

The Global Optimisation Problem: An Introduction

by

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Abstract

In this paper we introduce the papers of the volume paying special emphasis on the philosophy of the standard set of computational tests. A summary of the performance of the various algorithms in these tests is also given.

1. Practical Applications

Many important practical problems can be posed as mathematical programming problems. This has been internationally appreciated since 1944 and has led to major research activities in many countries, in all of which the aim has been to write efficient computer programs to solve subclasses of this problem. An important subclass that has proved very difficult to solve occurs in many practical engineering applications. Let us consider the design of a system that has to meet certain design criterion. The system will include features that may be varied by the designer within certain limits. The values given to these features will be the optimisation variables of the problem. Frequently when the system performance is expressed as a mathematical function of the optimisation variables, this function, which will sometimes be called the objective function, is not convex and possesses more than one local minimum. The problem of writing computer algorithms that distinguish between these local minima and locate the best local minimum is known as the global optimisation problem, and is the subject of this volume.

To emphasize the importance of having an efficient solution to this problem, we note that the different local minima can often correspond to different technologies. Finding the nearest local minimum to the current design is then equivalent to improving current technology and this can frequently only correspond to obtaining an improvement in performance of a few per cent. Moving to a different local minimum, i.e. a different technology, can imply a very significant change in performance.

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The difficulty inherent in such a statement is obvious in that it is difficult to model a new technology before it is implemented and that considerable faith is normally required to attempt to implement an unexpected type of solution to the problem.

One case in which the model of the system could be trusted as it was based on well established physical relations occurred in the design of optical filters. In this problem an examination of the objective function showed that it had many local minima and that the best did not correspond to a local improvement of the current design. In McKeown & Nag (1976) it is reported that when the new design was implemented the predicted improvement in performance occurred.

Other areas where the same phenomenon is well known occur in the design of electrical filters (Gutteridge (1972)), fitting experimental data (Amadori, Mika, and V. Studnitz (1976)), and system identification (Soderström (1974)).

A much less well known application of these techniques to the problem of the optimal design of a dry cooling tower for a thermal power plant is described in the paper by Archetti & Frontini. In this paper they also discuss some larger optical filter design problems.

Multi extremal problems also arise in the econometric sphere and the application of one of the global optimisation techniques to such a problem is described in the second paper by J. Gomulka. K. Cichocki in a paper presented at the workshop in Bergamo (1977) develops an identification model of the Polish economy which also has multiple minima.

2. A standard test series for comparing results

When the first volume of this series was published in 1975 it became apparent that an acceptable comparison was needed showing the relative performance of the many different methods that had been proposed for solving multiextremal problems. Most authors when presenting their method for publication include the results of tests on a number of standard functions. As these tests are frequently performed on differing machines, with different accuracy and termination requirements and also on a different set of functions, it is very difficult to draw meaningful conclusions on the relative efficiency of the different methods. In the field of unconstrained optimisation it is normal to base efficiency on the number of function (and/or gradient) evaluations required to minimise a set of nearly universally agreed test functions. This approach ignores the relative overhead cost of the different methods completely, on the basis that for small dimensional practical problems these are indeed usually ignorable. In contrast in the field of nonlinear programming Colville (1968) proposed a set of test functions as well as a

standard time test, he compared the algorithms both on the basis of the relative numbers of times the correct solution was located and also computed the relationship between the total computer time and the standard time. This method emphasises the relative overhead cost as the cost of a function evaluation of his test functions are often trivial compared to industrial problems.

In designing a test series for multiextremal functions it was decided to combine both criteria. The field of multiextremal problems was restricted to the much narrower field of essentially unconstrained problems in two to six dimensions. A subset of the standard test functions quoted in the literature was then selected. The functions chosen were all relatively cheap to evaluate so that a considerable proportion of the run time could be expected to be spent on algorithmic overheads. A standard computing problem was introduced on which the comparison study would be based. This problem was simply the evaluation of one of the test problems 1000 times.

Each of the contributors to the main section of this volume agreed to present their results on the standard set of function in the proposed format. The inclusion of the standard time test was, however, a late decision and unfortunately this implied that in one case the tests had already been completed and the times had not been recorded, so in that case the time comparison could not be made.

The details of the test series are given in Appendix 1 so that they will be available for the future use of other research workers in the area. The detailed results of each individual test series are given in each authors' contribution; a summary of the results is however given later in this paper.

