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SEQUENTIAL OPTIMIZATION OF MULTIMODAL DISCRETE FUNCTION WITH BOUNDED RATE OF CHANGE*†

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In this paper a search scheme for finding the maximum of a discrete objective function of several variables is proposed. The scheme is sequential and optimal in the sense that it determines the maximum with minimum number of measurements of the function. Unlike other sequential search techniques with this property the scheme places no condition on the number of possible maxima of the objective function. Instead it requires that some bounds on the rate of change of the function per each variable be known. The points at which measurements of the value of the objective function are to be taken are determined successively by maximization of an auxiliary function updated after each measurement. The algorithm is described in detail, and illustrated by two simple examples.

Introduction

Let us consider a linear network on Figure 1 consisting of m nodes connected by $m - 1$ branches. It may be convenient to think about the network as a traffic route, for instance a railroad or a controlled access highway with nodes representing stations or interchanges along the route. Assuming stationary state there is an average number of units of cargo entering and leaving each node in a unit of time. Denote these numbers at the ith node by $g^+(i)$ and $g^-(i)$ respectively. Denote further by $f(i)$ the flow (i.e., the average number of units per time being transported) through the branch connecting the ith and $(i + 1)$st nodes. Suppose now that for some purpose it is desired to determine the branch or branches where the flow reaches its extreme, say maximum value and/or that value.

Formally, this problem is nothing but maximization of an objective function f of a single discrete variable, $i = 1, \cdots, m$. However, objective functions arising from practical problems such as the one described will hardly be given algebraically. Rather its values $f(i)$ for various i will be obtained by actual measurements in a real situation. In our example this amounts to measuring the average flow in branches. On many occasions these measurements may be difficult or costly to perform. Thus we would like to avoid the necessity of measuring the flow in each branch, and rather seek a search scheme that would allow us to find the maximum with as few measurements as possible.

There are several search methods with this property, the Fibonacci search [2] and others more or less related to it ([1], [4] et al.). Unfortunately, all of them by their very nature require that the objective function be unimodal, that is if f has the maximum at some i^* then the function must be nondecreasing over $i = 1, \cdots, i^*$ and nonincreasing over $i = i^*, \cdots, m$. Clearly, we have no grounds to expect that this would be the case in our network example. On the other hand it is obvious that unless we replace the unimodality condition by some other assumption applicable to our case we will always be forced to make all the m measurements. The reason is that we have a chance of reducing the number of measurements only if each group of measurements

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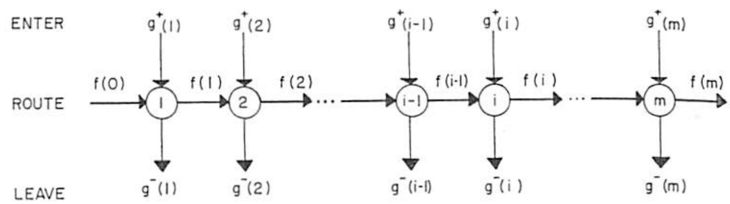

FIGURE 1

already performed provides us with some relevant information about the rest of the function. Such is the case with unimodality; we must, however, look for something else.

Returning to our network we notice that typically there is a natural bound on the number of units per time entering or leaving each node. This bound may be due to the limited capacity of the stations in handling the cargo or to the speed limit in entrances and exits to the highway. In symbols, there will be a known constant K such that $0 \leqq g^+(i) \leqq K$ and $0 \leqq g^-(i) \leqq K$ for all $i = 1, \cdots, m$. Since at each node $f(i) - f(i-1) = g^+(i) - g^-(i)$ the flow must satisfy the condition

$$|f(i) - f(i-1)| \leqq K, \qquad i = 1, \cdots, m.$$

There are many other optimization problems the objective function of which satisfies a condition of this kind rather than being unimodal. For instance $f(\mathbf{x})$ may be some performance criterion of a system with a discrete control parameter $\mathbf{x}$, possibly a vector. The mathematical description of the system, although known, may be very complicated and the actual evaluation of $f(\mathbf{x})$ for each particular $\mathbf{x}$ may require a large amount of computer time. At the same time it may be relatively easy to determine, either from the physical nature or from the mathematical description of the system, that if a single component of the control vector $\mathbf{x}$ is changed by one unit the performance $f(\mathbf{x})$ of the system cannot change by more than a certain amount.

