Zero to Hero: WiMo

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1 Analyse in 1 veranderlijke: enkele aspecten

1.1 Continuiteitseigenschappen van functies

Functie $f(x)$ is contine over $]a, b[$ als:

1. $f(x)$ bestaat in elk punt
2. de limiet van $f(x)$ bestaat in elk punt

Continue afgeleide: $f(x)$ is continue (zie hierboven) en $f'(x)$ bestaat in elk punt. Dit kan:

1. gladde functies zijn: elk afgeleide is continue
2. stuksgewijs: $f(x)$ heeft een singulariteit, maar het bestaat in deel intervallen $]a, c[\ ]c, b[$

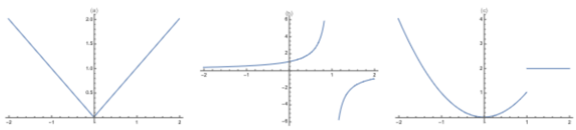

Figure 1: a) Continue functie, stukgewijs continue afgeleide (als je afleid krijg je een singulareit) b) Heeft een singulaireit, dus stukgewijs continue, stukgewijs glad continue afleidbaar c) Deze is glad stukgewijs continue afleidbaar, is ook stukgewijs continue

1.2 Taylorontwikkeling

We willen zaken gaan benaderen. Hiervoor gebruiken we:

$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...$

waarbij a het **werkpunt** is.

Veel voorkomende Taylorontwikkelingen:

See Figure 2.

Storingsrekening

Zie Figuur 4.

Of Maple solution 3.

1.3 Twee eenvoudige differentiaalvergelijkingen

1.3.1 Eerste orde differentiaalvergelijking

$y'(x) = \lambda y(x)$

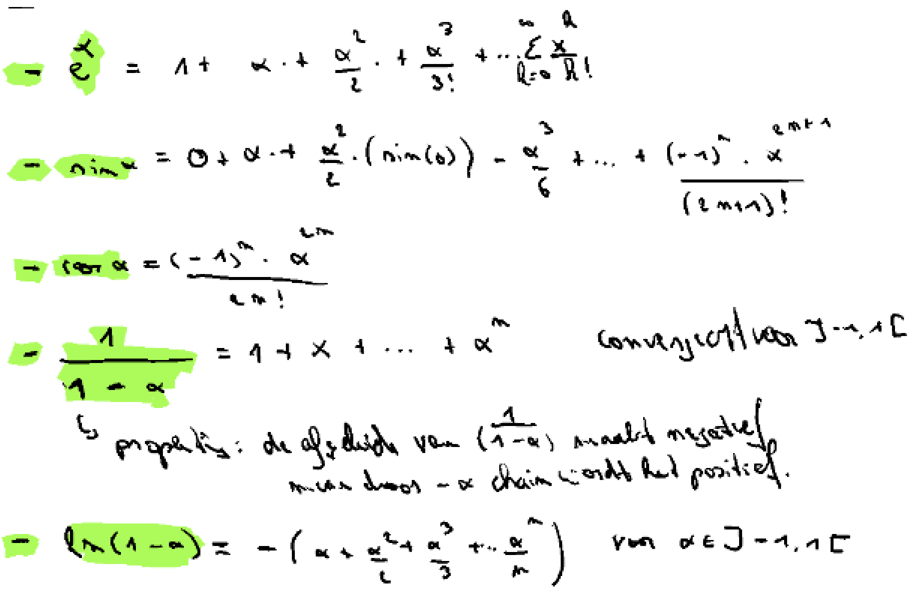

Figure 2: Simply use the formulas

Als we dit uitwerken krijgen we:

$ln(y(x)) = \lambda x + C$

$y(x) = e^{\lambda x + C} = e^C e^{\lambda x} = Ce^{\lambda x}$ met $C = y(0)$

$y(x) = y(0)e^{\lambda x}$

Radioactief verval

Zie Figuur 5.

1.3.2 Tweede orde differentiaalvergelijking

$y''(x) = \lambda y(x)$

Hierbij heb je 3 gevallen:

1. $\lambda > 0$: $y(t) = Ae^{\sqrt{\lambda}t} + Be^{-\sqrt{\lambda}t}$
2. $\lambda = 0$: $y(t) = A + Bt$
3. $\lambda < 0$: $y(t) = Acos(\sqrt{-\lambda}t) + Bsin(\sqrt{-\lambda}t)$

1.3.3 Complexe getallen

Algemene vorm: $z = a + bi$

waarbij a reeel, b imaginair en $i^2 = -1$

inverse: $(a + bi)^{-1} = \frac{a-bi}{a^2+b^2}$

complement: $z = a + bi \rightarrow z^* = a - bi$

modulus: $|z| = \sqrt{a^2 + b^2}$

in **polaire vorm**:

