RECENT ADVANCES IN OPTIMIZATION TECHNIQUES

Edited by:

Abrahim Lavi

Associate Professor of Electrical Engineering
Carnegie Institute of Technology

and

Thomas P. Vogl

Manager, Optical Physics
Westinghouse Research Laboratories

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A Comparison among Eight Known Optimizing Procedures*

A. LEON

Mental Health Research Institute
University of Michigan
Ann Arbor, Michigan

I. INTRODUCTION

It is intended here to present some results of research on optimization techniques. Eight known optimum seeking methods are used to optimize five simple two-variable unconstrained functions. Four of the problems are presented and analyzed by Witte and Holst [1] and the other one by Beale [2]. The type of optimization process reported here is that of locating a set of values of a set of variables that yields either a minimum or a maximum value for a function given in algebraic form.

Each one of the optimizing techniques is programmed in FORTRAN II language for IBM 709/7090 computers. These programs will subsequently be referred to as CODES.

This instructive exercise is carried out to acquire knowledge on the different operational characteristics of the computer codes. Familiarity with the program parameters of each one of the codes and understanding of their internal stopping rules is required to introduce the necessary changes for them to be used in connection with GROPE. GROPE is a Universal Adaptive Code for Optimization developed by Professor Merrill M. Flood and the author [3, 4] at the University of California, Berkeley. A Universal Adaptive Code for Optimization, as seen by the authors, is a general code which selects adaptively and sequentially among a group of several optimizing codes as each problem calculation progresses.

It is also found of interest to compare the behavior of these techniques under identical conditions, that is, with the same set of

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problems under the same computing system (University of California, Berkeley, Executive System).

Detailed descriptions of each technique are not included and the reader is referred to the proper references. The objective here is to comment on some results rather than present extensive descriptions or analysis of the codes.

II. THE OPTIMIZING TECHNIQUES

The optimization codes studied in this paper are VARMIT, MINFUM, STEP, LOOK, BEST UNIVAR, ITERATED PARTAN, CONTINUED PARTAN, and a version of the STEEPEST DESCENT method contained in the PARTAN code.

The eight techniques can be classified into two broad categories: techniques based on conventional mathematical methods and techniques of the DIRECT SEARCH type.

VARMINT, MINFUN, STEP, ITERATED PARTAN, CONTINUED PARTAN and STEEPEST DESCENT belong to the group of conventional mathematical methods (INDIRECT OPTIMIZATION). These techniques, in some way or another, use the gradient of the function to be optimized and so require the analytical or numerical evaluation of the partial derivatives of that function. The gradient brings the idea of the direction of fastest improvement toward a solution (either ascending or descending), obviously of great significance. It is enough to say here that the gradient vector points in the direction in which the function increases or decreases most rapidly and its length is the rate of increase or decrease in that direction.

BEST UNIVAR and LOOK are representatives of the DIRECT SEARCH type of optimizing procedures (DIRECT OPTIMIZATION). Hooke and Jeeves [5] have written that "DIRECT SEARCH is just sequential examination of trial solutions. Each trial solution is compared with the 'best' obtained up to that time, and there is a strategy for determining (as a function of earlier results) what the next trial solution will be."

Some relevant aspects of the eight codes are included in the following paragraphs.

A. VARMINT (Variable Metric Method for Minimization) [6, 7, 8]

Davidon says: "This is a method for determining numerically local minima of differentiable functions of several variables.

In the process of locating each minimum, a matrix which characterizes the behavior of the function about the minimum is determined. For a region in which the function depends quadratically on the variables, no more than N iterations are required, where N is the number of variables. By suitable choice of starting values and without modification of the procedure, linear constraints can be imposed upon the variables."

Fend and Chandler point out that, "Gradient methods basically analyze the changes in slope (i.e., vector components of the gradient) corresponding to changes in the position of the trial point. They attempt to drive the components of the slope to zero and thus obtain the location of the optimum point which is sought. Davidon's method systematically varies the metric which specifies the change in vector components of the gradient corresponding to changes in the location of the best point. In this respect it may be characterized as an adaptive procedure. It has a further advantage in that it uses an interpretative procedure once the optimum point is bracketed."

In the neighborhood of any one point the second derivatives of the function to be optimized, $F = f(X)$,* specify a linear mapping of changes in position, dX, onto changes in gradient $d\nabla$. These changes are expressed for a change in the ith derivative, for example, as

$$d\left(\frac{\partial F}{\partial X_i}\right) = \sum_{j=1}^{n} \frac{\partial^2 F}{\partial X^i \partial X^j} \, dX^j$$

$$d\left(\frac{\partial F}{\partial X_i}\right) = H^{ij} dX^j$$

where H^{ij} is the Hessian matrix. As we know, the optimum point will require that $\partial F/\partial X_i$ vanishes and so the desired change in X_i (under the assumption that the Hessian matrix is constant) will be

$$dX^r = \| H^{ij} \|^{-1} \left[- \frac{\partial F}{\partial X_i} \right]$$

In general, H^{ij} does not remain fixed and here lies the important contribution of VARMINT: correction of the Hessian matrix from iteration to iteration. This idea was mentioned by Crockett and

* F denotes the function to be optimized.

Chernoff [9] while discussing the differences between the Newton method and the gradient methods.

The matrix H can be visualized as an error matrix and must be a positive definite matrix. A suggested initial value for H is

$$H^{ii} = (\delta X_i)^2$$

$$H^{ij} = 0 \text{ for } i \neq j$$

where δX_i is estimated error in X_i. In the absence of a better estimate, H may be taken to be the identity matrix of order N.

This paper uses the version of VARMINT available at the Lawrence Radiation Laboratory, the University of California, Berkeley (Deck Z0E0Z013 - FORTRAN II).

B. MINFUN (A General Minimizing Routine)

Humphrey says [10]: "Briefly, the program is a FORTRAN control routine and two subroutines which are designed to be used with a function subroutine to be coded by the user. This group of programs uses the ravine stepping procedure to either explore the 'space' of the independent variables near the minimum or seek the actual set of variables at the minimum (at the option of the user). Provisions have been included to allow exclusion of regions of the variable space from the allowed steps."

