Applied Nonlinear Programming

David M. Himmelblau
The University of Texas, Austin, Texas

xi, 498p. : ill. ; 24 cm.

c1972

McGRAW-HILL BOOK COMPANY

New York St. Louis San Francisco Düsseldorf Johannesburg Kuala Lumpur
London Mexico Montreal New Delhi Panama
Rio de Janeiro Singapore Sydney Toronto

Applied Nonlinear Programming

Library of Congress Catalog Card Number 76-148127
07-028921-2

567890DODO798

This book was set in Times New Roman, and was printed and bound by R. R. Donnelley & Sons Company. The designer was Paula Tuerk. The editors were B. J. Clark and Madelaine Eichberg. John A. Sabella supervised production.

appendix a

Nonlinear programming problems and their solutions

Problem 1

Source: J. Bracken and G. P. McCormick, "Selected Applications of Nonlinear Programming," John Wiley & Sons, Inc., New York, 1968.

No. of variables: 2

No. of constraints: 1 linear equality constraint
1 nonlinear inequality constraint

Objective function:

Minimize: $f(\mathbf{x}) = (x_1 - 2)^2 + (x_2 - 1)^2$

Constraints:

$$h_1(\mathbf{x}) = x_1 - 2x_2 + 1 = 0$$

$$g_1(\mathbf{x}) = -\frac{x_1^2}{4} - x_2^2 + 1 \geq 0$$

Nonfeasible starting point:

$$\mathbf{x}^{(0)} = [2 \quad 2]^T$$

$$f(\mathbf{x}^{(0)}) = 1$$

Results:

$$f(\mathbf{x}^*) = 1.393$$
$$x_1^* = 0.823$$
$$x_2^* = 0.911$$

Problem 2

Source: H. H. Rosenbrock, An Automatic Method for Finding the Greatest and Least Value of a Function, *Computer J.*, **3**:175 (1960).

No. of variables: 2

No. of constraints: 0

Objective function:

$$\text{Minimize:}\quad f(\mathbf{x}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

Starting point:

$$\mathbf{x}^{(0)} = [-1.2 \quad 1]^T$$
$$f(\mathbf{x}^{(0)}) = 24.20$$

Results:

$$\mathbf{x}^* = [1 \quad 1]^T$$
$$f(\mathbf{x}^*) = 0$$

Problem 3

Source: G. K. Barnes, M.S. thesis, The University of Texas, Austin, Tex., 1967.

No. of variables: 2

No. of constraints: 3 nonlinear inequality constraints
4 bounds on independent variables

Objective function:

$$\begin{aligned}\text{Maximize:}\quad f(\mathbf{x}) = {} & 75.196 - 3.8112x_1 + 0.12694x_1^2 - 2.0567 \\ & \times 10^{-3}x_1^3 + 1.0345 \times 10^{-5}x_1^4 - 6.8306x_2 \\ & + 0.030234x_1x_2 - 1.28134 \times 10^{-3}x_2x_1^2 + 3.5256 \\ & \times 10^{-5}x_2x_1^3 - 2.266 \times 10^{-7}x_2x_1^4 + 0.25645x_2^2 \\ & - 3.4604 \times 10^{-3}x_2^3 + 1.3514 \times 10^{-5}x_2^4 \\ & - \frac{28.106}{x_2 + 1} - 5.2375 \times 10^{-6}x_1^2x_2^2 - 6.3 \\ & \times 10^{-8}x_1^3x_2^2 + 7 \times 10^{-10}x_1^3x_2^3 + 3.4054 \\ & \times 10^{-4}x_1x_2^2 - 1.6638 \times 10^{-6}x_1x_2^3 \\ & - 2.8673 \exp(0.0005x_1x_2)\end{aligned}$$

Constraints:

$$0 \le x_1 \le 75$$

$$0 \le x_2 \le 65$$

$$x_1 x_2 - 700 \ge 0$$

$$x_2 - 5\left(\frac{x_1}{25}\right)^2 \ge 0$$

$$(x_2 - 50)^2 - 5(x_1 - 55) \ge 0$$

Nonfeasible starting point:

$$\mathbf{x}^{(0)} = [90 \quad 10]^T$$

$$f(\mathbf{x}^{(0)}) = -82.828$$

Results:

$$\mathbf{x}^* = [75 \quad 65]^T$$

$$f(\mathbf{x}^*) = 58.903$$

Problem 4

Source: J. Bracken and G. P. McCormick, "Selected Applications of Nonlinear Programming," John Wiley & Sons, Inc., New York, 1968.

No. of variables: 10

No. of constraints: 3 linear equality constraints
10 bounds on independent variables

Problem 4 is a problem in the chemical equilibrium at constant temperature nd pressure.

Objective function:

$$\text{Minimize:} \quad f(\mathbf{x}) = \sum_{i=1}^{10} x_i \left(c_i + \ln \frac{x_i}{\sum_{j=1}^{10} x_j} \right)$$

here $c_1 = -6.089 \quad c_2 = -17.164 \quad c_3 = -34.054 \quad c_4 = -5.914$

$c_5 = -24.721 \quad c_6 = -14.986 \quad c_7 = -24.100 \quad c_8 = -10.708$

$c_9 = -26.662 \quad c_{10} = -22.179$

onstraints:

$$h_1(\mathbf{x}) = x_1 + 2x_2 + 2x_3 + x_6 + x_{10} - 2 = 0$$

$$h_2(\mathbf{x}) = x_4 + 2x_5 + x_6 + x_7 - 1 = 0$$

$$h_3(\mathbf{x}) = x_3 + x_7 + x_8 + 2x_9 + x_{10} - 1 = 0$$

$$x_i \ge 0 \qquad i = 1, \ldots, 10$$

Nonfeasible starting point:

$$\mathbf{x}_i^{(0)} = 0.1 \qquad i = 1, \ldots, 10$$
$$f(\mathbf{x}^{(0)}) = -20.961$$

Results: See Problem 4*a*.

Problem 4*a*

No. of independent variables: 10
No. of constraints: 3 nonlinear equality constraints
Objective function:

$$\text{Minimize:} \quad f(\mathbf{x}') = \sum_{i=1}^{10} \left\{ e^{x_i'} \left[c_i + x_i' - \ln\left(\sum_{i=1}^{10} e^{x_i'} \right) \right] \right\}$$

Constraints:

$$h_1(\mathbf{x}') = e^{x_1'} + 2e^{x_2'} + 2e^{x_3'} + e^{x_6'} + e^{x_{10}'} - 2 = 0$$
$$h_2(\mathbf{x}') = e^{x_4'} + 2e^{x_5'} + e^{x_6'} + e^{x_7'} - 1 = 0$$
$$h_3(\mathbf{x}') = e^{x_3'} + e^{x_7'} + e^{x_8'} + 2e^{x_9'} + e^{x_{10}'} - 1 = 0$$

Nonfeasible starting point:

$$x_i' = -2.3 \qquad i = 1, \ldots, 10$$

Results:

	NLP	*Flexible tolerance*	*GGS*	*GRG*	*SUMT*
$f(\mathbf{x})$	−47.751	−47.736	−47.656	−47.761	−47.761
x_1	0.0350	0.0128	0	0.0406	0.0407
x_2	0.1142	0.1433	0.1695	0.1477	0.1477
x_3	0.8306	0.8078	0.7536	0.7832	0.7832
x_4	0.0012	0.0062	0	0.0014	0.0014
x_5	0.4887	0.4790	0.5000	0.4853	0.4853
x_6	0.0005	0.0033	0	0.0007	0.0007
x_7	0.0209	0.0324	0	0.0274	0.0274
x_8	0.0157	0.0281	0	0.0180	0.0180
x_9	0.0289	0.0250	0.0464	0.0375	0.0373
x_{10}	0.0751	0.0817	0.1536	0.0969	0.0969
$h_1(\mathbf{x})$	3.E−12	3.E−05	0	1.E−06	−8.E−08
$h_2(\mathbf{x})$	3.E−12	2.E−05	0	1.E−06	−1.E−07
$h_3(\mathbf{x})$	2.E−11	9.E−05	0	1.E−06	−1.E−07

Problem 5

Source: D. A. Paviani, Ph.D. dissertation, The University of Texas, Austin, Tex., 1969.

No. of variables: 3

No. of constraints: 1 nonlinear equality constraint
1 linear equality constraint
3 bounds on independent variables

Objective function:

Minimize: $f(\mathbf{x}) = 1000 - x_1^2 - 2x_2^2 - x_3^2 - x_1x_2 - x_1x_3$

Constraints:

$$h_1(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 - 25 = 0$$
$$h_2(\mathbf{x}) = 8x_1 + 14x_2 + 7x_3 - 56 = 0$$
$$x_i \geq 0 \qquad i = 1,2,3$$

Nonfeasible starting point:

$$x_i^{(0)} = 2;\ \text{also } x_i^0 = 10 \qquad i = 1,2,3$$
$$f(\mathbf{x}^{(0)}) = 976$$

Results:

$f(\mathbf{x}^*)$	961.715
x_1	3.512
x_2	0.217
x_3	3.552
$h_1(\mathbf{x}^*)$	0
$h_2(\mathbf{x}^*)$	0

Problem 6

Source: A. P. Jones, "The Chemical Equilibrium Problem: An Application of SUMT," Research Analysis Corporation, McLean, Va., RAC-TP-272, 1967.

