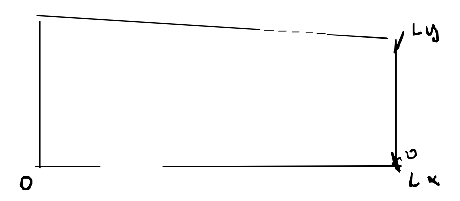
1)



punt: $u(x,y,t)$

$$\begin{cases} u(x,0,t)=0, \quad u(x,L_y,t)=0 \\ \frac{\partial}{\partial x} u(x,y,t)\Big|_{x=0} = 0, \quad \frac{\partial}{\partial x} u(x,y,t)\Big|_{x=L_x} = 0 \\ u(x,y,0) = f(x,y) \\ \frac{\partial}{\partial t} u(x,y,t)\Big|_{t=0} = g(x,y) \end{cases}$$

laplaciaan

i) 2D golfvgl oplossing: $c^2 \cdot \Delta u(x,y,t) = \partial_t^2 u(x,y,t)$

$$\Leftrightarrow c^2\left(\partial_x^2 u(x,y,t) + \partial_y^2 u(x,y,t)\right) = \partial_t^2 u(x,y,t)$$

In korte notatie: $\underline{c^2 \cdot (u_{xx} + u_{yy}) = u_{tt}}$

Randvw: boven, onder:

$$\begin{cases} u(x,0,t)=0 \\ u(x,L_y,t)=0 \end{cases}$$

links en rechts:

$$\begin{cases} \partial_x u(x,y,t)\big|_{x=0} = 0 \\ \partial_x u(x,y,t)\big|_{x=L_x} = 0 \end{cases}$$

Beginvw:

$$\begin{cases} u(x,y,0) = f(x,y) \\ \partial_t u(x,y,t)\big|_{t=0} = g(x,y) \end{cases}$$

Stap 1: scheiding van veranderlijken.

$$u(x,y,t) = X(x)\cdot Y(y)\cdot T(t)$$

we vullen dit in: $c^2 X''(x)\, Y(y)\cdot T(t) + c^2 \cdot Y''(y)\cdot X(x)\cdot T(t) = T''(t)\cdot X(x)\cdot Y(y)$

deel door $XYT \Rightarrow c^2\left(\underbrace{\frac{X''(x)}{X(x)}}_{\sigma_x} + \underbrace{\frac{Y''(y)}{Y(y)}}_{\sigma_y}\right) = \underbrace{\frac{T''(t)}{T(t)}}_{c^2\cdot(\sigma_x+\sigma_y)}$

dus we hebben dv's:

$X''(x) = X(x)\cdot\sigma_x$ (1)

$Y''(y) = Y(y)\cdot\sigma_y$ (2)

$T''(t) = c^2\cdot(\sigma_x+\sigma_y)\cdot T(t)$ (3)

we lossen (1) en (2) nu op via opgelegde randvoorwaarden:
De linker en rechter randvoorwaarden worden opgelegd op $X(x)$ en boven en onder op $Y(y)$. Links en rechts zijn Neumann voorwaarden.

$$\begin{cases} X''(x) = X(x)\cdot\sigma_x, \quad X'(0)=0, \quad X'(L_x)=0 \\ Y''(y) = Y(y)\cdot\sigma_y, \quad Y(0)=0, \quad Y(L_y)=0 \end{cases}$$

dus:

$X(x) = A\cos(kx) + B\sin(kx)$ met $B=0$, en $kL = m\pi$
$\Leftrightarrow k = \frac{m\pi}{L_x}$

dus $X_m(x) = \cos\left(\frac{m_x\pi}{L_x}\cdot x\right)$, $m_x = 0,1,2,\dots$

en $Y_m(x) = \sin\left(\frac{m_y\pi}{L_y}\cdot y\right)$, $m_y = 1,2,3,\dots$

met:

$$\sigma_{m_x} = -\left(\frac{m_x\pi}{L_x}\cdot x\right)^2, \quad \sigma_{m_y} = -\left(\frac{m_y\pi}{L_y}\cdot y\right)^2$$

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nu dat deze waarden kennen, kijken we naar (3):

$$T''_{m_x,m_y}(t) = -c^2\cdot\left(\left(\frac{m_x\pi}{L_x}x\right)^2 + \left(\frac{m_y\pi}{L_y}\cdot y\right)^2\right)\cdot T_{m_x,m_y}(t)$$

omdat dit negatief is, gebruiken we cosinus en sinus.

$$T_{m_x,m_y}(t) = S_{m_x,m_y}\cdot\cos\left(c\pi\cdot\sqrt{\frac{m_x^2}{L_x^2}+\frac{m_y^2}{L_y^2}}\,t\right) + V_{m_x,m_y}\cdot\sin\left(c\cdot\pi\cdot\sqrt{\frac{m_x^2}{L_x^2}+\frac{m_y^2}{L_y^2}}\,t\right)$$

dus elke eigen oplossing:

$$u_{m_x,m_y}(x,y,t) = X_{m_x}(x)\, Y_{m_y}(y)\cdot T_{m_x,m_y}(t)$$

$$= \cos\left(\frac{m_x\pi}{L_x}x\right)\cdot\sin\left(\frac{m_y\pi\cdot y}{L_y}\right)\cdot S_{m_x,m_y}\cdot\cos\left(c\pi\cdot\sqrt{\frac{m_x^2}{L_x^2}+\frac{m_y^2}{L_y^2}}\,t\right) + V_{m_x,m_y}\cdot\sin\left(c\cdot\pi\cdot\sqrt{\frac{m_x^2}{L_x^2}+\frac{m_y^2}{L_y^2}}\,t\right)$$

