \cos(3x - 3y) \sin(z + y) \sin(5x + 5y), 3 \sin(3x - 3y) \sin(z + y) \cos(5x + 5y) + \cos(3x - 3y) \cos(z + y) \cos(5x + 5y) - 5 \cos(3x - 3y) \sin(z + y) \sin(5x + 5y), \cos(3x - 3y) \cos(z + y) \cos(5x + 5y)]] \end{aligned} \quad (2)$$

$$> H := \text{eval}(H, [x = 0, y = 0, z = 0]);$$

$$H := \begin{bmatrix} 34 & 16 & 0 \\ 16 & 35 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad (3)$$

> # Now that we have the hessian, take the eigenvectors out

$$> J, Q := \text{JordanForm}(H, \text{output} = ['J', 'Q'])$$

$$\begin{aligned} J, Q := & \left[\left[\frac{(39520 + 54i\sqrt{1794374})^{1/3}}{3} + \frac{1894}{3(39520 + 54i\sqrt{1794374})^{1/3}}, \right. \right. \\ & \left. \left. + \frac{70}{3}, 0, 0 \right], \right] \end{aligned} \quad (4)$$

$$\begin{aligned}
& \left[0, -\frac{(39520 + 54 I \sqrt{1794374})^{1/3}}{6} - \frac{947}{3 (39520 + 54 I \sqrt{1794374})^{1/3}} \right. \\
& + \frac{70}{3} \\
& - \frac{1}{2} \left(I \sqrt{3} \left(\frac{(39520 + 54 I \sqrt{1794374})^{1/3}}{3} \right. \right. \\
& \left. \left. - \frac{1894}{3 (39520 + 54 I \sqrt{1794374})^{1/3}} \right) \right), 0 \Big], \\
& \left[0, 0, -\frac{(39520 + 54 I \sqrt{1794374})^{1/3}}{6} - \frac{947}{3 (39520 + 54 I \sqrt{1794374})^{1/3}} \right. \\
& + \frac{70}{3} \\
& + \frac{1}{2} \left(I \sqrt{3} \left(\frac{(39520 + 54 I \sqrt{1794374})^{1/3}}{3} \right. \right. \\
& \left. \left. - \frac{1894}{3 (39520 + 54 I \sqrt{1794374})^{1/3}} \right) \right) \Big], \left[\right. \\
& \left(2 (21423609 I (39520 + 54 I \sqrt{1794374})^{4/3} \sqrt{1794374} \right. \\
& - 533520 \sqrt{3} (39520 + 54 I \sqrt{1794374})^{5/3} \sqrt{1794374} \\
& - 1600560 I (39520 + 54 I \sqrt{1794374})^{5/3} \sqrt{1794374} \\
& + 7748504457015228 I (39520 + 54 I \sqrt{1794374})^{1/3} \sqrt{3} \\
& \left. \left. - 41286483 \sqrt{3} (39520 + 54 I \sqrt{1794374})^{4/3} \sqrt{1794374} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& - 3038689104 \operatorname{I} \sqrt{3} (39520 + 54 \operatorname{I} \sqrt{1794374})^{4/3} \\
& - 1376461569 (39520 + 54 \operatorname{I} \sqrt{1794374})^{5/3} + 137583055926 \operatorname{I} (39520 \\
& + 54 \operatorname{I} \sqrt{1794374})^{2/3} \sqrt{1794374} + 23584283082 (39520 \\
& + 54 \operatorname{I} \sqrt{1794374})^{2/3} \sqrt{1794374} \sqrt{3} - 97209607728 (39520 \\
& + 54 \operatorname{I} \sqrt{1794374})^{4/3} - 85685895709408 \operatorname{I} \sqrt{3} (39520 \\
& + 54 \operatorname{I} \sqrt{1794374})^{2/3} + 4402657789632 \operatorname{I} (39520 + 54 \operatorname{I} \sqrt{1794374})^{1/3} \\
& \sqrt[3]{1794374} + 4402657789632 \sqrt{3} \sqrt{1794374} (39520 \\
& + 54 \operatorname{I} \sqrt{1794374})^{1/3} + 458820523 \operatorname{I} \sqrt{3} (39520 + 54 \operatorname{I} \sqrt{1794374})^{5/3} \\
& + 347443077231792 \sqrt{3} \sqrt{1794374} - 100690414262880 (39520 \\
& + 54 \operatorname{I} \sqrt{1794374})^{2/3} + 5917589914101924 (39520 + 54 \operatorname{I} \sqrt{1794374})^{1/3}) \\
&) / (2841 (947 \operatorname{I} \sqrt{3} (39520 + 54 \operatorname{I} \sqrt{1794374})^{2/3} - 19760 \operatorname{I} (39520 \\
& + 54 \operatorname{I} \sqrt{1794374})^{1/3} \sqrt{3} - 27 \sqrt{3} \sqrt{1794374} (39520 + 54 \operatorname{I} \sqrt{1794374})^{1/3} \\
& - 27 \operatorname{I} (39520 + 54 \operatorname{I} \sqrt{1794374})^{1/3} \sqrt{1794374} + 947 (39520 \\
& + 54 \operatorname{I} \sqrt{1794374})^{2/3} + 19760 (39520 + 54 \operatorname{I} \sqrt{1794374})^{1/3} - 3587236) \\
& (39520 + 54 \operatorname{I} \sqrt{1794374})^{1/3} (19760 \operatorname{I} (39520 + 54 \operatorname{I} \sqrt{1794374})^{1/3} \sqrt{3}
\end{aligned}$$