$e^{i\theta} = cos(\theta) + isin(\theta)$ (Dit kan via Taylor bewezen worden (zie oefeningen))

Okey, nu nog een paar goniometrische formules:

> # x^3 + epsilon $\cdot x = 1$

>

> $f := x \to x^3 + \text{epsilon} \cdot x = 1$

$$f := x \mapsto x^3 + \mathrm{e} \cdot x = 1 \tag{1}$$

> # *When epsilon is null, we get x = 1*

> $g := 1 + u \cdot \text{epsilon} + v \cdot \text{epsilon}^2$

$$g := v\,\mathrm{e}^2 + u\,\mathrm{e} + 1 \tag{2}$$

> $f\_subs := subs(x = g, f(x))$

$$f\_subs := (v\,\mathrm{e}^2 + u\,\mathrm{e} + 1)^3 + \mathrm{e}\,(v\,\mathrm{e}^2 + u\,\mathrm{e} + 1) = 1 \tag{3}$$

> $f\_expand := expand(f\_subs)$

$$f\_expand := \mathrm{e}^6 v^3 + 3\,\mathrm{e}^5 u\,v^2 + 3\,\mathrm{e}^4 u^2 v + 3\,\mathrm{e}^4 v^2 + \mathrm{e}^3 u^3 + 6\,\mathrm{e}^3 u\,v + \mathrm{e}^3 v + 3\,\mathrm{e}^2 u^2 + \mathrm{e}^2 u + 3\,v\,\mathrm{e}^2 + 3\,u\,\mathrm{e} + \mathrm{e} + 1 = 1 \tag{4}$$

> # *First keep the left hand side*

> $left\_hand\_side := lhs(f\_expand)$

$$left\_hand\_side := \mathrm{e}^6 v^3 + 3\,\mathrm{e}^5 u\,v^2 + 3\,\mathrm{e}^4 u^2 v + 3\,\mathrm{e}^4 v^2 + \mathrm{e}^3 u^3 + 6\,\mathrm{e}^3 u\,v + \mathrm{e}^3 v + 3\,\mathrm{e}^2 u^2 + \mathrm{e}^2 u + 3\,v\,\mathrm{e}^2 + 3\,u\,\mathrm{e} + \mathrm{e} + 1 \tag{5}$$

