1) golfvgl: $u_{tt} = c^2 u_{xx}$

$\Leftrightarrow \partial_t^2 u(x,t) = c^2 \cdot \partial_x^2 u(x,t)$

$\hookrightarrow$ met $x \in [0,L]$, $t \in [0, +\infty[$

Beginvoorwaarden:

$$\begin{cases} u(x,0) = f(x) \\ u'(x,0) = g(x) \end{cases}$$

randvw:

$$\begin{cases} u_x(0,t) = 0 \rightsquigarrow \text{Neuman} \\ u(L,t) = h_0 \rightsquigarrow \text{dirichlet} \end{cases}$$

1) Scheiding van veranderlijken

$u_h(x,t) = X(x) \cdot T(t)$

$\partial_t^2 X(x) \cdot T(t) = c^2 \cdot \partial_x^2 X(x) \cdot T(t)$

$\Leftrightarrow \dfrac{T''(t)}{T(t)} = c^2 \cdot \dfrac{X''(x)}{X(x)} = \sigma$

$$\begin{cases} X''(x) = \sigma \cdot X(x) \\ T''(x) = \sigma \cdot c^2 \cdot T(t) \end{cases}$$

2) Bepaling normale modes

Homogene randvw

randvw.

$X''(x) = \sigma X(x)$, met $X'(0) = 0$, $X(L) = 0$

1) $\sigma > 1 \to X_1(x) = c_1 e^{\sqrt{\sigma} \cdot x} + c_2 \cdot e^{-\sqrt{\sigma} x}$

2) $\sigma = 0 \to X_2(x) = c_1 + c_2 \cdot x$

3) $\sigma < 1 \to c_1 \cos(kx) + c_2 \sin(kx)$, $k^2 = -\sigma$

1) $\left.\begin{array}{l} X_1'(0) = \sqrt{\sigma}(c_1 - c_2) = 0 \\ X_1(L) = c_1 + c_2 = 0 \end{array}\right\} \to c_1 = c_2 = 0 \to$ triviaal

2) $\left.\begin{array}{l} X_2'(0) = c_2 = 0 \\ X_2(L) = c_1 = 0 \end{array}\right\} \to$ triviaal

3) $X_3'(0) = -c_1 k \cdot \sin(kx) + c_2 \cdot k \cdot \cos(kx)$

$= c_2 \cdot k = 0$

$X_3(L) = c_1 \cdot \cos(kL) + \underbrace{c_2 \cdot \sin(kL)}_{\neq 0} = 0$

dus $c_1 = 0$ tenzij:

$k = \dfrac{(2n+1)}{L} \cdot \dfrac{\pi}{2}$

$\hookrightarrow$ niet triviaal!

$X_n(x) = c_1 \cdot \cos\left(\dfrac{(2n+1)}{L} \cdot \dfrac{\pi \cdot x}{2}\right)$

of nog:

$\sigma_n = -\left(\dfrac{(2n+1)\pi}{2L}\right)^2$

dus: $T_n(t) = s_n \cdot \cos\left(c \cdot \dfrac{(2n+1) \cdot \pi}{2L} \cdot t\right) + t_n \sin\left(c \cdot \dfrac{(2n+1)\pi t}{2L}\right)$

en dus

$$u_h(x,t) = \sum_{n=0}^{+\infty} \left(s_n \cdot \cos\left(c \cdot \frac{(2n+1)\pi \cdot t}{2L}\right) + t_n \cdot \sin\left(\frac{c \cdot (2n+1) \cdot \pi \cdot t}{2L}\right)\right) \cdot \cos\left(\frac{(2n+1) \cdot \pi \cdot x}{2L}\right)$$

inhomogene randvoorwaarden

$$\begin{cases} u'(0,t) = 0 \\ u(L,t) = h_0 \end{cases}$$

$\to u(x,t) = u_p(x) + u_h(x,t)$

$= h_0 + u_h(x,t)$

3. verwerking beginvw:

bepaal s_n, t_n

general solution:

$$u(x,t) = h_0 + \sum_{n=0}^{+\infty} \left(s_n \cdot \cos\left(c \cdot \frac{(2n+1)\pi \cdot t}{2L}\right) + t_n \cdot \sin\left(\frac{c \cdot (2n+1) \cdot \pi t}{2L}\right)\right) \cdot \cos\left(\frac{(2n+1)\pi x}{2L}\right)$$

dus if $u(x,0) = f(x) = h_0 + \sum_{n=0}^{+\infty} s_n \cdot \cos\left(\dfrac{(2n+1)\pi x}{2L}\right)$

$u'(x,0) = g(x) = \sum_{n=0}^{+\infty} t_n \cdot \dfrac{2n+\pi}{2L} \cdot \cos\left(\dfrac{(2n+1)\pi x}{2L}\right)$

$s_n = \dfrac{2}{L} \cdot \displaystyle\int_0^L \cos\left(\frac{(2n+1)\pi}{2L} \cdot x\right)(f_x - h_0) \cdot dx$

$t_n = \dfrac{4}{(2n+1)\pi c} \cdot \displaystyle\int_0^L \cos\left(\frac{(2n+1)\pi}{2L} x\right) \cdot g(x)\, dx$