

# Zero to Hero: WiMo

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## 1 Analyse in 1 veranderlijke: enkele aspecten

### 1.1 Continuiteitseigenschappen van functies

Functie  $f(x)$  is continue over  $]a, b[$  als:

1.  $f(x)$  bestaat in elk punt
2. de limiet van  $f(x)$  bestaat in elk punt

**Continue afgeleide:**  $f(x)$  is continue (zie hierboven) en  $f'(x)$  bestaat in elk punt. Dit kan:

1. gladde functies zijn: elk afgeleide is continue
2. stuksgewijs:  $f(x)$  heeft een singulariteit, maar het bestaat in deel intervallen  $]a, c[$   $]c, b[$

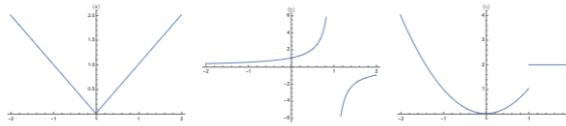


Figure 1: a) Continue functie, stuksgewijs continue afgeleide (als je afleid krijg je een singulareit) b) Heeft een singulaireit, dus stuksgewijs continu, stuksgewijs glad continu afleidbaar c) Deze is glad stuksgewijs continu afleidbaar, is ook stuksgewijs continu

### 1.2 Taylorontwikkeling

We willen zaken gaan benaderen. Hiervoor gebruiken we:

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \dots$$

waarbij  $a$  het **werkpunt** is.

Veel voorkomende Taylorontwikkelingen:

See Figure 2.

### Storingsrekening

Zie Figuur 4.

Of Maple solution 3.

### 1.3 Twee eenvoudige differentiaalvergelijkingen

#### 1.3.1 Eerste orde differentiaalvergelijking

$$y'(x) = \lambda y(x)$$

$$\begin{aligned}
 - e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots + \sum_{k=0}^n \frac{x^k}{k!} \\
 - \sin x &= 0 + x + \frac{x^2}{2} \cdot (\sin(0)) - \frac{x^3}{3!} + \dots + \frac{(-1)^{k+1} \cdot x^{2k+1}}{(2k+1)!} \\
 - \tan x &= \left( -\frac{1}{1-x} \right)^{\infty} \\
 - \frac{1}{1-x} &= 1 + x + \dots + x^n \quad \text{convergent voor } x \in ]-1, 1[ \\
 \hookrightarrow \text{propositie: de afgeleide van } \frac{1}{1-x} \text{ maakt negatief}\\
 \text{van door } -x \text{ daarin wordt het positief.} \\
 - \ln(1-x) &\approx - \left( x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^n}{n} \right) \quad \text{van } x \in ]-1, 1[
 \end{aligned}$$

Figure 2: Simply use the formulas

Als we dit uitwerken krijgen we:

$$\ln(y(x)) = \lambda x + C$$

$$y(x) = e^{\lambda x + C} = e^C e^{\lambda x} = C e^{\lambda x} \text{ met } C = y(0)$$

$$y(x) = y(0) e^{\lambda x}$$

### Radioactief verval

Zie Figuur 5.

### 1.3.2 Tweede orde differentiaalvergelijking

$$y''(x) = \lambda y(x)$$

Hierbij heb je 3 gevallen:

1.  $\lambda > 0$ :  $y(t) = A e^{\sqrt{\lambda}t} + B e^{-\sqrt{\lambda}t}$
2.  $\lambda = 0$ :  $y(t) = A + Bt$
3.  $\lambda < 0$ :  $y(t) = A \cos(\sqrt{-\lambda}t) + B \sin(\sqrt{-\lambda}t)$

### 1.3.3 Complexe getallen

Algemene vorm:  $z = a + bi$

waarbij  $a$  reeel,  $b$  imaginair en  $i^2 = -1$

$$\text{inverse: } (a + bi)^{-1} = \frac{a - bi}{a^2 + b^2}$$

$$\text{complement: } z = a + bi \rightarrow z^* = a - bi$$

$$\text{modulus: } |z| = \sqrt{a^2 + b^2}$$

in polaire vorm:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta) \quad (\text{Dit kan via Taylor bewezen worden (zie oefeningen)})$$

Okey, nu nog een paar goniometrische formules:

```

> # x^3 + epsilon · x = 1
> f := x → x^3 + epsilon · x = 1
f := x → x^3 + e · x = 1 (1)
> # When epsilon is null, we get x = 1
> g := 1 + u · epsilon + v · epsilon^2
g := v e^2 + u e + 1 (2)
> f_subs := subs(x = g, f(x))
f_subs := (v e^2 + u e + 1)^3 + e (v e^2 + u e + 1) = 1 (3)
> f_expand := expand(f_subs)
f_expand := e^6 v^3 + 3 e^5 u v^2 + 3 e^4 u^2 v + 3 e^4 v^2 + e^3 u^3 + 6 e^3 u v + e^3 v + 3 e^2 u^2
+ e^2 u + 3 v e^2 + 3 u e + e + 1 (4)
> # First keep the left hand side
> left_hand_side := lhs(f_expand)
left_hand_side := e^6 v^3 + 3 e^5 u v^2 + 3 e^4 u^2 v + 3 e^4 v^2 + e^3 u^3 + 6 e^3 u v + e^3 v
+ 3 e^2 u^2 + e^2 u + 3 v e^2 + 3 u e + e + 1 (5)
> # Extract coeff 1 and 2
> coeff_1 := coeff(left_hand_side, epsilon, 1)
coeff_1 := 3 u + 1 (6)
> coeff_2 := coeff(left_hand_side, epsilon, 2)
coeff_2 := 3 u^2 + u + 3 v (7)
> solve({coeff_1 = 0, coeff_2 = 0}, {u, v})
{u = -1/3, v = 0} (8)
> # Final result
> result := subs({u = -1/3, v = 0}, g)
result := -e/3 + 1 (9)

```

Figure 3: Maple solution

### Storingreeksing

$$\alpha^3 + \varepsilon u = 1$$

1) if  $\varepsilon = 0 \Rightarrow \alpha = 1$

$$\rightarrow \alpha_0 = 1 + \alpha_1 \varepsilon + \alpha_2 \varepsilon^2$$

Vul in:  $(1 + \alpha_1 \varepsilon + \alpha_2 \varepsilon^2)^3 + \varepsilon(1 + \alpha_1 \varepsilon + \alpha_2 \varepsilon^2) \approx 1$

$(1 + \alpha_1 \varepsilon + \alpha_2 \varepsilon^2)^3 \approx 1$

1) 1.

2)  $3\alpha_1 \varepsilon$  (fint en dan)

$\Rightarrow 3\alpha_1 \varepsilon^2 + 3\alpha_2 \varepsilon$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2$$

~~$\Rightarrow 3\alpha_1 \varepsilon + \varepsilon^2(2\alpha_1^2 + 3\alpha_1) + \varepsilon(1 + \alpha_1 \varepsilon + \alpha_2 \varepsilon^2) \approx 1$~~

$$\begin{cases} 1 = 1 \\ 3\alpha_1 + 1 = 0 \\ 0 = 0 \end{cases} \Rightarrow \alpha_1 = -\frac{1}{3}$$

$$\Rightarrow x = 1 - \frac{1}{3}\varepsilon$$

Figure 4: 1. Merk op dat als epsilon 0 is, dan is  $x = 1$ . Dus we benaderen value 1:  $1 + \varepsilon.u + \varepsilon^2.v$ . Vul dit in the main equation. Gebruik maple om dit op te lossen en vul  $u$  en  $v$  in  $x_1$

## Radioactie

$$N'(t) = -\lambda N(t)$$

$$\Leftrightarrow \frac{N'(t)}{N(t)} = -\lambda$$

$$\Leftrightarrow \ln(N(t)) = -\lambda t + C$$

$$\Leftrightarrow N(t) = e^{-\lambda t} \cdot e^C = N_0(t)$$

$$\Leftrightarrow N(t) = N_0(t) \cdot e^{-\lambda t}$$

$$\text{halveringstijd: } \frac{N_0}{2}$$

$$\Rightarrow \frac{N_0(0)}{2} = N_0(t) \cdot e^{-\lambda t}$$

$$\Leftrightarrow \frac{1}{2} = e^{-\lambda t_{1/2}}$$

$$\Leftrightarrow \frac{1}{2} = e^{-\lambda t_{1/2}}$$

$$\Leftrightarrow \ln\left(\frac{1}{2}\right) = -\lambda t_{1/2}$$

$$\Leftrightarrow \ln(2) = \lambda t_{1/2}$$

$$\Leftrightarrow \frac{\ln(2)}{\lambda} = t_{1/2}$$

Figure 5: Vindt eerst de differentiaalvergelijking (zie eerste differentiaalvergelijking). Dan kunnen we de oplossing gelijkstellen aan  $N_0/2$ . Werk dit uit en je hebt  $t_{1/2}$  gevonden

$$\begin{aligned}\sin^2(x) + \cos^2(x) &= 1 \\ \sin(x) &= \frac{e^{ix} - e^{-ix}}{2i} \\ \sin(2x) &= 2\sin(x)\cos(x) \\ \cos(x) &= \frac{e^{ix} + e^{-ix}}{2} \\ \cos(2x) &= \cos^2(x) - \sin^2(x)\end{aligned}$$

### 1.3.4 Hoofdstelling van de algebra

Als we een kwadratisch veelterm hebben:  $ax^2 + bx + c = 0$

Dan vinden we de nulpunten (oplossingen) met:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

met  $b = -4 * a * c$  vinden we de discriminant.

## 2 Lineare Algebra

### 2.1 Lineare onafhankelijkheid

**Lineare onafhankelijkheid** betekent dat de vectoren niet op een lijn liggen. Dit betekent dat de determinant van de matrix niet 0 is. Maar ook dat de vectoren niet een lineaire combinatie van elkaar zijn.

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**Voorbeeld:** beschouw de drie vectoren (we gebruiken hier al de kolom-notatie voor vectoren)

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad (2.1.3)$$

Figure 6: 1. Merk op dat als epsilon 0 is, dan is  $x = 1$ . Dus we benaderen value 1:  $1 + \epsilon.u + \epsilon^2.v$ . Vul dit in the main equation. Gebruik maple om dit op te lossen en vul  $u$  en  $v$  in  $x_1$

$v_1, v_2, v_3$  zijn lineair onafhankelijk. Maar  $v_1$  en  $v_2$  bijvoorbeeld, vormen een lineaire combinatie van  $v_3$

### 2.2 Inproduct, Norm, Orthogonaliteit

- **Inproduct:**  $uv = \sum_{i=1}^n u_i v_i$

- **Norm:**  $\|v\| = \sqrt{v \cdot v}$

Side note: om de **hoek** tussen 2 vectoren te vinden,  $\frac{uv}{\|u\|\|v\|} = \cos(\theta)$

### 2.3 Gramm-Schmidt

Dit gaat ons toelaten om een basis te vinden van vectoren.

$$v_1 = u_1 / \|u_1\|$$

$$v_2 = u_2 - \frac{u_2 \cdot v_1}{\|v_1\|^2} v_1$$

$$v_x = \dots$$

Ook definieerd het dat een vector  $v = v^{\parallel} + v^{\perp}$

Waarbij  $v^{\parallel}$  de projectie is van  $v$  op  $u$  en  $v^{\perp}$  de projectie op de orthogonale basis.

$$v^{\parallel} = (u_1 \dot{y})u_1 + (u_2 \dot{y})u_2 + \dots$$

### 2.3.1 Voorbeeld Gramm-Schmidt

Zie Figuur 7.

```

> with(LinearAlgebra):
> v_1 := Vector([1, 1, 0])
      v_1 := 
$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$
 (1)

> v_2 := Vector([0, 1, 1])
      v_2 := 
$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$
 (2)

> v_3 := Vector([1, 0, 1])
      v_3 := 
$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$
 (3)

> u_1 :=  $\frac{v_1}{\text{Norm}(v_1, 2)}$ 
      u_1 := 
$$\begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$
 (4)

> u_2 :=  $\frac{(v_2 - (u_1 \cdot v_2) \cdot u_1)}{\text{Norm}(v_2 - (u_1 \cdot v_2) \cdot u_1, 2)}$ 
      u_2 := 
$$\begin{bmatrix} -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \end{bmatrix}$$
 (5)

> u_3 :=  $(v_3 - ((u_2 \cdot v_3) \cdot u_2) - ((u_1 \cdot v_3) \cdot u_1)) / \text{Norm}(v_3 - ((u_2 \cdot v_3) \cdot u_2) - ((u_1 \cdot v_3) \cdot u_1), 2)$ ;
      (6)

```

Figure 7: Hier gebruiken we dus de iteratieve formule om de basis te vinden.

**Note:** Dit kan uitgebreid worden naar functieruimtes. Hiervoor gaan we een oefening zien in het werkcollege.

## 2.4 Matrices

Typische vorm:  $y = Ax$

Het **getransponeerde** is  $A^T$ , de rijen worden kolommen en omgekeerd.

Hier geldt dan dat:  $(AB)^T = B^T A^T$ , dus je draait de matrices om

## 2.5 Kolomruimte, rijruimte, nulruimte

- **Kolomruimte:** alle mogelijke lineaire combinaties van de kolommen ( $K(A)$ )
- **Rijruimte:** alle mogelijke lineaire combinaties van de rijen ( $K(A^T)$ )
- **Nulruimte:** alle vectoren die op 0 worden afgebeeld ( $N(A)$  of  $N(A^T)$ )

Zeer belangrijk is dat:

$N(A)$  complementair is aan  $K(A^T)$  en  $N(A^T)$  is complementair aan  $K(A)$

Bij de representatie van het vlak heb je een **rang**. Dit is basically de hoeveelheid kolommen van  $Q$  niet nul zijn.

### 2.5.1 Voorbeeld definities

Zie Figuur 8.

## 2.6 Matrix Inverse

$$x = A^{-1}y$$

Typisch gezien is dit gedaan via:  $\frac{1}{\det(A)} \text{adj}(A)$

Wat heel nuttig is is dat  $AA^{-1} = I$  en  $A^{-1}A = I$ . Dit kan heel wat schrijfwerk vermijden

Weer zoals bij transponeren geldt dat  $(AB)^{-1} = B^{-1}A^{-1}$

## 2.7 Projectie en kleinste kwadraten benadering

Zoals we eerder hebben gezien bij Gramm-Schmidt, kunnen we  $v = v^{\parallel} + v^{\perp}$  schrijven.

We kunnen er iets abstracter boven plakken en werken met een **Projector**.

Deze is gedefinieerd als:  $P = A(A^T A)^{-1} A^T$  (deze vorm is niet al te belangrijk).

Met de nieuwe definitie  $P$  kunnen we nu zeggen dat  $v = Pv + (I - P)v$

The following properties hold:

1.  $P^2 = P$
2.  $P^T = P$

### 2.7.1 Voorbeeld Projectie

Zie Figuur 9.

## 2.8 Kleinste kwadraten fit

$$x = (A^T A)^{-1} A^T y$$

Dit gaat ons toelaten om te fitten op data.

In Maple wordt dit gedaan via `LinearAlgebra:-LeastSquares`.

Vorlesung 5:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

max m:  
2 a 4:

Endlich linear aufg.  
holomorph!

$$K(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \quad n=2$$

$$N(A^T) = \overline{0}$$

$$K(A^T) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$N(A) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \quad \text{durch rechnen darf man nur:} \quad A\alpha = 0$$

$$A\vec{\alpha} = \vec{b}$$

$$\begin{cases} \alpha_1 + \alpha_3 = b_1 \\ \alpha_2 + \alpha_4 = b_2 \end{cases}$$

$$\vec{\alpha} = \frac{b_1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{b_2}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

linear Rcm:

$$\vec{\alpha} = \vec{\alpha}_p + \vec{\alpha}_o$$

orthogonale Vektoren  $\Delta = \cdot N(A)$

Figure 8:

```

[> with(LinearAlgebra):
=> A := Matrix([[1, 0], [0, 1], [0, c]])
      A := 
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & c \end{bmatrix}$$
 (1)
=> p := A.(Transpose(A).A)-1.Transpose(A)
      p := 
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{c^2+1} & \frac{c}{c^2+1} \\ 0 & \frac{c}{c^2+1} & \frac{c^2}{c^2+1} \end{bmatrix}$$
 (2)
=> subs(c = 0, p)
      
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
 (3)

```

Figure 9:

## 2.9 Vierkante matrices

### 2.9.1 Determinant

De determinant is super handig. Hoe berekenen we deze?

Let  $\mathbf{A}$  be a  $3 \times 3$  matrix:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The determinant of  $\mathbf{A}$ , denoted as  $\det(\mathbf{A})$  or  $|\mathbf{A}|$ , is calculated as:

$$\det(\mathbf{A}) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Expanding the terms, we have:

$$\det(\mathbf{A}) = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Properties:

1.  $\det(A) = \det(A^T)$
2.  $\det(AB) = \det(A)\det(B)$
3.  $\det(A) = 0 \rightarrow \text{linear dependent}$

## 2.10 Basic rotation matrix

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

So a typical transformation equation looks like:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ with the middle matrix being **inverse**.}$$

### 2.10.1 Example

See Figure 10 and 11.

**Voorbeeld: geroteerde basis.** Beschouw de basis  $\mathbf{v}_1 = \mathbf{e}_x = (1, 0)$ ,  $\mathbf{v}_2 = \mathbf{e}_y = (0, 1)$  en de geroteerde basis  $\mathbf{v}'_1 = \mathbf{e}'_x = (\cos \theta, \sin \theta)$ ,  $\mathbf{v}'_2 = \mathbf{e}'_y = (-\sin \theta, \cos \theta)$ . Wat is de transformatie, die voor een vector de componenten  $\mathbf{x}' = (x', y')$  t.o.v. de geroteerde basis geeft, in functie van de originele componenten  $\mathbf{x} = (x, y)$ .

Figure 10: Rotation opgave

## 2.11 Eigenvectoren, eigenwaarden, diagonalisatie en de Jordan-decompositie

$v_i$  is een eigenvector van  $A$  als  $Av_i = \lambda_i v_i$

$$p_A(\lambda) = \det(A - \lambda I) = 0$$

- geometrische multipliciteit: aantal lineair onafhankelijke eigenvectoren (je moet de eigenvalue invullen en row echelon reduced form verkrijgen. Dan zie je hoeveel eigenvectoren er degelijk zijn)

Figuur 12 toont een voorbeeld van de multipliciteit.

- algebraïsche multipliciteit: aantal keer dat de eigenwaarde voorkomt in de determinant

Indien alle geometrische multipliciteiten gelijk zijn aan de algebraïsche multipliciteiten, dan is de matrix diagonaliseerbaar.  $A = MDM^{-1}$

met  $D$  een diagonale matrix met de eigenwaarden op de diagonaal. en  $M$  een matrix met de eigenvectoren.

```

> restart: with(LinearAlgebra):
>
> M := Matrix([[cos(theta), -sin(theta)], [sin(theta), cos(theta)]])
M := 
$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$
 (1)

> inv := MatrixInverse(M)
inv := 
$$\begin{bmatrix} \frac{\cos(\theta)}{\cos(\theta)^2 + \sin(\theta)^2} & \frac{\sin(\theta)}{\cos(\theta)^2 + \sin(\theta)^2} \\ -\frac{\sin(\theta)}{\cos(\theta)^2 + \sin(\theta)^2} & \frac{\cos(\theta)}{\cos(\theta)^2 + \sin(\theta)^2} \end{bmatrix}$$
 (2)

> v := Vector([x, y])
v := 
$$\begin{bmatrix} x \\ y \end{bmatrix}$$
 (3)

> inv . v

$$\begin{bmatrix} \frac{\cos(\theta)x}{\cos(\theta)^2 + \sin(\theta)^2} + \frac{\sin(\theta)y}{\cos(\theta)^2 + \sin(\theta)^2} \\ -\frac{\sin(\theta)x}{\cos(\theta)^2 + \sin(\theta)^2} + \frac{\cos(\theta)y}{\cos(\theta)^2 + \sin(\theta)^2} \end{bmatrix}$$
 (4)

```

Figure 11: Rotation antwoord

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{aligned} & \xrightarrow{\text{det}} \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{pmatrix} = 0 = -\lambda(\lambda - 1) - 1(-\lambda - \lambda) + 1(1 + \lambda) \\ & = -\lambda^3 + \lambda + \lambda + 1 + \lambda \\ & = -\lambda^3 + 3\lambda + 1 \end{aligned}$$

$\hookrightarrow$  calculate eigenvalues:

$$\lambda = -1, -1, 2$$

$$\text{Eigen } \lambda_1 = -1$$

$$\xrightarrow{\text{det}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \xrightarrow{\text{row reduction}} \begin{matrix} x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \\ x_3 = 0 \end{matrix} \quad \begin{matrix} x_1 = -x_2 \\ x_2 = -x_3 \\ x_3 = 0 \end{matrix}$$

$$\xrightarrow{\text{det}} \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \quad \text{if } n=1, A=1$$

$$\Rightarrow \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \rightarrow \text{eigen vectors} \rightarrow \text{geometric mult.} = 2$$

Figure 12: Een voorbeeld van de multipliciteit

## 2.12 Jordan Form

$A = M J M^{-1}$  met  $J$  een Jordan matrix. Dit is nodig indien de dimensie van de eigenruimte (geometrische multiplicitet) kleiner is dan de algebraische multiplicitet. Aka, de matrix is niet diagonaliseerbaar.

$$\begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ * & * & * & \ddots & * \\ * & * & * & * & * \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}$$

Figure 13: Jordan matrix

This gives us:

### 2.12.1 Example

See Figure 16.

## 2.13 Matrixmachten en iteratieve matrixvergelijkingen

$A^k = MD^kM^{-1}$  voor diagonaliseerbare matrices

$A^k = MJ^kM^{-1}$  voor niet-diagonaliseerbare matrices

met diagonaal matrix:

**iteratieve matrixvergelijking:**  $x_k = MD^kM^{-1}x_0$  Op deze manier kun je telkens de  $k$ de stap berekenen.

Dit is enkel voor de diagonaliseerbare matrices. ( $A^k = MD^kM^{-1}$ )

**asymptotisch gedrag:**  $\lim_{k \rightarrow \infty} u_k = \lambda^k(v_L * u_0)v_1$

$v_1$  is een fixed point in het asymptotisch gedrag.

$$A = \begin{bmatrix} 3 & 1 & -3 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Figure 14: Jordan matrix example

Vinden we:

$$A = \left[ \begin{array}{c|cc|cc} 0 & -1 & 1/2+i & 1/2-i \\ 0 & 0 & 3/2+i/2 & 3/2-i/2 \\ 0 & 0 & 1/2+i & 1/2-i \\ 1 & 0 & 0 & 0 \end{array} \right] \left[ \begin{array}{cccc} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1-i & 0 \\ 0 & 0 & 0 & 1+i \end{array} \right] \left[ \begin{array}{ccc} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 2/5+i/5 & -1/5-3i/5 \\ 0 & 2/5-i/5 & -1/5+3i/5 \end{array} \right]$$

Hieruit lezen we de eigenwaarden af:  $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 1-i, \lambda_4 = 1+i$  en corresponderende (rechter)-eigenvectoren:

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1/2+i \\ 3/2+i/2 \\ 1/2+i \\ 0 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 1/2-i \\ 3/2-i/2 \\ 1/2-i \\ 0 \end{bmatrix}, \quad (2.3.120)$$

en linker-eigenvectoren:

$$\mathbf{v}_{L1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{v}_{L2} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_{L3} = \begin{bmatrix} 0 \\ 2/5+i/5 \\ -1/5-3i/5 \\ 0 \end{bmatrix} \quad \mathbf{v}_{L4} = \begin{bmatrix} 0 \\ 2/5-i/5 \\ -1/5+3i/5 \\ 0 \end{bmatrix}.$$

Figure 15: Jordan matrix solution

**Voorbeeld 2:** Geef voor de matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 2 \end{bmatrix}, \quad (2.3.122)$$

de eigenwaarden, en eigenvectoren. We vinden (opnieuw met Maple):

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Jordan-vorm}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{\text{Jordan-vorm}} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}. \quad (2.3.123)$$

Uit de Jordan-vorm (middenste matrix) zien we dat er twee verschillende eigenwaarden zijn:  $\lambda_1 = 1$  en  $\lambda_2 = 2$ , waarbij de algebraische multipliciteit van  $\lambda_2$  gelijk is aan drie en de geometrische multipliciteit gelijk aan twee. De eigenvector  $\mathbf{v}_1$  horende bij  $\lambda_1$  lezen we af in de eerste kolom van de eerste matrix ( $\mathbf{M}$ ), de twee eigenvectoren  $\mathbf{v}_2$  en  $\mathbf{v}_3$  horende bij  $\lambda_2$  lezen we af in kolom twee en kolom vier:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (2.3.124)$$

(Uit de structuur van de Jordan-vorm volgt dat de derde kolomvector geen eigenvector is.)

Figure 16: Jordan matrix example 2

$$\mathbf{D}^k = \begin{bmatrix} \lambda_1^k & 0 & 0 & \dots & 0 \\ 0 & \lambda_2^k & 0 & \dots & 0 \\ 0 & 0 & \lambda_3^k & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n^k \end{bmatrix}.$$

Figure 17: Diagonaal matrix

### 2.13.1 Voorbeeld

Zie Figuur 18.

**Toepassing: De Fibonacci rij:**  $0, 1, 1, 2, 3, 5, 8, 13, \dots$ , is een bekende recursieve rij die blijkbaar terug te vinden is in allerhande processen in de natuur. Ze wordt bepaald door de recursie-relatie  $x_k = x_{k-1} + x_{k-2}$ , met als startvoorwaarden  $x_0 = 0, x_1 = 1$ . We kunnen hier als volgt een iteratieve matrixvergelijking aan associëren. Stellen we  $\mathbf{u}_k = \begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix}$ , dan hebben we  $\mathbf{u}_k = \mathbf{A}\mathbf{u}_{k-1}$  met

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{1-\sqrt{5}}{2\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{bmatrix} (= \mathbf{MDM}^{-1}), \quad (2.3.132)$$

waarbij we de matrix in de laatste gelijkheid, bvb. via  $\text{Maple JordanForm}$  diagonaliseerden. Vervolgens vinden we uit (2.3.128):

$$\begin{aligned} \mathbf{u}_k = \mathbf{MD}^k \mathbf{M}^{-1} \mathbf{u}_0 &= \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^k & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^k \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix} \quad (2.3.133) \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1} \\ \left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k \end{bmatrix} \approx \frac{1}{\sqrt{5}} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} \\ \left(\frac{1+\sqrt{5}}{2}\right)^k \end{bmatrix}, \end{aligned}$$

waarbij de laatste gelijkheid benaderend geldt voor grote  $k$ . We vinden dus de uitdrukking

$$x_k = \frac{1}{\sqrt{5}} \left( \left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k \right) \approx \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^k \quad (2.3.134)$$

voor de Fibonacci-getallen, alsook de asymptotische ratio  $\lim_{k \rightarrow \infty} x_{k+1}/x_k = \frac{1+\sqrt{5}}{2}$  wat de beroemde gulden snede is.

Het asymptotisch gedrag (grote  $k$ ) kunnen we ook rechtstreeks uit de dominante eigenvectoren en corresponderende eigenwaarde halen. Uit (2.3.132) lezen we af dat:

$$\text{fixed point} \quad \mathbf{v}_1 = \left(\frac{1+\sqrt{5}}{2}, 1\right) \quad \mathbf{v}_{L1} = \left(\frac{1}{\sqrt{5}}, -\frac{1-\sqrt{5}}{2\sqrt{5}}\right) \quad \lambda_1 = \frac{1+\sqrt{5}}{2}. \quad (2.3.135)$$

Uit (2.3.131) vinden we dan onmiddellijk het asymptotisch gedrag

$$\begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} \approx \lambda_1^k (\mathbf{v}_{L1} \cdot \mathbf{u}_0) \mathbf{v}_1 = \left(\frac{1+\sqrt{5}}{2}\right)^k \left(\frac{1}{\sqrt{5}}\right) \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} \\ \left(\frac{1+\sqrt{5}}{2}\right)^k \end{bmatrix}, \quad (2.3.136)$$

wat precies overeenkomt met de laatste benaderde gelijkheid in (2.3.134). Wat de asymptotische ratio  $x_{k+1}/x_k$  betreft, vinden we ook onmiddellijk  $\mathbf{u}_k \approx \lambda_1 \mathbf{u}_{k-1}$ , waaruit  $\lim_{k \rightarrow \infty} x_{k+1}/x_k = \lambda_1 = \frac{1+\sqrt{5}}{2}$ , consistent met het eerdere resultaat.

Figure 18: Fibonacci voorbeeld

In het bovenstaande zien we dat  $\lambda_1$  de dominante eigenwaarde is. Daardoor kunnen we de fixed point berekenen met:

$$\lambda_1^k (v_{L1} \cdot u_0) v_1$$

Omdat  $\lambda_1$  dominant is, nemen we voor  $v_1$  de eerste kolom van  $M$  en voor  $v_{L1}$  de eerste rij van  $M^{-1}$

Dit kan toegepast worden in de zogezegde **Markov proces**

algemene vorm:  $u_k = P u_{k-1}$  waarbij  $P$  een matrix is die de overgangen tussen de verschillende states aangeeft met probabilitet.  $\sum p_{ij} = 1$

Ook goed om te weten is dat wanneer de matrix strikt positieve getallen heeft, dat matrix  $P$  een uniek dominante eigenwaarde  $\lambda_1 = 1$  heeft met  $v_1$  een positieve eigenvector. Deze  $v_1$  is dan ook een fixed point.

### 2.13.2 Voorbeeld Markov proces

Zie Figuur 19 and 20.

**Voorbeeld.** Neem volgend eenvoudig weermodel met 3 toestanden: zonnig, bewolkt en regen. En wanneer het zonnig is, een kans van 70% dat het de volgende dag ook zonnig is, 10% kans dat het bewolkt is, en 20% kans dat het regent. Wanneer het bewolkt is, 40% kans dat het zonnig wordt, 20% kans dat het bewolkt blijft, en 40% kans dat het de volgende dag regent. En wanneer het regent, 20% kans dat het de volgende dag zonnig wordt, 30% dat het bewolkt wordt, en 50% kans dat het zal regenen. Dit geeft de volgende Markov matrix:

Figure 19: Markov proces

## 2.14 Matrixexponent en lineaire differentiaalvergelijkingen

Hier gaan we een matrix plaatsen de exponent.

$e^{At} = M e^{Dt} M^{-1}$  -*j* concreet voorbeeld

**Algemeen:**

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

of

$$e^A = M e^{D} M^{-1} \quad e^A = M e^J M^{-1} \text{ (niet-diagonaliseerbare matrices)}$$

In matrix vorm zie je het volgende: 21

Note: Maple geeft de functie ‘MatrixExponential(A, t)’ om  $e^{At}$  te berekenen.

### 2.14.1 eerste-orde differentiaalvergelijking

$$y'(t) = Ay(t)$$

$$y(t) = e^{At} y(0)$$

### 2.14.2 n-de differentiaalvergelijkin

Hetzelfde als hierboven

### 2.14.3 Herschrijf de tweede-orde differentiaalvergelijking $y''(t) + w^2 y(t) = 0$

Zie Figuur 22.

## 2.15 Symmetrische matrices

-  $A = A^T$  -  $A$  heeft enkel reele eigenwaarden -  $A$  heeft orthogonale eigenvectoren -  $A = ODO^T$  met  $O$  een orthogonale matrix en  $D$  een diagonale matrix

Omdat  $O^T = O^{-1}$ , kunnen we zeggen dat  $O^T O = I$

Ook is het zo dat geometrische multiplicitet = algebraïsche multiplicitet. -*j*  $A$  is diagonaliseerbaar.

```

restart;
with(LinearAlgebra):
A := Matrix([[0.7, 0.4, 0.2], [0.1, 0.2, 0.3], [0.2, 0.4, 0.5]])
A := 
$$\begin{bmatrix} 0.7 & 0.4 & 0.2 \\ 0.1 & 0.2 & 0.3 \\ 0.2 & 0.4 & 0.5 \end{bmatrix}$$
 (1)

A := convert(A, rational)
A := 
$$\begin{bmatrix} \frac{7}{10} & \frac{2}{5} & \frac{1}{5} \\ \frac{1}{10} & \frac{1}{5} & \frac{3}{10} \\ \frac{1}{5} & \frac{2}{5} & \frac{1}{2} \end{bmatrix}$$
 (2)

J, Q := JordanForm(A, output = ['J', 'Q'])
J, Q := 
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{5} - \frac{\sqrt{5}}{10} & 0 \\ 0 & 0 & \frac{1}{5} + \frac{\sqrt{5}}{10} \end{bmatrix},$$
 (3)


$$\left[ \left[ \frac{28\sqrt{5}}{(-5+8\sqrt{5})(8+\sqrt{5})}, \frac{-71+31\sqrt{5}}{2(-5+8\sqrt{5})(8+\sqrt{5})}, \right. \right.$$


$$\left. \frac{7+3\sqrt{5}}{2(-5+8\sqrt{5})} \right],$$


$$\left[ \frac{11\sqrt{5}}{(-5+8\sqrt{5})(8+\sqrt{5})}, -\frac{-29+11\sqrt{5}}{2(-5+8\sqrt{5})(8+\sqrt{5})}, -\frac{3+\sqrt{5}}{2(-5+8\sqrt{5})} \right]$$


$$\left. \left[ \frac{20\sqrt{5}}{(-5+8\sqrt{5})(8+\sqrt{5})}, -\frac{-21+10\sqrt{5}}{(-5+8\sqrt{5})(8+\sqrt{5})}, -\frac{2+\sqrt{5}}{-5+8\sqrt{5}} \right] \right]$$

evalf(Q) (4)

```

Figure 20: Markov proces solution

$$e^{t\mathbf{J}_n(\lambda)} = \begin{bmatrix} e^{t\lambda} & te^{t\lambda} & \frac{t^2}{2}e^{t\lambda} & \dots & \frac{t^{n-1}}{(n-1)!}e^{t\lambda} \\ 0 & e^{t\lambda} & te^{t\lambda} & \dots & \frac{t^{n-2}}{(n-2)!}e^{t\lambda} \\ 0 & 0 & e^{t\lambda} & \dots & \frac{t^{n-3}}{(n-3)!}e^{t\lambda} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & e^{t\lambda} \end{bmatrix}$$

Figure 21: Matrix exponent

**Voorbeeld.** Herschrijf de tweede-orde differentiaalvergelijking

$$y''(t) + \omega^2 y(t) = 0, \quad (2.3.154)$$

als een lineair stelsel van eerste orde vergelijkingen, en los deze vervolgens op.

*Oplossing:* Met  $\mathbf{y}(t) = (y'(t), y(t))$ , kunnen we (2.3.154) herschrijven als:

$$\frac{d}{dt} \begin{bmatrix} y'(t) \\ y(t) \end{bmatrix} = \mathbf{A}\mathbf{y}(t) = \begin{bmatrix} 0 & -\omega^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y'(t) \\ y(t) \end{bmatrix}. \quad (2.3.155)$$

Verder vinden we:

$$e^{t\mathbf{A}} = \begin{bmatrix} \cos \omega t & -\omega \sin \omega t \\ \frac{1}{\omega} \sin \omega t & \cos \omega t \end{bmatrix}, \quad (2.3.156)$$

waaruit we de oplossing kunnen aflezen als tweede component van  $\mathbf{y} = e^{t\mathbf{A}}\mathbf{y}_0$ :

$$y(t) = y(0) \cos \omega t + \frac{y'(0)}{\omega} \sin \omega t, \quad (2.3.157)$$

wat inderdaad overeenkomt met de oplossing (1.3.23) die we in het vorige hoofdstuk poneerden.

Figure 22: Tweede-orde differentiaalvergelijking

## 2.16 SVD (Singular Value Decomposition)

$$A = U\Sigma V^T$$

of

$$A = \sum_{i=1}^r \sigma_i u_i v_i^T$$

met  $U$  en  $V$  orthogonale matrices en  $\Sigma$  een diagonale matrix met singular values

$U$  is mxm,  $V$  is nxn en  $\Sigma$  is mxn

### 2.16.1 Example SVD

Als we compressie willen uitvoeren moeten we essentially SVD uitvoeren, maar onze som wordt beperkt door een rang  $r'$

$$A = \sum_{i=1}^{r'} \sigma_i u_i v_i^T$$

## 3 Hoofdstuk 3: Integratie en afleiding in $R^n$

### 3.1 Partiele afgeleiden

$$D_i f(x_1, x_2, \dots, x_n) = \frac{\partial f}{\partial x_i}$$

#### 3.1.1 Example

---

**Voorbeeld.** Beschouw de functie  $f(x, y) = \sqrt{x^2 + y}$ . Bepaal hiervoor  $\partial_x f(x, y)$  en  $\partial_y f(x, y)$ .

Figure 23: Partial derivatives example

```

> f:= x->sqrt(x^2 + y)          f:= x-> $\sqrt{x^2 + y}$            (1)
=
> diff(f(x), x)                   $\frac{x}{\sqrt{x^2 + y}}$            (2)
=
> diff(f(x), y)                   $\frac{1}{2\sqrt{x^2 + y}}$          (3)
=

```

Figure 24: Partial derivatives example

#### Higher order partial derivatives:

Basically hetzelfde, doe het gewoon na elkaar, van binnen naar buiten

---

**Voorbeeld.** Beschouw dezelfde functie  $f(x, y) = \sqrt{x^2 + y}$  als hierboven. Bepaal nu de tweede orde partieel afgeleiden  $\partial_x \partial_x f(x, y) = \partial_x^2 f(x, y)$ ,  $\partial_x \partial_y f(x, y)$ ,  $\partial_y \partial_x f(x, y)$  en  $\partial_y \partial_y f(x, y) = \partial_y^2 f(x, y)$ . (Het is nuttig van dergelijke afgeleiden met de hand te kunnen berekenen. ©Maple kan dit natuurlijk ook, zie het diff commando.)

Figure 25: Partial derivatives example

```

> f:= x->sqrt(x^2 + y)          (1)
=
> diff(f(x), x)                 (2)
=                                     x
=                                     -----
> diff(f(x), y)                 (3)
=                                     1
=                                     -----
> diff(diff(f(x), x), x)        (4)
=                                     - x^2
=                                     -----
=                                     (x^2 + y)^3/2 + 1
=                                     -----
> diff(diff(f(x), y), y)        (5)
=                                     - 1
=                                     -----
=                                     4 (x^2 + y)^3/2
=>

```

Figure 26: Partial derivatives example

### 3.2 Kettingregel

When deriving, make sure to derive the respected variable as well.

### **3.3 Coordinaten transformaties**

Hier gaan we de coordinaten transformeren naar een andere coordinatenstelsel. (om het probleem zo gemakkelijk mogelijk te maken)

### 3.3.1 Example

**Voorbeeld.** (Een grotere oefening) Bij 2D golfproblemen, zoals de trilling van een drum, verschijnt de 2D Laplacian:

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \quad (3.1.14)$$

In het geval van een cirkelvormige drum is het aangewezen van poolcoördinaten  $(\rho, \theta)$  te gebruiken, bepaald door  $(x, y) = (\rho \cos \theta, \rho \sin \theta)$ . Druk deze 2D Laplacian uit in termen van deze polaire coördinaten.

*Oplossing:* Laat ons vooreerst de coördinatentransformatie inverteren. We vinden:

$$\rho = \sqrt{x^2 + y^2} \quad \theta = \arctan(y/x). \quad (3.1.15)$$

Hieruit halen we de partieel afgeleiden:

$$\begin{aligned} \frac{\partial \rho}{\partial x} &= \frac{x}{(x^2 + y^2)^{1/2}} = \cos \theta & \frac{\partial \theta}{\partial x} &= \frac{-y/x^2}{1 + (y/x)^2} = -\frac{\sin \theta}{\rho} \\ \frac{\partial \rho}{\partial y} &= \frac{y}{(x^2 + y^2)^{1/2}} = \sin \theta & \frac{\partial \theta}{\partial y} &= \frac{1/x}{1 + (y/x)^2} = \frac{\cos \theta}{\rho}. \end{aligned} \quad (3.1.16)$$

Toepassing van de kettingregel (3.1.13) levert dan:

$$\frac{\partial}{\partial x} = \cos \theta \frac{\partial}{\partial \rho} - \frac{\sin \theta}{\rho} \frac{\partial}{\partial \theta} \quad \frac{\partial}{\partial y} = \sin \theta \frac{\partial}{\partial \rho} + \frac{\cos \theta}{\rho} \frac{\partial}{\partial \theta}, \quad (3.1.17)$$

en hieruit krijgen we dan:

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= (\cos \theta \frac{\partial}{\partial \rho} - \frac{\sin \theta}{\rho} \frac{\partial}{\partial \theta})(\cos \theta \frac{\partial}{\partial \rho} - \frac{\sin \theta}{\rho} \frac{\partial}{\partial \theta}) \\ &= \cos^2 \theta \frac{\partial^2}{\partial \rho^2} + \frac{\cos \theta \sin \theta}{\rho^2} \frac{\partial}{\partial \theta} - \frac{\cos \theta \sin \theta}{\rho} \frac{\partial^2}{\partial \rho \partial \theta} + \frac{\sin^2 \theta}{\rho} \frac{\partial}{\partial \rho} \\ &\quad - \frac{\sin \theta \cos \theta}{\rho} \frac{\partial^2}{\partial \theta \partial \rho} + \frac{\sin \theta \cos \theta}{\rho^2} \frac{\partial}{\partial \theta} + \frac{\sin^2 \theta}{\rho^2} \frac{\partial^2}{\partial \theta^2} \end{aligned} \quad (3.1.18)$$

$$\begin{aligned} &= \cos^2 \theta \frac{\partial^2}{\partial \rho^2} + \frac{2 \cos \theta \sin \theta}{\rho^2} \frac{\partial}{\partial \theta} - \frac{2 \cos \theta \sin \theta}{\rho} \frac{\partial^2}{\partial \theta \partial \rho} \\ &\quad + \frac{\sin^2 \theta}{\rho} \frac{\partial}{\partial \rho} + \frac{\sin^2 \theta}{\rho^2} \frac{\partial^2}{\partial \theta^2}. \end{aligned} \quad (3.1.19)$$

Analoog vind je:

$$\begin{aligned} \frac{\partial^2}{\partial y^2} &= \sin^2 \theta \frac{\partial^2}{\partial \rho^2} - \frac{2 \cos \theta \sin \theta}{\rho^2} \frac{\partial}{\partial \theta} + \frac{2 \cos \theta \sin \theta}{\rho} \frac{\partial^2}{\partial \theta \partial \rho} \\ &\quad + \frac{\cos^2 \theta}{\rho} \frac{\partial}{\partial \rho} + \frac{\cos^2 \theta}{\rho^2} \frac{\partial^2}{\partial \theta^2}. \end{aligned} \quad (3.1.20)$$

Tellen we deze twee uitdrukkingen op dan daalt de rook neer en vinden we:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2}, \quad (3.1.21)$$

of dus

1. Calculeer de  $\rho$  en  $\theta$  naar  $x$  en  $y$
2. vorm de chain rule:  $\frac{\partial}{\partial x} = \frac{\partial \rho}{\partial x} \frac{\partial}{\partial \rho} + \frac{\partial \theta}{\partial x} \frac{\partial}{\partial \theta}$
3. doe hetzelfde voor met respect tot  $y$ .
4. Vermenigvuldig twee maal met elkaar door de dubbele afgeleide
5. som met elkaar
6. Je hebt nu de laplacian ( $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ )

### 3.4 Gradient en de differentiaal

**gradient operator:**  $\nabla = \sum_{i=1}^n e_i \partial_i$

Via deze gradient kunnen we de richtingsafgeleide berekenen (variatie van een functie langs een kromme)

$$\nabla f \cdot \frac{dx}{dt}$$

#### 3.4.1 Example

---

**Voorbeeld:** Bepaal voor het veld  $f(x, y) = x + y$  de gradient. Integreer vervolgens de richtingsafgeleide:  $df/dt = \nabla f \cdot \frac{dx}{dt}$  voor de kromme  $(x(t), y(t)) = (x_0 t, y_0 t)$ , van  $t = 0$

tot  $t = 1$  en toon expliciet aan dat je voor deze integraal inderdaad de uitdrukking  $f(x(1), y(1)) - f(x(0), y(0)) = x_0 + y_0$  krijgt.

Figure 28: Gradient example

Hier zien we dat de gradient  $\nabla = (\partial_x, \partial_y) = (1, 1)$  (want  $f(x, y) = x + y$ )

de richtingsafgeleide is dan  $\nabla f \cdot \frac{dx}{dt} = (1, 1) \cdot (x_0, y_0) = x_0 + y_0$

Als je de integraal pakt van  $t = 0$  naar  $t = 1$ :  $\int_0^1 x_0 + y_0 dt = x_0 + y_0$

De gradient operator en richtingsafgeleide wordt gebruikt bij zaken zoals stochastic gradient descent, waar we proberen de lokale minima te vinden.

### 3.4.2 Example

**Voorbeeld.** Bereken de gradient en toon expliciet aan dat deze orthogonaal staat op de veldlijnen voor  $f(x, y) = xy$ .