> # *Extract coeff 1 and 2*

> $coeff\_1 := coeff(left\_hand\_side, \text{epsilon}, 1)$

$$coeff\_1 := 3\,u + 1 \tag{6}$$

> $coeff\_2 := coeff(left\_hand\_side, \text{epsilon}, 2)$

$$coeff\_2 := 3\,u^2 + u + 3\,v \tag{7}$$

> $solve(\{coeff\_1 = 0, coeff\_2 = 0\}, \{u, v\})$

$$\left\{u = -\frac{1}{3}, v = 0\right\} \tag{8}$$

> # *Final result*

> $result := subs\left(\left\{u = -\frac{1}{3}, v = 0\right\}, g\right)$

$$result := -\frac{\mathrm{e}}{3} + 1 \tag{9}$$

>

Figure 3: Maple solution

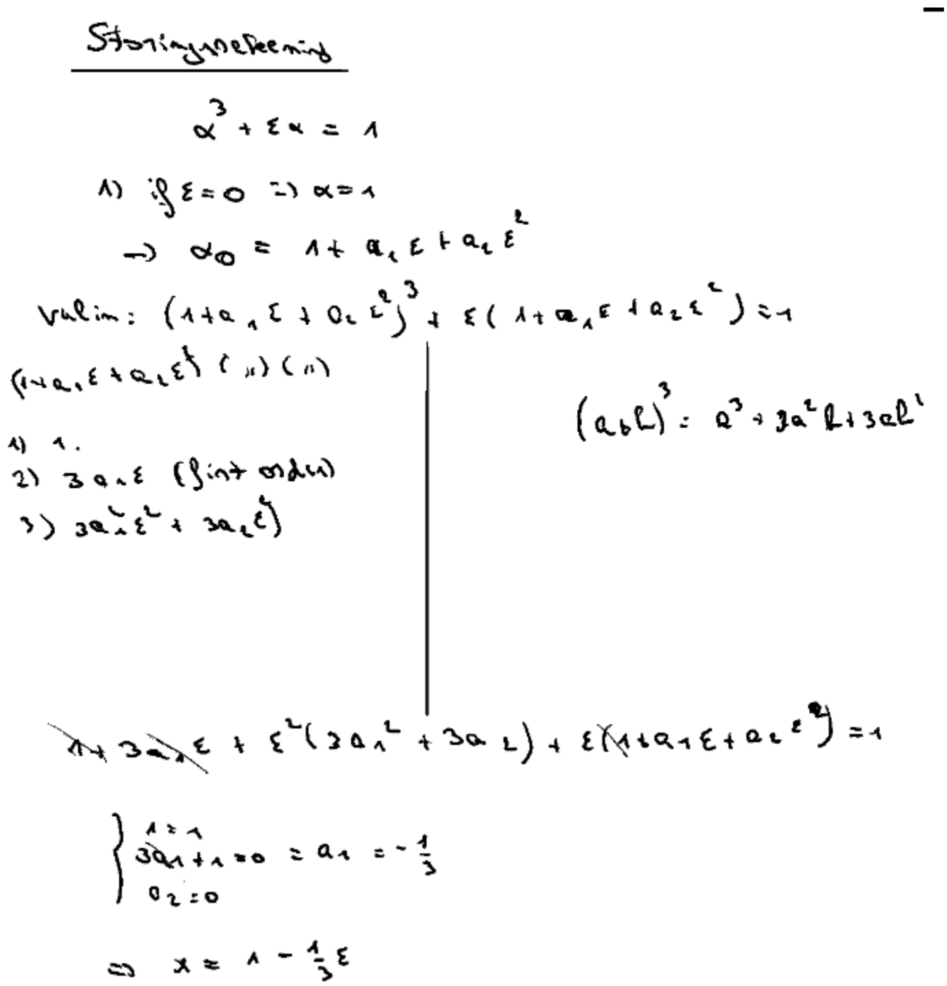

Figure 4: 1. Merk op dat als epsilon 0 is, dan is x = 1. Dus we benaderen value 1: $1 + \epsilon.u + \epsilon^2.v$. Vul dit in the main equation. Gebruik maple om dit op te lossen en vul u en v in x_1

radioactief

$$N'(t) = -\lambda N(t)$$

$$\Leftrightarrow \frac{N'(t)}{N(t)} = -\lambda$$

$$\Leftrightarrow \ln(N(t)) = -\lambda t + c$$

$$\Leftrightarrow N(t) = e^{-\lambda t} \cdot \underline{e^{c}} = N_0(t)$$

$$\Leftrightarrow N(t) = N_0(t) \cdot e^{-\lambda t}$$

halveringstijd: $\frac{N_0}{2}$

$$\Rightarrow \frac{N_0(t)}{2} = N_0(t) \cdot e^{-\lambda t}$$

$$\Leftrightarrow \frac{1}{2} = e^{-\lambda t_{1/2}}$$

$$\Leftrightarrow \overset{0}{\ln(1)} - \ln(2) = -\lambda t_{1/2}$$

$$\Leftrightarrow \ln(2) = \lambda t_{1/2}$$

$$\Leftrightarrow \frac{\ln(2)}{\lambda} = t_{1/2}$$

Figure 5: Vindt eerst de differentiaalvergelijking (zie eerste differentiaalvergelijking). Dan kunnen we de oplossing gelijkstellen aan $N_0/2$. Werk dit uit en je hebt $t_{1/2}$ gevonden

$sin^2(x) + cos^2(x) = 1$

$sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$

$sin(2x) = 2sin(x)cos(x)$

$cos(x) = \frac{e^{ix} + e^{-ix}}{2}$

$cos(2x) = cos^2(x) - sin^2(x)$

1.3.4 Hoofdstelling van de algebra

Als we een kwadratisch veelterm hebben: $ax^2 + bx + c = 0$

Dan vinden we de nulpunten (oplossingen) met:

$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$

$x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$

met $b = -4 * a * c$ vinden we de discriminant.

Formularium

Taylorontwikkeling

- $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...$
- $sin(x) = x$ voor kleine x

Differentiaalvergelijkingen

- $y'(x) = \lambda y(x)$
- $y''(x) = \lambda y(x)$ (hier werden 3 gevallen besproken)

Complexe getallen

- $z = a + bi$ (algemene vorm)
- $i^2 = -1$
- **inverse**: $(a+bi)^{-1} = \frac{a-bi}{a^2+b^2}$
- **complement**: $z = a + bi \rightarrow z^* = a - bi$
- **modulus**: $|z| = \sqrt{a^2+b^2}$
- $e^{i\theta} = cos(\theta) + isin(\theta)$
- $sin^2(x) + cos^2(x) = 1$
- $sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$
- $sin(2x) = 2sin(x)cos(x)$
- $cos(x) = \frac{e^{ix} + e^{-ix}}{2}$
- $cos(2x) = cos^2(x) - sin^2(x)$

