On Minimizing a Convex Function Subject to Linear Inequalities

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SUMMARY

The minimization of a convex function of variables subject to linear inequalities is discussed briefly in general terms. Dantzig's Simplex Method is extended to yield finite algorithms for minimizing either a convex quadratic function or the sum of the t largest of a set of linear functions, and the solution of a generalization of the latter problem is indicated. In the last two sections a form of linear programming with random variables as coefficients is described, and shown to involve the minimization of a convex function.

1. *Introduction*

Linear programming has been studied extensively in the last few years, as indicated by Vajda (1955). Various authors have mentioned the possibility of relaxing the requirement of linearity, but the practical problems of non-linear programming do not seem to have been considered in any detail.* This paper is concerned with some aspects of the simplest form of non-linear programming—the minimization of a convex function of variables subject to linear inequalities. In principle this can always be done using the method of steepest descents, but this will rarely be practical in its primitive form. We therefore consider special methods for some important particular classes of such functions.

In Section 2 the Simplex Method, originally developed by Dantzig (1951) for linear programming, is outlined in terms sufficiently general to cover the applications to non-linear programming considered in the next two sections.

In Section 3 we show how to minimize a convex quadratic function. This enables one to use what amounts to the Newton-Raphson Method for minimizing a well-behaved general convex function: one finds a feasible solution of the constraints, and at each stage minimizes the quadratic function whose first and second derivatives at the feasible solution are the same as those of the given function.

In Section 4 we show how to minimize the sum of the t largest of a set of linear functions. (When $t = 1$ this is the standard problem in the theory of games). This algorithm, and its generalization to the minimization of the sum of the t_1 largest linear forms from one group plus the sum of the t_2 largest linear forms from another group and so on, provides a more efficient numerical method for certain applications of the theory of games than the straightforward reduction to linear programming.

In Section 5 a generalization of linear programming in which the coefficients in the constraints are random variables is presented, and shown to involve the minimization of a convex function of the variables.

In Section 6 a special case of this problem is considered in more detail.

2. *Outline of the Simplex Method*

Suppose that we wish to minimize some convex function† C of n variables x_j that must be non-negative and satisfy the $m(< n)$ linearly independent equations, or constraints,

$$\sum_{j=1}^{n} A_{ij} x_j = B_i, \text{ for } i = 1, \ldots, m. \qquad (2.1)$$

* Kuhn & Tucker (1952) have shown that under fairly general conditions the solution of such a problem is equivalent to the solution of some Two-Person Zero-Sum Infinite Game. However this result does not readily lead to numerical methods for either problem.

† A convex function is one that is never underestimated by linear interpolation, i.e. is such that $f(\lambda x + (1 - \lambda)y) \leqslant \lambda f(x) + (1 - \lambda) f(y)$ if $0 \leqslant \lambda \leqslant 1$, where x and y are vectors.

We can, in general, solve (2.1) for any m of the variables in terms of the others, obtaining

$$x_h = a_{h0} + \sum_{l=1}^{n-m} a_{hl}\, x_{j_{m+l}}, \text{ for } h = j_1, \quad . \; . \; . \quad , j_m. \qquad (2.2)$$

Our system can therefore be regarded as having $n - m$ independent variables, which determine m dependent variables. The dependent variables are known as basic and the independent variables as nonbasic, the term "independent" being inappropriate since we do not retain the same set of nonbasic variables throughout the work.

Now if C has a finite minimum, it will be attained when the $n - m$ nonbasic variables satisfy $n - m$ equations, each representing either a boundary to the region of feasible solutions* or else the condition that some (partial) derivative of C with respect to one of the nonbasic variables either vanishes or is discontinuous. The first type of equation takes the form $x_j = 0$. We may write the other type in the form $u_k = 0$, where the new variables u_k are defined in terms of the x_j but differ from them in that they are not restricted to non-negative values. We therefore call the x_j "restricted" variables and the u_k "free" variables.

Assume that we can specify in advance a finite set of restricted and free variables, denoted collectively by $z_1, z_2, \; . \; . \; . \;$, with which we have to deal. An iteration must then terminate if it always proceeds from one feasible solution where at least $n - m$ of the variables vanish to another such feasible solution giving a smaller value of C. We shall arrange that it terminates only at a local minimum of C; and this must be an absolute minimum if C is convex, since the region of feasible solutions is also convex.

As yet, no particular restrictions have been made on the form of the free variables. However the special techniques used in linear programming, and in this paper, seem to be applicable only when they are linear functions of the restricted variables. Then, corresponding to any feasible solution considered in the iteration, we take $n - m$ of the variables that vanish as nonbasic variables and use (2.1) and the definitions of the nonbasic free variables to express the other restricted variables as linear functions of the nonbasic variables. This leads to a generalization of (2.2) which can be written (possibly after renumbering the nonbasic variables) in the form

$$x_h = a_{h0} + \sum_{l=1}^{n-m} a_{hl}\, z_l, \text{ for } h = j_1, \quad . \; . \; . \quad , j_{m+s}, \dagger \qquad (2.3)$$

where s is the number of nonbasic free variables. If each $a_{h0} \geqslant 0$ we associate with (2.3) the feasible solution in which each z_l vanishes and each x_h equals a_{h0}. The set of nonbasic variables will be known as the nonbasic set.

