Automatica, Vol. 5, pp. 347–355. Pergamon Press, 1969. Printed in Great Britain.

The Ψ-Transform for Solving Linear and Non-linear Programming Problems*

La transformation en Ψ pour la solution des problèmes de programmation linéaire et non-linéaire

Die Ψ-Transformierung zur Lösung linearer und nichtlinearer Programmierungsprobleme

Ψ-Превращение для решения проблем линейного и нелинейного программирования

V. K. CHICHINADZE †

The global extremum value, as well as its coordinates, of a non-linear multidimensional objective function may be found approximately, but practically, as the zero value of its Ψ transformation, a monotonically decreasing scalar function.

Summary—This paper is concerned with the problem of determining the global extremum value of a multidimensional, non-linear objective function which may have several extreme values. The problem is solved by transforming the objective function, through a particular Ψ transformation, into a function $\Psi(\zeta)$ of one new variable (ζ). The value of this transformed function is shown to decrease continuously to zero as the value of this new variable is increased, and the value of the variable when the transformed function equals zero is the global extremum of the original objective function. Methods of calculating the transformed function are discussed and examples of the technique are given. It is shown that the values of the system coordinates corresponding to the global extremum can also be determined.

1. INTRODUCTION

THERE are various methods for solving linear programming problems. The use of these methods, however, involves, sometimes, considerable difficulties owing to excessive complicacy of the calculation process. As for the non-linear programming problems, most work is presently concentrated in the field of solving a problem having a convex objective function, i.e. a function with one extremum. In practice, however, not every objective function is convex.

The present work suggests a method of solving for the global extremum rather easily. The use of such a method for solving a linear programming problem may also make the solution considerably easier, though in this case other difficulties may arise.

* Received 7 August, 1968; revised 3 October, 1968 and 20 December, 1968. The original version of this paper was presented at the IFAC Symposium held in Cleveland, Ohio, USA, in June 1968. The paper was considered for publication by associate editor H. Kwakernaak.

† Institute of Electronics, Automatics and Telemechanics Academy of Science Georgian S.S.R.

The basic idea of the suggested method is that a transformed function $\Psi(\zeta)$ of one variable (ζ) may be investigated instead of an objective function $f(X)$ in n-dimensional space. With such an approach, the problem of discovering the extremum of the objective function $f(X)$ may be considerably simplified. In the case of non-linear programming, the function $f(X)$ may prove to be many extremal. It is readily proved that whatever the number of extremums of the objective function $f(X)$, the function $\Psi(\zeta)$ is always a decreasing one as (ζ) increases. With the appropriate system restrictions being available, it is proved that the discovery of the zero of the function $\Psi(\zeta)$ permits the minimum or the maximum of objective function $f(X)$ to be determined. This extremum, determined as a scalar, is the value of (ζ) when $\Psi(\zeta)=0$. The corresponding values of the coordinates, $\mathbf{x}_1, \ldots, \mathbf{x}_n$, may also be found from an appropriate algorithm based on a method of search in some region of the extremum as suggested in this paper.

In comparing the value of the extremum, found using the $\mathbf{X}$ vector in the search, with the zero of function $\Psi(\zeta)$, it is possible to determine as one-valued both the value of the global extremum and the coordinates $\mathbf{x}_1, \ldots, \mathbf{x}_n$.

A simplified illustration of the Ψ transformation is given in Fig. 1, but it is only a very elementary indication of the principle involved. Actually, the nature of the transformation in n-dimensional space is much more general and involved, and the various methods of making the transformation as well as its basic properties, conditions for existence, and use in determining a global extremum are discussed in the following sections.

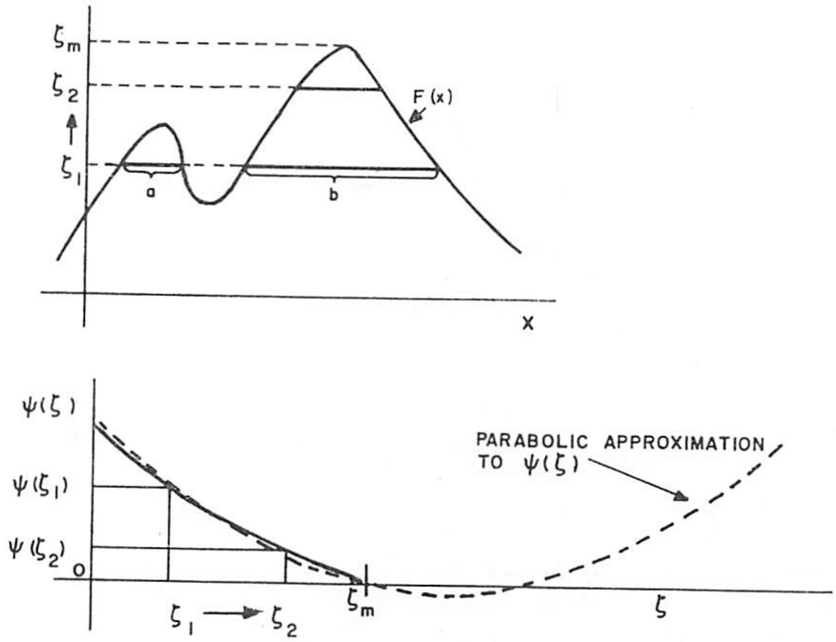

FIG. 1. A simplified illustration of the Ψ transformation where $\Psi(\zeta_1)=a+b$. If $f(\mathrm{x})$ were a function of one variable, $\Psi(\zeta_1)$ would be the sum of the cross sectional areas at a and b.