*Oplossing:* We vinden:

$$\nabla f(x, y) = (y, x). \quad (3.1.32)$$

Verder hebben we voor de veldlijnen  $xy = c$  voor een bepaalde constante  $c$ , ofwel  $y = c/x$ . Voor een infinitesimale verplaatsing langs een veldlijn hebben we dus  $dy = -\frac{c}{x^2}dx$ . De corresponderende vector  $\mathbf{dx}$  leest dan:

$$d\mathbf{x} = dx(1, -c/x^2) = dx(1, -y/x) \quad (3.1.33)$$

waar we voor de laatste uitdrukking  $c = xy$  invullen. Deze vector staat inderdaad orthogonaal op de gradient,  $d\mathbf{x} \cdot \nabla f = 0$ .

Figure 29: Gradient example

1. Calculate the gradient

$$\nabla f = (\partial_x, \partial_y) = (y, x)$$

onze veldlijn:  $c = xy$  Note: We willen uiteindelijk kunnen zeggen dat onze richtingsafgeleide = 0, we hebben dus  $\mathbf{dx}$  nodig

$$y = \frac{c}{x}$$

$$dy = -\frac{c}{x^2}dx$$

We weten ook dat onze displacement along the kromme  $dx = (dx, dy)$

We weten nu wel  $dy$  dus we vullen dit in:  $dx = (dx, -\frac{c}{x^2}dx) = dx(1, -\frac{c}{x^2})$

We weten ook wat  $c$  is:  $dx(1, -\frac{xy}{x^2}) = dx(1, -\frac{y}{x})$

Nu dat we de displacement hebben, vinden we de richtingsafgeleide:

$$\nabla f \cdot dx = (y, x) \cdot dx(1, -\frac{y}{x}) = ydx - ydx = 0$$

Dus hebben we bewezen dat de gradient  $\nabla f$  loodrecht staat op de kromme.

## 3.5 Taylorreeks

$$\begin{aligned} f(\mathbf{x}) &= f(\mathbf{x}_0) + \sum_i \left( \frac{\partial f}{\partial x_i} \right) \Delta x_i + \frac{1}{2} \sum_{ij} \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) \Delta x_i \Delta x_j + \dots \\ &\quad + \frac{1}{k!} \sum_{i_1, \dots, i_k} \left( \frac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}} \right) (\Delta x_{i_1} \dots \Delta x_{i_k}) + \dots , \end{aligned} \quad (3.1.34)$$

waarbij  $\Delta x_i = (x_i - x_{0i})$  en de partieel afgeleiden worden genomen in het punt  $\mathbf{x}_0$  waarrond ontwikkeld wordt.

Figure 30: Taylorreeks bij afgeleiden

Als  $\Delta x_i$  (tweede term van taylorreeks), we  $\nabla f \cdot \Delta x_i$

Dit is een **stationair punt** wanneer  $\nabla f = 0$

Er zijn hierbij 3 gevallen voor 1 variable:

- $\frac{\partial^2 f}{\partial x_i^2} > 0$  (minimum)
- $\frac{\partial^2 f}{\partial x_i^2} < 0$  (maximum)
- $\frac{\partial^2 f}{\partial x_i^2} = 0$  (saddle point (stationair buigpunt))

Voor meerdere variabelen, maken we gebruik van de **Hessiaan** dat voorkomt in de tweede orde term van de taylorreeks:

- Hessiaan is symmetrisch, dus  $H = ODO^T$

Voor meerdere variabelen is het mogelijke kwalitatieve gedrag rijker. In dat geval wordt het gedrag rond een stationair punt  $\mathbf{x}_0$  gevatt door de **Hessiaan**  $H_{ij}$  de matrix die optreedt in de tweede orde term van (3.1.34):

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \frac{1}{2} \sum_{ij} \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) \Delta x_i \Delta x_j = f(\mathbf{x}_0) + \frac{1}{2} \sum_{ij} \Delta x_i H_{ij} \Delta x_j. \quad (3.1.36)$$

Deze matrix  $\mathbf{H}$  is symmetrisch,  $H_{ij} = H_{ji} = \frac{\partial^2 f}{\partial x_i \partial x_j}$ , wegens de commutatie van de partiële afgeleiden. Uit het vorig hoofdstuk weten we dan dat die wordt gediagonaleerd door een orthogonale transformatie  $\mathbf{H} = \mathbf{O} \mathbf{D} \mathbf{O}^T$ . Zo krijgen we:

$$f(\mathbf{x}) \approx f(\mathbf{x}_0) + \frac{1}{2} \sum_{ijl} \Delta x_i O_{il} \lambda_l O_{jl} \Delta x_j = f(\mathbf{x}_0) + \frac{1}{2} \sum_l \lambda_l \Delta x_l'^2, \quad (3.1.37)$$

met  $O_{ij}$  de orthogonale matrixcomponenten,  $\lambda_l$  de (reële) eigenwaarden en  $\Delta x_l' = \sum_i O_{il} \Delta x_i$ . Dit leidt dan tot de volgende mogelijke kwalitatieve gedragingen (zie ook fig. 3.1):

Figure 31: Hessiaan

- als alle  $\lambda_i > 0$  (minimum)
- als alle  $\lambda_i < 0$  (maximum)
- als er zowel positieve als negatieve  $\lambda_i$  zijn (saddle point)
- als er eigenwaarden zijn die = 0, dan krijg je in sommige richtingen een stationair buigpunt gedefinieerd door de derde orde vorm van taylorreeks

### 3.5.1 Example

---

**Voorbeeld.** Bepaal de stationaire punten voor de functie  $f(x, y) = x^3 e^{-(x^2+y^2)}$ , en onderzoek hun karakter (minimum, maximum, ...) met behulp van de Hessiaan.

Figure 32: Hessiaan

$$e^f = \zeta A^k$$

$$f(x,y) = x^3 - e^{(x^2+y^2)}$$

$$\frac{\partial f(x,y)}{\partial x} = 3x^2 - e^{(x^2+y^2)} + x^2 \cdot e^{(x^2+y^2)} \cdot 2x \\ \Leftrightarrow e^{(x^2+y^2)} \cdot (3x^2 - 2x^3) = 0$$

$$\Leftrightarrow (\sqrt{3}/2, 0)$$

$$\frac{\partial f(x,y)}{\partial y} = 0 + x^3 \cdot e^{(x^2+y^2)} \cdot (-2y) \Leftrightarrow -2y x^3 \cdot e^{(x^2+y^2)} = 0 \\ \Leftrightarrow (-\sqrt{3}/2, 0)$$

■ stationaire punten  
we kijken naar 3 tweede orde afleiden:

$$-\frac{\partial^2 f}{\partial x^2} = (4x^5 - 16x^3 + 6x) e^{(x^2+y^2)}$$

$$-\frac{\partial^2 f}{\partial y^2} = x^3 (4y^2 - 2) e^{(x^2+y^2)}$$

$$-\frac{\partial^2 f}{\partial x \partial y} = 2x^2 y (2x^3 - 3) e^{(x^2+y^2)}$$

$$H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

high maar eigenwaarden van de stationaire punten en zie bovenstaande.

Figure 33: Hessiaan

```

[> restart: with(LinearAlgebra):
[> # Define the function
=> f := (x,y) ->x^3·exp(-(x^2 + y^2));
f := (x, y) ->x^3·e^-y^2-x^2
(1)

[> df_dx := diff(f(x,y),x);
df_dy := diff(f(x,y),y);

df_dx := 3 x^2 e^-x^2-y^2 - 2 x^4 e^-x^2-y^2
df_dy := -2 x^3 y e^-x^2-y^2
(2)

[> stationary_points := solve({df_dx = 0, df_dy = 0}, {x,y});
stationary_points := {x = 0, y = y}, {x = RootOf(2 Z^2 - 3), y = 0}
(3)

[> d2f_dx2 := diff(f(x,y),x,x);
d2f_dy2 := diff(f(x,y),y,y);
d2f_dxdy := diff(f(x,y),x,y);
Hessian := Matrix([[d2f_dx2, d2f_dxdy], [d2f_dxdy, d2f_dy2]]);

d2f_dx2 := 6 x e^-x^2-y^2 - 14 x^3 e^-x^2-y^2 + 4 x^5 e^-x^2-y^2
d2f_dy2 := -2 x^3 e^-x^2-y^2 + 4 x^3 y^2 e^-x^2-y^2
d2f_dxdy := -6 x^2 y e^-x^2-y^2 + 4 x^4 y e^-x^2-y^2
Hessian :=

$$\begin{bmatrix} 6 x e^{-x^2-y^2} - 14 x^3 e^{-x^2-y^2} + 4 x^5 e^{-x^2-y^2} & -6 x^2 y e^{-x^2-y^2} + 4 x^4 y e^{-x^2-y^2} \\ -6 x^2 y e^{-x^2-y^2} + 4 x^4 y e^{-x^2-y^2} & -2 x^3 e^{-x^2-y^2} + 4 x^3 y^2 e^{-x^2-y^2} \end{bmatrix}$$

(4)

[> Hessian_at_points := subs(stationary_points, Hessian);
Hessian_at_points :=  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ 
(5)

[> eigenvalues := LinearAlgebra:-Eigenvalues(Hessian_at_points);
eigenvalues :=  $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ 
(6)

[> # Analyse the behaviour along the y-axis if needed
[> df_dx_at_x0 := subs(x = 0, df_dx);
df_dy_at_x0 := subs(x = 0, df_dy);
df_dx_at_x0 := 0
df_dy_at_x0 := 0
(7)

```

Figure 34: Hessiaan

## 3.6 Integratie

### 3.6.1 De riemanniaanse integraal

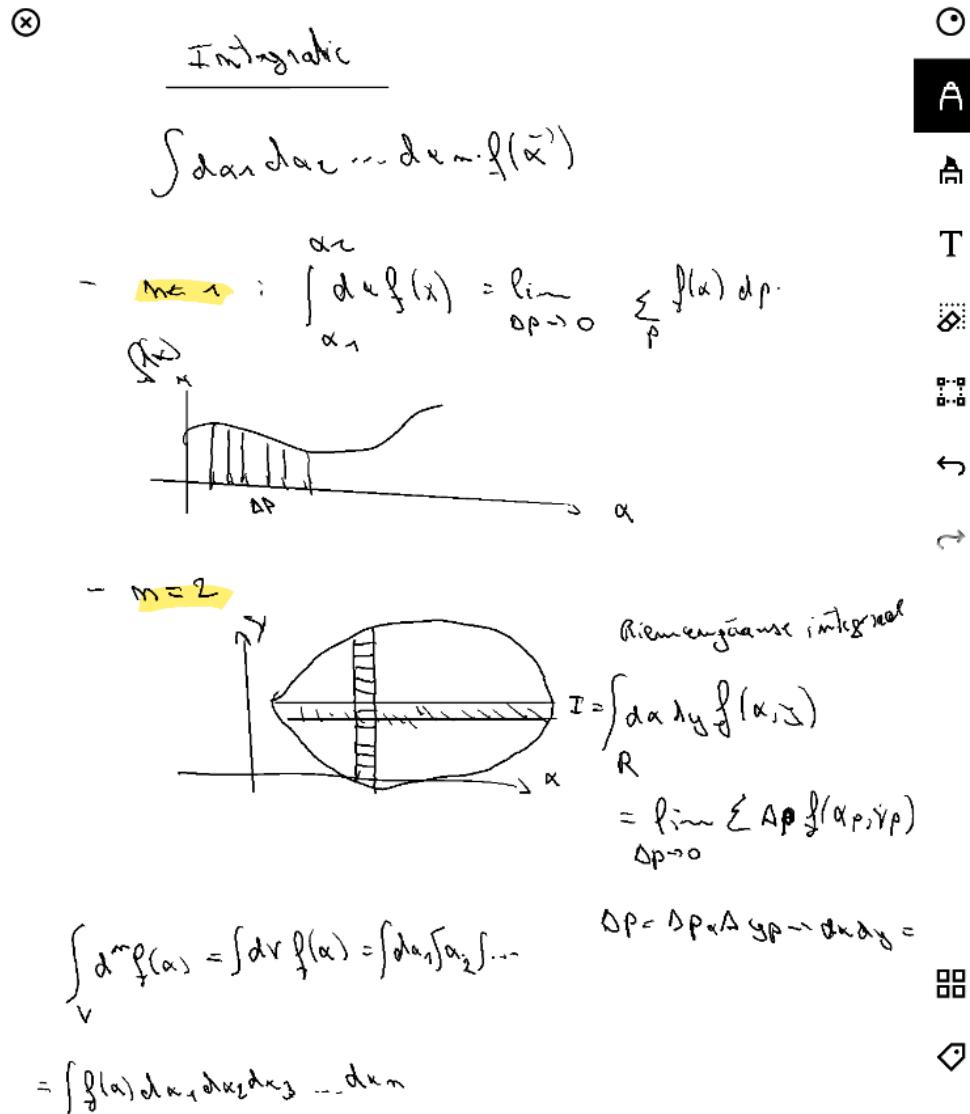


Figure 35: Integratie uitleg

### 3.6.2 Voorbeeld

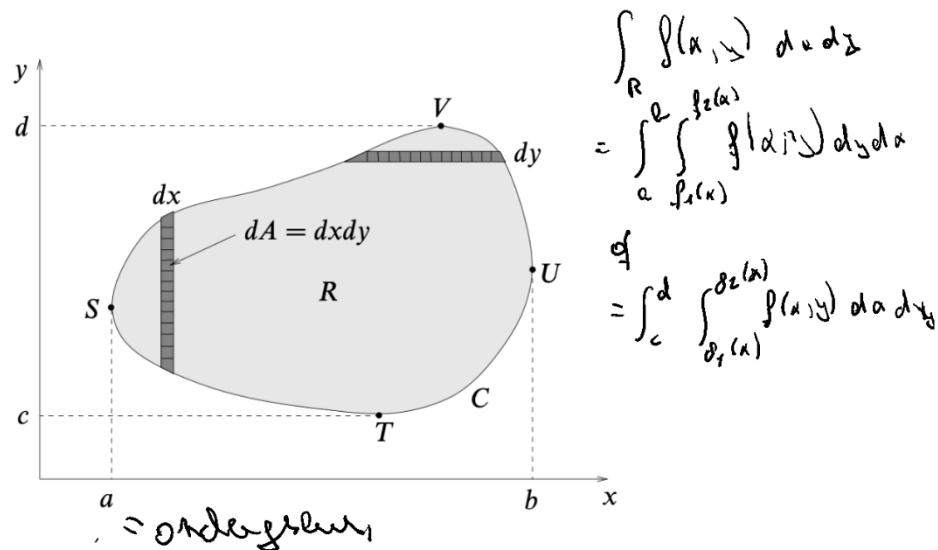


Figure 36: Integraal oppervlak

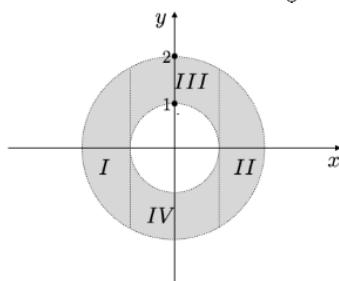
### 3.6.3 Voorbeeld

$$\textcircled{X} \quad \left[ \frac{x^2}{2} \right]_1^2 = 2 \cdot \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$$

66

### HOOFDSTUK 3. INTEGRATIE EN AFLEIDING IN $\mathbb{R}^N$

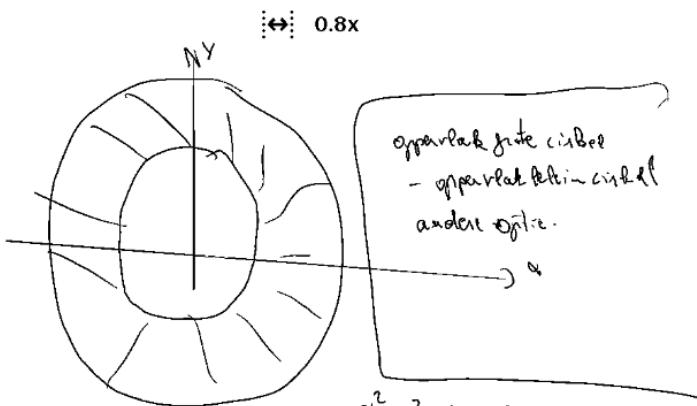
$$\int_0^{\pi} \int_0^1 r dr d\theta$$



simple polar coordinates

Figure 37: Cirkel oppervlak

Dit kan ook via de  $(x, y)$  coordinaten, waarbij de cirkel wordt voorgesteld door:  $x^2 + y^2 = r^2$ , dan krijg je:



$$I = I_1 + I_2 + I_3 + I_4$$

$$\alpha^2 + \gamma^2 = 4 \rightarrow \text{buiten}$$

$$\alpha^2 + \gamma^2 = 1 \rightarrow \text{binnen}$$

$$I_1 = \int_{-1}^1 d\alpha \int_{-\sqrt{4-\alpha^2}}^{\sqrt{4-\alpha^2}} dy = \int_{-1}^1 2\sqrt{4-\alpha^2} d\alpha$$

$$I_2 = \int_{-1}^1 d\alpha \int_{-\sqrt{1-\alpha^2}}^{\sqrt{1-\alpha^2}} dy = \sqrt{3} + \frac{\pi}{6}$$

$$\begin{aligned} & \text{Oppervlak cirkel } (\pi r^2) \\ & = \pi(4-1) = 3\pi \end{aligned}$$

$$I = 2I_1 + 2I_2 = 3\pi$$

Figure 38: Oppervlak cirkel x, y

We kunnen ook volumes berekenen, is basically een triple integraal...

### 3.6.4 Voorbeeld

---

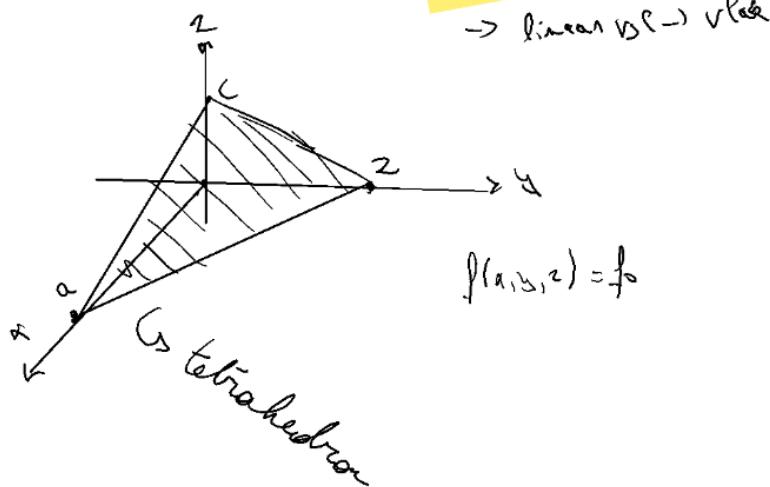
**Voorbeeld, volume-integraal.** Bepaal de massa  $M$  van de tetraedron  $R$ , begrenst door de coördinaatvlakken  $x = 0$ ,  $y = 0$ ,  $z = 0$  en door het vlak  $x/a + y/b + z/c = 1$ , waarbij de dichtheid gegeven wordt door  $\rho(x, y, z) = \rho_0(1 + x/a)$ .

Figure 39: Volume vraag

Volume:

$$\alpha=0, y=0, z=0, \frac{\alpha}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

→ linear  $\rightarrow V_{\text{Plat}}$



$$f(\alpha, y, z) = f_0 \left(1 + \frac{\alpha}{a}\right)$$

$$\begin{aligned} & \int_0^a \int_0^{b(1-\frac{\alpha}{a})} \int_0^{(1-\frac{\alpha}{a}-\frac{y}{b})c} f_0 \left(1 + \frac{x}{a}\right) dx dy dz \\ &= f_0 \int_0^a \int_0^{b(1-\frac{\alpha}{a})} \int_0^{(1-\frac{\alpha}{a}-\frac{y}{b})c} dz dy d\alpha = \frac{f_0 \cdot abc}{24} \end{aligned}$$

Figure 40: Volume solution

Een paar typische toepassingen zijn:

- Berekening oppervlakte of volume

$$A = \int_R dx dy \quad V = \int_R dx dy dz \quad (3.2.60)$$

- Berekening massa door de integratie van de oppervlakte- of volume dichtheid:

$$M = \int_R dx dy \rho(x, y) \quad M = \int_R dx dy dz \rho(x, y, z) \quad (3.2.61)$$

- Berekening massmiddelpunt  $\mathbf{x}_{mm}$ :

$$\begin{aligned} (x_{mm}, y_{mm}) &= \frac{1}{M} \int_R dx dy \rho(x, y) (x, y) \\ (x_{mm}, y_{mm}, z_{mm}) &= \frac{1}{M} \int_R dx dy dz \rho(x, y, z) (x, y, z). \end{aligned} \quad (3.2.62)$$

(De totale impuls van een star lichaam is dan  $M\dot{\mathbf{x}}_{mm} = M\mathbf{v}_{mm}$ . )

- Berekening inertiaal moment tensor  $I_{ij}$  (met  $(x_1, x_2, x_3) = (x, y, z)$ ):

$$I_{ij} = \int_R dx dy \rho(x, y) x_i x_j \quad I_{ij} = \int_R dx dy dz \rho(x, y, z) x_i x_j \quad (3.2.63)$$

(Voor het draaimoment  $\vec{L}$  van een star lichaam dat roteert rond de oorsprong, rond een draaiaas met eenheidsvector  $\vec{n}$  en met hoeksnelheid  $\omega$  hebben we dan  $L_i = \omega \sum_j I_{ij} n_j$ . Gegeven een krachtmoment (*torque*)  $m$  in de richting van  $\mathbf{n}$ , hebben we verder  $m = \dot{\omega}(\sum_{ij} n_i n_j I_{ij})$ )

Figure 41: Typische integralen

### 3.7 Verandering integratievariabelen

Basically we willen de integraal veranderen van coördinatenstelsel. Dit kan via de Jacobiaan.

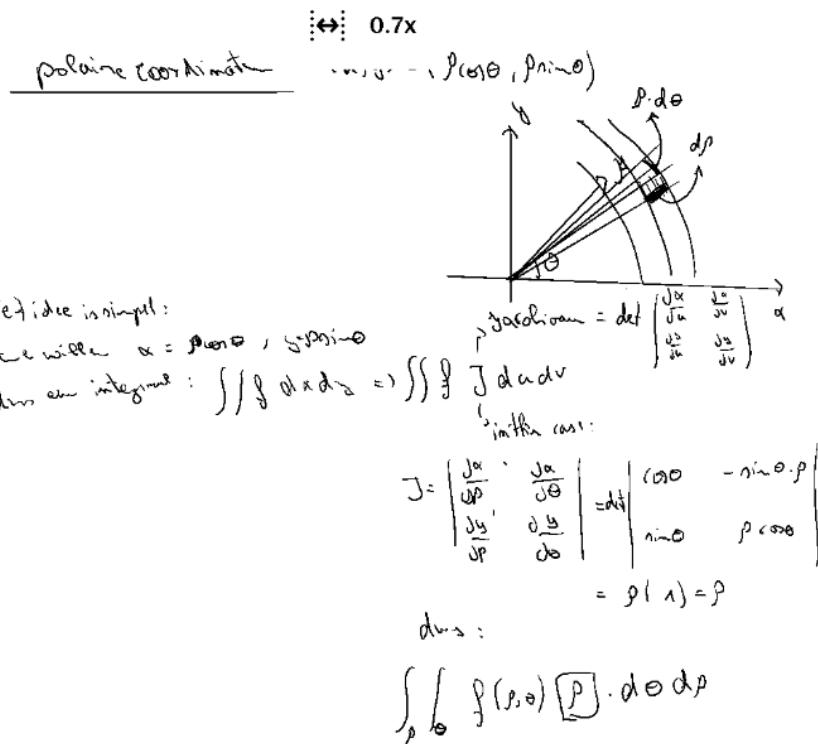


Figure 42: Voorbeeld coördinaten transformatie

## 3.8 Pool-, cilinder- en bolcoordinaten

- **Poolcoördinaten**

$$(x, y) = (r \cos \theta, r \sin \theta) \quad dA = r dr d\theta \quad (3.2.69)$$

Het coördinaatgebied  $\{0 < r < \infty, 0 \leq \theta < 2\pi\}$  beschrijft heel de Euclidische 2D ruimte op de oorsprong  $(x, y) = (0, 0)$  na, waarvoor  $\theta$  onbepaald is. Voor de radiële coördinaat  $r$  hebben we  $r = \sqrt{x^2 + y^2}$  (= afstand tot de oorsprong). Soms wordt ook nog  $\rho$  i.p.v.  $r$  gebruikt.

- **Cilindercoördinaten**

$$(x, y, z) = (r \cos \theta, r \sin \theta, z) \quad dV = r dr d\theta dz \quad (3.2.70)$$

Het coördinaatgebied  $\{0 < r < \infty, 0 \leq \theta < 2\pi, -\infty < z < +\infty\}$  beschrijft heel de Euclidische 3D ruimte op de  $z$ -as na,  $(x, y, z) = (0, 0, z)$ , waarvoor  $\theta$  onbepaald is. Voor de radiële coördinaat  $r$  hebben we opnieuw  $r = \sqrt{x^2 + y^2}$  (= afstand tot de  $z$ -as). Soms wordt ook nog  $\rho$  i.p.v.  $r$  gebruikt.

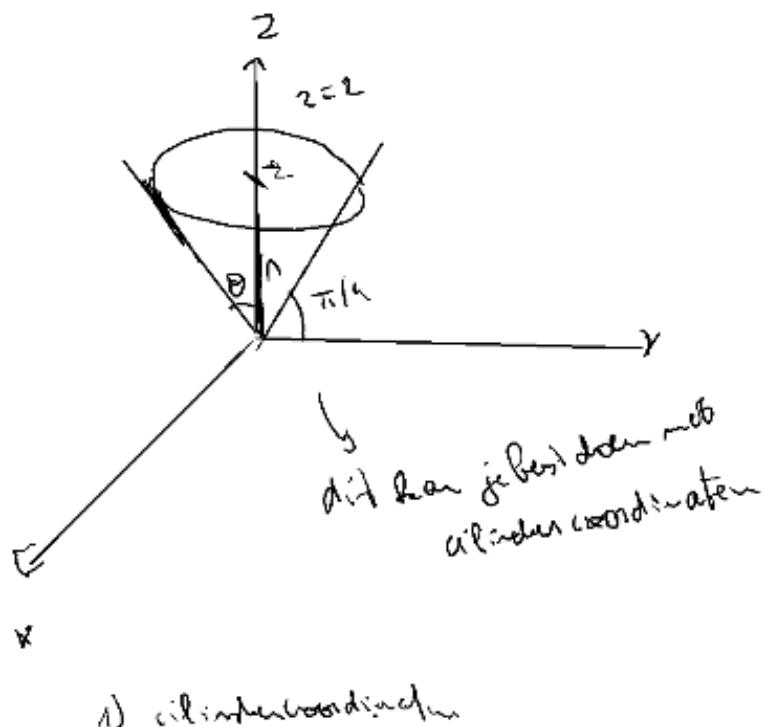
- **Bolcoördinaten**

$$(x, y, z) = (r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) \quad dV = r^2 \sin \theta dr d\theta d\varphi \quad (3.2.71)$$

Het coördinaatgebied  $\{0 < r < \infty, 0 < \theta < \pi, 0 \leq \varphi < 2\pi\}$  beschrijft heel de Euclidische 3D ruimte op de  $z$ -as na,  $(x, y, z) = (0, 0, z)$ , waarvoor  $\varphi$  onbepaald is. Nu hebben we voor de radiële coördinaat  $r$  dat  $r = \sqrt{x^2 + y^2 + z^2}$  (=afstand tot de oorsprong).

Figure 43: Pool, cilinder en bol coordinaten

### 3.8.1 Voorbeeld



$$\begin{aligned}
 & \int_R r \cdot dr \cdot dz \cdot d\phi \\
 & \int_0^2 dz \cdot \int_0^2 r dr \cdot \int_0^{2\pi} d\phi \\
 & = 2\pi \int_0^2 dz \int_0^2 r dr = 2\pi \int_0^2 dz \left[ \frac{r^2}{2} \right]_0^2 \\
 & = 2\pi \cdot \int_0^2 \frac{2^2}{2} dz \\
 & = \frac{1}{2} \pi \cdot 8 = \frac{8\pi}{3}
 \end{aligned}$$

Figure 44: Voorbeeld pool coordinaten

## 4 Hoofdstuk 4: Vectoranalyse in drie dimensies

### 4.1 Vectoren en vector bewerkingen

#### 4.1.1 Scalair product, norm, afstand

Chapter 4

$$\vec{v} \leftrightarrow 0.7x$$

$$v = (x, y, z)$$

$$= (\vec{e}_1, \vec{e}_2, \vec{e}_3)$$

$$= \sum_{i=1}^3 x_i \vec{e}_i$$

$$\vec{OP} : \vec{r} = \vec{r}_P$$

$$\vec{PQ} = \vec{r}_Q - \vec{r}_P$$

differentiation:  $\vec{v} = \frac{d\vec{r}(t)}{dt}$

Scalair product:  $\vec{v} \cdot \vec{w} = \vec{v} \cdot w$   
 $= \|v\| \|w\| \cos \theta$

norm:  $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$        $\vec{v} = \frac{\vec{v}}{\|\vec{v}\|} \rightarrow \text{eenheitsvektor}$

$\|\vec{PQ}\| = \|\vec{r}_Q - \vec{r}_P\|$   
 $\rightarrow \text{afstand}$

vectoriel product

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2) \vec{e}_1 + (u_3 v_1 - u_1 v_3) \vec{e}_2 + (u_1 v_2 - u_2 v_1) \vec{e}_3$$

geometrisch

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

$$\|\vec{u}\| \cdot \|\vec{v}\|$$

deutung produkt

$$\vec{u}(\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \vec{w}(\vec{u} \times \vec{v})$$

$\hookrightarrow$  Assoziativität

Figure 45: Definities

Belangrijk om te weten is dat  $w = u \times v$  de vector is die loodrecht staat op  $u$  en  $v$ .

**Triple product:**  $u \times (v \times w) = (u.w).v - (u.v).w$

## 4.2 Vector velden en vectoriele afleiding

### 4.2.1 Vector- en scalaire velden: Maxwell en Navier-Stokes

Hier een paar vergelijkingen die je **niet** moet kennen maar wel moet kunnen bewijzen:

$$\begin{aligned}\vec{\nabla} \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \text{Faraday} \quad \leftarrow \quad \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= \vec{0} \\ \text{ampere} \quad \leftarrow \quad \underbrace{\nabla \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}}_{\text{Ampere}} &= \mu_0 \vec{j}\end{aligned}$$

Figure 46: Maxwell laws

De navier-stokes vergelijkingen beschrijven gas/liquid flow en zijn als volgt:

**De Navier-Stokes vergelijkingen.** Deze liggen aan de basis van de beschrijving van het dynamisch gedrag van vloeistoffen en gassen. We schrijven zowel de **continuiteitsvergelijking** (zonder afvoer- of bronterm):

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \quad (4.2.23)$$

als de eigenlijke Navier-Stokes momentum vergelijking (voor het geval van constante dynamische viscositeit  $\mu$  en verdwijnende tweede viscositeitscoëfficiënt):

$$\rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right) = -\vec{\nabla} p + \mu \left( \vec{\nabla}^2 \vec{v} + \frac{1}{3} \vec{\nabla} (\vec{\nabla} \cdot \vec{v}) \right) + \rho \vec{g}. \quad (4.2.24)$$

De twee vectorvelden hier zijn:  $\vec{v}(x, y, z, t)$ , die de snelheid beschrijft van het vloeistof/gas-elementje op de betreffende positie en tijd, en  $\vec{g}(x, y, z)$ , het zwaartekracht-

Figure 47: Navier-Stokes

#### 4.2.2 Conservatieve velden

### Conservatieve vectorvelden

$$\vec{F} = \vec{\nabla} \phi = (\partial_x \phi, \partial_y \phi, \partial_z \phi)$$

$$\vec{\nabla} \times \vec{F} = \underbrace{\vec{\nabla} \times \vec{\nabla} \phi}_{=0} = 0$$

Figure 48: Conservatieve velden

Een vectorveld wordt conservatief genoemd wanneer deze kan herschreven worden als een gradient van een scalair veld  $\phi$

$$F = \nabla \phi$$

Samengevat: voor een ESG (enkelvoudig samenhangend gebied) is een vectorveld conservatief als de rotor = 0,  $\nabla \times F = 0$ . Hierbij hebben we dan ook het feit dat het vectorveld conservatief is:  $F = \nabla \phi$

**ESG** btw is een gebied die we kunnen herleiden tot een punt. Dus stel er is een paal midden in het gebied, is dit geen ESG want we de paal staat in de weg!

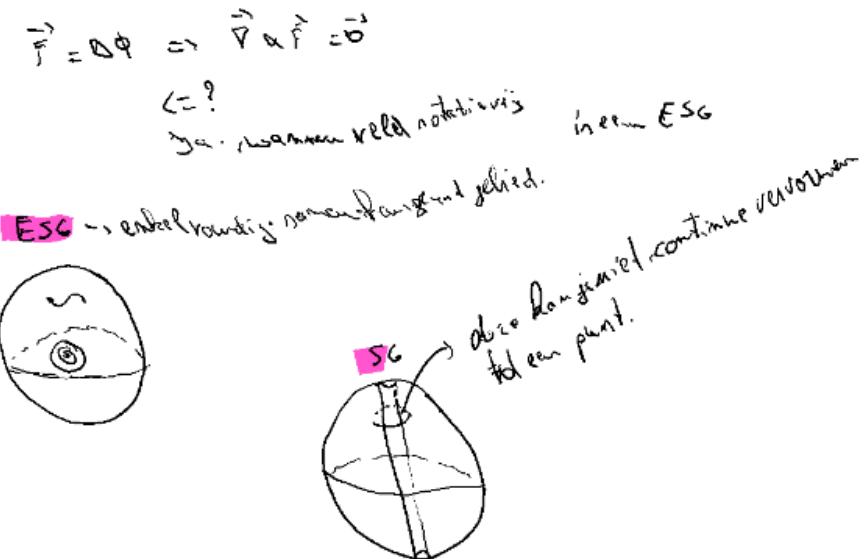


Figure 49: ESG

### 4.2.3 Voorbeeld

Voorbeeld:  $\vec{F} = \left( \frac{x}{z}, \frac{y}{z}, -\frac{xy}{z^2} \right)$

*rotoren*

$\nabla \times \vec{F} = \vec{0}$ ,  $F_x = \partial_z \phi = \frac{y}{z} \Rightarrow \phi = \frac{xy}{z} + f(y, z)$

$F_y = \partial_z \phi = \frac{x}{z} \Rightarrow \frac{xy}{z} + f(z)$

$F_z = \partial_z \phi = -\frac{xy}{z^2} \Rightarrow \frac{xy}{z} + c$

Als  $F$  twee keer afleidbaar is, dan is het veld conservatief, als de veld ESG is, dan is het veld conservatief.

Figure 50: Voorbeeld rotor conservatief

### 4.2.4 The Fab Four: gradient, divergentie, laplaciaan, rotor

1. Gradient:  $\nabla = (\partial_x, \partial_y, \partial_z)$
2. Divergentie: Als we de gradient combineren met een vector veld, krijgen we de divergentie. Dit is een scalaire waarde.  $\nabla \cdot E = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$
3. Laplacian: Is letterlijk gewoon gradient maar twee keer partieel afleiden per component...  $\nabla \cdot \nabla = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

// Note: De toepassing in de cursus wordt nooit op die manier gegeven... Skip

Nog iets heel belangrijk: **Rotor:**  $\nabla \times E = \begin{bmatrix} e_1 & e_2 & e_3 \\ \partial_x & \partial_y & \partial_z \\ E_x & E_y & E_z \end{bmatrix}$

Als rotor = 0, dan is het veld conservatief.

## 4.3 Nabla calculus

Dit zijn basically a bunch of formulas die je moet kunnen bewijzen:

allgemein  
 beweisen  
 beweisen

$$\begin{aligned}
 \vec{\nabla}(\phi + \chi) &= \vec{\nabla}\phi + \vec{\nabla}\chi \\
 \vec{\nabla} \cdot (\vec{F} + \vec{G}) &= \vec{\nabla} \cdot \vec{F} + \vec{\nabla} \cdot \vec{G} \\
 \vec{\nabla} \times (\vec{F} + \vec{G}) &= \vec{\nabla} \times \vec{F} + \vec{\nabla} \times \vec{G} \\
 \vec{\nabla} \cdot (\phi \vec{F}) &= (\vec{\nabla}\phi) \cdot \vec{F} + \phi \vec{\nabla} \cdot \vec{F} \\
 \vec{\nabla} \times (\phi \vec{F}) &= (\vec{\nabla}\phi) \times \vec{F} + \phi \vec{\nabla} \times \vec{F} \\
 \vec{\nabla} \cdot (\vec{F} \times \vec{G}) &= \vec{G} \cdot (\vec{\nabla} \times \vec{F}) - \vec{F} \cdot (\vec{\nabla} \times \vec{G}) \\
 \vec{\nabla} \times (\vec{\nabla}\phi) &= \vec{0} \\
 \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) &= 0 \\
 \vec{\nabla} \times (\vec{\nabla} \times \vec{F}) &= \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - (\vec{\nabla} \cdot \vec{\nabla}) \vec{F}.
 \end{aligned} \tag{4.2}$$

Figure 51: Bewijzen nabla calculus

$\frac{1}{\text{Alle bewijzen:}}$   
 $\vec{\nabla}(\varphi + \psi) = \left( \frac{\partial}{\partial x} (\varphi + \psi) + \frac{\partial}{\partial y} (\varphi + \psi) + \frac{\partial}{\partial z} (\varphi + \psi) \right)$   
 $= \left( \frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial x} + \frac{\partial \varphi}{\partial y} \right) + \left( \frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial y} + \frac{\partial \varphi}{\partial z} \right)$   
 $= \vec{\nabla} \varphi + \vec{\nabla} \psi$

$\frac{1}{\text{Som principle:}}$   
 $\vec{\nabla}(\vec{F} + \vec{G}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} + \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_x & G_y & G_z \end{vmatrix}$   
 $= \vec{\nabla} \vec{F} + \vec{\nabla} \vec{G}$

$\vec{\nabla} \cdot (\varphi \vec{F})$   
 $= \frac{\partial \varphi}{\partial x} F_x + \frac{\partial \varphi}{\partial y} F_y + \frac{\partial \varphi}{\partial z} F_z$   
 $= \left( \frac{\partial \varphi}{\partial x} F_x + \varphi \frac{\partial F_x}{\partial x} \right) + \dots$   
 $= \vec{\nabla}(\varphi) \cdot \vec{F} + \varphi \cdot \vec{\nabla} \vec{F}$

$\vec{\nabla} \times (\vec{\nabla} \varphi) = \vec{\nabla} \times \left( \frac{\partial \varphi}{\partial x} i + \frac{\partial \varphi}{\partial y} j + \frac{\partial \varphi}{\partial z} k \right)$   
 $= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial^2 \varphi}{\partial x^2} & \frac{\partial^2 \varphi}{\partial y^2} & \frac{\partial^2 \varphi}{\partial z^2} \end{vmatrix} = \begin{matrix} \cancel{\frac{\partial^2 \varphi}{\partial y \partial z}} - \cancel{\frac{\partial^2 \varphi}{\partial z \partial y}} \\ \cancel{\frac{\partial^2 \varphi}{\partial x \partial z}} - \cancel{\frac{\partial^2 \varphi}{\partial z \partial x}} \\ \cancel{\frac{\partial^2 \varphi}{\partial x \partial y}} - \cancel{\frac{\partial^2 \varphi}{\partial y \partial x}} \end{matrix} = 0$

Figure 52: Sommige bewijzen nabla calculus

#### 4.3.1 Solenoïdale vector velden

solenoidal is wanneer we het volgende hebben:  $F = \nabla \times A$  (Hierbij is  $F$  dus solenoidal)

Met  $A$  het **Vector potentiaal**.

Wanneer we  $\nabla \cdot F = 0$  dan is onze solenoidal vectorveld ook **divergentie vrij**.

Belangrijk volgende relatie:  $F = \nabla \times A$  (solenoidal)  $\equiv \nabla \cdot F = 0$  (divergentie vrij)

Zoals bij de Conservatieve velden, wanneer het veld niet ESG is, kunnen we niet convergeren naar 1 punt en dus  $F \neq \nabla \times A$

### 4.3.2 Voorbeeld

**Voorbeeld.** Beschouw het vectorveld  $\vec{F} = (z, x, y)$ . Is dit vectorveld solenoïdaal? Zoja, bepaal een vectorpotentiaal  $\vec{A}$ .

Figure 53: Sinusoidal example

It's very easy to see that the field is solenoidal, just take gradient and boem = (0, 0, 0)

Then we find  $A$ :

*Oplossing:* We zien onmiddellijk dat  $\nabla \cdot \vec{F} = 0$ , het vectorveld is dus inderdaad solenoïdaal.  $\vec{A}$  moet dan voldoen aan:

$$\partial_y A_z - \partial_z A_y = z \quad \partial_z A_x - \partial_x A_z = x \quad \partial_x A_y - \partial_y A_x = y \quad (4.2.50)$$

We gebruiken nu de vrijheid (4.2.49) om  $A_z$  gelijk aan nul te stellen. Dit kan inderdaad altijd: voor een gegeven  $(A_x, A_y, A_z)$  vinden we met  $\chi(x, y, z) = - \int_{z_0}^z dz' A_z(x, y, z')$  een nieuwe vectorpotentiaal  $\vec{A}' = (A'_x, A'_y, 0)$ .

The handwritten derivation shows the following steps:

- $A_z \rightarrow A_z + \int d\chi = 0$
- $\Rightarrow A_z = - \int d\chi$
- $r = \sqrt{x^2 + y^2} \Rightarrow x = r \cos \theta, y = r \sin \theta \Rightarrow d\chi = dz = -dz/r$
- $F_R = z \approx - \int dz A_z = A_z z = -z^2/2 + f(x, y)$
- $F_S = x \approx + \int d\chi A_x = A_x r \cos \theta = r \cos \theta + f(x, y)$
- $F_T = y \approx \int d\chi A_y = A_y r \sin \theta = r \sin \theta + f(x, y)$
- $\vec{A}' = (A'_x, A'_y, 0) = (r \cos \theta, r \sin \theta, -z^2/2 + f(x, y))$
- $\nabla \cdot \vec{A}' = 0, \nabla \times \vec{A}' = \vec{F}$

Figure 54: Solenoide solution

### 4.3.3 Helmholtz decompositie

In een ESG kunnen we een vectorveld decomponeren in een solenoidal en een conservatief veld.

$F = \nabla \phi + \nabla \times A$  (eerste term is conservatief, tweede term is solenoidal)

Note: Dit is een **poissonvergelijking**

Die je kunt berekenen door het volgende te doen:

$$\pi(x) = - \int \frac{\rho(x')}{||x-x'|| \cdot 4\pi} d^3 x'$$

### 4.3.4 Voorbeeld

**Voorbeeld:** Geef een Helmholtzontbinding voor het vectorveld  $\vec{F} = (x, z, y)$ .

Figure 55: Helmholtz example

```

with(VectorCalculus):
SetCoordinates(cartesian[x, y, z]):


# Define the vector field F
F := <Fx(x, y, z), Fy(x, y, z), Fz(x, y, z)>:


# Compute the solenoidal (divergence-free) component using the curl
Solenoidal := Curl(F):


# Compute the irrotational (curl-free) component using the gradient
Potential := int(Fx, x) + int(Fy, y) + int(Fz, z):
Irrotational := Gradient(Potential):


# Display the components
Solenoidal, Irrotational;

```

Figure 56: Helmholtz maple

## Helmholtz ontbinding

$\vec{F} = (\alpha, z, \omega)$

$\vec{F} = \nabla\phi + \nabla \times A$

A)  $\nabla F = \nabla\phi = \frac{\partial F_x}{\partial x} + \frac{\partial F_z}{\partial z} + \frac{\partial F_\omega}{\partial \omega} = 0$

en de oplossing is  $\phi = \frac{\alpha}{r}$  omdat  $(\frac{\alpha}{r}, z)$

$\nabla \times A = \vec{F} - \nabla\phi = (\alpha, z, \omega) - (\alpha, 0, 0) = (0, z, \omega)$

dus  $\nabla \times A = (0, z, \omega)$

$= \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$

$\begin{cases} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0 & \text{Annule } A_x = 0 \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = z \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = \omega \end{cases}$

$\Rightarrow \begin{cases} 0 = -\frac{\partial A_y}{\partial z} \\ z = -\frac{\partial A_z}{\partial x} \rightarrow A_z = \frac{1}{2}z^2 + f(y) \\ 0 = \frac{\partial A_y}{\partial x} = 0 = A_y \end{cases}$

$\therefore f(z) = -\frac{1}{2}z^2$

$\therefore \vec{A} = (0, 0, \frac{1}{2}(z^2 - \omega^2))$

Figure 57: Helmholtz solution

## 4.4 Integratie

### 4.4.1 Lijnintegralen

Is een integraal over een kromme  $C$  waarbij de lijnintegraal niet afhangt van de zin waarmee de kromme wordt doorlopen.

