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TEST PROBLEMS FOR CONSTRAINED NONLINEAR MATHEMATICAL PROGRAMMING ALGORITHMS

by

Larry W. Cornwell
Patricia A. Hutchison
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U of C-AUA-USDOE

APPLIED MATHEMATICS DIVISION

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Argonne, Illinois 60439

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Larry W. Cornwell+

Patricia A. Hutchison+

Michael Minkoff

Hilbert K. Schultz‡

Applied Mathematics Division

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+Dept. of Mathematics, Western Illinois University, Macomb, IL 61455.

‡College of Business Administration, University of Wisconsin, Oshkosh, WI 54901.

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TEST PROBLEMS FOR CONSTRAINED NONLINEAR MATHEMATICAL PROGRAMMING ALGORITHMS

by

Larry W. Cornwell, Patricia A. Hutchison, Michael Minkoff, Hilbert K. Schultz

ABSTRACT

The report presents a collection of constrained nonlinear programming problems for use in testing optimization algorithms. The problems vary in size from two variables to one hundred variables with various combinations of linear/nonlinear constraints and objective functions. IBM Fortran IV programs have been written to provide function values and gradients for the objective function and constraints. Each coded problem has been checked at several points against published results and a validation process was used to check the values of the objective function, constraints, and gradients.

The problems were collected from various sources and many of them have been used by other authors in published results of their algorithm testing. This report should also be useful in an educational setting to provide students with experience in nontrivial problems. Listings of the IBM Fortran code are included in this report.

I. INTRODUCTION

The extensive theoretical research and development in constrained optimization has resulted in a corresponding effort at comparing the computational efficiencies of various implementations -- see e.g. Colville [4]. This frequently results in authors searching the literature for appropriate "test problems", coding the problem and performing the testing. The authors have compiled, coded and tested 32 nonlinear problems from various sources. Some of these were used in testing optimization algorithms at Argonne National Laboratory and others were simply implemented. We present these problems for possible use by others in the hope of alleviating some of the difficulties and effort required in algorithm testing. The computer codes provide function and gradient information for all problems. Microfiche listings of the code are provided at the end of the report.

It should be emphasized that the collection of problems presented here is not intended to be exhaustive. In particular we envision revising this collection as needs arise.

II. USER GUIDE

The subroutines supplied for each problem provide values and gradients of the objective function and each constraint. We deal with the nonlinear programming problem

$$
\begin{aligned}
&\min f(x) \\
&\text{subject to} \\
&\qquad c_i(x) = 0 \qquad i = 1,\ldots,K \\
&\qquad c_i(x) \le 0 \qquad i = K+1,\ldots,M
\end{aligned}
\qquad \text{(II.1)}
$$

where $x \in R^N$ and there are M constraints, the first K of which are equality constraints. In the event the problem involves simple bound constraints, i.e. $\ell_i \le x_i \le u_i$ where ℓ_i and u_i are lower and upper bounds, we have made these the last constraints in the problem. Thus these routines can be used in conjunction with an algorithm implementation which handles simple bounds directly by reducing the value of M (not evaluating the simple bounds in these routines).

The software for problem number I is given by SUBROUTINE FVALI (which evaluates the objective or a constraint), SUBROUTINE GVALI (which evaluates the gradient of the objective or a constraint), and internal subroutines, if necessary.

To evaluate the objective function and the M constraints for a current x-vector, the subroutine FVALI must be called M+1 times. The call statement for FVALI is

```
CALL FVALI(N,X,VAL,IN)
```

where the arguments are defined

N - the number of variables (INTEGER)

X - current x-vector (DOUBLE PRECISION vector)

VAL - value of the objective function or constraint requested (DOUBLE PRECISION)

IN - indicator for objective function or constraint being requested (INTEGER)

IN = 0 (Objective function)
IN = 1 (First constraint)
IN = 2 (Second constraint)
etc.

The call statement for GVALI is

CALL GVALI(N,X,G,IN)

where the arguments are defined

N - number of variables (INTEGER)

X - current x-vector (DOUBLE PRECISION vector)

G - requested gradient vector (DOUBLE PRECISION vector)

IN - indicator for requested gradient (objective function or constraint) (INTEGER)

IN = 0 (Objective function
IN = 1 (First constraint)
IN = 2 (Second constraint)
etc.

The problems have been collected from several sources including D. M. Himmelblau's textbook and source deck. They were reformulated in the present structure for use in testing at Argonne National Laboratory. All subroutines have been written as double precision routines (IBM REAL*8). They can be converted to single precision by replacing the DOUBLE PRECISION statements with REAL statements, changing all library function names and converting the type of constants in the data statements and in-line code. Note that library functions used in a given routine are indicated in prologue comments of the routine. Also, in most cases, in-line use of constants involve the pattern ".0D0" and can thus be easily changed to ".0E0" via a text editor. The problems were taken from various sources and, where possible, existing computer codes were used in the programs supplied here. For this reason, no major attempt was made to optimize the programs although their validity was tested by the procedure described in the next section.

III. DESCRIPTION OF PROBLEMS

This report describes 32 problems which consists of problems from the original Himmelblau problem set and 8 additional problems. A simple testing program was written to check the values generated by FVALI and GVALI. The value of the objective function and all constraints were printed along with the analytic and numerical derivative. The numerical derivatives were calculated by taking a fixed step in each variable and computing the ratio of the difference of the function values and the step. The comment section after each problem describes the results of the testing program. It should be noted that the points tested are values obtained from the reference cited (usually Himmelblau) or, in the case of problems 15-21, from an augmented Lagrangian code [Schultz, Minkoff, and Cornwell]. The comments given apply to the objective value and feasibility of points stated to be starting or optimal points. A FORTRAN listing of the testing program and sample output is found in the Appendix I.

The problem set is made up of five types of problems and are numbered to allow for later expansion. Problems 1-21 are at least continuously differentiable and do not use any library functions. Problems 51-56 are also continuously differentiable but do involve library functions, e.g. DLOG, DEXP, DSIN. Problems 71 and 72 are continuously differentiable but are quite large compared to the previous problems. Problems 81 and 82 involve discontinuities. Problem 81 involves a discontinuous objective while Problem 82 involves a discontinuous objective derivative. Finally, Problem 91 provides an example of a problem with multiple local minima.

Table I presents a condensed description of each problem. More details such as the exact description, original source, problem number in terms of Himmelblau's and Colville's set, and comments are found in the remainder of this section. The problem formulations are given in a more general form than (II.1). In particular the inequality constraints may be expressed as "greater than or equal to" and simple bounds are given as the last constraints. However, the problems are converted to the form of (II.1) by means of multiplying by minus one and by directly coding simple bounds as two constraints, e.g. $x_i - u_i \leq 0$ and $-x_i + \ell_i \leq 0$.

As examples of the routines, FORTRAN listings of problems 7, 15, and 91 are given in Appendix II.

TABLE I

Problem Number	Number of Variables	Number of Constraints	Linear Equalities	Nonlinear Equalities	Linear Inequalities	Nonlinear Inequalities	Objective Functions
1	5	15	0	0	15	0	N
2	15	20	0	0	15	5	N
3	5	16	0	0	10	6	N
4	4	8	0	0	8	0	N
5	16	40	8	0	32	0	N
6	3	20	0	0	6	14	N
7	2	2	1	0	0	1	N
8	3	5	1	1	3	0	N
9	5	48	0	0	13	35	N
10	5	16	0	0	10	6	N
11	9	14	0	0	1	13	N
12	24	44	2	12	24	6	L
13	6	16	0	0	12	4	L
14	2	2	0	0	1	1	N
15	15	10	0	0	0	10	L
16	15	10	0	0	0	10	L
17	15	10	0	0	0	10	L
18	15	11	0	0	0	11	L
19	15	11	0	0	0	11	L
20	15	15	0	0	4	11	L
21	15	15	0	0	4	11	L
51	2	7	0	0	4	3	N
52	10	13	3	0	10	0	N
53	10	3	0	3	0	0	N
54	4	6	0	0	5	1	N
55	10	20	0	0	20	0	N
56	3	6	0	0	0	6	N
71	45	61	16	0	45	0	N
72	100	112	0	0	112	0	N
81	6	4	0	0	0	4	N
82	6	16	0	4	12	0	L
91	5	5	0	0	4	1	N

Problem Number 1

Sources: 1) Shell Development Co.

2) Himmelblau problem number 10, pp. 404-405.

3) Colville problem number 1, p. 21.

Objective function: Linear ___ Nonlinear X

Number of variables: 5

Number of constraints: 15 Inequalities: Linear 15 Nonlinear ___

Equalities: Linear ___ Nonlinear ___

Problem: Minimize: $f(\overline{x}) = \sum_{j=1}^{5} e_j x_j + \sum_{i=1}^{5} \sum_{j=1}^{5} c_{ij} x_i x_j + \sum_{j=1}^{5} d_j x_j^3$

Subject to: $\sum_{j=1}^{5} a_{ij} x_j - b_i \geq 0, \quad i=1,\ldots,10$

$x_j \geq 0, \quad j=1,\ldots,5$

where e_j, c_{ij}, d_j, a_{ij} and b_j are given in Table II.

Points tested and objective values [Himmelblau]:

1) Feasible starting point

$\overline{x} = (0,0,0,0,1) \quad f(\overline{x}) = 20$

2) Solution point

$\overline{x} = (0.3, 0.3335, 0.4, 0.4285, 0.224) \quad f(\overline{x}) = -32.349$

Comments: The testing program agreed with the published results.

TABLE II

j	1	2	3	4	5
e_j	-15	-27	-36	-18	-12
c_{1j}	30	-20	-10	32	-10
c_{2j}	-20	39	-6	-31	32
c_{3j}	-10	-6	10	-6	-10
c_{4j}	32	-31	-6	39	-20
c_{5j}	-10	32	-10	-20	30
d_j	4	8	10	6	2
a_{1j}	-16	2	0	1	0
a_{2j}	0	-2	0	0.4	2
a_{3j}	-3.5	0	2	0	0
a_{4j}	0	-2	0	-4	-1
a_{5j}	0	-9	-2	1	-2.8
a_{6j}	2	0	-4	0	0
a_{7j}	-1	-1	-1	-1	-1
a_{8j}	-1	-2	-3	-2	-1
a_{9j}	1	2	3	4	5
a_{10j}	1	1	1	1	1

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
-40	-2	-.25	-4	-4	-1	-40	-60	5	1

Problem Number 2

Sources: 1) Shell Development Company.

2) Himmelblau problem number 18, pp. 405, 417-417.

3) Colville problem number 2, pp. 22-23.

Objective function: Linear ___ Nonlinear X

Number of variables: 15

Number of constraints: 20 Inequalities: Linear 15 Nonlinear 5
Equalities: Linear ___ Nonlinear ___ .

Problem: Minimize:

$$f(\bar{x}) = -\sum_{i=1}^{10} b_i x_i + \sum_{j=1}^{5} \sum_{i=1}^{5} c_{ij} x_{10+i} x_{10+j} + 2\sum_{j=1}^{5} d_j x_{10+j}^3$$

Subject to:

$$2\sum_{i=1}^{5} c_{ij} x_{10+i} + 3d_j x_{10+j}^2 + e_j - \sum_{i=1}^{10} a_{ij} x_i \geq 0 \qquad j = 1,\ldots,5$$

$$x_i \geq 0, \quad i = 1,\ldots,15$$

where

e_j, c_{ij}, d_j, a_{ij} and b_j are defined in Table II.

Points tested and objective values [Himmelblau]:

1) Feasible starting point

$\bar{x}$ = (0.0001, 0.0001, 0.0001, 0.0001, 0.0001, 0.0001, 60., 0.0001, 0.0001, 0.0001, 0.0001, 0.0001, 0.0001, 0.0001, 0.0001) $f(\bar{x})$ = 2400.01

2) Nonfeasible starting point

$\bar{x}$ = (-40, -2, -0.25, -4, -4, -1, -40, -60, 5, 1, 0, 0, 0, 0, 1) $f(\bar{x})$ = -6829.06

3) Solution point

$\bar{x}$ = (0., 0., 5.174, 0., 3.0611, 11.8395, 0., 0., 0.1039, 0., 0.3, 0.3335, 0.4, 0.4283, 0.224)

$f(\bar{x})$= 32.386

Comments: The testing program obtained $f(\bar{x})$ = 32.3485 for the third point.

Problem Number 3

Sources: 1) Proctor and Gamble Co.

2) Himmelblau problem number 11, p. 406.

3) Colville problem number 3, p. 24.