2. n-DIMENSIONAL SPACE FUNCTIONS AND THE Ψ-TRANSFORMATION

The transformation of the objective function, $f(\mathbf{X})$ where $\mathbf{X}$ is an n-dimensional vector, is carried out by some non-linear operator

$$\mathscr{L}\{f(\mathbf{x})\}\to\Psi(\zeta) \tag{1}$$

where scalar ζ is a modular value of function $f(\mathbf{X})$. The value of function $f(\mathbf{X})$ in each point of ζ_i corresponds to the measure of some set E_i^* on which the value of function $f(\mathbf{X})$ is more (less) or equal to that of ζ_i

$$\mu^*=\Psi(\zeta) \tag{2}$$

such a transformation of function $f(\mathbf{X})$ is carried out by Lebesgue's division of $f(\mathbf{X})$. Let the upper and the lower borders of function $f(\mathbf{X})$ be

$$\zeta_o=\inf f(\mathbf{x}) \tag{3}$$

$$\zeta_M=\operatorname{Sup} f(\mathbf{x})\,. \tag{4}$$

Lebesgue's division of the objective function $f(\mathbf{X})$ divides $(\zeta_o;\,\zeta_M)$ into equal parts

$$\inf f(\mathbf{x})=\zeta_o<\zeta_1<\zeta_2$$

$$<\ldots<\zeta_i<\ldots<\zeta_M=\operatorname{Sup} f(\mathbf{x})\,. \tag{5}$$

Now the points on the hypersurface $\mathbf{x}_1,\ldots,\mathbf{x}_n$, where

$$|f(x_1,\ldots,x_n)|\geq\zeta_i \tag{6}$$

are in question. Let the points mentioned be considered to correspond to some set $E_i^*(|f(\mathbf{x})|\geq\zeta_i)$. Let E be the set on which the function $f(\mathbf{X})$ is determined. Then

$$E_i^*(|f(\mathbf{x})|\geq\zeta_i)\subset E\,. \tag{7}$$

Next two sets $\bar{E}$ and $\bar{E}^*$ are selected from the sets E and E^* such that

$$\bar{E}_i^*\subset\bar{E}\subset E\,. \tag{8}$$

Sets $\bar{E}$ and $\bar{E}^*$ are determined by a system of restrictions. The condition of the restrictions is that the vector $\mathbf{x}$ is in region $\bar{E}$ or $\bar{E}_i^*$. If the characteristic function of the type

$$\theta_i(x_1,\ldots,x_n)=\left|\begin{matrix}1 \text{ on } \bar{E}\\ 0 \text{ on } \bar{E}/\bar{E}_i^*\end{matrix}\right. \tag{9}$$

is considered, it is readily seen that the Riemann integral from this function on set $\bar{E}$ will be equal to the measure $\bar{\mu}^*$ of $\bar{E}^*$

$$(R)\int_{(\bar{E})}\cdots\int\theta(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n=\bar{\mu}^*\,. \tag{10}$$

In the case of non-linear programming, an objective function $f(\mathbf{X})$ being non-convex, set $\bar{E}^*$ may include several subsets $e_k^*(k=1,2\ldots l)$, where l is a number of extremums.

Thus, the Ψ-transformation and the construction of function $\Psi(\zeta)$ can be carried out by calculating the many dimensional integral (10).

While solving the problems of mathematical programming, an objective function of different types may be encountered as follows:

1. (a) An objective function which is linear

$$f(\mathbf{x})=\sum_{i=1}^{n}\beta_i x_i \tag{11}$$

and (b) with restrictions which are linear as well.

2. An objective function which, like all its derivatives, is a continuous one. Restrictions may be both linear and non-linear.

3. An objective function which is continuous, while its derivatives undergo the breaks of the first type. This is the case when $f(X)$ is described by different equations on different sections. The system of restrictions may be linear or non-linear.

4. An objective function which in some points of set E, the number of which is finite, undergoes the breaks of the first type.

5. An objective function which is described by different equations and, besides, in some points, the number of which is finite, undergoes the breaks of the first type.

In items 4 and 5, the restrictions are considered to be of any type.

Let the objective functions, adduced in items 1–5, be united into some class C. Pick out the class $C^{(0)}$ from class C. This class is characterized by the fact that the gradient of the objective function on some of the so-called solid subsets $\angle$ of set E is zero. The solid set $\angle$ is distinguished by having no point $y \notin \angle \setminus [\angle]$ in the neighborhood $\rho(m_\alpha, m_\beta)$ of any element m_α of this set.

Thus, the condition $f(X) \in C^{(0)}$ can be written down as follows:

$$\mathrm{d}f_{(b)} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \mathrm{d}x_i = 0\,. \tag{12}$$

Now let the class $C^{(k)} = C \setminus C^{(0)}$ be introduced. Class $\bar{C}^{(k)}$ including a continuous function $f(X)$, may be selected from class $C^{(k)}$.