[A] Booglengte, dichtheid, massamiddelpunt, ...:

Stel we willen een koord lengte definieren, dit kunnen we doen door:

$$ds = \sqrt{dx^2 + dy^2}$$

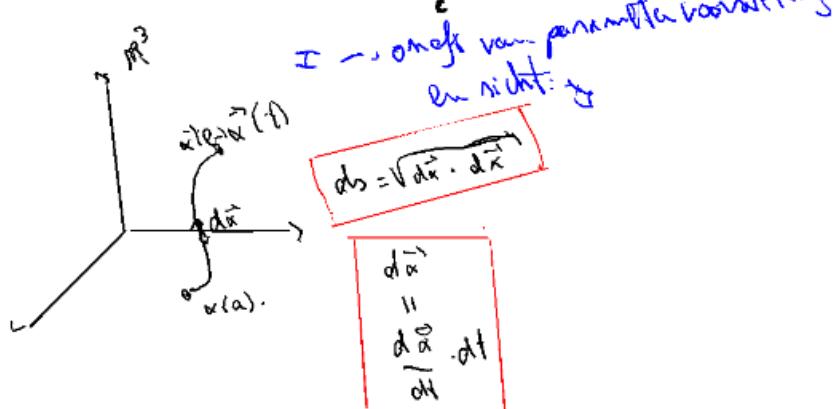
$$\text{met } dx = \frac{dx}{dt} dt$$

waarbij  $\frac{dx}{dt}$  de **raakvector** is.

Hierbij mijn notities over het concept:

## Lijn integraal

$$\int_C ds \dots, \int_C d\vec{x} \cdot \vec{F}, \int_C d\alpha \times \vec{F}$$



$$\text{II } \int_C d\vec{x} \cdot \vec{F} = \int_a^b dt \frac{d\vec{x}}{dt} \cdot \vec{F} = \int_a^b dt \in F_i(\vec{x}(t)) \frac{d\alpha_i(t)}{dt}$$

Figure 58: Notities lijn integraal

### 4.4.2 Voorbeeld

**Voorbeeld.** Bereken de lengte-integraal (4.3.65) voor het halve cirkelsegment

$$(x, y, z) = (\cos \theta, \sin \theta, 0), \quad \text{met } \theta \in [0, \pi], \quad (4.3.70)$$

voor 2 parametrisaties. Gebruik voor de eerste parametrisatie  $t = \theta$  en voor de tweede parametrisatie  $t = y$ , de cartesische  $y$ -coördinaat.

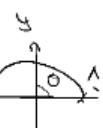


Figure 59: Voorbeeld lijn integraal

*met  $\lambda = 1$*

**Voorbeeld.** Bereken de lengte-integraal (4.3.65) voor het halve cirkelsegment

$$(x, y, z) = (\cos \theta, \sin \theta, 0), \quad \text{met } \theta \in [0, \pi], \quad (4.3.70)$$

voor 2 parametrisaties. Gebruik voor de eerste parametrisatie  $t = \theta$  en voor de tweede parametrisatie  $t = y$ , de cartesische  $y$ -coördinaat.

*Oplossing:* Voor de eerste parametrisatie vinden we  $\frac{d\vec{x}}{d\theta} = (-\sin \theta, \cos \theta)$  en daaruit:

$$\left( -\sin \theta, \cos \theta \right) \cdot \left( \cos \theta, \sin \theta \right) = \frac{d\vec{x}}{d\theta} \cdot \frac{d\vec{x}}{d\theta} = 1 \Rightarrow d\theta = \sqrt{\pi} \approx 1 \quad (4.3.71)$$

Zodoende wordt de lengte-integraal (4.3.65):

$$\ell = \int_0^{\pi} d\theta = \pi. \quad (4.3.72)$$

Voor de tweede parametrisatie  $t = y$  verdelen we vooreerst de gevraagde totale kromme in twee segmenten  $C_1$  en  $C_2$  links en rechts van de  $Y$ -as. We krijgen dan volgende parametrisaties:

$$\begin{aligned} C_1 : \quad (x, y, z) &= (-\sqrt{1-y^2}, y, 0) \quad \text{met } y \in [0, 1] \\ C_2 : \quad (x, y, z) &= (+\sqrt{1-y^2}, y, 0) \quad \text{met } y \in [0, 1] \end{aligned} \quad (4.3.73)$$

Nu hebben we:  $\frac{d\vec{x}}{dy} = (\pm \frac{y}{\sqrt{1-y^2}}, 1, 0)$  en zodus:

$$\frac{d\vec{x}}{dy} \cdot \frac{d\vec{x}}{dy} = \frac{y^2}{1-y^2} + 1 = \frac{1}{1-y^2}. \quad (4.3.74)$$

Zo vinden we dan:

$$\ell = \ell_1 + \ell_2 = 2 \times \int_0^1 dy \sqrt{\frac{1}{1-y^2}} = \pi, \quad \text{daarvan beide kanten} \quad (4.3.75)$$

Bemerkt dat het linker-segment  $C_1$  (als functie van  $y \in [0, 1]$ ) hier in tegengestelde richting wordt doorlopen als het segment  $C_2$  en als het door  $\theta$  geparametriserde volledige segment  $C$ . Bemerkt ook dat voor dit voorbeeld  $t = \theta$  duidelijk de slimmere keuze is, de  $y$ -parametrisatie werd hier genomen om expliciet te illustreren dat de lengte-integraal onafhankelijk is van de zin waarin de kromme wordt doorlopen.

*Gedownload!*

Figure 60: Lijnintegraal solution

**Voorbeeld.** Bereken de massa en het massamiddelpunt van de solenoïdale draad met uniforme dichtheid  $\rho$ , gelegen volgens de kromme:

$$(x, y, z) = (R \cos \theta, R \sin \theta, z_0 \theta) \quad \text{met} \quad \theta \in [0, \theta_0]. \quad (4.3.76)$$

$$\text{massa} = \int \rho \cdot d\vec{s} = \rho \int d\theta$$

*Oplossing:* We kiezen  $t = \theta$  en hebben dan:

$$\frac{d\vec{x}}{d\theta} \cdot \frac{d\vec{x}}{d\theta} = R^2 (\underbrace{\sin^2 \theta + \cos^2 \theta}_{=1}) + z_0^2 = R^2 + z_0^2. \quad (4.3.77)$$

Voor de massa vinden we dan:

$$M = \rho \int_0^{\theta_0} d\theta \sqrt{R^2 + z_0^2} = \rho \theta_0 \sqrt{R^2 + z_0^2}. \quad (4.3.78)$$

Verder vinden we dan voor het **massamiddelpunt**:

$$\begin{aligned} M(x_{m.m.}, y_{m.m.}, z_{m.m.}) &= \rho \int_0^{\theta_0} d\theta \sqrt{R^2 + z_0^2} (R \cos \theta, R \sin \theta, z_0 \theta) \\ &= \rho \sqrt{R^2 + z_0^2} (R \sin \theta_0, R(1 - \cos \theta_0), z_0 \frac{\theta_0^2}{2}), \end{aligned} \quad (4.3.79)$$

en daaruit tot slot:

$$\vec{x}_{m.m.} = \underbrace{(R \frac{\sin \theta_0}{\theta_0}, R \frac{1 - \cos \theta_0}{\theta_0}, z_0 \frac{\theta_0^2}{2})}_{\text{as-100-punt}}. \quad (4.3.80)$$

Figure 61: Example lijnintegraal massa middelpunt

[B] **Vectorvelden:**  $\int F \cdot dx, \int F \times dx$ :

Hier hetzelfde als bij punt [A], maar nu definieren we de volgende integraal:

$$\int_b^a dt \sum_i F_i(x_1, x_2, x_3) \frac{dx_i}{dt},$$

Figure 62:

nog een ander type integraal is de volgende:

$$\int_b^a dt (F_y \dot{z} - F_z \dot{y}, F_z \dot{x} - F_x \dot{z}, F_x \dot{y} - F_y \dot{x}),$$

Figure 63: Ingtegraal vector velden type 2

#### 4.4.3 Voorbeeld

**Voorbeeld.** Bereken de lijnintegraal (4.3.81) voor het vectorveld  $\vec{F} = (z, x, y)$  langs de solenoïdale kromme van het vorige voorbeeld (4.3.76) voor het eindpunt  $\theta_0 = \frac{3\pi}{2}$ .

*Oplossing:* Gebruiken we opnieuw de parametrisatie volgens  $\theta$  dan vinden we:

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{x} &= \int_0^{\frac{3\pi}{2}} d\theta (z_0\theta, R\cos\theta, R\sin\theta) \cdot (-R\sin\theta, R\cos\theta, z_0) \\ &= \int_0^{\frac{3\pi}{2}} d\theta R^2 \cos^2\theta + Rz_0 \sin\theta(1-\theta) \\ &= \frac{3}{4}\pi R^2 + 2z_0 R.\end{aligned}\quad (4.3.84)$$

Figure 64: Voorbeeld lijn integraal vectorveld

#### 4.4.4 Voorbeeld

Tornado voorbeeld

$$\begin{aligned}&\text{Handtekening voorbeeld} \\ &\text{Veld } \vec{v} \neq \vec{0} \quad \vec{v} = \vec{\nabla}\phi \\ &\vec{v} \times \vec{v} = \vec{0} \\ &\oint_C \vec{v} \cdot d\vec{x} \\ &\text{met } \vec{v} = \vec{\nabla}\phi \text{ zelfs punt} \\ &\phi(x(a)) - \phi(x(b)) = 0, \text{ dus kan niet!}\end{aligned}$$

$$\boxed{\Sigma \cdot n \cdot F = \oint_E \vec{E} \cdot d\vec{x}}$$

$$\text{III. } \int d\vec{x} \times \vec{F} = \int_a^b dt \left( \frac{d\vec{x}}{dt} \times \vec{F} \right)$$

Figure 65: Voorbeeld tornado

#### 4.4.5 Voorbeeld

Bij oost Sarant.

$$\vec{B}(\vec{x}) = \frac{\mu_0 I}{4\pi} \cdot \int \frac{d\vec{x}' \times (\vec{x} - \vec{x}')}{\|\vec{x} - \vec{x}'\|^3}$$

$$\vec{x}' = (0, 0, z') \quad d\vec{x}' = dz' (0, 0, 1)$$

$$d\vec{x}' \times (\vec{x} - \vec{x}') = dz' \begin{vmatrix} e_x & e_y & e_z \\ 0 & 0 & 1 \\ x & y & z - z' \end{vmatrix}$$

$$= dz' (-y, +x, 0)$$

$$\int_{-\infty}^{+\infty} dz' \frac{(-y, x, 0)}{\left( (x^2 + y^2 + (z-z')^2)^{1/2} \right)^3} = \frac{\mu_0 I}{2\pi} \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0 \right)$$

$$\int_{-\infty}^{+\infty} \frac{1}{(x^2 + y^2 + (z-z')^2)^{1/2}} dz' = \frac{2}{x^2 + y^2}$$

Figure 66:

## 4.5 Oppervlakteintegralen

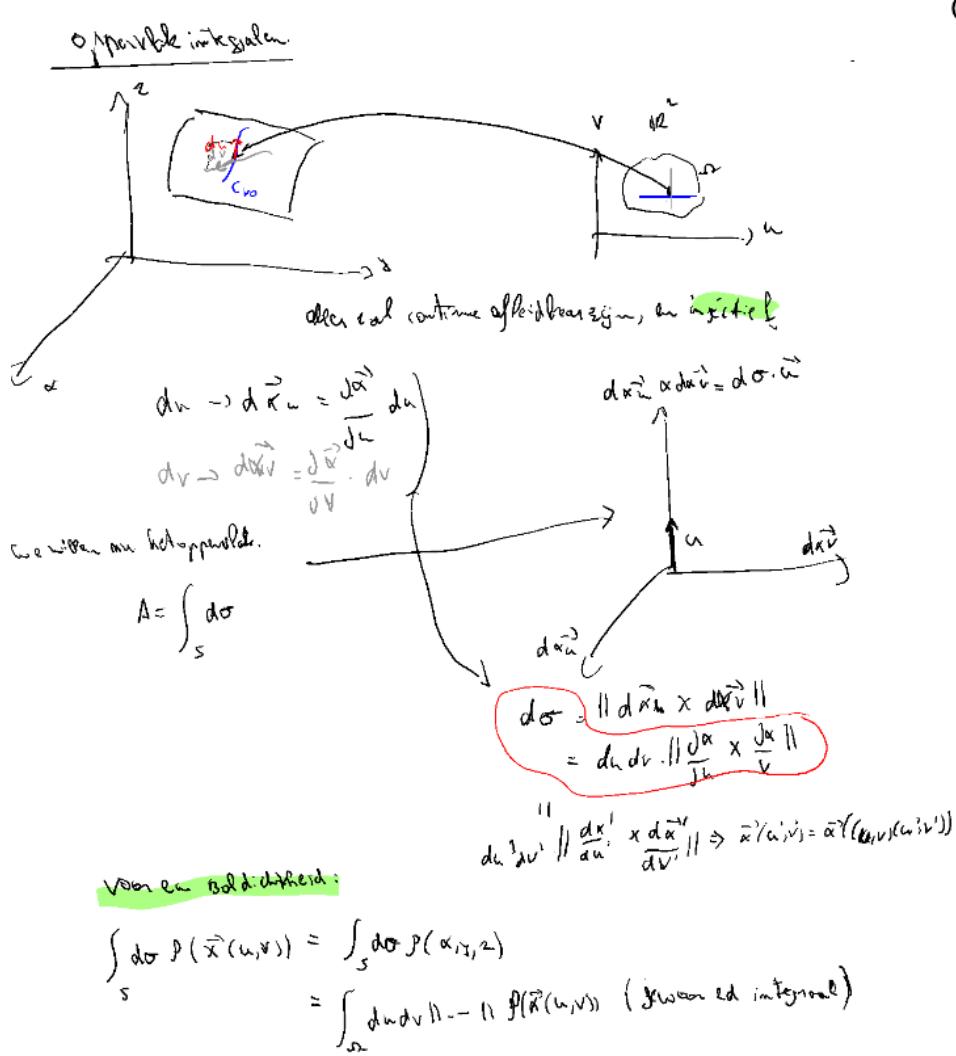


Figure 67: Oppervlakte integralen: maw, we gaan de oppervlakte in 3D berekenen door  $\int \int dudv \text{norm}(du \times dv)$

### 4.5.1 Voorbeeld

**Voorbeeld.** Bereken de oppervlakte voor het randoppervlak van een bol met straal  $R$ .

Figure 68: Voorbeeld boloppervlak

oplossing boloppervlak  $\rightarrow$  bolcoördinaten

$$\vec{r} = (R \cos\theta \cos\phi, R \cos\theta \sin\phi, R \sin\theta)$$

$$\Theta = [0, \pi], \phi : [0, 2\pi]$$

$$d\vec{r} ? \quad \frac{\partial \vec{r}}{\partial \theta} = (R \cos\theta \cos\phi, R \cos\theta \sin\phi, -R \sin\theta)$$

$$\frac{\partial \vec{r}}{\partial \phi} = (-R \sin\theta \cos\phi, R \sin\theta \sin\phi, 0)$$

$\rightarrow$  Jacobidam.

$$\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi} = \overbrace{R^2 \sin\theta}^{\text{eenheidsvector } \hat{n}} (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$$

$\rightarrow$  eenheidsvector  $\hat{n}$

$$\int_S d\sigma = \int_0^\pi d\theta \int_0^{2\pi} d\phi \underbrace{R^2 \sin\theta}_{4\pi R^2}$$

$$d\vec{\sigma} = d\vec{\Omega} \hat{n} = du dv \cdot \left( \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right)$$

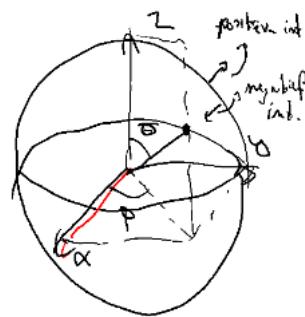
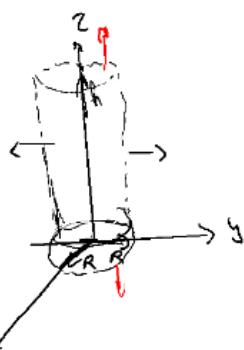


Figure 69: Voorbeeld boloppervlak oplossing

$$\int_S d\vec{\alpha} \cdot \vec{F}$$

$$\vec{r} = (x, y, 0)$$



$\vec{\alpha} = (R_{\cos\theta}, R_{\sin\theta}, z)$   
 $\theta \in [0, 2\pi]$

$$z = \sum_0 k_i$$

$$\vec{d\alpha} = (-R_{\sin\theta}, R_{\cos\theta}, 0)$$

$$d\theta$$

$$\vec{j\alpha} = (0, 0, 1)$$

$$\frac{\vec{d\alpha}}{d\theta} = \frac{d\vec{\alpha}}{dz} = (R_{\cos\theta}, R_{\sin\theta}, 0)$$

$$\begin{aligned} & \int_0^R \int_0^{2\pi} R (\cos\theta, \sin\theta, 0) (R_{\cos\theta}, R_{\sin\theta}, 0) \\ & \quad \text{Fluisveld in } F = (x, y, 0) \\ & = R^2 R \pi \cdot 2 \end{aligned}$$

*stelt orthogonaal op het oppervlak (mantel)*

Figure 70: Cilinder oppervlak oplossing

Dan is er ook een toepassing van flux doorheen een oppervlak die kan uitgelegd worden met de volgende afbeelding:

Lading:  $\vec{J}$  :  $\frac{\text{Lading}}{\text{m}^2 \cdot \text{n}}$

$$\text{Flux} = \frac{\text{Lading}}{\text{A}}$$

$$= \int_S d\vec{\sigma} \cdot \vec{J}$$

Voorbeeld:

$$\vec{J} = \left( 0, 0, J_0 t - \frac{p^z}{m_e c} \right)$$

$\rightarrow$  oppervlakstelement  
met (met) gesloten randen (delen)

$$\vec{\nabla} \cdot \vec{J} = 0 \rightarrow \text{divergentie-loos}$$

$$n = p = \sqrt{x^2 + y^2}$$

Figure 71: Flux voorbeeld

#### 4.6 Integratiestellingen

4.6 Integratiestellingen

$\int_a^b d\vec{x} \frac{df}{dx} = f(b) - f(a)$	$\int_C d\vec{x} \cdot \vec{\nabla} f = f(\vec{x}_b) - f(\vec{x}_a)$
$\xrightarrow{2D} \xrightarrow{1D}$	
$\boxed{\int_S d\vec{\sigma} (\vec{V} \times \vec{F}) = \int_C d\vec{\alpha} \cdot \vec{F}}$	

S.F. den

Figure 72: Integratie stellingen

Hier zien we **Stokes**, **divergentiestelling van Gauss**, en **Stelling van Green**. **Stokes** is een generalisatie van **Green**. Het idee van stokes is dat we een 2D integraal die de oppervlakte integraal is van een rotor **herleidt** tot een lijnintegraal over de rand.

$$\boxed{\int_S d\vec{S} \cdot (\vec{\nabla} \times \vec{F}) = \oint_{C+} \vec{F} \cdot d\vec{x}},$$

Figure 73: Stelling van Stokes

Nu we weten wat Stokes eigenlijk is, laten we een voorbeeld bezichtigen:

voerbeeld

$$\vec{A} = \left( z^2, x^2, y^2 \right)$$

$$\nabla \times \vec{A} = 3(y^2, x^2, z^2)$$

$$\int_{S_1} d\vec{\sigma} (\nabla \times \vec{A})$$

$$\frac{\partial \vec{\alpha}}{\partial \varphi} \times \frac{\partial \vec{\alpha}}{\partial \theta} = \vec{p} \Big| (0, 0, 1)$$

$$d\sigma = J \cdot \vec{p} \cdot d\Omega \quad (0, 0, 1)$$

$$\int_0^R d\varphi \cdot J \cdot \int_0^{2\pi} d\theta \cdot 3 \rho^2 \cos^2 \theta = \boxed{\frac{3}{4} \pi R^4}$$

$$\int_{C^1} d\vec{\alpha} \cdot \vec{A} = \int_0^{2\pi} d\theta R \cos^2 \theta (0^2, 0) = \frac{3}{4} \pi R^4$$

$\vec{\alpha} = (r, \rho \cos \theta, \rho \sin \theta, 0)$   
 $\frac{\partial \vec{\alpha}}{\partial \varphi} = (\cos \theta, \sin \theta, 0)$   
 $\frac{\partial \vec{\alpha}}{\partial \theta} = \rho (-\sin \theta, \cos \theta, 0)$   
 $\vec{p} = \rho (-\sin \theta, \cos \theta, 0)$   
 $d\Omega = d\theta R (-\sin \theta, \cos \theta, 0)$

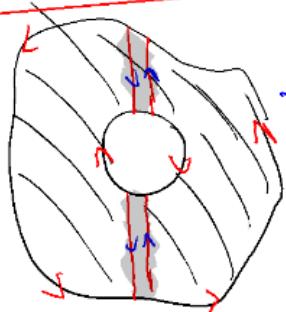
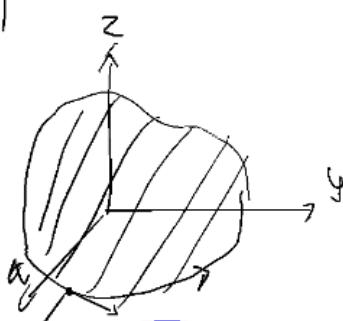
Figure 74: Stokes voorbeeld opgelost

Stelling van Green:

$$\vec{F} = (f_1(x, y), f_2(x, y))$$

$$\int_S d\vec{\alpha} \cdot (\vec{\nabla} \times \vec{F}) = \int_R dxdy (d\alpha f_2 - dy f_1)$$

$$\int_C^+ d\vec{\alpha} \cdot \vec{F} = \oint_C^+ (dx f_1 + dy f_2)$$



→ we kunnen ook  
totale integraal = sommen in tygaal

Figure 75: Uitleg Green

Aan de andere kant heb je de divergentiestelling, waarbij we een 3D volume integraal van de divergentie tot een oppervlak integraal over de randoppervlak kunnen herleiden.

## Hoe gaan Divergentiestelling

30 -> 20

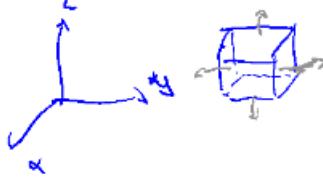
$$\begin{aligned}
 \int_V dV \vec{\nabla} \cdot \vec{F} &= \int_{\partial V} d\vec{\sigma} \cdot \vec{F} \\
 \downarrow & \\
 I &= \int_V d\vec{\sigma} \cdot \vec{n} \cdot \vec{F} && \text{Bewijz (NK)} \\
 &= dx dy dz (dxF_x + dyF_y + dzF_z) \\
 &= dx dy dz \left( \frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x} \right) \\
 &\quad + dy dz (f_x(x+\Delta x, y, z) - f_x(x, y, z)) \\
 &= \int_{\partial V} d\vec{\sigma} \cdot \vec{F} \\
 d\vec{\sigma} &= dx dy dz
 \end{aligned}$$


Figure 76: Divergentie stelling

Of gewoon in formule:

$$\boxed{\int_V dV \vec{\nabla} \cdot \vec{F} = \int_{\partial V} d\vec{\sigma} \cdot \vec{F} = \int_{\partial V} d\sigma \vec{n}^u \cdot \vec{F}},$$

Figure 77: Formule Divergentiestelling

Als je dan de stelling van Gauss wilt bekijken wordt het als volgt gedaan:

voorbeeld

Bol

$$\alpha^2 + y^2 + z^2 < R^2, \vec{F} = (\alpha, y, z)$$

$$\int_{\text{bol}} \sigma F = R^2$$

$$\nabla \vec{F} = 3$$

$$d\vec{\sigma} = R^2 d\theta d\phi \sin\theta \hat{\omega}$$

$$\vec{F} = \hat{\omega} R$$

$$\int_0^\pi \int_0^{2\pi} d\theta d\phi R^3 \sin\theta = 4\pi R^3$$

↓

*Hier zit de Jacobian erin.*

Figure 78: Voorbeeld Gauss toepassing

Zoals de prof het zei: "Ik ben trots op deze figuur". Hier gaan we zien dat de 2 infinitesimale oppervlaktes een shared boundary hebben, deze worden te niet gezien, waardoor je uiteindelijk 1 lijn integraal krijgt over die twee vlakjes, extrapolere nu naar de rest en zo bewijs je dus dat **Stokes** een Top G is. Btw, de rechterhand regel leert ons dan welke teken de lijnintegraal nodig heeft.

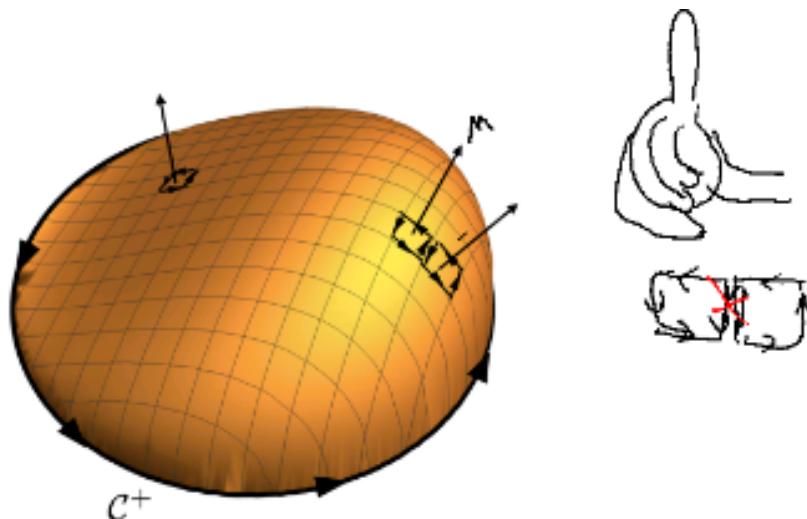


Figure 79: Inzicht integratie stellingen

## 4.7 Toepassing: Continuiteitsvergelijking

Het komt erop neer dat dit fysische toepassingen zijn waarbij we de wet van behoud van massa toepassen. Dit is een toepassing van de divergentiestelling. (een voorbeeld)

Het ander voorbeeld is behoud van lading:

4.7.3 Continuiteitsvergelijking

$$\text{Flux} = \frac{dQ}{dt} = \int_S d\sigma \cdot \vec{J}$$

$$\frac{d\sigma v}{dt} = - \int_V d\sigma \cdot \vec{J}$$

$$\int_V dv J_t \cdot g = - \int_V dv g \cdot \vec{J}$$

$$\int_V dv (J_t \cdot p + \nabla \cdot \vec{J}) = 0$$

Want aantal van lading veranderd  
dat er lading weggaat of komt.

Figure 80: Lading

## Formularium

### Taylorontwikkeling

- $f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \dots$
- $\sin(x) = x$  voor kleine  $x$

## Differentiaalvergelijkingen

- $y'(x) = \lambda y(x)$
- $y''(x) = \lambda y(x)$  (hier werden 3 gevallen besproken)

## Complexe getallen

- $z = a + bi$  (algemene vorm)
- $i^2 = -1$
- **inverse:**  $(a + bi)^{-1} = \frac{a - bi}{a^2 + b^2}$
- **complement:**  $z = a + bi \rightarrow z^* = a - bi$
- **modulus:**  $|z| = \sqrt{a^2 + b^2}$
- $e^{i\theta} = \cos(\theta) + i\sin(\theta)$
- $\sin^2(x) + \cos^2(x) = 1$
- $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$
- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$
- $\cos(2x) = \cos^2(x) - \sin^2(x)$

## Hoofdstelling van de algebra

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

met  $b = -4 * a * c$

## Lineare Algebra

$$\frac{u \cdot v}{\|u\| \|v\|} = \cos(\theta) \quad // \text{ Hoek tussen twee vectoren}$$

$$v^{\parallel} = (u_1 \cdot v) u_1 + (u_2 \cdot v) u_2 + \dots \quad // \text{ Projectie van } v$$

## Matrixen

- Jacobiaan:  $J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$

## Oefeningen

### Huis 1

```

> restart: with(LinearAlgebra):
> t := taylor(sqrt(x+4)^3,x=0,3);
v := sqrt(y+4)^3;

$$t := 8 + 3x + \frac{3}{16}x^2 + O(x^3)$$


$$v := (y+4)^{3/2} \tag{1}$$


> # voor  $5^{3/2}$  nemen we x = 1
> x_1 := evalf( $8 + 3 + \frac{3}{16}$ )

$$x_1 := 11.18750000 \tag{2}$$


> # Wat is de fout? Wel, dat zal de derde term zijn ( $O(x^3)$ )
> error_1 := abs( $\frac{x^3}{6} \cdot \text{diff}(v, y\$3)$ );
error_1 := evalf(subs(y=0, x=1, error_1))

$$\text{error}_1 := \frac{|x|^3}{16|y+4|^{3/2}}$$


$$\text{error}_1 := 0.007812500000 \tag{3}$$


> # voor  $6^{3/2}$  nemen we x = 2
> x_2 := evalf( $8 + 3 \cdot 2 + \frac{3}{16} \cdot 2^2$ )

$$x_2 := 14.75000000 \tag{4}$$


> error_2 := abs( $\frac{x^3}{6} \cdot \text{diff}(v, y\$3)$ )

$$\text{error}_2 := \frac{|x|^3}{16|y+4|^{3/2}} \tag{5}$$


> error_2 := evalf(subs(x=2, y=0, error_2))

$$\text{error}_2 := 0.0625000000 \tag{6}$$


> # Waarom nemen we y = 0? Omdat dit de grootste fout zou maken, we
nemen altijd max. Dus fout \element{0, 1, 2}

```

Figure 81: Exercise 1

```

> restart:
> # i
> limit( sin(3·x) / sinh(x), x = 0 )           3
(1)
> # ii
> limit( (tan(x) - tanh(x)) / (sinh(x) - x), x = 0 )   4
(2)
> # iii
> limit( (sqrt(1 - a·x) - sqrt(1 + a·x)) / x, x = 0 ) -a
(3)
> # IV
> limit( (ln(1 + x) + ln(1 - x)) / x^2, x = 0 ) -1
(4)

```

Figure 82: Exercise 2

$$\exp(i\theta) = \cos\theta + i\sin\theta \quad B:$$

$$\begin{aligned}
 e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \rightarrow e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} \\
 &= 1 + i\theta - \frac{i\theta^2}{2} - i\frac{\theta^3}{6} + \frac{\theta^4}{24} + \dots \\
 &= \underbrace{i\left(\theta - \frac{\theta^3}{6} + \dots\right)}_{\cos\theta} + \underbrace{-\frac{\theta^2}{2} + \frac{\theta^4}{24} + \dots}_{i\sin\theta} \\
 &= \cos\theta + i\sin\theta
 \end{aligned}$$

Figure 83: Exercise 3

$$\begin{aligned}
 a &= \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} \quad b = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \\
 a \cdot b &= 1 \cdot (-1) + (1 \cdot 1) + (-1 \cdot 0) = 0
 \end{aligned}$$

$$\begin{aligned}
 A < ab^T &= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}
 \end{aligned}$$

Figure 84: Exercise 4

## WC 1

```

> p := x→x·(x - 1) - a
      p := x→x·(x - 1) - a
                                         (1)

> # i
> p_i := subs(a = 0, p(x))
      p_i := x(x - 1)
                                         (2)

> solve(p_i = 0)
      0, 1
                                         (3)

> # Dus lambda = 0, en lambda = 1
> # Benader naar x = 0
> x_0 := 0 + u·a + v·a2
      x_0 := v a2 + u a
                                         (4)

> p_subs_0 := subs(x = x_0, p(x))
      p_subs_0 := (v a2 + u a) (v a2 + u a - 1) - a
                                         (5)

> p_expand_0 := expand(p_subs_0)
      p_expand_0 := a4 v2 + 2 a3 u v + a2 u2 - v a2 - u a - a
                                         (6)

> coeff_1_0 := coeff(p_expand_0, a, 1)
      coeff_1_0 := -u - 1
                                         (7)

> coeff_2_0 := coeff(p_expand_0, a, 2)
      coeff_2_0 := u2 - v
                                         (8)

> solve_0 := solve({coeff_1_0 = 0, coeff_2_0 = 0}, {u, v})
      solve_0 := {u = -1, v = 1}
                                         (9)

> result_0 := subs(u = rhs(solve_0[1]), v = rhs(solve_0[2]), x_0)
      result_0 := a2 - a
                                         (10)

>
> # Benader 1
> x_1 := 1 + u·a + v·a2
      x_1 := v a2 + u a + 1
                                         (11)

> p_subs_1 := subs(x = x_1, p(x))
      p_subs_1 := (v a2 + u a + 1) (v a2 + u a) - a
                                         (12)

> p_expand_1 := expand(p_subs_1)
      p_expand_1 := a4 v2 + 2 a3 u v + a2 u2 + v a2 + u a - a
                                         (13)

> coeff_1_1 := coeff(p_expand_1, a, 1)
      coeff_1_1 := u - 1
                                         (14)

> coeff_1_2 := coeff(p_expand_1, a, 2)
      coeff_1_2 := u2 + v
                                         (15)

> solve_1 := solve({coeff_1_1 = 0, coeff_1_2 = 0}, {u, v})
      solve_1 := {u = 1, v = -1}
                                         (16)

> result_1 := subs(u = rhs(solve_1[1]), v = rhs(solve_1[2]), x_1)

```

Figure 85: Exercise 1

```

> # Define the Taylor expansion for a forward approximation
Taylor_y := (h,t) → y(t) + h*diff(y(t),t) + (1/2)*h^2*diff(y(t),t$2) + (1/6)
    *h^3*diff(y(t),t$3) + (1/24)*h^4*diff(y(t),t$4):

# Define the Taylor expansion for a backward approximation
Taylor_g := (h,t) → y(t) - h*diff(y(t),t) + (1/2)*h^2*diff(y(t),t$2) - (1/6)
    *h^3*diff(y(t),t$3) + (1/24)*h^4*diff(y(t),t$4):

# Define the result expression
result := (h,t) → (Taylor_y(h,t) + Taylor_g(h,t) - 2*y(t)) / h^2:

# Simplify the result
simplified_result := simplify(result(h,t));


$$\text{simplified\_result} := \frac{h^2 \left( \frac{d^4}{dt^4} y(t) \right)}{12} + \frac{d^2}{dt^2} y(t) \quad (1)$$


>
> #i) kwadratisch
> #ii
> restart;
with(plots):
with(plottools):

t_val := evalf( $\left(\frac{\pi}{3}\right)$ );

# Define the function y and its 2nd derivative
y := t → -cos(t);

# Define the error function as a function of h for a specific t
err := (h) → abs( $y(t_{\text{val}})$ 
    -  $\frac{(\cos(h+t_{\text{val}}) + \cos(h-t_{\text{val}}) - 2\cos(t_{\text{val}}))}{h^2}$ );

# Plot the error as a function of h with log-log scale
loglogplot(err(h), h = 10^{-8} .. 10^3);
t_val := 1.047197551
y := t → -cos(t)
err := h →  $|y(t_{\text{val}}) - \frac{\cos(h+t_{\text{val}}) + \cos(h-t_{\text{val}}) - 2\cos(t_{\text{val}})}{h^2}|$ 

```

Figure 86: Exercise 2

$$3) \quad v_1 = \begin{pmatrix} 2i \\ -1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix}$$

$$\vec{v}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{(2i, -1, 0, i)^T}{\sqrt{(2i)^2 + (-1)^2}} = \frac{1}{\sqrt{2i \cdot (-1) + 1}} \cdot (2i, -1, 0, i)^T$$

$$= \left( \frac{2i}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, 0, 0 \right)^T$$

$$\vec{v}_2 = \vec{v}_2 - (\vec{v}_1 \cdot \vec{v}_2) \cdot \vec{v}_1 =$$

$$v_3 = v_3 - \sum_{k=1}^{n-1} \langle v_k \cdot v_3 \rangle \cdot v_k$$

⋮

$$2) \quad \text{open } \langle v_1, v_2 \rangle$$

$$\omega'' = (v_1 \cdot v_2) v_1 + (v_2 \cdot v_1) v_2$$

$$\text{Remember: } \omega = \omega'' + \omega^\perp$$

Figure 87: Exercise 3

```

#ii)
with(LinearAlgebra):
> v_1 := Vector([2·I, -1, 0, 0])

```

$$v_1 := \begin{bmatrix} 2I \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad (1)$$

```

> v_2 := Vector([I, 0, 3, 1])

```

$$v_2 := \begin{bmatrix} I \\ 0 \\ 3 \\ 1 \end{bmatrix} \quad (2)$$

```

> u_1 := v_1 / Norm(v_1, 2)

```

$$u_1 := \begin{bmatrix} \frac{2I}{\sqrt{5}} \\ -\frac{\sqrt{5}}{5} \\ 0 \\ 0 \end{bmatrix} \quad (3)$$

```

> u_2 := (v_2 - (u_1 · v_2) · u_1) / Norm(v_2 - (u_1 · v_2) · u_1, 2)

```

$$u_2 := \begin{bmatrix} \frac{I}{255} \sqrt{255} \\ \frac{2\sqrt{255}}{255} \\ \frac{\sqrt{255}}{17} \\ \frac{\sqrt{255}}{51} \end{bmatrix} \quad (4)$$

```

> w := Vector([3, 1 - I, 2 + I, 1])

```

$$(5)$$

Figure 88: Exercise 3

## Bord 1

```

> p := x→x·(x - 1)·(x - 2)·(x - 4) - a
      p := x→x·(x - 1)·(x - 2)·(x - 4) - a
                                         (1)
> x_0 := u·a + v·a2
      x_0 := v a2 + u a
                                         (2)
> p_subs := subs(x = x_0, p(x))
      p_subs := (v a2 + u a) (v a2 + u a - 1) (v a2 + u a - 2) (v a2 + u a - 4) - a
                                         (3)
> p_expand := expand(p_subs)
      p_expand := a8 v4 + 4 a7 u v3 + 6 a6 u2 v2 - 7 a6 v3 + 4 a5 u3 v - 21 a5 u v2 + a4 u4
      - 21 a4 u2 v + 14 a4 v2 - 7 a3 u3 + 28 a3 u v + 14 a2 u2 - 8 v a2 - 8 u a - a
                                         (4)
> # Only keep the 1st order and second order
>
> coeff_1 := coeff(p_expand, a, 1)
      coeff_1 := -8 u - 1
                                         (5)
> coeff_2 := coeff(p_expand, a, 2)
      coeff_2 := 14 u2 - 8 v
                                         (6)
> solve({coeff_1 = 0, coeff_2 = 0}, {u, v})
      {u = -1/8, v = 7/256}
                                         (7)
>

```

Figure 89: Exercise 1

$$2) \text{ i) } y'(t_n) \approx v_n = \frac{y_{n+1} - y_n}{h}$$

$$\begin{cases} c = t_n \\ t_{n+1} = t_n + h \end{cases}$$

$$y(t_{n+1}) = y(t_n) + h \cdot y'(t_n) + \frac{h^2}{2} \tilde{y}''(f) \quad f \in [t_n, t_{n+1}]$$

$$\Rightarrow v_n = \frac{y_{n+1} - y_n}{h} = \frac{y(t_n) + h \cdot y'(t_n) + \frac{h^2}{2} \tilde{y}''(f) - y(t_n)}{h} \\ = y'(t_n) + \frac{h}{2} \tilde{y}''(f)$$

found:  $|y'(t_n) - v_n| = \frac{h}{2} \tilde{y}''(f) = \frac{h}{6} c \rightarrow \text{linear.}$   
 $\hookrightarrow$  each division  $\leq 2$

ii)

$$y'(t_n) = v_n = \frac{y_{n+1} - y_{n-1}}{2h}$$

$$- y(t_{n+1}) = y(t_n) + h \cdot y'(t_n) + \frac{h^2}{2} \tilde{y}''(t_n) + \frac{h^3}{6} \tilde{y}'''(f) \quad f \in [t_n, t_{n+1}]$$

$$- y(t_{n-1}) = y(t_n) - h \cdot y'(t_n) + \frac{h^2}{2} \tilde{y}''(t_n) - \frac{h^3}{6} \tilde{y}'''(f) \quad f \in [t_{n-1}, t_n]$$

$$y'(t_n) = v_n = \frac{y(t_n) + h \cdot y'(t_n) + \frac{h^2}{2} \tilde{y}''(t_n) + \frac{h^3}{6} \tilde{y}'''(f_+) - (y(t_n) - h \cdot y'(t_n) + \frac{h^2}{2} \tilde{y}''(t_n) - \frac{h^3}{6} \tilde{y}'''(f_-))}{2h}$$

$$= y'(t_n) + \frac{h^2}{12} ( \tilde{y}'''(f_+) + \tilde{y}'''(f_-) )$$

$$|y'(t_n) - v_n| = \frac{h^2}{6} |\tilde{y}'''(f)| \leq \frac{h^2}{6} c \rightarrow \text{using maple you can see this.}$$

Figure 90: Exercise 2

```

> restart;
> with(plots):
>
> y := t->cos(t)
y := t-> cos(t)                                (1)
> exact_speed := t->-sin(t)
exact_speed := t-> -sin(t)                      (2)
> t_val := evalf( $\frac{\pi}{2}$ )
t_val := 1.570796327                            (3)
>
> forward_difference := (t, h)-> $\frac{y(t+h) - y(t)}{h}$ 
forward_difference := (t, h) ->  $\frac{y(t+h) - y(t)}{h}$           (4)
> central_difference := (t, h)-> $\frac{y(t+h) - y(t-h)}{2 \cdot h}$ 
central_difference := (t, h) ->  $\frac{y(t+h) - y(t-h)}{2 \cdot h}$       (5)
>
> forward_error := h->abs(exact_speed(t_val) - forward_difference(t_val, h))
forward_error := h -> |exact_speed(t_val) - forward_difference(t_val, h)| (6)
> central_error := h->abs(exact_speed(t_val) - central_difference(t_val, h))
central_error := h -> |exact_speed(t_val) - central_difference(t_val, h)| (7)
> loglogplot([central_error(h), forward_error(h)], h = 10-8 .. 1, color = [red, blue])

```

Figure 91: Exercise 2 part 2 Maple

$$3) \vec{v}_1 = (2, 3, 0) \text{ und } \vec{v}_2 = (1, -2, 3) \quad \text{inner product} = \text{dot product}$$

$$\text{i)} \vec{u}_1 = \vec{v}_1$$

$$2) \vec{u}_2 = \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \cdot \vec{u}_1$$

$$3) \vec{u}_3 = \vec{v}_3 - \frac{\langle \vec{v}_3, \vec{u}_1 \rangle}{\|\vec{u}_1\|^2} \cdot \vec{u}_1 - \frac{\langle \vec{v}_3, \vec{u}_2 \rangle}{\|\vec{u}_2\|^2} \cdot \vec{u}_2$$

$\left\langle \vec{v}_1, \vec{v}_2 \right\rangle = \vec{v}_1 \cdot \vec{v}_2 = 0 \rightarrow \text{daarvan verder doen}$

$$\boxed{\vec{u}_k = \vec{v}_k - \sum_{l=1}^{k-1} \frac{\langle \vec{v}_k, \vec{u}_l \rangle}{\|\vec{u}_l\|^2} \cdot \vec{u}_l}$$

$$\vec{v}_1 = \frac{(2, 3, 0)}{\sqrt{13}} = \left( \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, 0 \right) = \left( \frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}, 0 \right)$$

$$\vec{v}_2 = \frac{\vec{v}_2 - (\vec{v}_1 \cdot \vec{v}_2) \cdot \vec{v}_1}{\| \vec{v}_2 - (\vec{v}_1 \cdot \vec{v}_2) \cdot \vec{v}_1 \|}, \text{ in maple:}$$

$$\left( \frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}, 0 \right) \cdot (1, -2, 3)$$

$$= \left( \frac{2\sqrt{13}}{13}, -\frac{6\sqrt{13}}{13}, 0 \right) \cdot \left( \frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}, 0 \right)$$

$$= \left( \frac{52}{169}, -\frac{117}{169}, 0 \right)$$

ii) zit maple

iii) Rangieren  $\vec{u}_1$  und  $\vec{u}_2$  durch für Vektoren mit  
gepaartem Abstand von  $\vec{v}_1$  bzw.  $\vec{v}_2$  benennen we:

$$\vec{y} = (\vec{y} \cdot \vec{u}_1) \cdot \vec{u}_1 + (\vec{y} \cdot \vec{u}_2) \cdot \vec{u}_2$$

$$\text{dann } \vec{y}^\perp = \vec{y} - \vec{y}^{\parallel 75} \Rightarrow \vec{y} = \vec{y}^{\parallel} + \vec{y}^{\perp}$$

Figure 92: Exercise 3

```

restart;
with(plots): with(plottools);
{annulus, arc, arrow, circle, cone, cuboid, curve, cutout, cylinder, disk, dodecahedron, ellipse, ellipticArc, exportplot, extrude, getdata, hemisphere, hexahedron, homothety, hyperbola, icosahedron, importplot, line, octahedron, parallelepiped, pieSlice, point, polygon, polyhedronByName,
prism, project, reflect, rotate, scale, sector, semitorus, sphere, stellate, tetrahedron, torus, transform, translate, triangulate}

v1 := (2|3|0)
v2 := (1, -2, 3)

v1 := [ 2 3 0 ]
v2 := [ 1
         -2
         3 ]

u1 := (2*sqrt(13) | 3*sqrt(13) | 0)
u2 := (52 | -18*sqrt(13) | 0)

u1 := [ 2*sqrt(13) 3*sqrt(13) 0 ]
u2 := [ 4
         -18
         0 ]

origin := (0|0|0);

line_v1 := line(origin, v1, color=blue);
line_v2 := line(origin, v2, color=red);
line_u1 := line(origin, u1, color=green);
line_u2 := line(origin, u2, color=yellow);

# Display the lines together
display([line_v1, line_v2, line_u1, line_u2], axes=normal, scaling=constrained);

```

origin := [ 0 0 0 ]  
 line\_v1 := CURVES([ 0. 0. 0., 2. 3. 0. ], COLOUR(RGB, 0, 0, 1.0000000 ))  
 line\_v2 := CURVES([ 0. 0. 0., 1. -2. 3. ], COLOUR(RGB, 1.0000000, 0, 0. ))  
 line\_u1 := CURVES([ 0. 0. 0., 0.554700196225229 0.832050294337844 0. ], COLOUR(RGB, 0, 1.0000000, 0. ))  
 line\_u2 := CURVES([ 0. 0. 0., 0.307692307692308 -1.38461538461538 0. ], COLOUR(RGB, 1.0000000, 1.0000000, 0. ))

Figure 93: Exercise 3 - plot

Huis 2

Bord 2

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(t) g(t) dt$$

met definitie  $w_m = \text{proj}_m \langle \cos(kt), \sin(kt), k=0, 1, \dots, n \rangle$

1) eerst orthogonale we met gram-schmidt:

Aanv:  $\cos(0 \cdot t) = 1$

→ we applying gram-schmidt to sequences of  
 $\cos(kt)$  and  $\sin(kt)$

initial set of functions:

$$f_0 = 1, f_1 = \cos(t), f_2 = \sin(t), f_3 = \cos(2t), f_4 = \sin(2t), \dots$$

Als we nu  $f_0 = 1$ . We orthogonale we met  $\cos(kt)$

$$w_1 = \int_{-\pi}^{\pi} f_1 - (\mu_0 \cdot f_0) w_0 dt = \mu_1 = \beta_1 = \cos(t)$$

$\rightarrow$   $\cos$  and  $\sin$  are orthogonal to each other

$$w_2 = \int_{-\pi}^{\pi} f_2 - (\mu_0 \cdot f_1) w_0 - (\mu_1 \cdot f_2) w_1 dt = \sin(t)$$

; gekenmerkt weergeven in basisvektor vormen.