Objective function: Linear ___ Nonlinear X

Number of variables: 5

Number of constraints: 16 Inequalities: Linear 10 Nonlinear 6
Equalities: Linear ___ Nonlinear ___

Problem: Minimize: $f(\bar{x}) = 5.3578547x_3^2 + 0.8356891x_1x_5 + 37.293239x_1 - 40792.141$

Subject to:

$$0 \le 85.334407 + 0.0056858x_2x_5 + 0.0006262x_1x_4 - 0.0022053x_3x_5 \le 92$$

$$90 \le 80.51249 + 0.0071317x_2x_5 + 0.0029955x_1x_2 + 0.0021813x_3^2 \le 110$$

$$20 \le 9.300961 + 0.0047026x_3x_5 + 0.0012547x_1x_3 + 0.0019085x_3x_4 \le 25$$

$$78 \le x_1 \le 102$$
$$33 \le x_2 \le 45$$
$$27 \le x_3 \le 45$$
$$27 \le x_4 \le 45$$
$$27 \le x_5 \le 45$$

Points tested and objective values [Himmelblau]:

1) Feasible starting point
$\bar{x} = (78.62, 33.44, 31.07, 44.18, 35.32)$ $f(\bar{x}) = -30367$

2) Nonfeasible starting point
$\bar{x} = (78, 33, 27, 27, 27)$ $f(\bar{x}) = -32217$

3) Solution point
$\bar{x} = (78, 33, 29.995, 45, 36.776)$ $f(\bar{x}) = -30665.5$

Comments: The testing program disagreed with some of the values of the objective function. The values were $f(\bar{x}) = -30367.379$, $f(\bar{x}) = -32217.431$, and $f(\bar{x}) = -30665.609$, respectively.

Problem Number 4

Sources: 1) C. F. Wood, Westinghouse Research Laboratory.

2) Himmelblau problem number 8, p. 403.

3) Colville problem number 4, p. 25.

Objective function: Linear ___ Nonlinear X

Number of variables: 4

Number of constraints: 8 Inequalities: Linear 8 Nonlinear ___

Equalities: Linear ___ Nonlinear ___

Problem: Minimize: $f(\bar{x}) = 100(x_2-x_1^2)^2 + (1-x_1)^2 + 90(x_4-x_3^2)^2$
$+ (1-x_3)^2 + 10.1[(x_2-1)^2 + (x_4-1)^2]$
$+ 19.8(x_2-1)(x_4-1)$

Subject to: $-10 \leq x_i \leq 10 \quad i=1,2,3,4$

Points tested and objective values [Himmelblau]: 1) Feasible starting point

$\bar{x} = (-3,-1,-3,-1) \quad f(\bar{x}) = 19192.$

2) Solution point

$\bar{x} = (1,1,1,1) \quad f(\bar{x}) = 0.0$

Comments: The testing program agreed with the published values of the objective function.

Problem Number 5

Sources: 1) J. M. Gauthier, IBM France.

2) Himmelblau problem number 19, pp. 417-419.

3) Colville problem number 7, pp. 29-30.

Objective function: Linear ___ Nonlinear X

Number of variables: 16

Number of constraints: 40 Inequalities: Linear 32 Nonlinear ___
Equalities: Linear 8 Nonlinear ___

Problem: Minimize: $f(\bar{x}) = \sum_{i=1}^{16} \sum_{j=1}^{16} a_{ij}(x_i^2+x_i+1)(x_j^2+x_j+1)$

Subject to: $\sum_{j=1}^{16} b_{ij}x_j = c_i$, $i = 1,\ldots,8$

$0 \leq x_j \leq 5$, $j = 1,\ldots,16$

where a_{ij}, b_{ij} and c_i are defined in Table III.

Points tested and objective values [Himmelblau]:

1) Nonfeasible starting point

$\bar{x} = (10,10,10,10,10,10,10,10,10,10,10,10,10,10,10,10)$

$f(\bar{x}) = 209457$

2) Solution point

$\bar{x} = (0.04, 0.792, 0.203, 0.844, 1.270, 0.935, 1.682, 0.155, 1.568, 0, 0, 0, 0.66, 0, 0.674, 0)$

$f(\bar{x}) = 244.900$

Comments: The testing program disagreed with the values of the objective function for the first x-vector. The value found was $f(\bar{x}) = 566766$.

TABLE III

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_{1j}	1			1			1	1								1
a_{2j}		1	1				1			1						
a_{3j}			1				1		1	1				1		
a_{4j}				1			1				1				1	
a_{5j}					1	1				1		1				1
a_{6j}						1		1							1	
a_{7j}							1				1		1			
a_{8j}								1		1					1	
a_{9j}									1			1				1
a_{10j}										1				1		
a_{11j}											1		1			
a_{12j}												1		1		
a_{13j}													1	1		
a_{14j}														1		
a_{15j}															1	
a_{16j}																1
b_{1j}	0.22	0.20	0.19	0.25	0.15	0.11	0.12	0.13	1							
b_{2j}	-1.46		-1.30	1.82	-1.15		0.80			1						
b_{3j}	1.29	-0.89			-1.16	-0.96		-0.49			1					
b_{4j}	-1.10	-1.06	0.95	-0.54		-1.78	-0.41					1				
b_{5j}				-1.43	1.51	0.59	-0.33	-0.43					1			
b_{6j}		-1.72	-0.33		1.62	1.24	0.21	-0.26						1		
b_{7j}	1.12			0.31			1.12		-.36						1	
b_{8j}		0.45	0.26	-1.10	0.58		-1.03	0.10								1
c_i	2.5	1.1	-3.1	-3.5	1.3	2.1	2.3	-1.5								

Problem Number 6

Source: 1) A. R. Colville, A Comparative Study on Nonlinear Programming Codes, IBM N.Y. Sci. Center Rept. 320-2949, June 1968 (problem number 8).

2) Himmelblau problem number 7, pp. 401-402.

Objective function: Linear ___ Nonlinear X

Number of variables: 3

Number of constraints: 20 Inequalities: Linear 6 Nonlinear 14

Equalities: Linear ___ Nonlinear ___

Problem: Minimize: $f(\overline{x}) = -0.063y_2y_5 + 5.04x_1 + 3.36y_3 + 0.035x_2 + 10x_3$

where the y's are defined in the FORTRAN program below.

Subject to:

$$0 \le x_1 \le 2000$$
$$0 \le x_2 \le 16000$$
$$0 \le x_3 \le 120$$
$$0 \le y_2 \le 5000$$
$$0 \le y_3 \le 2000$$
$$85 \le y_4 \le 93$$
$$90 \le y_5 \le 95$$
$$3 \le y_6 \le 12$$
$$0.01 \le y_7 \le 4$$
$$145 \le y_8 \le 162$$

```
    Y(2) = 1.6*X(1)
 10 Y(3) = 1.22*Y(2) - X(1)
    Y(6) = (X(2) + Y(3))/X(1)
    Y2CALC = X(1)*(112. + 13.167*Y(6) - 0.6667*Y(6)**2)/100.
    IF(ABS(Y2CALC - Y(2)) - 0.001)30,30,20
 20 Y(2) = Y2CALC
    GO TO 10
 30 CONTINUE
    Y(4) = 93.
100 Y(5) = 86.35 + 1.098*Y(6) - 0.038*Y(6)**2 + 0.325*(Y(4) - 89.)
    Y(8) = -133. + 3.*Y(5)
    Y(7) = 35.82 - 0.222*Y(8)
    Y4CALC = 98000.*X(3)/(Y(2)*Y(7) + X(3)*1000.)
    IF(ABS(Y4CALC - Y(4)) - 0.0001)300,300,200
200 Y(4) = Y4CALC
    GO TO 100
300 CONTINUE
```

Points tested and objective values [Himmelblau]:

1) Feasible starting point

$\bar{x} = (1745, 12000, 110) \quad f(\bar{x}) = -868.6458$

2) Solution point

$\bar{x} = (1728.37, 16000, 98.13) \quad f(\bar{x}) = -1162.036$

Comments: The testing program agreed with the published values.

Problem Number 7

Source: 1) J. Bracken and G. P. McCormick, Selected Applications of Nonlinear Programming, John Wiley & Sons, Inc., New York, 1968, p. 19.

2) Himmelblau problem number 1, pp. 393-394.

Objective function: Linear ___ Nonlinear X

Number of variables: 2

Number of constraints: 2 Inequalities: Linear ___ Nonlinear 1

Equalities: Linear 1 Nonlinear ___

Problem: Minimize: $f(\bar{x}) = (x_1-2)^2 + (x_2-1)^2$

Subject to:

$$x_1 - 2x_2 + 1 = 0$$

$$\frac{x_1^2}{4} + x_2^2 - 1 \leq 0$$

Points tested and objective values [Himmelblau]:

1) Nonfeasible starting point

$\bar{x} = (2.,2.)$ $f(\bar{x}) = 1.$

2) Solution point

$\bar{x} = (0.8229,0.9114)$ $f(\bar{x}) = 1.3935$

Comments: The objective value at the solution point was 1.39341.

Problem Number 8

Source: 1) D. A. Paviani, Ph.D. dissertation, The University of Texas, Austin, Texas, 1969.

2) Himmelblau, problem number 5, p. 397.

Objective function: Linear ___ Nonlinear X

Number of variables: 3

Number of constraints: 5 Inequalities: Linear 3 Nonlinear ___

Equalities: Linear 1 Nonlinear 1

Problem: Minimize: $f(\overline{x}) = 1000 - x_1^2 - 2x_2^2 - x_3^2 - x_1x_2 - x_1x_3$

Subject to: $x_1^2 + x_2^2 + x_3^2 - 25 = 0$

$8x_1 + 14x_2 + 7x_3 - 56 = 0$

$x_i \geq 0, \quad i = 1,2,3$

Points tested and objective values [Himmelblau]:

1) Nonfeasible starting point

$\overline{x} = (2,2,2) \quad f(\overline{x}) = 976$

2) Nonfeasible starting point

$\overline{x} = (10,10,10) \quad f(\overline{x}) = 400$

3) Solution point

$\overline{x} = (3.512, 0.217, 3.552) \quad f(\overline{x}) = 961.715$

Comments: The function value at the solution point was 961.718.

Problem Number 9

Sources: 1) G. K. Barnes, M.S. thesis, The University of Texas, Austin, Texas, 1967.

2) Himmelblau problem number 12, pp. 407-410.

Objective function: Linear ___ Nonlinear X

Number of variables: 5

Number of constraints: 48 Inequalities: Linear 13 Nonlinear 35

Equalities: Linear ___ Nonlinear ___

Problem: Minimize:
$$f(\overline{x}) = -0.0000005843y_{17} + 0.000117y_{14} + 0.1365 + 0.00002358y_{13} + 0.000001502y_{16} + 0.0321y_{12} + 0.004324y_5 + 0.0001\frac{c_{15}}{c_{16}} + 37.48\frac{y_2}{c_{12}}$$

where

$$y_1 = x_2 + x_3 + 41.6$$
$$c_1 = 0.024x_4 - 4.62$$
$$y_2 = \frac{12.5}{c_1} + 12.0$$
$$c_2 = 0.0003535x_1^2 + 0.5311x_1 + 0.08705y_2x_1$$
$$c_3 = 0.052x_1 + 78 + 0.002377y_2x_1$$
$$y_3 = \frac{c_2}{c_3}$$
$$y_4 = 19y_3$$
$$c_4 = 0.04782(x_1-y_3) + \frac{0.1956(x_1-y_3)^2}{x_2} + 0.6376y_4 + 1.594y_3$$
$$c_5 = 100x_2$$
$$c_6 = x_1 - y_3 - y_4$$
$$c_7 = 0.950 - \frac{c_4}{c_5}$$
$$y_5 = c_6c_7$$
$$y_6 = x_1 - y_5 - y_4 - y_3$$

$$c_8 = (y_5+y_4)0.995$$

$$y_7 = \frac{c_8}{y_1}$$

$$y_8 = \frac{c_8}{3798}$$

$$c_9 = y_7 - \frac{0.0663y_7}{y_8} - 0.3153$$

$$y_9 = \frac{96.82}{c_9} + 0.321y_1$$

$$y_{10} = 1.29y_5 + 1.258y_4 + 2.29y_3 + 1.71y_6$$

$$y_{11} = 1.71x_1 - 0.452y_4 + 0.580y_3$$

$$c_{10} = \frac{12.3}{752.3}$$

$$c_{11} = (1.75y_2)(0.995x_1)$$

$$c_{12} = 0.995y_{10} + 1998.0$$

$$y_{12} = c_{10}x_1 + \frac{c_{11}}{c_{12}}$$

$$y_{13} = c_{12} - 1.75y_2$$

$$y_{14} = 3623.0 + 64.4x_2 + 58.4x_3 + \frac{146312.0}{y_9 + x_5}$$

$$c_{13} = 0.995y_{10} + 60.8x_2 + 48.0x_4 - 0.1121y_{14} - 5095.0$$

$$y_{15} = \frac{y_{13}}{c_{13}}$$

$$y_{16} = 148000.0 - 331000.0y_{15} + 40.0y_{13} - 61.0y_{15}y_{13}$$

$$c_{14} = 2324.0y_{10} - 28740000.0y_2$$

$$y_{17} = 14130000.0 - 1328.0y_{10} - 531.0y_{11} + \frac{c_{14}}{c_{12}}$$

$$c_{15} = \frac{y_{13}}{y_{15}} - \frac{y_{13}}{0.52}$$

$$c_{16} = 1.104 - 0.72 y_{15}$$

$$c_{17} = y_9 + x_5$$

Subject to:

$$y_4 - \frac{0.28}{0.72} y_5 \geq 0$$

$$1.5x_2 - x_3 \geq 0$$

$$21.0 - 3496 \frac{y_2}{c_{12}} \geq 0$$

$$\frac{62{,}212}{c_{17}} - 110.6 - y_1 \geq 0$$

$$213.1 \leq y_1 \leq 405.23$$
$$17.505 \leq y_2 \leq 1053.6667$$
$$11.275 \leq y_3 \leq 35.03$$
$$214.228 \leq y_4 \leq 665.585$$
$$7.458 \leq y_5 \leq 584.463$$
$$0.961 \leq y_6 \leq 265.916$$
$$1.612 \leq y_7 \leq 7.046$$
$$0.146 \leq y_8 \leq 0.222$$
$$107.99 \leq y_9 \leq 273.366$$
$$922.693 \leq y_{10} \leq 1286.105$$
$$926.832 \leq y_{11} \leq 1444.046$$
$$18.766 \leq y_{12} \leq 537.141$$
$$1072.163 \leq y_{13} \leq 3247.039$$
$$8961.448 \leq y_{14} \leq 26844.086$$
$$0.063 \leq y_{15} \leq 0.386$$
$$71{,}084.33 \leq y_{16} \leq 140{,}000$$
$$2{,}802{,}713 \leq y_{17} \leq 12{,}146{,}108$$
$$704.4148 \leq x_1 \leq 906.3855$$
$$68.6 \leq x_2 \leq 288.88$$
$$0 \leq x_3 \leq 134.75$$
$$193 \leq x_4 \leq 287.0966$$
$$25 \leq x_5 \leq 84.1988$$

Points tested and objective values [Himmelblau]:

1) Feasible starting point

$\bar{x} = (900, 80, 115, 267, 27)$ $\quad f(\bar{x}) = -0.939$

2) Solution point

$\bar{x} = (705.06, 68.6, 102.9, 282.341, 35.627)$

$f(\bar{x}) = -1.905$

Comments: The testing program agreed with the published results. However, for the solution point, constraint 33 was violated by 0.179802. The following point was found by an augmented Lagrangian algorithm:

$$\bar{x} = (705.174537, 68.6, 102.9, 282.324932, 37.584116)$$

with

$$f(\bar{x}) = -1.905155$$

and no constraint violated by more than 10^{-10}.