No. of variables: 45

No. of constraints: 16 linear equality constraints

Objective function:

$$\text{Minimize:} \quad f(\mathbf{x}) = \sum_{k=1}^{7} \left[\sum_{j=1}^{n_k} x_{jk} \left(c_{jk} + \ln \frac{x_{jk}}{\sum_{j=1}^{n_k} x_{jk}} \right) \right]$$

Note: See accompanying tables for n_k and c_{jk}.

Constraints:

$$h_i(\mathbf{x}) = \sum_{k=1}^{7} \left(\sum_{j=1}^{n_k} E_{ijk} x_{jk} \right) - b_i = 0 \qquad i = 1, \ldots, 16$$

$$x_{jk} \geq 0 \qquad [(j = 1, \ldots, n_k), k = 1, \ldots, 7]$$

Nonfeasible starting point:

$$x_{jk} = 0.1 \qquad [(j = 1, \ldots, n_k), k = 1, \ldots, 7]$$

$$f(\mathbf{x}^{(0)}) = -30.958$$

Results:

	NLP	*SUMT*	*Jones (SUMT)*
$f(\mathbf{x}^*)$	-1909.740	-1910.361	-79.108
x^*_{11}	7.854E − 07	6.599E − 06	6.440E − 01
x^*_{21}	8.078E − 02	2.512E − 01	2.590E − 01
x^*_{31}	3.706E − 00	3.705E − 00	3.705E − 00
x^*_{41}	8.855E − 02	2.535E − 01	2.997E − 01
x^*_{12}	6.894E − 01	6.529E − 01	5.617E − 05
x^*_{22}	3.020E − 02	1.235E − 03	6.880E − 04
x^*_{32}	1.398E − 04	3.667E − 04	2.062E − 04
x^*_{42}	1.626E − 04	2.794E − 06	1.101E − 06
x^*_{52}	0	5.441E − 06	2.433E − 06
x^*_{62}	2.782E − 02	7.363E − 02	5.715E − 02
x^*_{72}	7.950E − 02	8.791E − 02	7.938E − 02
x^*_{82}	3.421E − 02	3.542E − 02	3.231E − 03
x^*_{92}	2.486E + 01	4.458E + 01	2.839E − 01
$x^*_{10,2}$	3.873E − 02	2.669E − 02	1.388E − 02
$x^*_{11,2}$	1.500E − 04	7.709E − 06	3.283E − 06
$x^*_{12,2}$	1.170E − 05	3.764E − 05	1.738E − 05
$x^*_{13,2}$	1.550E − 02	1.550E − 02	1.155E − 02
x^*_{13}	0	9.900E − 07	5.956E − 05

	NLP	SUMT	Jones (SUMT)
x^*_{23}	2.649E − 02	5.077E − 05	4.419E − 04
x^*_{33}	1.251E − 04	3.107E − 05	2.205E − 04
x^*_{43}	1.064E − 01	1.546E − 06	1.095E − 06
x^*_{53}	0	3.102E − 06	1.852E − 06
x^*_{63}	5.253E − 02	6.416E − 03	2.291E − 02
x^*_{73}	8.710E − 03	2.202E − 04	8.751E − 03
x^*_{83}	1.471E − 02	1.287E − 02	4.506E − 02
x^*_{93}	4.735E − 02	2.165E − 00	1.832E − 01
$x^*_{10,3}$	9.208E − 02	2.675E − 00	6.396E − 03
$x^*_{11,3}$	3.119E − 04	3.437E − 06	2.855E − 06
$x^*_{12,3}$	1.560E − 02	1.400E − 05	7.806E − 06
$x^*_{13,3}$	2.421E − 02	1.927E − 02	2.113E − 02
$x^*_{14,3}$	2.448E − 03	1.855E − 03	7.429E − 06
$x^*_{15,3}$	8.398E − 03	3.264E − 06	3.017E − 05
$x^*_{16,3}$	5.285E − 03	7.579E − 07	5.056E − 05
$x^*_{17,3}$	0	3.510E − 07	4.871E − 05
$x^*_{18,3}$	1.601E − 03	2.513E − 07	2.142E − 03
x^*_{14}	4.968E − 07	0	2.337E − 06
x^*_{24}	1.978E − 02	4.200E − 07	1.821E − 04
x^*_{34}	6.271E − 03	7.063E − 06	8.583E − 05
x^*_{15}	5.328E − 02	0	2.355E − 05
x^*_{25}	0	0	1.251E − 03
x^*_{35}	0	1.305E − 06	7.573E − 03
x^*_{16}	2.510E − 02	1.465E − 05	3.038E − 04
x^*_{26}	1.220E − 06	1.382E − 05	3.902E − 05
x^*_{17}	0	2.872E − 06	2.879E − 02
x^*_{27}	0	2.476E − 06	1.499E − 03
$h_1(\mathbf{x}^*)$	5.118E − 02	2.529E − 07	−4.800E − 07
$h_2(\mathbf{x}^*)$	2.407E − 03	2.263E − 07	1.592E − 06
$h_3(\mathbf{x}^*)$	2.559E − 05	1.917E − 07	2.631E − 06
$h_4(\mathbf{x}^*)$	4.493E − 02	1.112E − 06	−4.624E + 01
$h_5(\mathbf{x}^*)$	2.389E − 02	−4.518E − 07	−4.624E + 01
$h_6(\mathbf{x}^*)$	3.100E − 04	−3.946E − 07	1.340E − 06
$h_7(\mathbf{x}^*)$	6.692E − 06	6.771E − 07	1.362E − 06
$h_8(\mathbf{x}^*)$	6.376E − 04	5.101E − 07	1.362E − 06
$h_9(\mathbf{x}^*)$	0	−1.869E − 07	−3.948E − 03
$h_{10}(\mathbf{x}^*)$	2.082E − 02	−1.950E − 07	2.280E − 03
$h_{11}(\mathbf{x}^*)$	−2.273E − 03	−2.273E − 03	−2.273E − 03
$h_{12}(\mathbf{x}^*)$	−1.380E − 02	−5.583E − 07	−2.699E − 04
$h_{13}(\mathbf{x}^*)$	6.946E − 04	6.001E − 07	−8.575E − 03
$h_{14}(\mathbf{x}^*)$	5.331E − 02	1.169E − 06	8.847E − 03
$h_{15}(\mathbf{x}^*)$	7.789E − 06	2.180E − 07	−5.529E − 07
$h_{16}(\mathbf{x}^*)$	3.528E − 09	1.284E − 07	−3.330E − 07

b_i's and c_{jk}'s for Problem 6

i	b_i	j	k	c_{jk}	j	k	c_{jk}
1	0.6529581	1	1	0.0	6	3	0.0
2	0.281941	2	1	−7.69	7	3	2.2435
3	3.705233	3	1	−11.52	8	3	0.0
4	47.00022	4	1	−36.60	9	3	−39.39
5	47.02972	1	2	−10.94	10	3	−21.49
6	0.08005	2	2	0.0	11	3	−32.84
7	0.08813	3	2	0.0	12	3	6.12
8	0.04829	4	2	0.0	13	3	0.0
9	0.0155	5	2	0.0	14	3	0.0
10	0.0211275	6	2	0.0	15	3	−1.9028
11	0.0022725	7	2	0.0	16	3	−2.8889
12	0.0	8	2	2.5966	17	3	−3.3622
13	0.0	9	2	−39.39	18	3	−7.4854
14	0.0	10	2	−21.35	1	4	−15.639
15	0.0	11	2	−32.84	2	4	0.0
16	0.0	12	2	6.26	3	4	21.81
		13	2	0.0	1	5	−16.79
		1	3	10.45	2	5	0.0
		2	3	0.0	3	5	18.9779
		3	3	−0.50	1	6	0.0
		4	3	0.0	2	6	11.959
		5	3	0.0	1	7	0.0
					2	7	12.899

E_{ijk} Data for Problem 6

x_{jk} \ i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
x_{11}	1															
x_{21}		1														
x_{31}			1													
x_{41}				1	1											
x_{12}	1															
x_{22}		1														
x_{32}			1													
x_{42}				1								1				
x_{52}					1							−1				
x_{62}						1						−1				
x_{72}							1					1				
x_{82}								1				1				
x_{92}				1	1											
$x_{10.2}$		1			1							−1				

x_{jk} \ i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$x_{11.2}$		1		1	1											
$x_{12.2}$		1		−1	1							−2				
$x_{13.3}$									1			−1				
x_{13}	1															
x_{23}		1														
x_{33}			1													
x_{43}				1												
x_{53}					1											
x_{63}						1										
x_{73}							1									
x_{83}								1								
x_{93}				1	1											
$x_{10.3}$		1			1											
$x_{11.3}$		1		1	1											
$x_{12.3}$		1		−1	1											
$x_{13.3}$										1						
$x_{14.3}$											1	−4				
$x_{15.3}$	1										1	−3	−1			
$x_{16.3}$	2										1	−2	−2			
$x_{17.3}$	3										1	−1	−3			
$x_{18.3}$	4										1		−4			
x_{14}				1									1			
x_{24}													1			
x_{34}				−1									1		−4	
x_{15}				1										1		
x_{25}														1		
x_{35}				−1										1		−4
x_{16}															1	
x_{26}		1		−1											1	
x_{17}																1
x_{27}		1		−1												1

Problem 7

Source: A. R. Colville, A Comparative Study on Nonlinear Programming Codes, *IBM N.Y. Sci. Center Rept.* 320–2949, June, 1968, p. 31.