Some properties, very important for Ψ-transformations, follow:

1. If $f(X) \in \bar{C}^{(k)}$, the transformed function $\Psi(\zeta)$ is continuous and monotonically decreasing. In fact, for any value there exists a limit

$$\lim_{\zeta \to \zeta_\alpha} \Psi(\zeta) = \Psi(\zeta_\alpha)\,. \tag{13}$$

For any two values $\zeta_{\alpha-\Delta}$ and $\zeta_{\alpha+\Delta}$ there are measures $\mu_{\alpha-\Delta}$ and $\mu_{\alpha+\Delta}$ of corresponding sets $E(f>\zeta_{\alpha-\Delta})$ and $E(f>\zeta_{\alpha+\Delta})$, which are determined as

$$\mu_{\alpha-\Delta} = \int_{E(f>\zeta_{\alpha-\Delta})} \cdots\cdots \int \theta(x_1, \ldots, x_n)\mathrm{d}x_1 \ldots \mathrm{d}x_n \tag{14}$$

and

$$\mu_{\alpha+\Delta} = \int_{E(f>\zeta_{\alpha+\Delta})} \cdots\cdots \int \theta(x_1, \ldots, x_n)\mathrm{d}x_1 \ldots \mathrm{d}x_n\,. \tag{15}$$

As $\Delta \to 0$, the values of integrals (14) and (15) are readily seen to approach one another as closely as needed.

2. If the function $f(X)$ is continuous but the derivative $f'(X)$ undergoes breaks of the first type, the transformed function $\Psi(\zeta)$ is continuous and monotonically decreasing. The proof of the continuity of $\Psi(\zeta)$ in the points of the breaks in the function $f'(X)$ must necessarily be shown. Suppose $f'(X)$ undergoes the breaks at a certain point where the value of $f(X)$ is equal to ζ_α. As the function $f(X)$ itself has no breaks at this point, there exists a limit (13). Measures $\mu_{\alpha-\Delta}$ and $\mu_{\alpha+\Delta}$, therefore, are close to each other at $\Delta \to 0$ and

$$|\mu_{\alpha-\Delta} - \mu_{\alpha+\Delta}| < \varepsilon \tag{16}$$

where ε is as small a number as is needed.

3. If $f(X) \in C^{(k)}$, $\Psi(\zeta)$ is continuous and decreasing. It will be recalled that if it belongs to class $C^{(k)}$, the function $f(X)$ on set $\bar{E}$ may have breaks of the first type. To prove the fact, let $f(X)$ have breaks of the first type on the set $\bar{E}_\alpha \subset \bar{E}$.

Let another set $E_{\alpha+\Delta}$ be considered, which is characterized by

$$\rho([E_{\alpha+\Delta}]\,;\,[E_\alpha]) < \varepsilon \tag{17}$$

where ρ is a distance symbol. Suppose $f(X)$ on set $E_{\alpha+\Delta}$ assumes the value $f_{\alpha+\Delta}$ and $|f_\alpha - f_{\alpha+\Delta}| > d$ where d is some number. This means that since there is no limit

$$\lim_{X \to X_\alpha} f(X) = f(X_\alpha) \tag{18}$$

in changing the value of the function within

$$\begin{aligned} f_\alpha(X_\alpha) &\le |f(X)| \\ &\le f_{\alpha+\Delta}(X) \end{aligned} \tag{19}$$

the multidimensional integral

$$\int_{E(f_\alpha<f<f_{\alpha+\Delta})} \cdots\cdots\cdots \int \theta(x_1, \ldots, x_n)\mathrm{d}x_1 \ldots \mathrm{d}x_n$$

will be a constant. The function on the interval $(f_\alpha; f_{\alpha+\Delta})$ is, therefore, constant.

Thus the fact that the transformed function $\Psi(\zeta)$ is continuous and decreasing, while the original $f(X)$ undergoes breaks of the first type, has been proved.

4. If $f(X) \in C^{(0)}$, the function $\Psi(\zeta)$ on the solid sets $\angle$ undergoes the first type of breaks. It will be recalled that the function belongs to class $C^{(0)}$ provided that equation (12) is fulfilled. According to [7]

$$\int_E f(X)\mathrm{d}x = \lim_{\Delta f \to 0} \Delta f \sum_\nu \mu_\nu\,. \tag{20}$$

Since $f(X)$ is a bounded function and has no points of the second type of breaks on set $\bar{E}$

$$\lim_{\Delta f \to 0} \Delta f \sum_\nu \mu_\nu = a$$

where a is some constant. The set $\bar{E}$ may be represented as the sum

$$\bar{E}=\bar{E}_\alpha U\bar{E}_\beta \tag{21}$$

where $\bar{E}=Ue_r$ is the sum of sets, and where

$$\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\mathrm{d}x_i=0$$

and $\bar{E}_\beta=\bar{E}\backslash\bar{E}_\alpha$ is the set where

$$\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\mathrm{d}x_i\neq 0$$

$$\bar{E}_\alpha\cap\bar{E}_\beta=\phi. \tag{22}$$

According to (21) it is possible to write

$$\mu_\nu=\int_{E_\alpha(f>\zeta_\nu)}\cdots\int\theta(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n + \int_{E_\beta(f>\zeta_\nu)}\cdots\int\theta(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n. \tag{23}$$

Consequently,

$$\sum_{\nu=1}^{M}\mu_\nu = \sum_{\nu=1}^{M}\left\{\int_{E_\beta(f>\zeta_\nu)}\cdots\int\theta(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n\right\} + \int_{E_\alpha(f>\zeta_\nu)}\cdots\int\theta(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n. \tag{24}$$

Since

$$\int_{E_\alpha(f>\zeta_\nu)}\cdots\int\theta(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n = \sum_{j=1}^{r}\int_{E_j(f>\zeta_\nu)}\cdots\int\theta(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n. \tag{25}$$