Problem Number 10

Sources: 1) M. J. Box, "A New Method of Constrained Optimization and a Comparison with Other Methods," Computer Journal, 8:42, 1965.

2) Himmelblau problem number 13, pp. 410-412.

Objective function: Linear ___ Nonlinear X

Number of variables: 5

Number of constraints: 16 Inequalities: Linear 10 Nonlinear 6
Equalities: Linear ___ Nonlinear ___

Problem: Minimize:
$$f(\bar{x}) = [-50y_1 - 9.583y_2 - 20y_3 - 15y_4 + 852960 + 38100(x_2+0.01x_3) - k_{31} - k_{32}x_2 - k_{33}x_3 - k_{34}x_4 - k_{35}x_5]x_1 + 24345 - 15x_6$$

where

$$x_6 = (k_1 + k_2x_2 + k_3x_3 + k_4x_4 + k_5x_5)x_1$$

$$y_1 = k_6 + k_7x_2 + k_8x_3 + k_9x_4 + k_{10}x_5$$

$$y_2 = k_{11} + k_{12}x_2 + k_{13}x_3 + k_{14}x_4 + k_{15}x_5$$

$$y_3 = k_{16} + k_{17}x_2 + k_{18}x_3 + k_{19}x_4 + k_{20}x_5$$

$$y_4 = k_{21} + k_{22}x_2 + k_{23}x_3 + k_{24}x_4 + k_{25}x_5$$

$$x_7 = (y_1+y_2+y_3)x_1$$

$$x_8 = (k_{26}+k_{27}x_2+k_{28}x_3+k_{29}x_4+k_{30}x_5)x_1 + x_6 + x_7$$

$$k_1 = -145{,}421.402$$

$$k_2 = 2{,}931.1506$$

$$k_3 = -40.427932$$

$$k_4 = 5{,}106.192$$

$$k_5 = 15{,}711.36$$

$$k_6 = -161{,}622.577$$

$$k_7 = 4{,}176.15328$$

$k_8 = 2.8260078$	$k_{22} = -306.262544$
$k_9 = 9{,}200.476$	$k_{23} = 16.243649$
$k_{10} = 13{,}160.295$	$k_{24} = -3{,}094.252$
$k_{11} = -21{,}686.9194$	$k_{25} = -5{,}566.2628$
$k_{12} = 123.56928$	$k_{26} = -26{,}237$
$k_{13} = -21.1188894$	$k_{27} = 99$
$k_{14} = 706.834$	$k_{28} = -0.42$
$k_{15} = 2{,}898.573$	$k_{29} = 1{,}300$
$k_{16} = 28{,}298.388$	$k_{30} = 2{,}100$
$k_{17} = 60.81096$	$k_{31} = 925{,}548.252$
$k_{18} = 31.242116$	$k_{32} = -61{,}968.8432$
$k_{19} = 329.574$	$k_{33} = 23.3088196$
$k_{20} = -2{,}882.082$	$k_{34} = -27{,}097.648$
$k_{21} = 74{,}095.3845$	$k_{35} = -50{,}843.766$

Subject to:
$$0 \le x_6 \le 294000$$
$$0 \le x_7 \le 294000$$
$$0 \le x_8 \le 277200$$
$$0 \le x_1 \le 5$$
$$1.2 \le x_2 \le 2.4$$
$$20 \le x_3 \le 60$$
$$9 \le x_4 \le 9.3$$
$$65 \le x_5 \le 7$$

Points tested and objective values [Himmelblau]:

1) Feasible starting point

$\bar{x} = (2.52,\ 2,\ 37.5,\ 9.25,\ 6.8)$ $\quad f(\bar{x}) = -2351243.5$

2) Solution point

$\bar{x} = (4.538,\ 2.4,\ 60,\ 9.3,\ 7)$ $\quad f(\bar{x}) = -5280254.$

Comments: The testing program disagreed on the value of the objective function for the solution point, $f(\bar{x}) = -5281000.38$, and the sixth constraint was violated by more than 34. This violation is not large in view of the scaling of the constraints. In fact, if the first component of the solution is changed to 4.537431, the maximum constraint violation is less than 10^{-4}.

Problem Number 11

Sources: 1) J. D. Pearson, "On Variable Metric Methods of Minimization," Research Analysis Corp. Rept. RAC-TP-302, McLean, Va., May, 1968.

2) Himmelblau problem number 16, p. 415.

Objective function: Linear ___ Nonlinear X

Number of variables: 9

Number of constraints: 14 Inequalities: Linear 1 Nonlinear 13

Equalities: Linear ___ Nonlinear ___

Problem: Minimize: $f(\bar{x}) = -0.5(x_1x_4 - x_2x_3 + x_3x_9 - x_5x_9 + x_5x_8 - x_6x_7)$

Subject to:

$$1 - x_3^2 - x_4^2 \geq 0$$
$$1 - x_9^2 \geq 0$$
$$1 - x_5^2 - x_6^2 \geq 0$$
$$1 - x_1^2 - (x_2-x_9)^2 \geq 0$$
$$1 - (x_1-x_5)^2 - (x_2-x_6)^2 \geq 0$$
$$1 - (x_1-x_7)^2 - (x_2-x_8)^2 \geq 0$$
$$1 - (x_3-x_5)^2 - (x_4-x_6)^2 \geq 0$$
$$1 - (x_3-x_7)^2 - (x_4-x_8)^2 \geq 0$$
$$1 - x_7^2 - (x_8-x_9)^2 \geq 0$$
$$x_1x_4 - x_2x_3 \geq 0$$
$$x_3x_9 \geq 0$$
$$- x_5x_9 \geq 0$$
$$x_5x_8 - x_6x_7 \geq 0$$
$$x_9 \geq 0$$

Points tested and objective values [Himmelblau]:

1) Nonfeasible starting point
$\bar{x} = (1,1,1,1,1,1,1,1,1)$ $f(\bar{x}) = 0$

2) Solution point
$\bar{x} = (0.9971, -0.0758, 0.553, 0.8331, 0.9981, -0.0623, 0.5642, 0.8256, 0.0000024)$
$f(\bar{x}) = -0.8660$

Comments: The testing program obtained $f(\bar{x}) = -.86589$ at the solution point.

Problem Number 12

Sources: 1) D. A. Paviani, Ph.D. dissertation, The University of Texas, Austin, Texas, 1969.

2) Himmelblau problem number 20, pp. 419-421.

Objective function: Linear X Nonlinear ___

Number of variables: 24

Number of constraints: 44 Inequalities: Linear 24 Nonlinear 6

Equalities: Linear 2 Nonlinear 12

Problem: Minimize: $f(\bar{x}) = \sum_{i=1}^{24} a_i x_i$

Subject to: $$\frac{x_{i+12}}{b_{i+12} \sum_{j=13}^{24} \frac{x_j}{b_j}} - \frac{c_i x_i}{40 b_i \sum_{j=1}^{12} \frac{x_j}{b_j}} = 0, \qquad i = 1,\dots,12$$

$$\sum_{i=1}^{24} x_i - 1 = 0$$

$$\sum_{i=1}^{12} \frac{x_i}{d_i} + (0.7302)(530)\left(\frac{14.7}{40}\right) \sum_{i=13}^{24} \frac{x_i}{b_i} - 1.671 = 0$$

$$-(x_i + x_{i+12}) + e_i \geq 0, \qquad i = 1,2,3$$

$$-(x_{i+3} + x_{i+15}) + e_i \geq 0, \qquad i = 4,5,6$$

$$x_i \geq 0, \qquad i = 1,\dots,24$$

where a_i, b_i, c_i, d_i and e_i are defined in Table IV.

Points tested and objective values [Himmelblau]:

1) Nonfeasible starting point

$\bar{x} = (0.04,\ldots,0.04)$ $f(\bar{x}) = 0.14696$

2) Flexible tolerance solution point

$\bar{x}$ = (7.804E-03, 1.121E-01, 1.136E-01, 0., 0., 0., 6.609E-02, 0., 0., 0., 1.914E-02, 6.009E-03, 5.008E-02, 1.844E-01, 2.693E-01, 0., 0., 0., 1.704E-01, 0., 0., 0., 8.453E-04, 1.98E-04)

$f(\bar{x}) = 0.057$

3) NLP solution point

$\bar{x}$ = (9.537E-07, 0., 4.215E-03, 1.039E-04, 0., 0., 2.072E-01, 5.979E-01, 1.298E-01, 3.35E-02, 1.711E-02, 8.427E-03, 4.657E-10, 0., 0., 0., 0., 0., 2.868E-04, 1.193E-03, 8.332E-05, 1.239E-04, 2.07E-05, 1.829E-05)

$f(\bar{x}) = 0.0967$

4) SUMT solution point

$\bar{x}$ = (9.109E-03, 3.739E-02, 8.961E-02, 1.137E-02, 4.155E-03, 4.184E-03, 5.98E-02, 1.554E-02, 1.399E-02, 8.78E-03, 1.231E-02, 1.153E-02, 7.57E-02, 7.997E-02, 2.797E-01, 1.168E-02, 2.347E-02, 6.368E-03, 2.028E-01, 7.451E-03, 4.547E-03, 1.01E-02, 1.22E-03, 1.81E-03)

$f(\bar{x}) = 0.07494$

Comments: The problem implements a correction to the 15^{th} through 20^{th} constraints (the ones involving e_i). The form given in [Himmelblau] involves a misprint pointed out to us by Himmelblau. The e_i should appear in the numerator, not the denominator. Applying this correction and the constraint $\sum_{i=1}^{24} x_i = 1$ we obtain the form given. The testing program agreed on the values of the objective function for the four x-vectors. At the second vector, only the second constraint is violated (by less than 10^{-4}). At the third vector, there are five significant constraint violations (the eighth constraint is violated by 0.55). At the fourth x-vector there are seven constraint violations, the largest being $8.7 \cdot 10^{-3}$. The following point was obtained by an augmented Lagrangian algorithm:

$\bar{x}$ = (3.936900E-11, 1.072478E-1, 1.113895E-1, 4.867737E-9, 1.106982E-8, 1.208312E-8, 7.554074E-2, 3.031136E-10, 8.255681E-9, -3.867616E-9, -4.861967E-9, 1.119520E-2, 1.266763E-9, 1.927522E-1, 2.886105E-1, 7.219007E-9, 2.605090E-8, 1.143271E-8, 2.128577E-1, 3.798788E-9, 7.448921E-9, 1.083373E-9, 2.457765E-10, 4.062254E-4)

with $f(\bar{x}) = .055658$

and no constraint violated by more than $2 \cdot 10^{-8}$.

TABLE IV

i	a_i	b_i	c_i	d_i	e_i
1	0.0693	44.094	123.7	31.244	0.1
2	0.0577	58.12	31.7	36.12	0.3
3	0.05	58.12	45.7	34.784	0.4
4	0.20	137.4	14.7	92.7	0.3
5	0.26	120.9	84.7	82.7	0.6
6	0.55	170.9	27.7	91.6	0.3
7	0.06	62.501	49.7	56.708	
8	0.10	84.94	7.1	82.7	
9	0.12	133.425	2.1	80.8	
10	0.18	82.507	17.7	64.517	
11	0.10	46.07	0.85	49.4	
12	0.09	60.097	0.64	49.1	
13	0.0693	44.094			
14	0.0577	58.12			
15	0.05	58.12			
16	0.20	137.4			
17	0.26	120.9			
18	0.55	170.9			
19	0.06	62.501			
20	0.10	84.94			
21	0.12	133.425			
22	0.18	82.507			
23	0.10	46.07			
24	0.09	60.097			

Problem Number 13

Sources: 1) U.S. Steel Company.