The problem must first be expressed in the form (2.3) with non-negative a_{h0}, if necessary by adding artificial variables and using the M-method. We ensure that the a_{h0} are always positive, and not zero, by using the ε-perturbations (applied only to the restricted variables) introduced by Charnes (1952). Both these devices are described for example by Vajda (1955).

Using (2.3), we express C in terms of the nonbasic variables, and start the iteration from the associated feasible solution.

A general step of this iteration will now be described.

We see if C can be reduced by making a (small) change in some nonbasic free variable, or failing that by increasing some nonbasic restricted variable, keeping the other nonbasic variables equal to zero. Having found such a variable, say z_p, we may assume that it is to be increased—if it is a free variable to be decreased we change its sign throughout.

From (2.3) we see that we must keep

$$z_p \leqslant \frac{a_{q0}}{-a_{qp}} = \min_{a_{hp} < 0} \frac{a_{h0}}{-a_{hp}} = Z_p, \text{ say}, \qquad (2.4)$$

otherwise we would have x_q negative. If C decreases all the time as z_p increases from 0 to Z_p, we make x_q a nonbasic variable in place of z_p. Otherwise we define a new free variable u_r that

* A feasible solution is a set of non-negative x_j satisfying (2.1).

† Note that the a_{hl}, and similarly the c_{kl} introduced later, represent numerical coefficients that vary throughout the computation. In particular those in (2.3) may not be numerically equal to those in (2.2).

vanishes when C ceases to be a decreasing function of z_p, and make u_r nonbasic in place of z_p. The transformation of the equations due to the change of nonbasic set is described by Vajda (1955).

Of course if all the a_{hp} are non-negative, Z_p and x_q are undefined, and z_p can be increased indefinitely without making any restricted variable negative. Then if u_r can be found in the usual way it is taken as the new nonbasic variable. Otherwise z_p can profitably be increased indefinitely, and in our applications this decreases C indefinitely.

If Z_p is defined, the ε-perturbations ensure both that x_q is defined uniquely, and that x_q and u_r do not vanish simultaneously. For the value of z_p when $u_r = 0$ depends only on C and is independent of ε, whereas the values of z_p when the basic restricted variables vanish each contain different nonzero terms in ε, as proved by Charnes (1952).

If C cannot be decreased by changing any single nonbasic variable, we must verify that no other change can decrease it. This must be so if C is a differentiable function of the z_l. After this the problem is solved.

Note that it is not normally necessary to retain the equation defining a free variable once that variable has ceased to be nonbasic. The equations for the basic restricted variables are only retained to prevent these variables from surreptitiously becoming negative.

3. *Minimizing a Convex Quadratic Function*

We now consider the application of the above routine when C is a quadratic function. We represent C by a symmetric matrix (c_{kl}) for $k, l, = 0,1, \quad . \ . \ . \quad , n-m$, such that

$$C = \sum_{k=0}^{n-m} \sum_{l=0}^{n-m} c_{kl} z_k z_l, \qquad . \quad . \quad . \quad . \quad . \quad . \quad (3.1)$$

where $z_0 = 1$.

From (3.1),

$$\tfrac{1}{2} \frac{\partial C}{\partial z_k} = c_{k0} + \sum_{l=1}^{n-m} c_{kl} z_l,$$

so the variable z_p to be removed from the nonbasic set is found by examining the coefficients c_{k0}. We can reduce C by changing any nonbasic free variable z_k such that $c_{k0} \neq 0$, or by increasing any nonbasic restricted variable z_k such that $c_{k0} < 0$. If there is no nonbasic variable that can profitably be altered in this way, it follows from the fact that C is a differentiable function of the nonbasic variables, that no small change in these variables can reduce C, and we must then be at an absolute minimum of C if it is convex.

Further $\partial C / \partial z_p < 0$ as long as

$$u_r = c_{p0} + \sum_{l=1}^{n-m} c_{pl} z_l \leqslant 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.2)$$

There is therefore no difficulty in applying the Simplex Method as described in the previous section; but it remains to prove that the iteration must terminate, since there is no obvious limit to the number of free variables that may be involved.

Before proving this, we require two simple lemmas based on the form of the transformed matrix (c_{kl}), which will now be investigated.