It follows

$$\sum_{\nu=1}^{M}\left\{\int_{E\beta(f>\zeta_\nu)}\cdots\int\theta(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n\right\} + \int_{E_\alpha(f>\zeta_\nu)}\cdots\int\theta(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n$$
$$= \sum_{\nu=1}^{M}\left\{\int_{E_\beta(f>\zeta_\nu)}\cdots\int\theta(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n\right\} + \sum_{\nu=1}^{M}\sum_{j=1}^{r}\int_{e_j(f>\zeta_\nu)}\cdots\int\theta(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n. \tag{26}$$

By multiplying the last equation by Δf, examining the limit as $\Delta f\to 0$ and taking into consideration that $a=\text{const}$, some values

$$\sum_{\nu=1}^{M}\left\{\int_{E_\beta(f>\zeta_\nu)}\cdots\int\theta(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n\right\} \tag{27}$$

and

$$\sum_{\nu=1}^{M}\sum_{j=1}^{r}\left\{\int_{e_j(f>\zeta_\nu)}\cdots\int\theta(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n\right\} \tag{28}$$

are obtained which are convergent. Consequently

$$\lim_{\nu\to\infty}\int_{E_\beta(f>\zeta_\nu)}\cdots\int\theta(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n=0. \tag{29}$$

Since on the sets e_j

$$\sum_{j=1}^{M}\frac{\partial f}{\partial x_i}\mathrm{d}x_i=0$$

then

$$\lim_{\nu\to\infty}\int_{e_j(f>\zeta_\nu)}\cdots\int\theta(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n = \begin{cases}\int_{e_j(f>\zeta_\nu)}\cdots\int\theta(x_1,\ldots,x_n)\mathrm{d}x_1\ldots\mathrm{d}x_n & \text{for } \nu=\sigma\\ 0 & \text{for } \nu>M\end{cases} \tag{30}$$

where σ is some number.

The function $\Psi(\zeta)$ at the point ζ_ν, therefore, undergoes the first type break.

On the basis of the above and also taking into consideration the results obtained in [7], the following may be concluded:

1. The function $\Psi(\zeta)$ is in all cases, except when $f(x)\in c^{(0)}$ is monotonically decreasing.
2. The function $\Psi(\zeta)$ when $f(x)\in c^{(0)}$ has the first type of breaks.
3. The zero of the function $\Psi(\zeta)$ corresponds to the scalar value of the global extremum of the function $f(\mathbf{X})$.

3. CONSTRUCTION OF THE TRANSFORMED FUNCTION $\Psi(\zeta)$ AND THE DETERMINATION OF THE GLOBAL EXTREMUM OF OBJECTIVE FUNCTION $f(\mathbf{X})$

An essential peculiarity of obtaining the function $\Psi(\zeta)$ is a difficulty of an analytical character, connected with the process of making the transformation of $f(\mathbf{X})$ into $\Psi(\zeta)$. Analytically only unimodal functions of three variables can be handled. Hence, from the practical point of view, the analytical method of making the transformation of the function $f(\mathrm{X})$ cannot be recommended. Nevertheless, the function $\Psi(\zeta)$ can be constructed at any point by calculating the multi-dimensional integral (10), but as noted the analytical calculation of the integral mentioned involves a fair number of difficulties. The determination of the multi-dimensional integral, therefore, is carried out practically by a method of statistical tests. An unknown quantity of measure of set $\bar{E}_i^*$ can be determined by carrying out a certain number of random tests. In this case the *a priori* probability of hitting the region $\bar{E}_i^*$ is equal to

$$P_i = \frac{\overset{\cup}{m}\bar{\mu}_r^*}{\bar{\mu}}. \tag{31}$$

According to the law of large numbers

$$\lim_{s\to\infty} P_r\left\{\left|\frac{\xi}{s} - P_i\right| > \varepsilon\right\} = 0 \tag{32}$$

for any $\varepsilon > 0$; here ξ is the number of points, having hit the region and s is the number of tests. For the estimation of $\bar{\mu}^*$, a certain number of such tests must be carried out, and in so doing a given system of restrictions must be fulfilled. Next it is necessary to calculate the number of points in the tests when $|f(\mathrm{x})| \geq \zeta_i$ and to calculate the quotient

$$\bar{\mu}^* = \frac{\xi}{s}. \tag{33}$$

The error $|(\xi/s) - p|$ will have the order $1/\sqrt{s}$. A Bernoulli scheme is employed to carry out the tests. The number of tests on each step of ζ_i depends on the required accuracy of calculating the many dimensional integral (10). The probability for realizing the set $\bar{E}^*$ at $|f(\mathrm{x})| \geq \zeta_i$ is determined as

$$\beta\left(\xi;\ s,\ \frac{\bar{\mu}^*}{\bar{\mu}}\right) = \binom{s}{\xi} \times \left(\frac{\bar{\mu}^*}{\bar{\mu}}\right)\left(1 - \frac{\bar{\mu}_i^*}{\bar{\mu}}\right)^{s-\xi}. \tag{34}$$

Such a method of determining the measure $\bar{\mu}^*$ is not exact, therefore, the process of the discovering the zero of $\Psi(\zeta)$ comes to that of discovering the zero of a regressive line of the random function $\bar{\Psi}(\zeta)$. It is impossible, however, to determine the value ζ_M with sufficient accuracy when $\bar{\Psi}(\zeta) = 0$, because in the neighborhood of this point, the probability of realizing the event $|f(\mathrm{x})| = \zeta_M$ is infinitesimally small. The way out is to approximate the function $\bar{\Psi}(\zeta)$ with a curve determined from points calculated for values of $\zeta < \zeta_M$. Subsequently, this curve is extrapolated to determine the zero of $\bar{\Psi}(\zeta)$ more accurately.

