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APPROXIMATE CALCULATION of INTEGRALS

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CHAPTER 10

Quadrature Formulas with Equal Coefficients

10.1. DETERMINING THE NODES

Quadrature formulas with equal coefficients

$$\int_a^b p(x) f(x)\, dx \approx c_n \sum_{k=1}^{n} f(x_k) \tag{10.1.1}$$

are very convenient for computations and in particular for graphical calculations. These formulas have been the subject of many investigations and in this chapter we will develop their theory.

Formula (10.1.1) contains the $n+1$ parameters $c_n, x_1, \ldots, x_n$ and we can choose these parameters so that the formula will be exact for all possible polynomials of degree $\leq n$.

The requirement that (10.1.1) be exact for $f(x) \equiv 1$ means that we must have

$$\int_a^b p(x)\, dx = nc_n$$

which determines the coefficient c_n:

$$c_n = \frac{1}{n} \int_a^b p(x)\, dx. \tag{10.1.2}$$

If we also require that (10.1.1) be exact for the monomials $f(x) = x$, $x^2, \ldots, x^n$ then we obtain the following system of equations for the nodes x_k:

$$x_1 + x_2 + \cdots + x_n = c_n^{-1} \int_a^b p(x)\, x\, dx$$

$$x_1^2 + x_2^2 + \cdots + x_n^2 = c_n^{-1} \int_a^b p(x)\, x^2\, dx$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (10.1.3)$$

$$x_1^n + x_2^n + \cdots + x_n^n = c_n^{-1} \int_a^b p(x)\, x^n\, dx$$

Let $\omega(x)$ be the polynomial of degree n which has the nodes $x_1, \ldots, x_n$ for its roots

$$\omega(x) = (x - x_1)(x - x_2)\cdots(x - x_n). \qquad (10.1.4)$$

Using equations (10.1.3) we can easily construct this polynomial. If we write $\omega(x)$ in the form

$$\omega(x) = x^n + A_1 x^{n-1} + A_2 x^{n-2} + \cdots + A_n \qquad (10.1.5)$$

then the coefficients $A_1, \ldots, A_n$ are the well-known elementary symmetric functions of the roots. On the other hand the left sides of equations (10.1.3) are the sums of powers of the roots:

$$s_k = x_1^k + x_2^k + \cdots + x_n^k \qquad (k = 1, 2, \ldots, n).$$

The right sides of (10.1.3) are the values of these functions for the polynomial (10.1.4).

In the theory of equations the relationship between the elementary symmetric functions A_i $(i = 1, \ldots, n)$ and the functions s_k $(k = 1, \ldots, n)$ is well known. This is given by the following equations which are often called *Newton's equations*[1]

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[1]The logarithmeric derivative of (10.1.4) leads to the following equation

$$\frac{\omega'(x)}{\omega(x)} = \sum_{i=1}^{n} \frac{1}{x - x_i}.$$

If $|x| > |x_i|$ then the fraction $\frac{1}{x - x_i}$ can be expanded in a power series in negative powers of x:

$$(x - x_i)^{-1} = \sum_{\nu=0}^{\infty} \frac{x_i^\nu}{x^{\nu+1}}.$$

Therefore if $|x| > |x_i|$ $(i = 1, \ldots, n)$ then the following expansion is valid:

$$\begin{aligned}
&s_1 + A_1 = 0\\
&s_2 + A_1 s_1 + 2A_2 = 0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (10.1.6)\\
&s_n + A_1 s_{n-1} + A_2 s_{n-2} + \cdots + nA_n = 0.
\end{aligned}$$

From these equations we can sequentially calculate the coefficients A_i $(i = 1, \ldots, n)$ from the values of s_k given by (10.1.3). From the A_i we can construct the polynomial $\omega(x)$ and calculating the roots of this polynomial gives the quadrature formula (10.1.1). If (10.1.1) is to be useful the x_k should all be real, distinct and should belong to the segment of integration. The possibility of constructing formula (10.1.1) which is exact for all polynomials of degree $\leq n$ is, therefore, determined by whether or not the roots of $\omega(x)$ satisfy the above requirements where the coefficients A_k in $\omega(x)$ are found from Newton's formulas.

We can construct another expression for the polynomial (10.1.4) by making use of a few results from the theory of analytic functions. Let us apply formula (10.1.1) to the fraction $f(x) = \dfrac{1}{z-x}$ which is the kernel of the Cauchy integral and consider the remainder:

$$R\left(\frac{1}{z-x}\right) = \int_a^b \frac{p(x)}{z-x}\,dx - c_n \sum_{k=1}^{n} \frac{1}{z-x_k} = \int_a^b \frac{p(x)}{z-x}\,dx - c_n \frac{\omega'(z)}{\omega(z)}.$$

We will find the expansion of the remainder in powers of z^{-1} for $|z|$ large. Let ρ be a number so large that the segment of integration $[a, b]$ and all of the nodes x_k lie in the circle $|z| \leq \rho$. Then for $|z| > \rho$

$$\frac{\omega'(x)}{\omega(x)} = \sum_{i=1}^{n} \sum_{\nu=0}^{\infty} \frac{x_i^{\nu}}{x^{\nu+1}} = \sum_{\nu=0}^{\infty} \frac{s_\nu}{x^{\nu+1}}$$

$$s_\nu = \sum_{i=1}^{n} x_i^{\nu}.$$

Multiplying both sides of this equation by $\omega(x)$ and replacing $\omega(x)$ by its representation (10.1.5) gives

$$nx^{n-1} + (n-1)A_1 x^{n-2} + (n-2)A_2 x^{n-3} + \cdots + A_{n-1} =$$

$$= (x^n + A_1 x^{n-1} + \cdots) \sum_{\nu=0}^{\infty} \frac{s_\nu}{x^{\nu+1}}.$$