76

Also we de norm tellen van  $\sqrt{\frac{1}{\pi} \int_{-\pi}^{\pi} w_k^2 dt}$

Basis vormen  $\frac{1}{\sqrt{\pi}}, \frac{\cos(t)}{\sqrt{\pi}}, \frac{\sin(t)}{\sqrt{\pi}}, \frac{\cos(2t)}{\sqrt{\pi}}, \dots$

```

> restart;
> with(LinearAlgebra): with(plottools): with(plots):
> #i
v_1 := Vector([2, 3, 0])
v_1 := 
$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$
 (1)

> v_2 := Vector([1, -2, 3])
v_2 := 
$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$
 (2)

> u_1 :=  $\frac{v_1}{\text{Norm}(v_1, 2)}$ 
u_1 := 
$$\begin{bmatrix} \frac{2\sqrt{13}}{13} \\ \frac{3\sqrt{13}}{13} \\ 0 \end{bmatrix}$$
 (3)

> u_2 :=  $\frac{(v_2 - (u_1 \cdot v_2) \cdot u_1)}{\text{Norm}(v_2 - (u_1 \cdot v_2) \cdot u_1, 2)}$ 
u_2 := 
$$\begin{bmatrix} \frac{21\sqrt{2158}}{2158} \\ -\frac{7\sqrt{2158}}{1079} \\ \frac{3\sqrt{2158}}{166} \end{bmatrix}$$
 (4)

> # ii
> null_vector := Vector([0, 0, 0])
null_vector := 
$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
 (5)

> line_v1 := line(null_vector, v_1, color = purple)
line_v1 := CURVES(
$$\begin{bmatrix} 0. & 0. & 0. \\ 2. & 3. & 0. \end{bmatrix}, \text{COLOUR(RGB, 0.50196078, 0., 1.)}$$
) (6)

```

Figure 94: Exercise 3

$$\left\{ \begin{array}{l} \underline{x + y + z = b_1} \\ \underline{2x - y + z = b_2} \\ 3x + 2z = b_3 \end{array} \right.$$

Naar,  $b_3$  is lineair comb van  $b_1$  en  $b_2$

Figure 95: Huis 2 Exercise 1

- 2) - moet voldoende zijn en rijen/colommen moeten  
 lineair onafhankelijk zijn. Wanneer determinaat nog  
 ook geen  $0^{2 \times 2}$ .

Figure 96: Huis 2 Exercise 2

3)

$$i) \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} k & k \\ k & k \end{pmatrix} = AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0} \bullet$$

ii)  $\det(AB) = 0 \neq \det(A) \cdot \det(B) = 0$   
 omdat A niet rechthoekig is, of B.

Figure 97: Huis 2 Exercise 3

$$4) A\vec{\alpha} = \vec{y}$$

$$\Leftrightarrow \begin{pmatrix} \cos(0) & \cos(0) \\ \cos(1) & \cos(1) \\ \cos(2) & \cos(2) \\ \cos(3) & \cos(3) \\ \cos(4) & \cos(4) \\ \cos(5) & \cos(5) \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0,04 \\ 1,05 \\ -0,2 \\ -2 \\ -0,2 \\ 1,41 \end{pmatrix}$$

$$\vec{\alpha} = (\Delta^T \cdot A)^{-1} \cdot \vec{b}^T = \begin{pmatrix} 1,04 \\ -0,34 \end{pmatrix}$$

draw using maple:

1) pointplot

2) plot(f1)

should be quite close

Figure 98: Huis 2 Exercise 4

```

# First define the data
> x := [0.04, 1.05, -0.2, -2, -0.2, 1.11];
x := [0.04, 1.05, -0.2, -2, -0.2, 1.11]

> t := [0, 1, 2, 3, 4, 5]
t := [0, 1, 2, 3, 4, 5]

> # Construct the matrix A
> A := Matrix([[cos(0), cos(0)], [cos(1), cos(2)], [cos(2), cos(4)], [cos(3), cos(6)], [cos(4), cos(8)], [cos(5), cos(10)]])
A := 
$$\begin{bmatrix} 1 & 1 \\ \cos(1) & \cos(2) \\ \cos(2) & \cos(4) \\ \cos(3) & \cos(6) \\ \cos(4) & \cos(8) \\ \cos(5) & \cos(10) \end{bmatrix}$$


> # Perform kleinste kwadraten fit: x = (A'A)^{-1}A^T
> c := LeastSquares(A, Vector(x))
c := 
$$\begin{bmatrix} 1.040593609 \\ -0.9362867375 \end{bmatrix}$$


> x_function := t->c[1]*cos(t) + c[2]*cos(2*t);
x_function := t → c1·cos(t) + c2·cos(2·t)

> plot_points := pointplot([seq([t[i], x[i]], i=1..6)], color=red):
plot_fit := plot(x_function(t_), t_=0..5, color=blue):
display(plot_points, plot_fit)

```

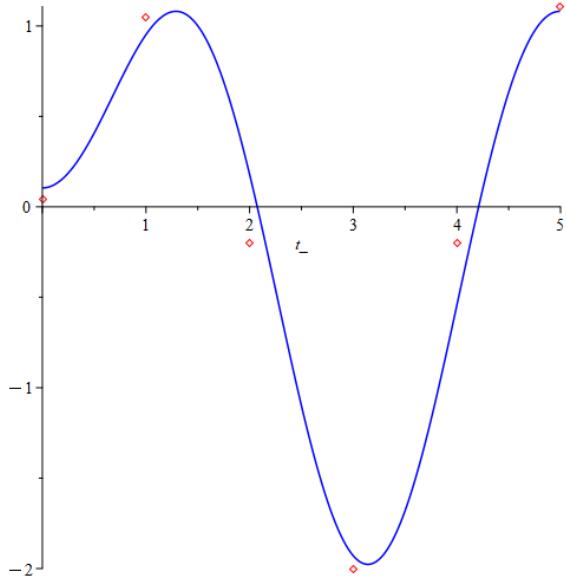


Figure 99: Huis 2 Exercise 4 Maple

```

> with(LinearAlgebra):
> p := Matrix([ [ 5/6, 1/6, 1/3 ], [ 1/6, 5/6, -1/3 ], [ 1/3, -1/3, 1/3 ] ])
      
$$p := \begin{bmatrix} \frac{5}{6} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & \frac{5}{6} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad (1)$$

> J, Q := JordanForm(p, output = ['J', 'Q'])
      
$$J, Q := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{6} & \frac{17}{6} & 2 \\ -\frac{1}{6} & \frac{1}{6} & 0 \\ -\frac{1}{3} & \frac{4}{3} & 1 \end{bmatrix} \quad (2)$$


```

Figure 100: Huis 2 Exercise 5: We zien dat de projector op het YZ vlak projecteert.

6) Hier moeten we matrix A vinden waervan twee vectoren

$v_1 = (1, 0, 1)$ ,  $v_2 = (1, 2, 0)$  in de rijruimte liggen,  
waarbij de nullruimte orthogonaal is (relatief) van de  
rijruimte.

$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$ , we moeten aantonen dat  
 $\{k_1, k_2\}$  lineaire combinatie is  
van  $K_A$ . Als dit het geval is is  
dit deel van rijruimte.

Wat is  $b$ ?

$$\begin{pmatrix} 0 \\ b \\ 2 \end{pmatrix} = c_1 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{cases} 0 = c_1 + c_2 \\ b = 2c_2 \rightarrow b = -4 \\ 2 = c_1 \end{cases}$$

$k_3$  in lineaire combinatie dan behoort niet tot  
nullruimte

Figure 101: Huis 2 Exercise 6: Uit de cursus weten we dat er geen basis kan gevonden worden voor  $K(A^T)$  en  $N(A)$

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{pmatrix}$$

1) einsetzen:

$$\det(A - \lambda I) = 0$$

$$\begin{pmatrix} 2-\lambda & 1 & 2 \\ 4 & 2-\lambda & 4 \\ 2 & 1 & 2-\lambda \end{pmatrix}$$

$$(2-\lambda)((2-\lambda)(2-\lambda) - 4) - 1 \cdot (4(2-\lambda) - 8) + 2(4 - 2(2-\lambda)) = (2-\lambda)(\cancel{4} - \cancel{4}\lambda^2 - 2\lambda + \cancel{\lambda^2} - \cancel{4}) - (\cancel{8} - \cancel{4}\lambda \cancel{+ 8}) + 2(\cancel{4} - \cancel{4}\lambda + \cancel{4}) = (2-\lambda)(-4\lambda + \lambda^2) + 4\lambda - 4\lambda = 0 = 1 \Rightarrow -8\lambda + 2\lambda^2 + 4\lambda - 4\lambda = \lambda^2 - 4\lambda = 0 \Rightarrow -\lambda^2 + 4\lambda = 0$$

$$\Rightarrow \lambda^2 - 4\lambda = 0$$

$$\Rightarrow \lambda^2(\lambda - 4) = 0$$

$$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 4$$

$\Rightarrow$  abunken finnen die Eigenvektoren.

Figure 102: Huis 2 Exercise 7

```

[> with(LinearAlgebra):
> A := Matrix([[2, 1, 2], [4, 2, 4], [2, 1, 2]])
      A := 
$$\begin{bmatrix} 2 & 1 & 2 \\ 4 & 2 & 4 \\ 2 & 1 & 2 \end{bmatrix}$$
 (1)

[> J, Q := JordanForm(A, output = ['J', 'Q'])
      J, Q := 
$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & -1 \\ -\frac{2}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & \frac{1}{3} & 1 \end{bmatrix}$$
 (2)

```

Figure 103: Huis 2 Exercise 7 Maple

$$8) \begin{array}{ll} v_1 = 0 & v \\ v_2 = 3 & v \\ v_3 = 5 & w \end{array}$$

i) Basis voor ruimte is span  $\{w\}$  omdat  $Aw = 0$ .  
Kolomruimte is dan  $\{v, w\}$

ii) 1 oplossing:  $Ax = b$   
 $\rightarrow x = \frac{1}{3}v + \frac{1}{5}w$ , alle oplossingen zijn veelvoud  
van mutant:  $x = \frac{1}{3}v + \frac{1}{5}w + k \cdot u$  met  $k \in \mathbb{R}$

iii)  $x = \alpha u + \beta v + \gamma w$   
 $= 3v + 5w$ ,  $w$  is niet in de vergelijkingen dus  
kan niet berdekt worden.

Figure 104: Huis 2 Exercise 8

$$\begin{aligned} 2) f_0'' &= \langle f(t), c_0 \rangle + \langle c_0, f(t) \rangle + \langle c_1, c_0 \rangle + \langle c_0, c_1 \rangle + \dots \\ \langle f(t), c_0 \rangle &= \int_{-\pi}^{\pi} \frac{t^2}{\sqrt{\pi}} \cdot dt = \frac{\sqrt{2}}{3} \pi^{5/2} \\ \langle f(t), c_k \rangle &= \int_{-\pi}^{\pi} t^2 \cos(kt) \cdot dt = \frac{4\sqrt{\pi}}{k^2} \frac{(-1)^k}{k} \\ \langle f(t), s_k \rangle &= \int_{-\pi}^{\pi} t^2 \sin(kt) \cdot dt = 0 \\ \Rightarrow f_0 &= \frac{\sqrt{2}}{3} \pi^{5/2} \cdot \frac{1}{\sqrt{2}} + \left\{ \frac{4\sqrt{\pi}}{k^2} \frac{(-1)^k}{k} \cdot \sin(kt) \right\} + 0 \end{aligned}$$

Figure 116: Bord 2 Exercise 5 Part 2

Door de hash te 'ondertekenen' met een privé-sleutel, kan de identiteit van het geverifieerd en kan de afzender niet ontkennen dat hij de data heeft.

entitale: identificeert de bron van de data.

aliteit: De data zelf wordt niet verborgen.  
Identiteit: Het heeft geen invloed op het verbergen van communicatie.

ation Code)

Verzekert dat de data niet is gewijzigd.  
Authenticatie: Verzekert dat het bericht van een specifieke entiteit komt (degene die).

Zowel zender als ontvanger hebben toegang tot de sleutel, dus het kan niet  
bericht heeft verzonden.  
Confidentialiteit of Traffic-flow confidentiality: Verborgt de inhoud of het bestaan  
nie.

Integriteit: Versleutelt de data om deze privé te houden.

Data-origin authenticatie, Entity authentication: Omdat de sleutel gedeeld  
wordt tussen zender en ontvanger.

Confidentialiteit: Verborgt niet het feit dat communicatie plaatsvindt.  
Integriteit en Data-origin authentication: Voor al wanneer gebruikt voor digitale  
handtekening. De eigenaar van de private sleutel kan geïdentificeerd worden als de enige  
die handtekening kan hebben gecreëerd.

Identiteit: Wanneer iets met de publieke sleutel wordt versleuteld, kan alleen de  
private sleutel het ontcijferen.  
Integriteit en Data-origin authentication: Voor al wanneer gebruikt voor digitale  
handtekening. De eigenaar van de private sleutel kan geïdentificeerd worden als de enige  
die handtekening kan hebben gecreëerd.

Confidentialiteit: Verborgt niet het feit dat communicatie plaatsvindt.

Figure 105: Huis 2 Exercise 8 second version

g)  $A = UV^T$

i) Rang 1 matrix omdat  $A = UV^T$ , dus alle kolommen van  $A$  zijn lineaire combinaties van elkaar.

ii)  $Au = \underbrace{d_u}_{\text{pijuwanted}}$

$$Au = (UV^T)u = u(V^Tu) = (V^Tu)u$$

$Au$  is een rechte vermenigvuldiging van  $u$  dus en dus is  $u$  een eigenvector

iii)  $l_1 = V^Tu$

$d_2 = 0 \rightarrow$  eigenschap van rang 1 - matrizen.

iv)

$\text{trace}(A) = \text{sum van diagonale elementen}$

$$\text{trace}(A) = a_{11} + a_{22} = \underbrace{V^T}_{l_1} \underbrace{u}_{l_1}$$

Figure 106: Huis 2 Exercise 9

```

> with(LinearAlgebra):
> u := Vector([u1, u2])

$$u := \begin{bmatrix} u1 \\ u2 \end{bmatrix} \quad (1)$$

> v := Transpose(Vector([v1, v2]))

$$v := \begin{bmatrix} v1 & v2 \end{bmatrix} \quad (2)$$

> A := u . v

$$A := \begin{bmatrix} u1 v1 & u1 v2 \\ u2 v1 & u2 v2 \end{bmatrix} \quad (3)$$

> J, Q := JordanForm(A, output = ['J', 'Q'])

$$J, Q := \begin{bmatrix} 0 & 0 \\ 0 & u1 v1 + u2 v2 \end{bmatrix}, \begin{bmatrix} \frac{u2 v2}{u1 v1 + u2 v2} & \frac{u1 v1}{u1 v1 + u2 v2} \\ -\frac{u2 v1}{u1 v1 + u2 v2} & \frac{u2 v1}{u1 v1 + u2 v2} \end{bmatrix} \quad (4)$$

> # Hier kunnen we zien dat de rang = 1
> Eigenvalues(A)

$$\begin{bmatrix} 0 \\ u1 v1 + u2 v2 \end{bmatrix} \quad (5)$$

> # We zien dus dat  $\lambda = u \cdot v^T$ 
> #  $\lambda_1 = 0, \lambda_2 = u1.v1 + u2.v2$ 
> Eigenvectors(A)

$$\begin{bmatrix} u1 v1 + u2 v2 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{u1}{u2} & -\frac{v2}{v1} \\ 1 & 1 \end{bmatrix} \quad (6)$$

> # We zien hier dat u een eigenvector is!
>
> # Als laatste, de trace is simpel, onze  $\lambda_2 = u1.v1 + u2.v2$ . Als je kijkt
    naar A, en je telt de diagonaal op, dan zie je hetzelfde uitkomen ;)

```

Figure 107: Huis 2 Exercise 9 Maple

$$(t_1, y_1) = (10, 10, 1)$$

$$(t_2, y_2) = (21, 7.4 \text{ m})$$

$$(t_3, y_3) = (31, -5, 1)$$

1)  $y(t) = \sqrt{t} - \frac{3}{2} t^2 \quad | A \propto = 3$

$$Q = \begin{pmatrix} 2 & 1 \\ 10 & 1 \\ -\frac{3}{2} & 1 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 3 \end{pmatrix}$$

$$\tilde{y} = \begin{pmatrix} 3 & 1 \\ 10.1 & 1 \\ 7.1 & 1 \\ -5.2 & 1 \end{pmatrix}$$

→ check in maple

2) proof:  $A^T (\tilde{y} - \tilde{y}) = 0$

$$\Leftrightarrow A^T \tilde{y} - A^T \tilde{y} = 0$$

$$\Leftrightarrow A^T \tilde{y} - A^T A (A^T A)^{-1} A^T \tilde{y} = 0$$

$$\Leftrightarrow A^T \tilde{y} - \underbrace{A^T A}_{A^T A^{-1}} (A^T A^{-1}) A^T \tilde{y} = 0$$

$$\Leftrightarrow A^T \tilde{y} - \underbrace{A^T}_{A^T A^{-1}} A^{-1} A^T \tilde{y} = 0$$

$$\Leftrightarrow 0 = 0 \quad \square$$

Figure 108: Bord 2 Exercise 1

```

> restart: with(LinearAlgebra): with(plots): with(plottools):
> # i)
> A := Matrix([[1, 1], [2, 4], [3, 9]])
A := 
$$\begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \end{bmatrix}$$
 (1)

> y := Vector([10.1, 7.4, -5.2])
y := 
$$\begin{bmatrix} 10.1 \\ 7.4 \\ -5.2 \end{bmatrix}$$
 (2)

> result := LeastSquares(A, y)
result := 
$$\begin{bmatrix} 15.3552631578947 \\ -5.71315789473684 \end{bmatrix}$$
 (3)

> v_0 := result[1]
v_0 := 15.3552631578947 (4)

> g := result[2] · (-1) · (2)
g := 11.4263157894737 (5)

> #ii) zie notes
> #iii)
> y_tilde := A . result;
y_tilde := 
$$\begin{bmatrix} 9.64210526315789 \\ 7.85789473684210 \\ -5.35263157894737 \end{bmatrix}$$
 (6)

> column_space_vector := Column(A, 2);
column_space_vector := 
$$\begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$$
 (7)

> # Define line representing the column space K(A)
K_A := plot([t → column_space_vector[1]*t, t → column_space_vector[2]*t,
            t = -1 .. 3], color = blue, thickness = 2, title = "Column Space K(A)");
Error, (in plot) expected a range but received t = -1 .. 3

> # Plot the original data points y
data_points := plot([seq([i, y[i]], i = 1 .. 3)], style = point, symbol = circle, color
                    = black, title = "Data Points and Column Space");

```

Figure 109: Bord 2 Exercise 1 Maple

# Wiskundige modellering in de ingenieurswetenschappen: Bordoefeningenles 2

## Oefening 1

```
[> restart: with(plots):with(LinearAlgebra):with(plottools):  
oorsprong := <0,0,0>;
```

Constructie matrix A en vector y:

```
[> K1 := <1,2,3>;  
K2 := <1,4,9>;  
A := <K1|K2>;  
y := <10.1,7.4,-5.2>;
```

$$\begin{aligned}
 K1 &:= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
 K2 &:= \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} \\
 A &:= \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \end{bmatrix} \\
 y &:= \begin{bmatrix} 10.1 \\ 7.4 \\ -5.2 \end{bmatrix} \tag{1.1}
 \end{aligned}$$

Heeft dit stelsel een oplossing?

```
[> solve(A.<v0, -g/2>)=y, {v0,g});  
> Determinant(<K1|K2|y>);
```

5.8

(1.2)

We bepalen de kleinste kwadraten benadering en fit:

De onbekenden x bepalen kan op 2 manieren:

- met een stelsel (meest efficiënt)

```
[> solve((A^%T.A).(<v0,-g/2>)=A^%T.y,{v0,g});  
{g = 11.42631579, v0 = 15.35526316}
```

(1.3)

- met behulp van de matrix inverse

```
[> x := MatrixInverse(A^%T.A).A^%T.y;
```

Figure 110: Bord 2 Exercise 1 Maple Version 2: Warning, pdf is not fully loaded, go look in my files

$$\begin{array}{l}
 l_1 = i, l_2 = \bar{l}_1 = -i, l_3 = 2 \\
 m_1 = 1 = 1 \quad m_2 = 1 \quad m_3 = 2 \\
 \tilde{m}_4 = 1 \quad \tilde{m}_2 = 1 \quad \tilde{m}_3 = 1
 \end{array}
 \rightarrow \text{dit is omdat we w-gorden}\newline
 \text{vorm halverwege}\newline
 \text{niet diagonaliseerbaar} \rightarrow \tilde{m} < m^2$$

Lijsevectoren:

$$\begin{pmatrix} -1 & 1 & 2 \\ -1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Figure 111: Bord 2 Exercise 2

(3) Uit de orthonormaliteit van de basis volgt dat

$$\begin{aligned}
 \|f(t) - f_0\|^2 &= \langle t^2 - f_0, t^2 - f_0 \rangle \\
 &= \langle t^2, t^2 \rangle - 2\langle f_0, t^2 \rangle + \langle f_0, f_0 \rangle \\
 &= \frac{2\pi^5}{5} - 2 \left( \frac{\pi^2}{3} \langle 1, t^2 \rangle + 4 \sum_{k=1}^n \frac{(-1)^k}{k^2} \langle \cos(kt), t^2 \rangle \right) \\
 &\quad + \frac{2\pi^5}{9} + 16 \sum_{k=1}^n \frac{1}{k^4} \langle \cos(kt), \cos(kt) \rangle \\
 &= \frac{8\pi^5}{45} - 32\pi \sum_{k=1}^n \frac{1}{k^4} + 16\pi \sum_{k=1}^n \frac{1}{k^4} \\
 &= 16\pi \left( \frac{\pi^4}{90} - \sum_{k=1}^n \frac{1}{k^4} \right)
 \end{aligned}$$

Als  $n \rightarrow +\infty$ , dan  $\|f(t) - f_0\| \rightarrow 0$ , want

$$\sum_{k=1}^{+\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

Figure 117: Bord 2 Exercise 5 Part 3

```

[> with(LinearAlgebra):
> A := Matrix([[0, -1, 3, 0], [1, 0, 0, 1], [0, 0, 3, -1], [0, 0, 1, 1]])
      A := 
$$\begin{bmatrix} 0 & -1 & 3 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$
 (1)

=> J, Q := JordanForm(A, output = ['J', 'Q'])
      J, Q := 
$$\begin{bmatrix} -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 & 1 \\ -\frac{I}{2} & \frac{I}{2} & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
 (2)
]

```

Figure 112: Bord 2 Exercise 2 Maple

$$3) \quad A = V_1 V_1^T + V_3 V_3^T$$

1)  $K(A) = \text{span} \{ V_1, V_3 \}$

2)  $K(A^T) = \text{span} \{ V_2, V_1 \}$

3)  $N(A) : A\alpha = (V_1 V_1^T + V_3 V_3^T) \alpha = 0$

$$= V_1 (V_1^T \alpha) + V_3 (V_3^T \alpha) = 0$$

These need to be null, no look in direction on  $V_3$ , thus

$$N(A) = \text{span} \{ V_3 \}$$

4)  $N(A^T) : \text{we want to solve:}$

$$A^T y = (V_2 V_1^T + V_1 V_3^T) y = 0$$

$$= V_2 (V_1^T y) + V_1 (V_3^T y) = 0$$

$$\Rightarrow N(A^T) = \{ V_2 \}$$

Figure 113: Bord 2 Exercise 3

4)

$$A = \frac{1}{\sqrt{2}} \cdot (e_1 e_2^T + e_2 e_1^T)$$

a. Is A symmetrisch?

$$A = \frac{1}{\sqrt{2}} \cdot (e_1 e_2^T + e_2 e_1^T) ; A^T = \frac{1}{\sqrt{2}} \cdot (e_2 e_1^T + e_1 e_2^T) \quad \checkmark$$

b.  $A^L = A^R$ ?

$$\begin{aligned} A^L &= \frac{1}{2} \cdot (e_1 e_2^T + e_2 e_1^T)^2 \\ &= \frac{1}{2} (e_1 e_2^T + e_2 e_1^T) (e_1 e_2^T + e_2 e_1^T) \\ &= \frac{1}{2} (e_1 e_2^T e_1 e_2^T + e_1 e_2^T \cdot e_2 e_1^T + e_2 e_1^T e_1 e_2^T \\ &\quad + e_2 e_1^T \cdot e_2 e_1^T) \\ &= \frac{1}{2} (e_1 e_2^T + e_2 e_1^T) \neq A \rightarrow \text{green projection} \end{aligned}$$

Figure 114: Bord 2 Exercise 4

WC 2

$$A) \quad l_m(u(t)) = l_m(a) - ct^2 - bt$$

$$A = \begin{pmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \\ 1 & t_4 & t_4^2 \\ 1 & t_5 & t_5^2 \end{pmatrix} \quad \vec{\alpha} = \begin{pmatrix} l_m(a) \\ -c \\ -b \end{pmatrix} \quad \vec{y} = \begin{pmatrix} l_m(y_1) \\ l_m(y_2) \\ l_m(y_3) \\ l_m(y_4) \\ l_m(y_5) \end{pmatrix}$$

Stoch im Graph:  $\vec{\alpha}' = (A^T A)^{-1} A^T \vec{y}$

$\therefore a = 5, 20, b = 0, 35, c = 0, 32$

plot same as homework.

Figure 118: WC 2 Exercise 1

```

> restart: with(LinearAlgebra):
> inproduct := (f, g) → integrate(f*g, t = -infinity ..infinity);
NORM   := f → sqrt(inproduct(f, f)):

inproduct := (f, g) → integrate(g·f, t = -∞ ..∞)          (1)
> f0 := exp(-t^2/2)/NORM(exp(-t^2/2)):

> w1 := t*exp(-t^2/2)-f0*inproduct(f0, t*exp(-t^2/2));
f1 := w1/NORM(w1):
w1 :=  $t e^{-\frac{t^2}{2}}$                                          (2)

> w2 := t^2*exp(-t^2/2)-f1*inproduct(t^2*exp(-t^2/2), f1)-f0
      *inproduct(t^2*exp(-t^2/2), f0);
f2 := w2/NORM(w2):
w2 :=  $t^2 e^{-\frac{t^2}{2}} - \frac{e^{-\frac{t^2}{2}}}{2}$            (3)

> w3 := t^3*exp(-t^2/2)-f2*inproduct(t^3*exp(-t^2/2), f2)-f1
      *inproduct(t^3*exp(-t^2/2), f1)-f0*inproduct(t^3*exp(-t^2/2),
      f0);
f3 := w3/NORM(w3):
w3 :=  $t^3 e^{-\frac{t^2}{2}} - \frac{3 t e^{-\frac{t^2}{2}}}{2}$            (4)

> w4 := t^4*exp(-t^2/2)-f3*inproduct(t^4*exp(-t^2/2), f3)-f2
      *inproduct(t^4*exp(-t^2/2), f2)-f1*inproduct(t^4*exp(-t^2/2),
      f1)-f0*inproduct(t^4*exp(-t^2/2), f0);
f4 := w4/NORM(w4):
w4 :=  $t^4 e^{-\frac{t^2}{2}} - 3 t^2 e^{-\frac{t^2}{2}} + \frac{3 e^{-\frac{t^2}{2}}}{4}$      (5)

> # i: de orthonormale basis is dus de span van <f_0, w_1, w_2, w_3, w_4, ...>
> # ii: zie notes

```

Figure 119: WC 2 Exercise 2

### Lange oplossing

```
> g0 := fn(0)*inproduct(fn(0),g);
g1 := fn(0)*inproduct(fn(0),g)+fn(1)*inproduct(fn(1),g);
g2 := fn(0)*inproduct(fn(0),g)+fn(1)*inproduct(fn(1),g)+fn(2)*inproduct(fn(2),g);
g3 := fn(0)*inproduct(fn(0),g)+fn(1)*inproduct(fn(1),g)+fn(2)*inproduct(fn(2),g)+fn(3)*inproduct(fn(3),g);
g4 := fn(0)*inproduct(fn(0),g)+fn(1)*inproduct(fn(1),g)+fn(2)*inproduct(fn(2),g)+fn(3)*inproduct(fn(3),g)+fn(4)*inproduct(fn(4),g);
```

### Korte oplossing (niet te kennen)

```
> g_n := n -> add(fn(i)*inproduct(fn(i),g),i=0..n);
Warning, (in g_n) `i` is implicitly declared local
```

Visualisatie

```
> app0 := plot(g_n(1), t=-5..5, view=[-5..5,-0.3..1.05], color=blue, legend="projectie op W[0]");
app1 := plot(g_n(1), t=-5..5, view=[-5..5,-0.3..1.05], color=black, linestyle="dot", legend="projectie op W[1]", thickness=3);
app2 := plot(g_n(2), t=-5..5, view=[-5..5,-0.3..1.05], color=red, legend="projectie op W[2]");
app3 := plot(g_n(3), t=-5..5, view=[-5..5,-0.3..1.05], color=purple, linestyle="dot", legend="projectie op W[3]", thickness=3);
app4 := plot(g_n(4), t=-5..5, view=[-5..5,-0.3..1.05], color=coral, legend="projectie op W[4]");
> gplot := plot(g, t=-5..5, thickness=1, color=black, legend="g(t)", thickness=2);
> display(gplot,app0, app1, app2, app3, app4);
```

Figure 120: WC 2 Exercise 2 Maple

```

> restart: with(LinearAlgebra):
> e_1 := Vector([1, 0, 0])

$$e_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (1)$$

> e_2 := Vector([0, 1, 0])

$$e_2 := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad (2)$$

> A :=  $\frac{1}{2} \cdot (e_1 \cdot \text{Transpose}(e_1) + e_2 \cdot \text{Transpose}(e_2) + e_1 \cdot \text{Transpose}(e_2) + e_2 \cdot \text{Transpose}(e_1))$ 

$$A := \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (3)$$

> # Is A = Transpose(A)?
> Transpose(A)

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (4)$$

> # YES!
> # Is A = A^2?
> A^2

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5)$$

> # YEHAAAHHH
> # The subspace is thus e_1 + e_2

```

Figure 121: WC 2 Exercise 3 Maple

## Huis 3

```

> restart: with(LinearAlgebra):
> A := Matrix([[3, 4], [1, 0]])

$$A := \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix} \quad (1)$$

> J, Q := JordanForm(A, output = [J'|Q'])

$$J, Q := \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{bmatrix} \quad (2)$$

> recursie_relatie := Q . MatrixPower(J, n) . Q-1 . Vector([1, 1])

$$\text{recursie\_relatie} := \begin{bmatrix} \frac{-3(-1)^n}{5} + \frac{84^n}{5} \\ \frac{3(-1)^n}{5} + \frac{24^n}{5} \end{bmatrix} \quad (3)$$

> limit_eq := recursie_relatie[1]

$$\text{limit\_eq} := \frac{-3(-1)^n}{5} + \frac{84^n}{5} \quad (4)$$

> limit(limit_eq, n = infinity)
4
> 
$$\lim_{n \rightarrow \infty} \left( \frac{-3(-1)^n}{5} + \frac{84^n}{5} \right) = 4 \quad (5)$$


```

Figure 122: Huis 3 Exercise 1

$$y_{n+1} = 3y_n + 4y_{n-1}$$

$$y_0 = y_1 = 1$$

$$\lim_{n \rightarrow \infty} \frac{y_{n+1}}{y_n}$$

$$\begin{pmatrix} y_{n+1} \\ y_n \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 4 \\ 1 & 0 \end{pmatrix}}_{\text{via maple}} \begin{pmatrix} y_n \\ y_{n-1} \end{pmatrix}$$

↓

via maple

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} \quad Q = \frac{1}{5} \begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

De recursieve relatie zegt:  $\begin{pmatrix} y_{n+1} \\ y_n \end{pmatrix} = Q D^{n-1} Q^{-1} \begin{pmatrix} y_1 \\ y_0 \end{pmatrix}$

(via maple)

$$\begin{matrix} \text{maple}, \\ = \dots \end{matrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{\text{---}} \begin{pmatrix} \text{---} \\ \text{---} \end{pmatrix}$$

$$\lim_{n \rightarrow \infty} \left( \begin{pmatrix} \text{---} \\ \text{---} \end{pmatrix} \right) = q$$

Figure 123: Huis 3 Exercise 1

$$\begin{aligned}
 & \det \begin{pmatrix} a & b & e & f \\ c & d & g & h \\ 0 & 0 & i & j \\ 0 & 0 & k & l \end{pmatrix} \\
 &= a \cdot \begin{pmatrix} d & g & h \\ 0 & i & j \\ 0 & k & l \end{pmatrix} - c \cdot \begin{pmatrix} b & e & f \\ 0 & i & j \\ 0 & k & l \end{pmatrix} \\
 &= ad \begin{pmatrix} i & j \\ k & l \end{pmatrix} - cd \begin{pmatrix} i & j \\ k & l \end{pmatrix} \\
 &= (ad - cd) \begin{pmatrix} i & j \\ k & l \end{pmatrix} = \det(a) \det(c) = \det(ac)
 \end{aligned}$$

Figure 124: Huis 3 Exercise 2

```

> with(LinearAlgebra):
> # First express the matrix ( N=2)
> A := Matrix([[a,b,e,f],[c,d,g,h],[0,0,i,j],[0,0,k,l]])
      
$$A := \begin{bmatrix} a & b & e & f \\ c & d & g & h \\ 0 & 0 & i & j \\ 0 & 0 & k & l \end{bmatrix}$$
 (1)

> Determinant(A)           
$$adil - adjk - bcil + bcjk$$
 (2)
> # This is the answer, the only thing now is to identify the result
> det_a := a·d - c·b       
$$det_a := ad - cb$$
 (3)
> det_b := e·h - g·f       
$$det_b := eh - gf$$
 (4)
> det_c := i·l - k·j       
$$det_c := il - kj$$
 (5)
> expand(det_a · det_c)    
$$adil - adjk - bcil + bcjk$$
 (6)
> # Look, same answer, thus it can be done like this.

```

Figure 125: Huis 3 Exercise 2

// Exercise 3

> # De recursie relatie is:  $x_k = x_{k-1} + x_{k-2}$

> # Als we dit schrijven in matrix vorm krijgen we:

> restart: with(LinearAlgebra):

> A := Matrix([[1, 1], [1, 0]])

$$A := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad (1)$$

> J, Q := JordanForm(A, output = ['J', 'Q'])

$$J, Q := \begin{bmatrix} -\frac{\sqrt{5}}{2} + \frac{1}{2} & 0 \\ 0 & \frac{\sqrt{5}}{2} + \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} - \frac{\sqrt{5}}{10} & \frac{(\sqrt{5}+1)\sqrt{5}}{10} \\ -\frac{\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{bmatrix} \quad (2)$$

> JK := Matrix( $\left[ \left[ \left( -\frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k, 0 \right], \left[ 0, \left( \frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k \right] \right]$ )

$$JK := \begin{bmatrix} \left( -\frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k & 0 \\ 0 & \left( \frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k \end{bmatrix} \quad (3)$$

> Q\_inverse := MatrixInverse(Q)

$$Q_{\text{inverse}} := \begin{bmatrix} 1 & -\frac{\sqrt{5}}{2} - \frac{1}{2} \\ 1 & -\frac{(-5+\sqrt{5})\sqrt{5}}{10} \end{bmatrix} \quad (4)$$

> x\_0 := 1

$$x_0 := 1 \quad (5)$$

> x\_1 :=  $\frac{(1-\sqrt{5})}{2}$

$$x_1 := -\frac{\sqrt{5}}{2} + \frac{1}{2} \quad (6)$$

> v\_1 := Vector([x\_1, x\_0])

$$v_1 := \begin{bmatrix} -\frac{\sqrt{5}}{2} + \frac{1}{2} \\ 1 \end{bmatrix} \quad (7)$$

> result := Q • JK • Q\_inverse • v\_1

$$\begin{aligned}
result := & \left[ \left[ \left( \left( \frac{1}{2} - \frac{\sqrt{5}}{10} \right) \left( -\frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k + \frac{(\sqrt{5}+1)\sqrt{5} \left( \frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k}{10} \right) \left( -\frac{\sqrt{5}}{2} + \frac{1}{2} \right) \right. \right. \\
& - \frac{\sqrt{5}}{2} + \frac{1}{2} \Big) + \left( \frac{1}{2} - \frac{\sqrt{5}}{10} \right) \left( -\frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k \left( -\frac{\sqrt{5}}{2} - \frac{1}{2} \right) \\
& \left. \left. - \frac{(\sqrt{5}+1) \left( \frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k (-5+\sqrt{5})}{20} \right] , \right. \\
& \left[ \left( -\frac{\sqrt{5} \left( -\frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k}{5} + \frac{\sqrt{5} \left( \frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k}{5} \right) \left( -\frac{\sqrt{5}}{2} + \frac{1}{2} \right) \right. \\
& \left. \left. - \frac{\sqrt{5} \left( -\frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k \left( -\frac{\sqrt{5}}{2} - \frac{1}{2} \right)}{5} - \frac{\left( \frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k (-5+\sqrt{5})}{10} \right] \right]
\end{aligned} \tag{8}$$

```

> # calculate the limit
> limit(result[1], k = infinity)

$$-\frac{\sqrt{5}}{2} + \frac{1}{2} \tag{9}$$


```

```

> # Et voila ;)
>
> # Ok but for b) we need to add a small epsilon to this shit
> epsilon :=
> x_0 := 1 + e

$$x_0 := 1 + e \tag{10}$$


```

```

> x_1 :=  $\frac{(1 - \text{sqrt}(5))}{2} + e$ 

$$x_1 := \frac{1}{2} - \frac{\sqrt{5}}{2} + e \tag{11}$$


```

```

> v_1 := Vector([x_1, x_0])

$$v_1 := \begin{bmatrix} \frac{1}{2} - \frac{\sqrt{5}}{2} + e \\ 1 + e \end{bmatrix} \tag{12}$$


```

```

> result := Q . JK . Q inverse . v_1

$$result := \left[ \left( \left( \frac{1}{2} - \frac{\sqrt{5}}{10} \right) \left( -\frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k + \frac{(\sqrt{5}+1)\sqrt{5} \left( \frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k}{10} \right) \left( \frac{1}{2} \right) \right] \tag{13}$$


```

$$\begin{aligned}
& - \frac{\sqrt{5}}{2} + e \Big) + \left( \left( \frac{1}{2} - \frac{\sqrt{5}}{10} \right) \left( -\frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k \left( -\frac{\sqrt{5}}{2} - \frac{1}{2} \right) \right. \\
& \left. - \frac{(\sqrt{5} + 1) \left( \frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k (-5 + \sqrt{5})}{20} \right) (1 + e), \\
& \left[ \left( -\frac{\sqrt{5} \left( -\frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k}{5} + \frac{\sqrt{5} \left( \frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k}{5} \right) \left( \frac{1}{2} - \frac{\sqrt{5}}{2} + e \right) + \right. \\
& \left. - \frac{\sqrt{5} \left( -\frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k \left( -\frac{\sqrt{5}}{2} - \frac{1}{2} \right)}{5} - \frac{\left( \frac{\sqrt{5}}{2} + \frac{1}{2} \right)^k (-5 + \sqrt{5})}{10} \right] (1 \\
& + e)
\end{aligned}$$

> # calculate the limit

>  $result := limit(result[1], result[2], k = infinity)$

$$result := \frac{3\sqrt{5} + 5}{5 + \sqrt{5}} \quad (14)$$

> evala(result)

$$\frac{\sqrt{5}}{2} + \frac{1}{2} \quad (15)$$

> # Hupa, correct bitches

```

> # We doen de uitdagende bijvraag niet aangezien dit wordt geskipped door
   professor.
> restart: with(LinearAlgebra):
> A := Matrix([ [ 8/10, 3/10 ], [ 2/10, 7/10 ] ])
      A := 
$$\begin{bmatrix} \frac{4}{5} & \frac{3}{10} \\ \frac{1}{5} & \frac{7}{10} \end{bmatrix}$$
 (1)

> J, Q := JordanForm(A, output = [ 'J', 'Q' ])
      J, Q := 
$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{3}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{2}{5} \end{bmatrix}$$
 (2)

> # Lambda_1 = 1, lam_2 =  $\frac{1}{2}$ ,
   obviously gaat lambda_1 domineren wanneer we exponentiële vorm
   nemen.
> # Aka,  $v_1 = \left[ \frac{3}{5}, \frac{2}{5} \right]$  is het asymptotische vector. Let's proof this shit
> JK := Matrix([ [ 1, 0 ], [ 0, (1/2)^k ] ])
      JK := 
$$\begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{1}{2}\right)^k \end{bmatrix}$$
 (3)

> result := Q * JK * MatrixInverse(Q)
      result := 
$$\begin{bmatrix} \frac{3}{5} + \frac{2\left(\frac{1}{2}\right)^k}{5} & \frac{3}{5} - \frac{3\left(\frac{1}{2}\right)^k}{5} \\ \frac{2}{5} - \frac{2\left(\frac{1}{2}\right)^k}{5} & \frac{2}{5} + \frac{3\left(\frac{1}{2}\right)^k}{5} \end{bmatrix}$$
 (4)

> # In this form, we can see if k -> infinity, then we get the answer we were
   looking for. Prove accepted.

```

Figure 126: Huis 3 Exercise 4

```

> restart : with(LinearAlgebra) :
> A := Matrix([ [ 98/100, 0, 0 ], [ 2/100, 97/100, 0 ], [ 0, 3/100, 1 ] ])
      
$$A := \begin{pmatrix} \frac{49}{50} & 0 & 0 \\ \frac{1}{50} & \frac{97}{100} & 0 \\ 0 & \frac{3}{100} & 1 \end{pmatrix}$$

> J, Q := JordanForm(A, output = ['J', 'Q'])
      
$$J, Q := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{49}{50} & 0 \\ 0 & 0 & \frac{97}{100} \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & -2 \\ 1 & -3 & 2 \end{pmatrix}$$

> MatrixPower(J, n)
      
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \left(\frac{49}{50}\right)^n & 0 \\ 0 & 0 & \left(\frac{97}{100}\right)^n \end{pmatrix}$$

> # Here we see that Lambda_1 is dominant, so eigenvector 1 is dominant
> v_1 := Vector([0, 0, 1])
      
$$v_1 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

> # everyone will die.
>

```

Figure 127: Exercise 5

6. Voor een Markov matrix is de som van de componenten van een vector  $\mathbf{x}$  gelijk aan de som van de componenten van  $\mathbf{Ax}$ . Bewijs dat voor eigenvectoren  $\mathbf{Av}_k = \lambda_k \mathbf{v}_k$ , met  $\lambda_k \neq 1$  de som van de componenten verdwijnt.

Verifieer ook voor de matrix  $A$  uit vraag 4.

**Oplossing:**

$$\sum_i (\mathbf{v}_k)_i = \sum_i (A\mathbf{v}_k)_i = \lambda_k \sum_i (\mathbf{v}_k)_i$$

$$(1 - \lambda_k) \sum_i (\mathbf{v}_k)_i = 0 \iff \lambda_k = 1 \text{ of } \sum_i (\mathbf{v}_k)_i = 0$$

Figure 128: Exercice 6

7. a) Schrijf de tweede-orde differentiaalvergelijking

$$\ddot{y} + a\dot{y} + by = 0, \quad \text{met } y(t), a, b \in \mathbb{R} \quad (2)$$

als een eerste-orde matrix differentiaalvergelijking  $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ , met  $\mathbf{y} = (\dot{y}, y)$ , en los deze vergelijking vervolgens op.