The dispersion $\bar{D}$ of the binomial law of distribution is known to be equal to

$$\bar{D} = s\frac{\bar{\mu}^*}{\bar{\mu}}\left(1 - \frac{\bar{\mu}^*}{\bar{\mu}}\right). \tag{35}$$

Such a dispersion will have a maximum value when

$$\bar{\mu}_i^* = 0{\cdot}5\bar{\mu}. \tag{36}$$

Hence, the function $\bar{\Psi}(\zeta)$ will have a maximum deflection from its mathematical expectation when $\Psi(\zeta) = \bar{\mu}/2$. The probability γ of the deflection of the value $\bar{\Psi}(\zeta)$ from its mathematical expectation is also of interest. A proper estimate of the probability mentioned can be obtained from a well-known Tchebyscheff formula. According to Tchebyscheff's inequality, the e is

$$\gamma(|\bar{\Psi}(\zeta) - M(\zeta)| \geq \Delta) \leq s\bar{\mu}_i^* \frac{\bar{\mu} - \bar{\mu}_i^*}{\bar{\mu}} \tag{37}$$

where Δ is a positive number and $M(\zeta)$ is the mathematical expectation of the function $\bar{\Psi}(\zeta)$. By using formulae (35) and (37), the accuracy of the Ψ-transformation of the function $f(\mathrm{X})$ may be estimated to some extent.

The curve approximating the function $\bar{\Psi}(\zeta)$ for points calculated within the half-segment $[0;\ \zeta_M]$ may be obtained in different ways; for example, by a trigonometric series

$$T(\zeta) = \tfrac{1}{2}b + \sum_{k=1}^{\infty} b_k \cos k\zeta \tag{38}$$

or by some polynomial

$$D(\zeta) = a_0\zeta^m + a_1\zeta^{m-1}. + \ldots + a_m. \tag{39}$$

Of course, the approximation is made to determine the zero of $\bar{\Psi}(\zeta)$ which approximates $\Psi(\zeta)$. Since ζ_M is an unknown quantity as well as the root of the equation $\Psi(\zeta)$, an extrapolation or prediction problem arises. A Taylor's series is known to have such an extrapolation property. The extrapolation can be carried out by the Gregory–Newton formula. With empiric, and more so with random functions, the use of Gregory–Newton and Taylor's formulae is known to have considerable restrictions arising from the fact that the error generated in the realization of $\Psi(\zeta)$ may cause large errors in the high order differences. The increase of such errors in determining the high order differences occurs according to the binomial law, and it may cause the results of approximation and, more so, of the extrapolation to lead to nothing. The use of the Gregory–Newton and Taylor formulae is, therefore, possible only for small quantities of dispersion, which is the case, when it is possible to carry out a great number of tests.

A sufficiently suitable accuracy for $\Psi(\zeta)$ can be realized by using methods of regressive analysis, a least squares technique.

The method of the least squares provides on the average a fairly good approximation. The use of this method involves the solution of some system of linear equations of the type

$$Y\bar{\mathbf{a}}=G \tag{40}$$

where $\bar{\mathbf{a}}$ is a column-vector, the elements of which are the coefficients of the polynomial (39).

The elements of the matrix Y are the values $[\zeta^k]$. According to the Gauss designation

$$[\zeta^k]=\sum_{i=1}^{m}\zeta_i^k$$

$$k=1, 2 \ldots m \tag{41}$$

where m is the order of matrix Y. Furthermore,

$$G=\begin{bmatrix}[\bar{\mu}]\\ [\zeta, \bar{\mu}]\\ \cdot\\ \cdot\\ \cdot\\ [\zeta^m, \bar{\mu}]\end{bmatrix}. \tag{42}$$

Since the function $\Psi(\zeta)$ is monotone, except in the case when $f(\mathrm{x})\in c^{(0)}$ such a function can be easily approximated by a parabola. In this case

$$Y=\begin{bmatrix}\alpha_4^*[\zeta]; & \alpha_3^*[\zeta]; & \alpha_2^*[\zeta]\\ \alpha_3^*[\zeta]; & \alpha_2^*[\zeta]; & \alpha_1^*[\zeta]\\ \alpha_2^*[\zeta]; & \alpha_1^*[\zeta]; & \alpha_0^*[\zeta]\end{bmatrix} \tag{43}$$

and

$$G=\begin{bmatrix}\alpha_{2\cdot 1}^*[\zeta, \bar{\mu}^*]\\ \alpha_{1\cdot 1}^*[\zeta, \bar{\mu}^*]\\ \alpha_{0\cdot 1}^*[\zeta, \bar{\mu}^*]\end{bmatrix} \tag{44}$$

and

$$D(\zeta)=a_0\zeta^2+a_1\zeta+a_2 \tag{45}$$

where

$$a=\begin{bmatrix}a_0\\ a_1\\ a_2\end{bmatrix} \tag{46}$$

Here, $\alpha_{ij}^*[\zeta, \bar{\mu}^*]$ are correlation moments which are calculated according to realized values of the function $\overline{\Psi}(\zeta)$. The roots of the equation (45) can be easily determined, after it is equated to zero. The least positive root of the equation will correspond to the global extremum of the objective function $f(\mathbf{X})$.