If the equation defining the new nonbasic variable in terms of the old ones is

$$z_q = d_{q0} + \sum_{l=1}^{n-m} d_{ql} z_l, \qquad . \quad . \quad . \quad . \quad . \quad . \quad (3.3)$$

t is clear that $d_{qp} \neq 0$, since we have made z_q vanish by increasing z_p from zero. So we have

$$\begin{aligned} z_p &= -\frac{d_{q0}}{d_{qp}} + \frac{1}{d_{qp}} z_q - \sum_{l \neq p} \frac{d_{ql}}{d_{qp}} z_l \\ &= e_0 + e_q z_q + \sum_{l \neq p} e_l z_l, \text{ say.} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.4) \end{aligned}$$

Now, retaining the notation $z_0 = 1$, we have

$$\begin{aligned} C = {} & (c_{00} z_0 + \ \ldots \ + c_{0p} z_p + \ \ldots \ + c_{0,\, n-m} z_{n-m}) z_0 \\ & + \ \ldots \\ & + (c_{p0} z_0 + \ \ldots \ + c_{pp} z_p + \ \ldots \ + c_{p,\, n-m} z_{n-m}) z_p \\ & + \ \ldots \\ & + (c_{n-m,\, 0} z_0 + \ \ldots \ + c_{n-m,\, p} z_p + \ \ldots \ + c_{n-m,\, n-m} z_{n-m}) z_{n-m}. \end{aligned}$$

If we first use (3.4) to substitute for the z_p inside these brackets, we obtain

$$\begin{aligned} C = {} & (c'_{00} z_0 + \ \ldots \ + c'_{0q} z_q + \ \ldots \ + c'_{0,\, n-m} z_{n-m}) z_0 \\ & + \ \ldots \\ & + (c'_{p0} z_0 + \ \ldots \ + c'_{pq} z_q + \ \ldots \ + c'_{p,\, n-m} z_{n-m}) z_p \\ & + \ \ldots \\ & + (c_{n-m,\, 0} z_0 + \ \ldots \ + c'_{n-m,\, q} z_q + \ \ldots \ + c'_{n-m,\, n-m} z_{n-m}) z_{n-m}, \end{aligned}$$

where

$$\left.\begin{aligned} c'_{kq} &= c_{kp} e_q \\ c'_{kl} &= c_{kl} + c_{kp} e_l, \text{ for } l \neq q. \end{aligned}\right\} \qquad (3.5)$$

Thus the c'_{kl} are derived from the old c_{kl} in the same way as the transformed a_{kl} are derived from the old a_{hl}. We now substitute for the remaining z_p and obtain

$$\begin{aligned} C = {} & (c''_{00} z_0 + \ \ldots \ + c''_{0q} z_q + \ \ldots \ + c''_{0,\, n-m} z_{n-m}) z_0 \\ & + \ \ldots \\ & + (c''_{q0} z_0 + \ \ldots \ + c''_{qq} z_q + \ \ldots \ + c''_{q,\, n-m} z_{n-m}) z_q \\ & + \ \ldots \\ & + (c''_{n-m,\, 0} z_0 + \ \ldots \ + c''_{n-m,\, q} z_q + \ \ldots \ + c''_{n-m,\, n-m} z_{n-m}) z_{n-m}, \end{aligned}$$

where

$$\left.\begin{aligned} c''_{ql} &= c_{pl} e_q, \\ c''_{kl} &= c'_{kl} + c'_{pl} e_k \text{ for } k \neq q. \end{aligned}\right\} \qquad (3.6)$$

We see from (3.5) and (3.6) that

$$\left.\begin{aligned} c''_{qq} &= c_{pp} e_q^2, \\ c''_{ql} &= c_{pl} e_q + c_{pp} e_q e_l, \\ c''_{kq} &= c_{kp} e_q + c_{pp} e_k e_q, \\ c''_{kl} &= c_{kl} + c_{kp} e_l + c_{pl} e_k + c_{pp} e_k e_l, \end{aligned}\right\} \qquad (3.7)$$

where $k, l \neq q$.

It follows that if (c_{kl}) is symmetric so is (c''_{kl}), and the latter is therefore the transformed (c_{kl}). It is most easily computed from (3.5) and (3.6).

We now come to the two lemmas mentioned above. Both follow immediately from (3.7).

Lemma 1.

If z_q is a free variable, then (3.3) becomes (3.2) and

$$e_q = 1/c_{pp},\ e_l = -c_{pl}/c_{pp} \text{ for } l \neq q,$$

so that

$$c''_{ql} = c''_{kq} = 0, \text{ for } k, l \neq q.$$

Lemma 2.

If for some l, $c_{kl} = c_{lk} = 0$ for all $k \neq l$, and if z_q is a free variable, then $e_l = 0$, so that $c''_{kl} = c''_{lk} = 0$ for all $k \neq l$.

