PRACNIQUES

The Techniques Department is interested in publishing short descriptions of Techniques which improve the logistics of information processing. To quote from the policy statement, Communications of the ACM 1 (Jan. 1958), 5: "It is preferable that techniques contributed be factual and in successful usage, rather than speculative or theoretical. One of the major criteria for acceptance and the question one should answer before submitting any material is—Can the reader use this tomorrow?" Clear, concise statments of fairly well-known but rarely documented methods will contribute significantly to raising the general level of professional competence.—C.L.McC.

CONSTRUCTION OF NONLINEAR PROGRAMMING TEST PROBLEMS

In order to test a nonlinear programming algorithm it is very useful to be able to construct test problems with known optimum solutions. The purpose of this note is to describe a simple procedure for constructing such test problems. A concave maximization problem subject to concave constraints is used as an example.

The concave maximization problem is

$$\max_x \{\phi(x) \mid h_i(x) \geqq 0, \quad i = 1, 2, \cdots, k\},$$

where $x \in E^m$, and $\phi(x)$ and $h_i(x)$ are real-valued concave functions of x. The procedure will be described for $\phi(x) = \theta(x) + c'x$, and $h_i(x) = q_i(x) + b_i$, $i = 1, \cdots k$, where $\theta(x)$ and $q_i(x)$, $i = 1, \cdots k$ are any selected differentiable concave functions of x, c is a vector $\in E^m$ and the b_i are scalars.

Step I. Choose any $x^0 \in E^m$ as a desired optimum point, and any set of $u_i^0 \geqq 0$, $i = 1, \cdots k$, as the corresponding optimum dual solution. That is, we first specify the primal and dual solution to the problem.

Step II. Choose b_i, $i = 1, \cdots k$, so that $h_i(x^0) = 0$ for $u_i > 0$ and $h_i(x^0) \geqq 0$ for $u_i = 0$. Note that $u_i > 0$ means that the ith constraint is active.

Step III. Let

$$c = -\nabla\theta(x^0) - \sum_{i=1}^{k} u_i^0 \nabla q_i(x^0).$$

This choice satisfies the Kuhn-Tucker condition

$$\nabla\phi(x^0) + \sum_{i=1}^{k} u_i^0 \nabla h_i(x^0) = 0,$$

and therefore ensures that x^0 is an optimum solution to the concave programming problem.

We illustrate this procedure below by applying it to the quadratic problem where $\theta(x) = x'Q_0x$, $q_i(x) = x'Q_ix + a_i'x$, and the Q_i, $i = 0, 1, \cdots k$, are negative semidefinite matrices.

Example (quadratic problem with four variables and three constraints). Let

$$Q_0 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad Q_3 = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$a_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad a_3 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

(Step I)

Let

$$x^0 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \end{pmatrix}$$

and

$$u^0 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

(Step II)

$$h_1(x^0) = x^{0\prime}Q_1x^0 + a_1'x^0 + b_1 = -8 + b_1$$

Since $u_1 > 0$, $b_1 = 8$.

$$h_2(x^0) = x^{0\prime}Q_2x^0 + a_2'x^0 + b_2 = -9 + b_2$$

since $u_2 = 0$, we choose $b_2 = 10$ so that $h_2(x^0) = 1 > 0$.

$$h_3(x^0) = x^{0\prime}Q_3x^0 + a_3'x^0 + b_3 = -5 + b_3$$

Since $u_3 > 0$, $b_3 = 5$.

(Step III)

$$c = \begin{pmatrix} 2x_1^0 \\ 2x_2^0 \\ 4x_3^0 \\ 2x_4^0 \end{pmatrix} + \begin{pmatrix} 2x_1^0 + 1 \\ 2x_2^0 - 1 \\ 2x_3^0 + 1 \\ 2x_4^0 - 1 \end{pmatrix} + 2\begin{pmatrix} 4x_1^0 + 2 \\ 2x_2^0 - 1 \\ 2x_3^0 \\ 0 \quad - 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 21 \\ -7 \end{pmatrix}.$$

The constructed problem is:

minimize

$$\phi = -x_1^2 - x_2^2 - 2x_3^2 - x_4^2 + 5x_1 + 5x_2 + 21x_3 - 7x_4$$

subject to

$$-x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_1 + x_2 - x_3 + x_4 + 8 \geqq 0$$

$$-x_1^2 - 2x_2^2 - x_3^2 - 2x_4^2 + x_1 + x_4 + 10 \geqq 0$$

$$-2x_1^2 - x_2^2 - x_3^2 - 2x_1 + x_2 + x_4 + 5 \geqq 0$$

and has as its optimum function value $\phi(x^0) = 44$.

J. B. Rosen
S. Suzuki
Stanford University
Stanford, California

Received September, 1964

Prepared under NASA grant Ns G 565 at Stanford University.