No. of variables: 3

No. of constraints: 14 nonlinear inequality constraints
6 bounds on independent variables

Problem 7 was typical of problems in which functions are described by a self-contained computer subroutine.

Objective function:

$$\text{Maximize:} \quad f(\mathbf{x}) = 0.063y_2y_5 - 5.04x_1 - 3.36y_3 - 0.035x_2 - 10x_3$$

Constraints:

$$\begin{aligned}
0 &\le x_1 \le 2000 \\
0 &\le x_2 \le 16{,}000 \\
0 &\le x_3 \le 120 \\
0 &\le y_2 \le 5000 \\
0 &\le y_3 \le 2000 \\
85 &\le y_4 \le 93 \\
90 &\le y_5 < 95 \\
3 &\le y_6 \le 12 \\
0.01 &\le y_7 \le 4 \\
145 &\le y_8 \le 162
\end{aligned}$$

Fortran description of the calculation of y_2 to y_8.

```
      Y(2) = 1.6*X(1)
 10   Y(3) = 1.22*Y(2) - X(1)
      Y(6) = (X(2) + Y(3))/X(1)
      Y2CALC = X(1)*(112. + 13.167*Y(6) - 0.6667*Y(6)**2)/100.
      IF(ABS(Y2CALC - Y(2)) - 0.001) 30,30,20
 20   Y(2) = Y2CALC
      GO TO 10
 30   CONTINUE
      Y(4) = 93.
100   Y(5) = 86.35 + 1.098*Y(6) - 0.038*Y(6)**2 + 0.325*(Y(4) - 89.)
      Y(8) = -133. + 3.*Y(5)
      Y(7) = 35.82 - 0.222*Y(8)
      Y4CALC = 98000.*X(3)/(Y(2)*Y(7) + X(3)*1000.)
      IF(ABS(Y4CALC - Y(4)) - 0.0001) 300,300,200
200   Y(4) = Y4CALC
      GO TO 100
300   CONTINUE
```

Feasible starting point:

$$\mathbf{x}^{(0)} = [1745 \quad 12000 \quad 110]^T$$

$$f(\mathbf{x}^{(0)}) = 868.6458$$

Results:

$$\mathbf{x}^* = [1728.37 \quad 16000 \quad 98.13]^T$$

$$f(\mathbf{x}^*) = 1162.036$$

Problem 8

Source: C. F. Wood, Westinghouse Research Laboratory (cited in Colville, *IBM N.Y. Sci. Center Rept.* 320–2949, June, 1968).
No. of variables: 4
No. of constraints: 8 bounds on the independent variables
Problem 8 was designed to have a nonoptimal stationary point at $f(\mathbf{x}) \approx 8$ that can cause premature convergence.
Objective function:

$$\text{Minimize:} \quad f(\mathbf{x}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 + 90(x_4 - x_3^2)^2 + (1 - x_3)^2 + 10.1[(x_2 - 1)^2 + (x_4 - 1)^2] + 19.8(x_2 - 1)(x_4 - 1)$$

Constraints:

$$-10 \le x_i \le 10 \qquad i = 1,2,3,4$$

Feasible starting point:

$$\mathbf{x}^{(0)} = [-3 \quad -1 \quad -3 \quad -1]^T$$
$$f(\mathbf{x}^{(0)}) = 19{,}192$$

Results:

$$\mathbf{x}^* = [1 \quad 1 \quad 1 \quad 1]^T$$
$$f(\mathbf{x}^*) = 0$$

Problem 9

Source: D. M. Himmelblau and R. V. Yates, A New Method of Flow Routing, *Water Resources Res.*, **4**:1193 (1968).
No. of variables: 4
No. of constraints: 1 nonlinear inequality constraint
Objective function:

$$\text{Minimize:} \quad f(\mathbf{x}) = \sum_{i=1}^{19} (y_{i,\text{cal}} - y_{i,\text{obs}})^2$$

$$y_{i,\text{cal}} = \frac{x_3 \beta^{x_2} \left(\dfrac{x_2}{6.2832}\right)^{\frac{1}{2}} \left(\dfrac{c_i}{7.658}\right)^{(x_2 - 1)} \exp\left(x_2 - \beta \dfrac{c_i x_2}{7.658}\right)}{1 + \dfrac{1}{12 x_2}}$$

$$+\frac{(1-x_3)\left(\dfrac{\beta}{x_4}\right)^{x_1}\left(\dfrac{x_1}{6.2832}\right)^{\frac{1}{2}}\left(\dfrac{c_i}{7.658}\right)^{x_1-1}\exp\left(x_1-\beta\dfrac{c_i x_1}{7.658x_4}\right)}{1+\dfrac{1}{12x_1}}$$

where $\beta = x_3 + (1 - x_3)x_4$. (*Note:* The c_i and $y_{i,\text{obs}}$ are given in the accompanying table.)

Constraints:

$$x_3 + (1 - x_3)x_4 \geq 0$$
$$x_4 \geq 0 \qquad i = 1, \ldots, 4$$
$$x_3 \leq 1$$

Starting point:

$$\mathbf{x}^{(0)} = [2 \quad 4 \quad 0.04 \quad 2]^T$$
$$f(\mathbf{x}^{(0)}) = 4.8024$$

Results:

$$\mathbf{x}^* = [12.277 \quad 4.632 \quad 0.313 \quad 2.029]^T$$
$$f(\mathbf{x}^*) = 0.0075$$

c_i and $y_{i,\text{obs}}$ for Test Problem 9

i	c	$y_{i,\text{obs}}$	i	c	$y_{i,\text{obs}}$
1	0.1	0.00189	11	10	0.702
2	1	0.1038	12	11	0.528
3	2	0.268	13	12	0.385
4	3	0.506	14	13	0.257
5	4	0.577	15	14	0.159
6	5	0.604	16	15	0.0869
7	6	0.725	17	16	0.0453
8	7	0.898	18	17	0.01509
9	8	0.947	19	18	0.00189
10	9	0.845			

Problem 10

Source: Shell Development Co. (cited in Colville, *IBM N.Y. Sci. Center Rept.* 320–2949, June, 1968, p. 21).

No. of variables: 5

No. of constraints: 10 linear inequality constraints
5 bounds on independent variables

Objective function:

$$\text{Minimize:}\quad f(\mathbf{x}) = \sum_{j=1}^{5} e_j x_j + \sum_{i=1}^{5}\sum_{j=1}^{5} c_{ij} x_i x_j + \sum_{j=1}^{5} d_j x_j^3$$

Constraints:

$$\sum_{j=1}^{5} a_{ij} x_j - b_i \ge 0 \qquad i = 1, \ldots, 10$$

$$x_j \ge 0 \qquad j = 1, \ldots, 5$$

(*Note:* The e_j, c_{ij}, d_j, a_{ij}, and b_j are given in the accompanying table.)
Feasible starting point:

$$\mathbf{x}^{(0)} = [0 \quad 0 \quad 0 \quad 0 \quad 1]^T$$

$$f(\mathbf{x}^{(0)}) = 20$$

Results:

$$\mathbf{x}^* = [0.3000 \quad 0.3335 \quad 0.4000 \quad 0.4285 \quad 0.224]^T$$

$$f(\mathbf{x}^*) = -32.349$$

Data for Test Problems 10 and 18

j	1	2	3	4	5
e_j	−15	−27	−36	−18	−12
c_{1j}	30	−20	−10	32	−10
c_{2j}	−20	39	−6	−31	32
c_{3j}	−10	−6	10	−6	−10
c_{4j}	32	−31	−6	39	−20
c_{5j}	−10	32	−10	−20	30
d_j	4	8	10	6	2
a_{1j}	−16	2	0	1	0
a_{2j}	0	−2	0	.4	2
a_{3j}	−3.5	0	2	0	0
a_{4j}	0	−2	0	−4	−1
a_{5j}	0	−9	−2	1	−2.8
a_{6j}	2	0	−4	0	0
a_{7j}	−1	−1	−1	−1	−1
a_{8j}	−1	−2	−3	−2	−1
a_{9j}	1	2	3	4	5
a_{10j}	1	1	1	1	1

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
−40	−2	−.25	−4	−4	−1	−40	−60	5	1

Problem 11

Source: Proctor and Gamble Co. (cited in Colville, *IBM N.Y. Sci.Center Rept.* 320–2949, June, 1968, p. 24).
No. of variables: 5
No. of constraints: 6 nonlinear inequality constraints
10 bounds on independent variables
Note that x_2 and x_4 are not included in the definition of $f(\mathbf{x})$.
Objective function:

$$\text{Minimize:}\quad f(\mathbf{x}) = 5.3578547x_3^2 + 0.8356891x_1x_5 + 37.293239x_1 - 40792.141$$

Constraints:

$$0 \le 85.334407 + 0.0056858x_2x_5 + 0.0006262x_1x_4 - 0.0022053x_3x_5 \le 92$$

$$90 \le 80.51249 + 0.0071317x_2x_5 + 0.0029955x_1x_2 + 0.0021813x_3^2 \le 110$$

$$20 \le 9.300961 + 0.0047026x_3x_5 + 0.0012547x_1x_3 + 0.0019085x_3x_4 \le 25$$

$$78 \le x_1 \le 102$$

$$33 \le x_2 \le 45$$

$$27 \le x_3 \le 45$$

$$27 \le x_4 \le 45$$

$$27 \le x_5 \le 45$$

Feasible starting point:

$$\mathbf{x}^{(0)} = [78.62 \quad 33.44 \quad 31.07 \quad 44.18 \quad 35.22]^T$$

$$f(\mathbf{x}^{(0)}) = -30367$$

Results:

$$x^* = [78.000 \quad 33.000 \quad 29.995 \quad 45.000 \quad 36.776]^T$$

$$f(\mathbf{x}^*) = -30665.5$$

Nonfeasible starting point:

$$\mathbf{x}^{(0)} = [78 \quad 33 \quad 27 \quad 27 \quad 27]^T$$

$$f(\mathbf{x}^{(0)}) = -32217$$

Results:

$f(\mathbf{x}^*) = -30665.5$ $\quad$ x_i's as for feasible starting point

Problem 12

Source: G. K. Barnes, M.S. thesis, The University of Texas, Austin, Tex., 1967. Adapted from C. W. Carroll, Ph.D. dissertation, The Institute of Paper Chemistry, Appleton, Wis., 1959.