The whole operation of MINFUN can be made clear by considering a hypothetical function F of two variables (x, y). Fig. 1 in Appendix A shows the schematic representation of the optimization process. At point 0, the initial point, the starting direction is taken as being along the gradient. A step is taken transverse to the line 0-1 from the point 1 to point 2. At point 2 the function is evaluated. Using the information available at points 1 and 2, a minimum is predicted along line 1-2 at point 3. The function is calculated at point 3 to verify the minimum at that point. To complete the cycle, a step is now taken along the line 0-3 to a point 1' and the operation repeats as described at point 1.

The author is indebted to Mr. W. E. Humphrey for a copy of the FORTRAN II deck of his program as well as for fruitful conversations concerning the use of MINFUN.

C. STEP (An Extremum Locating Algorithm)

The procedure used by STEP is designed to circumvent the existence of local cols (in the surface which is generated by the function) which point in directions other than that of the minimum. When such cols exist, the subroutine uses two points along the spine of the col for extrapolation (in the direction of descent) to a point from which is sought the next spinal point. If the minimum appears to be overshot, then an interpolation takes place. Following this, probing parameters are scaled down, and the whole procedure is iterated until either convergence occurs or the procedure exceeds the limit on the number of iterations.

Baer [11] has written that: "Roughly put, the algorithm consists of using alternately two procedures: EXPLORING and HOMING. Exploring consists of generating a sequence of restricted minima along the spine of the valley of the surface generated by the function. Homing consists of interpolation between appropriate restricted minima when there is an indication that the neighborhood of the required minimum has been overshot.

The efficiency of the procedure lies in the mode of generation of the restricted minima. Having obtained more than one of these, one generates the next by extrapolation (an appreciable distance) in the direction of the vector difference of the preceding two, and then relying on the gradient. Except for the first in this sequence of restricted minima, no great care need be taken in their determination, inasmuch as they need not be exact."

It is interesting to add what Baer means by restricted minimum. "If α, β are taken to be fixed vectors, and if t is a (real-valued) scaler, then the minimum (with respect to t) of $f(\alpha + t\beta)$ will be called a RESTRICTED MINIMUM."

A FORTRAN II version of STEP available through the IBM Share System is utilized in this research. The author is indebted to Dr. R. M. Baer of the Computing Center, University of California, Berkeley, for helpful instructions to work properly with his code.

D. Steepest Descent

There are many codes using in different ways the steepest descent (or ascent) ideas. A straightforward steepest descent procedure available in PARTAN is applied here (described in a subsequent section).

This version of the steepest descent method may be called optimum gradient because it locates the optimum in the gradient direction at each point. The code works as follows [12]: (See Fig. No. 2 in Appendix A)

1. Determine the direction of the gradient at the starting point P_0.
2. Locate the minimum on this "steepest descent" path; designate this point as P_2.
3. Determine the direction of the gradient at P_2.
4. Locate the minimum on this "steepest descent" path. Designate this point as P_3.
5. Continue this procedure to P_n.

"A simple algorithm using cubic interpolation is employed to estimate the minimum on any line $x + \lambda s$, where x is the origin of the line, s is the vector determining the direction, and λ is the step-size parameter to be estimated."

It is interesting to notice that [13] in principle the steepest descent method will not reach the optimum in a finite number of steps because the steps shorten as the point is approached. However, the optimum can be approached as closely as desired, and if the starting point is not too near the major axis, the neighborhood of the optimum is attained rapidly.

E. Iterated PARTAN

The general PARTAN code includes the version of steepest descent described previously together with two variations of the PARALLEL TANGENTS (PARTAN) technique as presented by B. V. Shah, et al. [14]. Both variations of PARTAN look for some sort of acceleration of the steepest descent search. This is an attempt to reduce to a finite number the "infinite" number of steps required to reach the optimum by means of the steepest descent procedure.

The authors of PARTAN say that in the two algorithms, one proceeds to optima of F on successive straight lines. The path directions are alternately determined by positions of points already reached or by certain gradient directions. They also say that all the theoretical results concern the "ideal" case, meaning by ideal:

1. F is quadratic;
2. F and its gradient direction can be determined without error at any specified point;
3. On any given line, the point at which F is an optimum can

be determined without error.

In the absence of error the procedure converges exactly to the optimum in (2N - 1) steps for a quadratic function.

ITERATED PARTAN operates in the following way (Fig. 3, Appendix A):

1. Connect P_1 and P_3 and locate the minimum on this extended line. Designate this point as P_4.
2. From here on re-do the steps involved in the steepest descent process plus the previous one using P_4 as the starting point.

F. CONTINUED PARTAN

The so-called CONTINUED PARTAN, as was said before, is a variation of the previous one and it involves the following steps (Fig. 4, Appendix A) [15]:

1. Determine the direction of the gradient at P_4;
2. Locate the minimum on this steepest descent path. Designate this point as P_5;
3. Connect P_2 and P_5 and locate the minimum on this line. Designate this point as P_6;
4. Repeat the previous steps until obtaining P_N; always taking a steepest descent direction at P_{2j}, $j = 2, 3, \ldots$ and connecting P_{2j-2} and P_{2j+1}, $j = 2, 3, \ldots$ for the PARTAN acceleration step."[14].

The author is indebted to Dr. O. Kempthorne and Mr. Thomas E. Doerfler, both from the Statistical Laboratory of Iowa State University, for a copy of the FORTRAN II deck of PARTAN and its operating instructions.

G. LOOK

LOOK is fully described by Hooke and Jeeves [5]. It may be described briefly as follows [16]:

"1. Initialization. A starting point for the search is calculated* and stored.

"2. Exploratory search. Various moves are made to determine a desirable direction for the search. Any move which is better than the reference value is kept and becomes the new reference value. On the initial entry or whenever the exploratory search is not immediately preceded by a pattern

* Or given as Data.

move, the reference value is the last base point. Following a pattern move, the reference is the value at the end of the pattern move.

"3. Success? If the best value found for the function during the exploratory search is better than its value at the last base point, a new base point is established. Otherwise, the last base point is restored.

"4. Save base point and make Pattern Move. The latest functional value replaces the previous value and the corresponding values of the independent variables do likewise. This establishes a new base point. The pattern move is generated by moving each independent variable away from the latest base point value by an amount equal to the difference between the old and new base point values. A pattern move is always followed immediately by an exploratory search.

"5. Restore last Base Point. The independent variables are set at the values corresponding to the last base point. The functional value for the same point becomes the initial reference for testing the individual moves of the exploratory search.

"6. Had Pattern Move just been made. If the exploratory search preceding the failure was itself preceded by a pattern move, perform another exploratory search. Otherwise, check for search completion.