$$\begin{aligned}
& + 27 \sqrt{3} \sqrt{1794374} (39520 + 54 I \sqrt{1794374})^{1/3} + 81 I (39520 \\
& + 54 I \sqrt{1794374})^{1/3} \sqrt{1794374} - 1793618 I \sqrt{3} - 59280 (39520 \\
& + 54 I \sqrt{1794374})^{1/3} - 5380854), (2 (72641529 I (39520 \\
& + 54 I \sqrt{1794374})^{4/3} \sqrt{1794374} - 533520 \sqrt{3} (39520 \\
& + 54 I \sqrt{1794374})^{5/3} \sqrt{1794374} + 1600560 I (39520 \\
& + 54 I \sqrt{1794374})^{5/3} \sqrt{1794374} - 915457271456652 I (39520 \\
& + 54 I \sqrt{1794374})^{1/3} \sqrt{3} + 9931437 \sqrt{3} (39520 + 54 I \sqrt{1794374})^{4/3} \\
& \sqrt{1794374} - 47085459312 I \sqrt{3} (39520 + 54 I \sqrt{1794374})^{4/3} \\
& + 1376461569 (39520 + 54 I \sqrt{1794374})^{5/3} - 170998159266 I (39520 \\
& + 54 I \sqrt{1794374})^{2/3} \sqrt{1794374} + 56999386422 (39520 \\
& + 54 I \sqrt{1794374})^{2/3} \sqrt{1794374} \sqrt{3} - 53162837520 (39520 \\
& + 54 I \sqrt{1794374})^{4/3} + 93188154986144 I \sqrt{3} (39520 \\
& + 54 I \sqrt{1794374})^{2/3} - 4402657789632 I (39520 + 54 I \sqrt{1794374})^{1/3} \\
& \sqrt{1794374} - 4402657789632 \sqrt{3} \sqrt{1794374} (39520 \\
& + 54 I \sqrt{1794374})^{1/3} + 458820523 I \sqrt{3} (39520 + 54 I \sqrt{1794374})^{5/3}
\end{aligned}$$