No. of variables: 5

No. of constraints: 4 linear inequality constraints
34 nonlinear inequality constraints (as listed—some can be eliminated)
10 bounds on independent variables

The objective function in Problem 12 was the net profit of a hypothetical wood-pulp plant. The constraints (or model) included the usual material and energy balances as well as several empirical equations.

Objective function:

$$\text{Maximize:}\quad f(\mathbf{x}) = 0.0000005843y_{17} - 0.000117y_{14} - 0.1365 - 0.00002358y_{13} - 0.000001502y_{16} - 0.0321y_{12} - 0.004324y_5 - 0.0001\frac{c_{15}}{c_{16}} - 37.48\frac{y_2}{c_{12}}$$

Calculation of y_i's and c_i's:

$$y_1 = x_2 + x_3 + 41.6$$

$$c_1 = 0.024x_4 - 4.62$$

$$y_2 = \frac{12.5}{c_1} + 12.0$$

$$c_2 = 0.0003535x_1^2 + 0.5311x_1 + 0.08705y_2x_1$$

$$c_3 = 0.052x_1 + 78 + 0.002377y_2x_1$$

$$y_3 = \frac{c_2}{c_3}$$

$$y_4 = 19y_3$$

$$c_4 = 0.04782(x_1 - y_3) + \frac{0.1956(x_1 - y_3)^2}{x_2} + 0.6376y_4 + 1.594y_3$$

$$c_5 = 100x_2$$

$$c_6 = x_1 - y_3 - y_4$$

$$c_7 = 0.950 - \frac{c_4}{c_5}$$

$$y_5 = c_6 c_7$$

$$y_6 = x_1 - y_5 - y_4 - y_3$$

$$c_8 = (y_5 + y_4)0.995$$

$$y_7 = \frac{c_8}{y_1}$$

$$y_8 = \frac{c_8}{3798}$$

$$c_9 = y_7 - \frac{0.0663 y_7}{y_8} - 0.3153$$

$$y_9 = \frac{96.82}{c_9} + 0.321 y_1$$

$$y_{10} = 1.29 y_5 + 1.258 y_4 + 2.29 y_3 + 1.71 y_6$$

$$y_{11} = 1.71 x_1 - 0.452 y_4 + 0.580 y_3$$

$$c_{10} = \frac{12.3}{752.3}$$

$$c_{11} = (1.75 y_2)(0.995 x_1)$$

$$c_{12} = 0.995 y_{10} + 1998$$

$$y_{12} = c_{10} x_1 + \frac{c_{11}}{c_{12}}$$

$$y_{13} = c_{12} - 1.75 y_2$$

$$y_{14} = 3623 + 64.4 x_2 + 58.4 x_3 + \frac{146{,}312}{y_9 + x_5}$$

$$c_{13} = 0.995 y_{10} + 60.8 x_2 + 48 x_4 - 0.1121 y_{14} - 5095$$

$$y_{15} = \frac{y_{13}}{c_{13}}$$

$$y_{16} = 148{,}000 - 331{,}000 y_{15} + 40 y_{13} - 61 y_{15} y_{13}$$

$$c_{14} = 2324 y_{10} - 28{,}740{,}000 y_2$$

$$y_{17} = 14{,}130{,}000 - 1328 y_{10} - 531 y_{11} + \frac{c_{14}}{c_{12}}$$

$$c_{15} = \frac{y_{13}}{y_{15}} - \frac{y_{13}}{0.52}$$

$$c_{16} = 1.104 - 0.72 y_{15}$$

$$c_{17} = y_9 + x_5$$

Constraints

$$y_4 - \frac{0.28}{0.72} y_5 \geq 0$$

$$1.5x_2 - x_3 \geq 0$$

$$21.0 - 3496 \frac{y_2}{c_{12}} \geq 0$$

$$\frac{62{,}212}{c_{17}} - 110.6 - y_1 \geq 0$$

$$213.1 \leq y_1 \leq 405.23$$

$$17.505 \leq y_2 \leq 1053.6667$$

$$11.275 \leq y_3 \leq 35.03$$

$$214.228 \leq y_4 \leq 665.585$$

$$7.458 \leq y_5 \leq 584.463$$

$$0.961 \leq y_6 \leq 265.916$$

$$1.612 \leq y_7 \leq 7.046$$

$$0.146 \leq y_8 \leq 0.222$$

$$107.99 \leq y_9 \leq 273.366$$

$$922.693 \leq y_{10} \leq 1286.105$$

$$926.832 \leq y_{11} \leq 1444.046$$

$$18.766 \leq y_{12} \leq 537.141$$

$$1072.163 \leq y_{13} \leq 3247.039$$

$$8961.448 \leq y_{14} \leq 26844.086$$

$$0.063 \leq y_{15} \leq 0.386$$

$$71{,}084.33 \leq y_{16} \leq 140{,}000$$

$$2{,}802{,}713 \leq y_{17} \leq 12{,}146{,}108$$

$$704.4148 \leq x_1 \leq 906.3855$$

$$68.6 \leq x_2 \leq 288.88$$

$$0 \leq x_3 \leq 134.75$$

$$193 \leq x_4 \leq 287.0966$$

$$25 \leq x_5 \leq 84.1988$$

Feasible starting point:

$$\mathbf{x}^{(0)} = [900 \quad 80 \quad 115 \quad 267 \quad 27]^T$$
$$f(\mathbf{x}^{(0)}) = 0.939$$

Results:

$$\mathbf{x}^* = [705.060 \quad 68.600 \quad 102.900 \quad 282.341 \quad 35.627]^T$$
$$f(\mathbf{x}^*) = 1.905$$

Problem 13

Source: M. J. Box, A New Method of Constrained Optimization and a Comparison with Other Methods, *Computer J.*, **8**:42 (1965).

This problem can be regarded as having (*a*) 12 variables, 7 equality constraints, and 16 upper and lower bounds, or more simply, (*b*) 5 variables, 3 nonlinear inequality constraints, and 10 upper and lower bounds.

Problem 13 is an example of determining parameters in highly nonlinear differential equations from experimental data. The objective function was the sum of squared residuals between experimental data and numerically integrated solutions of the differential equations.

Objective function:

$$\text{Maximize:} \quad f(\mathbf{x}) = [50y_1 + 9.583y_2 + 20y_3 + 15y_4 - 852{,}960 - 38{,}100(x_2 + 0.01x_3) + k_{31} + k_{32}x_2 + k_{33}x_3 + k_{34}x_4 + k_{35}x_5]x_1 - 24{,}345 + 15x_6$$

Calculation of x_6, the y_i's, and x_7, x_8:

$$x_6 = (k_1 + k_2x_2 + k_3x_3 + k_4x_4 + k_5x_5)x_1$$
$$y_1 = k_6 + k_1x_2 + k_8x_3 + k_9x_4 + k_{10}x_5$$
$$y_2 = k_{11} + k_{12}x_2 + k_{13}x_3 + k_{14}x_4 + k_{15}x_5$$
$$y_3 = k_{16} + k_{17}x_2 + k_{18}x_3 + k_{19}x_4 + k_{20}x_5$$
$$y_4 = k_{21} + k_{22}x_2 + k_{23}x_3 + k_{24}x_4 + k_{25}x_5$$
$$x_7 = (y_1 + y_2 + y_3)x_1$$
$$x_8 = (k_{26} + k_{27}x_2 + k_{28}x_3 + k_{29}x_4 + k_{30}x_5)x_1 + x_6 + x_7$$

where:

$$
\begin{aligned}
k_1 &= -145{,}421.402 & k_{19} &= 329.574 \\
k_2 &= 2{,}931.1506 & k_{20} &= -2{,}882.082 \\
k_3 &= -40.427932 & k_{21} &= 74{,}095.3845 \\
k_4 &= 5{,}106.192 & k_{22} &= -306.262544 \\
k_5 &= 15{,}711.36 & k_{23} &= 16.243649 \\
k_6 &= -161{,}622.577 & k_{24} &= -3{,}094.252 \\
k_7 &= 4{,}176.15328 & k_{25} &= -5{,}566.2628 \\
k_8 &= 2.8260078 & k_{26} &= -26{,}237 \\
k_9 &= 9{,}200.476 & k_{27} &= 99 \\
k_{10} &= 13{,}160.295 & k_{28} &= -0.42 \\
k_{11} &= -21{,}686.9194 & k_{29} &= 1{,}300 \\
k_{12} &= 123.56928 & k_{30} &= 2{,}100 \\
k_{13} &= -21.1188894 & k_{31} &= 925{,}548.252 \\
k_{14} &= 706.834 & k_{32} &= -61{,}968.8432 \\
k_{15} &= 2{,}898.573 & k_{33} &= 23.3088196 \\
k_{16} &= 28{,}298.388 & k_{34} &= -27{,}097.648 \\
k_{17} &= 60.81096 & k_{35} &= -50{,}843.766 \\
k_{18} &= 31.242116 & &
\end{aligned}
$$