"7. Can step size be reduced? If the step sizes for all the independent variables are at their minima, the search is complete. Otherwise, reduce step size and perform another exploratory search."

It can be seen that the final termination of the search is made when the step size is sufficiently small to ensure that the optimum has been closely approximated. In any case, the step size must be kept above a practical limit imposed by the means of computation. The search is stopped when two conditions occur at the same time, namely,

1. the step size is at minimum, and
2. the forward and reverse moves of all independent variables fail following a base point test failure.

As Hooke and Jeeves say, "In practice, pattern search has proved particularly successful in locating minima on hyper surfaces which contain 'sharp valleys.' On such surfaces classical techniques behave badly and can only be induced to approach the minimum slowly."

The author is indebted to Mr. C. F. Wood for a copy of the

FORTRAN II deck of LOOK.

II. BEST UNIVAR

This Direct Search Code uses one of the many possible strategies that might be employed to determine subsequent trials as a function of previous results.

BEST UNIVAR is fully described together with numerical examples in two papers written jointly by Professor Merrill M. Flood and the author [17,18]. BEST UNIVAR is available in FORTRAN II for 709/7090 computers operated either under the University of Michigan or the University of California Executive Systems. Changes were introduced recently and the code is also available now in FORTRAN IV for IBM 7090/7094 computers processed by the FORTRAN IV compiler, and 7090/7094 IBJOB Processor Component.

BEST UNIVAR may be described very briefly as follows:

1. Initialization. The optimization process is initiated by picking up, as the starting point, an arbitrary point inside the operating space.
2. Order of analysis. Once the function has been evaluated at the starting point, the independent variables to be changed are changed in an order selected initially by the experimenter.
3. One-at-a-time search. After deciding upon the order in which to search the one-at-a-time search is initiated. Let X_i be the first variable under study; this variable is incremented by an amount Δ_i, holding the other variables at their initial values. If the functional value at this point is better than the one at the preceding point, there is some reason for trying further in the same direction. A larger step size is now used, taken equal to $\lambda_i \Delta_i$ (where $\lambda_i > 1$), and if a better functional value (comparing against the immediately previous one) is obtained, a step of length $\lambda_i^2 \Delta_i$ is used next. This is continued in the same direction of powers of λ_i until no further improvement is obtained. Assume that step $\lambda_i^{h+1} \Delta_i$ was the first unsuccessful one; in this case the preceding base point is kept, namely, the one obtained by step $\lambda_i^h \Delta_i$ and a new sequence is started from this point with initial step size equal to Δ_i following the same scheme as before. If a step of Δ_i in the positive direction does not bring a better point, then a step of length Δ_i in the negative direction is tried; if this happens to be

a successful step, the $\lambda_i \Delta_i$ is tried in the same negative direction continuing in the same fashion as was done in the positive direction. Finally, a point is reached where no improvement is obtained by moving variable X_i either Δ_i or $-\Delta_i$; this point is considered to be the best temporarily for variable X_i. After the best point in the X_i direction is found the second variable in the list is ready to be analyzed. The process is repeated until the total number of variables to be analyzed has been studied and a point X' presenting the best functional value of the round is reached.

4. Pattern Move. If the functional value F' at the end of step 3 is better than the initial one F then the pattern move is tried. The coordinates of the F' point are incremented by an amount proportional to the change experienced for the coordinates in going from F to F'. This rate of change will be greater than one. If point F'', after the initial pattern move, happens to be better than F', a new step of length $(\lambda P)(\Delta P)$ is taken in the same direction. The role of λP here is identical to that of λ in the one-at-a-time portion of the process. The process here follows the same scheme explained in phase 3. As before, when a point is reached where no improvement is obtained by moving the vector either (ΔP) or $-(\Delta P)$, this point is considered the best of this series of pattern moves.

   If the point obtained after a series of pattern moves is better than the point at the beginning of the series (i.e., at the end of the one-at-a-time round), a new round of the one-variable-at-a-time phase, as it was previously described, is attempted, and the process is kept going until no better points are found. If the pattern move phase happens to be a failure, a one-at-a-time round will be tried, resulting either in the final point, i.e., the optimum searched (as far as the technique can tell), or in the continuation of the optimization calculation.

   It is easily seen from the above comments that the end point of the process will always be the starting point of a one-variable-at-a-time phase.

III. THE SAMPLE PROBLEMS

The techniques described previously are tested with a group of five two-variable unconstrained functions.

Three of the problems are by Witte and Holst [19]; the names given to these functions in the original paper are kept here so they will be called: SHALOW, STRAIT and CUBE. The fourth problem was presented for the first time by H. H. Rosenbrock [20] and also included by Witte and Holst who called it ROSIE. The fifth problem is one presented and analyzed by E. M. L. Beale [21] and by Shah, et al. [22]; it is called BEALE in this presentation.

The following are the algebraic expressions of the set of test problems (to be minimized):

$$\text{ROSIE} = 100\,(y-x^2)^2 + (1-x)^2$$

$$\text{SHALOW} = (y-x^2)^2 + (1-x)^2$$

$$\text{STRAIT} = (y-x^2)^2 + 100(1-x)^2$$

$$\text{CUBE} = 100\,(y-x^3)^2 + (1-x)^2$$

$$\text{BEALE} = \sum_{i=1}^{3} U_i^2 \text{ where } U_i = C_i - x(1-y^i)$$

and where $C_1 = 1.5$, $C_2 = 2.25$, $C_3 = 2.625$.

ROSIE has a minimum $F = 0$ at $(1, 1)$, with a steep valley along $y = x^2$, and a side valley along the negative y - axis.

SHALOW presents a minimum of $F = 0$ at $(1, 1)$ with valleys along $y = x^2$ and $x = 1$. SHALOW is similar to the function ROSIE, but has a shallow valley compared with the steep valley of ROSIE.

STRAIT has its minimum of $F = 0$ at $(1, 1)$ with a steep valley along $x = 1$.

CUBE presents a minimum $F = 0$ at $(1, 1)$ with a steep valley along $y = x^3$.

BEALE has a minimum of $F = 0$ at $(3, 0.5)$ with a narrow curving valley approaching the line $y = 1$.

Each one of the problems is solved beginning the optimization calculation at five different starting points so as to expose each procedure to a variety of topographical conditions. ROSIE, SHALOW, STRAIT and CUBE use the same initial points of Witte and Holst. BEALE uses five of the starting values tried by Shah et al., in fact the ones which appear to be most difficult ones.