Equating the coefficients of $x^{n-2}, x^{n-3}, \ldots,$ we then obtain Newton's equations.

$$\frac{1}{z-x} = \sum_{\nu=0}^{\infty} \frac{x^\nu}{z^{\nu+1}}$$

and

$$\int_a^b \frac{p(x)}{z-x}\,dx = \sum_{\nu=0}^{\infty} \frac{1}{z^{\nu+1}} \int_a^b p(x)\,x^\nu\,dx = \sum_{\nu=0}^{\infty} \frac{\mu_\nu}{z^{\nu+1}}.$$

Here μ_ν denotes the moment of order ν of the weight function $p(x)$. Similarly

$$\frac{1}{z-x_k} = \sum_{\nu=0}^{\infty} \frac{x_k^\nu}{z^{\nu+1}}$$

and

$$\frac{\omega'(z)}{\omega(z)} = \sum_{k=1}^{n} \frac{1}{z-x_k} = \sum_{\nu=0}^{\infty} \frac{s_\nu}{z^{\nu+1}},$$

$$R\left(\frac{1}{z-x}\right) = \sum_{\nu=0}^{\infty} \frac{\mu_\nu - c_n s_\nu}{z^{\nu+1}}. \tag{10.1.7}$$

Assuming that (10.1.1) is exact for the powers $x, x^2, \ldots, x^n$ then by (10.1.2) and (10.1.3) we have

$$\mu_\nu - c_n s_\nu = 0, \qquad \nu = 0, 1, \ldots, n$$

and the smallest exponent of $1/z$ in the last expansion will be $n+2$:

$$\int_a^b \frac{p(x)}{z-x}\,dx - c_n \frac{\omega'(z)}{\omega(z)} = \sum_{\nu=n+1}^{\infty} \frac{\mu_\nu - c_n s_\nu}{z^{\nu+1}}.$$

Integrating with respect to z and applying a simple transformation we obtain:

$$\omega(z)\exp\left(\sum_{\nu=n+1}^{\infty} \frac{s_\nu - c_n^{-1}\mu_\nu}{\nu z^\nu}\right) = A\exp\left(c_n^{-1}\int_a^b p(x)\ln(z-x)\,dx\right) \tag{10.1.8}$$

where A is a certain constant.

Since the expansion of $\exp\left(\sum_{\nu=n+1}^{\infty} \frac{s_\nu - c_n^{-1}\mu_\nu}{\nu z^\nu}\right)$ in powers of $1/z$ differs from unity by only powers of $1/z$ greater than n it is clear that the integer part of the expansion of the right side of (10.1.8) in powers of $1/z$ must coincide with $\omega(z)$ for large $|z|$:

$$\omega(z) = \text{integer part of } A \exp\left(c_n^{-1} \int_a^b p(x)\, ln\,(z - x)\, dx\right). \tag{10.1.9}$$

The constant A could be found by using the fact that the leading term of $\omega(z)$ is z^n. We will not need to calculate this factor since it does not affect the roots of the right side of (10.1.9).

As mentioned above the formula (10.1.1) which is exact for all polynomials of degree $\leq n$ is of interest only when the roots of $\omega(x)$ are real, distinct and lie inside the segment $[a, b]$[2]. The polynomial $\omega(x)$ is completely defined by the weight function $p(x)$ and we would like to know for what weight functions this polynomial has the properties we desire. The solution to this problem is not known in general. Below we discuss two weight functions for which the answer is known.

10.2. UNIQUENESS OF THE QUADRATURE FORMULAS OF THE HIGHEST ALGEBRAIC DEGREE OF PRECISION WITH EQUAL COEFFICIENTS

In Chapter 7 we discussed the quadrature formulas of the highest algebraic degree of precision for the weight function $p(x) = (1 - x^2)^{-\frac{1}{2}}$ on $[-1, 1]$. We obtained formula (7.3.2)

$$\int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}} f(x)\, dx \approx \frac{\pi}{n} \sum_{k=1}^{n} f\left(\cos \frac{2k - 1}{2n}\, \pi\right)$$

which is exact for all polynomials of degree $\leq 2n - 1$. In this formula the number of nodes n is arbitrary. It is remarkable that the coefficients in any one of these formulas are all equal.

We may ask whether these formulas are unique: does there exist on the segment $[-1, 1]$ another weight function $p(x)$ which is different from $(1 - x^2)^{-\frac{1}{2}}$ for which quadrature formulas of the highest algebraic degree of precision exist and which also have equal coefficients?

A negative answer to this question was first given by K. A. Posse and also later by N. Ia. Sonin.

Here we prove a more general theorem due to Ia. L. Geronimus from which the theorem of Posse easily follows.

Let us be given a weight function $p(x)$ which is almost everywhere positive on the segment $[-1, 1]$. Let us take the system of orthogonal polynomials $\omega_n(x) = x^n + \beta_n x^{n-1} + \gamma_n x^{n-2} + \cdots$ $(n = 0, 1, 2, \ldots)$ which correspond to this weight function. Let $x_k^{(n)}$ $(k = 1, \ldots, n)$ denote the

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[2]We assume that the integrand is only defined on $[a, b]$ and therefore do not consider formulas with nodes outside $[a, b]$.

roots of $\omega_n(x)$ and consider the quadrature formula with equal coefficients for which the nodes coincide with $x_k^{(n)}$:

$$\int_{-1}^{1} p(x) f(x)\, dx \approx c_n \sum_{k=1}^{n} f(x_k^{(n)}). \tag{10.2.1}$$

Theorem 1. *If for arbitrary values of $n = 1, 2, \ldots,$ there exists constants c_n such that formula (10.2.1) is exact for*[3] $f(x) = 1$, $f(x) = x$, $f(x) = x^2$, *then* $p(x)$ *coincides with the Chebyshev weight function* $(1 - x^2)^{-\frac{1}{2}}$.