We now define a standard form for C as one containing no linear term in any free variable. When C is in standard form, its value in the associated solution, c_{00}, cannot be decreased keeping the nonbasic restricted variables equal to zero. Therefore, since C decreases at every step, it can never return to a standard form with the same set of restricted nonbasic variables, even with a different set of free nonbasic variables. There is only a finite number of possible sets of restricted nonbasic variables, so the iteration must terminate if it always reaches a standard form in a finite number of steps whenever it is not already in standard form. We now prove this.

Our rules of procedure ensure that when C is not in standard form, a free variable will be removed from the nonbasic set. Therefore s, the number of nonbasic free variables, cannot increase. Moreover if the new nonbasic variable is free, Lemma 1 shows that the new nondiagonal elements of (c_{kl}) in the row and column associated with the new nonbasic variable must all vanish. It follows that C does not then contain a linear term in this variable, and Lemma 2 shows that it can never contain one unless some other restricted variable becomes nonbasic, thereby decreasing s. Therefore if C is not in standard form and $s = s_0$ say, then s cannot increase, and must decrease after at most s_0 steps, unless C meanwhile achieves standard form. Since C is always in standard form when $s = 0$, the required result follows.

4. *Minimizing the Sum of the* t *Largest of a Set of Linear Forms*

We now consider the application of the Simplex Method when C is the sum of the t largest of a set of g linear forms $L_f(x_1, \ldots, x_n)$ for $f = 1, \ldots, g$.

This problem might arise in the Theory of Games, if the enemy has to choose t out of a set of g possible actions, and L_f represents his average gain through using the f^{th}. It could be solved by expressing the condition that the sum of every combination of t forms must not exceed some quantity u. Ordinary linear programming methods could then be used to minimize u. However this involves $\binom{g}{t}$ additional constraints, which might make the problem unwieldy if $t > 1$ and g is large. The method of section 2 applied to a nonlinear C is therefore offered as providing a more compact algorithm.

We first introduce some terminology. Corresponding to any set of values of the variables, we imagine the linear forms arranged in decreasing order of magnitude, so that

$$L_{(1)} \geqslant \ldots \geqslant L_{(\alpha_0)} > L_{(\alpha_0+1)} = \ldots = L_{(\beta_0)} > L_{(\beta_0+1)} \geqslant \ldots \geqslant L_{(g)}, \text{ say,}$$

where $0 \leqslant \alpha_0 < t \leqslant \beta_0 \leqslant g$. We call $L_{(1)}, \ldots, L_{(\alpha_0)}$ the major set, $L_{(\alpha_0+1)}, \ldots, L_{(\beta_0)}$ the borderline set, and $L_{(\beta_0+1)}, \ldots, L_{(g)}$ the minor set. We refer to members of these sets as major, borderline and minor forms respectively. The t largest will be the α_0 major forms together with any $(t - \alpha_0)$ of the borderline forms.

We first find a suitable nonbasic set, and express the linear forms in terms of the nonbasic variables, writing

$$L_f = c_{f0} + \sum_{l=1}^{n-m} c_{fl}\, z_l, \text{ for } f = 1, \ldots, g. \qquad (4.1)$$

For the j^{th} largest form we write

$$L_{(j)} = c_{(j)0} + \sum_{l=1}^{n-m} c_{(j)l}\, z_l.$$

In order to apply the Simplex Method, two problems must be considered:

(*a*) Is it profitable to change some nonbasic variable from its present value zero, keeping the others equal to zero? If so, by how much should it be changed, and what new variable, having the value zero at the new feasible solution, should replace it in the nonbasic set? And

(*b*) In what circumstances can we be sure that C attains its minimum value when all the nonbasic variables vanish?

We consider (*a*) first, and ask what happens if the nonbasic variable z_p is increased. We consider the division of the linear forms into the major, borderline and minor sets when z_p is some arbitrarily small positive quantity. Then (4.1) shows that the borderline set will be such that

$$c_{(\alpha_0+1)0} = \ldots = c_{(\beta_0)0} \text{ and } c_{(\alpha_0+1)p} = \ldots = c_{(\beta_0)p}.$$

Therefore if we define some borderline form, say L_v, as the critical form, we have

$$C = \sum_{j=1}^{t} L_{(j)} = (A_0 + \tau c_{v0}) + (A_p + \tau c_{vp})\, z_p,$$

where

$$A_0 = \sum_{j=1}^{\alpha_0} c_{(j)0},\ A_p = \sum_{j=1}^{\alpha_0} c_{(j)p}, \text{ and } \tau = t - \alpha_0.$$

It is therefore profitable to increase z_p if and only if

$$\gamma_0 = A_p + \tau c_{vp} < 0.$$

If this is so, we find Z_p and x_q from (2.4), define $\zeta_0 = 0$, and proceed as follows for $\rho = 0, 1, \ldots$.