2) Himmelblau problem number 22, pp. 422-423.

Objective function: Linear X Nonlinear ___

Number of variables: 6

Number of constraints: 16 Inequalities: Linear 12 Nonlinear 4

Equalities: Linear ___ Nonlinear ___

Problem: Minimize:

$$f(\bar{x}) = 4.3x_1 + 31.8x_2 + 63.3x_3 + 15.8x_4 + 68.5x_5 + 4.7x_6$$

Subject to:

$$17.1x_1 + 38.2x_2 + 204.2x_3 + 212.3x_4 + 623.4x_5 + 1495.5x_6 - 169x_1x_3 - 3580x_3x_5 - 3810x_4x_5 - 18500x_4x_6 - 24300x_5x_6 \geq 4.97$$

$$17.9x_1 + 36.8x_2 + 113.9x_3 + 169.7x_4 + 337.8x_5 + 1385.2x_6 - 139x_1x_3 - 2450x_4x_5 - 16600x_4x_6 - 17200x_5x_6 \geq -1.88$$

$$-273x_2 - 70x_4 - 819x_5 + 26000x_4x_5 \geq -29.08$$

$$159.9x_1 - 311x_2 + 587x_4 + 391x_5 + 2198x_6 - 14000x_1x_6 \geq -78.02$$

$$0 \leq x_1 \leq 0.31$$
$$0 \leq x_2 \leq 0.046$$
$$0 \leq x_3 \leq 0.068$$
$$0 \leq x_4 \leq 0.042$$
$$0 \leq x_5 \leq 0.028$$
$$0 \leq x_6 \leq 0.0134$$

Point tested and objective value [Himmelblau]: 1) Solution point

$\bar{x} = (0, 0, 0, 0, 0, 0.00333)$ $f(\bar{x}) = 0.0156$

Comments: The testing program obtained $f(\bar{x}) = .015651$.

Problem Number 14

Sources: 1) J. Bracken and G. P. McCormick, Selected Applications of Nonlinear Programming, John Wiley & Sons, Inc., New York, 1968, p. 19.

2) Himmelblau problem number 24, p. 426.

Objective function: Linear ___ Nonlinear X

Number of variables: 2

Number of constraints: 2 Inequalities: Linear 1 Nonlinear 1
Equalities: Linear ___ Nonlinear ___

Problem: Minimize: $f(\overline{x}) = (x_1-2)^2 + (x_2-1)^2$

Subject to: $-x_1^2 + x_2 \geq 0$

$-x_1 - x_2 + 2 \geq 0$

Points tested and objective values [Himmelblau]:

1) Nonfeasible starting point
$\overline{x} = (2,2) \quad f(\overline{x}) = 1$

2) Solution point
$x = (1,1) \quad f(\overline{x}) = 1$

Comments: The testing program agreed with the published results.

Problem Number 15

Source: J. B. Rosen, Computer Science Department, University of Minnesota.

Objective function: Linear X Nonlinear ___

Number of variables: 15

Number of constraints: 10 Inequalities: Linear ___ Nonlinear 10

Equalities: Linear ___ Nonlinear ___

Problem: Minimize:

$$f(\overline{x}) = -486x_1 - 640x_2 - 758x_3 - 776x_4 - 477x_5 - 707x_6 - 175x_7 - 619x_8 - 627x_9 - 614x_{10} - 475x_{11} - 377x_{12} - 524x_{13} - 468x_{14} - 529x_{15}$$

Subject to:

$$\sum_{j=1}^{15} a_{ij}x_j^2 - b_i \leq 0, \qquad i = 1,2,\ldots,10$$

where the a_{ij} and b_i are defined in Table V.

Points tested and objective values (from augmented Lagrangian):

1) Feasible starting point

$\overline{x} = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$

$f(\overline{x}) = 0.0$

2) Solution point

$\overline{x} = (1.,1.,1.,1.,1.,1.,1.,1.,1.,1.,1.,1.,1.,1.,1.)$

$f(\overline{x}) = -8252.0$

TABLE V

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	b_i
a_{1j}	100	100	10	5	10	0	0	25	0	10	55	5	45	20	0	385
a_{2j}	90	100	10	35	20	5	0	35	55	25	20	0	40	25	10	470
a_{3j}	70	50	0	55	25	100	40	50	0	30	60	10	30	0	40	560
a_{4j}	50	0	0	65	35	100	35	60	0	15	0	75	35	30	65	565
a_{5j}	50	10	70	60	45	45	0	35	65	5	75	100	75	10	0	645
a_{6j}	40	0	50	95	50	35	10	60	0	45	15	20	0	5	5	430
a_{7j}	30	60	30	90	0	30	5	25	0	70	20	25	70	15	15	485
a_{8j}	20	30	40	25	40	25	15	10	80	20	30	30	5	65	20	455
a_{9j}	10	70	10	35	25	65	0	30	0	0	25	0	15	50	55	390
a_{10j}	5	10	100	5	20	5	10	35	95	70	20	10	35	10	30	460

Problem Number 16

Source: J. B. Rosen, Computer Science Department, University of Minnesota

Objective function: Linear X Nonlinear ___

Number of variables: 15

Number of constraints: 10 Inequalities: Linear ___ Nonlinear 10
Equalities: Linear ___ Nonlinear ___

Problem: Same problem as number 15 except:

$$a_{10,3} = 500 \quad \text{and} \quad b_{10} = 860$$

Points tested and objective values (from augmented Lagrangian):

1) Feasible starting point

$\bar{x} = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$

$f(\bar{x}) = 0.0$

2) Solution point

$\bar{x} = (.8609538, 0.9173613, 0.9197364, 0.8960056, 1.037295, 0.9730890, 0.8224363, 1.198722, 1.156335, 1.144387, 1.030568, 0.9094946, 1.082045, 0.8468238, 1.172372)$

$f(\bar{x}) = -8310.2591$

Problem Number 17

Source: J. B. Rosen, Computer Science Department, University of Minnesota

Objective function: Linear X Nonlinear ___

Number of variables: 15

Number of constraints: 10 Inequalities: Linear ___ Nonlinear 10
Equalities: Linear ___ Nonlinear ___

Problem: Same problem as number 15 except:

$$a_{9,9} = 500 \quad \text{and} \quad b_9 = 890$$

Points tested and objective values (from augmented Lagrangian):

1) Feasible starting point

$\bar{x} = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$

$f(\bar{x}) = 0.0$

2) Solution point

$\bar{x} = (0.813470, 1.132796, 1.086118, 0.998330,$
$1.075486, 1.068876, 0.627816, 1.092998,$
$0.913632, 1.861913, 1.004731, 0.877430,$
$0.986715, 1.041127, 1.186099)$

$f(\bar{x}) = -8315.2859$

Problem Number 18

Source: J. B. Rosen, Computer Science Department, University of Minnesota

Objective function: Linear X Nonlinear ___

Number of variables: 15

Number of constraints: 11 Inequalities: Linear ___ Nonlinear 11

Equalities: Linear ___ Nonlinear ___

Problem: Same problem as number 15 with the additional constraint:

$$-\tfrac{1}{2} \sum_{j=1}^{15} j(x_j - 2)^2 + 61 \leq 0$$

Points tested and objective values (from augmented Lagrangian):

1) Feasible starting point

$\bar{x} = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$

$f(\bar{x}) = 0.0$

2) Solution point

$\bar{x} = (1.012542, 1.015851, 1.030904, 0.9969702, 0.9852836, 1.036853, 0.9934936, 0.9720114, 0.9999409, 0.9954730, 0.9695388, 1.008057, 0.9823699, 0.9905799, 0.9776016)$

$f(\bar{x}) = -8250.1422$

Problem Number 19

Source: J. B. Rosen, Computer Science Department, University of Minnesota

Objective function: Linear X Nonlinear ___

Number of variables: 15

Number of constraints: 11 Inequalities: Linear ___ Nonlinear 11
Equalities: Linear ___ Nonlinear ___

Problem: Same problem as number 15 with the additional constraint:

$$-\tfrac{1}{2}\sum_{j=1}^{15} j(x_j-2)^2 + 70 \leq 0$$

Points tested and objective values (from augmented Lagrangian):

1) Feasible starting point

$\bar{x} = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$

$f(\bar{x}) = 0.0$

2) Solution point

$\bar{x} = (1.004273, 1.087117, 1.103379, 1.030719, 0.9285794, 1.256806, 0.7605842, 0.8568893, 1.089778, 0.9811951, 0.8510646, 0.9655595, 0.9064414, 0.8380401, 0.8093246)$

$f(\bar{x}) = -8164.3687$

Problem Number 20

Source: J. B. Rosen, Computer Science Department, University of Minnesota

Objective function: Linear X Nonlinear ___

Number of variables: 15

Number of constraints: 15 Inequalities: Linear 4 Nonlinear 11

Equalities: Linear ___ Nonlinear ___

Problem: Same problem as number 15 with the additional constraints:

$$-\tfrac{1}{2}\sum_{j=1}^{15} j(x_j-2)^2 + 193.121 \leq 0$$

$$\sum_{j=1}^{15} a_{ij}x_j - b_i \leq 0, \qquad i = 12,13,14,15$$

where the new a_{ij} and b_i are defined in Table VI.

Points tested and objective values (from augmented Lagrangian):

1) Feasible starting point

$\bar{x} = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$
$f(\bar{x}) = 0.0$

2) Solution point

$\bar{x} = (0.6222888, 1.428984, 1.462689, 0.7282862, 0.7842342, 1.215137, -1.137170, 1.058826, -0.1304257, 1.185717, 0.9624097, -0.8496205, 0.4839910, -0.3405321, 0.6845858)$

$f(\bar{x}) = -5819.9197$

TABLE VI

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	b_i
a_{12j}	1	2	3	4	5	6	7	8	9	10	15	16	17	18	19	70
a_{13j}	45	25	35	85	40	73	17	52	86	14	30	50	40	70	60	361
a_{14j}	53	74	26	17	25	25	26	24	85	35	14	23	37	56	10	265
a_{15j}	12	43	51	39	58	42	60	20	40	80	75	85	95	23	67	395

Problem Number 21

Source: J. B. Rosen, Computer Science Department, University of Minnesota

Objective function: Linear X Nonlinear ___

Number of variables: 15

Number of constraints: 15 Inequalities: Linear 4 Nonlinear 11

Equalities: Linear ___ Nonlinear ___

Problem: Same problem as number 15 with the additional constraints:

$$-\tfrac{1}{2}\sum_{j=1}^{15} j(x_j-2)^2 + 200 \le 0$$

$$\sum_{j=1}^{15} a_{ij}x_j - b_i \le 0, \qquad i = 12,13,14,15$$

where the new a_{ij} and b_i are defined in Table X.

Points tested and objective values (from augmented Lagrangian):

1) Feasible starting point

$\bar{x} = (0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$

$f(\bar{x}) = 0.0$

2) Solution point

$\bar{x} = (0.6713863,\ 1.388435,\ 1.467825,\ 0.7602664,$
$0.8289360,\ 1.164142,\ -1.257559,\ 0.9817774,$
$0.0683914,\ 1.147308,\ 0.9865662,\ -0.8805073,$
$0.5645052,\ -0.5814102,\ 0.7207766)$

$f(\bar{x}) = -5809.7124$

Problem Number 51

Source: 1) G. K. Barnes, M.S. thesis, The University of Texas, Austin, Texas, 1967.
2) Himmelblau problem number 3, pp. 394-395.

Objective function: Linear ___ Nonlinear X

Number of variables: 2

Number of constraints: 7 Inequalities: Linear 4 Nonlinear 3

Equalities: Linear ___ Nonlinear ___

Problem: Minimize

$$
\begin{aligned}
f(\bar{x}) = &- 75.196 + 3.8112x_1 - 0.12694x_1^2 \\
&+ 2.0567\cdot 10^{-3}x_1^3 - 1.0345\cdot 10^{-5}x_1^4 + 6.8306x_2 \\
&- 0.030234x_1x_2 + 1.28134\cdot 10^{-3}x_2x_1^2 \\
&- 3.5256\cdot 10^{-5}x_2x_1^3 + 2.266\cdot 10^{-7}x_2x_1^4 - 0.25645x_2^2 \\
&+ 3.4604\cdot 10^{-3}x_2^3 - 1.3514\cdot 10^{-5}x_2^4 + \frac{28.106}{x_2+1} \\
&+ 5.2375\cdot 10^{-6}x_1^2x_2^2 + 6.3\cdot 10^{-9}x_1^3x_2^2 \\
&- 7.\cdot 10^{-10}x_1^3x_2^3 - 3.4054\cdot 10^{-5}x_1x_2^2 \\
&+ 1.6638\cdot 10^{-6}x_1x_2^3 + 2.8673\exp(0.0005x_1x_2)
\end{aligned}
$$

Subject to:

$$0 \le x_1 \le 75$$

$$0 \le x_2 \le 65$$

$$x_1x_2 - 700 \ge 0$$

$$x_2 - 5\left(\frac{x_1}{25}\right)^2 \ge 0$$

$$(x_2-50)^2 - 5(x_1-55) \ge 0$$

Points tested and objective values [Himmelblau]:

1) Nonfeasible starting point

$\bar{x} = (90.,10.)$ $f(\bar{x}) = 82.828$

2) Solution point

$\bar{x} = (75.,65.)$ $f(\bar{x}) = -58.903$

Comments: The coefficient $6.3\cdot 10^{-9}$ is a correction (provided by D. M. Himmelblau) of the value $6.3\cdot 10^{-8}$ which is a misprint in [Himmelblau]. The testing program found differences in the value of the objective function for the two x-vectors, $f(\bar{x}) = 82.475209$ and $f(\bar{x}) = -58.928020$.