Proof. Without loss of generality we can assume

$$\mu_0 = \int_{-1}^{1} p(x)\, dx = 1.$$

The requirement that the quadrature formula be exact for $f(x) = 1$ then determines the constant c_n:

$$\int_{-1}^{1} p(x)\, dx = nc_n, \qquad c_n = \frac{1}{n}.$$

Assuming in turn that $f(x) = x$ and $f(x) = x^2$ we obtain the following equations

$$\mu_1 = \int_{-1}^{1} p(x)\, x\, dx = \frac{1}{n} \sum_{k=1}^{n} x_k^{(n)} = -\frac{1}{n} \beta_n, \qquad n = 1, 2, \ldots$$

$$\mu_2 = \int_{-1}^{1} p(x)\, x^2\, dx = \frac{1}{n} \sum_{k=1}^{n} [x_k^{(n)}]^2 = \frac{1}{n} \left\{ \left[\sum_{k=1}^{n} x_k^{(n)} \right]^2 - 2 \sum_{j<k} x_j^{(n)} x_k^{(n)} \right\}$$

$$= \frac{1}{n} (\beta_n^2 - 2\gamma_n), \qquad n = 2, 3, \ldots$$

Thus we can find the first two coefficients of $\omega_n(x)$:

$$\beta_n = -n\mu_1, \qquad n = 1, 2, \ldots$$

$$\gamma_1 = 0$$

$$\gamma_n = \frac{1}{2} [\beta_n^2 - n\mu_2] = \frac{n}{2} [n\mu_1^2 - \mu_2], \qquad n = 2, 3, \ldots.$$

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[3]The requirement that (10.2.1) be exact for $f(x) = x^2$ is only necessary for $n > 1$.

In Section 2.1 we showed that there is a recursion relation between three consecutive polynomials of an orthogonal sequence. If we denote by $P_n(x)$ the orthonormal polynomials for the weight function $p(x)$ then the recursion relation is given by (2.1.10). The polynomial $\omega_n(x)$ differs by only a constant multiple from the corresponding orthonormal polynomial $P_n(x)$ of the same degree. Using the fact that the leading coefficient of $\omega_n(x)$ is unity then the recursion relation for $\omega_n(x)$ can be written in the form

$$x\omega_0(x) = \omega_1(x) + \alpha_0$$

$$x\omega_n(x) = \omega_{n+1}(x) + \alpha_n\omega_n(x) + \lambda_n\omega_{n-1}(x) \qquad n = 1, 2, \ldots$$

Knowing β_n and γ_n we can find the coefficients α_n and λ_n. Indeed, equating the coefficients of x^n on opposite sides of the last equation we find

$$\beta_n = \beta_{n+1} + \alpha_n$$

$$\alpha_n = \beta_n - \beta_{n+1} = -n\mu_1 + (n+1)\mu_1 = \mu_1.$$

All the α_n $(n = 0, 1, \ldots)$ have the same value which for simplicity we denote by α:

$$\alpha_n = \alpha \qquad (n = 0, 1, \ldots).$$

Equating the coefficients of x^{n-1} in the same way we obtain:

$$\gamma_n = \gamma_{n+1} + \alpha_n\beta_n + \lambda_n$$

$$\underline{\lambda_n} = \gamma_n - \gamma_{n+1} - \alpha_n\beta_n$$

Introducing the quantity σ we can write λ_n as:

$$\lambda_1 = \mu_2 - \mu_1 = \frac{\sigma^2}{2},$$

$$\lambda_n = \frac{1}{2}[\mu_2 - \mu_1] = \frac{\sigma^2}{4}, \qquad n = 2, 3, \ldots.$$

Thus the recursion relation for the polynomials $\omega_n(x)$ is

$$\omega_0(x) = 1, \qquad \omega_1(x) = x - \alpha$$

$$x\omega_n(x) = \omega_{n+1}(x) + \alpha\omega_n(x) + \frac{\sigma^2}{4}\omega_{n-1}(x), \qquad n = 1, 2, \ldots \tag{10.2.2}$$

We recall now (see Section 2.3) that the Chebyshev polynomial of the first kind $T_n(x) = \cos(n \arccos x) = 2^{n-1}x^n + \cdots$ has the recursion relation

$$T_0(x) = 1, \quad T_1(x) = x$$

$$x\,T_n(x) = \frac{1}{2} T_{n+1}(x) + \frac{1}{2} T_{n-1}(x).$$

If we reduce the leading coefficient of $T_n(x)$ to unity we obtain the polynomial $T_n^*(x) = 2^{-n+1}\,T_n(x)$, $T_0^*(x) = T_0(x)$. The recursion relation for $T_n^*(x)$ is

$$T_0^*(x) = 1, \quad T_1^*(x) = x$$

$$x\,T_n^*(x) = T_{n+1}^*(x) + \frac{1}{4} T_{n-1}^*(x).$$

Finally if the variable x is replaced by $\dfrac{x-\alpha}{\sigma}$ and we introduce the polynomials $T_n^+(x) = \sigma^n\,T_n^*\left(\dfrac{x-\alpha}{\sigma}\right)$ then for these polynomials we obtain the recursion relation

