Ph20 - Assignment 3

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1 Explicit and implicit Euler methods

Analytic solution to equation of a mass on a spring:

$$\begin{aligned} F = ma &= -kx \\ a + \frac{k}{m}x &= 0 \\ \frac{d^2x}{dt^2} + \frac{k}{m}x &= 0 \end{aligned}$$

Let $\frac{k}{m} = 1$, the equation becomes:

$$\frac{d^2x}{dt^2} + x = 0$$

The characteristic equation: $r^2 + 1 = 0$ has complex roots $r_{1,2} = \pm i$.
Solutions to the differential equation are: $x_1(t) = e^{it} = \cos(t) + i\sin(t)$ and $x_2(t) = e^{-it} = \cos(t) - i\sin(t)$.
The general solution has the form: $x(t) = c_1x_1(t) + c_2x_2(t)$.

$$\begin{aligned} u(t) = \frac{1}{2}x_1(t) + \frac{1}{2}x_2(t) &= \cos(t) \\ v(t) = \frac{1}{2i}x_1(t) + \frac{-1}{2i}x_2(t) &= \sin(t) \end{aligned}$$

General solution:

$$\begin{aligned} x(t) &= c_1\cos(t) + c_2\sin(t) \\ \Rightarrow x_0 &= c_1 \\ x'(t) &= -c_1\sin(t) + c_2\cos(t) \\ \Rightarrow v_0 &= c_2 \end{aligned}$$

Equations of simple harmonic motion of a mass attached to a spring:

$$\begin{aligned} x(t) &= x_0\cos(t) + v_0\sin(t) \\ v(t) &= -x_0\sin(t) + v_0\cos(t) \end{aligned}$$

Figure 1: $x(t)$ and $v(t)$ of a mass attached to a spring in simple harmonic motion ($x_0 = -2, v_0 = 0$) computed using the analytic, explicit and implicit Euler methods.

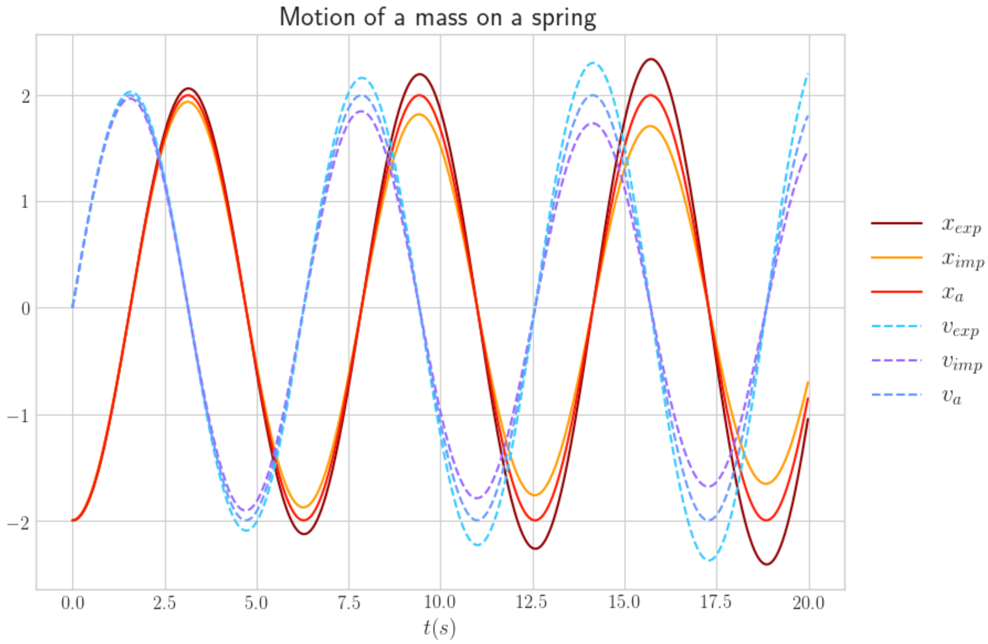
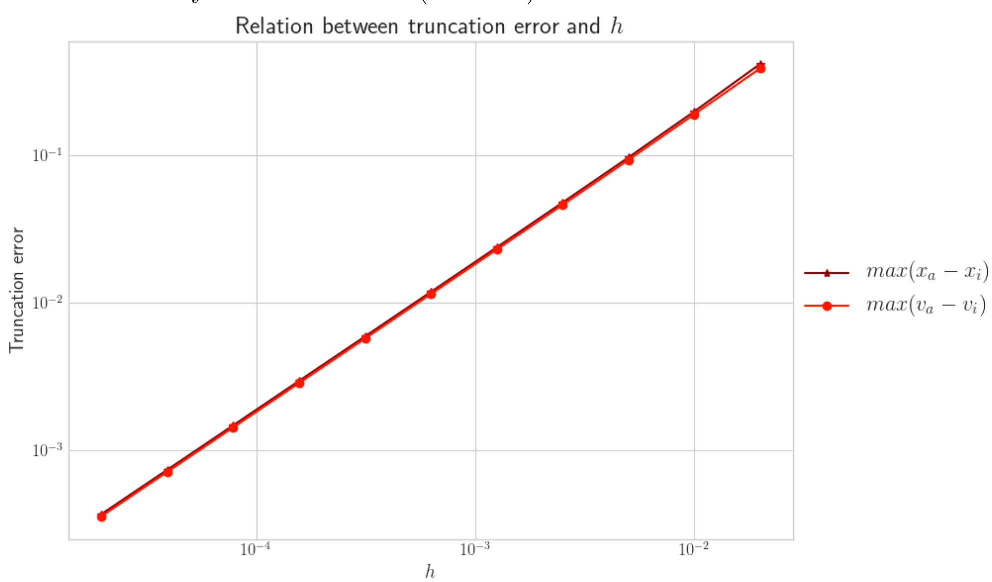