3. The two basic approaches

Algorithms for the solution of the global optimisation problem divide into two broad classes: deterministic and probabilistic. Even if the basic philosophies behind the two approaches are quite different in most advanced global optimisation algorithms the two basic approaches are combined.

A complete discussion of the relationships between the two classes of algorithms is presented in the first paper by Gomulka.

It must also be pointed out that certain algorithms are designed for the identification of the global minimiser of the function as well as of the absolute minimum function value, others are trying only to identify the absolute minimum function value.

Purely deterministic methods try to guarantee that a neighbourhood of the global

minimum will be located. It is now established that this is only possible for restricted classes of function. Well known examples for which deterministic methods can be fruitfully used include convex functions, and one dimensional rational or Lipschitz functions and polynomials. For n -dimensional Lipschitzian functions, whose Lipschitz constant is known, bounds on the possible error of the grid search technique have, of course, been known for a long time.

The grid search technique is an example of a "passive strategy" as defined by Archetti and Betro, i.e. an algorithm where each point of the minimising sequence does not depend on the function values at the previous points. In their paper they show that for the class of Lipschitzian functions two classes of optimal passive strategies exist. They further show that no sequential strategy i.e. no strategy where each point of the sequence depends on the function values at previous points, can guarantee a better value than a particular type of optimal passive strategy with the same number of points, they call this strategy the A-optimal passive strategy. They also prove that no sequential strategy can guarantee any prefixed accuracy with less points than those required by another particular type of optimal passive strategy. It follows from this result that the use of deterministic sequential strategies can only pay when the class of objective functions is more narrowly chosen and the additional properties actually used in the minimisation procedure. Clearly nothing can be said in this respect for functions for which the Lipschitz constant is not known.

A large number of deterministic methods have been investigated in this volume; in her third contribution J. Gomulka describes her experience in implementing two versions of Branin's trajectory method. Though her implementations are quite efficient and often locate the neighbourhood of the global minimum, no subclass of nonconvex functions has been singled out for which success is guaranteed.

In Szegö (1972) Treccani, Trabattoni and Szegö introduce a method for identifying one of the saddle points which must exist on the boundary of the region of attraction of a local minimum by growing a family of ellipses completely around the minimum and completely contained in its region of attraction. This technique requires certain strong assumptions on the objective function, but the convergence of the algorithm is proved. One can use this technique to construct a global minimisation algorithm which allows the identification of the global minimum through the search of all local minima of the function. An implementation of this method for two dimensional problems was described by Corles (1975*) and considerable success was achieved even if the simplifications in the original technique introduced by Corles make a general convergence proof impossible. A closer investigation of the numerical technique used discloses that, though not explicitly stated, major smoothness assumptions on the functions were implied in the imple-

mentation of the algorithm. In addition even on functions, on which the method is able to identify the global minimiser and the absolute minimum function value and for which convergence is guaranteed, the difficulties of implementing it in higher dimensions seem to be too great to be worth the effort.

Goldstein & Price (1972) introduced a technique with guaranteed convergence on one dimensional polynomial functions, based upon the idea of descent from local minimum i.e. upon the idea of analysing only that area corresponding to function values which are less than a given function value (for example a given local minimum). Szegö (1975) independently proposed a similar technique with the name of global descent method without however giving any numerical results. Treccani extends this approach to more general functions and provides counterexamples of this general idea. Along the same line of thought we should mention the numerical results presented by Levy (1977).

Beale and Forrest describe how the global minimisation problem for functions which satisfy certain assumptions (which are satisfied by all the functions in the comparison tests) can be transformed into a mathematical programming problem. Dependent on the size of the mesh used in the transformation their method is guaranteed to locate a point whose function value is in the neighbourhood of the absolute minimum.

We may conclude by pointing out that for practical purposes purely deterministic methods present considerable difficulties unless some strict assumptions on the objective function can be made and used.

4. Probabilistic Methods

The probabilistic approach to the global optimisation problem relies on the following result.

