Pixel area for polar stereographic projection

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ϕ ... latitude
λ ... longitude
x, y ... pixel coordinates measured from the north pole
R ... radius of globe (km)
k_0 ... scale at pole along meridian (pixels/km)

For a stereographic projection around the north pole,

$$\begin{aligned} x &= \rho \sin \lambda && (1) \\ y &= \rho \cos \lambda && (2) \\ \rho &= 2Rk_0 \frac{1 - \tan(\phi/2)}{1 + \tan(\phi/2)} && (3) \end{aligned}$$

The inverse relations are

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2} && (4) \\ \phi &= 2 \arctan \left(\frac{2Rk_0 - \rho}{2Rk_0 + \rho} \right) && (5) \\ \lambda &= \arctan(-x/y) && (6) \end{aligned}$$

A useful relation is

$$\cos \phi = \frac{4Rk_0 \rho}{4R^2 k_0^2 + \rho^2}$$

The area of a small pixel is

$$\Delta A = \left| \frac{d\lambda}{dy} \frac{d\phi}{dx} - \frac{d\lambda}{dx} \frac{d\phi}{dy} \right| \cos \phi \tag{7}$$

The derivatives can be obtained from equations (4)-(6), and the result is

$$\boxed{\Delta A = \frac{1}{k_0^2 \left(1 + \frac{x^2 + y^2}{4k_0^2 R^2} \right)^2}} \tag{8}$$

Without distortions, the term in parentheses would be unity. The farther from the pole, the smaller the corresponding area of a pixel.

Tests:

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Delta A \, dx \, dy &= 4\pi R^2 \qquad \checkmark \\
2\pi \int_0^{\infty} \Delta A \rho \, d\rho &= 4\pi R^2 \qquad \checkmark \\
2\pi \int_0^{\rho_0} \Delta A \rho \, d\rho &= \frac{4\pi R^2 \rho_0^2}{4k_0^2 R^2 + \rho_0^2} = 2\pi R^2 (1 - \sin\phi_0) \qquad \text{area of spherical cap} \qquad \checkmark
\end{aligned}
$$