$$\begin{aligned}
& + 347443077231792 \sqrt{3} \sqrt{1794374} + 279564464958432 (39520 \\
& + 54 I \sqrt{1794374})^{2/3} + 14581551642573804 (39520 \\
& + 54 I \sqrt{1794374})^{1/3}) / (2841 (947 I \sqrt{3} (39520 + 54 I \sqrt{1794374})^{2/3} \\
& - 19760 I (39520 + 54 I \sqrt{1794374})^{1/3} \sqrt{3} - 27 \sqrt{3} \sqrt{1794374} (39520 \\
& + 54 I \sqrt{1794374})^{1/3} - 27 I (39520 + 54 I \sqrt{1794374})^{1/3} \sqrt{1794374} \\
& + 947 (39520 + 54 I \sqrt{1794374})^{2/3} + 19760 (39520 + 54 I \sqrt{1794374})^{1/3} \\
& - 3587236) (39520 + 54 I \sqrt{1794374})^{1/3} (19760 I (39520 \\
& + 54 I \sqrt{1794374})^{1/3} \sqrt{3} + 27 \sqrt{3} \sqrt{1794374} (39520 + 54 I \sqrt{1794374})^{1/3} \\
& + 81 I (39520 + 54 I \sqrt{1794374})^{1/3} \sqrt{1794374} - 1793618 I \sqrt{3} \\
& - 59280 (39520 + 54 I \sqrt{1794374})^{1/3} - 5380854)), \\
& (458820523 I \sqrt{3} (39520 + 54 I \sqrt{1794374})^{4/3} - 533520 \sqrt{3} (39520 \\
& + 54 I \sqrt{1794374})^{4/3} \sqrt{1794374} + 533520 I (39520 \\
& + 54 I \sqrt{1794374})^{4/3} \sqrt{1794374} + 2175603031842 I \sqrt{3} (39520 \\
& + 54 I \sqrt{1794374})^{2/3} + 774842976 (39520 + 54 I \sqrt{1794374})^{2/3} \\
& \sqrt{1794374} \sqrt{3} + 458820523 (39520 + 54 I \sqrt{1794374})^{4/3} \\
& - 774842976 I (39520 + 54 I \sqrt{1794374})^{2/3} \sqrt{1794374} \\
& - 51473048478784 I (39520 + 54 I \sqrt{1794374})^{1/3} \sqrt{3} \\
& - 91722037284 I (39520 + 54 I \sqrt{1794374})^{1/3} \sqrt{1794374} \\
& + 2175603031842 (39520 + 54 I \sqrt{1794374})^{2/3} \\
& + 118599991320704 (39520 + 54 I \sqrt{1794374})^{1/3} - 4613271969911016)
\end{aligned}$$

$$\begin{aligned}
& \left/ \left(5095668738 \left(947 \sqrt{3} \left(39520 + 54 \sqrt{1794374} \right)^{2/3} \right. \right. \right. \\
& - 19760 \sqrt{3} \left(39520 + 54 \sqrt{1794374} \right)^{1/3} \sqrt{3} - 27 \sqrt{3} \sqrt{1794374} \left(39520 \right. \\
& + 54 \sqrt{1794374} \right)^{1/3} - 27 \sqrt{3} \left(39520 + 54 \sqrt{1794374} \right)^{1/3} \sqrt{1794374} \\
& + 947 \left(39520 + 54 \sqrt{1794374} \right)^{2/3} + 19760 \left(39520 + 54 \sqrt{1794374} \right)^{1/3} \\
& \left. \left. \left. - 3587236 \right) \right) \right], \\
& \left[- \left(181824 \sqrt{3} \left(156566 \sqrt{3} \left(39520 + 54 \sqrt{1794374} \right)^{1/3} - 603 \left(39520 \right. \right. \right. \right. \\
& + 54 \sqrt{1794374} \right)^{1/3} \sqrt{1794374} + 2758232 \sqrt{3} - 17046 \sqrt{1794374} \left. \right) \right) \right] / \\
& \left(\left(947 \sqrt{3} \left(39520 + 54 \sqrt{1794374} \right)^{2/3} - 19760 \sqrt{3} \left(39520 \right. \right. \right. \right. \\
& + 54 \sqrt{1794374} \right)^{1/3} \sqrt{3} - 27 \sqrt{3} \sqrt{1794374} \left(39520 + 54 \sqrt{1794374} \right)^{1/3} \\
& - 27 \sqrt{3} \left(39520 + 54 \sqrt{1794374} \right)^{1/3} \sqrt{1794374} + 947 \left(39520 \right. \\
& + 54 \sqrt{1794374} \right)^{2/3} + 19760 \left(39520 + 54 \sqrt{1794374} \right)^{1/3} - 3587236 \left. \right) \\
& \left(19760 \sqrt{3} \left(39520 + 54 \sqrt{1794374} \right)^{1/3} \sqrt{3} + 27 \sqrt{3} \sqrt{1794374} \left(39520 \right. \right. \\
& + 54 \sqrt{1794374} \right)^{1/3} + 81 \sqrt{3} \left(39520 + 54 \sqrt{1794374} \right)^{1/3} \sqrt{1794374} \\
& \left. \left. - 1793618 \sqrt{3} - 59280 \left(39520 + 54 \sqrt{1794374} \right)^{1/3} - 5380854 \right) \right), \\
& \left(90912 \left(156566 \sqrt{3} \left(39520 + 54 \sqrt{1794374} \right)^{1/3} \sqrt{3} \right. \right. \\
& - 603 \sqrt{3} \sqrt{1794374} \left(39520 + 54 \sqrt{1794374} \right)^{1/3} + 1809 \sqrt{3} \left(39520 \right. \\
& + 54 \sqrt{1794374} \right)^{1/3} \sqrt{1794374} + 2758232 \sqrt{3} - 17046 \sqrt{3} \sqrt{1794374} \\
& \left. \left. - 51138 \sqrt{1794374} + 469698 \left(39520 + 54 \sqrt{1794374} \right)^{1/3} - 82746966 \right) \right)
\end{aligned}$$