The results are recorded in special tables. The tables are not included here but they are available from the author. Each table contains, for a particular problem, the initlal values together with the following information pertaining to each one of the optimizing codes:

1. Final Values. The optimum functional value together with the corresponding vector.
2. Number of times the evaluating function is called. In some of the codes this subroutine is called to evaluate the partial derivatives at some point without functional evaluation at all; however, these two calls are not separated and both are recorded as functional evaluations.
3. Execution time in seconds. Internal clock readings are taken both at the beginning and end of each one of the problems by means of a library subroutine of the Berkeley System. This subroutine is for use on the BC 7090 equipped with the Delco clock on Channel H.
4. Number of cycles. A cycle has a different meaning in each one of the codes. A brief definition of cycle for each one of the techniques follows:

BEST UNIVAR. A complete cycle includes the one-variable at-a-time phase and the series of pattern moves following the previous phase.

LOOK. A cycle is defined here as the exploratory search plus the pattern move.

VARMINT. A cycle includes establishing a direction to search, determining if the local minimum has been sufficiently well located and the modification of the H matrix on the bases of previous information.

MINFUN. A cycle here is as follows: determination of the gradient direction, step transverse to the gradient direction at the end of the previous step, prediction and verification of a minimum, step in the direction of the vector initial point → actual minimum.

STEP. Each iteration involves the necessary operations to locate a new restricted minimum.

ITERATED PARTAN. The cycle includes the operations (1) and (2) as explained in the description of this optimizing code.

CONTINUED PARTAN. A cycle here is understood as one including steps (1), (2), (3) and (4) as explained in the description of this optimizing procedure.

STEEPEST DESCENT. A Cycle is defined here as the phase of the optimization process including steps (1), (2), (3), and (4) of the description of the code.

IV. CONCLUDING REMARKS

The following general conclusions seem to be appropriate in view of the results obtained.

VARMINT presents the most consistent behavior among the group of techniques based on conventional mathematical methods. Similar results are reported by M. J. Box [23] working with a different group of techniques on a different set of problems. Box says that "Whilst these results are open to various interpretations,* the conclusion reached is that Davidon's method is a more efficient optimizer than Rosenbrock's method, perhaps by a factor of 2 or 3." Box's results together with the results of this work are in accordance with the following remarks by Fletcher and Powell [24]: "Davidon's work has been little publicized, but in our opinion constitutes a considerable advance over current alternatives." Fletcher and Powell add in the same reference that their results with a variety of numerical tests "confirm that the method is probably the most powerful general procedure for finding a local minimum which is known at the present time."

STEP shows good results with functions ROSIE, SHALOW, STRAIT and CUBE. It does not work too well with BEALE, especially approaching the narrow curving " valley" from the almost flat region at the lower right corner of the space.

BEST UNIVAR of the direct search type of techniques exhibits the most consistent behavior of its group. LOOK requires lower computing time and presents better numerical values when it approaches the true optimum; however, its work on CUBE and BEALE is quite unsatisfactory.

One of the essential characteristics of direct search procedures for optimization is that they do not require derivatives of the objective function. These techniques are then adequate to treat difficult optimization problems involving functions with many variables. Furthermore, algebraic expressions for the optimization function are not necessary. All that is required is a way of finding functional values but without having the function available in algebraic form at all. In cases involving complex functions or having no algebraic expressions for the objective function, the application of VARMINT or similar procedures would be difficult or impossible.

It is interesting to note the similarities in behavior between STEP and LOOK while optimizing BEALE. Further analysis of these results will be attempted in the near future.

* The same as the results reported here.

CUBE proves to be the most difficult function for all the techniques. It takes the longest computing time to reach its true solution if compared against the other functions. MINFUN, ITERATED PARTAN, STEEPEST DESCENT and LOOK do not even approach the optimum of CUBE in most of the cases.

Some experiences with the same group of codes testing the effects of dimensionality and the problems introduced by numerical evaluation of derivatives will be reported elsewhere. The presence of constraints will be analyzed in future work also.