Since a condition of this kind imposes a restriction on the rate of change of the objective function it is conceivable that a scheme for evaluating the function could be designed that achieves a substantial reduction in the number of evaluations or measurements required. In what follows we address ourselves to the problem of designing such a scheme. The principle is the same as in [3] where the case of an objective function of a single continuous variable has been studied. For the discrete case investigated in this paper, however, the algorithm is slightly different and generalizes readily to functions of several discrete variables.

Needless to add that the scheme applies as well to minimization of f or to location of roots of an equation $f(x) = a$, which can be easily converted into maximization problems by taking $-f$ and $-|f - a|$ respectively.

The Scheme

Let $f(i_1, \cdots, i_p)$ be a function of p integer-valued variables defined on some finite subset I of the set of all p-tuples of integers. The function satisfies the following condition: Denote by $\mathbf{e}_j$ the p-tuple with 1 at the jth place and zeros elsewhere. Then there are p constants K_j such that

$$\text{(C)} \qquad |f(\mathbf{x} \pm \mathbf{e}_j) - f(\mathbf{x})| \leqq K_j$$

for all $\mathbf{x} = (i_1, \cdots, i_p) \in I$ and $j = 1, \cdots, p$ provided $\mathbf{x} \pm \mathbf{e}_j \in I$.

In other words, if the jth of the p variables is changed by one the function on its feasible region cannot change its value by more than K_j.

The problem is to design a scheme for successive measurements of the values of the function f that would eventually determine the maximum $y^* = \max f(\mathbf{x})$ and one or all points $\mathbf{x}^* \in I$ at which the maximum is attained. We further want the number of measurements required by the scheme to be as small as possible.

We are now going to describe such a scheme. The main idea consists in successive maximization of an auxiliary function, which is updated after each measurement.

Denote by $\mathbf{t}_0, \mathbf{t}_1, \mathbf{t}_2, \cdots$ the sequence of points in the domain I at which the successive measurements $y_n = f(\mathbf{t}_n)$, $n = 0, 1, 2, \cdots$, are taken. The first measurement y_0 is taken at an arbitrary feasible point $\mathbf{t}_0 = (t_{01}, \cdots, t_{0p}) \in I$. An auxiliary function

$$F_1(\mathbf{x}) = y_0 + \textstyle\sum_{j=1}^{p} K_j| i_j - t_{0j} |, \qquad \mathbf{x} = (i_1, \cdots, i_p) \in I,$$

is then computed and the next measurement y_1 is taken at a point $\mathbf{t}_1 = (t_{11}, \cdots, t_{1p})$ which maximizes the function F_1 over the domain I, that is,

$$F_1(\mathbf{t}_1) = \max_{\mathbf{x} \in I} F_1(\mathbf{x}).$$

Then the function F_1 is updated by replacing its values $F_1(\mathbf{x})$ at each point $\mathbf{x} = (i_1, \cdots, i_p) \in I$ such that

$$y_1 + \textstyle\sum_{j=1}^{p} K_j| i_j - t_{1j} | < F_1(\mathbf{x})$$

by the value of the left-hand side of the above inequality. New auxiliary function $F_2(\mathbf{x})$ is thus obtained which is again maximized to determine $\mathbf{t}_2$ and so on.

The process continues as long as the maximum $F_n(\mathbf{t}_n)$ of the nth auxiliary function F_n is greater than $Z_n = \max \{y_0, y_1, \cdots, y_{n-1}\}$, the largest measurement obtained so far. As soon as $F_n(\mathbf{t}_n) = Z_n$ the process is stopped. The maximum y^* of the function f is then equal to that last Z_n and occurs at the point where the largest measurement was obtained.