Stochastic approximation is also of great interest in solving this type of problem.

4. A QUASI-EXTREMAL REGION AND THE DISCOVERY OF COORDINATES OF THE GLOBAL EXTREMUM

The Ψ-transformation, as already mentioned, permits the global extremum to be determined as a scalar. In so doing the values $x_1, \ldots, x_n$ at which the extremum of the objective function is realized, remain unknown. This problem, however, can be solved as one-valued by comparing a certain scalar quantity of the global extremum with the extremum in some quasi-extremal region, determined by methods in [1], [4], [5], [6].

To determine the coordinates of the global extremum, the largest (the least) quantities of the objective function, which were discovered as a result of a random search, are used. Such a search, as noted, was necessary for the determination of many dimensional integral (10). As the dispersion of the random function depends on the number of tests carried out, it is, therefore, desirable to carry out as many such tests as possible. The number of such tests is limited by a space measure, where the function $f(\mathbf{X})$ is given, and by the capabilities of a computer as well. Up-to-date high-speed computers permit the region of the global extremum to be discovered with much greater probability. Let the region of the global extremum be estimated by some norm $\|\Delta\|$

$$\|\Delta\|=(\int|\zeta_c-\zeta|^2 d\Omega)^{\frac{1}{2}} \tag{47}$$

where ζ_M is equal to the global extremum, ζ_c to the local extremum, which is in the immediate proximity to the global one. The hypersurface, drawn across the local extremum, together with the restrictions, determines the set $\bar{E}_c^*$ on which the following condition is satisfied

$$|f(\mathrm{x})|\geq\zeta_c. \tag{48}$$

According to [7] the measure of the set $\bar{E}_c^*$ is determined as

$$\bar{\mu}_c^*=-\frac{\bar{\mu}\lambda}{\tau v}\log(1-\beta) \tag{49}$$

where β is a probability determined according to (34), τ is the working time of the computer without failure, v is an average speed of the computer, and λ is the number of commands in the program.

In the case where β, τ, ν are small, the probability of hitting the region of the global extremum is also small and the method suggested cannot guarantee the discovery of the coordinates of the global extremum. In this case, one must be satisfied with the determination of the global extremum value.

The coordinates of the global extremum will be found only provided that one of these two inequalities is fulfilled:

$$\bar{\mu}_c^* \geq -\frac{\bar{\mu}\lambda}{\tau\nu}\log(1-\beta) \tag{50}$$

or

$$\inf \rho(\zeta_{rM}^j, \zeta_M) < \delta \tag{51}$$

and when

$$\bar{\mu}_{j-1}^* \geq -\frac{\bar{\mu}\lambda}{\tau\nu}\log(1-\beta) \tag{52}$$

where j is an ordinal number of local extremums ζ_{rM}, arranged in ascending (descending) order: $\zeta_{rM}^1 < \zeta_{rM}^2 < \ldots < \zeta_{rM}^j < \ldots < \zeta_M$; δ is an allowable error.

In case neither condition (44) or (45) is fulfilled, only the value of the global extremum at the unknown $x_1, \ldots, x_n$ may be determined.

5. SUCCESSIVE OPERATIONS FOR SOLVING THE PROBLEMS OF MATHEMATICAL PROGRAMMING BY THE Ψ-TRANSFORMATION METHOD

Let a simple algorithm for solving the problems of mathematical programming be considered. Suppose the maximum of a linear or non-linear objective function $f(\mathrm{X})$ is to be found when the restrictions or constraints on the system variables are:

$$Q(\mathrm{x}) \leq B_j \qquad j=1, 2 \ldots \tag{53}$$

The function $Q(\mathrm{X})$ may also be either linear or non-linear. To solve the problem the following must be done:

1. Determine an approximate range of the change of function $f(\mathrm{X})$. With this end in view, let random tests be carried out. Assume the least value, ζ_1, of the function $f(\mathrm{X})$ corresponding to a trail point on the ζ axis of function $\Psi(\zeta)$.

2. To obtain this assumed least value ζ_1 of $f(\mathrm{X})$ let statistical tests, s in number, be carried out. The number s depends on the capabilities of the computer. The statistical test may be made by adding a special generator of random numbers to the computer or with a program of random numbers. Thanks to this program, or to the generator, the values $x_1, \ldots, x_n$ are chosen from a changing region of these values. However, a check of the restricting, or constraint, equations (53) must follow each random selection of $x_1, \ldots, x_n$, and the function $f(\mathrm{X})$ is calculated only if the restrictions mentioned are satisfied. Then the calculated value of $f(\mathrm{X})$ is compared with ζ_1. In case the restriction system is not satisfied another set of random values $x_1, \ldots, x_n$ is chosen and so forth until the condition (53) is satisfied. After condition (53) is satisfied s times, the number of points, ξ, for which $|f(\mathrm{x})| \geq \zeta_1$ is determined. Then, according to (33), a point $\bar{\mu}_1^*$ of the function $\Psi(\zeta)$ is determined along with a corresponding set of restricting equations.

3. Similarly the statistical tests are carried out for another value, perhaps $\zeta_2 = 2\zeta_1$. A point $\bar{\mu}_2^*$ of the function $\Psi(\zeta_1)$ is determined in a similar way.