Hoofdstelling van de algebra

$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$

$x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$

met $b = -4 * a * c$

Oefeningen

Huis 1

> $restart : with(LinearAlgebra) :$

> $t := taylor(\mathrm{sqrt}(x+4)^3, x = 0, 3);$
$v := \mathrm{sqrt}(y+4)^3;$

$$t := 8 + 3\,x + \frac{3}{16}\,x^2 + \mathrm{O}(x^3)$$

$$v := (y+4)^{3/2} \qquad (1)$$

> # voor $5^{\frac{3}{2}}$ nemen we $x = 1$

> $x\_1 := evalf\left(8 + 3 + \frac{3}{16}\right)$

$$x\_1 := 11.18750000 \qquad (2)$$

> # Wat is de fout? Wel, dat zal de derde term zijn ($O(x^3)$)

> $error\_1 := \mathrm{abs}\left(\frac{x^3}{6} \cdot diff(v, y\$3)\right);$
$error\_1 := evalf(subs(y = 0, x = 1, error\_1))$

$$error\_1 := \frac{|x|^3}{16\,|y+4|^{3/2}}$$

$$error\_1 := 0.007812500000 \qquad (3)$$

> # voor $6^{\frac{3}{2}}$ nemen we $x = 2$

> $x\_2 := evalf\left(8 + 3 \cdot 2 + \frac{3}{16} \cdot 2^2\right)$

$$x\_2 := 14.75000000 \qquad (4)$$

> $error\_2 := \mathrm{abs}\left(\frac{x^3}{6} \cdot diff(v, y\$3)\right)$

$$error\_2 := \frac{|x|^3}{16\,|y+4|^{3/2}} \qquad (5)$$

> $error\_2 := evalf(subs(x = 2, y = 0, error\_2))$

$$error\_2 := 0.06250000000 \qquad (6)$$

> # Waarom nemen we y = 0? Omdat dit de grootste fout zou maken, we nemen altijd max. Dus fout \element {0, 1, 2}

>

Figure 6: Exercise 1

```
> restart:
> # i
> limit(sin(3·x)/sinh(x), x = 0)
```

$$3 \tag{1}$$

```
> # ii
> limit((tan(x) − tanh(x))/(sinh(x) − x), x = 0)
```

$$4 \tag{2}$$

```
> # iii
> limit((sqrt(1 − a·x) − sqrt(1 + a·x))/x, x = 0)
```

$$-a \tag{3}$$

```
> # IV
> limit((ln(1 + x) + ln(1 − x))/x^2, x = 0)
```

$$-1 \tag{4}$$

```
>
```

Figure 7: Exercise 2

$$\exp(i\theta) = \cos\theta + i\sin\theta \qquad B:$$

$$e^x = \sum \frac{x^n}{n!} \rightarrow e^{i\theta} = \frac{\sum (i\theta)^n}{n!}$$

$$= 1 + i\theta - \frac{\theta^2}{2} - i\cdot\frac{\theta^3}{6} + \frac{\theta^4}{24} + \cdots$$

$$= \underbrace{i\left(\theta - \frac{\theta^3}{6} + \cdots\right)}_{\sin\theta} + \underbrace{-\frac{\theta^2}{2} + \frac{\theta^4}{24} + \cdots}_{\cos\theta}$$

$$= \cos\theta + i\sin\theta$$

Figure 8: Exercise 3

$$a = \overset{3\times 1}{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}} \quad b = \overset{3\times 1}{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}$$

$$a\cdot b = 1\cdot(-1) + (1\cdot 1) + (-1\cdot 0) = 0$$

$$A = ab^T = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \cdot (-1, 1, 0)$$