The entire algorithm can be summarized as follows:

Start. Choose any $\mathbf{t}_0 = (t_{01}, \cdots, t_{0p}) \in I$ and measure $y_0 = f(\mathbf{t}_0)$. Then compute the function

$$F_1(\mathbf{x}) = y_0 + \textstyle\sum_{j=1}^{p} K_j | i_j - t_{0j} | \quad \text{for all} \quad \mathbf{x} = (i_1, \cdots, i_p) \in I.$$

Set $(\boldsymbol{\theta}, Z_1) = (\mathbf{t}_0, y_0)$, set $n = 1$, go to *Maximize.*

Update. Using the current value y_n measured at the point $\mathbf{t}_n = (t_{n1}, \cdots, t_{np})$ update the auxiliary function F_n by the formula:

$$\begin{aligned} F_{n+1}(\mathbf{x}) &= y_n + K \textstyle\sum_{j=1}^{p} | i_j - t_{nj} | \quad \text{if} \quad y_n + K \textstyle\sum_{j=1}^{p} | i_j - t_{nj} | < F_n(\mathbf{x}), \\ &= F_n(\mathbf{x}) \quad \text{if} \quad y_n + K \textstyle\sum_{j=1}^{p} | i_j - t_{nj} | \geqq F_n(x), \end{aligned}$$

where $\mathbf{x} = (i_1, \cdots, i_p) \in I$.

Also update

$$\begin{aligned} (\boldsymbol{\theta}_{n+1}, Z_{n+1}) &= (\boldsymbol{\theta}_n, Z_n) \quad \text{if} \quad y_n < Z_n, \\ &= (\mathbf{t}_n, y_n) \quad \text{if} \quad y_n = Z_n. \end{aligned}$$

Set $n = n + 1$, go to *Maximize.*

Maximize. Find any $\mathbf{t}_n \in I$ such that $F_n(\mathbf{t}_n) = \max_{\mathbf{x} \in I} F_n(\mathbf{x})$. Go to *Compare.*

Compare. If $F_n(\mathbf{t}_n) = Z_n$ go to *Stop*, otherwise go to *Measure.*

Measure. Measure $y_n = f(\mathbf{t}_n)$, go to *Update.*

Stop. The function f has the maximum $y^* = Z_n$ at the point $\mathbf{x}^* = \mathbf{t}_n$.

If it is desired to find *all* points $\mathbf{x}^*$ such that $f(\mathbf{x}^*) = y^*$ additional measurements must be taken after the algorithm has come to *Stop* at all other points $\mathbf{t} \in I$ (if any) such that $F_n(\mathbf{t}) = Z_n$ updating the auxiliary function as before. The points $\mathbf{x}^*$ are exactly those of the above $\mathbf{t}$'s at which $f(\mathbf{t}) = Z_n$.

The algorithm also provides at each iteration an interval estimate of the maximum value y^*, in particular $Z_n \leqq y^* \leqq F_n(\mathbf{t}_n)$. Hence if only an interval estimate of the maximum is sought the algorithm can be stopped as soon as the difference $F_n(\mathbf{t}_n) - Z_n$ decreases below the desired length of the interval thus further reducing the number of measurements needed.

To see that these conclusions are true notice first that the condition (C) implies that $f(\mathbf{x}) \leqq F_n(\mathbf{x})$ for all $\mathbf{x} \in I$ and $n = 1, 2, \ldots$. Futhermore $f(\boldsymbol{\theta}_n) = Z_n \leqq y^* \leqq F_n(\mathbf{t}_n) = \max_{\mathbf{x} \in I} F_n(\mathbf{x})$. Hence as soon as $Z_n = F_n(\mathbf{t}_n)$ for some n, which must happen sooner or later since all $\mathbf{t}_n$ are different and the number of points in I is finite, we must have $Z_n = y^*$ and $f(\mathbf{x}) = y^*$ if and only if $f(\mathbf{x}) = \max_{\mathbf{x} \in I} F_n(\mathbf{x})$.