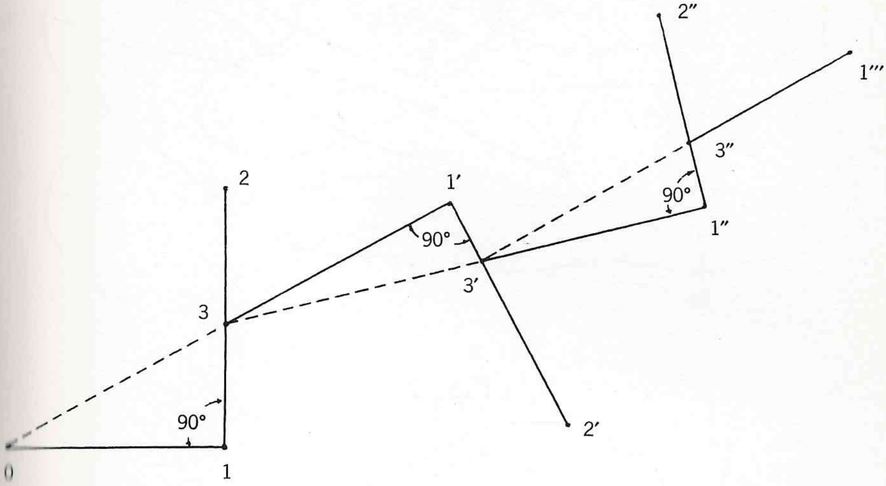

Fig. 1

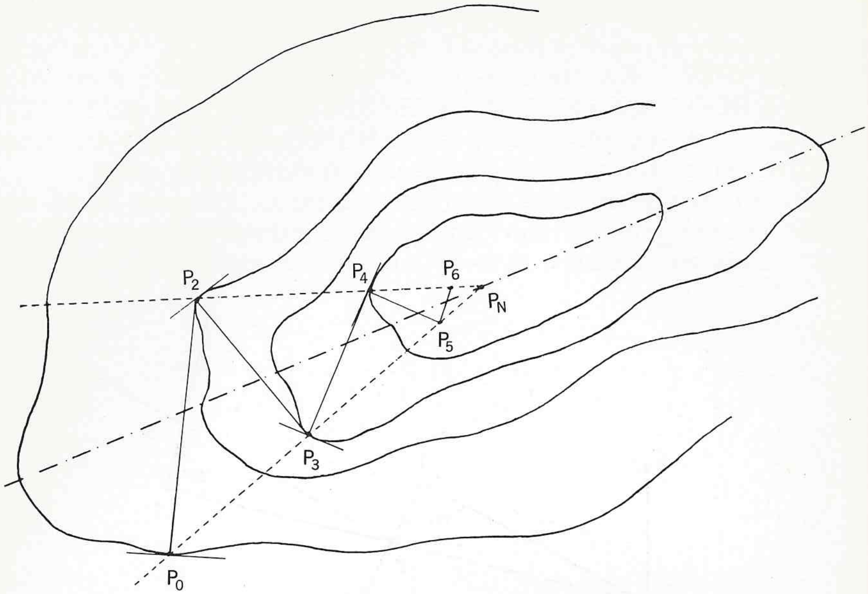

Fig. 2

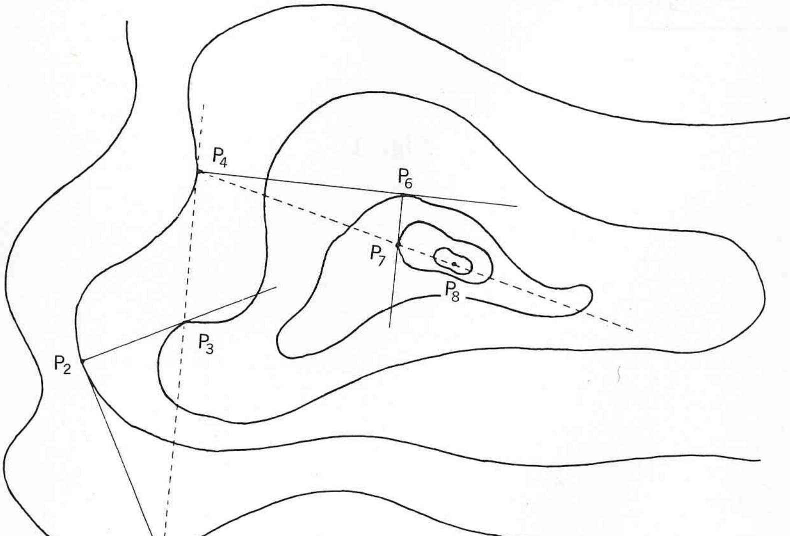

Fig. 3

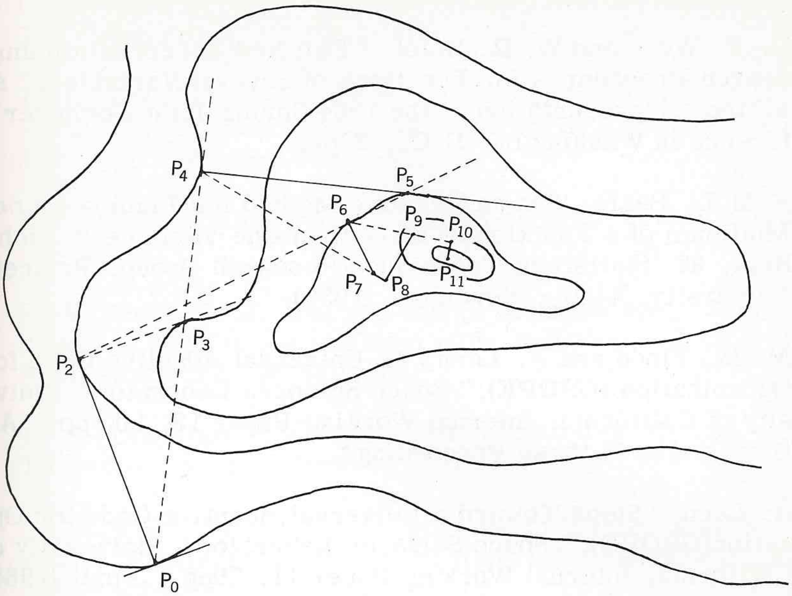

Fig. 4

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A Classified Bibliography on Optimization

A. LEON
Mental Health Research Institute
University of Michigan
Ann Arbor, Michigan

I. INTRODUCTION

A classified bibliography on optimization is presented which, by virtue of the extent of the field, cannot pretend to be exhaustive. It should, rather, be considered an extensive sampling of the field, and the inclusion or exclusion of a particular paper does not reflect on its relative quality. The selection of the material as well as the classifications are the personal judgment of the author. Therefore, as in all cases of judgment, others might have decided differently.

In the compilation of this bibliography attention is focussed primarily on optimization techniques. The type of optimization of interest here is that of finding the values of a set of variables that yield either a maximum or a minimum value for an objective, merit, or utility function. This function might be given in algebraic form or might be a simulation of some process that yields functional values directly. Other related but well known subjects like mathematical programming, the ordinary calculus, the calculus of variations, and aspects of automatic control (Pontryagin's Principle) are touched upon only briefly.

The classification consists of four major sections: General Comments on Optimization, Optimization Techniques, Stochastic Optimization, and Problem Formulation. Some of the material overlaps and in these cases cross-references are made. Thus a reference may appear in more than one section. The references in each section are divided into two groups. Group a) papers are concerned directly and primarily with the subject matter of the specific section. Group b) papers just mention the section topic.

When appropriate, specific comments are included at the beginning of subsections.

II. GENERAL COMMENTS ON OPTIMIZATION

Included in this section are papers and books which treat optimization in general, papers describing several optimization techniques, papers comparing techniques and reviews of optimization procedures. Also included are papers presenting applications to engineering problems, mathematics, decision theory, control processes, communications and prediction.

II. 01 Generalities

a) 4, 10, 24, 37, 39, 66, 87, 90, 102, 111, 137, 173, 182, 187, 210, 232, 237, 256, 293, 294, 310, 319, 348, 349, 352, 368.

b) 42, 51, 99, 141, 165, 180, 201, 229, 238, 297, 316, 346, 350, 369.

II. 02 Theoretical Contributions

a) 7, 42, 48, 122, 133, 139, 188, 189, 223, 263, 285.

b) 6, 8, 9, 10, 15, 18, 19, 24, 46, 62, 63, 64, 65, 74, 75, 76, 83, 90, 92, 93, 101, 102, 109, 123, 124, 125, 129, 136, 151, 154, 155, 156, 157, 166, 177, 183, 190, 199, 203, 214, 226, 230, 241, 247, 268, 269, 274, 293, 308, 317, 323, 326, 328, 329, 333, 348, 373, 374, 375.