Assume the area of interest in the global optimisation problem is the finite set S . In the test series this was defined to be a hypercube defined by upper and lower bounds on each variable

$$S = \{ \underline{x} : l_i \leq x_i \leq u_i, i = 1, \dots, n \}.$$

If A is any subset S with measure m and

$$\frac{m(A)}{m(S)} \geq \alpha > 0$$

and if $p(A, N)$ is the probability that at least one point of a sequence of N points, drawn randomly from a uniform distribution over S , falls in A

$$\text{then } \lim_{N \rightarrow \infty} p(A, N) = 1.$$

The simplest probabilistic algorithm is the Pure Random Search Algorithm - P.R.S.
 STEP 1 Evaluate $F(x)$ at N points drawn from a uniform distribution over S . This algorithm has been extensively discussed by Brooks (1958) and Anderssen (1972) who show that if

$$\frac{m(A)}{m(S)} = \alpha$$

then $p(A_1 N) = 1 - (1-\alpha)^N$.

More properties of this method which is also known as the Monte Carlo Method are described in Rubinstein & Weissman (1977).

The number of trial points N required to have a high probability of entering a small region, $\alpha = .0001$ say, round the global minimum, is very high. For this reason most probabilistic methods, combine this type of result with deterministic methods.

A very popular method is the multistart method M.S.

- STEP 1 Select $x^{(0)}$ at random
- STEP 2 start a local minimisation algorithm from $x^{(0)}$ with stopping criteria ϵg
- STEP 3 test whether this is probably a global minimum and if so STOP
- STEP 4 return to step 1.

The behaviour of local minimisation algorithms used in this way and the conditions that must be built into them for theoretical results to be possible are discussed in detail in Dixon et al (1976). In Dixon (1977) it is also shown that modifications typified by Hartman's S2 algorithm (1972) can lessen the probability of success on particular types of function.

Hartman's S2 algorithm consists of five steps

- STEP 1 $v^+ = +\infty$
- STEP 2 select $x^{(0)}$ at random
- STEP 3 if $F(x^{(0)}) > v^+$ return to 2
- STEP 4 perform a local minimisation from $x^{(0)}$ to M_j and set $v^+ = F(M_j)$
- STEP 5 return to 2.

In this algorithm an additional local search is only initiated if a point is found randomly that is better than the current local minimum. On many functions, including some in the test series, this is very unlikely. An alternative strategy to this is to locate a local minimum and to then seek an improved point by performing a number of line searches out from this point. These line searches can be performed with varying degrees of accuracy. One single completely random step is equivalent to Hartman's S2 method. Other versions that were tested included that suggested by Bremmermann (1972) of fitting a quartic through 5 equispaced points

along the line, where the step used is random, and also the powerful one-dimensional implementation due to Zilinskas of the Bayesian method described in Motskus et al. This method was found in tests at Hatfield to be an exceedingly reliable method for finding the global minimum along the line. However, on some of the test functions the relative size of the angle subtended on a hypersphere round a local minimum by its level set was so small, that no improved point was found in 2000 linear searches. For this reason probabilistic methods of improving on a local minimum point also seem doomed to be very inefficient on many functions.

The same test series also indicated that methods which undertake local line searches from random points, and accept the best point found along the line as the start for the next line search, frequently locate local minima very accurately. They then effectively revert to the algorithms just discussed above and have the same inherent disadvantage.

It was for these reasons amongst others that the multistart method has remained so popular. It has the further advantage that when features of the objective function are known these can be used to guide the choice of the initial selection which is rarely taken purely at random in practice. When few features of the objective function are known and random points must be taken, it can however be very expensive on function evaluations, as indicated by the results obtained by this method by Mockus et al and reported in their paper.

To overcome the high cost of the M.S. method, Becker & Lago (1970) introduced the concept of clustering into global optimisation. The basic assumption in clustering is that the points with low function values will be clustered in the neighbourhood of local minima. As further searches from points in the same neighbourhood should all lead to the same local minimum, a saving in cost can be obtained by continuing the search from a reduced number of points. Törn has devised a successful implementation of this approach based on a random search algorithm, while Price reports on an implementation based on the Nelder & Mead Simplex approach. Gomulka also reports on her experience using Törn's algorithm and on her own modification using a variable metric algorithm for the basic searches. All these algorithms are consistently successful on the functions in the standard test series, but Gomulka found that the variable metric based algorithm could succeed on more difficult econometric functions on which the random search based algorithm failed. When running these algorithms considerable additional information about the function can be obtained which can frequently be very valuable.