Problem Number 52

Source: 1) J. Bracken and G. P. McCormick, Selected Applications of Nonlinear Programming, John Wiley & Sons, Inc., New York, 1968, pp. 46-49.

2) Himmelblau problem number 4, pp. 395-396.

Objective function: Linear ___ Nonlinear X

Number of variables: 10

Number of constraints: 13 Inequalities: Linear 10 Nonlinear ___
Equalities: Linear 3 Nonlinear ___

Problem: Minimize: $$f(\bar{x}) = \sum_{i=1}^{10} x_i \left(c_i + \ln \frac{x_i}{\sum_{j=1}^{10} x_j} \right)$$

where $c_1 = -6.089$ $c_2 = -17.164$ $c_3 = -34.054$
$c_4 = -5.914$ $c_5 = -24.721$ $c_6 = -14.986$
$c_7 = -24.100$ $c_8 = -10.708$ $c_9 = -26.662$
$c_{10} = -22.179$

Subject to:
$$x_1 + 2x_2 + 2x_3 + x_6 + x_{10} - 2 = 0$$
$$x_4 + 2x_5 + x_6 + x_7 - 1 = 0$$
$$x_3 + x_7 + x_8 + 2x_9 + x_{10} - 1 = 0$$
$$x_i \geq 0, \quad i = 1,\ldots,10$$

Points tested and objective values [Himmelblau]:

1) Nonfeasible starting point
$\bar{x} = (0.1,0.1,0.1,0.1,0.1,0.1,0.1,0.1,0.1,0.1)$
$f(\bar{x}) = -20.961$

2) NLP solution point
$\bar{x} = (0.0350,0.1142,0.8306,0.0012,0.4887,0.0005,$
$0.0209,0.0157,0.0289,0.0751)$ $f(\bar{x}) = -47.751$

3) Flexible tolerance solution point
$\bar{x} = (0.0128,0.1433,0.8078,0.0062,0.4790,0.0033,0.0324,$
$0.0281,0.0250,0.0817)$ $f(\bar{x}) = -47.736$

4) GGS solution point
$\bar{x} = (0.,0.1695,0.7536,0.,0.5,0.,0.,0.,0.0464,0.1536)$
$f(\bar{x}) = -47.656$

5) GRG solution point

$\bar{x}$ = (0.0406,0.1477,0.7832,0.0014,0.4853,0.0007,0.0274, 0.0180,0.0375,0.0969) $f(\bar{x}) = -47.761$

6) SUMT solution point

$\bar{x}$ = (0.0407,0.1477,0.7832,0.0014,0.4853,0.0007,0.0274, 0.0180,0.0373,0.0969) $f(\bar{x}) = -47.761$

Comments: The testing program found slight differences in the values of the objective function for each x-vector. The values found are $f(\bar{x}) = -20.960285$, $f(\bar{x}) = -47.754120$, $f(\bar{x}) = -47.732986$, $f(\bar{x})$ undefined, $f(\bar{x}) = -47.769998$, and $f(\bar{x}) = -47.764888$, respectively. The fourth x-vector contained zero components causing the log function to fail. Note that the objective function can only be evaluated at feasible points which have x_i strictly positive.

Problem Number 53

Source: 1) J. Bracken and G. P. McCormick, Selected Applications of Nonlinear Programming, John Wiley & Sons, Inc., New York, 1968, pp. 46-49.

2) Himmelblau problem number 4A, p. 396.

Objective function: Linear ___ Nonlinear ___

Number of variables: 10

Number of constraints: 3 Inequalities: Linear ___ Nonlinear ___

Equalities: Linear ___ Nonlinear 3

Problem: Minimize: $f(x) = \sum_{i=1}^{10} \left\{ e^{x_i} \left[c_i + x_i - \ell n \left(\sum_{j=1}^{10} e^{x_j} \right) \right] \right\}$

where the c_i's are defined in problem 4.

Subject to: $e^{x_1} + 2e^{x_2} + 2e^{x_3} + e^{x_6} + e^{x_{10}} - 2 = 0$

$e^{x_4} + 2e^{x_5} + e^{x_6} + e^{x_7} - 1 = 0$

$e^{x_3} + e^{x_7} + e^{x_8} + 2e^{x_9} + e^{x_{10}} - 1 = 0$

Points tested: All points used in problem 52 were tested in problem 53. The points were transformed using the transformation:

$\bar{x}_{53} = \ell n(\bar{x}_{52})$

Comments: The testing program found slight differences in the values of the objective function for each x-vector. The values found are $f(\bar{x}) = -20.960285$, $f(\bar{x}) = -47.754120$, $f(\bar{x}) = -47.732986$, $f(\bar{x})$ undefined, $f(\bar{x}) = -47.769998$, and $f(\bar{x}) = -47.764888$, respectively. The fourth x-vector contained zero components which caused a failure in the transformation $\bar{x}_{53} = \ell n(\bar{x}_{52})$. Unlike problem 52, the log function does not have difficulties since e^{x_j} is positive.

Problem Number 54

Sources: 1) D. M. Himmelblau and R. V. Yates, "A New Method of Flow Routing," Water Resources, 4:1193(1968).

2) Himmelblau problem number 9, p. 403.

Objective function: Linear ___ Nonlinear X

Number of variables: 4

Number of constraints: 6 Inequalities: Linear 5 Nonlinear 1

Equalities: Linear ___ Nonlinear ___

Problem:

Minimize: $f(\bar{x}) = \sum_{i=1}^{19} \left(y_{i,cal} - y_{i,obs}\right)^2$

$$\text{where} \quad y_{i,cal} = \frac{x_3 \beta^{x_2} \left(\frac{x_2}{6.2832}\right)^{\frac{1}{2}} \left(\frac{c_i}{7.658}\right)^{(x_2-1)} \exp\left(x_2 - \beta \frac{c_i x_2}{7.658}\right)}{1 + \frac{1}{12x_2}}$$

$$+ \frac{(1-x_3)\left(\frac{\beta}{x_4}\right)^{x_1} \left(\frac{x_1}{6.2832}\right)^{\frac{1}{2}} \left(\frac{c_i}{7.658}\right)^{(x_1-1)} \exp\left(x_1 - \beta \frac{c_i x_1}{7.658 x_4}\right)}{1 + \frac{1}{12x_1}}$$

$$\beta = x_3 + (1-x_3)x_4 .$$

The c_i's and $y_{i,obs}$'s are defined in Table VII.

Subject to: $x_3 + (1-x_3)x_4 \geq 0$

$x_i \geq 0$, $i=1,2,3,4$

$x_3 \leq 1$

Points tested and objective values [Himmelblau]:

1) Feasible starting point

$\bar{x} = (2,4,0.04,2)$ $f(\bar{x}) = 4.8024$

2) Solution point

$\bar{x} = (12.277, 4.632, 0.313, 2.029)$ $f(\bar{x}) = 0.0075$

TABLE VII

i	c_i	$y_{i,obs}$
1	0.1	0.00189
2	1	0.1038
3	2	0.268
4	3	0.506
5	4	0.577
6	5	0.604
7	6	0.725
8	7	0.898
9	8	0.947
10	9	0.845
11	10	0.702
12	11	0.528
13	12	0.385
14	13	0.257
15	14	0.159
16	15	0.0869
17	16	0.0453
18	17	0.01509
19	18	0.00189

Comments: The testing program disagreed on the value of the objective function for the starting x-vector. The value found was $f(\overline{x}) = 0.98185961$ and $f(\overline{x}) = .0074985354$, respectively.

Problem Number 55

Sources: 1) D. A. Paviani, Ph.D. dissertation, The University of Texas, Austin, Texas, 1969.

2) Himmelblau, problem number 17, p. 416.

Objective function: Linear ___ Nonlinear X

Number of variables: 10

Number of constraints: 20 Inequalities: Linear 20 Nonlinear ___

Equalities: Linear ___ Nonlinear ___

Problem: Minimize: $$f(\overline{x}) = \sum_{i=1}^{10} \left\{ [\ell n(x_i - 2)]^2 + [\ell n(10 - x_i)]^2 \right\} - \left(\prod_{i=1}^{10} x_i \right)^{0.2}$$

Subject to: $2.001 < x_i < 9.999, \quad i=1,\ldots,10$

Points tested and objective values [Himmelblau]:

1) Feasible starting point

$\overline{x} = (9,9,9,9,9,9,9,9,9,9) \quad f(\overline{x}) = -43.134$

2) Solution point

$\overline{x} = (9.351,\ 9.351,\ 9.351,\ 9.351,\ 9.351,\ 9.351,\ 9.351,\ 9.351,\ 9.351,\ 9.351)$

$f(\overline{x}) = -45.778$

Comments: The testing program agreed with the published results.

Problem Number 56

Sources: 1) A. G. Holzman, SRCC Rept. 113, University of Pittsburgh, Pittsburgh, PA, 1969.

2) Himmelblau problem number 21, p. 422.

Ojbective function: Linear ___ Nonlinear X

Number of variables: 3

Number of constraints: 6 Inequalities: Linear 6 Nonlinear ___

Equalities: Linear ___ Nonlinear ___

Problem: Minimize: $f(\bar{x}) = \sum_{i=1}^{99} \left(\exp - \frac{(u_i - x_2)^{x_3}}{x_1} - 0.01i \right)^2$

where $u_i = 25 + (-50 \ln(0.01i))^{2/3}$

Subject to: $0.1 \leq x_1 \leq 100.$

$0.0 \leq x_2 \leq 25.6$

$0.0 \leq x_3 \leq 5.0$

Points tested and objective values [Himmelblau]:

1) Feasible starting point

$\bar{x} = (100,12.5,3)$ $f(\bar{x}) = 32.835$

2) Solution point

$\bar{x} = (50,25,1.5)$ $f(\bar{x}) = 0.0$

Comments: The testing program agreed with the published results.

Problem 71

Source: 1) J. C. DeHaven and E. C. Deland, "Reactions of Hemoglobin and Steady States in the Human Respiratory System: An Investigation Using Mathematical Models and an Electronic Computer," RM-3212-PR, The RAND Corporation, Dec. 1962.

2) Himmelblau problem number 6, pp. 397-401.

Objective function: Linear ___ Nonlinear X

Number of variables: 45

Number of constraints: 61 Inequalities: Linear 45 Nonlinear ___

Equalities: Linear 16 Nonlinear ___

Problem: Minimize:
$$f(\bar{x}) = \sum_{k=1}^{7}\left[\sum_{j=1}^{n_k} x_{jk}\left(c_{jk} + \ln \frac{x_{jk}}{\sum_{i=1}^{n_k} x_{ik}}\right)\right]$$

Subject to:
$$\sum_{k=1}^{7}\left(\sum_{j=1}^{n_k} E_{ijk}x_{jk}\right) - b_i = 0, \quad i=1,\ldots,16$$

$$x_{jk} \geq 0, \quad ((j=1,\ldots,n_k),\ k=1,\ldots,7)$$

where n_k, c_{jk}, E_{ijk}, and b_i are defined in Tables VIII and IX.

Points tested and objective values [Himmelblau]:

1) Nonfeasible starting point

$\bar{x} = (0.1, 0.1, \ldots, 0.1)$ $f(\bar{x}) = -30.958$

2) NLP solution point

$\bar{x}$ = (7.854E-7, 8.078E-2, 3.706, 8.855E-2, 6.894E-1, 3.02E-2, 1.398E-4, 1.626E-4, 0., 2.782E-2, 7.95E-2, 3.421E-2, 2.486E+1, 3.873E-2, 1.5E-4, 1.17E-5, 1.55E-2, 0., 2.649E-2, 1.251E-4, 1.064E-1, 0., 5.253E-2, 8.71E-3, 1.471E-2, 4.735E-2, 9.208E-2, 3.119E-4, 1.56E-2, 2.421E-2, 2.448E-3, 8.398E-3, 5.285E-3, 0., 1.601E-3, 4.968E-7, 1.978E-2, 6.271E-3, 5.328E-2, 0., 0., 2.51E-2, 1.22E6, 0., 0.)