$$= \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$$

Figure 9: Exercise 4

WC 1

```
> p := x→x·(x − 1) − a
                    p := x ↦ x·(x − 1) − a                              (1)
> # i
> p_i := subs(a = 0, p(x))
                    p_i := x (x − 1)                                    (2)
> solve(p_i = 0)
                    0, 1                                                (3)
> # Dus lambda = 0, en lambda = 1
> # Benader naar x = 0
> x_0 := 0 + u·a + v·a^2
                    x_0 := v a^2 + u a                                  (4)
> p_subs_0 := subs(x = x_0, p(x))
          p_subs_0 := (v a^2 + u a) (v a^2 + u a − 1) − a               (5)
> p_expand_0 := expand(p_subs_0)
          p_expand_0 := a^4 v^2 + 2 a^3 u v + a^2 u^2 − v a^2 − u a − a (6)
> coeff_1_0 := coeff(p_expand_0, a, 1)
                    coeff_1_0 := −u − 1                                 (7)
> coeff_2_0 := coeff(p_expand_0, a, 2)
                    coeff_2_0 := u^2 − v                                (8)
> solve_0 := solve({coeff_1_0 = 0, coeff_2_0 = 0}, {u, v})
                    solve_0 := {u = −1, v = 1}                          (9)
> result_0 := subs(u = rhs(solve_0[1]), v = rhs(solve_0[2]), x_0)

                    result_0 := a^2 − a                                 (10)
>
> # Benader 1
> x_1 := 1 + u·a + v·a^2
                    x_1 := v a^2 + u a + 1                              (11)
> p_subs_1 := subs(x = x_1, p(x))
          p_subs_1 := (v a^2 + u a + 1) (v a^2 + u a) − a               (12)
> p_expand_1 := expand(p_subs_1)
          p_expand_1 := a^4 v^2 + 2 a^3 u v + a^2 u^2 + v a^2 + u a − a (13)
> coeff_1_1 := coeff(p_expand_1, a, 1)
                    coeff_1_1 := u − 1                                  (14)
> coeff_1_2 := coeff(p_expand_1, a, 2)
                    coeff_1_2 := u^2 + v                                (15)
> solve_1 := solve({coeff_1_1 = 0, coeff_1_2 = 0}, {u, v})
                    solve_1 := {u = 1, v = −1}                          (16)
> result_1 := subs(u = rhs(solve_1[1]), v = rhs(solve_1[2]), x_1)
```

Figure 10: Exercise 1

```
> # Define the Taylor expansion for a forward approximation
  Taylor_y := (h, t) → y(t) + h*diff(y(t), t) + (1/2)*h^2*diff(y(t), t$2) + (1/6)
      *h^3*diff(y(t), t$3) + (1/24)*h^4*diff(y(t), t$4) :

  # Define the Taylor expansion for a backward approximation
  Taylor_g := (h, t) → y(t) − h*diff(y(t), t) + (1/2)*h^2*diff(y(t), t$2) − (1/6)
      *h^3*diff(y(t), t$3) + (1/24)*h^4*diff(y(t), t$4) :

  # Define the result expression
  result := (h, t) → (Taylor_y(h, t) + Taylor_g(h, t) − 2*y(t)) / h^2 :

  # Simplify the result
  simplified_result := simplify(result(h, t));
```

$$simplified\_result := \frac{h^2 \left(\frac{d^4}{dt^4} y(t) \right)}{12} + \frac{d^2}{dt^2} y(t) \qquad (1)$$

```
>
  #i) kwadratisch
> #ii
> restart;
  with(plots) :
  with(plottools) :

  t_val := evalf(Pi/3);

  # Define the function y and its 2nd derivative
  y := t → −cos(t);

  # Define the error function as a function of h for a specific t
  err := (h) → abs(y(t_val)
      − (cos(h + t_val) + cos(h − t_val) − 2*cos(t_val))/h^2);

  # Plot the error as a function of h with log-log scale
  loglogplot(err(h), h = 10^(−8) .. 10^3);
```

$$t\_val := 1.047197551$$

$$y := t \mapsto -\cos(t)$$

$$err := h \mapsto \left| y(t\_val) - \frac{\cos(h + t\_val) + \cos(h - t\_val) - 2 \cdot \cos(t\_val)}{h^2} \right|$$

Figure 11: Exercise 2

3) $v_1 = \begin{pmatrix} 2i \\ -1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} i \\ 0 \\ 1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1-i \\ 0 \\ 1 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix}$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{(2i, -1, 0, 0)^T}{\sqrt{(2i)^2 + (-1)^2}} = \frac{''}{\sqrt{2i \cdot (-2i) + 1}}$$

$$= \left(\frac{2i}{\sqrt{5}}, \frac{-1}{\sqrt{5}}, 0, 0\right)^T$$

$$\vec{u}_2 = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2) \cdot \vec{u}_1 =$$

$$u_3 = v_3 - \sum_{k=1}^{n-1} \langle u_k \cdot v_k \rangle \cdot u_k$$

$\vdots$

---

2) span $\langle u_1, u_2 \rangle$

$$w_0^{\parallel} = \langle u_1 \cdot w \rangle u_1 + \langle u_2 \cdot w \rangle u_2$$

Remember: $w = w^{\parallel} + w^{\perp}$

Figure 12: Exercise 3

> #ii)

> with(LinearAlgebra) :