Je vindt drie types kwalitatief verschillend gedrag van de oplossing, voor  $a^2 > 4b$ ,  $a^2 = 4b$  en  $a^2 < 4b$ . Verklaar dit gedrag vanuit de Jordan-vorm van  $\mathbf{A}$ .

Oplossing:

$$\begin{aligned}\ddot{y} &= -a\dot{y} - by \\ \frac{d}{dt} \begin{pmatrix} \dot{y} \\ y \end{pmatrix} &= \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{y} \\ y \end{pmatrix}\end{aligned}$$

- $a^2 > 4b$ :

We krijgen een diagonale Jordan vorm met 2 verschillende eigenwaarden. De basisoplossingen zijn  $e^{\lambda_1 t}$  en  $e^{\lambda_2 t}$ .

- $a^2 = 4b$ : In dit geval krijgen we een niet-diagonale Jordan vorm met  $\lambda_1 = \lambda_2 = -a/2$ . De basisoplossingen zijn  $e^{\lambda_1 t}$  en  $te^{\lambda_1 t}$ .

- $a^2 < 4b$ : De Jordanvorm is opnieuw diagonaal. We krijgen 2 complexe eigenwaarden. De basisoplossingen zijn gedempte oscillaties.

Figure 129: Exercise 7

```

> # A) is basically gwn Jordan Form en kijken wat je eigenwaarden
   eigenvectoren zijn lolz.
> restart: with(LinearAlgebra):
> A := Matrix([[-2, 2, 2], [-5, 4, 3], [0, 0, 2]])
      A := 
$$\begin{bmatrix} -2 & 2 & 2 \\ -5 & 4 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$
 (1)

> J, Q := JordanForm(A, output = [J,Q])
      J, Q := 
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1-I & 0 \\ 0 & 0 & 1+I \end{bmatrix}, \begin{bmatrix} 1 & -\frac{1}{5} - \frac{2I}{5} & -\frac{1}{5} + \frac{2I}{5} \\ 1 & -\frac{1}{2} - \frac{I}{2} & -\frac{1}{2} + \frac{I}{2} \\ 1 & 0 & 0 \end{bmatrix}$$
 (2)

> # Hierboven is a)
> # Ok nu b) Het idee is simpel, we willen alpha1, alpha2 en y(t)
y0 := Vector([1, 1, 1]):
> constants := solve(Q.Vector([alpha1, alpha2, conjugate(alpha2)]) = y0,
   [alpha1, alpha2]);
      constants := [[alpha1 = 1, alpha2 = 0]] (3)

> # nu nog y(t). y(t) = Q · exp(D) · Q-1 · y0
> D_exp := Matrix([[exp(2*t), 0, 0], [0, exp((1-I)*t), 0], [0, 0, exp((1+I)*t)]])
      D_exp := 
$$\begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{(1-I)t} & 0 \\ 0 & 0 & e^{(1+I)t} \end{bmatrix}$$
 (4)

> y := Q · D_exp · MatrixInverse(Q) · y0
      y := 
$$\begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$
 (5)

> # Et voila.

```

Figure 130: Exercise 8

// Exercise 9

> *restart: with(LinearAlgebra):*

>  $A := \text{Matrix}([[0, -1, 0, -1], [1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0]])$

$$A := \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (1)$$

>  $J, Q := \text{JordanForm}(A, \text{output} = [\text{'J}', \text{'Q'}])$

$$J, Q := \left[ \begin{array}{cccc} -\frac{1}{2} - \frac{I\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} + \frac{I\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} - \frac{I\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} + \frac{I\sqrt{3}}{2} \end{array} \right], \left[ \begin{array}{c} -\frac{1}{-3 + I\sqrt{3}}, \frac{\frac{I}{3}\sqrt{3}}{-1 + I\sqrt{3}}, -\frac{1 + I\sqrt{3}}{(-3 + I\sqrt{3})(-1 + I\sqrt{3})}, -\frac{1}{-3 + I\sqrt{3}} \\ -\frac{-1 + I\sqrt{3}}{2(-3 + I\sqrt{3})}, \frac{1}{-3 + I\sqrt{3}}, -\frac{1}{-3 + I\sqrt{3}}, \frac{-1 + I\sqrt{3}}{2(-3 + I\sqrt{3})} \\ -\frac{2}{(-3 + I\sqrt{3})(-1 + I\sqrt{3})}, \frac{2}{(-3 + I\sqrt{3})(-1 + I\sqrt{3})}, \\ -\frac{2}{(-3 + I\sqrt{3})(-1 + I\sqrt{3})}, \frac{1 + I\sqrt{3}}{2(-3 + I\sqrt{3})} \\ -\frac{1}{-3 + I\sqrt{3}}, -\frac{1 + I\sqrt{3}}{(-3 + I\sqrt{3})(-1 + I\sqrt{3})}, \frac{1 + I\sqrt{3}}{(-3 + I\sqrt{3})(-1 + I\sqrt{3})}, \\ \frac{1}{-3 + I\sqrt{3}} \end{array} \right] \quad (2)$$

>  $v := \text{Vector}([1, 0, 2, 1])$

$$v := \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \quad (3)$$

>  $result\_1 := simplify(MatrixExponential(t \cdot A) \cdot v)[4]$

$$result\_1 := \frac{2 \left( e^{-\frac{t}{2}} + \frac{3 e^{\frac{t}{2}}}{2} \right) \sqrt{3} \sin\left(\frac{t\sqrt{3}}{2}\right)}{3} + e^{-\frac{t}{2}} \cos\left(\frac{t\sqrt{3}}{2}\right) \quad (4)$$

=>

## Bord 3

```

[> restart: with(LinearAlgebra):
[> A := Matrix([[2, 3], [1, 0]])

$$A := \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \quad (1)$$

[> J, Q := JordanForm(A, output = ['J', 'Q'])

$$J, Q := \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} \quad (2)$$

[> # Dit is i) lambda_1 = -1, lambda_2 = 3
[> # ii)
[> v_1 := Q[ .., 1]

$$v_1 := \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix} \quad (3)$$

[> v_2 := Q[ .., 2]

$$v_2 := \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix} \quad (4)$$

[> v_3 := Vector([1, 2])

$$v_3 := \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad (5)$$

[> eq := <1, 2> = c1 * v_1 + c2 * v_2;

$$eq := \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{c1}{4} + \frac{3c2}{4} \\ -\frac{c1}{4} + \frac{c2}{4} \end{bmatrix} \quad (6)$$

[> solve(eq, {c1, c2})

$$\{c1 = -5, c2 = 3\} \quad (7)$$

[>
```

Figure 131: Exercice 1

```

> restart: with(LinearAlgebra):
> # Oefening 2 vind ik maar brol, wordt nooit gevraagd
> # i)
> # Leer gwn matrixExponential te gebruiken
> A := Matrix([[1, 1], [1, 1]])
      A := 
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
 (1)
> MatrixExponential(A)
      
$$\begin{bmatrix} \frac{1}{2} + \frac{e^2}{2} & \frac{e^2}{2} - \frac{1}{2} \\ \frac{e^2}{2} - \frac{1}{2} & \frac{1}{2} + \frac{e^2}{2} \end{bmatrix}$$
 (2)

```

Figure 132: Exercise 2

// Exercise 3

```
[> restart: with(LinearAlgebra): with(plots): with(plottools):  
[> #i)
```

```
[> P := Matrix([ [0, 0, 1, 0, 0], [1/2, 0, 0, 0, 0], [1/2, 1, 0, 0, 0], [0, 0, 0, 1/2, 0], [0, 0, 0, 1/2, 1] ]])
```

$$P := \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix} \quad (1)$$

```
[> T := 1/5 Matrix([[1, 1, 1, 1, 1], [1, 1, 1, 1, 1], [1, 1, 1, 1, 1], [1, 1, 1, 1, 1], [1, 1, 1, 1, 1]])
```

$$T := \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix} \quad (2)$$

```
[> G := a→a·P + (1 - a)·T
```

$$G := a \mapsto a \cdot P + (1 - a) \cdot T \quad (3)$$

```
[> #ii)
```

```
[> J, Q := JordanForm(G(a), output = [ 'J', 'Q' ])
```

(4)

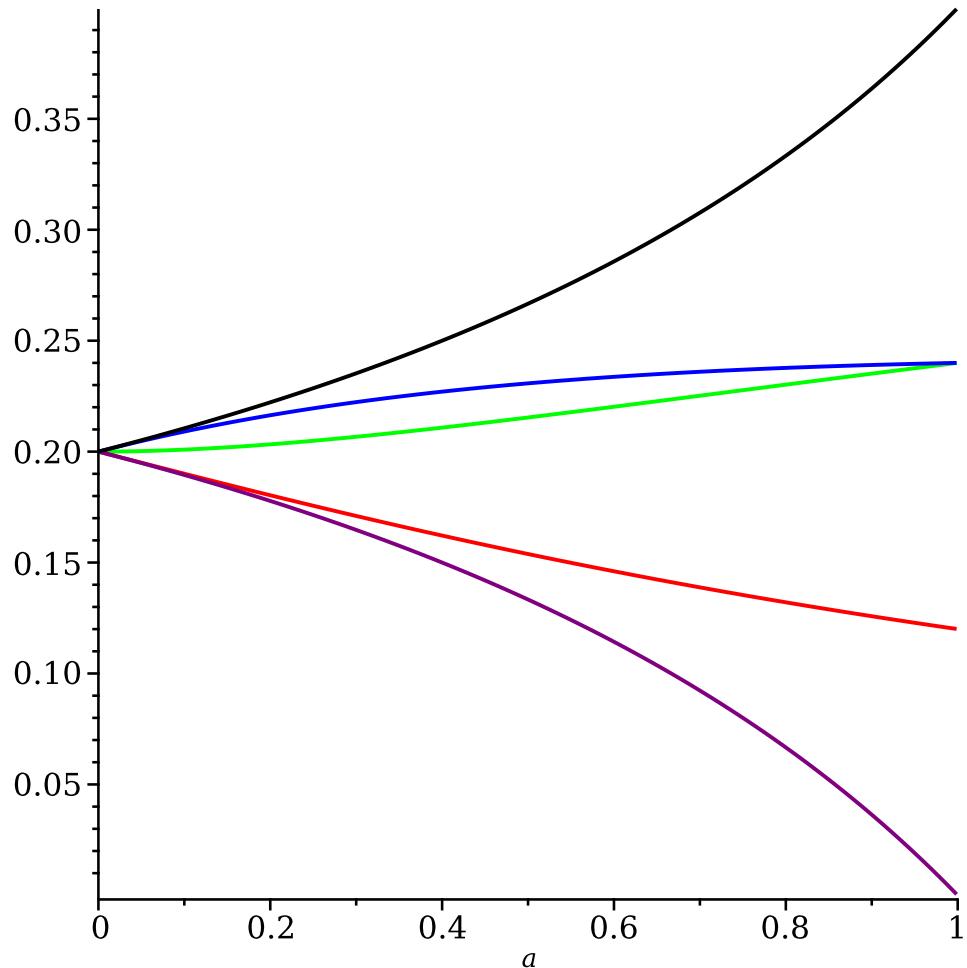
$$\boxed{J, Q := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & \left(-\frac{1}{2} - \frac{I}{2}\right)a & 0 \\ 0 & 0 & 0 & \left(-\frac{1}{2} + \frac{I}{2}\right)a \\ 0 & 0 & 0 & \frac{a}{2} \end{bmatrix}, \begin{bmatrix} \left[\frac{2(a^2 + a + 1)}{5(a^2 + 2a + 2)}, \frac{4}{25}, \right. \\ \left. \frac{3Ia + 10 + 2I + 5a}{10(2Ia + 3 + I + a)}, \frac{-\frac{I}{50}(11Ia^2 + 32Ia - 7a^2 - 4 + 32I - 14a)}{a^2 + 2a + 2}, 0 \right], \\ \left[ \frac{a+2}{5(a^2 + 2a + 2)}, \frac{2}{25}, \frac{Ia - 6 + 4I - 4a}{10(2Ia + 3 + I + a)}, \right. \\ \left. \frac{\frac{I}{50}(2Ia^2 + 9Ia - 9a^2 - 18 + 14I - 23a)}{a^2 + 2a + 2}, 0 \right], \\ \left[ \frac{a^2 + 3a + 2}{5(a^2 + 2a + 2)}, \frac{4}{25}, -\frac{4Ia + 4 + 6I + a}{10(2Ia + 3 + I + a)}, \right. \\ \left. \frac{\frac{I}{50}(9Ia^2 + 23Ia + 2a^2 + 14 + 18I + 9a)}{a^2 + 2a + 2}, 0 \right], \\ \left[ \frac{2(-1+a)}{5(a-2)}, 0, 0, 0, -\frac{2(-1+a)}{5(a-2)} \right], \\ \left[ -\frac{2}{5(a-2)}, -\frac{2}{5}, 0, 0, \frac{2(-1+a)}{5(a-2)} \right] \end{bmatrix}}$$

> # Hier zien we alvast dat lambda 1 dominant is, dus eigenvector 1 is sexy  
 > sol := Q[ .., 1]

(5)

$$sol := \begin{bmatrix} \frac{2(a^2 + a + 1)}{5(a^2 + 2a + 2)} \\ \frac{a + 2}{5(a^2 + 2a + 2)} \\ \frac{a^2 + 3a + 2}{5(a^2 + 2a + 2)} \\ \frac{2(-1 + a)}{5(a - 2)} \\ -\frac{2}{5(a - 2)} \end{bmatrix} \quad (5)$$

> `plot(sol, a = 0..1, color = [green, red, blue, purple, black]);`  
`evalf(subs({a = 1}, sol));`



(6)

$$\begin{bmatrix} 0.2400000000 \\ 0.1200000000 \\ 0.2400000000 \\ 0. \\ 0.4000000000 \end{bmatrix} \quad (6)$$

> # we zien dus dat website 5 het meest zal worden bezocht

> #iii)

>  $J, Q := JordanForm(G(1), output = ['J', 'Q'])$

$$J, Q := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} - \frac{I}{2} & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} + \frac{I}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad (7)$$

$$\begin{bmatrix} \frac{2}{5} & 0 & \frac{3}{10} - \frac{I}{10} & \frac{3}{10} + \frac{I}{10} & -\frac{2}{5} \\ \frac{1}{5} & 0 & -\frac{1}{10} + \frac{I}{5} & -\frac{1}{10} - \frac{I}{5} & -\frac{1}{5} \\ \frac{2}{5} & 0 & -\frac{1}{5} - \frac{I}{10} & -\frac{1}{5} + \frac{I}{10} & -\frac{2}{5} \\ 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

> # antwoord is dan ofcourse v\_1 en v\_5 letsgo

> #iv)

>  $state\_n := (n, x0) \rightarrow Q \cdot J^n \cdot MatrixInverse(Q) \cdot x0 :$

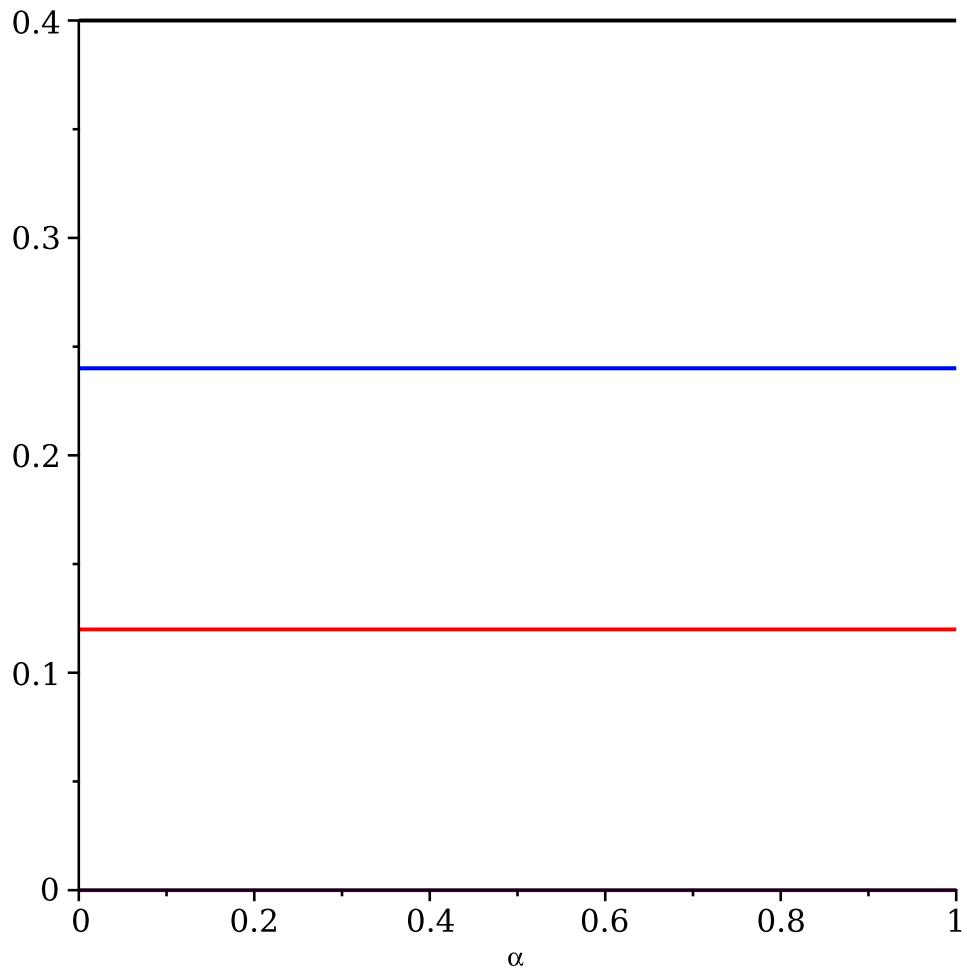
> # eerst uniform dus 0.2 prob per sprong

>  $x0 := \langle 0.2, 0.2, 0.2, 0.2, 0.2 \rangle;$

>  $state\_20 := state\_n(20, x0) :$

>  $plot1 := plot(state\_20, alpha = 0..1, color = [green, red, blue, purple, black]);$

$$x0 := \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix}$$

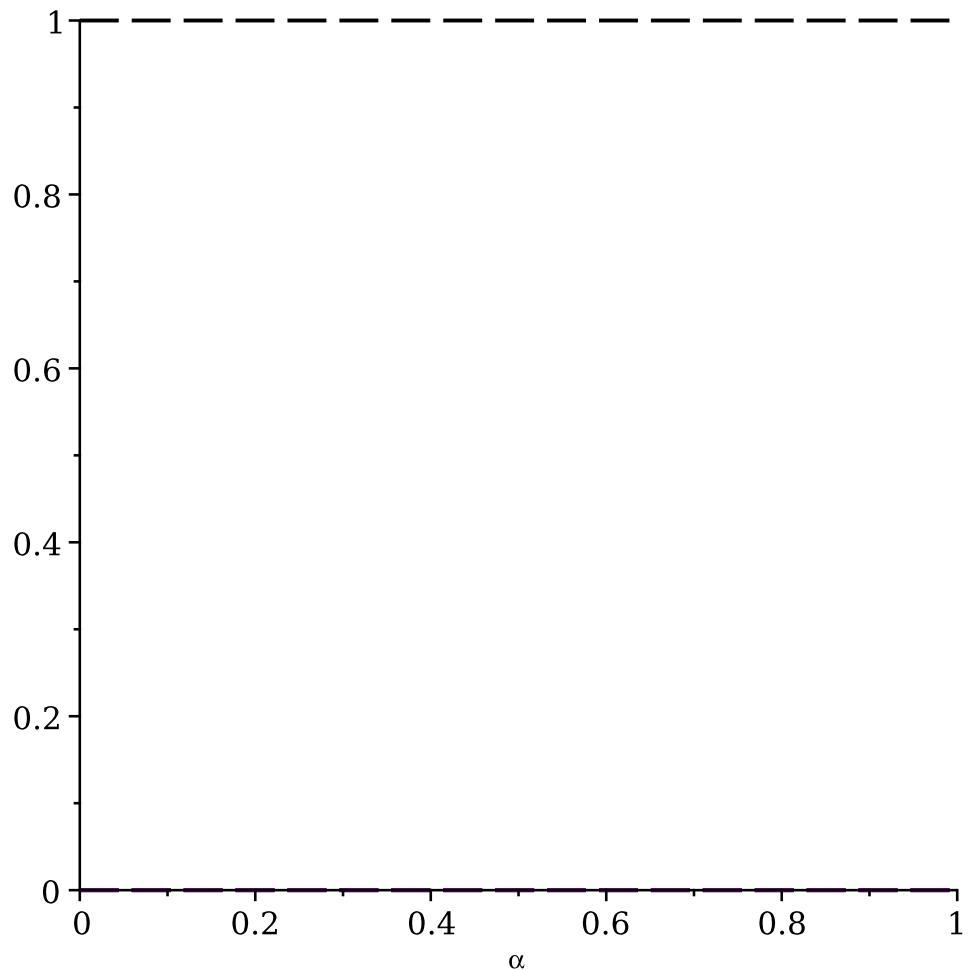


```

> # ik denk dat mijn Q en J niet goed zijn ingeladen, anyways I dont care
> # Nu kijken naar bezetting van website 4 en 5, dus 50 50
> x0 := <0, 0, 0, 0.5, 0.5>;
state_50 := state_n(20, x0) :
plot2 := plot(state_50, alpha = 0..1, color = [green, red, blue, purple, black],
linestyle = "dash");

```

$$x0 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \\ 0.5 \end{bmatrix}$$

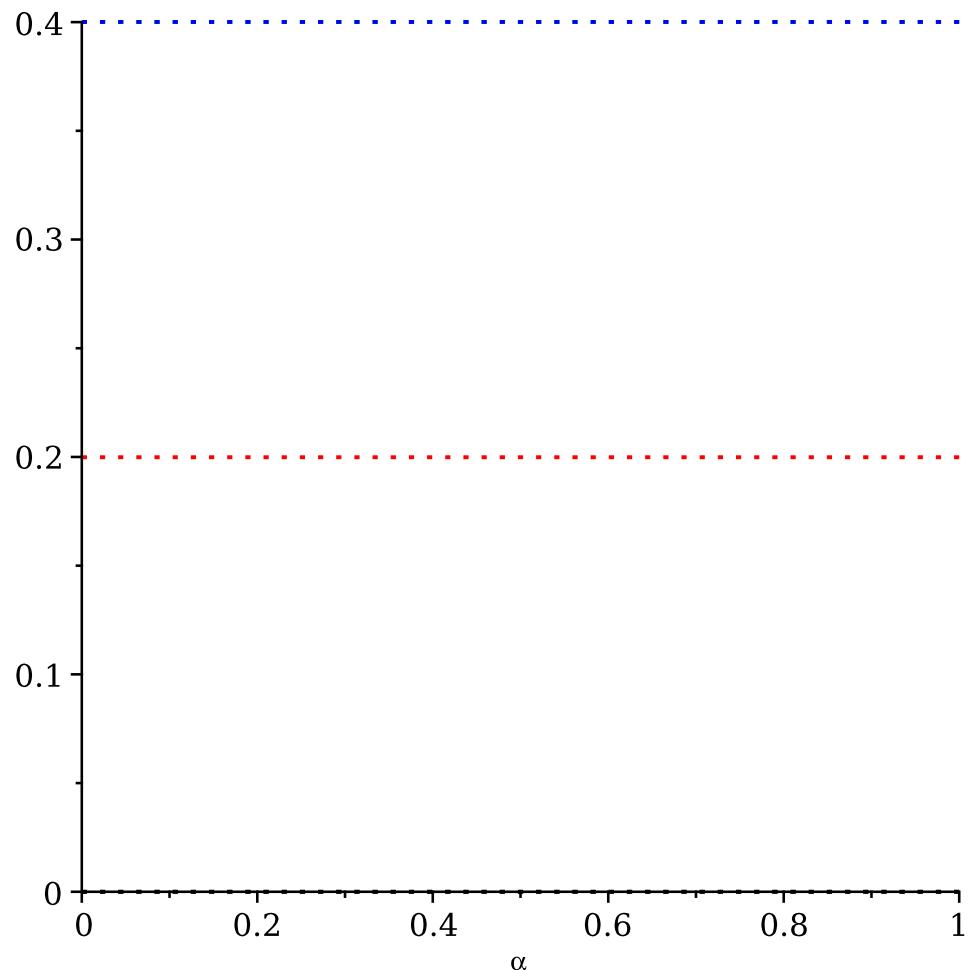


```

> # Als laatste de bezetting van website 1, 2 en 3
> x0 := <1/3, 1/3, 1/3, 0, 0>;
state_50 := state_n(20, x0):
plot3 := plot(state_50, alpha = 0..1, color = [green, red, blue, purple, black],
linestyle = "dot");

```

$$x_0 := \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \\ 0 \end{bmatrix}$$



> #Warning, die plots kloppen niet, maar moet gewoonweg gevult worden  
met de juiste  $J$



## WC 3

// Exercise 1

# Wiskundige modellering in de ingenieurswetenschappen: Werkcollege 3

## Oefening 1

> restart:with(plots):with(LinearAlgebra):

We definiëren de (6x6)-matrix A als functie van Ex, Ey en B

>  $A := (Ex, Ey, B) \rightarrow \langle \langle 0|0|B|0|Ex\rangle, \langle 1|0|0|0|0\rangle, \langle -B|0|0|0|Ey\rangle, \langle 0|0|1|0|0\rangle, \langle 0|0|0|0|0\rangle \rangle;$   
 $A(Ex,Ey,B);$

$A := (Ex, Ey, B) \mapsto \langle \langle 0|0|B|0|Ex\rangle, \langle 1|0|0|0|0\rangle, \langle -B|0|0|0|Ey\rangle, \langle 0|0|1|0|0\rangle, \langle 0|0|0|0|0\rangle \rangle$

$$\begin{bmatrix} 0 & 0 & B & 0 & Ex \\ 1 & 0 & 0 & 0 & 0 \\ -B & 0 & 0 & 0 & Ey \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (1.1)$$

Ook de beginwaarden steken we in een vector(functie)

>  $X0 := (x0, y0, vx0, vy0) \rightarrow \langle vx0, x0, vy0, y0, 1 \rangle:$   
 $X0(x0, y0, vx0, vy0);$

$$\begin{bmatrix} vx0 \\ x0 \\ vy0 \\ y0 \\ 1 \end{bmatrix} \quad (1.2)$$

De algemene oplossing wordt gevonden als

>  $sol := \text{MatrixExponential}(t^*A(Ex,Ey,B)).X0(x0, y0, vx0, vy0);$   
 $x\_sol := sol[2];$   
 $y\_sol := sol[4];$   
 $sol := \left[ \left[ \cos(tB) vx0 + \sin(tB) vy0 + \frac{Ex \sin(tB) - Ey \cos(tB) + Ey}{B} \right], \right.$   
 $\left[ \frac{\sin(tB) vx0}{B} + x0 + \frac{(-\cos(tB) + 1) vy0}{B} \right.$   
 $\left. + \frac{BEyt - Ex \cos(tB) - Ey \sin(tB) + Ex}{B^2} \right],$   
 $\left. \left[ -\sin(tB) vx0 + \cos(tB) vy0 + \frac{Ex \cos(tB) + Ey \sin(tB) - Ex}{B} \right], \right]$

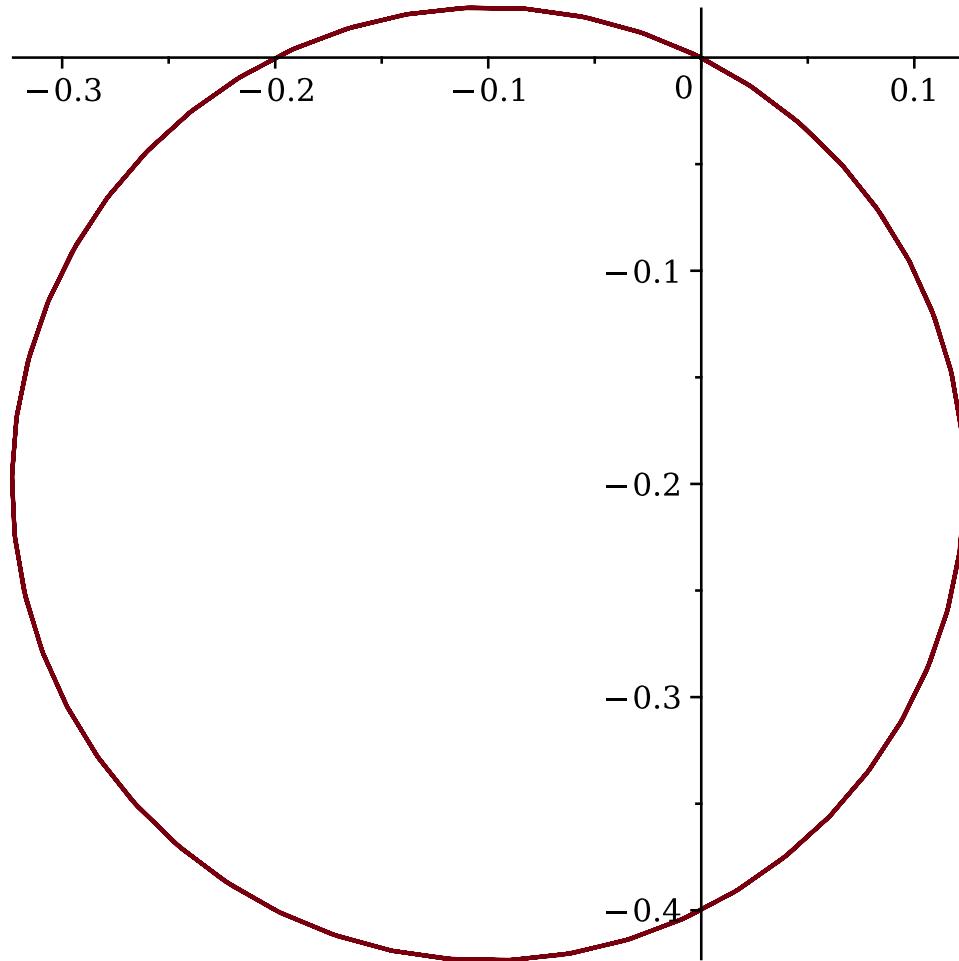
$$\begin{aligned}
& \left[ \frac{(\cos(tB) - 1) vx0}{B} + \frac{\sin(tB) vy0}{B} + y0 \right. \\
& \left. + \frac{-B Ex t + Ex \sin(tB) - Ey \cos(tB) + Ey}{B^2} \right] \\
& \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
x_{sol} &:= \frac{\sin(tB) vx0}{B} + x0 + \frac{(-\cos(tB) + 1) vy0}{B} \\
&+ \frac{BEyt - Ex \cos(tB) - Ey \sin(tB) + Ex}{B^2} \\
y_{sol} &:= \frac{(\cos(tB) - 1) vx0}{B} + \frac{\sin(tB) vy0}{B} + y0 \\
&+ \frac{-B Ex t + Ex \sin(tB) - Ey \cos(tB) + Ey}{B^2}
\end{aligned} \tag{1.3}$$

Kijken we nu naar de concrete beginvoorwaarden voor  $(Ex, Ey) = (0,0)$ ,  $B=10$

```

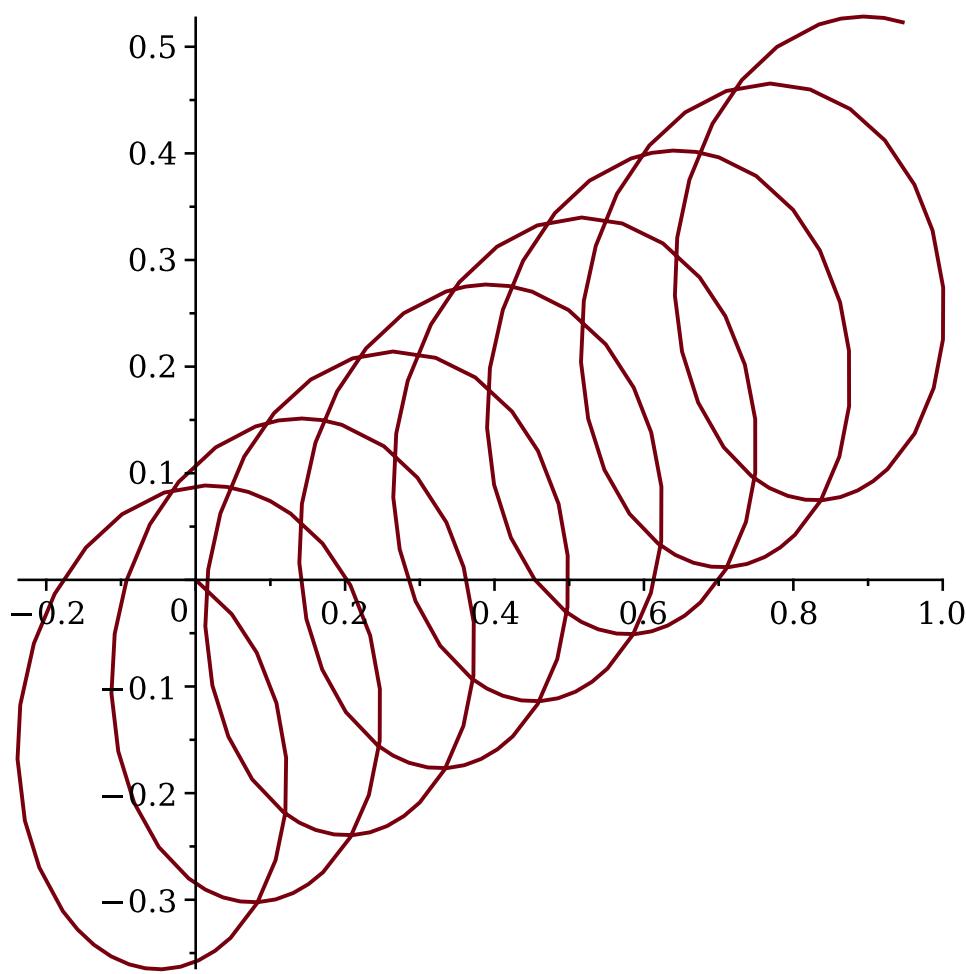
> sol_E0 := MatrixExponential(t*A(0,0,10)).X0(0,0,2,-1):
  x_sole0 := sol_E0[2]:
  y_sole0 := sol_E0[4]:
> plot([x_sole0, y_sole0, t=0..5]);

```



Analoog voor  $(Ex, Ey) = (-1, 2)$

```
> sol_E := MatrixExponential(t*A(-1,2,10)).X0(0,0,2,-1):
  x_sole := sol_E[2]:
  y_sole := sol_E[4]:
> plot([x_sole, y_sole,t=0..5]);
```



Het qualitatieve verschil is te zien in de Jordanvorm van de matrix A: indien E verschillend is van 0 verschijnen er Jordan-blokken, wat zal zorgen voor een verschillend gedrag wanneer men e exponentiële  $\exp(t^*A)$  neemt

```
> JE0 := JordanForm(A(0,0,B));
JE := JordanForm(A(Ex,Ey,B));
```

$$JE0 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -IB & 0 & 0 & 0 \\ 0 & 0 & IB & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$JE := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -IB & 0 & 0 \\ 0 & 0 & 0 & IB & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (1.4)$$

► **Oefening 3**

► **Oefening 2**

```

> restart: with(LinearAlgebra):
> A := Matrix([ [1, -1, 0], [0, 1/2, 0], [-3/4, 7/4, 1/4] ])
      
$$A := \begin{bmatrix} 1 & -1 & 0 \\ 0 & \frac{1}{2} & 0 \\ -\frac{3}{4} & \frac{7}{4} & \frac{1}{4} \end{bmatrix}$$
 (1)
> J, Q := JordanForm(A, output = [ 'J', 'Q' ])
      
$$J, Q := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & 2 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{3} & 1 & -\frac{2}{3} \end{bmatrix}$$
 (2)
> cosD := Matrix([[ (cos(Pi * J[1, 1])), 0, 0], [0, (cos(Pi * J[2, 2])), 0], [0, 0, (cos(Pi * J[3, 3]))]]);
      
$$cosD := \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$
 (3)
> result := Q * cosD * MatrixInverse(Q)
      
$$result := \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \\ 1 + \frac{\sqrt{2}}{2} & -2 - \frac{3\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$
 (4)
> # Et voila

```

Figure 133: Exercise 2

```

[> restart: with(LinearAlgebra):
[> # First we define the relationship as matrix
[> A := Matrix([[a,b],[1,0]])

$$A := \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \quad (1)$$


[> J, Q := JordanForm(A, output = [ 'J','Q'])

$$J, Q := \begin{bmatrix} \frac{a}{2} - \frac{\sqrt{a^2 + 4b}}{2} & 0 \\ 0 & \frac{a}{2} + \frac{\sqrt{a^2 + 4b}}{2} \end{bmatrix}, \quad (2)$$


$$\begin{bmatrix} \frac{\sqrt{a^2 + 4b} - a}{2\sqrt{a^2 + 4b}} & \frac{a + \sqrt{a^2 + 4b}}{2\sqrt{a^2 + 4b}} \\ -\frac{1}{\sqrt{a^2 + 4b}} & \frac{1}{\sqrt{a^2 + 4b}} \end{bmatrix}$$


[> result := Q . MatrixPower(J, n) . MatrixInverse(Q) . Vector([1, 1]):
[> limit_ := simplify(result[1] / result[2])

$$limit_ := \left( (\sqrt{a^2 + 4b} - a - 2b) \left( \frac{a}{2} - \frac{\sqrt{a^2 + 4b}}{2} \right)^n + (\sqrt{a^2 + 4b} + a + 2b) \left( \frac{a}{2} + \frac{\sqrt{a^2 + 4b}}{2} \right)^n \right) / \left( (\sqrt{a^2 + 4b} + a - 2) \left( \frac{a}{2} - \frac{\sqrt{a^2 + 4b}}{2} \right)^n - \left( \frac{a}{2} + \frac{\sqrt{a^2 + 4b}}{2} \right)^n (a - \sqrt{a^2 + 4b} - 2) \right) \quad (3)$$


[> limit(limit_, n = infinity)

$$\lim_{n \rightarrow \infty} \left( (\sqrt{a^2 + 4b} - a - 2b) \left( \frac{a}{2} - \frac{\sqrt{a^2 + 4b}}{2} \right)^n + (\sqrt{a^2 + 4b} + a + 2b) \left( \frac{a}{2} + \frac{\sqrt{a^2 + 4b}}{2} \right)^n \right) / \left( (\sqrt{a^2 + 4b} + a - 2) \left( \frac{a}{2} - \frac{\sqrt{a^2 + 4b}}{2} \right)^n - \left( \frac{a}{2} + \frac{\sqrt{a^2 + 4b}}{2} \right)^n (a - \sqrt{a^2 + 4b} - 2) \right) \quad (4)$$


```

Figure 134: Exercise 3

Huis 4

$$3y = z^3 + 2z \cdot 2$$

i)  $\frac{\partial}{\partial x} z(x,y) : 3 = 2z^2 \cdot \frac{\partial z}{\partial x} + 2x \cdot \frac{\partial z}{\partial x}$   
 $\Leftrightarrow 1 = \frac{\partial z}{\partial x} (z^2 + x)$   
 $\Leftrightarrow \frac{1}{z^2 + x} = \frac{\partial z}{\partial x}$

$$\frac{\partial}{\partial x} z(x,y) : 0 = 2z^2 \cdot \frac{\partial z}{\partial x} + 2z + 2 \cdot x \cdot \frac{\partial z}{\partial x}$$
 $\Leftrightarrow -2z = \frac{\partial z}{\partial x} (z^2 + x)$ 
 $\Leftrightarrow -\frac{z}{z^2 + x} = \frac{\partial z}{\partial x}$ 

ii)  $\frac{\partial}{\partial y} z(x,y) \frac{\partial (z^2 + x)^{-1}}{\partial x} = -1 \cdot (z^2 + x) \cdot 2z \cdot \frac{\partial z}{\partial x}$   
 $= -\frac{2z}{(z^2 + x)^2 \cdot (z^2 + x)} = -\frac{2z}{(z^2 + x)^3}$

$$\begin{aligned} \frac{\partial}{\partial x} z(x,y) &= \frac{\partial}{\partial x} \left( -\frac{z}{z^2 + x} \right) = -\frac{\left( \frac{\partial z}{\partial x} \cdot (z^2 + x) - 2 \cdot (z^2 + x) \cdot \frac{\partial z}{\partial x} + 1 \right)}{(z^2 + x)^2} \\ &= \frac{z + z \left( -2 \left( \frac{-z}{z^2 + x} \right) + 1 \right)}{(z^2 + x)^2} \end{aligned}$$

$$= z + z - \frac{2z^3}{z^2 + x} = \frac{z(1/z^2 + x) - 2z^3}{(z^2 + x)^3} = \frac{z^2 x}{(z^2 + x)^3}$$

so:  $x \cdot \frac{\partial}{\partial y} z(x,y) + \frac{\partial}{\partial x} z(x,y) = 0 \blacksquare$

Figure 135: Exercise 1

$$2) \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial t^2} = (x^2, y^2) \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right)$$

$$x = e^{i\omega t} \quad y = e^{i\omega t}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial x} + \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial x}$$

$$= \frac{\partial^2 f}{\partial x^2} \cdot e^{i\omega t} + \frac{\partial^2 f}{\partial y^2} \cdot i\omega e^{i\omega t}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial y} + \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial y}$$

$$= -\frac{\partial^2 f}{\partial x^2} \cdot e^{i\omega t} + \frac{\partial^2 f}{\partial y^2} \cdot e^{i\omega t}$$

$$\frac{\partial^2 f}{\partial t^2} = \left( \frac{\partial^2 f}{\partial x^2} \cdot e^{i\omega t} + \frac{\partial^2 f}{\partial y^2} \cdot i\omega e^{i\omega t} \right) \left( \frac{\partial^2 f}{\partial x^2} \cdot e^{i\omega t} + \frac{\partial^2 f}{\partial y^2} \cdot e^{i\omega t} \right)$$

$$= \frac{\partial^2 f}{\partial x^2} (e^{i\omega t})^2 + \frac{\partial^2 f}{\partial y^2} e^{i\omega t} \cdot e^{i\omega t} + \frac{\partial^2 f}{\partial x^2} i\omega e^{i\omega t} \cdot e^{i\omega t} + \frac{\partial^2 f}{\partial y^2} (i\omega e^{i\omega t})^2$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} e^{i\omega t} + \frac{\partial^2 f}{\partial y^2} e^{i\omega t} + \frac{\partial^2 f}{\partial y^2} e^{i\omega t} + \frac{\partial^2 f}{\partial x^2} e^{i\omega t}$$

$$\frac{\partial^2 f}{\partial y^2} = \left( -\frac{\partial^2 f}{\partial x^2} \cdot e^{i\omega t} + \frac{\partial^2 f}{\partial y^2} \cdot i\omega e^{i\omega t} \right) \left( -\frac{\partial^2 f}{\partial x^2} \cdot e^{i\omega t} + \frac{\partial^2 f}{\partial y^2} \cdot e^{i\omega t} \right)$$

$$= \frac{\partial^2 f}{\partial x^2} e^{i\omega t} + \frac{\partial^2 f}{\partial y^2} e^{i\omega t} - \frac{\partial^2 f}{\partial x^2} e^{i\omega t} - \frac{\partial^2 f}{\partial y^2} e^{i\omega t} - \frac{\partial^2 f}{\partial x^2} e^{i\omega t} + \frac{\partial^2 f}{\partial y^2} e^{i\omega t}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial t} = \frac{\partial^2 f}{\partial x^2} \left( e^{i\omega t} + e^{i\omega t} \right)$$

$$= e^{i\omega t} \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right)$$

$$= (x^2 + y^2) \cdot 0$$

Figure 136: Exercise 2

$$f(x,y) = x^2 - 12xy + 48x + 6y^2, \quad b \neq 0$$

1) calculate partial derivatives  $\frac{\partial f}{\partial x}$ ,  $\frac{\partial f}{\partial y}$ , and equal to zero

$$-\frac{\partial f}{\partial x} = 3x^2 - 12y + 48 = 0$$

↓  
to find stationary  
points.

$$-\frac{\partial f}{\partial y} = -12x + 2by = 0$$

$$\therefore 12x = 2by \Rightarrow 12a$$

$$\therefore y = \frac{6a}{b}$$

$$\therefore 3x^2 - 12 \cdot \frac{6a}{b} + 48 = 0$$

$$\therefore 3x^2 - \frac{72a}{b} + 48 = 0$$

$$\therefore x^2 - \frac{24a}{b} + 16 = 0$$

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$x_{\pm} = \frac{12 + 4\sqrt{9 - 6a^2}}{b}$$

$$\begin{aligned} D_{\text{local}} &= \left(\frac{\partial f}{\partial x}\right)^2 - b \cdot a \cdot 11 \\ &= 576a^2 - 64 \end{aligned}$$

- } als  $D > 0$ : sogenannte optima  
 als  $D < 0$ : false optima -  
 als  $D = 0$ : 1 Optimum.