Stap 2: normale modes

$$\sum_{m_x=0}^{+\infty}\sum_{m_y=0}^{+\infty}\cos\left(\frac{m_x\pi}{L_x}x\right)\cdot\sin\left(\frac{m_y\pi\cdot y}{L_y}\right)\cdot S_{m_x,m_y}\cdot\cos\left(c\pi\cdot\sqrt{\frac{m_x^2}{L_x^2}+\frac{m_y^2}{L_y^2}}\,t\right) + V_{m_x,m_y}\cdot\sin\left(c\cdot\pi\cdot\sqrt{\frac{m_x^2}{L_x^2}+\frac{m_y^2}{L_y^2}}\,t\right)$$

Stap 3: beginvoorwaarden:

(1) $f(x,y) = \sum_{m_x=0}^{+\infty}\sum_{m_y=1}^{+\infty}\cos\left(\frac{m_x\pi}{L_x}x\right)\cdot\sin\left(\frac{m_y\pi}{L_y}y\right)S_{m_x,m_y}$

(2) $g(x,y) = \sum_{m_x=0}^{+\infty}\sum_{m_y=1}^{+\infty}\cos\left(\frac{m_x\pi}{L_x}x\right)\sin\left(\frac{m_y\pi}{L_y}y\right)\cdot c\pi\sqrt{\frac{m_x^2}{L_x^2}+\frac{m_y^2}{L_y^2}}\; V_{m_x,m_y}$

[ok nu kunnen we orthogonaliteitsrelaties opstellen

(1): we vermenigvuldigen $\cos\left(\frac{m\pi}{L_x}x\right)$, integreren over $0 \to L$:

$$\int_0^{L_x} f(x,y)\cdot\cos\left(\frac{m\pi}{L_x}x\right)dx = \sum_{m_x=0}^{+\infty}\sum_{m_y=1}^{+\infty}\left(\int_0^{L_x}\cos\left(\frac{m_x\cdot\pi}{L_x}x\right)\cos\left(\frac{m\pi}{L_x}x\right)dx\right)\sin\left(\frac{m_y\pi}{L_y}y\right)S_{m_x,m_y}$$

$$= \frac{L_x}{2}\cdot\sum_{m_y=1}^{+\infty}\sin\left(\frac{m_y\pi y}{L_y}\right)S_{m,m_y}$$

Nu gaan we in de y-directie:

$\vdots$

$$= \frac{L_x}{2}\frac{L_y}{2}S_{m,k}$$

we krijgen dus: $S_{m,n} = \frac{4}{L_xL_y}\cdot\int_0^{L_y}\int_0^{L_x}\left(f(x,y)\cos\left(\frac{m\pi}{L_x}x\right)\sin\left(\frac{n\pi}{L_y}y\right)\right)dx\,dy$

$$S_{0,n} = \frac{4}{L_xL_y}\cdot\int_0^{L_x}\int_0^{L_y}\left(f(x,y)\cdot\sin\left(\frac{n\pi y}{L_y}\right)\right)dx\,dy$$

$$V_{m,n} = \frac{4}{L_xL_y}\cdot\frac{1}{c\cdot\pi}\cdot\left(\sqrt{\frac{m^2}{L_x^2}+\frac{n^2}{L_y^2}}\right)^{-1}\int_0^{L_y}\int_0^{L_x}\left(g(x,y)\cos\left(\frac{m\pi}{L_x}x\right)\cdot\sin\left(\frac{n\pi}{L_y}y\right)\right)dx\,dy$$

$V_{0,n} = \dots$

ii) als $g(x,y)=0 \to V_{m,n}=0$!

$$S_{m,n} = \frac{1}{3}\cdot\int_0^3\int_0^4\left(f(x,y)\cos\left(\frac{m\pi}{4}x\right)\cdot\sin\left(\frac{n\pi}{3}y\right)\right)dx\,dy$$

$$S_{0,n} = \frac{1}{6}\cdot\int_0^3\int_0^4\left(f(x,y)\cdot\sin\left(\frac{n\pi}{3}y\right)\right)dx\,dy$$

de totale oplossing luidt:

$$u(x,y,t) = \sum_{m_x=0}^{+\infty}\sum_{m_y=1}^{+\infty}\cos\left(\frac{m_x\pi}{4}x\right)\sin\left(\frac{m_y\pi}{3}y\right)\left(S_{m_x,m_y}\cos\left(\pi\cdot\sqrt{\frac{m_x^2}{16}+\frac{m_y^2}{9}}\,t\right)\right)$$