Another approach to the problem of combining stochastic and deterministic methods is outlined in the paper by Fagioli et al. In this method the region of interest is covered by predetermined set of cells and a stochastic automaton set up to

govern the choice of cell. The authors claim their method of constructing stochastic automata is more efficient than the related method by McMurtry & Fu.

Essentially after a set of initial samples have been taken, a point is selected in the cell that is most likely to contain an unfound global optima. A local constrained minimum in that cell is obtained by recursive quadratic programming (described in the paper by Bartholomew-Biggs), the stochastic automata is then updated and the method repeated till the probability of further improvements is significantly reduced.

While the overheads involved in the clustering algorithms and the stochastic automata approach are obviously higher than most of the simpler strategies discussed earlier they are still insignificant when compared to even small industrial problems. As industrial problems become larger so it becomes easier to justify even higher overheads if these can reduce the number of function evaluations still further. The Bayesian approach described by Mockus et al is designed to solve problems where the cost of the function evaluation is very large, and therefore large overheads can also be ignored. As will be seen later it is very successful in this aim.

Both the stochastic automata approach and the Bayesian approach have in built statistical assumptions from which the probability that the algorithm has already located a neighbourhood of the global optimum can be obtained. These probabilities are then used as terminating criteria. In the first volume of this series Archetti proposed a sampling technique based upon an idea of Chichinadze (1967). This has been developed into an algorithm which is described in the paper by De Biase & Frontini; in this algorithm an approximation is constructed to the probability distribution of the objective function, i.e. $P(v)$ is the probability that $F(x) < v$. An estimate is then made of the root $P(v) = 0$ and considerable theoretical results are established to guarantee the uniform convergence of the approximation to $P(v)$ and hence of the ultimate accuracy of the root. An efficient local minimisation algorithm is incorporated to determine the position of the global optimum accurately. The algorithm seems quite efficient at locating the global optimum, but unfortunately the theoretical convergence of the root to the global minimum value does not seem to occur from the results reported in their paper.

5. Numerical results

The detailed numerical results of each contributor on the standard test function will, of course, be found in their papers in this volume. It did seem sensible to gather this data together for comparison purposes, and to include some data on the random line search methods that would not otherwise be included in this

volume though they are reported elsewhere, Dixon (1977).

The least successful algorithms tested on the standard functions were these random line search routines, as they frequently became trapped by local minima. Results are included in the table for three variants, Bremmermann's original implementation, a slight modification that accepted the best point found along the line even if it was not the predicted minimum and a similar approach when the line search was replaced by a Bayesian method due to Zilinskas. Considerable testing showed that this was highly reliable in finding the global minimum along the line requiring about 30 function evaluations. But that from the local minimum of SQRN5 the window of the global minimum was so small that 2000 random directions did not penetrate it.

Two tables of results are given, in the first the number of function evaluations required to find the global minimum or terminate is recorded. In the second the time required, divided by the standard time of that machine, is recorded. An L indicates a local minimum was accepted. The standard time was obtained by calling the subroutine for evaluating SQRN5 1000 times at the point (4, 4, 4, 4).

Table 1 Function Evaluations

	SQRN5	SQRN7	SQRN10	HARTMAN 3	HARTMAN 6	GP	RCOS
<u>Line search</u>							
Bremmermann	340L	1700L	2500L	505L	L	210L	250
Mod Bremmermann	375L	405L	336L	420L	515	300	160
Zilinskas	L	12121L	8892L	8641			5129
<u>Trajectory</u>							
Gomulka/Branin	5500	5020	4860				
<u>Clustering</u>							
T8rn	3679	3606	3874	2584	3447	2499	1558
Gomulka/T8rn	6654	6084	6144				
Gomulka/V.M.	7085	6684	7352	6766	11125	1495	1318
Price	3800	4900	4400	2400	7600	2500	1800
<u>Sampling</u>							
Fagioli	2514	2519	2518	513	2916	158	1600
De Biase/Frontini	620	788	1160	732	807	378	597
Mockus	1174	1279	1209	513	1232	362	189

The Beale-Forrest approach can not be included in this table as it does not evaluate the function.