> $v\_1 := Vector([2\cdot I, -1, 0, 0])$

$$v\_1 := \begin{bmatrix} 2\,\mathrm{I} \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad (1)$$

> $v\_2 := Vector([I, 0, 3, 1])$

$$v\_2 := \begin{bmatrix} \mathrm{I} \\ 0 \\ 3 \\ 1 \end{bmatrix} \quad (2)$$

> $u\_1 := \dfrac{v\_1}{Norm(v\_1, 2)}$

$$u\_1 := \begin{bmatrix} \frac{2\,\mathrm{I}}{5}\sqrt{5} \\ -\frac{\sqrt{5}}{5} \\ 0 \\ 0 \end{bmatrix} \quad (3)$$

> $u\_2 := \dfrac{(v\_2 - (u\_1 \cdot v\_2) \cdot u\_1)}{Norm(v\_2 - (u\_1 \cdot v\_2) \cdot u\_1, 2)}$

$$u\_2 := \begin{bmatrix} \frac{\mathrm{I}}{255}\sqrt{255} \\ \frac{2\sqrt{255}}{255} \\ \frac{\sqrt{255}}{17} \\ \frac{\sqrt{255}}{51} \end{bmatrix} \quad (4)$$

> $w := Vector([3, 1 - I, 2 + I, 1])$

(5)

Figure 13: Exercise 3

Bord 1

```
> p := x→x·(x − 1)·(x − 2)·(x − 4) − a
```
$$p := x \mapsto x\cdot(x-1)\cdot(x-2)\cdot(x-4) - a \tag{1}$$

```
> x_0 := u·a + v·a^2
```
$$x\_0 := v\,a^2 + u\,a \tag{2}$$

```
> p_subs := subs(x = x_0, p(x))
```
$$p\_subs := (v\,a^2 + u\,a)\,(v\,a^2 + u\,a - 1)\,(v\,a^2 + u\,a - 2)\,(v\,a^2 + u\,a - 4) - a \tag{3}$$

```
> p_expand := expand(p_subs)
```
$$p\_expand := a^8 v^4 + 4\,a^7 u\,v^3 + 6\,a^6 u^2 v^2 - 7\,a^6 v^3 + 4\,a^5 u^3 v - 21\,a^5 u\,v^2 + a^4 u^4 - 21\,a^4 u^2 v + 14\,a^4 v^2 - 7\,a^3 u^3 + 28\,a^3 u\,v + 14\,a^2 u^2 - 8\,v\,a^2 - 8\,u\,a - a \tag{4}$$

```
> # Only keep the 1st order and second order
>
> coeff_1 := coeff(p_expand, a, 1)
```
$$coeff\_1 := -8\,u - 1 \tag{5}$$

```
> coeff_2 := coeff(p_expand, a, 2)
```
$$coeff\_2 := 14\,u^2 - 8\,v \tag{6}$$

```
> solve({coeff_1 = 0, coeff_2 = 0}, {u, v})
```
$$\left\{u = -\frac{1}{8},\, v = \frac{7}{256}\right\} \tag{7}$$

```
>
```

Figure 14: Exercise 1

2) i) $y'(t_n) \approx v_n = \frac{y_{n+1} - y_n}{h}$

$$\begin{cases} c = t_n \\ t_{n+1} = t_n + h \end{cases}$$

$$y(t_{n+1}) = y(t_n) + h \cdot y'(t_n) + \frac{h^2}{2} y''(\xi) \quad \xi \in [t_n, t_{n+1}]$$

$$\Rightarrow v_n = \frac{y_{n+1} - y_n}{h} = \frac{\cancel{y(t_n)} + h\, y'(t_n) + \frac{h^2}{2} y''(\xi) - \cancel{y(t_n)}}{h}$$

$$= y'(t_n) + \frac{h}{2} y''(\xi)$$

fout: $|y'(t_n) - v_n| = \frac{h}{2} y''(\xi) = \frac{h}{2} C \rightarrow$ linear.