Figure 137: Exercise 3

$$e) f(x(t), y(t)) = c$$

$$\Rightarrow \frac{x^3 - y^3}{x^3 + y^3} = c$$

$\rightarrow$  schrijf  $\partial A$  dan in relatieve vorm:

$$x^3 - y^3 = c(x^3 + y^3)$$

$$\Leftrightarrow x^3 - y^3 = cx^3 + cy^3$$

$$\Leftrightarrow x^3 - cx^3 = y^3 + cy^3$$

$$\Leftrightarrow x^3(1-c) = y^3(1+c)$$

$$\Rightarrow \frac{x^3(1-c)}{(1+c)} = y^3$$

$$\Leftrightarrow x \sqrt[3]{\frac{(1-c)}{(1+c)}} = y. \quad \text{N.B. } x(t) \neq 0$$

$$\Rightarrow y(t) = t \cdot \sqrt[3]{\frac{(1-c)}{(1+c)}}$$

$$\text{dus } (x(t), y(t)) = \left( t, t^3 \sqrt[3]{\frac{1-c}{1+c}} \right)$$

$$f) \text{ toon aan } \nabla f(x,y) \cdot \left( \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t} \right) = 0$$

$$\underbrace{\frac{\partial f}{\partial x} = \frac{6x^2y^3}{(x^3+y^3)^2}, \quad \frac{\partial f}{\partial y} = \frac{-6y^2x^3}{(x^3+y^3)^2}}$$

$\nabla f$  = gradient.

ok. partiële derivatieve voor  $(x(t), y(t))$

$$\frac{\partial x}{\partial t} = 1, \quad \frac{\partial y}{\partial t} = \sqrt[3]{\frac{1-c}{1+c}} \quad \rightarrow \text{Via rekening}$$

$$\text{dus m.a.: } \nabla f(x,y) \cdot \left( \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t} \right) = 0 \quad \blacksquare$$

Figure 138: Exercise 4

```

> restart: with(LinearAlgebra):
> V:=-cos(3·(x-y))·cos(z+y)·cos(5·(x+y))
       $V := -\cos(3x - 3y) \cos(z + y) \cos(5x + 5y)$  (1)
> H:=Matrix([[diff(V,x,x),diff(V,x,y),diff(V,x,z)], [diff(V,y,x),diff(V,y,y),
      diff(V,y,z)], [diff(V,z,x),diff(V,z,y),diff(V,z,z)]]):
> H:=eval(H,[x=0,y=0,z=0])
      
$$H := \begin{bmatrix} 34 & 16 & 0 \\ 16 & 35 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
 (2)
> J,Q:=JordanForm(H,output = ['J','Q']):
> evalf(J)
[[50.51822645 - 5. × 10-9 I, 0., 0.],
 [0., 18.51981170 + 1.932050808 × 10-8 I, 0.],
 [0., 0., 0.961961840 - 1.532050808 × 10-8 I]] (3)
> # dus w_12 = 50.51, w_22 = 18.51, w_32 = 0.96
>

```

Figure 139: Exercise 5

## Bord 4

$$f(x, y) = \sin(x) e^y + y^2$$

i)  $\vec{v} = \left\{ \cos(x) e^y, \sin(x) \cdot e^y + 2y \right\}$

plot is done with matlab.

ii) we kiezen een horizonte waarde  $f(x, y) = c$

$$\sin(x) e^y + y^2 = c$$

$$(c) \sin(x) = \frac{c - y^2}{e^y}$$

$$\therefore \arcsin\left(\frac{c - y^2}{e^y}\right) \approx x$$

$$\Rightarrow x = \arcsin\left(\frac{c - y^2}{e^y}\right) + 2k\pi, k \in \mathbb{Z}$$

iii) Maple.

iv) gradient toont richting aan:  $(\alpha_1, \alpha_2)$

$$\Rightarrow \left\{ \cos(x) e^y, \sin(x) \cdot e^y + 2y \right\}$$

$$\Theta = \arctan\left(\frac{\alpha_2}{\alpha_1}\right) = \arctan\left(\frac{\sin(x) \cdot e^y + 2y}{\cos(x) \cdot e^y}\right) \in 71,09^\circ$$

$-71,09^\circ$

want we  
richting naar buiten

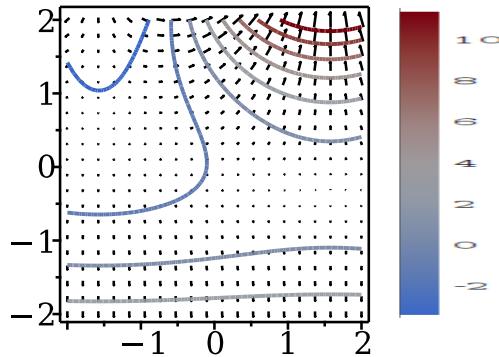
Figure 140: Exercise 1

```

> restart: with(LinearAlgebra) : with(VectorCalculus) : with(plots) :
  with(plottools) :
  SetCoordinates(cartesian[x,y]) :
> f := (x,y)→sin(x)·exp(y) + y2:
> gradient := Gradient(f(x,y)) :
> fig1:=gradplot(f(x,y),x=-2..2,y=-2..2,      axes=boxed,scaling=
  constrained):
> display(fig1):
>
> # ii)
>
> solve(f(x,y) = c, x) :
> fig2 := contourplot(f(x, y), x = -2..2, y = -2..2, axes = boxed, scaling
  = constrained) :
display(fig2) :

> # Now simply display them together
> display(fig1,fig2)

```



```

> # Here we confirmed that the gradient is pointing to the rising direction
> # iv)
> gradient_adapted_1 := cos(1)· exp(1) :

> gradient_adapted_2 := sin(1)·exp(1) + 2
                                gradient_adapted_2 := sin(1) e + 2

```

(1)

>  $\text{angle} := \text{convert}\left(\text{evalf}\left(\arctan\left(\frac{\text{gradient\_adapted\_2}}{\text{gradient\_adapted\_1}}\right)\right), \text{degrees}\right)$   
angle := 71.09038174 degrees

(2)

```

> restart: with(LinearAlgebra): with(VectorCalculus): with(plots):
SetCoordinates(cartesian[x,y]):
```

> # Define the function  
 $f := (x, y, z, t) \rightarrow 4*x*y*z + t^2 - x^2 - y^2 - z^2;$

# Set the coordinates for partial derivatives  
SetCoordinates(cartesian[x,y,z,t]);

# Calculate the gradient  
grad\_f := Gradient(f(x,y,z,t));

# Solve for critical points by setting each component of the gradient to zero  
critical\_points := solve({grad\_f[1] = 0, grad\_f[2] = 0, grad\_f[3] = 0, grad\_f[4] = 0}, {x, y, z, t});

# Calculate the Hessian matrix of f  
hessian\_f := Hessian(f(x,y,z,t));  

$$f := (x, y, z, t) \mapsto 4*x*y*z + t^2 + (-x^2) + (-y^2) + (-z^2)$$

$$\text{cartesian}_{x,y,z,t} \quad (1)$$

> # Now for every point we need to calculate the eigenvalues and explore these  
eval\_hessian := subs({x = 0, y = 0, z = 0, t = 0}, hessian\_f);

$$\text{eval\_hessian} := \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad (2)$$

> # For point (0,0,0,0) we get a saddle point cuz lambda {-2, 2}, thus pos and negative  
eval\_hessian := Eigenvalues(subs({x = 1/2, y = 1/2, z = 1/2, t = 0}, hessian\_f));

$$\text{eval\_hessian} := \begin{bmatrix} 2 \\ -4 \\ 2 \\ -4 \end{bmatrix} \quad (3)$$

> # saddle point, and so forth...

Figure 141: Exercise 2

$$\left\{ \begin{array}{l} (\Delta_t^2 - \Delta_\alpha^2) \cdot g(t, \alpha) = 0 \rightarrow \Delta_u \cdot \Delta_v \cdot g(u, v) = 0 \\ u = t - \alpha, v = t + \alpha \end{array} \right.$$

i) Wenn dann gilt  $(\Delta_t^2 - \Delta_\alpha^2) = 4 \Delta_u \cdot \Delta_v$

$$\Delta_t^2 = \frac{\partial^2 u}{\partial t^2} \cdot \Delta_u + \frac{\partial^2 v}{\partial t^2} \cdot \Delta_v = (\Delta_u + \Delta_v)^2$$

$$\Delta_\alpha^2 = \frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 v}{\partial \alpha^2} = (-\Delta_u + \Delta_v)^2$$

Daraus:  $\Delta_t^2 - \Delta_\alpha^2 = 4 \Delta_u \cdot \Delta_v$

**Beweis:**  $g(t, \alpha) = f_1(t - \alpha) + f_2(t + \alpha)$   
 $= f_1(u) + f_2(v) = g(u, v)$

en  $\Delta_u \cdot \Delta_v \cdot g(u, v) = 0 \rightarrow$  d.h. ghorig  
 von Soffrig.

Figure 142: Exercise 3

The following exercise is quite difficult, however we simply take the SVD and look at the diagonal matrix to see the dim of the dominant singular values.

# Wiskundige modellering in de ingenieurswetenschappen: Bordoeefingenenles 4

## ► Oefening 1

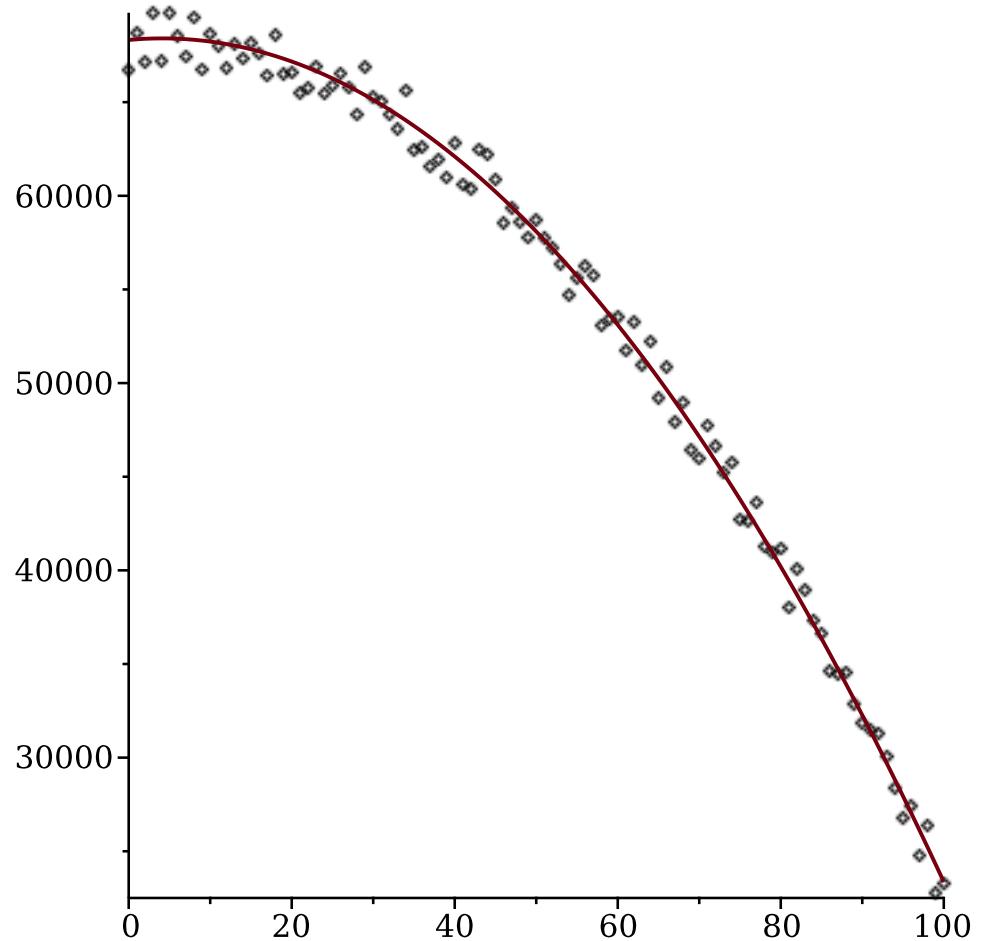
## ► Oefening 2

## ► Oefening 3

## ▼ Oefening 4

### Noisy Freefall

```
> restart:with(LinearAlgebra):with(plots):with(MTM):
Rijken en kolommen (# datapunten per meting en # metingen)
> mmax := 100:
nmax := 200:
Functies: constante functie, lineaire en kwadratische functie
> f1 := t -> t^0:
f2 := t -> t^1:
f3 := t -> t^2:
Tijdpunten (equidistant array/sequence)
> ts := [seq(t, t=0..mmax, 1)]:
Random generators: R1, R2 voor random beginsposities, beginsnelheden
H voor random noise per meetpunt
R1f := rand(-100000.0..100000.0):
R2f := rand(-1000.0..1000.0):
H := rand(-1600.0..1600.0):
lijst van beginposities en beginsnelheden
> R1 := [seq(R1f(), i=0..nmax)]:
R2 := [seq(R2f(), i=0..nmax)]:
Datamatrix (metingen in kolommen van M)
> M := Matrix(mmax+1, nmax, (i, j) -> R1[j]*f1(ts[i]) + R2[j]*
f2(ts[i])-9.8/2*f3(ts[i])+H()):
visualisatie meting i
> i := 101;
pointplot_i := pointplot(ts, M[.,i]):
curveplot_i := plot(ts, R1[i]*map(f1,ts)+R2[i]*map(f2,ts)
-9.8/2*map(f3,ts)):
display(pointplot_i, curveplot_i);
i := 101
```



SVDecomposition

> **U,S,V := svd(M);**

Singular values: 3 grote (orde  $10^6$ ,  $10^5$ ), rest zijn kleiner (orde 20)

> **Diagonal(S);**

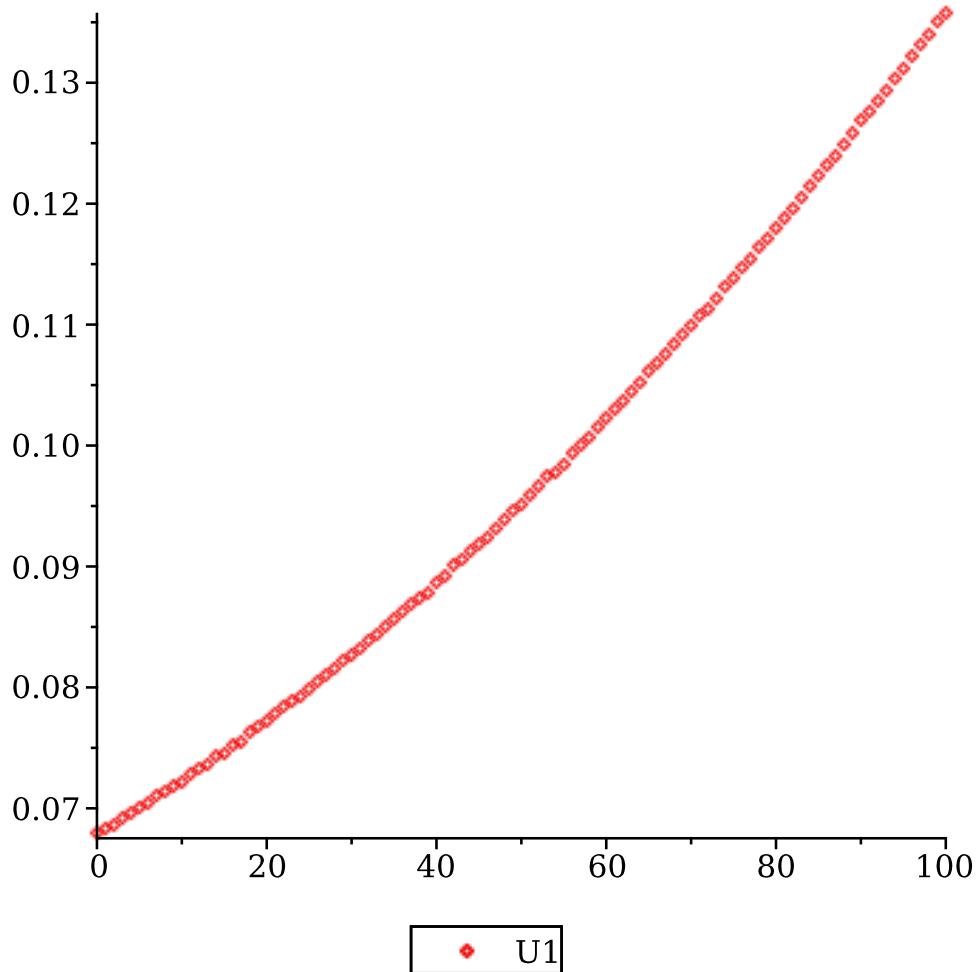
(4.1)

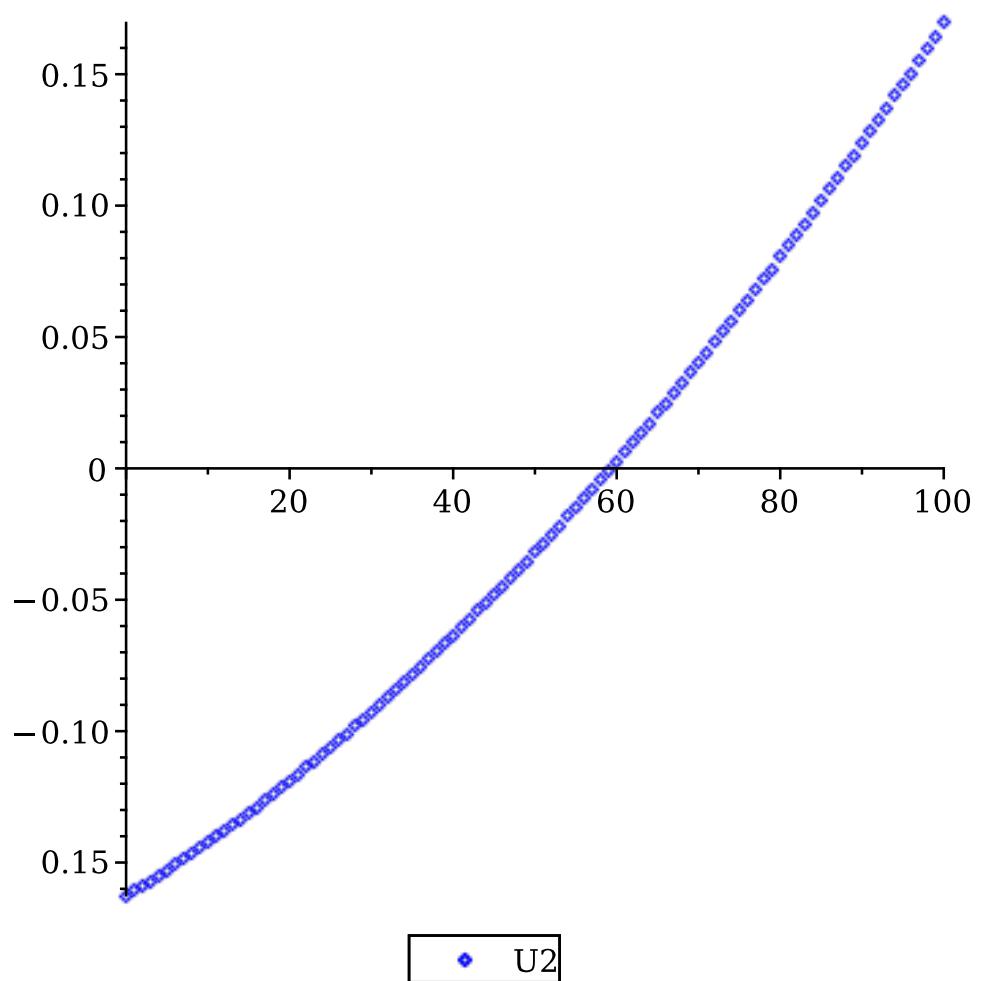
$$\begin{bmatrix} 9.64056374121872 \times 10^6 \\ 2.73723132122265 \times 10^6 \\ 396624.491513215 \\ 22195.3215958040 \\ 21414.8781379108 \\ 20743.5564482751 \\ 20522.3370051525 \\ 20028.2625394204 \\ 19723.4939016135 \\ 19679.6542984042 \\ \vdots \end{bmatrix} \quad (4.1)$$

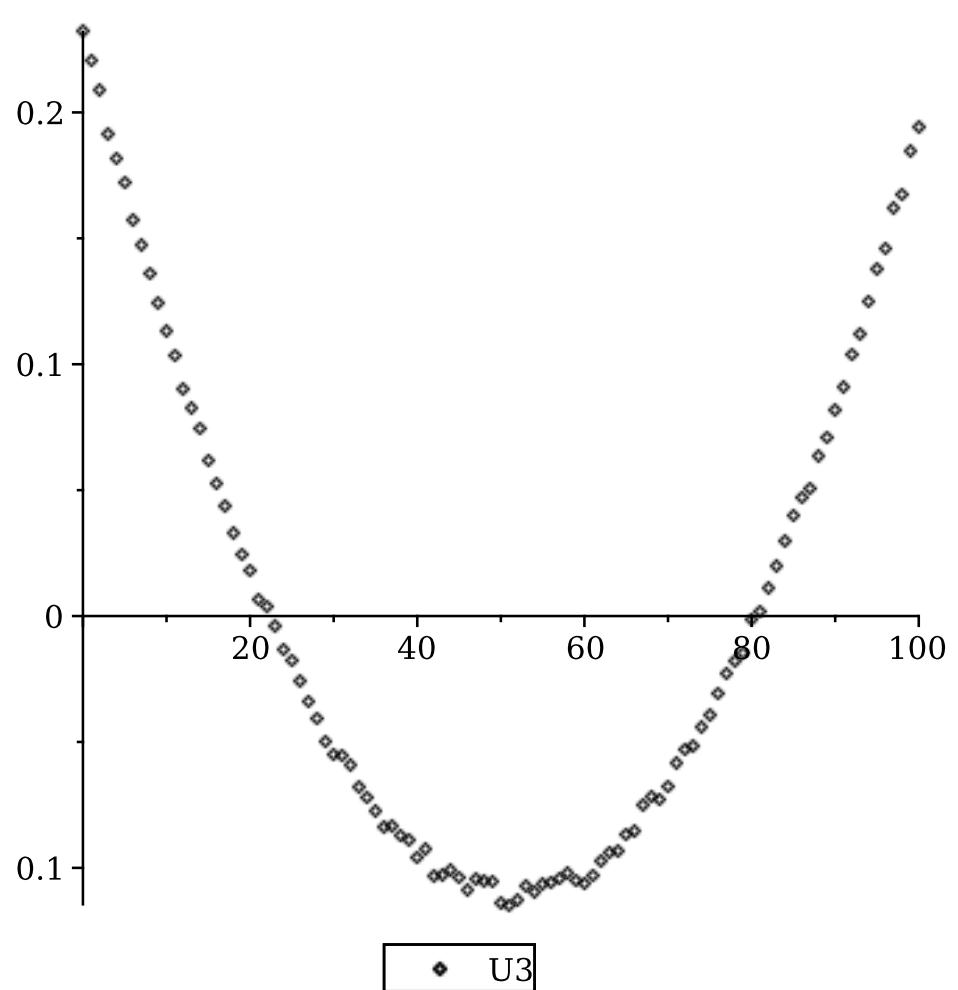
101 element Vector[column]

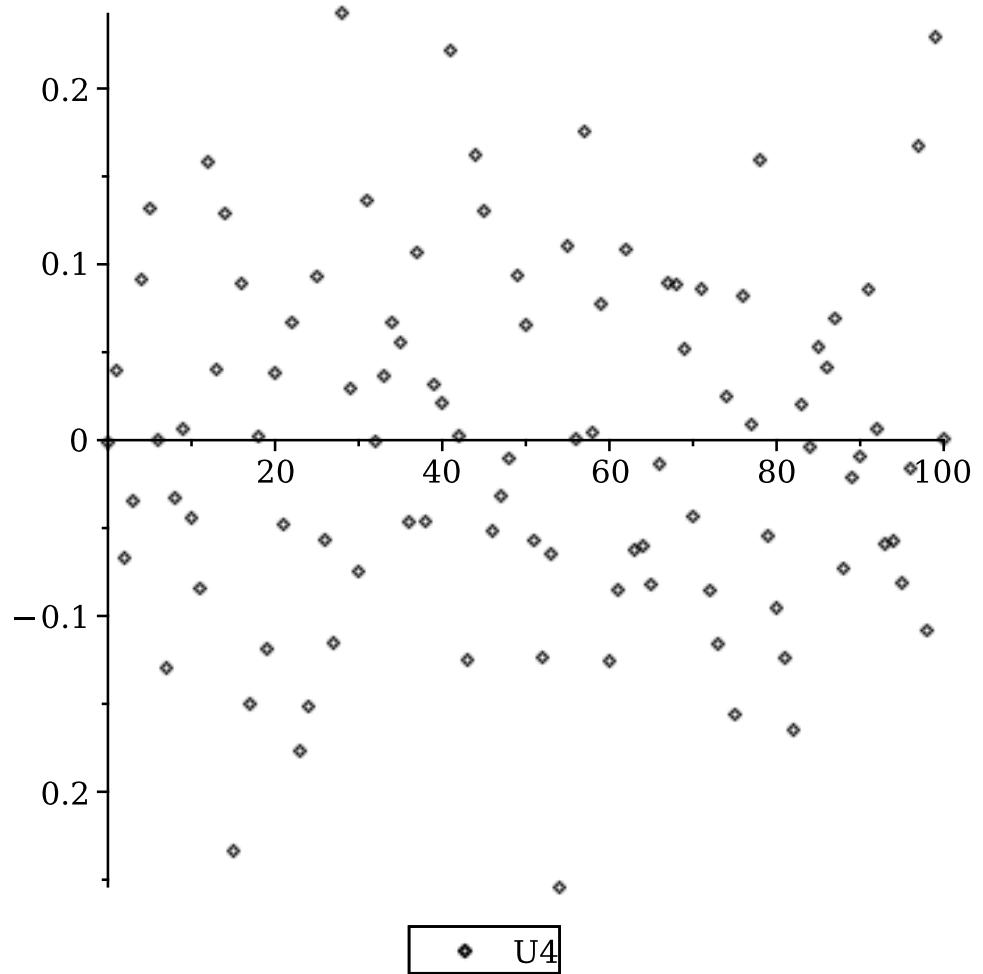
Plot U vectoren bij eerste 3 singular values

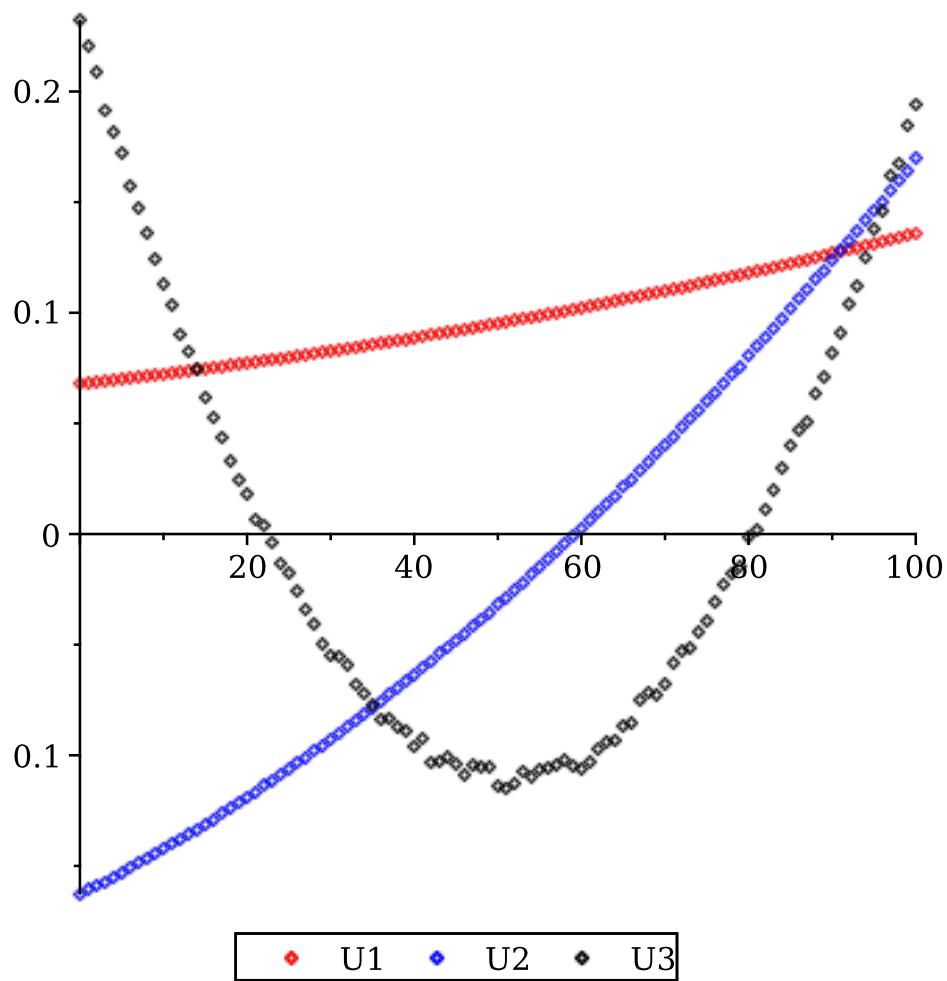
```
> U1plot := pointplot(ts, U[...], color=red, legend="U1");
  U2plot := pointplot(ts, U[...], color=blue, legend="U2");
  U3plot := pointplot(ts, U[...], color=black, legend="U3");
  U4plot := pointplot(ts, U[...], color=black, legend="U4");
  display(U1plot,U2plot, U3plot);
```











```
> i := 55;
```

```

pointplot_i := pointplot(ts, M[..,i]):
curveplot_i := plot(ts, R1[i]*map(f1,ts)+R2[i]*map(f2,ts)
-9.8/2*map(f3,ts)):

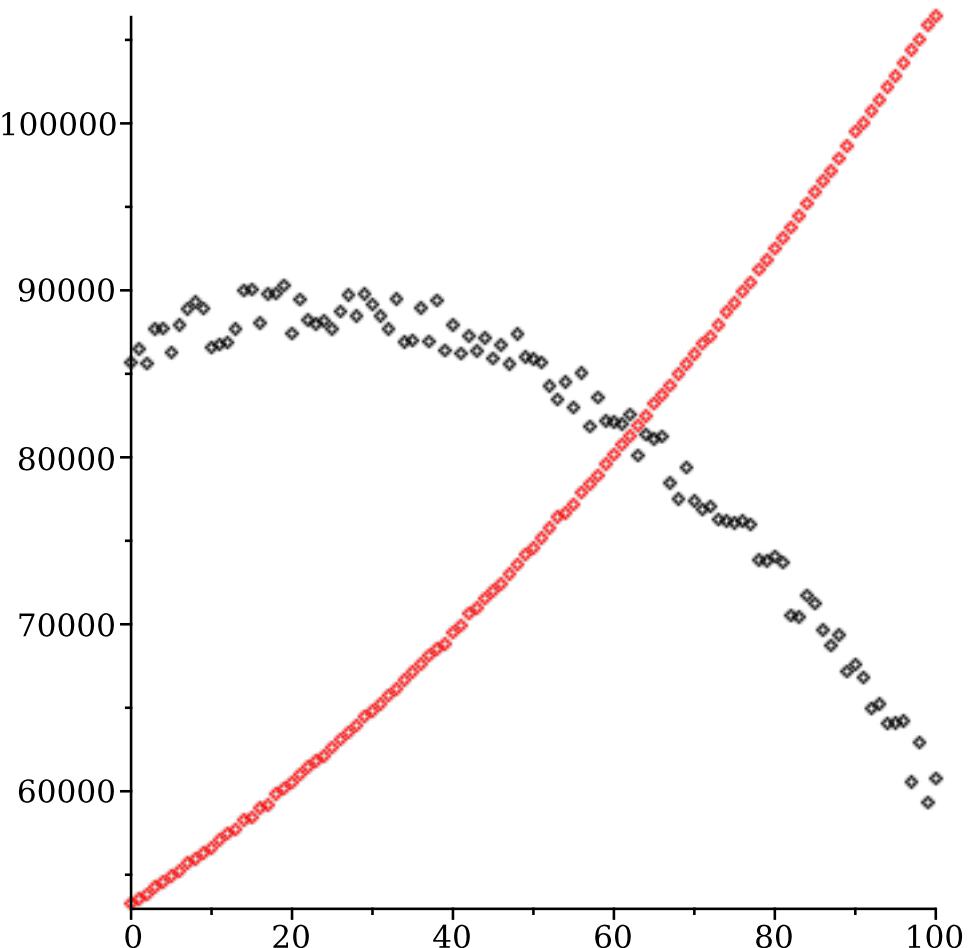
proj1 := (U[..,1].M[..,i]).U[..,1]:
proj2 := (U[..,1].M[..,i]).U[..,1]+(U[..,2].M[..,i]).U[..,2]:
proj3 := (U[..,1].M[..,i]).U[..,1]+(U[..,2].M[..,i]).U[..,2]+(U
[..,3].M[..,i]).U[..,3]:
proj4 := (U[..,1].M[..,i]).U[..,1]+(U[..,2].M[..,i]).U[..,2]+(U
[..,3].M[..,i]).U[..,3]+(U[..,4].M[..,i]).U[..,4]:

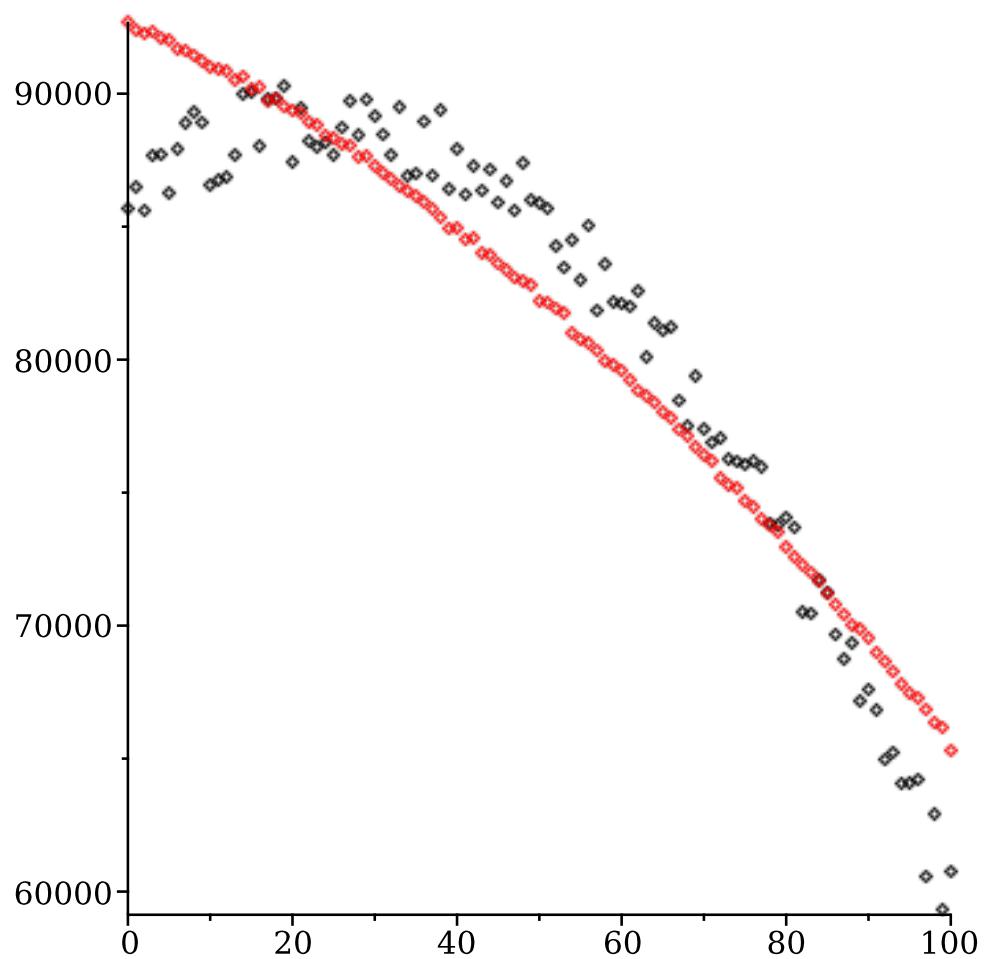
projectieplot1 := pointplot(ts, proj1, color=red):
projectieplot2 := pointplot(ts, proj2, color=red):
projectieplot3 := pointplot(ts, proj3, color=red):
projectieplot4 := pointplot(ts, proj4, color=red):

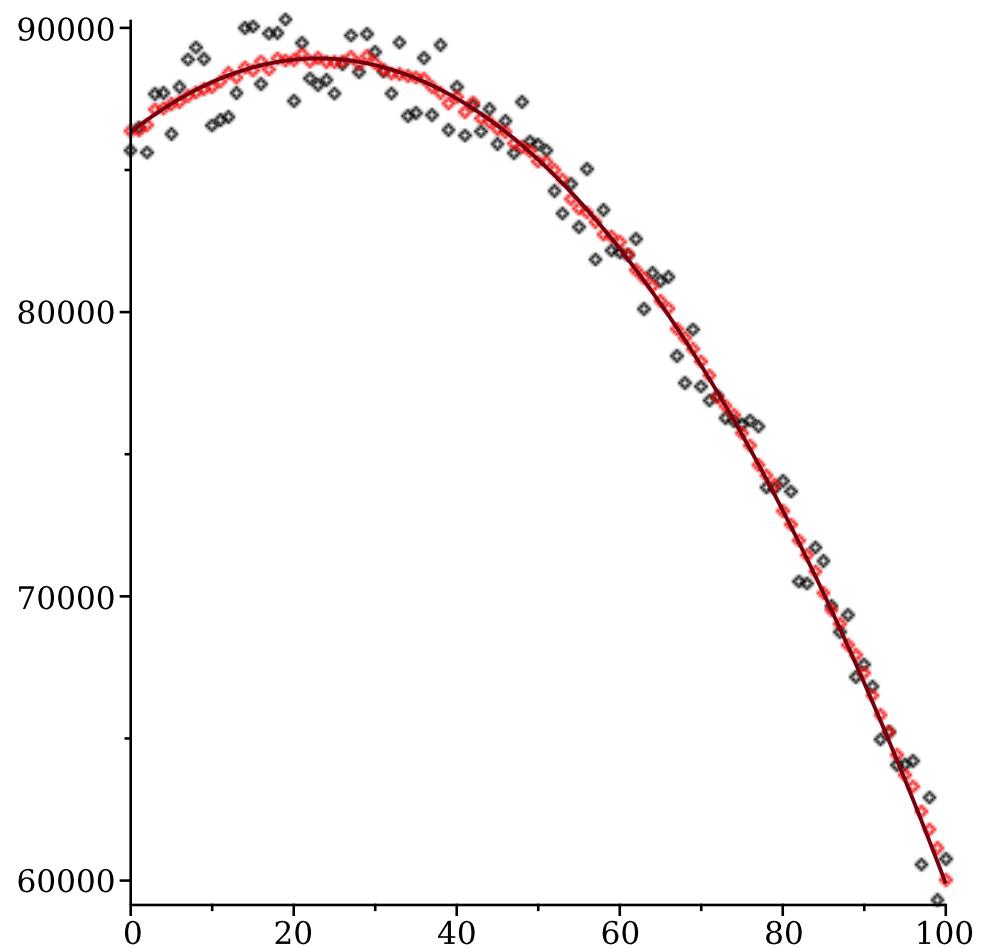
display(pointplot_i, projectieplot1);
display(pointplot_i, projectieplot2);
display(pointplot_i, projectieplot3, curveplot_i);

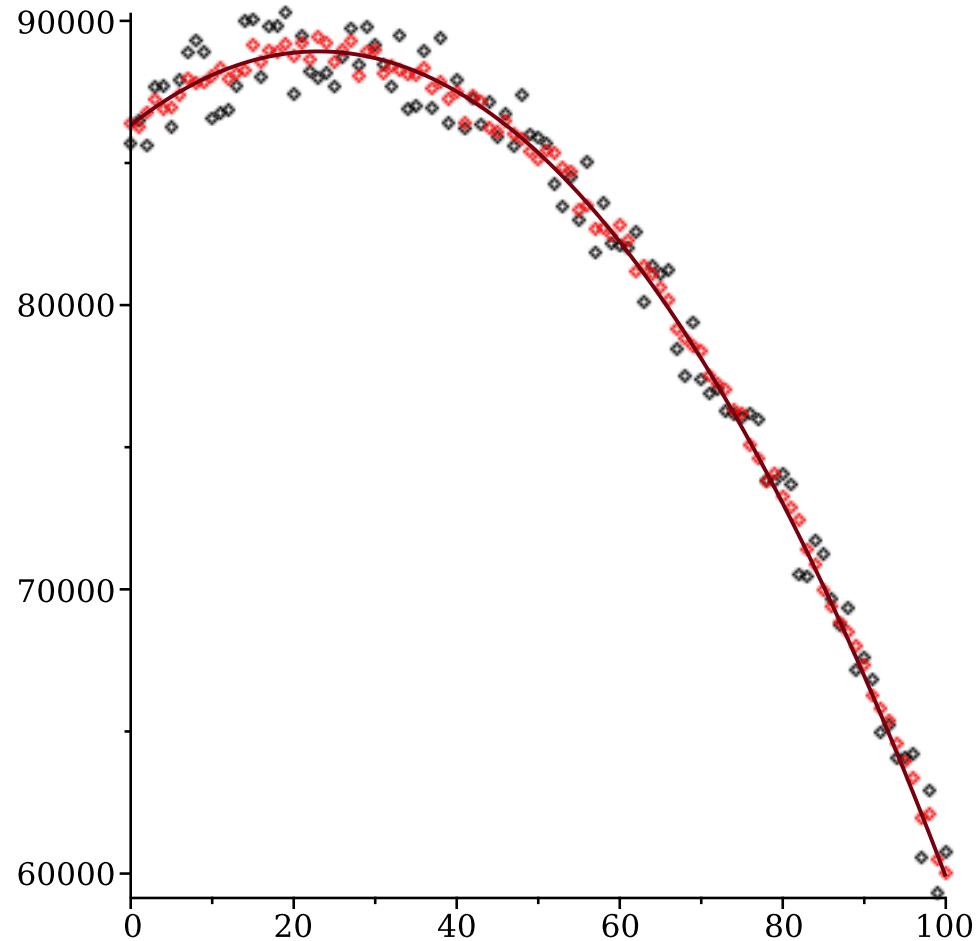
```

```
display(pointplot_i, projectieplot4, curveplot_i);  
i := 55
```









## Noisy oscillator

```

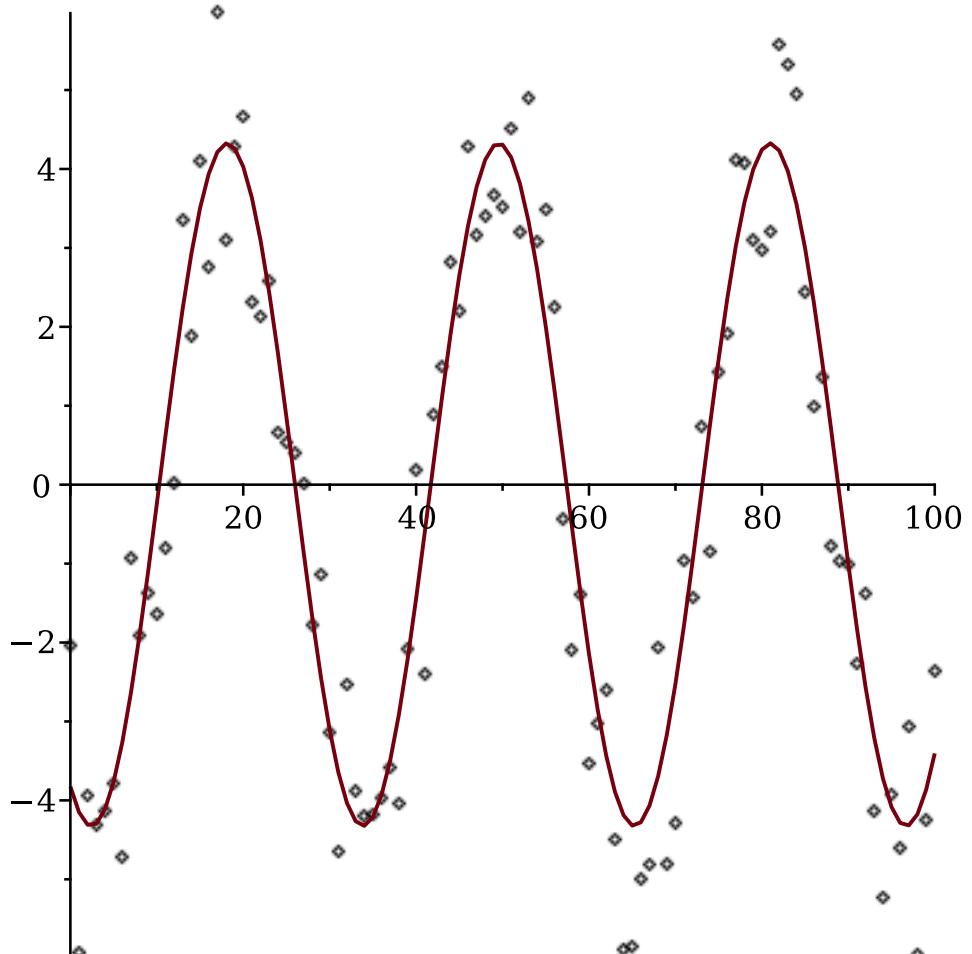
> restart:with(plots):with(LinearAlgebra):with(MTM):
Rijen en kolommen (# datapunten and # metingen)
> mmax := 100:
nmax := 200:
Functies: cosinus en sinus
> f1 := t -> cos(0.2*t):
f2 := t -> sin(0.2*t):
Tijdpunten (equidistant array/sequence)
> ts := [seq(t, t=0..mmax, 1)]:
Random functies: R voor random amplitudes, H voor random noise per
meetpunt
> A := 4.0:
R := rand(-A..A):
H := rand(-1.8..1.8):
lijst van amplitudes voor elke meting
> R1 := [seq(R(), i=0..nmax)]:
R2 := [seq(R(), i=0..nmax)]:
> M := Matrix(mmax+1, nmax, (i, j) -> R1[j]*f1(ts[i]) + R2[j]*f2