It remains to show that the scheme accomplishes the task with minimum number of measurements. The latter concepts, however, require some clarification since for every function there is a trivial scheme that requires only one measurement if that measurement happens to be the actual maximum. Clearly such schemes are pathological, and must be excluded. Let Φ be the class of all functions f defined on the same domain I and satisfying the condition (C) with the same constants K_j. Call a scheme admissible if it will determine the maximum of f for any function $f \in \Phi$ and if the points, where measurements are to be taken, depend only on the information available, i.e., on previous measurements and K_j's. Denote for a while

$$F'_n(\mathbf{x}) = \min_{k=0,\ldots,n-1} \{y_k + \textstyle\sum_{j=1}^{p} I_j | i_j - t_{pj} |\}$$

and $Z_n = \max\{y_0, \cdots, y_{n-1}\}$ for every $n = 1, 2, \cdots$. Clearly,

$$Z_n \leqq y^* \leqq \max_{\mathbf{x} \in I} F'_n(\mathbf{x})$$

so that any admissible scheme must keep taking measurements until the difference $D_n = \max_{\mathbf{x} \in I} F'_n(\mathbf{x}) - Z_n$ decreases to zero. Thus the total number of measurements will be the smallest one if the difference D_n decreases at each step by as much as possible. Since the determination of the next point of measurement $\mathbf{t}_n$ must not depend on the yet unknown value y_n and $Z_{n+1} = \max\{Z_n, y_n\}$ this amounts to choosing $\mathbf{t}_n$ to achieve the largest possible decrease of $\max_{\mathbf{x} \in I} F'_{n+1}(\mathbf{x})$ only. Writing

$$F'_{n+1}(\mathbf{x}) = \min\{y_n + \textstyle\sum_{j=1}^{p} K_j | i_j - t_{nj} |, F'_n(\mathbf{x})\}$$

it is easy to see that $\mathbf{t}_n$ should be placed at the point where $F'_n(\mathbf{x})$ reaches its maximum. However this is exactly what our scheme does since the function F'_n is in fact identical to the auxiliary function F_n used in the algorithm.

Examples and Concluding Remarks

To illustrate the algorithm we present two simple examples. In the first one we consider the linear network from Figure 1 with the number of nodes $m = 30$, the maximum number K of units entering or leaving a node equal to 5, the initial flow $f(0) = 0$ and no crossflow through a node, i.e., $g^-(i) \leqq f(i-1)$, $i = 1, \cdots, m$. The data i, $g^+(i)$, $g^-(i)$, $f(i)$ together with the auxiliary functions $F_n(i)$ of the algorithm are shown in Table 1. The first measurement is taken at $t_0 = 1$. To break the ambiguity in determination of t_n resulting from possible multiple maxima of the nth auxiliary function an ad hoc rule is adopted to take always the leftmost maximum. The points t_n are underlined in the table. The numbers $(\boldsymbol{\theta}_n, Z_n)$ of the algorithm are indicated by circling the

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
g^+	3	4	2	2	1	5	0	3	2	0	4	1	4	0	5
g^-	0	1	5	3	0	2	5	2	1	3	0	4	0	0	2
f	3	6	3	2	3	6	1	2	3	0	4	1	5	5	8
F_1	3	8	13	18	23	28	33	38	43	48	53	58	63	68	73
F_2	3	8	13	18	23	28	33	38	43	48	53	58	63	68	73
F_3	3	8	13	18	23	28	33	38	43	40	35	30	25	20	15
F_4	3	8	13	18	23	20	15	10	5	0	5	10	15	20	15
F_5	3	8	13	18	23	20	15	10	5	0	5	10	15	20	15
F_6	3	8	13	8	3	8	13	10	5	0	5	10	15	20	15
F_7	3	8	13	8	3	8	13	10	5	0	5	10	10	5	10
F_8	3	8	13	8	3	8	13	10	5	0	5	10	10	5	10
F_9	3	8	13	8	3	8	13	10	5	0	5	10	10	5	10
F_{10}	3	8	13	8	3	8	13	10	5	0	5	10	10	5	10
F_{11}	3	8	13	8	3	8	13	10	5	0	5	10	10	5	10
F_{12}	3	8	13	8	3	8	13	10	5	0	5	10	10	5	10
F_{13}	3	8	3	8	3	8	13	10	5	0	5	10	10	5	10
F_{14}	3	8	3	8	3	8	1	6	5	0	5	10	10	5	10