4. Successively larger values of ζ are assumed until the condition $|f(\mathrm{x})| \geq \zeta_i$ is not fulfilled. Then the problem of approximating and extrapolating the $\Psi(\zeta)$ function from the calculated points, $\bar{\mu}^*$, must be solved.

5. The values $\alpha_{ij}^*[\zeta, \mu]$, the correlation moments for (44) where $i=0, 1, 2, 3, 4$; $j=0, 1, 2$, are calculated by certain formulae as indicated previously.

6. a_0, a_1 and a_2 are then calculated from (40).

7. The roots of the corresponding quadratic equation

$$a_0\zeta^2 + a_1\zeta + a_2 = 0$$

are determined, one of which, the least positive, is equal to the global extremum of function $f(\mathrm{X})$.

8. In the process of making the statistical tests, the largest value of the function $f(\mathrm{X})$ is retained in the computer memory. The maximum value of $f(\mathrm{X})$ is determined in the neighborhood of this value by one of several well known techniques: the gradient method, the steepest descent method, a method of a successive search and the like.

9. A comparison of values of $f(\mathrm{X})$, determined in items 7 and 8, is carried out. If coinciding values are found, the problem is considered to be completely solved. If the value $f(\mathrm{X})$, determined according to item 8, is less than the one determined according to item 7, the statistical test continues until a larger value of $f(\mathrm{X})$ is discovered. Then the maximum in the neighborhood of this quantity is again defined in terms of the coordinates and the last one is compared with the quantity, defined according to item 7.

It should be noted that in some cases, the calculation of the initial part of function $\Psi(\zeta)$ in the process of its approximation and extrapolation may bring about an error. Such a case may occur in some problems of linear programming. The way out is to omit those parts and to approximate the rest of function $\Psi(\zeta)$. The necessity of so doing may be judged from the change of the derivative.

6. ILLUSTRATIVE EXAMPLES

Consider two following examples, one with a linear objective function and the other with a non-linear objective function.

a. The minimum of the linear objective function below is to be found.

$$f(x_1, x_2)=3x_1-x_2+8x_3+2x_4-x_5+9x_6$$

with the system of restrictions:

$$-6x_1+9x_2+3x_3+3x_2-2x_5-x_6\leqslant 12$$
$$-4x_2+3x_3-3x_4+x_5-x_6\leqslant 5$$
$$2x_1+8x_2-5x_3+6x_4-8x_5+4x_6\leqslant 20$$
$$-x_1-3x_2-4x_3-8x_4+4x_6\leqslant 10$$
$$5x_1+x_2+2x_3+4x_4+9x_5+5x_6\leqslant 24.$$

The value of the function $\Psi(\zeta)$ in each point of ζ_i was determined by the Monte-Carlo method.* The number of tests was $S_i=1000$. The function is shown in Fig. 2.

The approximation and extrapolation of function $\overline{\Psi}(\zeta)$ permits the maximum value of objective function $f_M=55{\cdot}0$ to be determined. The exact value is $f_{MT}=57{\cdot}0$. The error is of the order 3·5 per cent.

b. The maximum of the multi-extremal non-linear objective function below is to be found.

$$f(x_1, x_2)=-x_1^2+12x_1-11-10\cos\frac{\pi}{2}x_1-8\sin\pi x_1+\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x_2-0{\cdot}5)^2}{2}\right)$$

with the restrictions:

$$0\leqslant x_1\leqslant 10$$
$$0\leqslant x_2\leqslant 5$$
$$0{\cdot}5x_1+2x_2\leqslant 8.$$

This problem was solved by the method described. With this end in view, the number of tests, $S_i=100$ for each point of ζ_i, was not large.

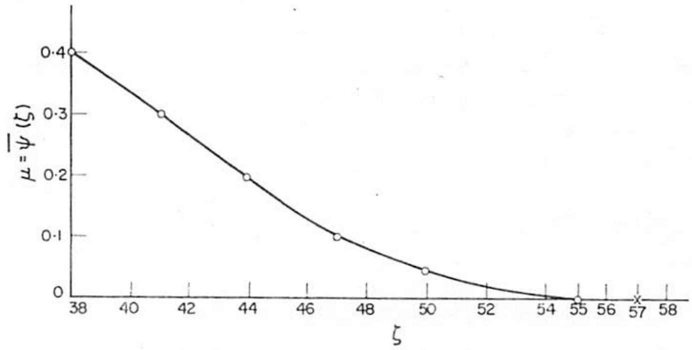

FIG. 2. Convergence of the transformed function $\Psi(\zeta)$ corresponding to the linear objective function in example 1.

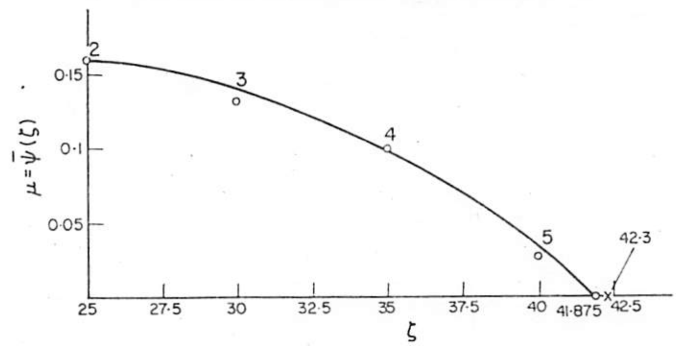

FIG. 3. Convergence of the transformed function $\Psi(\zeta)$ corresponding to the non-linear objective function in example 2.