```

```

 $(ts[i])+H():$ 
visualisatie meting i
> i := 30;
pointplot_i := pointplot(ts, M[..,i]):
curveplot_i := plot(ts, R1[i]*map(f1,ts)+R2[i]*map(f2,ts)):
display(pointplot_i,curveplot_i);
i := 30

```



```

SVDecomposition
> U,S,V := svd(M):
Singular values: 2 grote (orde 250), rest zijn kleiner (orde 20)
> Diagonal(S);

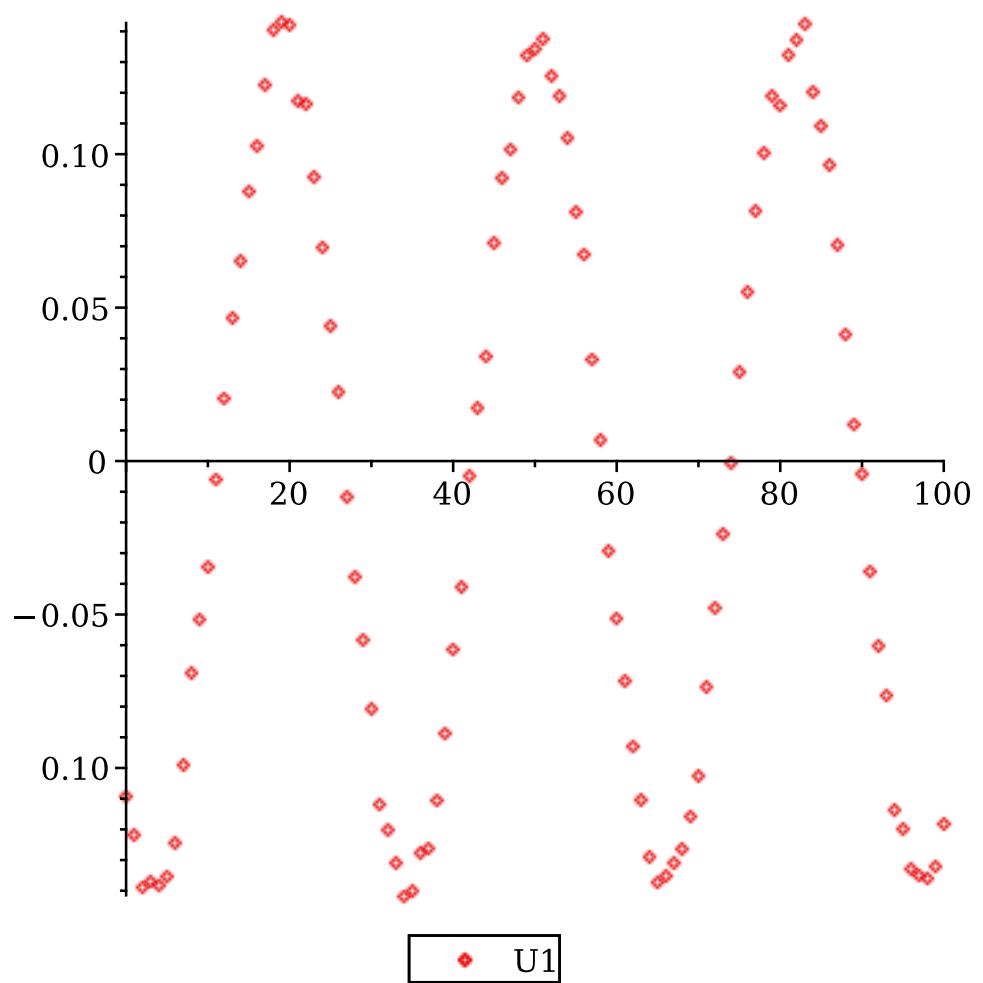
```

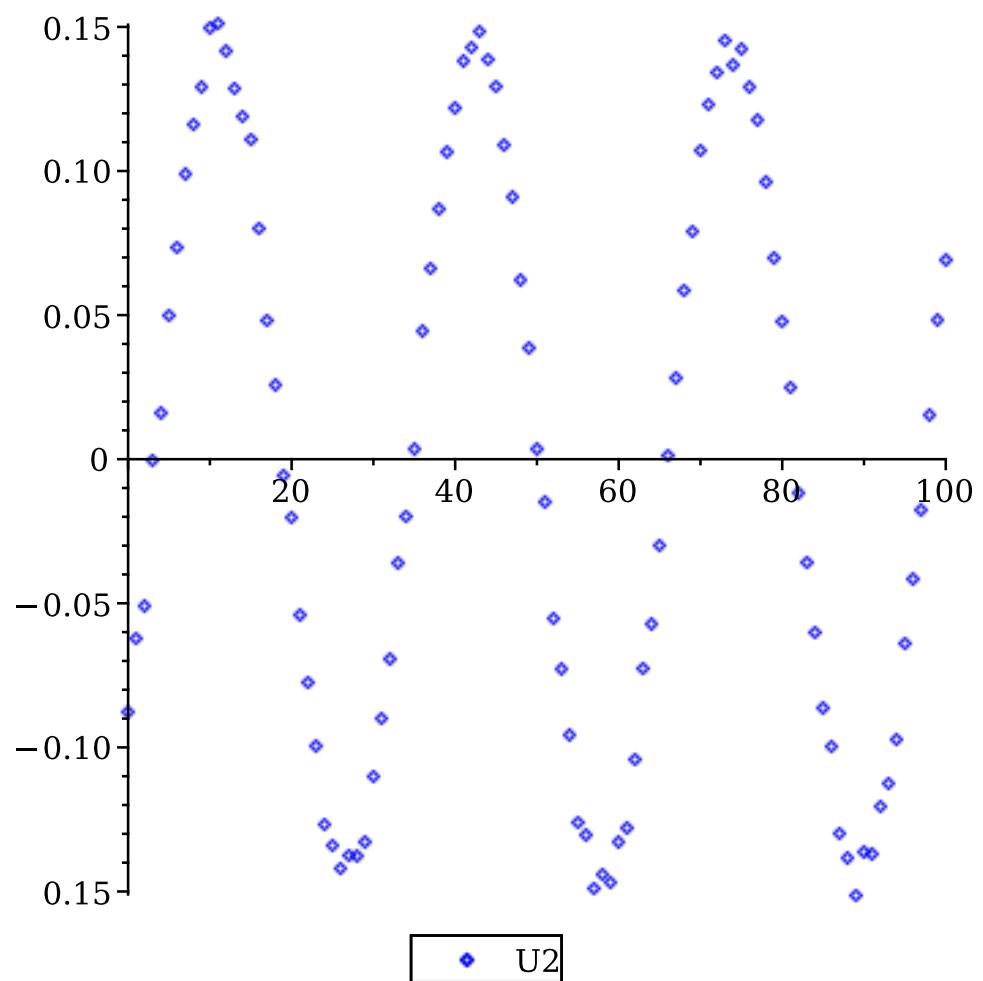
$$\begin{bmatrix}
 255.451392358734 \\
 213.264462492594 \\
 25.1582279645861 \\
 24.0507702090590 \\
 23.9080041033373 \\
 23.1593145814381 \\
 22.9934092535045 \\
 22.1853405869249 \\
 21.8719264275831 \\
 21.6988494211531 \\
 \vdots
 \end{bmatrix} \quad (4.2)$$

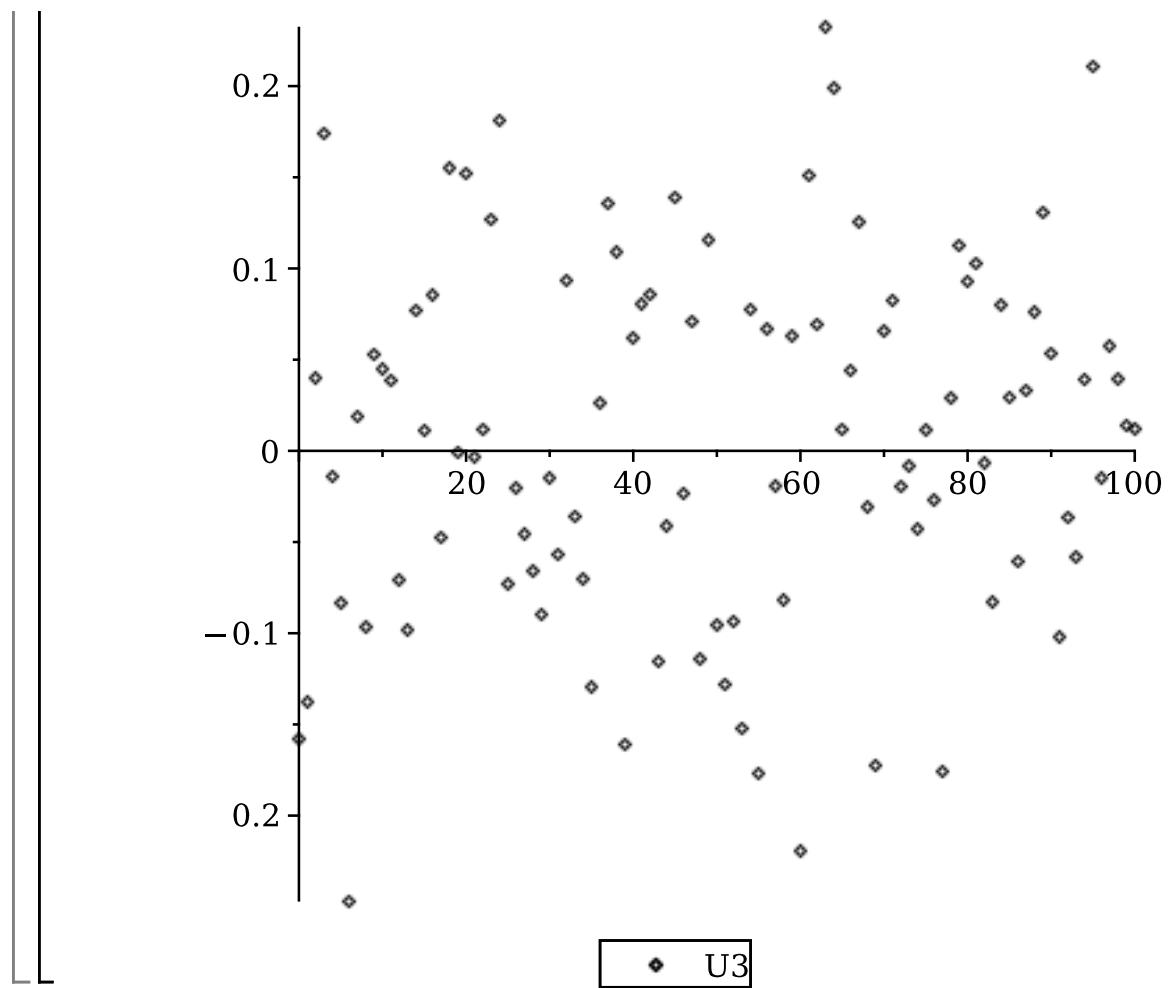
101 element Vector[column]

Plot U vectoren bij eerste 3 singular values

```
> U1plot := pointplot(ts, U[..,1], color=red, legend="U1");
U2plot := pointplot(ts, U[..,2], color=blue, legend="U2");
U3plot := pointplot(ts, U[..,3], color=black, legend="U3");
```







## WC 4

```

> restart: with(LinearAlgebra): with(plots): with(VectorCalculus):
> f := (x,y)→x·y·(x2 - y2 - 1)
      f := (x, y) ↪ x·y·(x2 + (-y2) - 1)                                (1)

> SetCoordinates(cartesian[x, y]):
> grad := Gradient(f(x, y))
      grad := (y(x2 - y2 - 1) + 2x2y)ēx + (x(x2 - y2 - 1) - 2xy2)ēy      (2)

> stationary_points := solve({grad[1] = 0, grad[2] = 0}, {x, y})
stationary_points := {x = 0, y = 0}, {x = 0, y = RootOf(_Z2 + 1)}, {x = 1, y = 0}, {x = -1, y = 0}, {x = -1/2, y = RootOf(_Z2 + 1)/2}, {x = 1/2, y = RootOf(_Z2 + 1)/2}      (3)

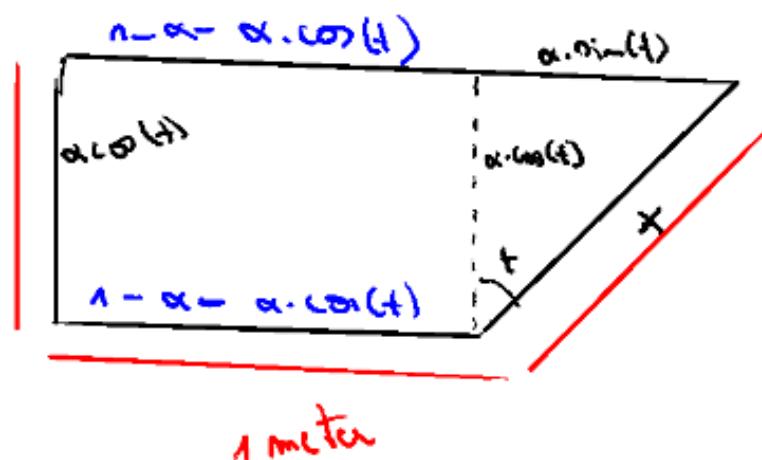
> hessian := Hessian(f(x, y))
      hessian := [ 6xy   3x2 - 3y2 - 1 ]
                  [ 3x2 - 3y2 - 1   -6xy ]                                (4)

> # So now for the first critical point for example:
> Eigenvalues(subs(x = 0, y = 0, hessian))
      [ 1 ]
      [ -1 ]                                (5)

> # So this is a saddle
> # And so forth...

```

Figure 143: Exercise 1



$$opp = \frac{(l + L) \cdot h}{2} = \text{via Maple}$$

Figure 144: Exercise 2

```

> restart: with(LinearAlgebra): with(VectorCalculus):
SetCoordinates(cartesian[x, t]):
> area := 
$$\frac{(1 - x - x \cdot \cos(t) + 1 - x - x \cdot \cos(t) + x \cdot \sin(t)) \cdot x \cdot \cos(t)}{2}$$


$$area := \frac{(2 - 2x - 2x \cos(t) + x \sin(t)) x \cos(t)}{2} \quad (1)$$

> gradient := simplify(Gradient(area))
gradient := 
$$(-2 \cos(t)^2 x + (x \sin(t) - 2x + 1) \cos(t)) \hat{\mathbf{e}}_x + \left( (2x^2 \cos(t) + x^2$$


$$- x) \sin(t) + x^2 \left( \cos(t)^2 - \frac{1}{2} \right) \right) \hat{\mathbf{e}}_t \quad (2)$$

> # Seems like we do not have a direction in the y direction, whatever
> critical_points := (solve({gradient[1] = 0, gradient[2] = 0}, {x, t}))
critical_points := 
$$\left\{ t = \frac{\pi}{2}, x = 0 \right\}, \left\{ t = \arctan(1, \text{RootOf}(\text{Z}^2 - 3)), x$$


$$= \frac{4 \text{RootOf}(\text{Z}^2 - 3)}{3} - 2 \right\}, \left\{ t = \frac{\pi}{2}, x = 2 \right\}, \left\{ t = -\frac{\pi}{2}, x = \frac{2}{3} \right\} \quad (3)$$

> # X needs to be J0, 1[ and t J0, pi:2]
> # seems like point 2 is the only valid one
> # Hessian
> hessian := Hessian(area):
> result := evalf(subs(x =  $\frac{4 \text{RootOf}(\text{Z}^2 - 3)}{3} - 2$ , t = arctan(1, RootOf(_Z^2
- 3)), hessian))
result := 
$$\begin{bmatrix} -2.799038106 & 0.5000000010 \\ 0.5000000010 & -0.1722201661 \end{bmatrix} \quad (4)$$

> Eigenvalues(result)

$$\begin{bmatrix} -2.89099141081943 + 0.1 \\ -0.0802668612805653 + 0.1 \end{bmatrix} \quad (5)$$

> # These two values are < 0, so they define the maximum, so the second
critical point is the maximum area.
> max_area := evalf(subs(t = arctan(1, RootOf(_Z^2 - 3)), x
=  $\frac{4 \text{RootOf}(\text{Z}^2 - 3)}{3} - 2$ , area))
max_area := 0.1339745961 \quad (6)

```

Figure 145: Exercise 2

Huis 5

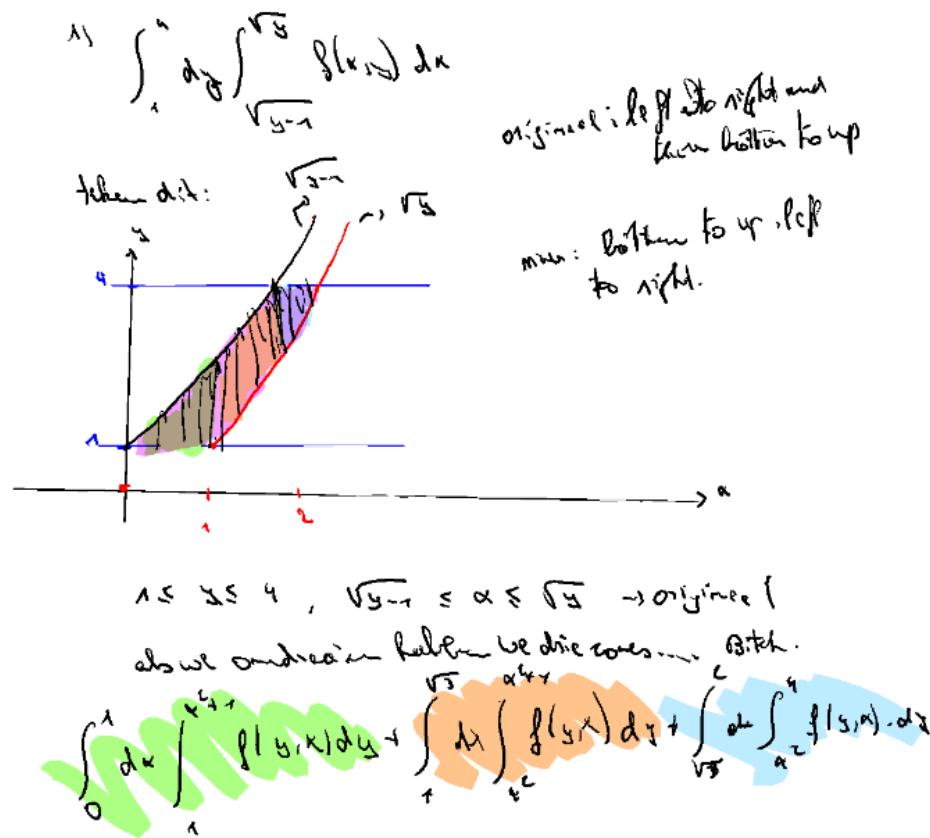


Figure 146: Exercise 1

```

> restart: with(VectorCalculus): with(LinearAlgebra):
> J:=Jacobian([r*sin(theta)*cos(phi), r*sin(theta)*sin(phi), r*cos(theta)], [r,
theta, phi])

$$J := \begin{bmatrix} \sin(\theta) \cos(\phi) & r \cos(\theta) \cos(\phi) & -r \sin(\theta) \sin(\phi) \\ \sin(\theta) \sin(\phi) & r \cos(\theta) \sin(\phi) & r \sin(\theta) \cos(\phi) \\ \cos(\theta) & -r \sin(\theta) & 0 \end{bmatrix} \quad (1)$$

> det := abs(Determinant(J))
det := |\sin(\theta)^3 \cos(\phi)^2 r^2 + \sin(\theta)^3 \sin(\phi)^2 r^2 + \sin(\theta) \cos(\phi)^2 \cos(\theta)^2 r^2
+ \sin(\theta) \sin(\phi)^2 \cos(\theta)^2 r^2|
> simplify(det)
|r|^2 |\sin(\theta)| \quad (2)
> # et voila ;0
>

```

Figure 147: Exercise 2

```

> with(VectorCalculus): with(LinearAlgebra):
> J := Jacobian([a·r·sin(theta)·cos(phi), b·r·sin(theta)·sin(phi), c·r·cos(theta)],
  [r, theta, phi])
      
$$J := \begin{bmatrix} a \sin(\theta) \cos(\phi) & a r \cos(\theta) \cos(\phi) & -a r \sin(\theta) \sin(\phi) \\ b \sin(\theta) \sin(\phi) & b r \cos(\theta) \sin(\phi) & b r \sin(\theta) \cos(\phi) \\ c \cos(\theta) & -c r \sin(\theta) & 0 \end{bmatrix} \quad (1)$$

> J := Determinant(J)
J := a \sin(\theta)^3 \cos(\phi)^2 b r^2 c + b \sin(\theta)^3 \sin(\phi)^2 a r^2 c
      + \sin(\theta) \cos(\phi)^2 \cos(\theta)^2 a b c r^2 + \sin(\theta) \sin(\phi)^2 \cos(\theta)^2 a b c r^2 \quad (2)
> result := simplify(J)
      result := a b c r^2 \sin(\theta) \quad (3)
> # Dit is de jacobiaan bitches
> # In een sferische situatie is r: 0..1, phi:0..2pi, theta:0..pi
> output := int(int(int(result, r = 0..1), theta = 0..Pi), phi = 0..2·Pi)
      output :=  $\frac{4 a b c \pi}{3}$  \quad (4)
> simplify(output)
      
$$\frac{4 a b c \pi}{3} \quad (5)$$


```

Figure 148: Exercise 3

$$I = \int_{-\infty}^{+\infty} \exp(-x^2) dx$$

gelende Umwandlung:  $r = x^2 + y^2$

$$I^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy$$

$$= \int_0^{2\pi} \int_0^{+\infty} e^{-r^2} \cdot r dr d\theta$$

setz  $r^2 = u$   
 $2r dr = du$   
 bounds blieben bestehen

$$= \frac{2\pi}{2} \int_0^{+\infty} e^{-u} \underbrace{du}_{\approx 1}$$

$$I^2 = \pi$$

$$\Rightarrow I = \sqrt{\pi}$$

Figure 149: Exercise 4

```
[> with(VectorCalculus): with(LinearAlgebra):
[> integral := int(exp(-x^2), x = -infinity..infinity)
[> integral :=  $\sqrt{\pi}$  (1)
```

Figure 150: Exercise 4 Maple

$$1 \quad \overbrace{x^2 + y^2}^g + z^2 = a^2$$

$$z = \rho \cos \theta$$

$$z = \epsilon$$

↳ cilinderconen kast  $\Rightarrow \int p \, dp \, dz \, d\theta$

$$\int_0^{2\pi} \left( \int_0^r \int_0^{\sqrt{a^2 - z^2}} p \, dp \, dz \, d\theta \right)$$

Conventie

$$= \pi \left( a^2(r - \epsilon) - \frac{r^2}{2} + \frac{\epsilon^3}{3} \right)$$

via Maple

Figure 151: Exercise 5

```

[> restart: with(LinearAlgebra):
[> integral := int(int(int(rho,rho = 0..sqrt(a^2 - z^2)),z = c..b),theta = 0..2·Pi)
      integral := a^2 (b - c) π -  $\frac{b^3 \pi}{3} + \frac{c^3 \pi}{3}$                                 (1)

[> simplify(integral)
       $\pi \left( a^2 (b - c) - \frac{b^3}{3} + \frac{c^3}{3} \right)$                            (2)

[>

```

Figure 152: Exercise 5

Bord 5

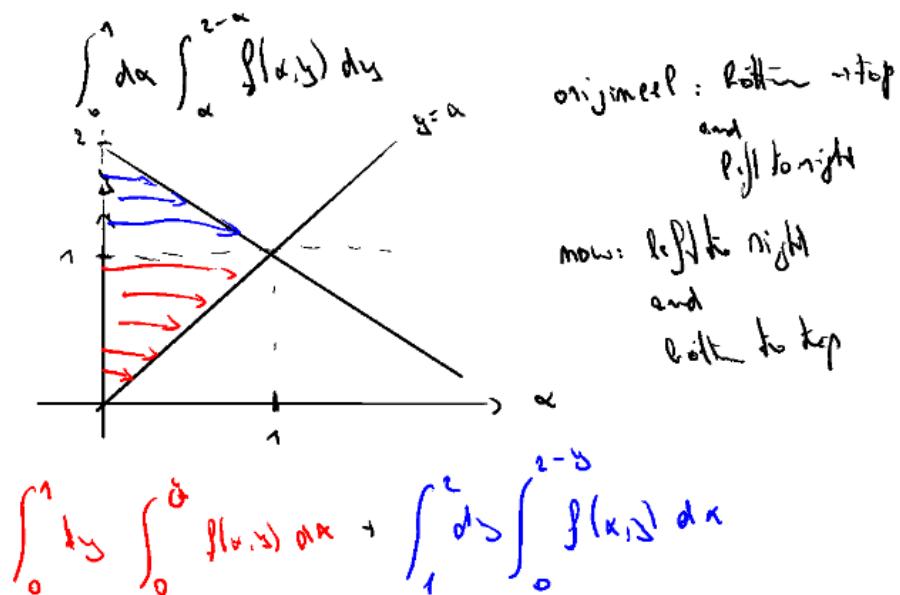
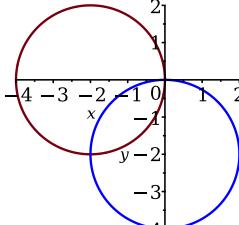


Figure 153: Exercise 1

```

> restart: with(plots):
> cr1:=implicitplot((x+2)^2+y^2=4,x=-4..0,y=-2..2,scaling=
  constrained):
cr2:=implicitplot(x^2+(y+2)^2=4,x=-2..2,y=-4..0,scaling=
  constrained, color=blue):
> display(cr1,cr2);


> # dus dy(dx), y gaat van -2 naar 0, x gaat van blauw naar bruin
> # bepaal limieten voor dx
> eq := y^2 + (x + 2)^2 = 4:
> solve(eq,y):
> # we nemen de negatieve versie
> # hierboven is de lowerbound, nu upperbound
> eq := x^2 + (y + 2)^2 = 4:
> solve(eq,y):
> # pak eerste versie, we gaan naar boven
> # dus result gebruikmakend van cartetische coords is:
> result := int(int((x^2 + y^2)^(3/4), x = -sqrt(-x^2 - 4*x) .. (-2 + sqrt(-x^2 + 4))), y
  = -2 .. 0):
> # Maple kan dit niet calculeren, druk pool coords... [ZIE NOTES]
> evalf(int(int(r*r^(3/2), r = 0 .. -4*cos(theta)), theta = Pi/4 .. 3*Pi/2) + int(int(r
  *r^(3/2), r = 0 .. -4*sin(theta)), theta = Pi .. 5*Pi/4))
4.270011865

```

(1)

Figure 154: Exercise 2

$$\iint_E (x^2 + y^2)^3 \, dx \, dy$$

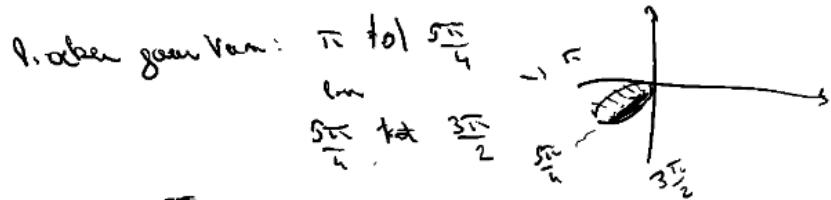
$\downarrow$

$$\begin{aligned} & \text{1) } x^2 + y^2 \leq 4 \quad \text{bzw. } x^2 + (y+2)^2 \leq 4 \\ & \text{2) } x = r \cos \theta \\ & y = r \sin \theta \end{aligned}$$

$$\begin{aligned} & \text{1) } r^2 \cos^2 \theta + 4 \cdot r \cos \theta + 4 + r^2 \sin^2 \theta \leq 4 \\ & \Leftrightarrow r^2 + 4 \cdot r \cos \theta + 4 \leq 4 \\ & \Leftrightarrow r^2 + 4 \cdot r \cos \theta \leq 0 \\ & \Leftrightarrow r \leq -4 \cos \theta \quad \text{2) } r^2 \cos^2 \theta + r^2 \sin^2 \theta + r \sin \theta \cdot 4 + 4 \leq 4 \\ & \Leftrightarrow r^2 \leq -r \sin \theta \cdot 4 \\ & \Leftrightarrow r \leq 4 \sin \theta \rightarrow \text{rechteckig} \end{aligned}$$

linken  
linker  
rechten  
rechter

$r = \sqrt{x^2 + y^2}$



$$\text{dws: } \int_{\pi}^{\frac{5\pi}{4}} d\theta \int_0^{-4 \cos \theta} r \cdot r^3 \, dr + \int_{\frac{5\pi}{4}}^{\frac{3\pi}{2}} d\theta \int_0^{4 \sin \theta} r \cdot r^3 \, dr$$

= 4, 27 of 20.

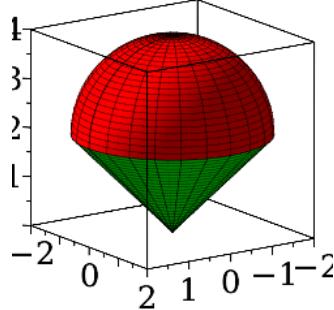
Figure 155: Exercise 2

// Exercise 3

```

> restart: with(plots):
> # first draw the plots
> pkegel:=plot3d([r,phi,Pi/4],r=0..2*sqrt(2),phi=0..2*Pi,coords=spherical,color=green):
pbol:=plot3d([4*cos(theta),phi,theta],phi=0..2*Pi,theta=0..Pi/4,coords=spherical,color=red):
> display(pkegel,pbol)

```



```
> # Eerst sferische coordinaten
```

$$x := r \cdot \sin(\theta) \cdot \cos(\phi) \quad (1)$$

$$y := r \cdot \sin(\theta) \cdot \sin(\phi) \quad (2)$$

$$z := r \cdot \cos(\theta) \quad (3)$$

$$j := r^2 \cdot \sin(\theta) \quad (4)$$

$$\begin{aligned} & \# hoe is onze r? just fill in into sphere eq \\ & sphere\_eq := x^2 + y^2 + (z - a)^2 = a^2 \\ & \quad sphere\_eq := r^2 \sin(\theta)^2 \cos(\phi)^2 + r^2 \sin(\theta)^2 \sin(\phi)^2 + (r \cos(\theta) - a)^2 = a^2 \end{aligned} \quad (5)$$

$$\begin{aligned} & result := solve(sphere\_eq, r) \\ & result := 0, \frac{2 \cos(\theta) a}{\sin(\theta)^2 \cos(\phi)^2 + \sin(\theta)^2 \sin(\phi)^2 + \cos(\theta)^2} \end{aligned} \quad (6)$$

$$\begin{aligned} & simplify(result[2]) \\ & \quad 2 \cos(\theta) a \end{aligned} \quad (7)$$

```

> # so r goes from 0 to 2*cos(theta)*a
> # theta goes from zero to 45 degrees
> # phi is 360 degrees
> # so the mass is:

```

$$\begin{aligned} & M := \text{rho} \cdot \text{int}\left(\text{int}\left(\text{int}(j, r = 0 .. 2 \cdot \cos(\theta) \cdot a), \theta = 0 .. \frac{\pi}{4}\right), \phi = 0 .. 2 \cdot \pi\right) \\ & M := \rho a^3 \pi \end{aligned} \quad (8)$$

```

> # de massa middelpunten zijn dan (zie formules)
>  $x_- := \frac{1}{M} \cdot \text{int}\left(\text{int}\left(\text{int}(\rho \cdot j \cdot x, r = 0..2 \cdot a \cdot \cos(\theta)), \theta = 0..\frac{\pi}{4}\right), \phi = 0..2\pi\right)$ 
>  $x_- := 0$  (9)

>  $y_- := \frac{1}{M} \cdot \text{int}\left(\text{int}\left(\text{int}(\rho \cdot j \cdot y, r = 0..2 \cdot a \cdot \cos(\theta)), \theta = 0..\frac{\pi}{4}\right), \phi = 0..2\pi\right)$ 
>  $y_- := 0$  (10)

>  $z_- := \frac{1}{M} \cdot \text{int}\left(\text{int}\left(\text{int}(\rho \cdot j \cdot z, r = 0..2 \cdot a \cdot \cos(\theta)), \theta = 0..\frac{\pi}{4}\right), \phi = 0..2\pi\right)$ 
>  $z_- := \frac{7a}{6}$  (11)

> # Nu met cilinder coords, basically hetzelfde.
> restart:
>  $x := r \cdot \cos(\theta)$  (12)
>  $y := r \cdot \sin(\theta)$  (13)
>  $z := z$  (14)
>  $j := r$  (15)

> # De theta doet 360
> # De r is van 0 -> a (simply see formula)
> # voor z, steken we de variabelen in de sphere equation
>  $sphere\_eq := x^2 + y^2 + (z - a)^2 = a^2$ 
>  $sphere\_eq := r^2 \cos(\theta)^2 + r^2 \sin(\theta)^2 + (z - a)^2 = a^2$  (16)

>  $result := solve(sphere\_eq, z)$ 
>  $result := a + \sqrt{a^2 - r^2}, a - \sqrt{a^2 - r^2}$  (17)

>  $upperbound\_z := result[1]$ 
>  $upperbound\_z := a + \sqrt{a^2 - r^2}$  (18)

> # dus z is van a tot upperbound_z
> # vul in in de vorige equations

```

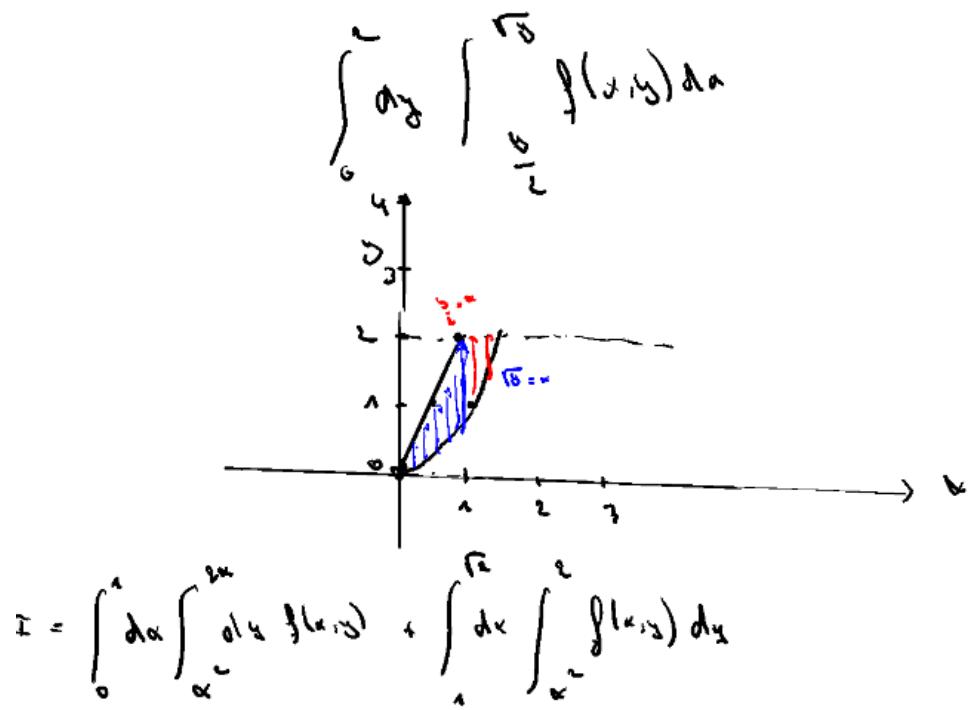


Figure 157: Exercise 1

```

> restart: with(plots): with(plottools):
> A := int(int(c*sin(y)/y, y = x..Pi), x = 0..Pi);
                                         A := 2 c
(1)
> solve(A = 1, c)
                                         1
                                         -
                                         2
(2)
> # I don't understand the issue given in the exercise, this is rather simple...
>

```

Figure 158: Exercise 2

$$x = a \cos \theta$$

$$y = b \sin \theta$$

$$J = \begin{vmatrix} 1 & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ 1 & \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

→ via maple:

10241

$$\int_0^{2\pi} \int_0^a abr \, dr \, d\theta = ab\pi \checkmark$$

Figure 159: Exercise 3

```

> restart: with(VectorCalculus): with(LinearAlgebra):
> j:=Jacobian([a·r·cos(theta), b·r·sin(theta)], [r, theta])
      
$$j := \begin{bmatrix} a \cos(\theta) & -a r \sin(\theta) \\ b \sin(\theta) & b r \cos(\theta) \end{bmatrix}$$
 (1)
> j:=Determinant(j)
      
$$j := a \cos(\theta)^2 b r + a r \sin(\theta)^2 b$$
 (2)
> simplify(j)
      
$$a b r$$
 (3)
> int(int(a·b·r, r = 0..1), theta = 0..2·Pi)
      
$$a b \pi$$
 (4)
>

```

Figure 160: Exercise 3 Maple

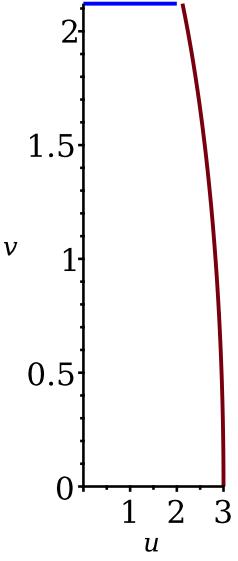
// Exercise 4

```

> restart : with(VectorCalculus) : with(LinearAlgebra) : with(plots) :
> x := r·cos(theta) :
> y := r·sin(theta) :
> z := 'z':
> # Find z upperbound
> z_eq := z =  $\frac{(x^2 - y^2)}{3}$  :
> solve(z_eq, z)

$$\frac{r^2 \cos(\theta)^2}{3} - \frac{r^2 \sin(\theta)^2}{3} \quad (1)$$

> # dit is eigenlijk:  $(r^2 \cdot \cos(2\cdot\theta))$  divided by 3 (ignore this shit formatting)
> j := r:
> plot_1 := implicitplot(u = sqrt(-v^2 + 9), u = 0 .. 3, v = 0 ..  $\frac{3}{\sqrt{2}}$ ) :
> plot_2 := plot(3/sqrt(2), u = 0 .. 2, color = blue):
    # Not fully aligned but you get the point lol
> display(plot_1, plot_2)


> # De boog (dat rechter stuk) is van 0 tot 3, dit is r.
> # dus per definitie is deze cirkel:  $x^2 + y^2 = 9$  ( $r = 3$ )
> # theta gaat van 0 tot de grens van  $3:\sqrt{2}$ 
> arctan( $\frac{\frac{3}{\sqrt{2}}}{\frac{3}{\sqrt{2}}}$ )

$$\frac{\pi}{4} \quad (2)$$

> # voor de blauwe kromme zetten we de coords in y:(3:sqrt(2))

```

>  $r\_blauw\_eq := y = \frac{3}{\sqrt{2}}$

>  $r\_blauw\_eq := r \sin(\theta) = \frac{3\sqrt{2}}{2}$  (3)

>  $solve(r\_blauw\_eq, r)$

$\frac{3\sqrt{2}}{2 \sin(\theta)}$  (4)

> # dus r gaat van 0 tot het bovenste  
 > # voor theta zou dit redelijk duidelijk moeten zijn, we gaan van pi:4 tot pi:2  
 > #Dus alles samengenomen (we nemen dz dr dtheta voor de simplificatie)  
 >  $result := int(int(int(r \cdot j, z = 0 .. \frac{r^2 \cdot \cos(2 \cdot \theta)}{3}), r = 0 .. 3), \theta = 0 .. \frac{\pi}{4}) + int(int(int(r \cdot j, z = 0 .. \frac{r^2 \cdot \cos(2 \cdot \theta)}{3}), r = 0 .. \frac{3\sqrt{2}}{2 \sin(\theta)}), \theta = \frac{\pi}{4} .. \frac{\pi}{2})$   
 >  $result := \frac{243}{32} + \frac{81\sqrt{2} \ln(\sqrt{2} - 1)}{64}$  (5)

Huis 6

$$\text{i) } \vec{\nabla}(\vec{F} \cdot \vec{G}) = J_x(F_x \cdot G_x - G_x F_x) + J_y(F_y \cdot G_y - G_y F_y) \\ + J_z(F_z \cdot G_z - G_z F_z) \\ = 0$$

$$\text{ii) } \vec{\nabla} \cdot (\vec{\nabla} \phi) = 0 \\ = \begin{vmatrix} \vec{\nabla}^2 & & \\ J_x & J_y & J_z \\ J_x \phi & J_y \phi & J_z \phi \end{vmatrix} = \vec{\epsilon}_x^T (\underbrace{J_y J_z \phi - J_z J_y \phi}_{\text{hier zijn de gelijke delen van de matrix}}) + \dots$$

$$\text{iii) } \vec{\nabla} \cdot (f(|\vec{x}|) \vec{x}) \\ = \vec{\nabla} \cdot f(|\vec{x}|) \cdot \vec{x} + f(|\vec{x}|) \cdot \vec{\nabla} \cdot \vec{x} \\ = f'(|\vec{x}|) \cdot \vec{x} + f(|\vec{x}|) \frac{\vec{\nabla} \cdot \vec{x}}{|\vec{x}|} \\ = f'(|\vec{x}|) \cdot |\vec{x}| + f(|\vec{x}|) \cdot 3 \quad (\text{Note: } \vec{x} = (x, y, z)) \\ = f'(|\vec{x}|) \cdot |\vec{x}| + 3 \cdot f(|\vec{x}|)$$

Figure 161: Exercise 1

the wave equation:  $c^2 \nabla^2 \vec{U} = \frac{\partial \vec{U}}{\partial t^2}$  mit  $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$

voon E  $\nabla \times (\vec{\nabla} \times \vec{E}) = -\partial_t (\nabla \times \vec{B})$

$$\Leftrightarrow \vec{\nabla} \cdot (\vec{\nabla} \times \vec{E}) - \vec{\nabla} \cdot \vec{E} = -\partial_t \mu_0 \epsilon_0 \partial_t \vec{E}$$

$$\Leftrightarrow \vec{\nabla} \cdot \vec{E} = -\partial_t \mu_0 \epsilon_0 \vec{E}$$

$$\Leftrightarrow \vec{\nabla} \cdot \vec{E} = \frac{\vec{E}}{\partial t} \cdot \mu_0 \epsilon_0 \rightarrow \text{harm. well. } L = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

$$\Leftrightarrow c^2 \vec{\nabla} \cdot \vec{E} = \frac{\vec{E}}{\partial t^2} \blacksquare$$

voon B  $\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \mu_0 \epsilon_0 \partial_t \cdot (\vec{\nabla} \times \vec{E})$

$$\Leftrightarrow \vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) - \vec{\nabla} \cdot \vec{B} = -\mu_0 \epsilon_0 \partial_t^2 \cdot \vec{B}$$

$$\Leftrightarrow \vec{\nabla} \cdot \vec{B} = \mu_0 \epsilon_0 \partial_t^2 \cdot \vec{B}$$

$$\Leftrightarrow c^2 \cdot \vec{\nabla} \cdot \vec{B} = \frac{\vec{B}}{\partial t^2} \blacksquare$$

Figure 162: Exercise 2

In de stationaire geval vinden we het volgende:

$$\left\{ \begin{array}{l} \vec{v} \cdot \vec{E} = 0 \\ \vec{v} \cdot \vec{B} = 0 \\ \vec{v} \times \vec{E} = 0 \\ \vec{v} \times \vec{B} = 0 \end{array} \right.$$

We zien ook dat de graad van  $\|B\|$  sterk afneemt.

$$\text{Dus } \|B\| = B \cdot 0$$

De trage van de Russische wortel:

$$J_2^L(B \cdot 0) + J_0^L(B \cdot 0)^2 = (B \cdot B) = A \cdot (B \cdot 0)$$

Hier zie je dat dit moet negatief worden, dus de eigenwaarden blijven positief, dus er vindt geen minimum.

Figure 163: Exercise 3

$\vec{v} \times \vec{B} = 0$  = geen rotatie aan brandpunt.

concreet als  $a_b = 0$  in ESG!

In dit geval kunnen we dit niet normaal pressen tot 1 punt!

Omdat de draad in de wegstaat.