$$\begin{aligned}
&) / ((947I\sqrt{3} (39520 + 54I\sqrt{1794374})^{2/3} - 19760I(39520 \\
& + 54I\sqrt{1794374})^{1/3}\sqrt{3} - 27\sqrt{3}\sqrt{1794374} (39520 + 54I\sqrt{1794374})^{1/3} \\
& - 27I(39520 + 54I\sqrt{1794374})^{1/3}\sqrt{1794374} + 947(39520 \\
& + 54I\sqrt{1794374})^{2/3} + 19760(39520 + 54I\sqrt{1794374})^{1/3} - 3587236) \\
& (19760I(39520 + 54I\sqrt{1794374})^{1/3}\sqrt{3} + 27\sqrt{3}\sqrt{1794374} (39520 \\
& + 54I\sqrt{1794374})^{1/3} + 81I(39520 + 54I\sqrt{1794374})^{1/3}\sqrt{1794374} \\
& - 1793618I\sqrt{3} - 59280(39520 + 54I\sqrt{1794374})^{1/3} - 5380854)), \\
& (8(19760I\sqrt{3} (39520 + 54I\sqrt{1794374})^{2/3} + 27(39520 \\
& + 54I\sqrt{1794374})^{2/3}\sqrt{1794374}\sqrt{3} - 27I(39520 \\
& + 54I\sqrt{1794374})^{2/3}\sqrt{1794374} - 1793618I(39520 \\
& + 54I\sqrt{1794374})^{1/3}\sqrt{3} + 19760(39520 + 54I\sqrt{1794374})^{2/3} \\
& + 1793618(39520 + 54I\sqrt{1794374})^{1/3} - 240344812)) / \\
& (947(947I\sqrt{3} (39520 + 54I\sqrt{1794374})^{2/3} - 19760I(39520 \\
& + 54I\sqrt{1794374})^{1/3}\sqrt{3} - 27\sqrt{3}\sqrt{1794374} (39520 + 54I\sqrt{1794374})^{1/3} \\
& - 27I(39520 + 54I\sqrt{1794374})^{1/3}\sqrt{1794374} + 947(39520 \\
& + 54I\sqrt{1794374})^{2/3} + 19760(39520 + 54I\sqrt{1794374})^{1/3} - 3587236)) \\
&], \\
& \left[(181824\sqrt{3} (19760I(39520 + 54I\sqrt{1794374})^{1/3} + 27(39520 \\
& + 54I\sqrt{1794374})^{1/3}\sqrt{1794374} - 1793618I)) / ((947I\sqrt{3} (39520 \\
& + 54I\sqrt{1794374})^{2/3} - 19760I(39520 + 54I\sqrt{1794374})^{1/3}\sqrt{3}
\end{aligned}$$