$$T_0^+(x) = 1, \quad T_1^+(x) = x - \alpha$$

$$(x-\alpha)\,T_n^+(x) = T_{n+1}^+(x) + \frac{\sigma^2}{4} T_{n-1}^+(x).$$

These coincide with (10.2.2) and because these equations completely determine $\omega_n(x)$ $(n = 0, 1, \ldots)$ then

$$\omega_n(x) = T_n^+(x) = \sigma^n\,T_n^*\left(\frac{x-\alpha}{\sigma}\right) = \frac{\sigma^n}{2^{n-1}}\,T_n\left(\frac{x-\alpha}{\sigma}\right) \qquad n = 1, 2, \ldots.$$

The roots of the polynomial $T_n(x)$ are $\cos \dfrac{2k-1}{2n}\pi$ $(k = 1, 2, \ldots, n)$. They lie inside the segment $[-1, 1]$ and as n increases they become dense in this segment. Hence it follows that the roots of $\omega_n(x)$ lie in the segment $[\alpha - \sigma, \alpha + \sigma]$ and these also become dense in this segment.

On the other hand we showed in Chapter 2 that the roots of polynomials of an arbitrary orthogonal system corresponding to a positive weight function lie inside the segment of orthogonality. From the theory of orthogonal polynomials it is also known that the roots of a sequence of orthogonal polynomials become dense in the segment of orthogonality.[4] Therefore the roots of the polynomials $\omega_n(x)$ belong to $[-1, 1]$ and form a dense set.

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[4] The more general theorem is known: *If the segment of orthogonality is* $[-1, 1]$ *and if the function* $p(x)$ *is summable and almost everywhere positive there, then the limiting distribution function of the zeros of the orthogonal polynomials coincides with the Chebyshev distribution function* $\mu(x) = \dfrac{1}{\pi}\displaystyle\int_{-1}^{x} \frac{dt}{\sqrt{1-t^2}}$.

We must therefore have $\alpha = 0$ and $\sigma = 1$ and

$$\omega_0(x) = T_0(x) = 1$$

$$\omega_n(x) = 2^{-n+1} T_n(x) \qquad (n = 1, 2, \ldots)$$

The polynomials $T_n(x)$ form an orthogonal system on $[-1, 1]$ with respect to the weight function $(1 - x^2)^{-\frac{1}{2}}$ and to complete the proof of the theorem there only remains to show that for a finite segment of integration and a given weight function the corresponding orthogonal polynomials are unique up to a constant multiple and up to their values on a set of points of measure zero.

Suppose that the $\omega_n(x)$ are orthogonal on $[-1, 1]$ with respect to both $p_1(x)$ and $p_2(x)$. If necessary we can multiply these weight functions by constants so that

$$\int_{-1}^{1} p_1(x)\, dx = \int_{-1}^{1} p_2(x)\, dx = 1.$$

By the orthogonality of $\omega_n(x)$ we must have

$$\int_{-1}^{1} p_1(x)\,\omega_n(x)\, dx = \int_{-1}^{1} p_2(x)\,\omega_n(x)\, dx = 0, \qquad (n = 1, 2, \ldots).$$

Thus the difference $\phi(x) = p_1(x) - p_2(x)$ must satisfy

$$\int_{-1}^{1} \phi(x)\,\omega_n(x)\, dx = 0, \qquad (n = 0, 1, 2, \ldots)$$

which is equivalent to

$$\int_{-1}^{1} \phi(x)\, x^n\, dx = 0 \qquad (n = 0, 1, 2, \ldots).$$

It is known[5] that the system of powers x^n $(n = 0, 1, 2, \ldots)$ is complete in L and thus from the last equation it follows that $\phi(x)$ is equivalent to zero.

10.3. INTEGRALS WITH A CONSTANT WEIGHT FUNCTION

In this section we turn our attention to the much investigated case of a constant weight function. Let us assume that the segment of integration has been transformed into $[-1, 1]$ and consider the quadrature formula

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[5] See, for example, I. P. Natanson, *Constructive Theory of Functions*, Gostekhizdat, Moscow, Chap. 3, Sec. 1 (Russian).

$$\int_{-1}^{1} f(x)\,dx \approx c_n \sum_{k=1}^{n} f(x_k). \tag{10.3.1}$$

The coefficient c_n and nodes x_k are to be chosen so that the formula is exact for all polynomials of degree $\leq n$. The coefficient c_n is determined from the requirement that (10.3.1) be exact for $f(x) \equiv 1$ and has the value

$$c_n = \frac{2}{n}.$$

Since

$$\int_{-1}^{1} x^k\,dx = \frac{1 - (-1)^{k+1}}{k+1}$$

the system of equations (10.3.1) which the nodes $x_1, \ldots, x_n$ must satisfy is:

$$\begin{aligned}
s_1 &= x_1 + x_2 + \cdots + x_n = 0\\
s_2 &= x_1^2 + x_2^2 + \cdots + x_n^2 = \frac{n}{3}\\
s_3 &= x_1^3 + x_2^3 + \cdots + x_n^3 = 0\\
s_4 &= x_1^4 + x_2^4 + \cdots + x_n^4 = \frac{n}{5}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
s_n &= x_1^n + x_2^n + \cdots + x_n^n = \frac{n}{2}\left[\frac{1 - (-1)^{n+1}}{n+1}\right]
\end{aligned} \tag{10.3.2}$$