Constraints:

$$
\begin{aligned}
0 &\le x_1 \le 5 \\
1.2 &\le x_2 \le 2.4 \\
20 &\le x_3 \le 60 \\
9 &\le x_4 \le 9.3 \\
6.5 &\le x_5 \le 7 \\
0 &\le x_6 \le 294{,}000 \\
0 &\le x_7 \le 294{,}000 \\
0 &\le x_8 \le 277{,}200
\end{aligned}
$$

Starting point:

$$
\begin{aligned}
\mathbf{x}^{(0)} &= [2.52 \quad 2 \quad 37.5 \quad 9.25 \quad 6.8]^T \\
f(\mathbf{x}^{(0)}) &= 2{,}351{,}243.5
\end{aligned}
$$

Results:

$$\mathbf{x}^* = [4.538 \quad 2.400 \quad 60.000 \quad 9.300 \quad 7.000]^T$$
$$f(\mathbf{x}^*) = 5{,}280{,}254$$
$$x_6^* = 75{,}570$$
$$x_7^* = 198{,}157$$
$$x_8^* = 277{,}200$$

Problem 14

Source: M. A. Efroymson, Esso Research and Engineering Co. (cited in Colville, *IBM N.Y. Sci. Center Rept.* 320–2949, June, 1968, p. 26).

No. of variables: 6

No. of constraints: 4 nonlinear inequality constraints

Problem 14 was obtained from an actual "refinery heat integration" problem, and it contained a myriad of local optima of many different values.

Objective function:

$$\text{Minimize:} \quad f(\mathbf{x}) = \sum_{i=1}^{4} c(x_i) + \sum_{i=5}^{6} 100c(x_i)$$

Constraints:

$$t_3 - 300 \geq 0$$
$$t_4 - 300 \geq 0$$
$$280 - T_5 \geq 0$$
$$250 - T_6 \geq 0$$

Calculation of $c(x_i)$, t_i's, and T_i's:

$$c(x_i) = 2.7x_i + 1300 \qquad \left(\text{smallest integer} \geq \frac{x_i}{2000}\right)$$

$$T_1 = \frac{0.0285x_1 + 300}{1 + 0.0001425x_1}$$
$$t_1 = 500 - T_1$$
$$\alpha_2 = -0.0001665x_2$$
$$T_2 = \frac{200 - 350e^{-\alpha_2}}{1 - 1.5e^{-\alpha_2}}$$

$$T_4 = \frac{t_2 + (70 - t_2)e^{-\alpha_4}}{1 - 0.8e^{-\alpha_4}}$$
$$t_4 = 350 + (t_2 - T_4)e^{\alpha}{}_4$$
$$T_{j2} = 0.8T_3 + 0.2T_4$$
$$\alpha_5 = 0.000375x_5$$

$$t_2 = 300 + (200 - T_2)e^{\alpha_2}$$
$$\alpha_3 = (0.085)(9.36)10^{-5}x_3$$
$$T_3 = \frac{t_1 + (29.75 - t_1)e^{-\alpha_3}}{1 - 0.915e^{-\alpha_3}}$$
$$t_3 = 350 + (t_1 - T_3)e^{\alpha_3}$$
$$\alpha_4 = 0.00025x_4$$
$$T_5 = 80 + (T_{j2} - 80)e^{-\alpha_5}$$
$$T_{j1} = 0.7T_1 + 0.3T_2$$
$$\alpha_6 = 0.0003x_6$$
$$T_6 = 80 + (T_{j1} - 80)e^{-\alpha_6}$$

Starting point:

$$\mathbf{x}^{(0)} = [8000 \quad 3000 \quad 14000 \quad 2000 \quad 300 \quad 10]^T$$
$$f(\mathbf{x}^{(0)}) = 459{,}100$$

Results with flexible tolerance

$$\mathbf{x}^* = [11{,}884 \quad 3288 \quad 20{,}000 \quad 4000 \quad 114.18 \quad -155.03]^T$$
$$f(\mathbf{x}^*) = 250{,}799.9$$

Values of $f(\mathbf{x})$ at the optimum $\mathbf{x}$ vector reported by Colville:

Feasible starting point	*Code*	*Nonfeasible starting point**
255,303.5	Generalized gradient search	266,754.0
389,858.0	POP–360	
132,518.0	Optim (Mobile Oil)	125,578.0

*Nonfeasible starting vector:

$$\mathbf{x}^0 = [8000 \quad 3000 \quad 10{,}000 \quad 2000 \quad 200 \quad 10]^T$$

Problem 15

Source: P. Huard, *Electricité de France*, Directions des Études et Recherches (cited in Colville, *IBM N.Y. Sci. Center Rept.* 320–2949, June, 1968, p. 28).

No. of variables: 6 total variables; 2 independent

No. of constraints: 4 nonlinear equality constraints
2 constraints on derivatives (discontinuous)
6 lower and 6 upper bounds on the variables

This problem was the optimization of an electrical network.

Objective function:

Minimize: $f(\mathbf{x}) = f_1(x_1) + f_2(x_2)$

Constraints:

$$f_1(x_1) = \begin{cases} 30x_1 & 0 \le x_1 < 300 \\ 31x_1 & 300 \le x_1 < 400 \end{cases}$$

$$f_2(x_2) = \begin{cases} 28x_2 & 0 \le x_2 < 100 \\ 29x_2 & 100 \le x_2 < 200 \\ 30x_2 & 200 \le x_2 < 1000 \end{cases}$$

$$x_1 = 300 - \frac{x_3x_4}{131.078}\cos(1.48577 - x_6) + \frac{0.90798x_3^2}{131.078}\cos(1.47588)$$

$$x_2 = -\frac{x_3x_4}{131.078}\cos(1.48477 + x_6) + \frac{0.90798x_4^2}{131.078}\cos(1.47588)$$

$$x_5 = -\frac{x_3x_4}{131.078}\sin(1.48477 + x_6) + \frac{0.90798x_4^2}{131.078}\sin(1.47588)$$

$$200 - \frac{x_3x_4}{131.078}\sin(1.48477 - x_6) + \frac{0.90798}{131.078}x_3^2\sin(1.47588) = 0$$

$$0 \le x_1 \le 400$$

$$0 \le x_2 \le 1000$$

$$340 \le x_3 \le 420$$

$$340 \le x_4 \le 420$$

$$-1000 \le x_5 \le 1000$$

$$0 \le x_6 \le 0.5236$$

Starting point:

$$\mathbf{x}^{(0)} = [390 \quad 1000 \quad 419.5 \quad 340.5 \quad 198.175 \quad 0.5]^T$$

$$f(\mathbf{x}^{(0)}) = 9074.14$$

Results: Different results can be obtained, depending upon the precision of the **x** vector; the discontinuity in the two derivative constraints forces jump changes in $f(\mathbf{x})$ and $\mathbf{x}^*$.

	High precision	*Moderate precision*
x_1^*	107.81	201.78
x_2^*	196.32	100.00
x_3^*	373.83	383.07
x_4^*	420.00	420.00
x_5^*	21.31	−10.907
x_6^*	0.153	0.07314
$f(\mathbf{x}^*)$	8,927.5888	8,853.44 or 8,953,40

Problem 16

Source: J. D. Pearson, On Variable Metric Methods of Minimization, *Research Analysis Corp. Rept.* RAC-TP-302, McLean, Va., May, 1968.

No. of variables: 9

No. of constraints: 13 nonlinear inequality constraints
1 upper bound

The problem was to maximize the area of a hexagon in which the maximum diameter was unity.

Objective function:

$$\text{Maximize:}\quad f(\mathbf{x}) = 0.5(x_1x_4 - x_2x_3 + x_3x_9 - x_5x_9 + x_5x_8 - x_6x_7)$$

Constraints:

$$1 - x_3^2 - x_4^2 \geq 0$$
$$1 - x_9^2 \geq 0$$
$$1 - x_5^2 - x_6^2 \geq 0$$
$$1 - x_1^2 - (x_2 - x_9)^2 \geq 0$$
$$1 - (x_1 - x_5)^2 - (x_2 - x_6)^2 \geq 0$$
$$1 - (x_1 - x_7)^2 - (x_2 - x_8)^2 \geq 0$$
$$1 - (x_3 - x_5)^2 - (x_4 - x_6)^2 \geq 0$$
$$1 - (x_3 - x_7)^2 - (x_4 - x_8)^2 \geq 0$$
$$1 - x_7^2 - (x_8 - x_9)^2 \geq 0$$
$$x_1x_4 - x_2x_3 \geq 0$$
$$x_3x_9 \geq 0$$
$$-x_5x_9 \geq 0$$
$$x_5x_8 - x_6x_7 \geq 0$$
$$x_9 \geq 0$$

Starting point:

$$x_i^{(0)} = 1 \qquad i = 1, \ldots, 9$$
$$f(\mathbf{x}^{(0)}) = 0$$

Results:

$$\mathbf{x}^* = [0.9971 \quad -0.0758 \quad 0.5530 \quad 0.8331 \quad 0.9981 \quad -0.0623 \quad 0.5642 \quad 0.8256 \quad 0.0000024]^T$$

$$f(\mathbf{x}^*) = 0.8660$$

Problem 17

Source: D. A. Paviani, Ph.D. dissertation, The University of Texas, Austin, Tex., 1969.