↳ zal ook visueel te zien.

ii)

$$y'(t_n) = v_n = \frac{y_{n+1} - y_{n-1}}{2h}$$

$$- \; y(t_{n+1}) = y(t_n) + h \cdot y'(t_n) + \frac{h^2}{2} y''(t_n) + \frac{h^3}{6} y'''(\xi) \quad \xi \in [t_n, t_{n+1}]$$

$$- \; y(t_{n-1}) = y(t_n) - h\, y'(t_n) + \frac{h^2}{2} y''(t_n) - \frac{h^3}{6} y'''(\xi) \quad \xi \in [t_{n-1}, t_n]$$

$$y'(t_n) = v_n = \frac{\cancel{y(t_n)} + h\, y'(t_n) + \cancel{\frac{h^2}{2} y''(t_n)} + \frac{h^3}{6} y'''(\xi_+) - \left(\cancel{y(t_n)} - h\, y'(t_n) + \cancel{\frac{h^2}{2} y''(t_n)} - \frac{h^3}{6} y'''(\xi_-)\right)}{2h}$$

$$= y'(t_n) + \frac{h^2}{12}\left(y'''(\xi_+) + y'''(\xi_-) \right)$$

$$|y'(t_n) - v_n| = \frac{h^2}{6} |y'''(\xi)| = \frac{h^2}{6} C \rightarrow$$ hoping maybe you can see this.

Figure 15: Exercise 2

> $restart;$

> $with(plots):$

>

> $y := t \rightarrow \cos(t)$

$$y := t \mapsto \cos(t) \tag{1}$$

> $exact\_speed := t \rightarrow -\sin(t)$

$$exact\_speed := t \mapsto -\sin(t) \tag{2}$$

> $t\_val := evalf\left(\frac{\mathrm{Pi}}{2}\right)$

$$t\_val := 1.570796327 \tag{3}$$

>

> $forward\_difference := (t, h) \rightarrow \frac{(y(t+h) - y(t))}{h}$

$$forward\_difference := (t, h) \mapsto \frac{y(t+h) - y(t)}{h} \tag{4}$$

> $central\_difference := (t, h) \rightarrow \frac{(y(t+h) - y(t-h))}{2 \cdot h}$

$$central\_difference := (t, h) \mapsto \frac{y(t+h) - y(t-h)}{2 \cdot h} \tag{5}$$

>

> $forward\_error := h \rightarrow \mathrm{abs}(exact\_speed(t\_val) - forward\_difference(t\_val, h))$

$$forward\_error := h \mapsto |exact\_speed(t\_val) - forward\_difference(t\_val, h)| \tag{6}$$

> $central\_error := h \rightarrow \mathrm{abs}(exact\_speed(t\_val) - central\_difference(t\_val, h))$

$$central\_error := h \mapsto |exact\_speed(t\_val) - central\_difference(t\_val, h)| \tag{7}$$

> $loglogplot([central\_error(h), forward\_error(h)], h = 10^{-8}..1, color = [red, blue])$

Figure 16: Exercise 2 part 2 Maple

3) $\vec{v_1} = (2,3,0)$ en $\vec{v_2} = (1,-2,3)$ inner product = dot product

i)

1) $\vec{u_1} = \vec{v_1}$

2) $$\vec{u_2} = \vec{v_2} - \frac{\langle \vec{v_2}, \vec{u_1} \rangle}{|\vec{u_1}|^2} \cdot \vec{u_1}$$

3) $$\vec{u_3} = \vec{v_3} - \frac{\langle \vec{v_3}, \vec{u_1} \rangle}{|\vec{u_1}|^2} \cdot \vec{u_1} - \frac{\langle \vec{v_3} \cdot \vec{u_2} \rangle}{|u_2|^2} \cdot \vec{u_2}$$

< Zijn haar ook $v_1 \times v_2 = v_3$ → daarmee verder doen.

$$\boxed{\vec{u_k} = \vec{v_k} - \sum_{k=1}^{k-1} \frac{\langle \vec{v_k} \cdot \vec{u_k} \rangle}{|\vec{u_k}|^2} \cdot u_k}$$

$$\vec{u_1} = \frac{(2,3,0)}{\sqrt{4+9}} = \left(\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, 0\right) = \left(\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}, 0\right.$$

$$\vec{u_2} = \frac{v_2 - (u_1 \cdot v_2) \cdot \vec{u_1}}{\| v_2 - (u_1 \cdot v_2) \cdot u_1 \|} = \text{in maple:}$$

$$\left(\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}, 0\right) \cdot (1,-2,3)$$

$$= \left(\frac{2\sqrt{13}}{13}, -\frac{6\sqrt{13}}{13}, 0\right) \cdot \left(\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}, 0\right)$$

$$= \left(\frac{52}{169}, \frac{-13 \cdot 13}{169}, 0\right)$$

ii) zie maple

iii) aangezien $\vec{u_1}$ en $\vec{u_2}$ deze van vectoren die opgespannen als $\vec{v_1}$ en $\vec{v_2}$ kunnen we:

$$\vec{y}^{\parallel} = (\vec{y} \cdot \vec{u_1}) \cdot \vec{u_1} + (\vec{y} \cdot \vec{u_2}) \cdot \vec{u_2}$$

dan $\vec{y}^{\perp} = \vec{y} - \vec{y}^{\parallel} \Leftrightarrow \vec{y} = \vec{y}^{\parallel} + \vec{y}^{\perp}$