Figure 164: Exercise 4

$\vec{v} = (x, y, z)$

$\partial_\alpha V = 0$

$V(x,y,z) = \left( -\frac{\alpha y}{\lambda_0^L}, \frac{\alpha x}{\lambda_0^L}, 0 \right)$  voor  $\alpha^L + y^L < \lambda_0^L$

$\therefore = \left( \frac{-\alpha y}{\alpha^L + y^L}, \frac{\alpha x}{\alpha^L + y^L}, 0 \right)$  voor  $\alpha^L + y^L > \lambda_0^L$

- i) op de plek waar de tornado het maximale is  $N = 70$
  - ii) Maple
  - iii) omdat binnen de tornado geen ESG is. Omdat binnen de tornado wordt gezien als een gat!
  - IV) Maple

Figure 165: Exercise 5

```

> restart: with(LinearAlgebra): with(VectorCalculus):
> #ii)
> SetCoordinates(cartesian[x,y,z]):
r := sqrt(x^2 + y^2):

> vector_field_outside := VectorField([[-alpha*y/r^2], [alpha*x/r^2], 0]):
> simplify(Curl(vector_field_outside))
(0)ēx + (0)ēy + (0)ēz (1)

> # Here we see its not = 0 !
> vector_field_inside := VectorField([-alpha*x/(x^2 + y^2), alpha*x/(x^2 + y^2), 0]):
> Curl(vector_field_inside)
(0)ēx + (0)ēy +  $\left(\frac{\alpha}{x^2 + y^2} - \frac{2\alpha x^2}{(x^2 + y^2)^2} - \frac{2\alpha xy}{(x^2 + y^2)^2}\right)\hat{e}_z$  (2)

> # Not zero!
> # IV
> restart: with(LinearAlgebra): with(VectorCalculus):
SetCoordinates(cartesian[x,y,z]):
> # Define the components of the velocity field symbolically
vx := -alpha*y/(x^2 + y^2):
vy := alpha*x/(x^2 + y^2):
vz := 0:

# Parameterize x and y for the circular path in terms of r and theta
x_expr := r*cos(theta):
y_expr := r*sin(theta):

# Substitute x and y expressions into the vector field components
vx_param := subs(x = x_expr, y = y_expr, vx):
vy_param := subs(x = x_expr, y = y_expr, vy):

# Compute dr as the derivative of the parameterized curve
dr := diff([x_expr, y_expr, 0], theta):

# Calculate the integrand as the dot product of [vx_param, vy_param, vz]
# and dr
integrandum := simplify(vx_param*dr[1] + vy_param*dr[2] + vz*dr[3]):

# Evaluate the integral over theta from 0 to 2*Pi
result := int(integrandum, theta = 0 .. 2*Pi);
result := 2απ (3)

```

Figure 166: Exercise 5

Abs  $\vec{f}$  continu diff is lin  $\nabla \times \vec{F} = 0$  im ESG

$\Rightarrow$  conservatief veld

$$\vec{F}(x,y) = x(x^2-y^2)\vec{e}_x - y(x^2-y^2)\vec{e}_y$$

$\hookrightarrow$  het domein van dit veld is  $R^2$  dus is het een  
ESG (name van  $R$  en  $R^3$ )

Roter in Maple:  $\vec{F} \rightarrow$  conservative

$$\left. \begin{array}{l} \frac{\partial}{\partial x} \Phi_x = x \cdot (x^2-y^2) \\ \frac{\partial}{\partial y} \Phi_y = y \cdot (x^2-y^2) \end{array} \right\} \Rightarrow \Phi = \int x(x^2-y^2) dx$$

$$\left. \begin{array}{l} \frac{\partial}{\partial x} \Phi_x = x \cdot (x^2-y^2) \\ \frac{\partial}{\partial y} \Phi_y = y \cdot (x^2-y^2) \end{array} \right\} \Rightarrow \Phi = \int y(x^2-y^2) dy$$

werd uitgewerkt in Maple.

Figure 167: Exercise 6

```

> restart: with(LinearAlgebra): with(VectorCalculus):
    SetCoordinates(cartesian[x,y,z]):
> f := VectorField([x*(x^2-y^2), -y*(x^2-y^2), 0]):
    f := (x(x^2-y^2))e_x + (-y(x^2-y^2))e_y + (0)e_z      (1)
> Curl(f)
    (0)e_x + (0)e_y + (0)e_z                                (2)
> # Here we proved that the Rotor is nul, dus we hebben een conservatief
   veld!
> # Nu berekenen we phi
  restart: with(LinearAlgebra): with(VectorCalculus):
  SetCoordinates(cartesian[x,y,z]):
> phi_x := int(x^3 - x*y^2, x);

phi_x :=  $\frac{1}{4}x^4 - \frac{1}{2}x^2y^2$                                 (3)

> phi := (x^4/4) - (x^2*y^2/2) + f(y);
    phi :=  $\frac{x^4}{4} - \frac{x^2y^2}{2} + f(y)$                 (4)

> eq := diff(phi, y) = -x^2*y + y^3;
    eq :=  $-x^2y + \frac{d}{dy}f(y) = -x^2y + y^3$             (5)

> f_prime := solve(eq, diff(f(y), y));
    f_prime :=  $y^3$                                          (6)

> f_y := int(f_prime, y) # + Const
    f_y :=  $\frac{y^4}{4}$                                        (7)

> phi := (x^4/4) - (x^2*y^2/2) + f_y; # +Const
    phi :=  $\frac{1}{4}x^4 - \frac{1}{2}x^2y^2 + \frac{1}{4}y^4$     (8)

```

Figure 168: Exercise 6

$$\vec{F}(x_1, x_2, z) = [z^2, x_2, 5]$$

↳ ist kontinuierlich differenzierbar

↳ ESG want  $\mathbb{R}^3$

$$\nabla \cdot \vec{F} = 0 + 0 + 0 = 0$$

$$\rightarrow \vec{F} \text{ ein solenoidal!}$$

würde man fragen ob  $A_1, A_2$  Ablöse ob

$$\left\{ \begin{array}{l} i_x = x_1 \\ i_y = x_2 \\ i_z = 0 \end{array} \right. \quad \left\{ \begin{array}{l} \frac{\partial}{\partial x} A_3 - \frac{\partial}{\partial y} A_2 = z^2 \\ \frac{\partial}{\partial z} A_1 - \frac{\partial}{\partial x} A_2 = 0 \\ \frac{\partial}{\partial y} A_1 - \frac{\partial}{\partial z} A_2 = 0 \end{array} \right.$$

$$\left\{ \begin{array}{l} A_1 = \int x_2 \, dz = \frac{x_2^2}{2} + C_1 \\ A_2 = - \int z^2 \, dz = -\frac{z^3}{3} + C_2 = 0 \end{array} \right.$$

$$(3) \Rightarrow \frac{\partial}{\partial z} \left( \frac{x_2^2}{2} + C_1 \right) - \frac{\partial}{\partial x} \left( -\frac{z^3}{3} + C_2 \right) = y^2$$

~~red~~  $C_1 = 0$

$$(\text{or}) \quad C_1 = \int y^2 \cdot dz = \boxed{-\frac{y^3}{3}}$$

$$\text{dann } A = \left[ \frac{x_2^2}{2} - \frac{y^3}{3}, -\frac{z^3}{3}, 0 \right]$$

↳ ist das divergenzfrei?

$$\nabla \cdot A = \frac{\partial}{\partial x} \cdot 0 = 0 \text{ das stimmt.}$$

Mean  $A' = \vec{A} + \Delta \Phi$  ist wel divergenzfrei als  $\vec{A}$

$$\vec{a} = \vec{\nabla} \cdot A' = \vec{\nabla} \cdot \vec{A} + \Delta \Phi = \frac{z^2}{2} + \Delta \Phi \text{ mit } \Delta \Phi = -\frac{z^2}{2}$$

$$\Phi = -\frac{z^2}{24}, \quad \text{dann } \nabla \Phi = \left[ 0, 0, -\frac{z^2}{6} \right]$$

$$\text{dann } A = \left[ \frac{x_2^2}{2} - \frac{y^3}{3}, -\frac{z^3}{3}, -\frac{z^2}{6} \right]$$

Figure 169: Exercise 7

$$\vec{F} = \left[ x^2(a+xz), y^4(b+z-a), z^2(c+u-z) \right]$$

we have  $\vec{\nabla} \cdot \vec{A} = 0$

$$\vec{\nabla} = \vec{\nabla} \phi + \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \cdot \vec{F} = \Delta \phi = 3(x^2 + y^2 + z^2)$$

$$\left\{ \begin{array}{l} \text{Lagrange} \\ \text{div} \end{array} \right. \phi = \left\{ \begin{array}{l} (x^2 + y^2 + z^2) \\ \nabla \phi = \{x^2, y^2, z^2\} \end{array} \right.$$

$$\nabla \times \vec{A} = \vec{F} - \nabla \phi = \left\{ \begin{array}{l} -J_2 A_2 = x^2(b-z) \\ J_2 A_1 = y^2(z-a) \\ J_3 A_1 - J_2 A_2 = z^2(a-y) \end{array} \right.$$

$$\left\{ \begin{array}{l} A_1 = \frac{y^2 z^2}{2} - \frac{x^2 z^2}{2} + \frac{g(x,y)}{2} = 0 \\ A_2 = -\frac{y^2 z^2}{2} + \frac{x^2 z^2}{2} + \frac{g(x,y)}{2} = 0 \\ A_3 = 0 \end{array} \right.$$

$$\vec{A} = \left[ \frac{y^2 z^2}{2} - \frac{x^2 z^2}{2}, -\frac{y^2 z^2}{2} + \frac{x^2 z^2}{2}, 0 \right]$$

$$\hookrightarrow \text{is it divergent free? } \nabla \cdot \vec{A} = 0$$

$$-\frac{y^2 z}{2} - \frac{x^2 z}{2} \neq 0$$

$$\text{we choose } A' = \vec{A} + \nabla \phi \Rightarrow \nabla \cdot A' + \Delta \phi = 0$$

$$\nabla \cdot \vec{A}' + \Delta \phi = 0$$

$$\Rightarrow -\frac{y^2 z}{2} - \frac{x^2 z}{2} + \Delta \phi = 0$$

$$\hookrightarrow \Delta \phi = \frac{y^2 z}{2} + \frac{x^2 z}{2} = \nabla \phi$$

$$\Rightarrow \phi = \frac{1}{12}(x^4 + y^4)$$

$$\nabla \phi = \left\{ \frac{1}{3}x^2 z^3, \frac{1}{3}y^2 z^3, \frac{1}{12}(x^4 + y^4) \right\}$$

$$\vec{A}' = \left[ \frac{y^2 z^2}{2} - \frac{x^2 z^2}{2} + \frac{1}{3}x^2 z^3 - \frac{y^2 z^2}{2} + \frac{x^2 z^2}{2} + \frac{1}{3}y^2 z^3, \frac{1}{12}(x^4 + y^4) \right]$$

then is well divergent free.

Figure 170: Exercise 8

$$\int_C \vec{f} \cdot d\vec{\alpha}$$

with:  $\vec{f}(x, y, z) = -y \cdot \vec{e}_x + x \cdot \vec{e}_y - \frac{z}{2} \vec{e}_z$

$\vec{\alpha}(t) = \cos t \vec{e}_x + \sin(t) \vec{e}_y + 2t \vec{e}_z, t \in [0, \pi]$

$$\frac{d\alpha \cdot dt}{dt} = dt (\cos t \vec{e}_x + \sin(t) \vec{e}_y + 2t \vec{e}_z) \cdot \underline{\underline{J}}$$

$$d\alpha = dt \cdot (-\sin t \vec{e}_x + \cos t \vec{e}_y + 2 \vec{e}_z)$$

Rech.  $\vec{\alpha}(t)$  in  $\vec{f}$

$$\Rightarrow -\sin(t) \vec{e}_x + \cos t \vec{e}_y - \frac{2t}{2} \vec{e}_z$$

$$\Leftrightarrow -\sin(t) \vec{e}_x + \cos t \vec{e}_y - t \vec{e}_z$$

Wu F.  $\frac{d\alpha}{dt} = (-\sin t, -\cos t, \cos t + 2 \cdot (-t)) dt$

$$\Leftrightarrow (1 - 2t) dt$$

$$\Rightarrow \int_0^\pi (1 - 2t) dt = \pi - 2 \cdot \left[ \frac{t^2}{2} \right]_0^\pi$$

$$= 2\pi - 4\pi^2$$

Figure 171: Exercise 9

$$\int_C \vec{f} \cdot d\vec{\omega}$$

$$\vec{F}(x,y) = (x^2 - 2xy) \vec{e}_x + (y^2 - 2xy) \vec{e}_y$$

note  $y = x^2$ , var(-2,4)  $\Rightarrow t \in (-1,1)$

$$\text{note } \alpha = t \Rightarrow y = t^2$$

$$\rightarrow n(t) = t \cdot \vec{e}_x + t^2 \cdot \vec{e}_y$$

$$\boxed{\frac{d}{dt} n(t) = \vec{e}_x + 2t \vec{e}_y}$$

$$\begin{aligned} F(n(t)) \cdot \frac{d}{dt} n(t) &= (t^2 - 2 \cdot t \cdot t^2) \vec{e}_x + (t^4 - 2 \cdot t \cdot t^2) \vec{e}_y \\ &\quad \cdot (\vec{e}_x + 2t \vec{e}_y) \\ &= (t^2 - 2t^3) + (2t^5 - 4t^4) \\ &= \underline{t^2 - 2t^3 - 4t^4 + 2t^5} \end{aligned}$$

$$\text{ans: } \int_{-1}^1 (t^2 - 2t^3 - 4t^4 + 2t^5) dt$$

via maple:  
 $\frac{-365}{10}$

Figure 172: Exercise 10

④

$$\text{M3} \quad \int_C z \cdot \underline{dx} = \sqrt{\underline{dx} \cdot \underline{dx}} = \sqrt{\frac{dx}{dt} \cdot \frac{dx}{dt} \cdot dt^2} = \left\| \frac{dx}{dt} \right\| \cdot dt$$

$$C: \vec{z}(t) = t \cdot (\cos t \vec{e_1} + \sin t \vec{e_2}) + t \cdot \vec{e_3}, \quad 0 \leq t \leq 1$$

$$\wedge \quad z = t$$

$$\Rightarrow \frac{dx(t)}{dt} = (\cos t - t \cdot \sin t) \vec{e_1} + (\sin t + t \cdot \cos t) \vec{e_2} + \vec{e_3}$$

$$\int_0^1 t \cdot \sqrt{(\cos t - t \cdot \sin t)^2 + (\sin t + t \cdot \cos t)^2 + 1} \, dt$$

Run op Vival Maple.

Figure 173: Exercise 11

```
[> restart: with(LinearAlgebra): with(VectorCalculus):  
=> simplify(int(t*sqrt((cos(t)-sin(t)*t)^2 + (sin(t)+t*cos(t))^2 + 1), t = 0..1))  
=> -2*sqrt(2)/3 + sqrt(3) (1)
```

Figure 174: Exercise 11 Maple

$$12) \rho = a, \alpha(\theta) = a(1 + \cos\theta), 0 < \theta \leq \pi, a > 0$$

$$\vec{R}_{CM} = \frac{1}{m} \cdot \int_{\Gamma} \vec{\alpha} \rho(\vec{r}) dV$$

Hier is een driedimensionaal  $\Gamma$ , om dat geschreven we ds

$$\Rightarrow \vec{R}_{CM} = \frac{1}{m} \int_C \vec{\alpha} \rho(\vec{r}) ds$$

$$C: \vec{\alpha}(\theta) = a(\cos\theta(1 + \cos\theta), (\sin\theta)(1 + \cos\theta)), 0 \leq \theta \leq \pi, a > 0$$

$$ds = \left| \left| \frac{d\vec{\alpha}}{d\theta} \right| \right| d\theta$$

$$\frac{d\vec{\alpha}}{d\theta} = \begin{pmatrix} a(-\sin\theta(1 + \cos\theta) - \sin^2\theta \cos\theta) \\ \cos\theta(1 + \cos\theta) - \sin^2\theta \end{pmatrix},$$

$$\int_0^\pi \alpha(\theta) \cdot \vec{p} \cdot \frac{d\vec{\alpha}}{d\theta} d\theta = \int_0^\pi \alpha(\theta) \cdot \left| \left| \frac{d\vec{\alpha}}{d\theta} \right| \right| \cdot d\theta$$

*Nog dit in Maple (daar voor we een  
matrachtig stappenplan.)*

Figure 175: Exercise 12

```

> restart: with(LinearAlgebra): with(VectorCalculus):
> # Step 1: Define parameters and parametric equations
a := 'a': # Keep 'a' as symbolic
x_theta := a*cos(theta)*(1 + cos(theta));
y_theta := a*sin(theta)*(1 + cos(theta));

# Step 2: Calculate the tangent vector components
dx_dtheta := diff(x_theta, theta);
dy_dtheta := diff(y_theta, theta);

# Step 3: Calculate the magnitude of the tangent vector
ds := sqrt(dx_dtheta^2 + dy_dtheta^2);

# Step 4: Define integrals for M, x_integral, and y_integral
M := int(ds, theta = 0 .. Pi):
x_integral := int(x_theta * ds, theta = 0 .. Pi):
y_integral := int(y_theta * ds, theta = 0 .. Pi):

# Step 5: Calculate the center of mass coordinates
x_cm := simplify(x_integral / M):
y_cm := simplify(y_integral / M):

# Step 6: Display the result for the center of mass
Center_of_Mass := [x_cm, y_cm];

x_theta := a cos(theta) (1 + cos(theta))
y_theta := a sin(theta) (1 + cos(theta))
dx_dtheta := -a sin(theta) (1 + cos(theta)) - a cos(theta) sin(theta)
dy_dtheta := a cos(theta) (1 + cos(theta)) - a sin(theta)^2
Center_of_Mass := [  $\frac{4a}{5}$ ,  $\frac{4a}{5}$  ] (1)

```

Figure 176: Exercise 12 Maple

$$\vec{B}(\vec{x}) = \frac{\mu_0 I}{4\pi r} \cdot \int_C \frac{d\alpha' \alpha (\vec{x} - \vec{\alpha}')}{||\vec{\alpha} - \vec{x}'||^3}$$

$$C: \alpha'(t) = (0, 0, t), t \in ]-\infty, +\infty[$$

$$d\alpha' = (0, 0, 1)$$

$$(\vec{\alpha} - \vec{\alpha}') = (\alpha, y, z-t)$$

$$||\vec{\alpha} - \vec{\alpha}'||^3 = (\alpha^2 + y^2 + (z-t)^2)^{3/2}$$

$$d\alpha' \alpha (\vec{x} - \vec{\alpha}') = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ \alpha & y & z-t \end{vmatrix}$$

$$= \left\{ y, -\alpha, 0 \right\}$$

$$\vec{B}_x = \frac{\mu_0 I}{4\pi r} \cdot \int_{-\infty}^{+\infty} \frac{y}{(\alpha^2 + y^2 + (z-t)^2)^{3/2}} dt$$

$$\vec{B}_y = \textcolor{teal}{\sim} \int -\frac{\alpha}{\dots}$$

$$\vec{B}_z = \textcolor{violet}{\sim} \int 0$$

$$\vec{B} = \left\{ \textcolor{red}{y}, \textcolor{blue}{-\alpha}, 0 \right\}$$

Figure 177: Exercise 13

$$(\alpha = 0, \psi = 0, z = r)$$

$$\zeta: \vec{\alpha}(t) = (R \cos \theta, R \sin \theta, 0), \theta \in [0, 2\pi]$$

$$\dot{\alpha} = (-R \sin \theta, R \cos \theta, 0)$$

$$\vec{\alpha} - \vec{\alpha}' = (0 - R \cos \theta, 0 - R \sin \theta, 2)$$

$$= (-R \cos \theta, -R \sin \theta, 2)$$

$$\frac{d\alpha(t)}{dt}, \alpha(\vec{\alpha} - \vec{\alpha}') = (2R \cos \theta, 2R \sin \theta, R^2)$$

$$\|\vec{\alpha} - \vec{\alpha}'\|^2 = (R^2 + 2^2)^{1/2}$$

$$B_x = \dots$$

$$B_y = \dots$$

$$B_z = \dots$$

$$B = \left\{ \dots, \dots, \dots \right\}$$

Figure 178: Exercise 14

$$\text{ellipse: } x^2 + y^2 + \frac{z^2}{r^2} = 1$$

$$S: \alpha(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

$$\theta \in [-\pi, \pi], \phi \in [0, 2\pi]$$


---

$$I = \int_0^{2\pi} \int_{-\pi}^{\pi} \left| \left| \frac{\partial \alpha}{\partial \theta} \times \frac{\partial \alpha}{\partial \phi} \right| \right| d\theta d\phi$$

↳ zic - opte

$$\approx 42,95$$

Figure 179: Exercise 1

$$\int_S \mathbf{F} \cdot \mathbf{n} d\sigma$$

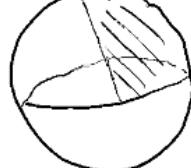
$\mathbf{F}(x, y, z) = x \mathbf{i} + y \mathbf{j}$   
 $x^2 + y^2 + z^2 = 1$  (Kugel obenan)  
 $n = \mathbf{k}$  BTW.

$\mathbf{F} = (x, y, 0)$ , we get the following combination:

$$(x, y, 0) \rightarrow \begin{cases} \text{nimmt } \varphi, \text{ nimmt } r, \text{ nimmt } \theta \\ dV = r^2 \sin \theta dr d\theta d\varphi \rightarrow r \text{ wird Weggeometrie} \\ \text{stehen} \end{cases}$$

$\Rightarrow$  particular case.

$$dr \cdot \Theta = \left[ 0, \frac{\pi}{2} \right] \quad \varphi = \left[ 0, \frac{\pi}{2} \right]$$



$$\iint_S \mathbf{f} \cdot d\mathbf{s} = \iint_S \mathbf{F} \cdot \mathbf{n} d\sigma$$

(it will be calculated via the notes right).

$$m_a = \begin{vmatrix} i & j & k \\ \cos \theta \sin \varphi & \sin \theta \sin \varphi & -\sin \varphi \\ \cos \theta \cos \varphi & \sin \theta \cos \varphi & 0 \end{vmatrix}$$

$$\frac{\partial \mathbf{a}}{\partial \theta} = \frac{\partial \mathbf{r}}{\partial \theta}$$

$$\approx i(\sin \theta \cos \varphi) - j(\sin \theta \sin \varphi) + (\cos \theta \cos \varphi + \sin \theta \sin \varphi)k$$

$$= i(i) - j(j) + k(\cos \theta) \rightarrow \sqrt{a^2 + r^2 + z^2}$$

$$\|\mathbf{m}\| = \sin \theta$$

$$\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma = \int_0^{\pi/2} \int_0^{\pi/2} [\sin \theta \cos \varphi, \sin \theta \sin \varphi, 0] \cdot \frac{M}{\sin \theta} \cdot \sin \theta d\theta d\varphi$$

$$\boxed{\iint_S \frac{\mathbf{F} \cdot \mathbf{n}}{\|\mathbf{m}\|} d\mathbf{s} = \int_0^{\pi/2} \int_0^{\pi/2} \mathbf{F} \cdot \mathbf{n} d\theta d\varphi}$$

jacobian

$$= \frac{\pi}{3}$$

Figure 180: Exercise 2

```

> with(LinearAlgebra): with(VectorCalculus): SetCoordinates(spherical[r,
theta,phi]):
> F := VectorField([sin(theta)*cos(phi), sin(theta)*sin(phi), cos(theta)]):
> n := VectorField([sin^2(theta)*cos(phi), -sin^2(theta)*sin(phi), cos(theta)
*sin(theta)]):
> n_norm := sqrt(n[1]^2 + n[2]^2 + n[3]^2)
n_norm :=  $\sqrt{\sin(\theta)^4 \cos(\phi)^2 + \sin(\theta)^4 \sin(\phi)^2 + \cos(\theta)^2 \sin(\theta)^2}$  (1)
> n := simplify( $\frac{n}{n\_norm}$ )
n := ( $\cos(\phi)$ ) $\hat{e}_r$  + (- $\sin(\phi)$ ) $\hat{e}_{\theta}$  + ( $\cot(\theta)$ ) $\hat{e}_{\phi}$  (2)
> dot_product := simplify(DotProduct(F, n))
dot_product := (2  $\sin(\phi)^2 \cos(\theta)^2$  + 2  $\cos(\phi)^2$  - 1)  $\csc(\theta)$  (3)
> int(int(dot_product * sin^2(theta), theta = 0 ..  $\frac{\text{Pi}}{2}$ ), phi = 0 ..  $\frac{\text{Pi}}{2}$ )
int( $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (2 \sin(\phi)^2 \cos(\theta)^2 + 2 \cos(\phi)^2 - 1) \csc(\theta) \sin^2(\theta) d\theta d\phi$ , (4)

```

Figure 181: Exercise 2

$$\int_C \vec{F} \cdot d\vec{x}$$

$$\vec{F}(x, y, z) = (0, -z, y)$$

$$\alpha(x, y) = (x \cdot \cos(u), x \sin(u), v), 0 \leq u \leq \frac{\pi}{2}, 0 \leq v \leq a$$

$$\int_C \vec{F} \cdot d\vec{x} = \int_S (\nabla \times \vec{F}) \cdot \vec{n} \cdot d\sigma \rightarrow \text{Richtung vom Stoßen}$$

$\downarrow$   
normalvektor folgt aus der Rektan-  
glehnen Regel.

$$\begin{aligned} \vec{n} &= \begin{vmatrix} i & j & k \\ -x \sin(u) + \cos(u) & x \sin(u) + x \cos(u) & 0 \\ 0 & 0 & 1 \end{vmatrix} \\ &\approx (x \sin(u) + x \cos(u), x \sin(u) - x \cos(u), 0) \end{aligned}$$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -z & y \end{vmatrix} = (0, 0, 0)$$

$$\text{daraus } \int_0^{\pi/2} \int_0^u (0, 0, 0) \cdot (x \sin(u) + x \cos(u), x \sin(u) - x \cos(u), 0) dv du$$

$\left( \text{Maple} \right)$

$$= \frac{\pi a^2}{2} - 2$$

Figure 182: Exercise 3

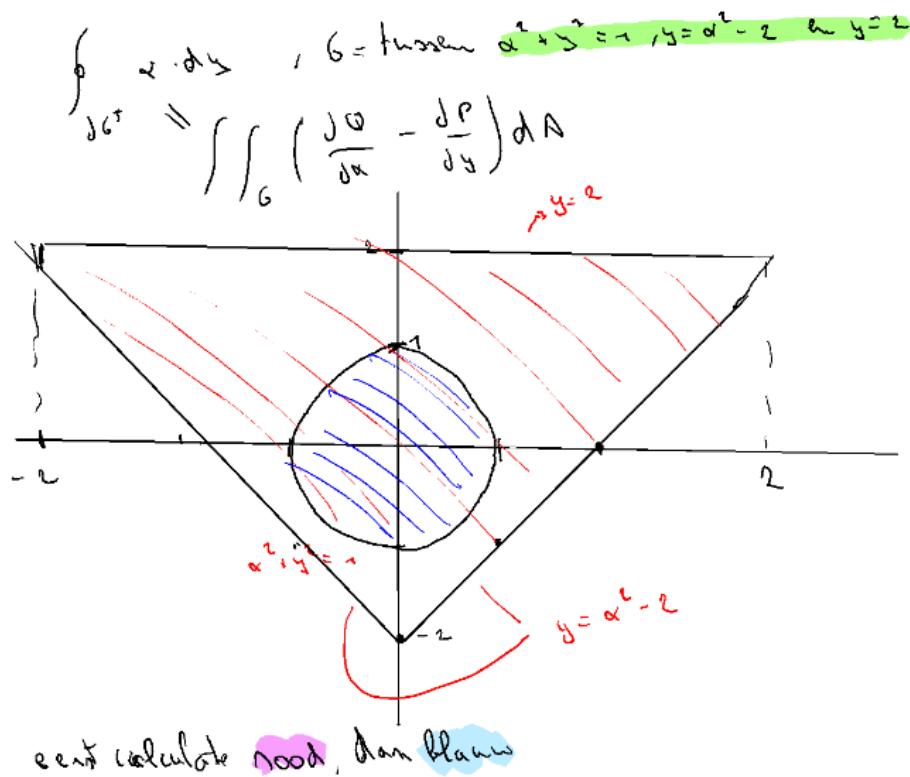
```

> restart: with(LinearAlgebra): with(VectorCalculus):
> SetCoordinates(cartesian[x,y,z]):
> # First define n_k = normaal vector met rechterhand
> v_1 := VectorField([-u.sin(u) + cos(u), sin(u) + u.cos(u), 0]):
> v_2 := VectorField([0, 0, 1]):
> n := CrossProduct(v_1, v_2)
n := ( $\sin(u) + u \cos(u)$ ) $\hat{e}_x$  + ( $u \sin(u) - \cos(u)$ ) $\hat{e}_y$  + (0) $\hat{e}_z$  (1)
> # Nu grad x F
> F := VectorField([0, -z, y]):
> curl := Curl(F)
curl := (2) $\hat{e}_x$  + (0) $\hat{e}_y$  + (0) $\hat{e}_z$  (2)
> # Dus
> int(int(DotProduct(curl, n), v = 0..u), u = 0.. $\frac{\text{Pi}}{2}$ )
-2 +  $\frac{\pi^2}{2}$  (3)
> # et voila
>

```

Figure 183: Exercise 3 Maple

Bord 7



$$\iint_A \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \iint_A 1 \cdot dA \rightarrow \text{area of } G$$

■  $A_1 = \int_{-2}^2 (2 - (x^2 - 2)) dx = \frac{32}{3}$

$\downarrow$   
Want van  $y = x^2$ ,  $y = x^2 - 2$

■  $A_2 = \int_0^{\pi} \int_0^{r(\theta)} r dr d\theta = \int_0^{\pi} \frac{1}{2} r^2 d\theta = \pi$

dim  $\frac{32}{3} - \pi$

Figure 184: Exercise 1

### Problem Overview

1. You have a sphere  $S$  defined by  $x^2 + y^2 + z^2 = \alpha^2$ .
2. This sphere is cut by a plane  $\eta$  with the equation  $y = \beta$ , where  $0 < \beta < \alpha$ . This cutting plane creates a spherical cap on the surface of  $S$ .
3. Points on  $S$  are constrained by  $y \geq \beta$ .
4. The surface  $D$  is the projection of  $S$  onto the plane  $\eta$ .
5.  $G$  is the volume enclosed between  $S$  and  $D$ .
6.  $\mathbf{n}^u$  represents the outward-pointing unit normal vector on the surfaces.

The exercise asks you to calculate the following integrals, where  $\mathbf{x} = (x, y, z)$ :

1.  $\int_S \mathbf{x} \cdot \mathbf{n}^u d\sigma$
2.  $\int_D \mathbf{x} \cdot \mathbf{n}^u d\sigma$
3.  $\int_{\partial G} \mathbf{x} \cdot \mathbf{n}^u d\sigma$

For part (iii), you also need to compute the integral explicitly using the Divergence Theorem.

### Solution Approach

(i) Integral over  $S$ :  $\int_S \mathbf{x} \cdot \mathbf{n}^u d\sigma$

For the spherical cap surface  $S$ :

- Since  $\mathbf{n}^u$  is the outward unit normal vector to  $S$ , we can use the fact that  $\mathbf{x} \cdot \mathbf{n}^u$  is simply the radial component of  $\mathbf{x}$ , and for a sphere of radius  $\alpha$ ,  $\mathbf{x} \cdot \mathbf{n}^u = \alpha$ .
- Therefore,  $\mathbf{x} \cdot \mathbf{n}^u = \alpha$  is constant over  $S$ .

To find  $\int_S \mathbf{x} \cdot \mathbf{n}^u d\sigma$ , integrate  $\alpha$  over the area of the spherical cap surface  $S$ .

(ii) Integral over  $D$ :  $\int_D \mathbf{x} \cdot \mathbf{n}^u d\sigma$

For the surface  $D$ , which is the projection onto the plane  $y = \beta$ :

- Here,  $\mathbf{n}^u$  would point perpendicular to the plane  $y = \beta$  along the  $y$ -axis.
- The integral  $\int_D \mathbf{x} \cdot \mathbf{n}^u d\sigma$  would evaluate the component of  $\mathbf{x}$  in the  $y$ -direction over the circular area in the  $y = \beta$  plane (a circle of radius derived from the intersection with the sphere).

(iii) Integral over  $\partial G$ :  $\int_{\partial G} \mathbf{x} \cdot \mathbf{n}^u d\sigma$

This integral represents the flux of  $\mathbf{x}$  across the boundary  $\partial G$ , which is the union of the surfaces  $S$  and  $D$ . You can evaluate this integral directly using the Divergence Theorem, which states:

$$\int_{\partial G} \mathbf{x} \cdot \mathbf{n}^u d\sigma = \iiint_G (\nabla \cdot \mathbf{x}) dV$$

Since  $\nabla \cdot \mathbf{x} = 3$  (the divergence of  $\mathbf{x} = (x, y, z)$  is  $\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$ ), this becomes:

$$\iiint_G 3 dV = 3 \text{ Volume}(G)$$

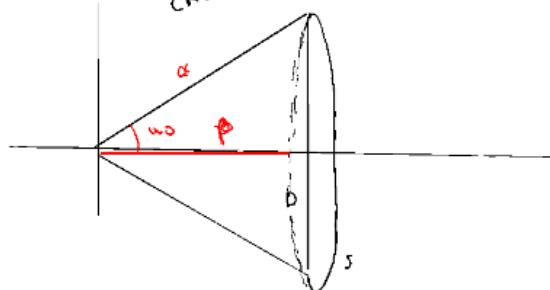
Figure 185: Alternative explanation for Exercise 2

x)

$$\Leftrightarrow 0.8x$$

$$\int_S \alpha \cdot n^m d\sigma \quad \Rightarrow \alpha(u_0) = \frac{\beta}{a}$$

CAS: (G) in outflow  
Minim.



i)  $S: \alpha(u, v) = \left[ \alpha \sin(u) \cos(v), \alpha \cos(u) \cos(v), \alpha \sin(u) \sin(v) \right]$

$$\|\frac{\partial \alpha}{\partial u} \times \frac{\partial \alpha}{\partial v}\| = a^2 \sin u$$

$$= \int_0^{\pi} \int_0^{\arcsin(\frac{\beta}{a})} a^2 \sin u \cdot \frac{1}{a} \cdot du dv = 2\pi a^2 (\alpha - \beta)$$

jacobian.

ii)  $\int_S x \cdot n^m d\sigma$

$\theta: \alpha(u, \omega) = [\alpha \cos(u), \beta, \alpha \sin(u)]$

$$\frac{\partial \alpha}{\partial u} \times \frac{\partial \alpha}{\partial \omega} = \begin{vmatrix} i & j & k \\ \cos(u) & 0 & \sin(u) \\ \sin(u) & 0 & \cos(u) \end{vmatrix} = [0, -\sin(u), 0]$$

$$\int_S \alpha \cdot n^m d\sigma = \int_0^{\pi} \int_0^{\sqrt{\alpha^2 - \beta^2}} [ \alpha \cos(u), \beta, \alpha \sin(u) ] \cdot [0, -\sin(u), 0] du dr$$

$$= \int_0^{\pi} \int_0^{\sqrt{\alpha^2 - \beta^2}} -\beta dr du = \beta \pi (\alpha^2 - \beta^2)$$

iii)  $\int_G (\nabla \cdot \alpha) dV = \int_0^{\pi} \int_0^{\sqrt{\alpha^2 - \beta^2}} \int_{\cos^{-1}(\beta/\alpha)}^{\alpha} 3 \alpha^2 \sin \theta d\theta d\omega d\phi$

jacobian

Figure 186: Exercise 2

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}^k d\sigma$$

$$\mathbf{F} = (y, z, x)$$

$$C: \alpha(u, v) = (\cos(u), \sin(u), v) \quad \text{and } \left\{ 0 \leq u \leq \pi, 0 \leq v \leq 1 + \sin(u) \right\}$$


---

$$\begin{matrix} \mathbf{M} \\ \text{def} \\ \frac{\partial}{\partial u} \frac{\partial}{\partial v} \end{matrix} = \begin{vmatrix} i & j & k \\ -\sin(u) & \cos(u) & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos u, \sin u, 0)$$

$$\nabla \times \mathbf{F} = [-1, -1, -1]$$

$$\text{ans: } \int_0^\pi \int_0^{1+\sin(u)} (-1, -1, -1) \cdot (\cos u, \sin u, 0) dv du$$

$$\approx -\frac{\pi^2}{6} - 2$$

Figure 187: Exercise 3

```

[> restart: with(LinearAlgebra): with(VectorCalculus):
[> SetCoordinates(cartesian[x,y,z]):
[> v_1 := VectorField([-sin(u), cos(u), 0]):
[> v_2 := VectorField([0, 0, 1]):
[> n_k := CrossProduct(v_1, v_2)
    n_k := ( $\cos(u)$ ) $\hat{e}_x$  + ( $\sin(u)$ ) $\hat{e}_y$  + (0) $\hat{e}_z$           (1)
[> F := VectorField([y, z, x]):
[> curl := Curl(F)
    curl := (-1) $\hat{e}_x$  + (-1) $\hat{e}_y$  + (-1) $\hat{e}_z$           (2)
[> # dus
[> int(int(DotProduct(curl, n_k), v = 0..1 + sin(u)), u = 0..Pi)
    -2 -  $\frac{\pi}{2}$           (3)

```

Figure 188: Exercise 3 Maple

Huis 7

$$\text{ellips: } x^2 + y^2 + \frac{z^2}{r^2} = 1$$

$$S: \alpha(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

$$\theta \in [-\pi, \pi], \phi \in [0, 2\pi]$$


---

$$I = \int_0^{2\pi} \int_{-\pi}^{\pi} \left| \left| \frac{\partial \alpha}{\partial \theta} \times \frac{\partial \alpha}{\partial \phi} \right| \right| d\theta d\phi$$

↳ zichtbaar

$$\approx 42,95$$

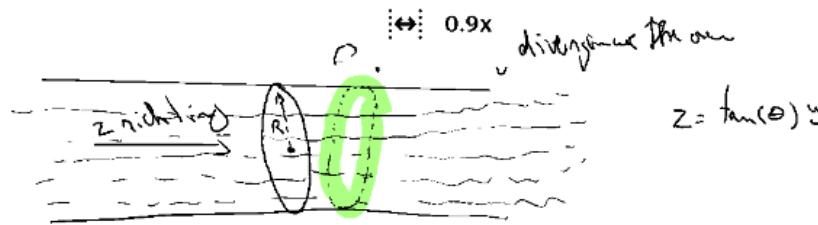
Figure 189: Exercise 1

```

> restart: with(LinearAlgebra): with(VectorCalculus):
> SetCoordinates(cartesian[r,theta,phi]):
> v_1 := VectorField([diff(sin(theta)·cos(phi),theta), diff(sin(theta)·sin(phi),
theta), diff(2·cos(theta),theta)])
      v_1 := ( $\cos(\theta) \cos(\phi)$ ) $\hat{e}_r$  + ( $\cos(\theta) \sin(\phi)$ ) $\hat{e}_{\theta}$  + (- $2 \sin(\theta)$ ) $\hat{e}_{\phi}$           (1)
> v_2 := VectorField([diff(sin(theta)·cos(phi),phi), diff(sin(theta)·sin(phi),
phi), diff(2·cos(theta),phi)])
      v_2 := (- $\sin(\theta) \sin(\phi)$ ) $\hat{e}_r$  + ( $\sin(\theta) \cos(\phi)$ ) $\hat{e}_{\theta}$  + (0) $\hat{e}_{\phi}$           (2)
> n := CrossProduct(v_1,v_2):
> n_norm := sqrt(n[1]^2 + n[2]^2 + n[3]^2):
> result := int(int(n_norm, theta = -Pi..Pi), phi = 0..2·Pi)
      result :=  $4\pi + \frac{16\sqrt{3}\pi^2}{9}$           (3)
> result_float := evalf(result)
      result_float := 42.95687068          (4)

```

Figure 190: Exercise 1



$$\vec{F}(x, y, z) = z - \tan \theta y = 0$$

$$\hat{n} = \frac{\nabla \vec{F}}{\|\nabla \vec{F}\|} \quad \hat{n} = \left\{ 0, -\tan \theta, 1 \right\}$$

$$|\hat{n}| = \sqrt{\tan^2 \theta + 1} = \frac{1}{\cos \theta}$$

$$\hat{n} = \left\{ 0, -\sin \theta, \cos \theta \right\}$$

$$\vec{v}(\text{is pmu z direktie}) = \left( 0, 0, v_0 \left( 1 - \frac{p^2}{R^2} \right) \right)$$

$$\vec{v} \cdot \hat{n} = v_0 n_z = \left( 0, 0, v_0 \left( 1 - \frac{p^2}{R^2} \right) \cos \theta \right)$$

$$dS = \frac{dx dy}{\cos \theta}$$

$$D = \int p_0 (v \cdot \hat{n}) dS = \int_{x^2 + y^2 \leq R^2} p_0 \left[ v_0 \left( 1 - \frac{x^2 + y^2}{R^2} \right) \cos \theta \right] \frac{dx dy}{\cos \theta}$$

werk dit nu afh van  $\theta$ !!!

$$D = p_0 v_0 \alpha \frac{\pi R^2}{2}$$

Nu bewijzen we dat met divergentie:

$$\boxed{\int_S \vec{v} \cdot \hat{n} \cdot dS \rightarrow \int_V \nabla \cdot \vec{v} \cdot dV} \rightarrow \nabla \cdot \vec{v} = 0$$

fully independent of  $\theta$ !

Figure 191: Exercise 2

$$\int \exp(x) \cos y \, dx - \exp x \sin y \, dx$$

Gren wgt :  $\iint \frac{P}{Jx} - \frac{Q}{Jy} \, dA$

$$= \iint_{-\pi}^{\pi} ((\exp(x) \cdot \cos y - \exp(x) \cos y)) \, dx \, dy$$

$\approx 0$

$I = 0$

Figure 192: Exercise 3

$$x^2 + y^2 + z^2 = a^2$$

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

Bol coordinate definition

$$\text{Mass} = \iiint r \cdot z \, dV$$

$\int_0^a \int_{\pi/2}^{\pi} \int_0^{2\pi} r \cos\theta \, j \, d\phi \, d\theta \, dr$   
 cte.  $= \frac{1}{4} \pi r^4$ , omdat het evenredig is met de hoogte



boven rechthoek ( $x, y$ ) vlak is  $M_x = 0, M_y = 0$ !

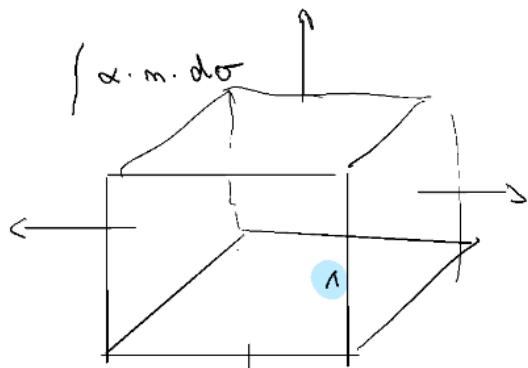
$$M_z = \frac{1}{M} \int_0^a \int_0^{\pi} \int_0^{2\pi} (0) \cdot j \, d\phi \, d\theta \, dr$$

Omdat we "z" volgen  
X blijft een  $\frac{2a}{3}$  ??

$$= M_{x,y,z}(0,0,1)\left(\frac{2a}{3}\right)$$

Figure 193: Exercise 4

⑥



gauß divergence theorem:

$$\int \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint (\nabla \cdot \mathbf{f}) \cdot dV$$

$$\mathbf{f} = (x, y, z) = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\iiint z \, dV = \int_0^1 \int_0^1 \int_0^1 z \, dx \, dy \, dz$$

$$= 3$$

Figure 194: Exercise 5

$$x^2 + y^2 + z^2 = r^2$$

$$u(x,y,z) = x^3 - y^3 + z^2, \nabla u(x,y,z) = (x^2, -y^2, 2z)$$

•

$$\int_S (\nabla u \times \nabla v) \cdot n \cdot d\sigma$$

$$\nabla u \times \nabla v = \nabla \times (u \nabla v) \quad \text{, } \rightarrow \text{ Stokes}$$

$$\int_S (\nabla u \times \nabla v) \cdot n \cdot d\sigma = \int_C \overbrace{u \nabla v}^{\text{green circle}} \cdot ds.$$

$$\left. \begin{array}{l} x = r \cos \theta \\ y = r \sin \theta \\ z = 0 \\ \theta \in [0, 2\pi] \end{array} \right\}$$

$$\bullet \quad d\gamma = \text{diff}((\cos \theta, \sin \theta, 0)) = \underline{(-\sin \theta, \cos \theta, 0)}$$

$$u = (r^3 \cos^3 \theta, -r^3 \sin^3 \theta)$$

$$\nabla v = \{1, 1, 1\}$$

zurück alles in Maple

Figure 195: Exercise 6

```

[> restart: with(VectorCalculus): With(LinearAlgebra):
[> SetCoordinates(spherical[r,theta,phi]):
[> u := cos(theta)^3 -sin(theta)^3:
[> v_curl := VectorField([1,1,1]):
[> dr := VectorField([-sin(theta),cos(theta),0]):
[> integrandum := u.v_curl.dr
    integrandum := -(cos(theta)^3 - sin(theta)^3) sin(theta) + (cos(theta)^3 - sin(theta)^3) cos(theta)  (1)
[> int(integrandum, theta = 0..2.Pi)
    
$$\frac{3\pi}{2}$$
                                (2)
[>

```

Figure 196: Exercise 6 Maple



Figure 156: Sigma Boss