$f(\bar{x}) = -1909.74$

3) SUMT solution point

$\bar{x}$ = (6.599E-6, 2.512E-1, 3.705, 2.53E-1, 6.529E-1,
1.235E-3, 3.667E-4, 2.794E-6, 5.441E-6, 7.363E-2,
8.791E-2, 3.542E02, 4.458E+1, 2.669E-2, 7.709E-6,
3.764E-5, 1.55E-2, 9.9E-7, 5.077E-5, 3.107E-5,
1.546E-6, 3.102E-6, 6.416E-3, 2.202E-4, 1.287E-2,
2.165, 2.675, 3.437E-6, 1.4E-5, 1.927E-2, 1.855E-3,
3.264E-5, 7.579E-7, 3.51E-7, 2.513E-7, 0., 4.2E-7,
7.063E-6, 0., 0., 1.305E-6, 1.465E-5, 1.382E-5,
2.872E-6, 2.476E-6)

$f(\bar{x})$ = -1910.361

4) Jones (SUMT) solution point

$\bar{x}$ = (6.44E-1, 2.59E-1, 3.705, 2.997E-1, 5.617E-5, 6.88E-4,
2.062E-4, 1.101E-6, 2.433E-6, 5.715E-2, 7.938E-2,
3.231E-3, 2.839E-1, 1.388E-2, 3.283E-6, 1.738E-5,
1.155E-2, 5.956E-5, 4.419E-4, 2.205E04, 1.095E-6,
1.852E-6, 2.291E-2, 8.751E-3, 4.506E-2, 1.832E-1,
6.396E-3, 2.855E-6, 7.806E-6, 2.113E-2, 7.429E-6,
3.017E-5, 5.056E-5, 4.871E-5, 2.142E-3, 2.337E-6,
1.821E-4, 8.583E-5, 2.355E-5, 1.251E-3, 7.573E-3,
3.038E-4, 3.902E-5, 2.879E-2, 1.499E-3)

$f(\bar{x})$ = -79.108

Comments: The testing program disagreed on the value of the objective function for all four x-vectors except the first x-vector. The values found for the second, third and fourth x-vectors were $f(\bar{x})$ = -1045.1338, $f(\bar{x})$ = -1971.0602, and $f(\bar{x})$ = -79.031796, respectively. The three solution vectors produced significant violations of equality constraints 2, 4, and 5. For example, the fourth point violated the fourth and fifth constraints by more than 46. Notice that the log function in the objective actually implies that feasible points have x_{jk} strictly positive.

TABLE VIII

i	b_i	k	n_k
1	0.6529581	1	4
2	0.281941	2	13
3	3.705233	3	18
4	47.00022	4	3
5	47.02972	5	3
6	0.08005	6	2
7	0.08813	7	2
8	0.04829		
9	0.0155		
10	0.0211275		
11	0.0022725		
12	0.0		
13	0.0		
14	0.0		
15	0.0		
16	0.0		

TABLE IX

x_{jk}	c_{jk}	E_{ijk}															
	$i =$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$x_{1,1}$	0.0	1															
$x_{2,1}$	-7.69		1														
$x_{3,1}$	-11.52			1													
$x_{4,1}$	-36.60				1	1											
$x_{1,2}$	-10.94	1															
$x_{2,2}$	0.0		1														
$x_{3,2}$	0.0			1													
$x_{4,2}$	0.0				1								1				
$x_{5,2}$	0.0					1							-1				
$x_{6,2}$	0.0						1						-1				
$x_{7,2}$	0.0							1					1				
$x_{8,2}$	2.5966								1				1				
$x_{9,2}$	-39.39				1	1											
$x_{10,2}$	-21.35		1			1							-1				
$x_{11,2}$	-32.84		1		1	1											
$x_{12,2}$	6.26		1		-1	1							-2				
$x_{13,2}$	0.0									1			-1				
$x_{1,3}$	10.45	1															
$x_{2,3}$	0.0		1														
$x_{3,3}$	-0.50			1													
$x_{4,3}$	0.0				1												
$x_{5,3}$	0.0					1											
$x_{6,3}$	0.0						1										
$x_{7,3}$	2.2435							1									
$x_{8,3}$	0.0								1								
$x_{9,3}$	-39.39				1	1											
$x_{10,3}$	-21.49		1			1											
$x_{11,3}$	-32.84		1		1	1											
$x_{12,3}$	6.12		1		-1	1											
$x_{13,3}$	0.0										1						
$x_{14,3}$	0.0											1		-4			
$x_{15,3}$	-1.9028	1										1		-3	-1		
$x_{16,3}$	-2.8889	2										1		-2	-2		
$x_{17,3}$	-3.3622	3										1		-1	-3		
$x_{18,3}$	-7.4854	4										1			-4		
$x_{1,4}$	-15.639				1									1			
$x_{2,4}$	0.0													1			
$x_{3,4}$	21.81				-1									1		-4	
$x_{1,5}$	-16.79				1										1		
$x_{2,5}$	0.0														1		
$x_{3,5}$	18.9779				-1										1		-4
$x_{1,6}$	0.0															1	
$x_{2,6}$	11.959		1		-1											1	
$x_{1,7}$	0.0																1
$x_{2,7}$	12.899		1		-1												1

Problem Number 72

Sources: 1) J. Bracken and G. P. McCormick, Selected Applications of Nonlinear Programming, John Wiley & Sons, Inc., New York, 1968, p. 26.

2) Himmelblau problem number 23, pp. 423-425.

Objective function: Linear ___ Nonlinear X

Number of variables: 100

Number of constraints: 112 Inequalities: Linear 112 Nonlinear ___

Equalities: Linear ___ Nonlinear ___

Problem: Minimize: $$f(\bar{x}) = \sum_{j=1}^{20} u_j \left(\prod_{i=1}^{5} a_{ij}^{x_{ij}} - 1 \right)$$

Subject to: $$\sum_{i=1}^{5} x_{ij} - b_j \geq 0 \qquad j=1,6,10,14,15,16,20$$

$$-\sum_{j=1}^{20} x_{ij} - c_i \geq 0 \qquad i=1,\ldots,5$$

$$x_{ij} \geq 0 \qquad i=1,\ldots,5 \quad j=1,\ldots,20$$

where a_{ij}, b_j, c_i, and u_j are defined in Table X.

Points tested and objective values [Himmelblau]: 1) Feasible solution (Holzman)

$\bar{x}$ = (0., 1., 0., 0., 47., 24., 8., 9., 0., 5., 0., 2., 0., 0., 36., 0., 16., 0., 0., 12., 0., 11., 29., 0., 0., 32., 0., 62., 0., 6., 37., 0., 0., 0., 0., 28., 0., 0., 0., 0., 22., 0., 0., 0., 0., 0., 0., 0., 0., 50., 0., 0., 0., 9., 42., 0., 0., 0., 39., 0., 0., 0., 0., 0., 51., 0., 0., 0., 58., 0., 5., 29., 35., 0., 1., 0., 9., 0., 44., 0., 0., 21., 17., 0., 0., 52., 0., 25., 0., 0., 0., 0., 62., 0., 0., 0., 0., 60., 0., 0.)

$f(\bar{x})$ = 1732

2) Feasible solution (Bracken and McCormick).

$\overline{x}$ = (0., 0., 0., 0., 50., 16., 0., 0., 0., 46., 0., 0., 0., 0., 47., 0., 23., 0., 0., 0., 0., 20., 0., 0., 0., 100., 0., 0., 0., 0., 38., 0., 0., 0., 0., 26., 0., 0., 0., 0., 20., 0., 0., 0., 0., 0., 0., 0., 0., 50., 0., 0., 0., 0., 57., 0., 0., 0., 39., 0., 0., 0., 0., 50., 0., 0., 0., 0., 57., 0., 0., 25., 45., 0., 0., 0., 31., 0., 4., 0., 0., 1., 76., 0., 0., 0., 0., 56., 0., 0., 0., 0., 62., 0., 0., 0., 0., 61., 0., 0.)

$f(\overline{x}) = 1732$

Comments: The testing program differed with the values of the objective function for the two x-vectors in sign. The values were $f(\overline{x}) = -1731.8048$ and $f(\overline{x}) = -1732.4431$.

TABLE X

j \ i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	c_i
								a_{ij}													
1	1	.95	1	1	1	.85	.90	.85	.80	1	1	1	1	1	1	1	1	.95	1	1	200
2	.84	.83	.85	.84	.85	.81	.81	.82	.80	.86	1	.98	1	.88	.87	.88	.85	.84	.85	.85	100
3	.96	.95	.96	.96	.96	.90	.92	.91	.92	.95	.99	.98	.99	.98	.97	.98	.95	.92	.93	.92	300
4	1	1	1	1	1	1	1	1	1	.96	.91	.92	.91	.92	.98	.93	1	1	1	1	150
5	.92	.94	.92	.95	.95	.98	.98	1	1	.90	.95	.96	.91	.98	.99	.99	1	1	1	1	250
b_j	30					100				40				50	70	35				10	
u_j	60	50	50	75	40	60	35	30	25	150	30	45	125	200	200	130	100	100	100	150	

Problem Number 81

Sources: 1) M. A. Efroymson, Esso Research and Engineering Co.

2) Himmelblau problem number 14, pp. 412-413.

3) Colville problem number 5, pp. 26-27.

Objective function: Linear ___ Nonlinear X

Number of variables: 6

Number of constraints: 4 Inequalities: Linear ___ Nonlinear 4

Equalities: Linear ___ Nonlinear ___

Problem: Minimize: $f(\overline{x}) = \sum_{i=1}^{4} c(x_i) + \sum_{i=5}^{6} 100c(x_i)$

Subject to:
$$t_3 - 300 \geq 0$$
$$t_4 - 300 \geq 0$$
$$280 - T_5 \geq 0$$
$$250 - T_6 \geq 0$$

where
$$c(x_i) = 2.7x_i + 1300\left(\text{smallest integer} \geq \frac{x_i}{2000}\right)$$
$$\alpha_2 = -.0001665x_2$$
$$\alpha_3 = (.085)(9.36)10^{-5}x_3$$
$$\alpha_4 = .00025x_4$$
$$\alpha_5 = .000375x_5$$
$$\alpha_6 = .0003x_6$$
$$T_1 = \frac{.0285x_1 + 300}{1 + .0001425x_1}$$
$$t_1 = 500 - T_1$$
$$T_2 = \frac{200 - 350e^{-\alpha_2}}{1 - 1.5e^{-\alpha_2}}$$
$$t_2 = 300 + (200-T_2)e^{\alpha_2}$$
$$T_3 = \frac{t_1 + (29.75-t_1)e^{-\alpha_3}}{1 - .915e^{-\alpha_3}}$$

$$t_3 = 350 + (t_1 - T_3)e^{\alpha_3}$$

$$T_4 = 350 + (t_2 - T_4)e^{\alpha_4}$$

$$t_4 = 350 + (t_2 - T_4)e^{\alpha_4}$$

$$T_{j1} = .7T_1 + .3T_2$$

$$T_{j2} = .8T_3 + .2T_4$$

$$T_5 = 80 + (T_{j2} - 80)e^{-\alpha_5}$$

$$T_6 = 80 + (T_{j1} - 80)e^{-\alpha_6}$$

Points tested and objective values [Himmelblau]:

1) Starting point

$\bar{x} = (8000, 3000, 14000, 2000, 300, 10)$ $f(\bar{x}) = 459100$

2) Solution point

$\bar{x} = (11884, 3288, 20000, 4000, 114.18, -155.03)$

$f(\bar{x}) = 250799.9$

Comments: The testing program disagreed on the value of the objective function. The values were $f(\bar{x}) = 434800.00$ and $f(x) = 250734.90$, respectively. The discontinuous objective function caused the gradient check to fail in the testing program.

Problem Number 82

Sources: 1) P. Huard, Electricité de France, directions des Études et Recherches.

2) Himmelblau problem number 15, pp. 413-414.

3) Colville problem number 6, p. 28.

Objective function: Linear X Nonlinear ___

Number of variables: 6

Number of constraints: 16 Inequalities: Linear 12 Nonlinear ___

Equalities: Linear ___ Nonlinear 4

Problem: Minimize: $f(\bar{x}) = f_1(x_1) + f_2(x_2)$

where $f_1(0) = 0, \quad f_2(0) = 0$

$$\frac{df_1}{dx_1} = \begin{cases} 30 & \text{if} \quad 0 \le x_1 < 300 \\ 31 & \text{if} \quad 300 \le x_1 \le 400 \end{cases}$$

$$\frac{df_2}{dx} = \begin{cases} 28 & \text{if} \quad 0 \le x_2 < 100 \\ 29 & \text{if} \quad 100 \le x_2 < 200 \\ 30 & \text{if} \quad 200 \le x_2 < 1000 \end{cases}$$

Subject to:

$$x_1 = 300 - \frac{x_3 x_4}{131.078} \cos(1.48477 - x_6) + \frac{0.90798 x_3^2}{131.078} \cos(1.47588)$$

$$x_2 = - \frac{x_3 x_4}{131.078} \cos(1.48477 + x_6) + \frac{0.90798 x_4^2}{131.078} \cos(1.47588)$$

$$x_5 = - \frac{x_3 x_4}{131.078} \sin(1.48477 + x_6) + \frac{0.90798 x_4^2}{131.078} \sin(1.47588)$$

$$0 = 200 - \frac{x_3 x_4}{131.078} \sin(1.48477 - x_6) + \frac{0.90798 x_3^2}{131.078} \sin(1.47588)$$

$$0 \le x_1 \le 400$$

$$0 \le x_2 \le 1000$$

$$340 \le x_3 \le 420$$

$$340 \le x_4 \le 420$$

$$-1000 \le x_5 \le 1000$$

$$0 \le x_6 \le .5236$$

Points tested and objective values [Himmelblau]:

1) Nonfeasible starting point

$\bar{x}$ = (390, 1000, 419.5, 340.5, 198.175, 0.5)

$f(\bar{x})$ = 9074.14

2) Solution point 1

$\bar{x}$ = (107.81, 196.32, 373.83, 420., 21.31, 0.153)

$f(\bar{x})$ = 8927.5888

3) Solution point 2

$\bar{x}$ = (201.78, 100., 383.07, 420., -10.907, 0.07314)

$f(\bar{x})$ = 8853.44 or 8953.4

Comments: The testing program disagreed on the values of the objective function. The values were $f(\bar{x})$ = 41490.00, $f(\bar{x})$ = 8827.5800, and $f(\bar{x})$ = 8853.4000, respectively. Solution point 1 caused a violation of 0.34 in the first constraint. Solution point 2 caused a violation of .014 and .0027 in constraints 1 and 3.