Table 2

	Standard Time						
<u>Line search</u>	SQRN5	SQRN7	SQRN10	HARTMAN 3	HARTMAN 6	GP	RCOS
Bremmermann	1 L	8 L	17 L	2 L	L	0.5L	1 L
Mod Bremmermann	1.5L	1.5 L	2 L	2 L	3	0.7	0.5
Zilinskas	L	282 L	214 L	175			80
<u>Trajectory</u>							
Gomulka/Branin	9	8.5	9.5				
<u>Clustering</u>							
Törn	10	13	15	8	16	4	4
Gomulka/Törn	17	15	20				
Gomulka/V.M.	19	23	23	17	48	2	3
Price	14	20	20	8	46	3	4
<u>Sampling</u>							
Fagioli	7	9	13	5	100	0.7	5
De Biase/Frontini	23	20	30	16	21	15	14
<u>M.P. Package</u>							
Beale/Forrest	96	258	1059	117	255	1407	1.5

It should be stressed at this point that the Beale-Forrest method is not designed to solve this type of problem. These are all small highly nonlinear functions while it is meant for large mainly linear problems. It did, however, successfully solve all the problems, as did the clustering methods and the sampling techniques.

6. Local optimisation algorithms

It will have been noticed that most of the efficient global optimisation algorithms discussed in section 4 were based upon well known local routines. Random search routines, Nelder & Mead simplex and the variable metric method were amongst those mentioned. Improvements in the understanding of the behaviour of unconstrained algorithms and in their performance are both important to the future development of algorithms for the global optimisation problems. The second part of this volume therefore again contains selected papers on this problem.

Spedicato presents two papers, in the first he reviews the present state of quasi-Newton variable metric methods, which are generally accepted to be the most efficient class of general purpose unconstrained optimisation algorithms. In his second paper he gives details of his computational experience using these methods on moderately sized problems, thus indicating the probable choice of algorithm for using in combination with, say, clustering on problems larger than those included in our test series. Crouch's note introduces another modification to the quasi-

Newton method which maintains conjugate directions without exact line searches being undertaken. Two previous ways of achieving this have been suggested by Dixon (1973) and Nazareth (1977) and the properties of such modifications are the subject of continuing investigation. The paper by Gaviano investigates the necessary and sufficient conditions for gradient related algorithms to converge to the minima of convex functions or stationary points of nonconvex functions, whilst that by Resta & Sutti discusses this problem for more general algorithms. Both papers mark a considerable advance on the previous position in these areas. A new algorithm for seeking the minimum of a differentiable function without estimating the gradient had been presented by Carla Sutti in the first volume of this series. The convergence of this algorithm can be established by this method. Numerical results obtained with her algorithms are contained in her own paper in this volume and compare favourably with those from other related methods for which such convergence properties have not been obtained.

The final paper in part two approaches the problem of stochastic optimisation, i.e. the problem where each evaluation of the objective function is subject to random error. Few implementable algorithms have been suggested that can theoretically solve this problem and the one presented here is a major advance as the authors, Betro' & De Biase, show that it is unnecessary to estimate the gradient and Hessian independently. This had been a necessary feature of previous proofs of convergence and had made the implementations very inefficient. The prospect of being able to incorporate such an algorithm with a clustering algorithm to give a global stochastic optimisation routine is quite exciting.

7. Constrained Optimisation

There have as yet been few attempts to investigate the global constrained optimisation problem except for the introduction of lower and upper bounds on the variables. When more general equality or inequality constraints are included, modification of efficient nonlinear programming algorithms will be required. The performance of two of the most efficient recent methods of nonlinear programming are compared in the paper by Bartholomew-Biggs. The two chosen methods are the ideal penalty function algorithm developed by Fletcher (1972) and his own recursive equality quadratic programming algorithm. An analysis is given which indicates in which circumstances each algorithm should be preferred.

Another recent development in this area is introduced in McKeown's paper. The method which he terms functional linear programming was originally devised to solve a problem in structural engineering which contained many features which often prove difficult or insurmountable for more orthodox algorithms. These include variables that can only take integer values, regions in which the objective function does not

exist, multiple finite minima. The problem also had the unusual feature that the total number of variables was itself a variable of the problem. The functional linear programming approach overcomes all these difficulties on the particular structural problem and on any other problem that can be posed in this form. Unexpectedly it can also be shown that the dual of any convex programming problem can be posed as a functional linear programming problem. In the paper by Resta, Sideri & Treccani a mathematical treatment is given of a convex programming method based on the approach used in functional linear programming. They have unfortunately not yet completed the convergence proof for problems in more than three dimensions. In the final paper of the volume Resta presents his numerical experience of solving convex programming problems by the functional linear programming approach. He has used two of the problems tested by Barthomolew-Biggs and the results in the two papers can be compared.