* It should be noted that the Monte Carlo method is not an essential feature of the Ψ-transformation. As indicated previously, it is sometimes possible to carry out the Ψ-transform analytically, and in this case the Monte Carlo method need not be used at all. In other cases, the Ψ-transformation may be carried out by statistical tests and by Bernoulli formulae. Thus the transformation is not any extension of the Monte Carlo method.

In the first test, ζ_1 was assumed to be 30, then the initial value of the argument of function $\Psi(\zeta_1)=\mu_1^*$ was determined. The value $\Delta\zeta=5$ was chosen as the step. Thus ζ_2 for the second trial was assumed to be 35, and so on. In Fig. 3, the function $\Psi(\zeta)=\bar{\mu}^*$ is shown. According to the extrapolation and approximation of the points, the global extremum of the non-linear objective function is

$$ {}_M=\zeta_M=41{\cdot}875 $$

when $\Psi(\zeta)=0$.

As a result of the statistical tests, the coordinates corresponding to the extremal value were found as indicated in section 58, and thus in the neighborhood of the calculated extremal point, f_M, the exact value of the global extremum, f_{MT}, was found to be

$$ f_{MT}(\mathrm{X}_1, \mathrm{X}_2)=42{\cdot}3 $$

with the coordinates

$$ X_1=6{\cdot}3 $$

$$ X_2=0{\cdot}5\,. $$

Note that the error between the values of f_M and f_{MT} above is less than 1·0 per cent.

7. CONCLUSIONS

The method suggested can be used for solving problems of non-linear and linear programming.

It should be noted that in solving these problems by the Ψ-transformation method, the time is basically spent on the calculation of the objective function $f(\mathbf{X})$ and also on checking-up the system of inequalities. Such calculations should follow each step. From this point of view, the use of a hybrid computer, the analogue part of which evaluates $f(\mathbf{X})$ and the restriction relationships (53) is worth bearing in mind. In cases where sufficient accuracy is available, the calculation of $f(\mathbf{X})$ and the check of system restrictions (53) can be carried out instantly. It permits the class of problems, which are solved according to the method given, to be considerably widened.

REFERENCES

[1] Д. Б. Юдин: Методы количестьенного анализа сложных систем. Техническая кибернетика Но. 1 1965г. и Но. 1 1966г.

[2] S. H. Brooks: Discussion of random methods for seeking maxima. *Operation Research* (1958), March-April, v. 6. (1958).

[3] D. J. Wilde: *Optimum Seeking Methods*. Prentice-Hall, Inc., (1964).

[4] С. М. Мовшович: Некоторые оценки времени случайного поиска глобального эксперимента .Сборник-Прикладные задачи технической кибернетики. Советское радио. Москъа, (1966г).

[5] Л. А. Растригин: Случайный поиск. Изд-во "Знание" Рига (1965г).

[6] А. А. Фельдбаум: Вычислительные устройства авюматических систем. Физматгиз (1959г).

[7] В. К. Чичинадзе: Об одном способе использования случайного поиска для определения экстремума функций нескольких переменных. Техническая кибернетика. Но. 1, (1967г).

Résumé—Cet article se réfère au problème de détermination de la valeur de l'extrémum global d'une fonction objective non-linéaire de variables multiples qui peut comporter plusieurs valeurs extrêmes. Le problème est résolu en transformant la fonction objective, au moyen d'une transformation particulière en Ψ, en une fonction Ψ (ζ) d'une seule variable nouvelle (ζ). Il est montré que la valeur de cette fonction transformée décroît continuellement à zéro avec l'augmentation de la valeur de cette nouvelle variable et la valeur de la variable pour laquelle la fonction transformée atteint zéro constitue l'extrémum global de la fonction objective originale. L'article discute des methodes de calcul de la fonction transformée et donne des exemples de la technique employée. Il est montré que les valeurs des variables du système correspondant a l'extrémum global peuvent également être determinées.

Zusammenfassung—Behandelt wird das Problem der Bestimmung des globalen Extremalwerts einer mehrdimensionalen nichtlinearen Zielfunktion, die mehrere Extremwerte besitzen kann. Das Problem wird durch Transformation der Zielfunktion gelöst, und zwar durch eine partikuläre Transformation in eine Funktion Ψ (ζ) der neuen Variablen ζ. Der Wert dieser transformierten Funktion nimmt, wie gezeigt wird, kontinuierlich auf Null ab, wenn der Wert der neuen Variablen anwächst. Andererseits nimmt der Wert der Variablen, wenn die transformierte Funktion gleich Null wird, das globale Extremum der Originalzielfunktion an. Methoden zur Berechnung der transformierten Funktion werden diskutiert und Beispiele angegeben. Nachgewiesen wird, daß auch die Werte der den globalen Extremum entsprechenden Koordinaten bestimmt werden können.

Резюме—Эта статья относится к проблеме определения значения общего экстремума многокоординатной нелинейной объективной функции могущей иметь несколько крайних значений. Задача решена превращая объективную функцию, при помощи особого Ψ—превращения, в функцию Ψ(ζ) единственной новой переменной (ζ). Показывается что значение этой превращенной функции постоянно убывает до нуля с увеличением значения этой новой переменной и значение переменной для которого превращенная функция достигает нуля составляет общий экстремум первоначалной объективной функции. Статья обсуждает методы вычисления превращенной функции и дает примеры используемой техники. Показывается что значения координат системы соответствующие общему экстремуму могут быть также определены.