The def for DFT is:

$$\tilde{\alpha}_k = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} \alpha_m e^{-i\frac{2\pi k m}{N}}$$

$$\rightarrow \tilde{\alpha}_k = \frac{1}{\sqrt{N}} \cdot \sum_{m=0}^{N-1} \underbrace{\sin(\omega m \Delta t)}_{\frac{e^{i\omega m \Delta t} - e^{-i\omega m \Delta t}}{2i}} e^{-i\frac{2\pi k m}{N}}$$

$$= \frac{1}{2i\sqrt{N}} \cdot \sum_{m=0}^{N-1} \left(e^{i\omega m \Delta t} - e^{-i\omega m \Delta t}\right) e^{-i\frac{2\pi k m}{N}}$$

$$= \text{''} \left(e^{i\left(\omega \Delta t - \frac{2\pi k}{N}\right) m} - e^{-i\left(\omega \Delta t + \frac{2\pi k}{N}\right) m} \right)$$

we gebruiken $\displaystyle\sum_{m=0}^{N-1} z^m = \frac{1 - z^N}{1 - z}$

dus: $$\tilde{\alpha}_k = \frac{1}{2i\sqrt{N}} \cdot \left(\frac{1 - z^N}{1 - z} - \frac{1 - z^N}{1 - z} \right)$$

stel $\alpha = \omega \Delta t - \frac{2\pi m k}{N}$, $\beta = \omega \Delta t + \frac{2\pi k}{N}$

$z_1 = e^{i\alpha}$, $z_2 = e^{-i\beta..}$

$$\Leftarrow \tilde{\alpha}_k = \frac{1}{2i\sqrt{N}} \cdot \left(\frac{1 - e^{i\alpha_1 \cdot N}}{1 - e^{i\alpha}} - \frac{1 - e^{-i\beta \cdot N}}{1 - e^{-i\beta}} \right)$$

Remember: $$1 - e^{i\alpha N} = e^{i\alpha \cdot \frac{N}{2}} \left(e^{-i\alpha \cdot \frac{N}{2}} - e^{i\alpha \cdot \frac{N}{2}} \right)$$

$$= e^{i\alpha\frac{N}{2}} \cdot \left(-2i \sin\left(\frac{\alpha \cdot N}{2}\right)\right)$$

$$1 - e^{i\alpha} = e^{i\frac{\alpha}{2}} \cdot \left(e^{-\alpha/2} + e^{i\alpha/2} \right)$$

$$= e^{i\frac{\alpha}{2}} \cdot \left(-2i \cdot \sin\left(\frac{\alpha}{2}\right)\right)$$

$$\Rightarrow \tilde{\alpha}_k = \frac{1}{2i\sqrt{N}} \cdot \left(\frac{e^{i\alpha\frac{N}{2}} \cdot \sin\left(\alpha \frac{N}{2}\right)}{e^{i\alpha/2} \cdot \sin\left(\frac{\alpha}{2}\right)} - \frac{e^{i\beta\frac{N}{2}} \cdot \sin\left(\beta\frac{N}{2}\right)}{e^{i\beta/2} \sin\left(\frac{\beta}{2}\right)} \right)$$

$$= \frac{1}{2i\sqrt{N}} \cdot \left(e^{i\frac{\alpha}{2}\cdot(N-1)} \cdot \frac{\sin\left(\frac{\alpha \cdot N}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} - e^{i\frac{\beta}{2}(N-1)} \cdot \frac{\sin\left(\frac{\beta N}{2}\right)}{\sin(\beta/2)} \right)$$

$\blacksquare$