Appendix 1 Test Functions for Global Optimisation

1. Shekel's family (SQRIN)

$$f(x) = - \sum_{i=1}^m \frac{1}{(x - a_i)^T (x - a_i) + c_i} \quad x = (x_1, \dots, x_n)^T$$

$$a_i = (x_{i1}, \dots, x_{in})^T$$

$$c_i > 0$$

Region of interest: $0 \leq x_j \leq 10 \quad j = 1, \dots, n.$

Consider 3 cases from the table below with

Data: $n=4, m=5, 7, 10$

i	a_i				c_i
1	4.	4.	4.	4.	.1
2	1.	1.	1.	1.	.2
3	8.	8.	8.	8.	.2
4	6.	6.	6.	6.	.4
5	3.	7.	3.	7.	.4
6	2.	9.	2.	9.	.6
7	5.	5.	3.	3.	.3
8	8.	1.	8.	1.	.7
9	6.	2.	6.	2.	.5
10	7.	3.6	7.	3.6	.5

2. Hartman's family

$$f(x) = - \sum_{i=1}^m c_i \exp \left(- \sum_{j=1}^n a_{ij}(x_j - p_{ij})^2 \right)$$

where $x = (x_1, \dots, x_n)$, $p_i = (p_{i1}, \dots, p_{in})$, $a_i = (a_{i1}, \dots, a_{in})$. p_i is an approximate location of i 'th local minimum, a_i is proportional to eigenvalues of Hessian at i 'th local minimum, $c_i > 0$ is the height (depth?) of i 'th local minimum (assuming that the interference of different local minima is not too strong).

Data $0 \leq x_i \leq 1$. (1) $m = 4, n = 3$

i	a_{ij}	c_i	p_{ij}
1	3. 10. 30.	1.	0.3689 0.1170 0.2673
2	.1 10. 35.	1.2	0.4699 0.4387 0.7470
3	3. 10. 30.	3.	0.1091 0.8732 0.5547
4	.1 10. 35.	3.2	0.03815 0.5743 0.8828

(2) $m = 4, n = 6$

i	a_{ij}	c_i
1	10. 3. 17. 3.5 1.7 8.	1.
2	.05 10. 17. .1 8. 14	1.2
3	3. 3.5 1.7 10. 17. 8.	3.
4	17. 8. .05 10. .1 14.	3.2

	p_{ij}					
i=1	0.1312	0.1696	0.5569	0.0124	0.8283	0.5886
i=2	0.2329	0.4135	0.8307	0.3736	0.1004	0.9991
i=3	0.2348	0.1451	0.3522	0.2883	0.3047	0.6650
i=4	0.4047	0.8828	0.8732	0.5743	0.1091	0.0381

3. Branin (RCOS)

$$f(x_1, x_2) = a (x_2 - b x_1^2 + c x_1 - d)^2 + e (1-f) \cos x_1 + e$$

$$a = 1, b = 5.1(4\pi^2), c = 5/\pi, d = 6, e = 10, f = 1/(8\pi)$$

Region of interest =

$$\begin{aligned} -5 \leq x_1 \leq 10 \\ 0 \leq x_2 \leq 15 \end{aligned}$$

There are three minima, all global, in this region.

4. Goldstein & Price (GOLDPR)

$$f(x_1, x_2) = [1 + (x_1 + x_2 + 1)^2 (19 - 14x_1 + 5x_1^2 - 14x_2 + 6x_1 x_2 + 5x_2^2)] \\ [30 + (2x_1 - 3x_2)^2 (18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1 x_2 + 27x_2^2)].$$

Region of
Interest =

$$-2 \leq x_{1,2} \leq 2$$

Four local minima. Global minima
at (0, -1) with the value f=3.

5. Standard Time

Evaluate Shekel (SQRIN) n = 4, m = 5 as a subroutine. Calling the subroutine 1000 times with $x_j = 4$, on the same machine/compiler/language etc. as that used in the test.

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References in the list below marked * refer to papers in the first volume, "Towards Global Optimisation", North-Holland (1975).

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