Figure 2: Truncation error in computing $x(t)$ of a simple harmonic oscillator using the explicit Euler method is proportional to h for reasonably small values of h ($h < 0.02$).



Equations for implicit Euler method:

$$\begin{cases} x_{i+1} - hv_{i+1} = x_i \\ hx_{i+1} + v_{i+1} = v_i \end{cases}$$

$$\begin{cases} x_{i+1} - hv_{i+1} = x_i \\ h^2x_{i+1} + hv_{i+1} = hv_i \end{cases}$$

$$
\begin{aligned}
(1+h^2)x_{i+1} &= x_i + hv_i \\
\Rightarrow x_{i+1} &= \frac{x_i + hv_i}{1+h^2} \\
v_{i+1} &= v_i - hx_{i+1} \\
\Rightarrow v_{i+1} &= v_i - h\frac{x_i + hv_i}{1+h^2} \\
\Rightarrow v_{i+1} &= \frac{v_i - hx_i}{1+h^2}
\end{aligned}
$$

Figure 3: Evolution of global errors with time of $x(t)$ ($h = 0.02$) of a simple harmonic oscillator computed using the explicit and implicit Euler method.

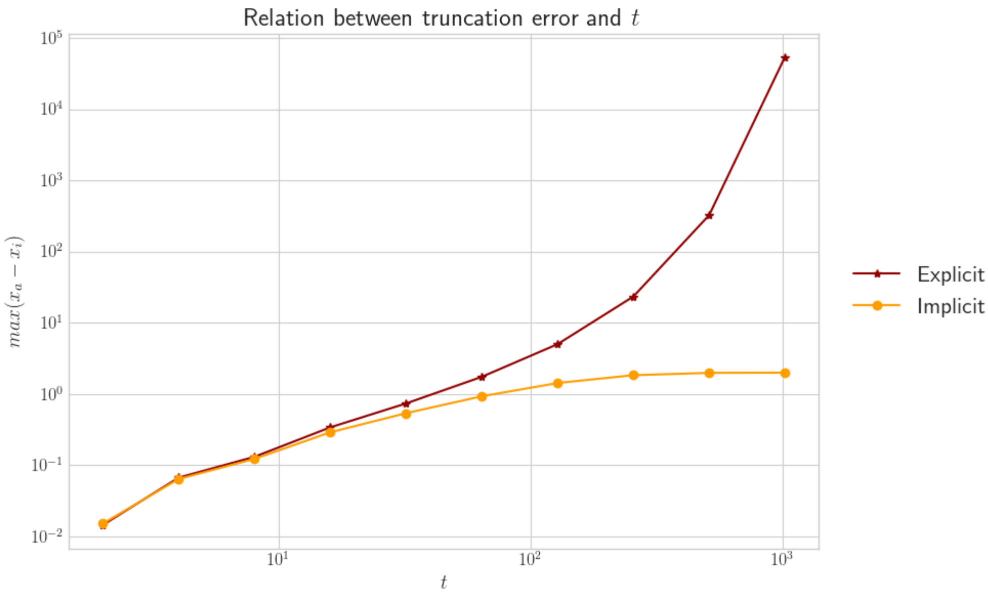
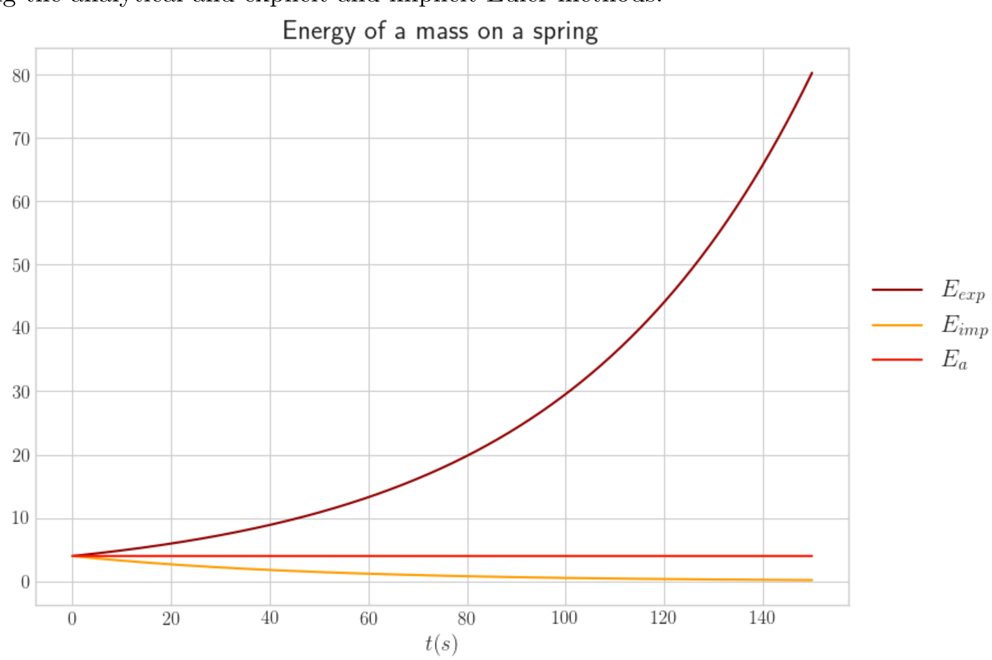


Figure 4: Evolution of the total energy with time of a mass attached to a spring ($x_0 = -2$, $v_0 = 0$, $h = 0.02$) computed using the analytical and explicit and implicit Euler methods.



Method	Long-range trend for E	Evolution of global error
Explicit Euler	Greater than E_a Rate of deviation from E_a increases with time	Error grows as a cubic function of time
Implicit Euler	Smaller than E_a Rate of deviation from E_a decreases with time Energy stops decreasing after sufficiently large time	Error grows as a 1/3 power function of time Error stops increasing after sufficiently large time

2 Symplectic Euler method

Total energy obtained using the symplectic Euler method varies periodically compared to the total energy obtained using the analytic solution. Since $E = x^2 + v^2$, $\sqrt{E}$ is the radius of the circle centered at the origin of the phase space diagram.

One circle corresponds to 2 periods. Over 2 periods,

- the two sections of E_{sym} where $E_{sym} > E_a$ correspond to two bulges (increased 'radius' from the circle of the phase space of the analytic solutions).
- the two sections of E_{sym} where $E_{sym} < E_a$ correspond to two dents (decreased 'radius' from the circle of the phase space of the analytic solutions)

Figure 5: Phase-space of a simple harmonic oscillator ($t = 20$) showing the positions and velocities of a mass attached to a spring computed using the analytic, explicit, implicit and symplectic Euler methods.