The coefficients of the polynomial $\omega(x) = (x - x_1)\cdots(x - x_n)$ must be found from the system of equations (10.1.6) which is in this case:

$$\begin{aligned}
A_1 &= 0\\
\frac{n}{3} + 2A_2 &= 0\\
A_3 &= 0\\
\frac{n}{5} + \frac{n}{3}A_2 + 4A_4 &= 0\\
A_5 &= 0\\
\frac{n}{7} + \frac{n}{5}A_2 + \frac{n}{3}A_4 + 6A_6 &= 0\\
A_7 &= 0\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{10.3.3}$$

Here all the A_k with odd subscripts are zero and the polynomial $\omega(x)$ has the form

$$\omega(x) = x^n + A_2 x^{n-2} + A_4 x^{n-4} + \cdots .$$

The roots of $\omega(x)$ are the nodes of the formula (10.3.1) and they are symmetrically located on $[-1, 1]$ with respect to the point $x = 0$. If n is odd then one of the nodes coincides with $x = 0$.

It should be noted that if n is an even number $n = 2m$ then the x_k satisfy the equations

$$\begin{gathered} x_1 + x_2 + \cdots + x_n = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_1^n + x_2^n + \cdots + x_n^n = \frac{n}{n+1}. \end{gathered}$$

Since $n + 1 = 2m + 1$ is an odd number and since the x_k are symmetrically located with respect to $x = 0$ then the nodes will also satisfy

$$x_1^{n+1} + x_2^{n+1} + \cdots + x_n^{n+1} = 0.$$

In this case formula (10.3.1) will be exact for one higher degree, that is it will be exact for all polynomials of degree $\leq n + 1$.

We will now construct formula (10.3.1) for low values of n.

For $n = 1$ we have $\omega(x) = x$ and $c_1 = 2$

$$\int_{-1}^{1} f(x)dx \approx 2f(0).$$

For $n = 2$ the coefficient is $c_2 = 1$ and the system of equations for A_1, A_2 is

$$A_1 = 0$$

$$\frac{2}{3} + 2A_2 = 0.$$

Thus

$$\omega(x) = x^2 - \frac{1}{3}$$

$$x_1 = -\frac{\sqrt{3}}{3}, \qquad x_2 = \frac{\sqrt{3}}{3}$$

$$\int_{-1}^{1} f(x)\,dx \approx f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right).$$

For $n = 3$ we have $c_3 = \frac{2}{3}$ and

$$A_1 = 0$$

$$1 + 2A_2 = 0$$

$$A_3 = 0$$

$$\omega(x) = x^3 - \frac{1}{2}x$$

and the formula is then

$$\int_{-1}^{1} f(x)dx \approx \frac{2}{3}\left[f\left(\frac{-\sqrt{2}}{2}\right) + f(0) + f\left(\frac{\sqrt{2}}{2}\right)\right].$$

For $n = 4$ we have $c_4 = \frac{1}{2}$ and the following system of equations for the A_k:

$$A_1 = 0$$

$$\frac{4}{3} + 2A_2 = 0$$

$$A_3 = 0$$

$$\frac{4}{5} + \frac{4}{3}A_2 + 4A_4 = 0$$

Thus we obtain

$$A_2 = -\frac{2}{3}, \qquad A_4 = \frac{1}{45}$$

$$\omega(x) = x^4 - \frac{2}{3}x^2 + \frac{1}{45}$$

which has the roots

$$-x_1 = x_4 = \sqrt{\frac{5 + 2\sqrt{5}}{15}}$$

$$-x_2 = x_3 = \sqrt{\frac{5 - 2\sqrt{5}}{15}}.$$

In a similar way we obtain the following polynomials:

$$n = 5, \quad \omega(x) = x^5 - \frac{5}{6}x^3 + \frac{7}{72}x$$

$$n = 6, \quad \omega(x) = x^6 - x^4 + \frac{1}{5}x^2 - \frac{1}{105}$$

$$n = 7, \quad \omega(x) = x^7 - \frac{7}{6}x^5 + \frac{119}{360}x^3 - \frac{149}{6480}x$$

$$n = 9, \quad \omega(x) = x^9 - \frac{3}{2}x^7 + \frac{27}{40}x^5 - \frac{57}{560}x^3 + \frac{53}{22400}x.$$

For $n = 8$ two of the roots of $\omega(x)$ are complex and it is impossible to construct a Chebyshev formula (10.3.1) in this case with real roots. Here we tabulate the decimal values of the nodes in (10.3.1) for[6] $n = 1(1)7, 9$.

$n = 1$	
0.00000 00000	
$n = 2$	
0.57735 02691	
$n = 3$	
0.00000 00000	
0.70710 67812	
$n = 4$	
0.18759 24741	
0.79465 44723	
$n = 5$	
0.00000 00000	
0.37454 14096	
0.83249 74870	
$n = 6$	
0.26663 54015	
0.42251 86538	
0.86624 68181	
$n = 7$	
0.00000 00000	
0.32391 18105	
0.52965 67753	
0.88386 17008	
$n = 9$	
0.00000 00000	
0.16790 61842	
0.52876 17831	
0.60101 86554	
0.91158 93077	

We could also calculate the nodes for the Chebyshev formulas for $n > 9$ but in every case it turns out that some of the roots of $\omega(x)$ will be complex and it will be impossible to construct formula (10.3.1) with real nodes. The general question as to the existence of Chebyshev formulas for $n > 9$ with all real nodes remained unanswered until S. N. Bernstein proved that such formulas do not exist. The remainder of this chapter is devoted to a somewhat simplified presentation of his results.

