Derivation of invariant parameters for surface temperatures of airless body

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Thermal inertia and thermal diffusivity are invariants of the one-dimensional heat equation with Stefan-Boltzmann Law boundary condition. Here, a formal derivation is given, along with a correction to the Standard Thermal Model for asteroids.

On an airless body, the governing equations are the surface energy balance and the heat equation:

$$(1-A)Q + k \left.\frac{\partial T}{\partial z}\right|_{z=0} = \left.\epsilon\sigma T^4\right|_{z=0} \tag{1}$$

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) \tag{2}$$

Where $Q(t)$ is the incoming solar irradiance, A albedo, k thermal conductivity, $T(z,t)$ temperature, z the vertical coordinate, ϵ the infrared emissivity, σ the Stefan-Boltzmann constant ρ density, c specific heat capacity, and t time. The lower boundary condition will be discussed below. Spatially uniform and time-constant thermal properties are used in the following.

Widely used parameter combinations are the thermal inertia $\Gamma = \sqrt{k\rho c}$ and the thermal diffusivity $\kappa = k/(\rho c)$. Substitute $z = \zeta\sqrt{\kappa}$, then

$$(1-A)Q + \sqrt{k\rho c}\left.\frac{\partial T}{\partial \zeta}\right|_{\zeta=0} = \left.\epsilon\sigma T^4\right|_{\zeta=0} \tag{3}$$

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial \zeta^2} \tag{4}$$

This identifies the thermal inertia Γ as the combination of thermal parameters that determines the surface temperature variations, i.e., any combination of k, ρ, and c that result in the same value for Γ leads to the same surface temperatures. Hence, Γ can be determined from remotely sensed surface temperatures, which explains the popularity of this parameter in planetary science.

In a plane-parallel approximation, the lower boundary condition is

$$Q_{\text{geotherm}} = -k\left.\frac{\partial T}{\partial z}\right|_{z=z_0}$$

but this is not invariant under rescaling of z. Hence, the invariance is strictly valid only for vanishing geothermal (interior) heat flux. However, the interior heat flux is practically negligible for the surface temperature.

The invariants are helpful for two reasons. First, they reduce three parameters (k, ρ, c) to two parameters (Γ, κ). Second, the combination of thermal parameters that determines the surface temperature variations is $\Gamma = \sqrt{k\rho c}$ and the depth profile scales with $\sqrt{\kappa} = \sqrt{k/(\rho c)}$.

When Q is periodic with a period $2\pi/\omega$, i.e., ω is the angular rotation rate, the thermal skin depth $\delta = \sqrt{2\kappa/\omega}$.

A more general rescaling is achieved by non-dimesionalizing t and z with $t = t'/\omega$ and $z = z'\sqrt{\kappa/\omega} = z'\delta/\sqrt{2}$. The governing equations (1) and (2) become

$$(1-A)Q + \Gamma\sqrt{\omega}\,\frac{\partial T}{\partial z'}\bigg|_{z'=0} = \epsilon\sigma T^4\Big|_{z'=0} \tag{5}$$

$$\frac{\partial T}{\partial t'} = \frac{\partial^2 T}{\partial z'^2} \tag{6}$$

Now introduce noontime variables

$$(1-A)Q_{\rm noon} = \epsilon\sigma T_{eq}^4$$

where T_{eq} is the equilibrium temperature at noon at the pertinent latitude. A non-dimensional insolation $q = Q/Q_{\rm noon}$ and a non-dimensional temperature $\theta = T/T_{\rm eq}$ can be introduced. The surface energy balance (5) becomes

$$q + \frac{\Gamma\sqrt{\omega}}{\epsilon\sigma T_{eq}^3}\,\frac{\partial\theta}{\partial z'}\bigg|_{z'=0} = \theta^4\Big|_{z'=0} \tag{7}$$

which means the invariant is

$$\Theta = \frac{\Gamma\sqrt{\omega}}{\epsilon\sigma T_{eq}^3} \tag{8}$$

and T_{eq} is the equilibrium temperature at noon, specific to a location. It could be chosen at any fixed local time, but local noon (maximum solar elevation) is a convenient choice. Spencer et al. (1989) consider Θ to be "a property of the whole" and set T_{eq} to the subsolar temperature, but the above derivation does not demonstrate such a relation, because the invariance is defined for each location separately.

To fix this problem they introduce a 'beaming factor' η, which would need to be set such that $\eta\, T_{\rm subsolar}^3$ equals some sort of disk average of T_{eq}^3. See also Lagerros (1996).

References

J. R. Spencer, L. A. Lebofsky, and M. V. Sykes. Systematic biases in radiometric diameter determinations. *Icarus*, 78:337–354, 1989.

J. S. V. Lagerros. Thermal physics of asteroids I. effects of shape, heat conduction and beaming. *Astronomy and Astrophysics*, 310:1011–1020, 1996.