II. 03. 01 Applications-Engineering

a) 50, 68, 69, 82, 85, 99, 103, 107, 118, 145, 147, 152, 158, 193, 213, 218, 229, 231, 238, 242, 267, 271, 279, 280, 297, 302, 308, 317, 332, 333, 345, 347, 367.

b) 10, 16, 41, 42, 49, 52, 55, 56, 61, 75, 76, 80, 84, 86, 88, 89, 96, 104, 106, 129, 134, 145, 159, 163, 173, 191, 207, 210, 224, 230, 233, 236, 246, 254, 255, 262, 298, 305, 316, 318, 321, 335, 338, 342, 343, 370.

II. 03. 02 Applications-Mathematics

a) 36, 97, 123, 124, 125, 129, 201, 230, 234, 304, 371.

b) 10, 122, 133, 207, 308, 312, 343.

II.03.03 Applications-Adaptive Processes and Control

a) 12, 49, 71, 110, 143, 191, 192, 194, 224, 256, 271, 321.
b) 159, 175, 267, 288, 294.

II.03.04 Applications-Decision Theory, Communications, Prediction, Search Theory

a) 81, 86, 160, 164, 219, 220, 221, 247, 253, 370.
b) 126, 134, 294, 343, 351.

III. OPTIMIZATION TECHNIQUES

Optimization Techniques as defined in the Introduction are divided into four general groups:

III.01 Mathematical Techniques (Indirect Optimization)
III.02 Non-Mathematical Techniques (Direct Optimization)
III.03 Constraint Handling
III.04 Special Methods for Special Problems

Indirect optimization makes use of necessary conditions for an optimum (e.g., the value of X that causes the first derivatives of f(X) with respect to X to vanish). Direct optimization looks for the value of X that makes f(X) optimum. These methods depend upon direct comparisons of the values of the function at two or more points inside the operating space.

III.01 Mathematical Techniques (Indirect Optimization)

There are several conventional mathematical methods for optimization. All of these methods require the existence of some explicity formulated objective function in algebraic form together with expressions for the constraints whenever they have to be considered.

We shall divide the Mathematical Techniques into two main sections:

III.01.01 Formal Mathematical Techniques
III.01.02 Methods Based on Formal Mathematical Techniques

The author believes that the Formal Mathematical Techniques can be grouped into four broad categories: root finding (or critical points), gradient methods, calculus of variations and

Pontryagin's Principle. The well known ordinary methods of differential calculus fall into the critical points category. Once the derivatives are found and are equated to zero the optimization problem has changed to one which is algebraic in nature. The algebraic problem is one of finding the roots, or acceptable approximations, of a given equation or a system of simultaneous equations (not necessarily a system of linear equations). A problem of practical interest is that of optimizing a function of several variables, when the variables are related by one or more equations. This type of constrained optimization is solved using the so called "Lagrange Multipliers" and forming the Lagrangian Function. However, the optimization of the Lagrangian Function is obtained by solving the system of simultaneous equations resulting from the partial derivatives of that function with respect to the variables and multipliers, equated to zero. At the end, one has another root finding problem.

The second section contains methods which are constructed having as their basis the ideas presented in the previous section.

III. 01. 01 Formal Mathematical Techniques

III. 01. 01. 01 Generalities

a) 51, 101, 151, 175, 196, 214, 274.
b) 24, 52, 187, 260, 294, 310, 349, 368.

III. 01. 01. 02 Root-Finding Procedures

a) 135, 168, 211, 251, 314, 337, 364.
b) 28, 45, 65, 73, 87, 101, 165, 180, 284, 293, 294, 304, 310, 317, 319, 349.

III. 01. 01. 03 Gradient Methods

a) 6, 73, 88, 121, 207, 281.
b) 9, 24, 28, 34, 41, 42, 45, 47, 72, 78, 87, 89, 105, 111, 119, 130, 138, 171, 187, 215, 288, 293, 297, 310, 315, 317, 327, 337, 348, 349.

III. 01. 01. 04 Calculus of Variations

a) 170, 195, 233, 254, 255, 313, 318.
b) 201, 211, 222, 251, 294.

III. 01. 01. 05 Pontryagin's Principle

a) 222.
b) 195, 240, 271, 294, 349.

III. 01. 02 Methods Based on Formal Mathematical Techniques

III. 01. 02. 01 Steepest Ascent (Descent)

a) 47, 52, 53, 72, 84, 180, 298, 327.
b) 24, 27, 37, 38, 39, 40, 42, 66, 78, 87, 88, 89, 130, 148, 149, 163, 175, 187, 207, 237, 288, 300, 310, 311, 319, 337, 348, 349, 368, 371, 372.

III. 01. 02 . 02 Least Ascents (Descents)

a) 150, 151.
b) 37, 39, 348, 349, 371.

III. 01. 02. 03 Kron's Interpolative Method

a) 111.
b) 348.

III. 01. 02. 04 Humphrey's MINFUN

a) 185.
b) 237, 297.

III. 01. 02. 05 PARTAN

a) 46, 299, 300.
b) 24, 87, 310, 349, 368.

III. 01. 02. 06 Baer's STEP

a) 11.
b) 237, 297.

III. 01. 02. 07 Davidon's Variable Metric

a) 77.
b) 37, 121, 237, 297, 310, 349.

III. 01. 02. 08 Witte and Holst's Linear and Circular Search

a) 353.
b) 237.

III. 01. 02. 09 Brown's Gradient Method

a) 41, 42.
b) 310.

III. 01. 02. 10 Optimum and Accelerated Gradient

a) 132.
b) 42, 66, 310, 349.

III. 01. 02. 11 Beale's Minimization Technique

a) 14.

III. 01. 02. 12 Powell's Minimizer

a) 276.

III. 01. 02. 13 Fletcher and Powell's Iterative Procedure

a) 120.
b) 121.

III. 02 Non-Mathematical Techniques

The section on Non-Mathematical Techniques for Optimization is divided into two parts:

III. 02. 01 Simultaneous Optimization (Experimental Design Techniques)
III. 02. 02 Sequential Experimentation

D. J. Wilde has said that "search plans fall naturally into two mutually exclusive classes which we shall call simultaneous and sequential. Plans specifying the location of every experiment before any results are known will be called simultaneous, while a plan permitting the experimenter to base future experiments on past outcomes will be called sequential" [349]*.

* The number in parentheses refers to the numbered bibliography appearing at the end of the paper.