We prove four preliminary lemmas.

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[6]These values are from the paper by Salzer, *J. Math. Phys.*, Vol. 26, 1947, pp. 191–194.

Lemma 1. *Let the formula*

$$\int_{-1}^{1} f(x)dx \approx \frac{2}{n}\sum_{k=1}^{n} f(x_k) \tag{10.3.4}$$

be exact for all polynomials of degree $\leq 2m - 1$ *where* $m < n$. *Let* ξ_m *denote the largest root of the* m^{th} *degree Legendre polynomial* $P_m(x)$. *Then, assuming that the* x_k *are enumerated in order of size:*

$$x_n > \xi_m.$$

Proof. Consider

$$f(x) = \frac{P_m^2(x)}{x - \xi_m}.$$

The function $P_m(x)/(x - \xi_m)$ is a polynomial of degree $m - 1$ and since $P_m(x)$ is orthogonal on $[-1, 1]$ to all polynomials of lower degree then

$$\int_{-1}^{1} f(x)dx = 0.$$

On the other hand $f(x)$ is a polynomial of degree $2m - 1$ and equation (10.3.4) must be exact for this function. Therefore

$$\sum_{k=1}^{n} f(x_k) = 0.$$

The polynomial $f(x) = P_m^2(x)/(x - \xi_m)$ has m distinct roots and therefore not all terms in the last sum can be zero. Thus this sum must contain positive and negative terms. But $f(x)$ takes on positive values only for $x > \xi_m$. Thus we can find a node x_k for which $x_k > \xi_m$ and hence the largest node must also be greater than ξ_m.

The following arguments are based on comparisons of (10.3.4) with Gauss quadrature formulas with m nodes

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{m} A_i f(\xi_i)$$

$$P_m(\xi_i) = 0 \qquad (i = 1, \ldots, m) \qquad A_i = \frac{2}{(1-\xi_i^2)[P_m'(\xi_i)]^2}. \tag{10.3.5}$$

Lemma 2. *If formula* (10.3.4) *is exact for all polynomials of degree* $\leq 2m - 1$ *where* $m < n$ *then*

$$A_m > \frac{2}{n}. \tag{10.3.6}$$

Proof. Let

$$f(x) = \left[\frac{P_m(x)}{(x - \xi_m)P'_m(\xi_m)}\right]^2.$$

Then $f(\xi_m) = 1$ and at the other ξ_i, $i < m$, $f(\xi_i) = 0$. Therefore for $f(x)$ the quadrature sum (10.3.5) becomes:

$$A_m f(\xi_m) = A_m.$$

The function $f(x)$ is a polynomial of degree $2m - 2$ and both (10.3.4) and (10.3.5) must be exact for this function. Therefore

$$\frac{2}{n}\sum_{k=1}^{n} f(x_k) = A_m.$$

Because $f(x) \geq 0$ for all x it follows that

$$\frac{2}{n} f(x_n) \leq A_m. \tag{10.3.7}$$

Writing

$$f(x) = [P'_m(\xi_m)]^{-2}(x - \xi_1)^2 \cdots (x - \xi_{m-1})^2$$

we see that, for $x \geq \xi_m$, $f(x)$ is an increasing function of x and since $x_n > \xi_m$ we have $f(x_n) > f(\xi_m) = 1$. Combining this with (10.3.7) proves the lemma.

In order to estimate the coefficient

$$A_m = \frac{2}{(1 - \xi_m^2)[P'_m(\xi_m)]^2}$$

in formula (10.3.5) we will obtain estimates for ξ_m and $P'_m(\xi_m)$.

Lemma 3. *For any value of m the largest root ξ_m of $P_m(x)$ satisfies the inequality*

$$1 - \xi_m < \frac{3}{m(m + 1)}. \tag{10.3.8}$$

Proof. We begin with the differential equation satisfied by $P_m(x)$:

$$\frac{d}{dx}[(1 - x^2)P'_m(x)] + m(m + 1)P_m(x) = 0.$$

Integrating both terms in this equation between the limits ξ_m and 1 we obtain

$$(1 - \xi_m^2)P'_m(\xi_m) = m(m + 1) \int_{\xi_m}^{1} P_m(x)dx.$$

Let us replace the polynomial $P_m(x)$ in this integral by its expansion in terms of powers of $x - \xi_m$

$$P_m(x) = \sum_{i=1}^{m} \frac{(x - \xi_m)^i}{i\,!} P_m^{(i)}(\xi_m).$$

Carrying out the integration gives

$$(1 - \xi_m^2)P'_m(\xi_m) = m(m + 1) \sum_{i=1}^{m} \frac{(1 - \xi_m)^{i+1}}{(i + 1)!} P_m^{(i)}(\xi_m)$$

Between each pair of adjacent roots ξ_j, ξ_{j+1} of the polynomial $P_m(x)$ there lies a root of $P'_m(x)$. There are $m - 1$ such roots of $P'_m(x)$ and no others. The $m - 2$ roots of the second derivative of $P_m(x)$ lie between adjacent roots of $P'_m(x)$ and so forth. Thus for any i all the roots of $P_m^{(i)}(x)$ lie in the interval $[\xi_1, \xi_m]$ and none of these roots are greater than ξ_m. Therefore $P_m^{(i)}(\xi_m) > 0$ and all terms on the right side of the last equation are positive. For a sufficiently precise estimate for ξ_m we can replace this sum by only its first two terms. Then dividing both sides by $1 - \xi_m$ we obtain the inequality

$$(1 + \xi_m)P'_m(\xi_m) > m(m + 1) \times$$

$$\times \left[\frac{1}{2}(1 - \xi_m)P'_m(\xi_m) + \frac{1}{6}(1 - \xi_m)^2 P''_m(\xi_m)\right].$$