No. of variables: 10

No. of constraints: 20 lower and upper bounds

The objective function was undefined outside the feasible region.

Objective function:

$$\text{Minimize:}\quad f(\mathbf{x}) = \sum_{i=1}^{10} \{[\ln(x_i - 2)]^2 + [\ln(10 - x_i)]^2\} - \left(\prod_{i=1}^{10} x_i\right)^{0.2}$$

Constraints:

$$2.001 < x_i < 9.999 \qquad i = 1, \ldots, 10$$

Starting point:

$$x_i^{(0)} = 9 \qquad i = 1, \ldots, 10$$

Results:

$$\mathbf{x}^* = [9.351 \quad 9.351 \quad 9.351 \quad 9.351 \quad 9.351 \quad 9.351 \quad 9.351 \quad 9.351 \quad 9.351 \quad 9.351]^T$$

$$f(\mathbf{x}^*) = -45.778$$

Problem 18

Source: Shell Development Co. (cited in Colville, *IBM N.Y. Sci. Center Rept.* 320–2949, June, 1968, p. 22).

No. of variables: 15

No. of constraints: 5 nonlinear inequality constraints
15 bounds on independent variables

This problem is the dual of Problem 10.

Objective function:

$$\text{Maximize:}\quad f(\mathbf{x}) = \sum_{i=1}^{10} b_i x_i - \sum_{j=1}^{5}\sum_{i=1}^{5} c_{ij} x_{10+i} x_{10+j} - 2\sum_{j=1}^{5} d_j x_{10+j}^3$$

Constraints:

$$2\sum_{i=1}^{5} c_{ij} x_{10+i} + 3 d_j x_{10+j}^2 + e_j - \sum_{i=1}^{10} a_{ij} x_i \geq 0 \qquad j = 1, \ldots, 5$$

$$x_i \geq 0 \qquad i = 1, \ldots, 15$$

Note: The e_j, c_{ij}, d_j, a_{ij}, and b_j were given in Problem 10.
Feasible starting point:

$$x_i^{(0)} = 0.0001 \qquad i = 1, \ldots, 15, i \neq 7$$
$$x_7^{(0)} = 60$$
$$f(\mathbf{x}^{(0)}) = -2400.01$$

Results:

$$\mathbf{x}^* = [0.0000 \quad 0.0000 \quad 5.1740 \quad 0.0000 \quad 3.0611 \quad 11.8395 \quad 0.0000 \quad 0.0000 \quad 0.1039 \quad 0.0000 \quad 0.3000 \quad 0.3335 \quad 0.4000 \quad 0.4283 \quad 0.2240]^T$$

$$f(\mathbf{x}^*) = -32.386$$

Nonfeasible starting point:

$$x_i^{(0)} = b_i^{(0)} \qquad i = 1, \ldots, 10$$
$$x_i^{(0)} = 0 \qquad i = 11, \ldots, 14$$
$$x_i^{(0)} = 1 \qquad i = 15$$
$$f(\mathbf{x}^{(0)}) = 6829.06$$

Results: Same $\mathbf{x}^*$ vector and value of $f(\mathbf{x}^*)$ as for feasible starting point.

Problem 19

Source: J. M. Gauthier, IBM France (cited in Colville, *IBM N.Y. Sci. Center Rept.* 320–2949, June, 1968, p. 29).
No. of variables: 16
No. of constraints: 8 linear equality constraints
32 upper and lower bounds on the variables
Objective function:

$$\text{Maximize:} \quad f(\mathbf{x}) = -\sum_{i=1}^{16} \sum_{j=1}^{16} a_{ij}(x_i^2 + x_i + 1)(x_j^2 + x_j + 1)$$

Constraints:

$$\sum_{j=1}^{16} b_{ij}x_j = c_i \qquad i = 1, \ldots, 8$$
$$0 \leq x_j \leq 5 \qquad j = 1, \ldots, 16$$

ta for Problem 19

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
a_{1j}	1			1			1	1								1
a_{2j}		1	1				1			1						
a_{3j}			1				1		1	1				1		
a_{4j}				1			1				1				1	
a_{5j}					1	1				1		1				1
a_{6j}						1		1							1	
a_{7j}							1				1		1			
a_{8j}								1		1					1	
a_{9j}									1			1				1
a_{10j}										1				1		
a_{11j}											1		1			
a_{12j}												1		1		
a_{13j}													1	1		
a_{14j}														1		
a_{15j}															1	
a_{16j}																1
b_{1j}	0.22	0.20	0.19	0.25	0.15	0.11	0.12	0.13	1.00							
b_{2j}	−1.46		−1.30	1.82	−1.15		0.80			1.00						
b_{3j}	1.29	−0.89			−1.16	−0.96		−0.49			1.00					
b_{4j}	−1.10	−1.06	0.95	−0.54		−1.78	−0.41					1.00				
b_{5j}				−1.43	1.51	0.59	−0.33	−0.43					1.00			
b_{6j}		−1.72	−0.33		1.62	1.24	0.21	−0.26						1.00		
b_{7j}	1.12			0.31			1.12		−0.36						1.00	
b_{8j}		0.45	0.26	−1.10	0.58		−1.03	0.10								1.00
c_i	2.5	1.1	−3.1	−3.5	1.3	2.1	2.3	−1.5								

Note: The a_{ij}, b_{ij}, and c_i are given in the accompanying table.
Nonfeasible starting point:

$$x_i^{(0)} = 10 \qquad i = 1, \ldots, 16$$
$$f(\mathbf{x}^{(0)}) = -209{,}457$$

Results:

$$\mathbf{x}^* = [0.040 \quad 0.792 \quad 0.203 \quad 0.844 \quad 1.270 \quad 0.935 \quad 1.682 \quad 0.155$$
$$1.568 \quad 0.000 \quad 0.000 \quad 0.000 \quad 0.660 \quad 0.000 \quad 0.674 \quad 0.000]^T$$
$$f(\mathbf{x}^*) = -244.900$$

Problem 20

Source: D. A. Paviani, Ph.D. dissertation, The University of Texas, Austin, Tex., 1969.
No. of variables: 24
No. of constraints: 12 nonlinear equality constraints
2 linear equality constraints
6 nonlinear inequality constraints
24 bounds on independent variables

This problem represents the minimization of the cost of blending multi-component mixtures.
Objective function:

$$\text{Minimize:} \quad f(\mathbf{x}) = \sum_{i=1}^{24} a_i x_i$$

Note: See accompanying tables for the a_i's, b_i's, c_i's, d_i's, e_i's.
Constraints:

$$h_i(\mathbf{x}) = \frac{x_{(i+12)}}{b_{(i+12)} \sum_{j+13}^{24} \frac{x_j}{b_j}} - \frac{c_i x_i}{40 b_i \sum_{j=1}^{12} \frac{x_j}{b_j}} = 0 \qquad i = 1, \ldots, 12$$

$$h_{13}(\mathbf{x}) = \sum_{i=1}^{24} x_i - 1 = 0$$

$$h_{14}(\mathbf{x}) = \sum_{i=1}^{12} \frac{x_i}{d_i} + f \sum_{i=13}^{24} \frac{x_i}{b_i} - 1.671 = 0$$

where $$f = (0.7302)(530)\left(\frac{14.7}{40}\right)$$

$$\frac{-[x_i + x_{(i+12)}]}{\sum_{j=1}^{24} x_j + e_i} \geq 0 \qquad i = 1,2,3$$

$$\frac{-[x_{(i+3)} + x_{(i+15)}]}{\sum_{j=1}^{24} x_j + e_i} \geq 0 \qquad i = 4,5,6$$

$$x_i \geq 0 \qquad i = 1, \ldots, 24$$

Nonfeasible starting point:

$$x_i^{(0)} = 0.04 \qquad i = 1, \ldots, 24$$

$$f(\mathbf{x}^{(0)}) = 0.14696$$

Results:

	Flexible tolerance	*NLP*	*SUMT*
$f(\mathbf{x})$	0.05700	0.09670	0.07494
x_1^*	7.804E − 03	9.537E − 07	9.109E − 03
x_2^*	1.121E − 01	0	3.739E − 02
x_3^*	1.136E − 01	4.215E − 03	8.961E − 02
x_4^*	0	1.039E − 04	1.137E − 02
x_5^*	0	0	4.155E − 03
x_6^*	0	0	4.184E − 03
x_7^*	6.609E − 02	2.072E − 01	5.980E − 02
x_8^*	0	5.979E − 01	1.554E − 02
x_9^*	0	1.298E − 01	1.399E − 02
x_{10}^*	0	3.350E − 02	8.780E − 03
x_{11}^*	1.914E − 02	1.711E − 02	1.231E − 02
x_{12}^*	6.009E − 03	8.427E − 03	1.153E − 02
x_{13}^*	5.008E − 02	4.657E − 10	7.570E − 02
x_{14}^*	1.844E − 01	0	7.997E − 02
x_{15}^*	2.693E − 01	0	2.797E − 01
x_{16}^*	0	0	1.168E − 02
x_{17}^*	0	0	2.347E − 02
x_{18}^*	0	0	6.368E − 03
x_{19}^*	1.704E − 01	2.868E − 04	2.028E − 01
x_{20}^*	0	1.193E − 03	7.451E − 03
x_{21}^*	0	8.332E − 05	4.547E − 03
x_{22}^*	0	1.239E − 04	1.010E − 02
x_{23}^*	8.453E − 04	2.070E − 05	1.220E − 03
x_{24}^*	1.980E − 04	1.829E − 05	1.810E − 03
$h_1(\mathbf{x}^*)$	0	4.908E − 07	−1.182E − 03
$h_2(\mathbf{x}^*)$	0	0	−4.329E − 04