Problem Number 91

Source: H. K. Schultz, College of Business Administration, University of Wisconsin - Oshkosh.

Objective function: Linear ___ Nonlinear X

Number of variables: 5

Number of constraints: 5 Inequalities: Linear 4 Nonlinear 1

Equalities: Linear ___ Nonlinear ___

Problem: Minimize:
$$f(\bar{x}) = x_1x_2x_3x_4 - 3x_1x_2x_4 - 4x_1x_2x_3 + 12x_1x_2 - x_2x_3x_4 + 3x_2x_4 + 4x_2x_3 - 12x_2 - 2x_1x_3x_4 + 6x_1x_4 + 8x_1x_3 - 24x_1 + 2x_3x_4 - 6x_4 - 8x_3 + 24 + 1.5x_5^4 - 5.75x_5^3 + 5.25x_5^2$$

Subject to:
$$x_1 \geq 1$$
$$x_2 \geq 2$$
$$x_3 \geq 3$$
$$x_4 \geq 4$$
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \leq 34$$

Points tested and objective values (obtained from testing program):

1) Feasible starting point

$\bar{x} = (1.1, 2.1, 3.1, 4.1, -1.)$

$f(\bar{x}) = 12.500100$

2) Relative minimum

$\bar{x} = (1.0, 2.1, 3.1, 4.1, 0.)$

$f(\bar{x}) = 0.0$

3) Global minimum

$\bar{x} = (1., 2., 3., 4., 2.)$

$f(\bar{x}) = -1.$

Comments: This problem has local minima with $f(\bar{x}) = 0$ at the following points:

1) $\bar{x} = (1,a,b,c,d)$

2) $\bar{x} = (a,2,b,c,d)$

3) $\bar{x} = (a,b,3,c,d)$

4) $\bar{x} = (a,b,c,4,d)$

where a, b, and c are arbitrary, $d = 0$, and the constraints are satisfied. The problem has global minima with $f(\bar{x}) = -1$ at points as above but with $d = 2$. The problem has saddle points when $d = 7/8$.

APPENDIX I

A. FORTRAN listing of testing program for test problem 7.

B. Output of testing program for test problem 7.

```
      INTEGER I,IN,J,M,MM,N,NUMPT
      DOUBLE PRECISION DEL,VAL,VAL1,VAL2
      DOUBLE PRECISION X(50),G(50),Y(50),GPROX(50)
C     **********
C
C     THIS PROGRAM READS NUMPT POINTS AND EVALUATES THE
C     OBJECTIVE VALUE AND CONSTRAINTS AT EACH POINT.  ALSO,
C     THE GRADIENT OF THE OBJECTIVE AND CONSTRAINTS IS EVALUATED
C     AND COMPARED WITH A DIVIDED DIFFERENCE APPROXIMATION.
C
C     **********
C
C     SET STEP SIZE FOR DIFFERENCE APPROXIMATIONS TO GRADIENTS
C
      DEL=.000001D0
C
C     READ THE NUMBER OF POINTS TO BE CHECKED
C
      READ(5,1001) NUMPT
   10 WRITE(6,1002) NUMPT
C
C     READ THE DIMENSION OF THE PROBLEM (N), THE NUMBER OF
C     CONSTRAINTS (M), AND THE COORDINATES OF THE CURRENT POINT (X)
C
      READ(5,1001) N,M,(X(I),I=1,N)
      WRITE(6,1003) N,M,(I,X(I),I=1,N)
      MM=M+1
C     **********
C
C     EVALUATE AND PRINT THE OBJECTIVE VALUE AND ALL CONSTRAINTS
C
C     **********
      DO 20 I=1,MM
         IN=I-1
         CALL FVAL7(N,X,VAL,IN)
         WRITE(6,1004) IN,VAL
   20    CONTINUE
C
C     COPY CURRENT POINT COORDINATES INTO Y
C
      DO 30 J=1,N
         Y(J)=X(J)
   30    CONTINUE
      DO 60 J=1,MM
         IN=J-1
         DO 40 I=1,N
C           **********
C
C           PERTURB THE ITH COORDINATE IN Y
C
C           **********
            Y(I)=Y(I)+DEL
            CALL FVAL7(N,X,VAL1,IN)
            CALL FVAL7(N,Y,VAL2,IN)
            Y(I)=X(I)
C           **********
C
C           COMPUTE DIVIDED DIFFERENCE APPROXIMATION
C
```

```
C            **********
             GPROX(I)=(VAL2-VAL1)/DEL
   40        CONTINUE
C
C         EVALUATE AND PRINT GRADIENT
C
          CALL GVAL7(N,X,G,IN)
          WRITE(6,1005) IN
          DO 50 I=1,N
C            **********
C
C            IF DIFFERENCE APPROXIMATION TO GRADIENT IS ACCURATE
C            ENOUGH, DO NOT PRINT THEM
C
C            **********
             IF(G(I).EQ.GPROX(I)) GO TO 50
             WRITE(6,1006) I,G(I),I,GPROX(I)
   50        CONTINUE
   60     CONTINUE
C
C     DECREMENT NUMPT TO INDICATE NEXT POINT NUMBER
C
      NUMPT=NUMPT-1
      IF(NUMPT.GT.0) GO TO 10
C
C     FORMAT STATEMENTS
C
 1001 FORMAT(2I5,/(6D10.5))
 1002 FORMAT('1','NUMBER OF POINT ',I5)
 1003 FORMAT(1X,'NO. OF VARIABLES =',I5,/,' NO. OF CONSTRAINTS =',I5,/,
     1      ' X-VALUES',//,(' X(',I2,') =',D25.16))
 1004 FORMAT(//,(' IN =',I5,'       VAL =',D25.16))
 1005 FORMAT('0GRADIENT TEST', I5,//)
 1006 FORMAT(' G(',I2,') =',D25.16,3X,'GPROX(',I2,') =',D25.16)
      STOP
C
C     LAST CARD IN TEST PROGRAM
C
      END
```

```
NUMBER OF POINT      2
NO. OF VARIABLES =    2
NO. OF CONSTRAINTS =    2
X-VALUES

X( 1) =    0.2000000000000000D+01
X( 2) =    0.2000000000000000D+01


IN =     0       VAL =    0.1000000000000000D+01


IN =     1       VAL =   -0.1000000000000000D+01


IN =     2       VAL =    0.4000000000000000D+01

GRADIENT TEST     0

G( 1) =    0.0                         GPROX( 1) =    0.9998668559774160D-06
G( 2) =    0.2000000000000000D+01   GPROX( 2) =    0.2000000999702323D+01

GRADIENT TEST     1

G( 1) =    0.1000000000000000D+01   GPROX( 1) =    0.9999999999177334D+00
G( 2) =   -0.2000000000000000D+01   GPROX( 2) =   -0.1999999999835467D+01

GRADIENT TEST     2

G( 1) =    0.1000000000000000D+01   GPROX( 1) =    0.1000000249717914D+01
G( 2) =    0.4000000000000000D+01   GPROX( 2) =    0.4000000999537790D+01
```

```
NUMBER OF POINT      1
NO. OF VARIABLES =     2
NO. OF CONSTRAINTS =     2
X-VALUES

X( 1) =    0.8229000000000000D+00
X( 2) =    0.9114000000000000D+00


IN =     0        VAL =    0.1393414369999999D+01


IN =     1        VAL =    0.1000000000000306D-03


IN =     2        VAL =   -0.5893750000000864D-04

GRADIENT TEST    0


G( 1) =   -0.2354200000000000D+01   GPROX( 1) =   -0.2354198999965362D+01
G( 2) =   -0.1772000000000000D+00   GPROX( 2) =   -0.1771990001397938D+00

GRADIENT TEST    1


G( 1) =    0.1000000000000000D+01   GPROX( 1) =    0.9999999999871223D+00
G( 2) =   -0.2000000000000000D+01   GPROX( 2) =   -0.1999999999835467D+01

GRADIENT TEST    2


G( 1) =    0.4114500000000000D+00   GPROX( 1) =    0.4114502499957817D+00
G( 2) =    0.1822800000000000D+01   GPROX( 2) =    0.1822800999973229D+01
```