$$\begin{aligned}
& -27\sqrt{3}\sqrt{1794374}(39520+54I\sqrt{1794374})^{1/3}-27I(39520 \\
& +54I\sqrt{1794374})^{1/3}\sqrt{1794374}+947(39520+54I\sqrt{1794374})^{2/3} \\
& +19760(39520+54I\sqrt{1794374})^{1/3}-3587236)(19760I(39520 \\
& +54I\sqrt{1794374})^{1/3}\sqrt{3}+27\sqrt{3}\sqrt{1794374}(39520+54I\sqrt{1794374})^{1/3} \\
& +81I(39520+54I\sqrt{1794374})^{1/3}\sqrt{1794374}-1793618I\sqrt{3} \\
& -59280(39520+54I\sqrt{1794374})^{1/3}-5380854)), \\
& -(90912(19760I(39520+54I\sqrt{1794374})^{1/3}\sqrt{3} \\
& +27\sqrt{3}\sqrt{1794374}(39520+54I\sqrt{1794374})^{1/3}-81I(39520 \\
& +54I\sqrt{1794374})^{1/3}\sqrt{1794374}-1793618I\sqrt{3}+59280(39520 \\
& +54I\sqrt{1794374})^{1/3}+5380854)))/((947I\sqrt{3}(39520 \\
& +54I\sqrt{1794374})^{2/3}-19760I(39520+54I\sqrt{1794374})^{1/3}\sqrt{3} \\
& -27\sqrt{3}\sqrt{1794374}(39520+54I\sqrt{1794374})^{1/3}-27I(39520 \\
& +54I\sqrt{1794374})^{1/3}\sqrt{1794374}+947(39520+54I\sqrt{1794374})^{2/3} \\
& +19760(39520+54I\sqrt{1794374})^{1/3}-3587236)(19760I(39520 \\
& +54I\sqrt{1794374})^{1/3}\sqrt{3}+27\sqrt{3}\sqrt{1794374}(39520+54I\sqrt{1794374})^{1/3} \\
& +81I(39520+54I\sqrt{1794374})^{1/3}\sqrt{1794374}-1793618I\sqrt{3} \\
& -59280(39520+54I\sqrt{1794374})^{1/3}-5380854)), -90912/
\end{aligned}$$

$$\begin{aligned} & (947\mathrm{I}\sqrt{3} (39520 + 54\mathrm{I}\sqrt{1794374})^{2/3} - 19760\mathrm{I} (39520 \\ & + 54\mathrm{I}\sqrt{1794374})^{1/3}\sqrt{3} - 27\sqrt{3}\sqrt{1794374} (39520 + 54\mathrm{I}\sqrt{1794374})^{1/3} \\ & - 27\mathrm{I} (39520 + 54\mathrm{I}\sqrt{1794374})^{1/3}\sqrt{1794374} + 947 (39520 \\ & + 54\mathrm{I}\sqrt{1794374})^{2/3} + 19760 (39520 + 54\mathrm{I}\sqrt{1794374})^{1/3} - 3587236) \end{aligned}$$

> J

$$\begin{aligned} & \left[\left[\frac{(39520 + 54\mathrm{I}\sqrt{1794374})^{1/3}}{3} + \frac{1894}{3(39520 + 54\mathrm{I}\sqrt{1794374})^{1/3}} + \frac{70}{3}, 0, (5) \right. \right. \\ & \left. \left. 0 \right], \right. \\ & \left[0, -\frac{(39520 + 54\mathrm{I}\sqrt{1794374})^{1/3}}{6} - \frac{947}{3(39520 + 54\mathrm{I}\sqrt{1794374})^{1/3}} \right. \\ & \left. + \frac{70}{3} \right. \\ & \left. - \frac{1}{2} \left(\mathrm{I}\sqrt{3} \left(\frac{(39520 + 54\mathrm{I}\sqrt{1794374})^{1/3}}{3} \right. \right. \right. \\ & \left. \left. - \frac{1894}{3(39520 + 54\mathrm{I}\sqrt{1794374})^{1/3}} \right) \right), 0 \right], \\ & \left[0, 0, -\frac{(39520 + 54\mathrm{I}\sqrt{1794374})^{1/3}}{6} - \frac{947}{3(39520 + 54\mathrm{I}\sqrt{1794374})^{1/3}} \right. \\ & \left. + \frac{70}{3} \right. \\ & \left. + \frac{1}{2} \left(\mathrm{I}\sqrt{3} \left(\frac{(39520 + 54\mathrm{I}\sqrt{1794374})^{1/3}}{3} \right. \right. \right. \\ & \left. \left. - \frac{1894}{3(39520 + 54\mathrm{I}\sqrt{1794374})^{1/3}} \right) \right) \right] \end{aligned}$$

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> evalf(J)
[[ 50.51822645 - 5. × 10-9 I, 0., 0.],
 [ 0., 18.51981170 + 1.932050808 × 10-8 I, 0.],
 [ 0., 0., 0.961961840 - 1.532050808 × 10-8 I]]

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(6)

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> w_1 := 50.51
w_1 := 50.51

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(7)

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> w_2 := 18.51
w_2 := 18.51

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(8)

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> w_2 := 0.961
w_2 := 0.961

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(9)

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>

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