If there is any limit to the amount to which z_p can be increased before some other form becomes equal to L_v, we denote it by $\zeta_{\rho+1}$. In view of (4.1),

$$\zeta_{\rho+1} = -\frac{c_{f\rho 0} - c_{v0}}{c_{f\rho p} - c_{vp}}, \quad . \quad . \quad . \quad . \quad . \quad (4.2)$$

where f_ρ is chosen from $1, \ldots, g$ so that $c_{f\rho p} \neq c_{vp}$, and so that $\zeta_{\rho+1} > \zeta_\rho$ but is otherwise as small as possible.

If $\zeta_{\rho+1}$ is either undefined or greater than Z_p, we make x_q the new nonbasic variable in place of z_p. However if $\zeta_{\rho+1} < Z_p$, we must consider whether or not it is profitable to increase z_p beyond $\zeta_{\rho+1}$, this being the first point at which $\partial C/\partial z_p$ may change. Let there be σ_1 possible choices of f_ρ satisfying (4.2) with $c_{f\rho p} < c_{vp}$, and σ_2 with $c_{f\rho p} > c_{vp}$. The borderline set for $z_p = \zeta_{\rho+1}$ is then augmented by the $\sigma_1 + \sigma_2$ corresponding forms, and consists of

$$L_{(\alpha_\rho - \sigma_1+1)}, \quad \ldots \quad, L_{(\beta_\rho+\sigma_2)}.$$

If these are arranged in decreasing order of c_{fp}, they will be in decreasing order of magnitude when z_p is just greater than $\zeta_{\rho+1}$. It is then profitable to increase z_p further if and only if

$$\gamma_{\rho+1} = \gamma_\rho + \sum_{j=\alpha_\rho-\sigma_1+1}^{t} c_{(j)p} - \sum_{j=\beta_\rho+\sigma_2-t+\alpha_\rho-\sigma_1+1}^{\beta_p+\sigma_2} c_{(j)p} < 0. \quad . \quad . \quad (4.3)$$

If this is so, the new borderline forms will be all those for which $c_{f0} = c_{(t)0}$ and $c_{fp} = c_{(t)p}$. If they are denoted by $L_{(\alpha_{\rho+1}+1)}, \ldots, L_{(\beta_{\rho+1})}$, this defines $\alpha_{\rho+1}$ and $\beta_{\rho+1}$. The process can now be repeated for the next value of ρ, using a new critical form from the new borderline set if and only if the old one is not in this set.

Since

$$c_{(j)p} = c_{vp} \text{ for } \alpha_\rho - \sigma_1 + \sigma_2 < j \leqslant \beta_\rho - \sigma_1 + \sigma_2,$$

(4.3) and the definitions of $\alpha_{\rho+1}$ and $\beta_{\rho+1}$ can be simplified in various ways depending on the relative values of $\alpha_\rho - \sigma_1 + \sigma_2$, t, and $\beta_\rho - \sigma_1 + \sigma_2$. The more important special cases are as follows:

$$\left.\begin{array}{l}
\text{If } \sigma_1 = 1,\ \sigma_2 = 0,\ \beta_\rho > t, \text{ then} \\
\qquad \gamma_{\rho+1} = \gamma_\rho - c_{f\rho p} + c_{vp},\ \alpha_{\rho+1} = \alpha_\rho - 1,\ \beta_{\rho+1} = \beta_\rho - 1. \\
\text{If } \sigma_1 = 1,\ \sigma_2 = 0,\ \beta_\rho = t, \text{ then} \\
\qquad \gamma_{\rho+1} = \gamma_\rho,\ \alpha_{\rho+1} = t - 1,\ \beta_{\rho+1} = t, \\
\text{and } L_{f\rho} \text{ must be taken as the new critical form.} \\
\text{If } \sigma_1 = 0,\ \sigma_2 = 1,\ \alpha_\rho < t - 1, \text{ then} \\
\qquad \gamma_{\rho+1} = \gamma_\rho + c_{f\rho p} - c_{vp},\ \dot{\alpha}_{\rho+1} = \alpha_{\rho+1},\ \beta_{\rho+1} = \beta_\rho + 1. \\
\text{If } \sigma_1 = 0,\ \sigma_2 = 1,\ \alpha_\rho = t - 1, \text{ then} \\
\qquad \gamma_{\rho+1} = \gamma_\rho + c_{f\rho p} - c_{vp},\ \alpha_{\rho+1} = t - 1,\ \beta_{\rho+1} = t,
\end{array}\right\} \quad (4.4)$$

and $L_{f\rho}$ must be taken as the new critical form.

If $\gamma_{\rho+1} \geqslant 0$, we take some f_ρ satisfying (4.3), define the free variable $u_{vf\rho}$ by

$$u_{vf\rho} = L_v - L_{f\rho},$$

and make this the new nonbasic variable in place of z_p.