III. 02. 01 Simultaneous Optimization

a) 70, 78, 108, 205, 369.
b) 187, 236, 349, 351.

III. 02. 01. 01 Single Factor at a Time

a) 70, 78, 108.
b) 4.

III. 02. 01. 02 Complete Factorial Designs

a) 29, 70, 78, 108, 275.
b) 4, 39, 173, 260.

III. 02. 01. 03 Fractionally Replicated Designs

a) 70, 78, 108.
b) 173.

III. 02. 01. 04 Confounded Designs

a) 70, 78, 108.
b) 173.

III. 02. 01. 05 Rotatable Designs

a) 32, 202.
b) 176.

III. 02. 01. 06 Sectioned Centroid

a) 111.

III. 02. 01. 07 Simple Random Procedure

a) 37, 38, 39, 141, 178, 258, 264.
b) 128, 173, 248, 249, 250, 260, 310, 346, 372.

III. 02. 01. 08 Case Study Method

a) 319.

III. 02. 02 Sequential Experimentation

Continuous optimization is ideal for problems about which little is known. Since the entire set of observations may be determined prior to any experimentation, the strategy is parallel in nature even though the physical structure of the process under analysis might require sequential optimization.

Due to limitations of storage capacity in electronic computers there is a great amount of information wasted in continuous optimization procedures. In general, only the information concerning the best point so far is retained. Some of the difficulties already mentioned strongly suggest that an efficient method ought to be sequential; in other words, that the experience gained from each trial may be utilized in those which follow.

The differences among sequential techniques lie in the way previous information about the objective function is utilized to the benefit of posterior action. For practical purposes what is necessary is to "direct the pattern of search in promising directions; to select new trial points which are in some sense 'like', or 'similar to,' or 'in the same direction as' those which have given the best previous results. We must sometimes tie together points which are heuristically related" [256]. The sequential procedures apply, in some way or another, the principle by Minsky quoted above.

III. 02. 02. 01 One Variable at-a-Time

a) 138.
b) 4, 24, 37, 38, 39, 87, 173, 248, 249, 250, 260, 277, 310, 349.

III. 02. 02. 02 Down-Hill Procedures (Up-Hill)

a) 111, 298.
b) 55, 56, 173, 349.

III. 02. 02. 03 Westervelt's Optimizer

a) 344.
b) 343.

III. 02. 02. 04 Box-Wilson Techniques (EVOP)

a) 30, 33, 34, 35, 60, 79, 186, 217.
b) 3, 4, 78, 87, 111, 176, 236, 248, 249, 250, 288, 311, 349.

III. 02. 02. 05 Sectioned Centroid

a) 111.

III. 02. 02. 06 Moment Rosetta

a) 111.

III. 02. 02. 07 Search Techniques

The presence of high speed digital computers has stimulated the interest in a class of optimizing procedures which are here called "Direct Search Techniques." R. Hooke and T. A. Jeeves have said that "roughly speaking, Direct Search is just sequential examination of trial solutions. Each trial solution is compared with the 'best' obtained up to that time, and there is a strategy for determining (as a function of earlier results) what the next trial solution will be" [179].

For the present purpose, the above definition is modified in the following way: Direct Search is just sequential examination of trial solutions which are obtained by direct numerical functional evaluations. Each trial solution is compared with the 'best' obtained up to that time, and there is a strategy for determining what the next trial solution will be. Thus this definition establishes the fundamental differences between the mathematical techniques described in an earlier section of this paper and Direct Search procedures. These differences can be seen from the essential elements of Direct Search techniques:

a. A way to obtain functional values at trial points inside the operating space (evaluation of an algebraic expression, simulation of a given system, experiments performed on an existing process, etc.);
b. A strategy to determine the sequence of trial solutions;
c. Criteria to decide when a solution point should be considered better than previous ones.

Thus, salient features of Direct Search procedures are:

a. Use of numerical rather than analytic techniques;
b. Direct type of optimization as opposed to indirect one;

c. General enough for use in connection with any class of optimization problems;
d. Very well adapted to use on electronic computers, since they tend to use repeated identical arithmetic operations with simple logic.

The emphasis on numerical and direct procedures when discussing Direct Search techniques should be noted because it is precisely the use of direct optimization and numerical methods that makes Direct Search techniques different from analytical sequential methods.

Search techniques are divided into three main categories [142]:

III. 02. 02. 07. 01 Blind Search
III. 02. 02. 07. 02 Local Search
III. 02. 02. 07. 03 Non-Local Search

In Blind Search Procedures the trial points are selected at random. However, there also exists some built-in strategy which incorporate advantages of sequential experimentation and differentiates these techniques from the simple random method.

In the second class, Local Search, the working point* moves continuously in the operating space. Each subsequent experiment must take place only in a small neighborhood of the parameter values of the preceding experiment.

It seems natural that a combination of blind and local search used in some intelligent way will provide the ideal technique. The third category, Non-Local Search, is supposed to help by jumping out of local optima and eventually locating the global optimum or at least a better local point. The distinctive feature of non-local search techniques lies in the fact that the curve describing the position of the operating point* inside the working space is no longer continuous. "Now the size of the region examined in a unit of time is much greater and it now becomes possible to make use of the individual characteristics of the objective function; thus considerable speeding up of the optimization process is achieved"[142].

It is important to note that the six techniques already described under the heading of Sequential Optimization together with the Ridge Analysis, Fibonaccian Search, and Golden Section included at the end of this section, may also be considered as Direct Search procedures. However, it is preferred to consider them independently for the following reasons:

* The sequence of trial solutions.

1. All of the local search procedures are based on the ideas of Friedman and Savage's one-at-a-time method.
2. Some of the local techniques make use of the down-hill (up-hill) concepts. The method still has some mathematical flavor.
3. Westervelt's optimizer is a variation of the down-hill (up-hill) techniques.
4. Box-Wilson techniques already have a name of their own as leaders of the Evolutionary Operation Movement (EVOP).
5. The Sectioned Centroid approach may better be considered as a way of reducing a multi-dimensional problem to a one-dimensional optimization search.
6. The Moment Rosetta is an extension of the Centroid ideas.

III. 02. 02. 07. 01 Blind Search

a) 38.
b) 349.

III. 02. 02. 07. 01. 01 Shrinkage Random

a) 38, 249, 250.
b) 128, 248, 349.

III. 02. 02. 07. 01. 02 Creeping Random

a) 38.