i	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
g^+	3	2	0	4	0	0	0	2	2	1	4	1	3	5	0
g^-	1	0	1	4	5	5	1	0	4	0	1	0	3	4	2
f	10	12	11	11	6	1	0	2	0	1	4	5	5	6	4
F_1	78	83	88	93	98	103	108	113	118	123	128	133	138	143	148
F_2	74	69	64	59	54	49	44	39	34	29	24	19	14	9	4
F_3	10	15	20	25	30	35	40	39	34	29	24	19	14	9	4
F_4	10	15	20	25	30	35	40	39	34	29	24	19	14	9	4
F_5	10	15	20	15	10	5	0	5	10	15	20	19	15	9	4
F_6	10	15	20	15	10	5	0	5	10	15	20	19	14	9	4
F_7	10	15	20	15	10	5	0	5	10	15	20	19	14	9	4
F_8	10	15	11	15	10	5	0	5	10	15	20	19	14	9	4
F_9	10	15	11	15	10	5	0	5	10	9	4	9	14	9	4
F_{10}	10	12	11	15	10	5	0	5	10	9	4	9	14	9	4
F_{11}	10	12	11	11	10	5	0	5	10	9	4	9	14	9	4
F_{12}	10	12	11	11	10	5	0	5	10	9	4	9	5	9	4
F_{13}	10	12	11	11	10	5	0	5	10	9	4	9	5	9	4
F_{14}	10	12	11	11	10	5	0	5	10	9	4	9	5	9	4

TABLE 1

f(i,j)		j									
		1	2	3	4	5	6	7	8	9	10
i	1	2	3	2	3	-	-	-	-	-	-
	2	3	4	3	4	3	-	-	-	-	-
	3	4	5	4	3	4	3	4	-	-	-
	4	5	6	5	4	3	4	5	4	-	-
	5	4	5	4	3	4	5	6	-	-	-
	6	3	4	3	2	3	4	-	-	-	-
	7	2	3	4	3	4	-	-	-	-	-
	8	1	2	3	4	-	-	-	-	-	-
	9	2	1	2	-	-	-	-	-	-	-
	10	1	0	-	-	-	-	-	-	-	-

TABLE 2

corresponding entry of F_n. The single maximum $y^* = 12$ at $i^* = 17$ is found after 14 measurements, thus saving 16 measurements.

In the second example the algorithm is applied to an integer-valued function $f(i, j)$ of two discrete variables $i = 1, \cdots, 10$, $j = 1, \cdots, 10$, subject to two additional constraints $i + j \leqq 12$, $2j - 3i \leqq 6$ and satisfying the condition (C) with $K_1 = K_2 = 1$. The function f on its feasible domain is given in Table 2. Several of the auxiliary func-

F_1	j							
	2	3	4	5				
	3	4	5	6	7			
	4	5	6	7	8	9	10	
	5	6	7	8	9	10	11	12
	6	7	8	9	10	11	12	
i	7	8	9	10	11	12		
	8	9	10	11	12			
	9	10	11	12				
	10	11	12					
	11	12						

F_2	j							
	2	3	4	5				
	3	4	5	6	7			
	4	5	6	7	8	9	10	
	5	6	7	8	9	10	11	12
	6	5	6	7	8	9	10	
i	5	4	5	6	7	8		
	4	3	4	5	6			
	3	2	3	4				
	2	1	2					
	1	0						

F_3	j							
	2	3	4	5				
	3	4	5	6	7			
	4	5	6	7	8	7	6	
	5	6	7	8	7	6	5	4
	6	5	6	7	8	7	6	
i	5	4	5	6	7	8		
	4	3	4	5	6			
	3	2	3	4				
	2	1	2					
	1	0						