The value of $P''_m(\xi_m)$ is easily found from the equation

$$(1 - x^2)P''_m(x) - 2xP'_m(x) + m(m + 1)P_m(x) = 0$$

by substituting $x = \xi_m$:

$$P''_m(\xi_m) = \frac{2\xi_m}{1 - \xi_m^2} P'_m(\xi_m). \tag{10.3.9}$$

Substituting this value in the inequality and cancelling the factor $P'_m(\xi_m)$ gives:

$$1 + \xi_m > m(m + 1)\left[\frac{1}{2}\ (1 - \xi_m) + \frac{1}{3}\ \frac{\xi_m(1 - \xi_m)}{1 + \xi_m}\right].$$

This inequality is made stronger if in the second term inside the brackets we replace $1 + \xi_m$ by the larger value 2:

$$1 + \xi_m > m(m + 1)\left[\frac{1}{2}(1 - \xi_m) + \frac{1}{6}\xi_m(1 - \xi_m)\right].$$

Setting $\lambda = m(m + 1)$ we can write this equation as

$$\lambda\xi_m^2 + 2(3 + \lambda)\xi_m + 6 - 3\lambda > 0. \tag{10.3.10}$$

Let us form the equations

$$\lambda z^2 + 2(3 + \lambda)z + 6 - 3\lambda = 0.$$

$$z = \frac{\pm\sqrt{4\lambda^2 + 9} - 3 - \lambda}{\lambda}.$$

If ξ_m satisfies the inequality (10.3.10) then ξ_m must be larger than the positive value of z:

$$\xi_m > \frac{\sqrt{4\lambda^2 + 9} - 3 - \lambda}{\lambda} > \frac{\sqrt{4\lambda^2} - 3 - \lambda}{\lambda}.$$

This gives

$$\xi_m > 1 - \frac{3}{\lambda} = 1 - \frac{3}{m(m + 1)}$$

$$1 - \xi_m < \frac{3}{m(m + 1)}.$$

This proves lemma 3.

Lemma 4. *The value of the derivative $P'_m(\xi_m)$ of the Legendre polynomial $P_m(x)$ at the largest root $x = \xi_m$ satisfies the inequality*

$$P'_m(\xi_m) > \frac{2}{3(1 - \xi_m)}\left[1 - \frac{\Gamma(m + 4)}{288\,\Gamma(m - 2)}(1 - \xi_m)^3\right]. \tag{10.3.11}$$

Proof. Making use of Taylor's series with two terms and the integral form of the remainder:

$$P_m(x) = P'_m(\xi_m)(x - \xi_m) + {}$$

$$+ \frac{1}{2}P''_m(\xi_m)(x - \xi_m)^2 + \frac{1}{2}\int_{\xi_m}^{x} P_m^{(3)}(t)(x - t)^2 dt.$$

For $x = 1$, using $P_m(1) = 1$, this becomes:

$$1 = P'_m(\xi_m)(1 - \xi_m) + \frac{1}{2}P''_m(\xi_m)(1 - \xi_m)^2 +$$

$$+ \frac{1}{2}\int_{\xi_m}^{1} P_m^{(3)}(t)(1 - t)^2 dt. \qquad (10.3.12)$$

Consider $P_m^{(3)}(t)$. In the proof of Lemma 3 we showed that all the roots of $P''_m(x)$ are less than ξ_m. Therefore $P_m^{(3)}(x)$ is a monotonically increasing function on $[\xi_m, 1]$ and its greatest value is achieved for $x = 1$. The value of $P_m^{(3)}(1)$ can be easily found using the differential equation

$$(1 - x^2)P''_m(x) - 2xP'_m(x) + m(m + 1)P_m(x) = 0.$$

Setting here $x = 1$ we find

$$P'_m(1) = \frac{m(m + 1)}{2}.$$

Differentiating gives

$$(1 - x^2)P_m^{(3)}(x) - 4xP''_m(x) + (m + 2)(m - 1)P'_m(x) = 0$$

and again setting $x = 1$ gives

$$P''_m(1) = \frac{(m + 2)(m - 1)}{4} P'_m(1) = \frac{(m + 2)(m + 1)m(m - 1)}{8}.$$

Differentiating once more

$$(1 - x^2)P_m^{(4)}(x) - 6xP_m^{(3)}(x) + (m + 3)(m - 2)P''_m(x) = 0$$

and substituting $x = 1$:

$$P_m^{(3)}(1) = \frac{(m + 3)(m - 2)}{6} P''_m(1) =$$

$$= \frac{(m + 3)(m + 2)\cdots(m - 2)}{48} = \frac{\Gamma(m + 4)}{48\Gamma(m - 2)}.$$

Substituting in (10.3.12) for $P''_m(\xi_m)$ its expression (10.3.9) and for $P_m^{(3)}(t)$ its upper bound on $[\xi_m, 1]$ leads to the inequality:

$$P'_m(\xi_m)(1 - \xi_m)\left[1 + \frac{\xi_m}{1 + \xi_m}\right] + \frac{\Gamma(m + 4)}{48\Gamma(m - 2)} \frac{(1 - \xi_m)^3}{3\,!} > 1.$$

This, together with $\dfrac{\xi_m}{1 + \xi_m} < \dfrac{1}{2}$, establishes (10.3.11).