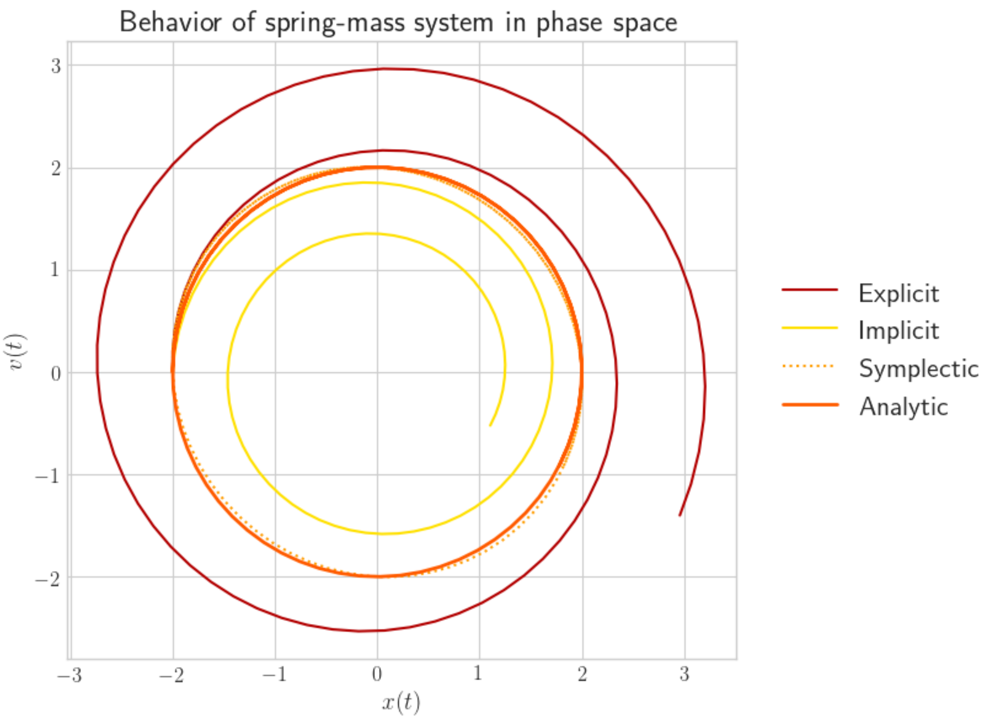


Figure 6: Evolution of total energy of a mass attached to a spring under simple harmonic motion computed using the symplectic Euler method in comparison with the total energy derived analytically.

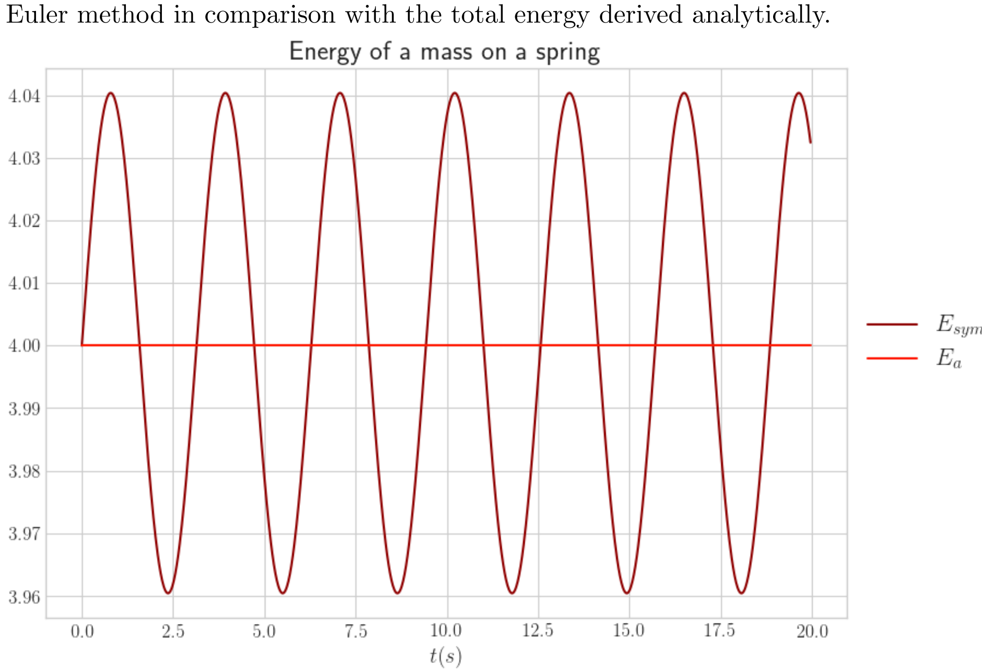


Figure 7: Long-term evolution of $x(t)$ and $v(t)$ of a simple harmonic oscillator computed using the symplectic Euler method. At large t, there is a visible lag between the the solutions computed by the symplectic Euler method and those computed analytically.