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APPENDIX II

A. FORTRAN listing of Problem Number 7

1. FVAL7

2. GVAL7

B. FORTRAN listing of Rosen's Problem Number 15

1. FVAL15

2. GVAL15

C. FORTRAN listing of Schultz's Problem Number 91

1. FVAL91

2. GVAL91

```
      SUBROUTINE FVAL7(N,X,VAL,IN)
      INTEGER N,IN
      DOUBLE PRECISION VAL
      DOUBLE PRECISION X(N)
C     **********
C
C     SUBROUTINE FVAL7(N,X,VAL,IN)
C
C     THIS SUBROUTINE RETURNS FUNCTION AND CONSTRAINT VALUES FOR
C     TEST PROBLEM 7 OF ARGONNE NATIONAL LABORATORY APPLIED MATHEMATICS
C     DIVISION TECHNICAL MEMORANDUM NO. 320 BY L. W. CORNWELL,
C     P. A. HUTCHISON, M. MINKOFF, AND H. K. SCHULTZ.
C
C     THE SUBROUTINE STATEMENT IS
C
C     SUBROUTINE FVAL7(N,X,VAL,IN)
C
C     WHERE
C
C     N IS THE DIMENSION OF THE PROBLEM
C
C     X IS THE POINT EVALUATED
C
C     VAL IS THE VALUE RETURNED ON OUTPUT
C
C     IN IS THE NUMBER OF THE CONSTRAINT TO BE EVALUATED.  THE
C       OBJECTIVE FUNCTION IS EVALUATED IF IN=0
C
C     **********
      DOUBLE PRECISION TEMP1,TEMP2
      IF (IN.NE.0) GO TO 5
      TEMP1=X(1)-2.D0
      TEMP2=X(2)-1.D0
      VAL=TEMP1*TEMP1+TEMP2*TEMP2
      RETURN
    5 CONTINUE
      GO TO (10,20),IN
   10 CONTINUE
      VAL=X(1)-2.D0*X(2)+1.D0
      RETURN
   20 CONTINUE
      VAL=(X(1)*X(1))/4.D0+X(2)*X(2)-1.D0
      RETURN
C
C     LAST CARD OF SUBROUTINE FVAL7
C
      END
      SUBROUTINE GVAL7(N,X,G,IN)
      INTEGER N,IN
      DOUBLE PRECISION X(N),G(N)
C     **********
C
C     SUBROUTINE GVAL7(N,X,G,IN)
C
C     THIS SUBROUTINE RETURNS FUNCTION AND CONSTRAINT GRADIENTS FOR
C     TEST PROBLEM 7 OF ARGONNE NATIONAL LABORATORY APPLIED MATHEMATICS
C     DIVISION TECHNICAL MEMORANDUM NO. 320 BY L. W. CORNWELL,
C     P. A. HUTCHISON, M. MINKOFF, AND H. K. SCHULTZ.
C
```

```
C      THE SUBROUTINE STATEMENT IS
C
C      SUBROUTINE GVAL7(N,X,G,IN)
C
C      WHERE
C
C      N IS THE DIMENSION OF THE PROBLEM
C
C      X IS THE POINT EVALUATED
C
C      G IS THE GRADIENT RETURNED ON OUTPUT
C
C      IN IS THE NUMBER OF THE CONSTRAINT TO BE EVALUATED.  THE
C        OBJECTIVE GRADIENT IS EVALUATED IF IN=0
C
C      **********
       IF(IN.NE.0)GO TO 5
       G(1)=2.D0*X(1)-4.D0
       G(2)=2.D0*X(2)-2.D0
       RETURN
     5 CONTINUE
       GO TO (10,20),IN
    10 CONTINUE
       G(1)=1.D0
       G(2)=-2.D0
       RETURN
    20 CONTINUE
       G(1)=X(1)/2.D0
       G(2)=2.D0*X(2)
       RETURN
C
C      LAST CARD OF SUBROUTINE GVAL7
C
       END
```

```
      SUBROUTINE FVAL15(N,X,VAL,IN)
      INTEGER N,IN
      DOUBLE PRECISION VAL
      DOUBLE PRECISION X(N)
C     **********
C
C     SUBROUTINE FVAL15(N,X,VAL,IN)
C
C     THIS SUBROUTINE RETURNS FUNCTION AND CONSTRAINT VALUES FOR
C     TEST PROBLEM 15 OF ARGONNE NATIONAL LABORATORY APPLIED MATHEMATICS
C     DIVISION TECHNICAL MEMORANDUM NO. 320 BY L. W. CORNWELL,
C     P. A. HUTCHISON, M. MINKOFF, AND H. K. SCHULTZ.
C
C     THE SUBROUTINE STATEMENT IS
C
C     SUBROUTINE FVAL15(N,X,VAL,IN)
C
C     WHERE
C
C     N IS THE DIMENSION OF THE PROBLEM
C
C     X IS THE POINT EVALUATED
C
C     VAL IS THE VALUE RETURNED ON OUTPUT
C
C     IN IS THE NUMBER OF THE CONSTRAINT TO BE EVALUATED.  THE
C       OBJECTIVE FUNCTION IS EVALUATED IF IN=0
C
C     **********
      INTEGER J
      DOUBLE PRECISION A(10,15),B(10),C(15)
      DATA A/1.D2,9.D1,7.D1,2*5.D1,4.D1,3.D1,2.D1,1.D1,5.D0,2*1.D2,5.D1,
     1       0.D0,1.D1,0.D0,6.D1,3.D1,7.D1,3*1.D1,2*0.D0,7.D1,5.D1,
     2       3.D1,4.D1,1.D1,1.D2,5.D0,3.5D1,5.5D1,6.5D1,6.D1,9.5D1,9.D1,
     3       2.5D1,3.5D1,5.D0,1.D1,2.D1,2.5D1,3.5D1,4.5D1,5.D1,0.D0,
     4       4.D1,2.5D1,2.D1,0.D0,5.D0,2*1.D2,4.5D1,3.5D1,3.D1,2.5D1,
     5       6.5D1,5.D0,2*0.D0,4.D1,3.5D1,0.D0,1.D1,5.D0,1.5D1,0.D0,
     6       1.D1,2.5D1,3.5D1,5.D1,60.D0,35.D0,60.D0,25.D0,10.D0,
     7       30.D0,35.D0,0.D0,55.D0,2*0.D0,65.D0,2*0.D0,80.D0,0.D0,
     8       95.D0,10.D0,25.D0,30.D0,15.D0,5.D0,45.D0,70.D0,20.D0,0.D0,
     9       70.D0,55.D0,20.D0,60.D0,0.D0,75.D0,15.D0,20.D0,30.D0,
     A       25.D0,20.D0,5.D0,0.D0,10.D0,75.D0,100.D0,20.D0,25.D0,
     1       30.D0,0.D0,10.D0,45.D0,40.D0,30.D0,35.D0,75.D0,0.D0,70.D0,
     2       5.D0,15.D0,35.D0,20.D0,25.D0,0.D0,30.D0,10.D0,5.D0,15.D0,
     3       65.D0,50.D0,10.D0,0.D0,10.D0,40.D0,65.D0,0.D0,5.D0,15.D0,
     4       20.D0,55.D0,30.D0/
      DATA B/3.85D2,4.7D2,5.6D2,5.65D2,6.45D2,4.3D2,4.85D2,4.55D2,3.9D2,
     1       4.6D2/
C     **********
C
C     C IS AN ARRAY OF THE NEGATIVE OF THE COEFFICIENTS IN THE OBJECTIVE
C       FUNCTION
C
C     **********
      DATA C/486.D0,640.D0,758.D0,776.D0,477.D0,707.D0,175.D0,619.D0,
     1       627.D0,614.D0,475.D0,377.D0,524.D0,468.D0,529.D0/
      IF(IN.NE.0) GO TO 20
      VAL=0.D0
      DO 10 J=1,15
```

```
         VAL=VAL-C(J)*X(J)
   10    CONTINUE
      RETURN
   20 VAL=0.D0
      DO 30 J = 1,15
         VAL=VAL+A(IN,J)*X(J)*X(J)
   30    CONTINUE
      VAL=VAL-B(IN)
      RETURN
C
C     LAST CARD OF SUBROUTINE FVAL15
C
      END
      SUBROUTINE GVAL15(N,X,G,IN)
      INTEGER N,IN
      DOUBLE PRECISION X(N),G(N)
C     **********
C
C     SUBROUTINE GVAL15(N,X,G,IN)
C
C     THIS SUBROUTINE RETURNS FUNCTION AND CONSTRAINT GRADIENTS FOR
C     TEST PROBLEM 15 OF ARGONNE NATIONAL LABORATORY APPLIED MATHEMATICS
C     DIVISION TECHNICAL MEMORANDUM NO. 320 BY L. W. CORNWELL,
C     P. A. HUTCHISON, M. MINKOFF, AND H. K. SCHULTZ.
C
C     THE SUBROUTINE STATEMENT IS
C
C     SUBROUTINE GVAL15(N,X,G,IN)
C
C     WHERE
C
C     N IS THE DIMENSION OF THE PROBLEM
C
C     X IS THE POINT EVALUATED
C
C     G IS THE GRADIENT RETURNED ON OUTPUT
C
C     IN IS THE NUMBER OF THE CONSTRAINT TO BE EVALUATED.  THE
C       OBJECTIVE GRADIENT IS EVALUATED IF IN=0
C
C     **********
      INTEGER J
      DOUBLE PRECISION A(10,15),C(15)
      DATA A/1.D2,9.D1,7.D1,2*5.D1,4.D1,3.D1,2.D1,1.D1,5.D0,2*1.D2,5.D1,
     1       0.D0,1.D1,0.D0,6.D1,3.D1,7.D1,3*1.D1,2*0.D0,7.D1,5.D1,
     2       3.D1,4.D1,1.D1,1.D2,5.D0,3.5D1,5.5D1,6.5D1,6.D1,9.5D1,9.D1,
     3       2.5D1,3.5D1,5.D0,1.D1,2.D1,2.5D1,3.5D1,4.5D1,5.D1,0.D0,
     4       4.D1,2.5D1,2.D1,0.D0,5.D0,2*1.D2,4.5D1,3.5D1,3.D1,2.5D1,
     5       6.5D1,5.D0,2*0.D0,4.D1,3.5D1,0.D0,1.D1,5.D0,1.5D1,0.D0,
     6       1.D1,2.5D1,3.5D1,5.D1,60.D0,35.D0,60.D0,25.D0,10.D0,
     7       30.D0,35.D0,0.D0,55.D0,2*0.D0,65.D0,2*0.D0,80.D0,0.D0,
     8       95.D0,10.D0,25.D0,30.D0,15.D0,5.D0,45.D0,70.D0,20.D0,0.D0,
     9       70.D0,55.D0,20.D0,60.D0,0.D0,75.D0,15.D0,20.D0,30.D0,
     A       25.D0,20.D0,5.D0,0.D0,10.D0,75.D0,100.D0,20.D0,25.D0,
     1       30.D0,0.D0,10.D0,45.D0,40.D0,30.D0,35.D0,75.D0,0.D0,70.D0,
     2       5.D0,15.D0,35.D0,20.D0,25.D0,0.D0,30.D0,10.D0,5.D0,15.D0,
     3       65.D0,50.D0,10.D0,0.D0,10.D0,40.D0,65.D0,0.D0,5.D0,15.D0,
     4       20.D0,55.D0,30.D0/
C     **********
```

```
C
C     C IS AN ARRAY OF THE NEGATIVE OF THE COEFFICIENTS IN THE OBJECTIVE
C       FUNCTION
C
C     **********
      DATA C/486.D0,640.D0,758.D0,776.D0,477.D0,707.D0,175.D0,619.D0,
     1       627.D0,614.D0,475.D0,377.D0,524.D0,468.D0,529.D0/
      IF(IN.NE.0) GO TO 20
      DO 10 J=1,15
         G(J)=-C(J)
   10    CONTINUE
      RETURN
   20 DO 30 J=1,15
         G(J)=2.D0*A(IN,J)*X(J)
   30    CONTINUE
      RETURN
C
C     LAST CARD OF SUBROUTINE GVAL15
C
      END
```

```
      SUBROUTINE FVAL91(N,X,VAL,IN)
      INTEGER N,IN
      DOUBLE PRECISION VAL
      DOUBLE PRECISION X(N)
C     **********
C
C     SUBROUTINE FVAL91(N,X,VAL,IN)
C
C     THIS SUBROUTINE RETURNS FUNCTION AND CONSTRAINT VALUES FOR
C     TEST PROBLEM 91 OF ARGONNE NATIONAL LABORATORY APPLIED MATHEMATICS
C     DIVISION TECHNICAL MEMORANDUM NO. 320 BY L. W. CORNWELL,
C     P. A. HUTCHISON, M. MINKOFF, AND H. K. SCHULTZ.
C
C     THE SUBROUTINE STATEMENT IS
C
C     SUBROUTINE FVAL91(N,X,VAL,IN)
C
C     WHERE
C
C     N IS THE DIMENSION OF THE PROBLEM
C
C     X IS THE POINT EVALUATED
C
C     VAL IS THE VALUE RETURNED ON OUTPUT
C
C     IN IS THE NUMBER OF THE CONSTRAINT TO BE EVALUATED.  THE
C       OBJECTIVE FUNCTION IS EVALUATED IF IN=0
C
C     **********
      DOUBLE PRECISION XINM1
      IF(IN.NE.0) GO TO 10
      VAL=X(1)*X(2)*X(3)*X(4)-3.D0*X(1)*X(2)*X(4)  - 4.D0*X(1)*X(2)*X(3)
     1    +12.D0*X(1)*X(2)-X(2)*X(3)*X(4)+3.D0*X(2)*X(4)+4.D0*X(2)*X(3)
     2    -12.D0*X(2)-2.D0*X(1)*X(3)*X(4)+6.D0*X(1)*X(4)+8.D0*X(1)*X(3)
     3    -24.D0*X(1)+2.D0*X(3)*X(4)-6.D0*X(4)-8.D0*X(3)
     4    +24.D0+1.5D0*X(5)**4-5.75D0*X(5)**3+5.25D0*X(5)*X(5)
      RETURN
   10 IF(IN.NE.1) GO TO 20
      VAL=X(1)*X(1)+X(2)*X(2)+X(3)*X(3)+X(4)*X(4)+X(5)*X(5)-34.D0
      RETURN
   20 XINM1=IN-1
      VAL = XINM1 - X(IN-1)
      RETURN
C
C     LAST CARD OF SUBROUTINE FVAL91
C
      END
      SUBROUTINE GVAL91(N,X,G,IN)
      INTEGER N,IN
      DOUBLE PRECISION X(N),G(N)
C     **********
C
C     SUBROUTINE GVAL91(N,X,G,IN)
C
C     THIS SUBROUTINE RETURNS FUNCTION AND CONSTRAINT GRADIENTS FOR
C     TEST PROBLEM 91 OF ARGONNE NATIONAL LABORATORY APPLIED MATHEMATICS
C     DIVISION TECHNICAL MEMORANDUM NO. 320 BY L. W. CORNWELL,
C     P. A. HUTCHISON, M. MINKOFF, AND H. K. SCHULTZ.
C
```

```
C     THE SUBROUTINE STATEMENT IS
C
C     SUBROUTINE GVAL91(N,X,G,IN)
C
C     WHERE
C
C     N IS THE DIMENSION OF THE PROBLEM
C
C     X IS THE POINT EVALUATED
C
C     G IS THE GRADIENT RETURNED ON OUTPUT
C
C     IN IS THE NUMBER OF THE CONSTRAINT TO BE EVALUATED.  THE
C       OBJECTIVE GRADIENT IS EVALUATED IF IN=0
C
C     **********
      INTEGER I
      IF(IN.NE.0) GO TO 10
      G(1)=X(2)*X(3)*X(4)-3.D0*X(2)*X(4)+12.D0*X(2)-4.D0*X(2)*X(3)
     1     -2.D0*X(3)*X(4)+6.D0*X(4)+8.D0*X(3)-24.D0
      G(2)=X(1)*X(3)*X(4)-3.D0*X(1)*X(4)-4.D0*X(1)*X(3)+12.D0*X(1)
     1     -X(3)*X(4)+3.D0*X(4)+4.D0*X(3)-12.D0
      G(3)=X(1)*X(2)*X(4)-4.D0*X(1)*X(2)-X(2)*X(4)+4.D0*X(2)
     1     -2.D0*X(1)*X(4)+8.D0*X(1)+2.D0*X(4)-8.D0
      G(4)=X(1)*X(2)*X(3)-3.D0*X(1)*X(2)-X(2)*X(3)+3.D0*X(2)
     1     -2.D0*X(1)*X(3)+6.D0*X(1)+2.D0*X(3)-6.D0
      G(5)=6.D0*X(5)**3-17.25D0*X(5)*X(5)+10.5D0*X(5)
      RETURN
   10 IF(IN.NE.1) GO TO 30
      DO 20 I = 1,5
         G(I)=2.D0*X(I)
   20    CONTINUE
      RETURN
   30 DO 40 I = 1,5
         G(I)=0.D0
   40    CONTINUE
      G(IN-1)=-1.D0
      RETURN
C
C     LAST CARD OF SUBROUTINE GVAL91
C
      END
```

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