Figure 17: Exercise 3

```
restart;
with(plots) : with(plottools)
```

[annulus, arc, arrow, circle, cone, cuboid, curve, cutin, cutout, cylinder, disk, dodecahedron, ellipse, ellipticArc, exportplot, extrude, getdata, hemisphere, hexahedron, homothety, hyperbola, icosahedron, importplot, line, octahedron, parallelepiped, pieslice, point, polygon, polygonbyname, prism, project, rectangle, reflect, rotate, scale, sector, semitorus, sphere, stellate, tetrahedron, torus, transform, translate, triangulate]

$v1 := \langle 2|3|0 \rangle$

$$v1 := \begin{bmatrix} 2 & 3 & 0 \end{bmatrix}$$

$v2 := \langle 1, -2, 3 \rangle$

$$v2 := \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$u1 := \left\langle \frac{2 \cdot \mathrm{sqrt}(13)}{13} \middle| \frac{3 \cdot \mathrm{sqrt}(13)}{13} \middle| 0 \right\rangle$

$$u1 := \begin{bmatrix} \frac{2\sqrt{13}}{13} & \frac{3\sqrt{13}}{13} & 0 \end{bmatrix}$$

$u2 := \left\langle \frac{52}{169} \middle| -\frac{18 \cdot 13}{169} \middle| 0 \right\rangle$

$$u2 := \begin{bmatrix} \frac{4}{13} & -\frac{18}{13} & 0 \end{bmatrix}$$

```
origin := ⟨0|0|0⟩;

line_v1 := line(origin, v1, color = blue);
line_v2 := line(origin, v2, color = red);
line_u1 := line(origin, u1, color = green);
line_u2 := line(origin, u2, color = yellow);

# Display the lines together
display(line_v1, line_v2, line_u1, line_u2, axes = normal, scaling = constrained);
```

$$origin := \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$line\_v1 := CURVES\left(\begin{bmatrix} 0. & 0. & 0. \\ 2. & 3. & 0. \end{bmatrix}, COLOUR(RGB, 0., 0., 1.00000000)\right)$$

$$line\_v2 := CURVES\left(\begin{bmatrix} 0. & 0. & 0. \\ 1. & -2. & 3. \end{bmatrix}, COLOUR(RGB, 1.00000000, 0., 0.)\right)$$

$$line\_u1 := CURVES\left(\begin{bmatrix} 0. & 0. & 0. \\ 0.554700196225229 & 0.832050294337844 & 0. \end{bmatrix}, COLOUR(RGB, 0., 1.00000000, 0.)\right)$$

$$line\_u2 := CURVES\left(\begin{bmatrix} 0. & 0. & 0. \\ 0.307692307692308 & -1.38461538461538 & 0. \end{bmatrix}, COLOUR(RGB, 1.00000000, 1.00000000, 0.)\right)$$

Figure 18: Exercise 3 - plot

```
> restart;
> with(LinearAlgebra) : with(plottools) : with(plots) :
> #i
  v_1 := Vector([2, 3, 0])
```

$$v\_1 := \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \tag{1}$$

```
> v_2 := Vector([1, -2, 3])
```

$$v\_2 := \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \tag{2}$$

$$> u\_1 := \frac{v\_1}{Norm(v\_1, 2)}$$

$$u\_1 := \begin{bmatrix} \frac{2\sqrt{13}}{13} \\ \frac{3\sqrt{13}}{13} \\ 0 \end{bmatrix} \tag{3}$$

$$> u\_2 := \frac{(v\_2 - (u\_1 \cdot v\_2) \cdot u\_1)}{Norm(v\_2 - (u\_1 \cdot v\_2) \cdot u\_1, 2)}$$

$$u\_2 := \begin{bmatrix} \frac{21\sqrt{2158}}{2158} \\ -\frac{7\sqrt{2158}}{1079} \\ \frac{3\sqrt{2158}}{166} \end{bmatrix} \tag{4}$$

```
> # ii
> null_vector := Vector([0, 0, 0])
```

$$null\_vector := \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

```
> line_v1 := line(null_vector, v_1, color = purple)
```

$$line\_v1 := CURVES\left(\left[\begin{bmatrix} 0. & 0. & 0. \\ 2. & 3. & 0. \end{bmatrix}\right], COLOUR(RGB, 0.50196078, 0., \right. \tag{6}$$

Figure 19: Exercise 3