III. 02. 02. 07. 01. 03 Stratified Random

a) 38.
b) 39.

III. 02. 02. 07. 01. 04 Satterthwaite (REVOP)

a) 296.
b) 128, 248, 249, 250.

III. 02. 02. 07. 01. 05 Directed Random

a) 306, 307.
b) 87.

III. 02. 02. 07. 01. 06 History Vector

a) 346.
b) 111.

III. 02. 02. 07. 02 Local Search

III. 02. 02. 07. 02. 01 Hooke-Jeeves Method

a) 179, 342, 365, 366.
b) 24, 66, 128, 148, 149, 215, 237, 297, 310, 349, 367, 368.

III. 02. 02. 07. 02. 02 Best Univar

a) 126, 127.
b) 237, 297.

III. 02. 02. 07. 02. 03 Glass Sequential Search

a) 148, 149.
b) 87.

III. 02. 02. 07. 02. 04 Adaptive Constrained Technique

a) 174, 265.

III. 02. 02. 07. 02. 05 Rosenbrock's Method

a) 291, 316.
b) 317, 349, 368.

III. 02. 02. 07. 02. 06 Parallel Cord

a) 111.

III. 02. 02. 07. 02. 07 Finkel's Method

a) 117.
b) 310.

III. 02. 02. 07. 02. 08 Powell's Without Derivatives

a) 277.

III. 02. 02. 07. 03 Non-Local Search

III. 02. 02. 07. 03. 01 Gelfand and Tsetlin's Non-Local Techniques

a) 142.
b) 24.

III. 02. 02. 07. 03. 02 Gradient Search

a) 372.

III. 02. 02. 07. 03. 03 Adaptive Sequential Optimization (Grope)

a) 128.

III. 02. 02. 07. 03. 04 Gibson's Piece-Wise Optimization

a) 144.

III. 02. 02. 07. 03. 05 AID

a) 58.

III. 02. 02. 08 Ridge Analysis

a) 176, 236.
b) 24.

III. 02. 02. 09 Fibonaccian Search

a) 190, 203, 225, 268, 269, 350.
b) 24, 310, 349.

III. 02. 02. 10 Golden Section

a) 349.
b) 310.

III. 03 Handling Constraints

a) 158.

III. 03. 01 Hemstitching

a) 24.
b) 101, 198, 199, 288.

III. 03. 02 Riding the Constraint

a) 24.
b) 288.

III. 03. 03 Multiple-Gradient Summation

a) 215.
b) 198, 200.

III. 03. 04 Glass Sequential Search

a) 148, 149, 199.

III. 03. 05 Stopping at the Boundary

a) 179.

III. 03. 06 Satterthwaite

a) 296.
b) 128, 248, 249, 250.

III. 03. 07 Klingman and Himmelblau

a) 215.
b) 198, 200.

III. 04 Special Methods for Special Problems

Included under this heading are special problems like systems of simultaneous equations, mathematical programming, regression analysis and least-squares estimation.

III. 04. 01 Least-Squares Estimation

a) 31, 165, 238, 244, 278.
b) 87, 180, 257, 352.

III. 04. 02 Systems of Simultaneous Equations

a) 22, 25, 27, 28, 45, 105, 157, 169, 171, 245, 273, 284, 315, 320, 322, 326, 334.
b) 37, 46, 121, 131, 165, 293, 314, 364.

III. 04. 03 Linear Programming

a) 61, 62, 63, 74, 76, 91, 96, 106, 140, 161, 177, 184, 208, 252, 270, 282, 283, 286, 301, 328, 329, 331, 335, 336, 338, 356, 360.
b) 1, 9, 13, 19, 24, 64, 75, 82, 134, 146, 167, 293, 294, 310, 312, 341, 348, 355, 361, 376, 377.

III. 04. 04 Non-Linear Programming

a) 1, 9, 19, 64, 65, 82, 92, 94, 95, 156, 162, 166, 167, 215, 216, 226, 241, 272, 281, 312, 328, 331, 335, 338, 341, 354, 356, 358, 359, 360, 361.
b) 76, 113, 114, 115, 116, 159, 252, 263, 293, 308, 348, 376, 377.

III. 04. 04. 01 Quadratic Programming

a) 15, 26, 172, 183, 235, 243, 323, 355, 362.
b) 76, 136.

III. 04. 04. 02 Integer Programming

a) 155, 181, 325.
b) 76.

III. 04. 04. 03 Gradient Projection Method

a) 289, 290.
b) 281, 294, 310.

III. 04. 04. 04 Arrow-Hurowicz Method

a) 281.

III. 04. 04. 05 Mugele's Poor Man Optimizer

a) 260, 261, 262.
b) 348.

III. 04. 04. 06 Kelly's Cutting Plane

a) 209, 343, 357.
b) 236, 257, 310.

III. 04. 04. 07 Method of Feasible Directions

a) 376, 377.
b) 134, 294.

III. 04. 04.08 Beale's Non-Linear Method

a) 13.
b) 294.

III. 04. 05 Dynamic Programming

a) 5, 16, 17, 18, 162, 292.
b) 24, 134, 194, 201, 293, 305, 317.

III. 04. 06 Regression Analysis

a) 168, 343.
b) 180, 344, 352.

IV. STOCHASTIC OPTIMIZATION

a) 20, 21, 23, 40, 44, 67, 100, 119, 150, 206, 212, 227, 228, 246, 248, 249, 250, 266, 287, 295, 311, 324, 339, 340, 351, 363.
b) 29, 34, 60, 66, 141, 205, 296, 349.

V. PROBLEM FORMULATION

Included here are some of the many aids available to help in the solution of some classes of problems. The author does not think of them as optimization techniques but rather as ways of formulating the problem and getting it ready to be handled by one, or more than one, optimizing technique.

V.01 Created Response Surface Technique

a) 55, 56, 112.
b) 113, 114, 115, 116, 281, 294.

V.02 Lagrange Multipliers

a) 8, 93, 104, 130, 226, 303, 330.
b) 7, 9, 24, 139, 189, 194, 293, 310, 349, 361.

V.03 Relaxation or Penalty Functions

a) 2, 54, 80, 259.
b) 27, 37, 131, 284, 293, 309, 310, 322, 349, 364.

V.04 Zener-Duffin

a) 98, 109, 373, 374, 375.

V.05 Fiacco-McCormick

a) 113, 114, 115, 116.
b) 127.

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