	Flexible tolerance	*NLP*	*SUMT*
$h_3(\mathbf{x}^*)$	0	0	3.467E − 03
$h_4(\mathbf{x}^*)$	0	0	2.217E − 04
$h_5(\mathbf{x}^*)$	0	0	−2.550E − 04
$h_6(\mathbf{x}^*)$	0	0	−7.368E − 04
$h_7(\mathbf{x}^*)$	0	−2.209E − 08	1.982E − 03
$h_8(\mathbf{x}^*)$	0	−8.521E − 08	−2.334E − 05
$h_9(\mathbf{x}^*)$	0	−5.854E − 09	1.629E − 03
$h_{10}(\mathbf{x}^*)$	0	8.137E − 08	−4.397E − 04
$h_{11}(\mathbf{x}^*)$	0	−2.596E − 08	9.431E − 04
$h_{12}(\mathbf{x}^*)$	0	5.766E − 08	1.853E − 03
$h_{13}(\mathbf{x}^*)$	0	0	−1.741E − 02
$h_{14}(\mathbf{x}^*)$	N	0	8.743E − 03

Data for Test Problems 10 and 18

i	a_i	b_i	c_i	d_i	e_i
1	0.0693	44.094	123.7	31.244	0.1
2	0.0577	58.12	31.7	36.12	0.3
3	0.05	58.12	45.7	34.784	0.4
4	0.20	137.4	14.7	92.7	0.3
5	0.26	120.9	84.7	82.7	0.6
6	0.55	170.9	27.7	91.6	0.3
7	0.06	62.501	49.7	56.708	
8	0.10	84.94	7.1	82.7	
9	0.12	133.425	2.1	80.8	
10	0.18	82.507	17.7	64.517	
11	0.10	46.07	0.85	49.4	
12	0.09	60.097	0.64	49.1	
13	0.0693	44.094			
14	0.0577	58.12			
15	0.05	58.12			
16	0.20	137.4			
17	0.26	120.9			
18	0.55	170.9			
19	0.06	62.501			
20	0.10	84.94			
21	0.12	133.425			
22	0.18	82.507			
23	0.10	46.07			
24	0.09	60.097			

Problem 21

Source: A. G. Holzman, *SRCC Rept.* 113, University of Pittsburgh, Pittsburgh, Pa., 1969.
No. of independent variables: 3
No. of constraints: 6 bounds on independent variables
Objective function:

$$\text{Minimize:} \quad f(\mathbf{x}) = \sum_{i=1}^{99}\left[\exp - \frac{(u_i - x_2)^{x_3}}{x_1} - 0.01i\right]^2$$

$$u_i = 25 + \left(-50 \ln 0.01i\right)^{\frac{1}{1.5}}$$

Constraints:

$$0.1 \le x_1 \le 100.0$$
$$0.0 \le x_2 \le 25.6$$
$$0.0 \le x_3 \le 5.0$$

Feasible starting point:

$$\mathbf{x}^{(0)} = [100.0 \quad 12.5 \quad 3.0]^T$$

Results:

$$\mathbf{x}^* = [50.0 \quad 25.0 \quad 1.5]^T$$
$$f(\mathbf{x}^*) = 0.0$$

Problem 22

Source: U.S. Steel Co. (cited by Holzman, *SRCC Rept.* 113, 1969).
No. of independent variables: 6
No. of constraints: 4 nonlinear inequality constraints
12 bounds on independent variables
Objective function:

$$\text{Minimize:} \quad f(\mathbf{x}) = 4.3x_1 + 31.8x_2 + 63.3x_3 + 15.8x_4 + 68.5x_5 + 4.7x_6$$

Constraints:

$$17.1x_1 + 38.2x_2 + 204.2x_3 + 212.3x_4 + 623.4x_5 + 1495.5x_6 - 169x_1x_3 - 3580x_3x_5 - 3810x_4x_5 - 18{,}500x_4x_6 - 24{,}300x_5x_6 \geq b_1$$

$$17.9x_1 + 36.8x_2 + 113.9x_3 + 169.7x_4 + 337.8x_5 + 1385.2x_6 - 139x_1x_3 - 2450x_4x_5 - 16{,}600x_4x_6 - 17{,}200x_5x_6 \geq b_2$$

$$-273x_2 - 70x_4 - 819x_5 + 26{,}000x_4x_5 \geq b_3$$

$$159.9x_1 - 311x_2 + 587x_4 + 391x_5 + 2198x_6 - 14{,}000x_1x_6 \geq b_4$$

$$0 \leq x_1 \leq 0.31 \qquad 0 \leq x_4 \leq 0.042$$

$$0 \leq x_2 \leq 0.046 \qquad 0 \leq x_5 \leq 0.028$$

$$0 \leq x_3 \leq 0.068 \qquad 0 \leq x_6 \leq 0.0134$$

Starting point:

For:				*Results:*						
b_1	b_2	b_3	b_4	x_1	x_2	x_3	x_4	x_5	x_6	$f(\mathbf{x}^*)$
4.97	−1.88	−29.08	−78.02	0	0	0	0	0	0.00333	0.0156
4.97	−1.88	−69.08	−118.02	0	0	0	0	0	0.00332	0.0156
32.97	25.12	−29.08	−78.02	0	0	0.0633	0	0	0.0134	4.070
32.97	25.12	−124.08	−173.02	0	0	0.0633	0	0	0.0134	4.070

Problem 23

Source: J. Bracken and G. P. McCormick, "Selected Applications of Nonlinear Programming," John Wiley & Sons, Inc., New York, 1968, p. 26. (A weapon assignment problem.)

No. of independent variables: 100

No. of constraints: 12 linear constraints
100 lower bounds on the independent variables

Objective function:

$$\text{Minimize:} \quad f(\mathbf{x}) = \sum_{j=1}^{20} u_j \left(\prod_{i=1}^{5} a_{ij}^{x_{ij}} - 1 \right)$$

Constraints:

$$\left(\sum_{i=1}^{5} x_{ij} \right) - b_j \geq 0 \qquad j = 1,6,10,14,15,16,20$$

$$-\left(\sum_{j=1}^{20} x_{ij} \right) - c_i \geq 0 \qquad i = 1, \ldots, 5$$

ta for Problem 23

i \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	c_1: *no. of weapons i available*
	a_{ij}: probability that weapon i will not damage target j																				
1	1	.95	1	1	1	.85	.90	.85	.80	1	1	1	1	1	1	1	1	.95	1	1	200
2	.84	.83	.85	.84	.85	.81	.81	.82	.80	.86	1	.98	1	.88	.87	.88	.85	.84	.85	.85	100
3	.96	.95	.96	.96	.96	.90	.92	.91	.92	.95	.99	.98	.99	.98	.97	.98	.95	.92	.93	.92	300
4	1	1	1	1	1	1	1	1	1	.96	.91	.92	.91	.92	.98	.93	1	1	1	1	150
5	.92	.94	.92	.95	.95	.98	.98	1	1	.90	.95	.96	.91	.98	.99	.99	1	1	1	1	250

b_j: minimum no. of weapons to be assigned to target j

30					100				40				50	70	35				10

u_j: military value of target j

60	50	50	75	40	60	35	30	25	150	30	45	125	200	200	130	100	100	100	150

Feasible solutions obtained for weapons assignment problem x_{ij}

Weapon type i	Target j																				Total
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
1		24 (16)				32 (100)	37 (38)	28 (26)	22 (20)						5			52			200 (200)
2	1	8	2	18 (23)	11 (20)										29 (25)	9 (31)	21 (1)				99 (100)
3		9			29	62									35 (45)		17 (76)	25 (56)	62 (62)	60 (61)	299 (300)
4											9	39 (39)	(50)	58 (57)		44 (4)					150 (150)
5	47 (50)	5 (46)	36 (47)	12		6				50 (50)	42 (57)		51		1						250 (250)
Total	48 (50)	46 (62)	38 (47)	30 (23)	40 (20)	100 (100)	37 (38)	28 (26)	22 (20)	50 (50)	51 (57)	39 (39)	51 (50)	58 (57)	70 (70)	53 (35)	38 (77)	77 (56)	62 (62)	60 (61)	

Notes: No.( ) from Holzman; with ( ) from Bracken and McCormick.
$f(\mathbf{x}) = 1732$.