We now turn to the problem (*b*) mentioned above. This is not entirely trivial, because although C can easily be shown to be convex, it is not everywhere differentiable. However the two parts of the following theorem virtually settle the matter.

Theorem 1 (*a*)

Suppose that

$$A = A_0 + \sum_l A_l z_l + \sum_{f=1}^{s} \varphi_f u_f,$$

$$L_0 = c_{00} + \sum_l c_{0l} z_l + \sum_{f=1}^{s} \theta_f u_f,$$

$$L_f = L_0 - u_f, \text{ for } f = 1, \quad \ldots \quad, s,$$

and that C equals A plus the sum of the τ largest of $L_0, L_1, \quad \ldots \quad, L_s$, where $\tau \leqslant s$.

If now all the u_f and (perhaps) some of the z_l are free variables, the other z_l being restricted to non negative values, then C is minimized when all the z_l and u_f vanish if and only if

$$\left.\begin{array}{l}
A_l + \tau c_{0l} \geqslant 0 \text{ for all } l, \\
A_l + \tau c_{0l} = 0 \text{ for all } l \text{ such that } z_l \text{ is free}, \\
0 \leqslant \varphi_f + \tau\theta_f \leqslant 1 \text{ for all } f, \\
\tau - 1 \leqslant \sum_{f=1}^{s} (\varphi_f + \tau\theta_f) \leqslant \tau.
\end{array}\right\} \quad . \quad . \quad . \quad (4.5)$$

and

If these conditions are not all satisfied, C can be decreased as follows:

If $A_l + \tau c_{0l} < 0$, increase z_l from zero.

If $A_l + \tau c_{0l} > 0$, decrease z_l from zero if it is free.

If $\varphi_f + \tau\theta_f < 0$, increase u_f from zero.

If $\varphi_f + \tau\theta_f > 1$, decrease u_f from zero.

If $\sum_{f=1}^{s} (\varphi_f + \tau\theta_f) < \tau - 1$, increase $u_1, \quad \ldots \quad, u_s$ equally from zero.

If $\sum_{f=1}^{s} (\varphi_f + \tau\theta_f) > \tau$, decrease $u_1, \quad \ldots \quad, u_s$ equally from zero.

Proof.—The second half of the theorem is obvious. It is also obvious that there is no essential asymmetry between L_0 and the other linear forms, since the possible suggested changes in the u_f all amount to keeping all but one of the forms equal and making the other either greater or smaller than them. We now prove the first half by assuming that we have a set of values for the variables, say z'_l, u'_f, in which at least τ of $L_1, \ldots, L_s$ are not less than L_0, and for which $C < A_0 + \tau . c_{00}$, and show that this implies that (4.5) is not satisfied.

We may assume without loss of generality that

$$u'_1 \leqslant u'_2 \leqslant \ldots \leqslant u'_s.$$

Then

$$C = A_0 + \tau c_{00} + \sum_l (A_l + \tau c_{0l}) z'_l + \sum_{f=1}^{\tau} (\varphi_f + \tau\theta_f - 1) u'_f + \sum_{f=\tau+1}^{s} (\varphi_f + \tau\,\theta_f) u'_f$$

$$= A_0 + \tau c_{00} + \sum_l (A_l + \tau c_l) z'_l + \sum_{f=1}^{\tau} (\varphi_f + \tau\theta_f - 1)(u'_f - u'_\tau)$$

$$+ \sum_{f=\tau+1}^{s} (\varphi_f + \tau\theta_f)(u'_f - u'_\tau) + \{ \sum_{f=1}^{s} (\varphi_f + \tau\theta_f) - \tau \} u'_\tau.$$

Now, by hypothesis, $u'_f - u'_\tau \leqslant 0$ for $f \leqslant \tau$, and $u'_f - u'_\tau \geqslant 0$ for $f > \tau$. Further $u'_\tau \leqslant 0$, since L_0 is not one of the τ largest forms. Therefore C can only be less than $A_0 + \tau c_{00}$ if either $A_l + \tau\, c_{0l}$ has the opposite sign to z'_l for some l,

or $\varphi_f + \tau\theta_f - 1 > 0$ for some $f \leqslant \tau$,

or $\varphi_f + \tau\theta_f < 0$ for some $f > \tau$,

or $\sum_{f=1}^{s} (\varphi_f + \tau\theta_f) - \tau > 0$.

All these possibilities violate (4.5), so the theorem is proved.

Theorem 1 (*b*)

Given the same situation as that considered in Theorem 1 (*a*) except that $\tau = s + 1$, then C is minimized when all the z_l and u_f vanish if and only if (4.5) is satisfied and in addition

$$\varphi_f + \tau\theta_f - 1 = 0, \text{ for all } f.$$

Otherwise C can be decreased by giving u_f some value with the opposite sign to this quantity. The proof is trivial.