We can now easily find an estimate for $A_m = \dfrac{2}{(1 - \xi_m^2)[P'_m(\xi_m)]^2}$.

$1 - \frac{1}{1+\xi_m} < \frac{1}{2}$

Substituting for $P'_m(\xi_m)$ its smaller value from (10.3.11)

$$A_m < \frac{9(1 - \xi_m)}{2(1 + \xi_m)} \left[1 - \frac{\Gamma(m + 4)}{288\Gamma(m - 2)}(1 - \xi_m)^3\right]^{-2}.$$

It will suffice to use a cruder inequality for A_m for $m \geq 6$. As m increases the value of ξ_m also increases and since $\xi_6 = 0.93246..$ we are justified in assuming $1 + \xi_m > 1.93$. We also replace $1 - \xi_m$ by the larger value $\frac{3}{m(m + 1)}$. We now estimate the value inside the brackets

$$(m + 3)(m - 2) = m(m + 1) - 6 < m(m + 1)$$

$$(m + 2)(m - 1) = m(m + 1) - 2 < m(m + 1)$$

$$\frac{\Gamma(m + 4)}{\Gamma(m - 2)} = (m + 3)(m + 2)(m + 1)m(m - 1)(m - 2) < m^3(m + 1)^3$$

$$1 - \frac{\Gamma(m + 4)}{288\,\Gamma(m - 2)}(1 - \xi_m)^3 > 1 - \frac{m^3(m + 1)^3}{288} \frac{3^3}{m^3(m + 1)^3} = \frac{29}{32}$$

$$A_m < \frac{27\,(32)^2}{2\,(1.93)\,(29)^2} \frac{1}{m\,(m + 1)} \approx \frac{8.517}{m(m + 1)}. \tag{10.3.13}$$

Theorem 2. *For $n \geq 10$ there is no formula* (10.3.4) *with all real roots which is exact for all polynomials of degree $\leq n$.*

Proof. Let us consider those values of n for which formula (10.3.4) exists. Let us suppose that n is an odd integer: $n = 2m - 1$. Then $m = \frac{1}{2}(n + 1)$ and $m(m + 1) = \frac{1}{4}(n + 1)(n + 3)$ and A_m must satisfy the inequality $A_m < \frac{4\,(8.517)}{(n + 1)(n + 3)}$. By Lemma 2 we must have

$$\frac{4\,(8.517)}{(n + 1)(n + 3)} > \frac{2}{n}$$

or

$$n^2 - (13.034)n + 3 < 0$$

$$n < 13.$$

Thus formula (10.3.4) does not exist for $n \geq 13$. But for $n = 11$ it also does not exist because then $m = 6$, $A_6 = 0.173\ldots$, $\frac{2}{11} = 0.1818\ldots$ and the inequality $\frac{2}{11} < A_6$ is not satisfied.

Suppose now that n is even. Then (10.3.4) must be exact for polynomials of degree $\leq n + 1$. Set $n + 1 = 2m - 1$, $m = \frac{1}{2}(n + 2)$. By (10.3.13) and (10.3.6) we must have

$$\frac{4\,(8.517)}{(n + 2)\,(n + 4)} > \frac{2}{n}$$

and hence

$$n < 11.$$

This means that for $n > 10$ formula (10.3.4) does not exist. For $n = 10$ it also does not exist because the inequality

$$A_6 = 0.173\ldots > \frac{2}{10} = 0.2$$

is clearly not valid.

REFERENCES

S. N. Bernstein, "On quadrature formulas of Cotes and Chebyshev," *Dokl. Akad. Nauk SSSR*, Vol. 14, 1937, pp. 323-26; Collected Works, Vol. 2, pp. 200–04 (Russian).

S. N. Bernstein, "On a system of indeterminate equations," Collected Works, Vol. 2, pp. 236–42 (Russian).

P. L. Chebyshev, "On Quadratures," *J. Math. Pures Appl.*, (2) Vol. 19, 1874, pp. 19–34; Collected Works, Vol. 3, Moscow, 1948, p. 49–62.

Ia. L. Geronimus, "On Gauss' and Chebyshev's quadrature formulas," *Dokl. Akad. Nauk SSSR*, Vol. 51, 1946, pp. 655–58 (Russian).

Ia. L. Geronimus, *Theory of Orthogonal Polynomials*, Gostekhizdat, Moscow, 1950.

V. I. Krylov, "Mechanical quadratures with equal coefficients for the integrals $\int_0^\infty e^{-x} f(x)dx$ and $\int_{-\infty}^\infty e^{-x^2} f(x)dx$," *Dokl. Akad. Nauk SSSR*, Vol. 2, 1958, pp. 187–92 (Russian).

R. O. Kuz′min, "On the distribution of roots of polynomials connected with quadratures of Chebyshev," *Izv. Akad. Nauk SSSR, Ser. Mat.*, Vol. 4, 1938, pp. 427–44 (Russian).

K. A. Posse, "Sur les quadratures," *Nouv. Ann. de Math.*, (2) Vol. 14, 1875, pp. 49–62.

H. E. Salzer, "Equally weighted quadrature formulas over semi-infinite and infinite intervals," *J. Math. Phys.*, Vol. 34, 1955, pp. 54–63.

N. Ia. Sonin, "On the approximate evaluation of definite integrals and on the related integral functions," *Warshawskia Universitetskia Izvestia*, Vol. 1, 1887, pp. 1–76 (Russian).

H. S. Wilf, "The possibility of Tschebycheff quadrature on infinite intervals," *Proc. Nat. Acad. Sci.*, Vol. 47, 1961, pp. 209–13.