Problem 24

Source: J. Bracken and G. P. McCormick, "Selected Applications of Nonlinear Programming," John Wiley & Sons, Inc., New York, 1968, p. 19.
No. of independent variables: 2
No. of constraints: 1 nonlinear inequality constraint
1 linear inequality constraint
Objective function:

Minimize: $f(\mathbf{x}) = (x_1 - 2)^2 + (x_2 - 1)^2$

Constraints:

$$g_1(\mathbf{x}) = -x_1^2 + x_2 \geq 0$$
$$g_2(\mathbf{x}) = -x_1 - x_2 + 2 \geq 0$$

Nonfeasible starting point:

$$\mathbf{x}^{(0)} = [2 \quad 2]^T$$
$$f(\mathbf{x}^{(0)}) = 1$$

Results:

$$f(\mathbf{x}^*) = 1$$
$$\mathbf{x}^* = [1 \quad 1]^T$$

Problem 25

Source: Unpublished
No. of independent variables: 2
No. of constraints: None
Objective function:

Minimize: $f(\mathbf{x}) = 4(x_1 - 5)^2 + (x_2 - 6)^2$

Starting point:

$$\mathbf{x}^{(0)} = [8 \quad 9]^T$$
$$f(\mathbf{x}^{(0)}) = 45$$

Results:

$$\mathbf{x}^* = [5 \quad 6]^T$$
$$f(\mathbf{x}^*) = 0$$

Problem 26

Source: M. J. D. Powell, *Computer J.*, **5**: 147 (1962).
No. of independent variables: 4
No. of constraints: None
Objective function:

$$\text{Minimize:}\quad f(\mathbf{x}) = (x_1 + 10x_2)^2 + 5(x_3 - x_4)^2 + (x_2 - 2x_3)^4 + 10(x_1 - x_4)^4$$

Starting point:

$$\mathbf{x}^{(0)} = [3 \quad -1 \quad 0 \quad 1]^T \qquad \text{and} \qquad [1 \quad 1 \quad 1 \quad 1]^T$$
$$f(\mathbf{x}^{(0)}) = 215 \qquad\qquad f(\mathbf{x}^{(0)}) = 125$$

Results:

$$\mathbf{x}^* = [0 \quad 0 \quad 0 \quad 0]^T$$
$$f(\mathbf{x}^*) = 0$$

Problem 27

Source: Unpublished
No. of independent variables: 2
No. of constraints: None
Objective function:

$$\text{Minimize:}\quad f(\mathbf{x}) = (x_1x_2)^2(1 - x_1)^2[1 - x_1 - x_2(1 - x_1)^5]^2$$

Starting point:

$$\mathbf{x}^{(0)} = [-1.2 \quad 1]^T$$
$$f(\mathbf{x}^{(0)}) = 26{,}656$$

Results:

$$\mathbf{x}^* = [1 \quad \text{unbounded}]^T \text{ or } [0 \quad \text{unbounded}]^T \text{ or } [\text{unbounded} \quad 0]^T$$
$$f(\mathbf{x}^*) = 0$$

Problem 28

Source: Unpublished; solution of a set of equations
No. of independent variables: 2
No. of constraints: None

Objective function:

Minimize: $f(\mathbf{x}) = (x_1^2 + x_2 - 11)^2 + (x_1 + x_2^2 - 7)^2$

Starting point:

$$\mathbf{x}^{(0)} = [1 \quad 1]^T$$
$$f(\mathbf{x}^{(0)}) = 106$$

Results:

$$\mathbf{x}^* = [3.58443 \quad -1.84813]^T \qquad \text{and} \qquad [3 \quad 2]^T$$
$$f(\mathbf{x}^*) = 0$$

Note: All the computer codes tested from $\mathbf{x}^0 = [1 \quad 1]^T$ yielded the second solution.

Problem 29

Source: Unpublished
No. of independent variables: 2
No. of constraints: None
Objective function:

Minimize: $f(\mathbf{x}) = (x_1^2 + 12x_2 - 1)^2 + (49x_1^2 + 49x_2^2 + 84x_1 + 2324x_2 - 681)^2$

Starting point:

$$\mathbf{x}^{(0)} = [1 \quad 1]^T$$
$$f(\mathbf{x}^{(0)}) = 3.3306 \times 10^6$$

Results:

$$\mathbf{x}^* = [0.28581 \quad 0.27936]^T$$
$$f(\mathbf{x}^*) = 5.9225$$

or

$$\mathbf{x}^* = [-21.026653 \quad -36.760090]^T$$
$$f(\mathbf{x}^*) = 0$$

Problem 30

Source: Unpublished
No. of independent variables: 3
No. of constraints: None

Objective function:

$$\text{Minimize:} \quad f(\mathbf{x}) = 100\left[x_3 - \left(\frac{x_1 + x_2}{2}\right)^2\right]^2 + (1 - x_1)^2 + (1 - x_2)^2$$

Starting point:

$$\mathbf{x}^{(0)} = [-1.2 \quad 2 \quad 0]^T$$
$$f(\mathbf{x}^{(0)}) = 8.40$$

Results:

$$\mathbf{x}^* = [1 \quad 1 \quad 1]^T$$
$$f(\mathbf{x}^*) = 0$$

Problem 31

Source: J. Bracken and G. P. McCormick, "Selected Applications of Nonlinear Programming," John Wiley & Sons, Inc., New York, 1968, p. 18.

No. of independent variables: 2

No. of constraints: None

Objective function:

$$\text{Minimize:} \quad f(\mathbf{x}) = (x_1 - 2)^2 + (x_2 - 1)^2 + \frac{0.04}{g_1(\mathbf{x})} + \frac{h_1^2(\mathbf{x})}{0.2}$$

$$g_1(\mathbf{x}) = -\frac{x_1^2}{4} - x_2^2 + 1 \qquad \text{and} \qquad h_1(\mathbf{x}) = x_1 - 2x_2 + 1$$

Starting point:

$$\mathbf{x}^{(0)} = [2 \quad 2]^T$$
$$f(\mathbf{x}^{(0)}) = 5.99$$

Results:

$$\mathbf{x}^* = [1.7954 \quad 1.3779]^T \qquad \text{(local minimum)}$$
$$f(\mathbf{x}^*) = 0.16904$$

Note: Other starting vectors yield other results; the solution vector reported in the source was not obtained, nor was the solution vector given in the source obtained from the source starting vector. Too large an initial step from $\mathbf{x}^{(0)}$ will bypass the local minimum and drive $f(\mathbf{x})$ to the global minimum at $g_1(\mathbf{x}) = -0$, where $f(\mathbf{x}) \to -\infty$.

Problem 32

Source: Unpublished (estimation of coefficients from experimental data by least squares)
No. of independent variables: 4
No. of constraints: None
Objective function:

$$\text{Minimize:}\quad f(\mathbf{x}) = 10^4 \sum_{i=1}^{7} \left(\frac{\dfrac{x_1^2 + x_2^2 a_i + x_3^2 a_i^2}{1 + x_4^2 a_i} - b_i}{b_i} \right)$$

Starting point:

$$\mathbf{x}^{(0)} = [2.7 \quad 90 \quad 1500 \quad 10]$$

$$f(\mathbf{x}^{(0)}) = 2.905 \times 10^4$$

Results:

$$\mathbf{x}^* = [2.714 \quad 140.4 \quad 1707 \quad 31.51]$$

$$f(\mathbf{x}^*) = 318.572$$

i	a_i	b_i
1	0.0	7.391
2	0.000428	11.18
3	0.00100	16.44
4	0.00161	16.20
5	0.00209	22.20
6	0.00348	24.02
7	0.00525	31.32

Problem 33

Source: Unpublished
No. of independent variables: 2
No. of constraints: None
Objective function:

$$\text{Maximize:}\quad f(\mathbf{x}) = e^{-x_1 - x_2}(2x_1^2 + 3x_2^2)$$

Starting point:

$$\mathbf{x}^{(0)} = [2.5 \quad 2.5]$$

$$f(\mathbf{x}^{(0)}) = 2.3299 \times 10^{-5}$$

Results:

$$\mathbf{x}^* = [0 \quad 1]^T \qquad \text{or} \qquad [0 \quad -1]^T$$
$$f(\mathbf{x}^*) = 1.1036$$

Problem 34

Source: R. Fletcher and M. J. D. Powell, *Computer J.*, **6**:33 (1963). (A helical valley in the x_3 direction.)
No. of independent variables: 3
No. of constraints: None
Objective function:

$$\text{Minimize:} \quad f(\mathbf{x}) = 100\{[x_3 - 10\theta(x_1,x_2)]^2 + [(x_1^2 + x_2^2)^{\frac{1}{2}} - 1]^2\} + x_3^2$$

$$\theta(x_1,x_2) = \begin{cases} \dfrac{1}{2\pi}\tan^{-1}\dfrac{x_2}{x_1} & x_1 > 0 \\ \dfrac{1}{2} + \dfrac{1}{2\pi}\tan^{-1}\dfrac{x_2}{x_1} & x_1 < 0 \end{cases}$$

Starting point:

$$\mathbf{x}^{(0)} = [-1 \quad 0 \quad 0]^T$$

Results:

$$\mathbf{x}^* = [1 \quad 0 \quad 0]^T$$
$$f(\mathbf{x}^*) = 0$$

Problem 35

Source: Unpublished
No. of independent variables: 2
No. of constraints: None
Objective function:

$$\text{Minimize:} \quad f(\mathbf{x}) = u_1^2 + u_2^2 + u_3^2$$
$$u_i = c_i - x_1(1 - x_2^i) \qquad c_1 = 1.5,\ c_2 = 2.25,\ c_3 = 2.625$$

Starting point:

$$\mathbf{x}^{(0)} = [2 \quad 0.2]^T$$
$$f(\mathbf{x}^{(0)}) = 0.52978$$

Results:

$$\mathbf{x}^* = [3.0000 \quad 0.5000]^T$$
$$f(\mathbf{x}^*) = 0$$