Now let A represent the sum of the α_0 major forms, L_0 the critical form, and $L_1, \ldots, L_s$ the forms introduced into the borderline set when the free variables, here denoted by $u_1, \ldots, u_s$, representing the differences between these forms and the critical form, were introduced into the nonbasic set. Theorem 1 then shows that if, as is normally the case, there are no other borderline forms, and if C cannot be reduced by changing one of the nonbasic variables in the usual way, then C must be already minimized unless the critical form can profitably be altered leaving the other borderline forms all equal. To do this one must of course take a new critical form. In the notation of Theorem 1, if L_1 is selected we replace the u_f by new nonbasic free variables u'_f representing the differences between L_1 and the L_f, defined by

$$u'_0 = -u_1,\ u'_f = u_f - u_1, \text{ for } f = 2, \ldots, s,$$

and try the effect of changing u'_0 in the usual way.

Now it is clear from (4.4) that if $\sigma_1 + \sigma_2 = 1$, then whenever one has to change the critical form the new one will always be the only borderline form, and further the free variable representing the difference between the critical form and any new borderline form will always be introduced into the nonbasic set. It follows that there can be no borderline forms other than those of the type considered in Theorem 1 if we start with a unique t^{th} largest form and never have $\sigma_1 + \sigma_2 > 1$. Therefore the existence of such other borderline forms must represent a numerical accident. This is called degeneracy in linear programming terminology, and can be overcome in the usual way, for instance by adding ε^j to the j^{th} such form, making it formally a major form. For hand computation at any rate, it seems sensible to introduce these ε's only when they are needed, and to drop them as soon as a genuinely different feasible solution has been found. However one could introduce them at the outset, thereby avoiding any complications due to having $\sigma_1 + \sigma_2 > 1$.

Finally we note that if we have a more general problem, with several groups of linear forms such that we wish to minimize the sum of the t_1 largest from the first group plus the sum of the t_2 largest from the second group, and so on, we can use the algorithm just described, generalized in the obvious way. In each group we define major, borderline and minor sets, and select a critical form. When increasing z_p, we must stop to consider the situation when any form becomes equal to the critical form in its own group. And so on, the generalization of Theorem 1 being cumbersome to state but quite easy to prove. This generalized routine may prove useful in the numerical study of problems in the Theory of Games in which the enemy's strategies can be suitably subdivided into independent factors.

5. *Linear Programming with Random Coefficients**

We now turn to a possible source of problems of the general type considered above.

Any linear programming problem can be expressed in the following form:

Minimize $C = \sum_{j=1}^{n} c_j x_j$ for non-negative x_j satisfying (2.1).

In this section we discuss some problems that arise when the coefficients in (2.1) are not known precisely but are random variables with known distributions. It is then hardly reasonable to require that (2.1) be satisfied exactly; but some loss, which can be incorporated in C, can perhaps be associated with any discrepancy between the left- and right-hand sides of (2.1) in the following way:

Given constants c_j, f_k, d_{ik}, choose non-negative x_j to minimize the mean value $E(C)$ of

$$C = \sum_{j=1}^{n} c_j x_j + \sum_{k=1}^{p} f_k y_k, \quad . \quad . \quad . \quad . \quad . \quad (5.1)$$

where the y_k are non-negative and are chosen to minimize C subject to the equations

$$\sum_{j=1}^{n} \alpha_{ij} x_j + \sum_{k=1}^{p} d_{ik} y_k = \beta_i, \text{ for } i = 1, \quad . \quad . \quad . \quad , m, \quad . \quad . \quad . \quad (5.2)$$

where the α_{ij} and β_i are random variables whose distributions are known when the x_j are chosen and whose actual values are known when the y_k are chosen.

This can be illustrated with reference to the transportation problem, first formulated by Hitchcock (1941). Suppose we have a quantity t_i of material at source S_i ($i = 1, \ . \ . \ . \ , m$) and wish to supply a quantity b_j to destination D_j ($j = 1, \ . \ . \ . \ , n$). If it costs c_{ij} to transport each unit from S_i to D_j, what is the cheapest way of supplying all the destinations?

If the c_{ij} are random variables, the mean cost for any given scheme is obtained by replacing each c_{ij} by its mean value, so the problem is not essentially altered.