F_7	j							
	2	3	4	5				
	3	4	5	4	3			
	4	5	6	5	4	5	6	
	5	6	5	4	5	6	5	4
i	4	5	6	5	6	5	6	
	5	4	5	6	5	4		
	4	3	4	5	6			
	3	2	3	4				
	2	1	2					
	1	0						

F_8	j							
	2	3	4	5				
	3	4	5	4	3			
	4	5	6	5	4	5	6	
	5	6	5	4	5	6	5	4
i	4	5	6	5	6	5	6	
	5	4	5	6	5	4		
	4	3	4	5	6			
	3	2	3	4				
	2	1	2					
	1	0						

F_{10}	j							
	2	3	4	5				
	3	4	5	4	3			
	4	5	4	5	4	5	6	
	5	6	5	4	5	6	5	4
i	4	5	4	5	6	5	6	
	5	4	5	6	5	4		
	4	3	4	5	6			
	3	2	3	4				
	2	1	2					
	1	0						

F_{11}	j							
	2	3	4	5				
	3	4	5	4	3			
	4	5	4	5	4	5	6	
	5	6	5	4	5	6	5	4
	4	5	4	3	4	5	6	
i	5	4	3	2	3	4		
	4	3	4	3	4			
	3	2	3	4				
	2	1	2					
	1	0						

F_{13}	j							
	2	3	4	5				
	3	4	5	4	3			
	4	5	4	5	4	5	4	
	5	6	5	4	5	4	5	4
	4	5	4	3	4	5	6	
j	5	4	3	2	3	4		
	4	3	4	3	4			
	3	2	3	4				
	2	1	2					
	1	0						

F_{14}	j							
	2	3	4	5				
	3	4	5	4	3			
	4	5	4	5	4	5	4	
	5	6	5	4	5	4	5	4
	4	5	4	3	4	5	6	
i	5	4	3	2	3	4		
	4	3	4	3	4			
	3	2	3	4				
	2	1	2					
	1	0						

TABLE 3

tions F_n computed by the algorithm are shown in Table 3. The algorithm begins with the initial choice $\mathbf{t}_0 = (1, 1)$. As before, whenever the maximum of F_n occurs at more than one point always that one with the smallest i and j (in that order) is taken for $\mathbf{t}_n$. The maximum $y^* = 6$ is found after the eighth measurement at $\mathbf{x}^* = (4,2)$. This time, however, the maximum may also occur at eight other points—those where $F_8(\mathbf{x}) = 6$. Hence if all points of the maximum are to be located at most eight further measurements are needed. After the fourteenth measurement another $\mathbf{x}^* = (5,7)$ is found. At this instance it is seen that f cannot have the maximum at any other point since $F_{14}(\mathbf{x}) < 6$ anywhere else. Thus 36 out of the 50 possible measurements are saved.

In general the percentage of measurements saved may vary from zero (if $f =$ const.) to almost one hundred. To obtain some idea about the average number of measurements required we generated a sample of 500 functions $f(i)$; $i = 1, \cdots, 100$; $f(0) = 0$, the increments $\alpha_i = f(i) - f(i-1)$ of each being generated as independent, uniformly distributed random digits $0, \pm 1, \cdots, \pm 9$. The algorithm (with $K = 10$) was applied to each of these functions and the number of necessary measurements to find an $\mathbf{x}^*$ recorded. The average percentage of measurements saved was found to be 72.11% and the sample standard deviation of the percentage was 7.67%. This indicates a substantial savings in the number of measurements.

Of course, the percentage also depends on the constants K_j. Generally it could be said that on the average the percentage of measurements saved will decrease with decreasing degree of restriction placed upon the objective function by the condition (C). Hence it will decrease with increasing K_j's, at least for integer-valued functions.

Finally let us mention that the scheme can be easily extended to the case where the bounds K_j in (C) depend on $\mathbf{x}$. In the traffic problem discussed in the introduction this would correspond to various stations along the route having different limits on their capacities. These limits must, of course, be known.

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