However it is altered if for example the requirements b_j are replaced by random variables β_j. If we assume that there is a loss of f_j for each unit by which the supply at D_j falls short of the demand, and a loss of f_{-j} for each unit by which it exceeds the demand, and if x_{ij} denotes the amount sent from S_i to D_j, the problem is to minimize $E(C)$ where

$$C = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{j=1}^{n} f_j y_j + \sum_{j=1}^{n} f_{-j} y_{-j},$$

* Note added in proof: Ideas very similar to those of this section and some of the next have been put forward by Dantzig (1955) in Report P-596 of the Rand Corporation, Santa Monica, California, entitled "Linear Programming Under Uncertainty."

and (5.2) becomes

$$\sum_{i=1}^{m} x_{ij} + y_j - y_{-j} = \beta_j, \text{ for } j = 1, \ldots, n,$$

and

$$\sum_{j=1}^{n} x_{ij} + x_{i0} = t_i, \text{ for } i = 1, \ldots, m.$$

For theoretical purposes it is convenient to rewrite (5.1) and (5.2) in matrix notation as

$$C = c'x + f'y, \qquad (5.3)$$

and

$$Ax + Dy = \beta. \qquad (5.4)$$

Now this problem can be regarded as a generalization of linear programming. However it can also be regarded as a special case of linear programming if the random variables have discrete distributions; for if there is a probability p_r that $A = A_r$ and $\beta = \beta_r$, the problem is to choose nonnegative x and y_r to minimize

$$E(C) = c'x + \sum_r p_r f' y_r, \qquad (5.5)$$

where

$$A_r x + D y_r = \beta_r, \qquad (5.6)$$

for all r.

The special features are the form of (5.6), and the fact that we are not directly interested in the values of any of the variables except x.

Two general results can easily be proved.

Theorem 2. $E(C)$ is a convex function of x.

Proof.—Consider any fixed values of the random variables A and β, and denote the value of y that minimizes C for any given value of x by $y(x)$, and the corresponding minimum value of C by $C(x)$.

Then if x_1 and x_2 are two values of x, (5.3) and (5.4) are satisfied by

$$x = x_\tau,\ y = y(x_\tau),\ C = C(x_\tau) \text{ for } \tau = 1 \text{ or } 2.$$

Since they are linear equations they are also satisfied by

$$x = \lambda_1 x_1 + \lambda_2 x_2,\ y = \lambda_1 y(x_1) + \lambda_2 y(x_2),\ C = \lambda_1 C(x_1) + \lambda_2 C(x_2),$$

if $\lambda_1 + \lambda_2 = 1$. Further if λ_1 and λ_2 are both non-negative, then these x and y are non-negative as long as x_1 and x_2 are non-negative. Therefore if $x = \lambda_1 x_1 + \lambda_2 x_2$ we can find a value of y satisfying the required conditions and making $C = \lambda_1 C(x_1) + \lambda_2 C(x_2)$, and we may be able to find a y giving an even smaller value of C. So

$$C(\lambda_1 x_1 + \lambda_2 x_2) \leqslant \lambda_1 C(x_1) + \lambda_2 C(x_2),$$

for any fixed values of A and β. The theorem follows on integration over all values of these variables.

Theorem 3. Given x, C is a convex function of A and β.

The proof closely resembles that of Theorem 2 and is omitted.

This theorem shows that, for any x, $E(C)$ cannot decrease if a random component with zero mean is added to any coefficient. It follows that if $E(C)$ is now regarded as a function of x, its minimum value cannot be decreased by the introduction of this random component.

6. *A Special Case*

The problem can be considerably simplified in an important special case, namely when we have a set of linear equations of the form (2.1), with known left-hand sides but unknown right-hand sides, and for each equation there is a loss for each unit by which the left-hand side exceeds

the right-hand side, and a (possibly different) loss for each unit by which the left-hand side falls short of the right-hand side. This can be called the case of independent equations with known left-hand sides, where the term independent refers to the losses, and not to the probability distributions of the coefficients, whose correlations do not affect the issue. The problem may be written as follows:
Minimize the mean value of

$$C = \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} f_i y_i + \sum_{i=1}^{m} f_{-i} y_{-i},$$

where

$$\sum_{j=1}^{n} A_{ij} x_j + y_i - y_{-i} = \beta_i, \text{ for } i = 1, \ldots, m. \qquad (6.1)$$

However the y_i can easily be eliminated from the problem, which then becomes:
Minimize

$$\left.\begin{aligned} E(C) &= \sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} H_i(u_i), \\ \text{where} & \\ \sum_{j=1}^{n} A_{ij} x_j &= u_i, \text{ for } i = 1, \ldots, m. \end{aligned}\right\} \qquad (6.2)$$

The function $H_i(u_i)$ depends on the distribution of β_i and will now be studied. Writing (6.1) in the form

$$y_i - y_{-i} = \beta_i - u_i,$$

we see that one solution is

$$\begin{aligned} y_i &= \beta_i - u_i, \; y_{-i} = 0, && \text{if } \beta_i - u_i \geqslant 0, \\ y_i &= 0, \qquad y_{-i} = u_i - \beta_i, && \text{if } \beta_i - u_i \leqslant 0. \end{aligned}$$

This is the solution envisaged in our verbal formulation of the problem, and it minimizes C unless $f_i + f_{-i} = g_i$, say, is negative. We therefore assume that $g_i \geqslant 0$. If $F_i(\beta_i)$ denotes the cumulative distribution function of β_i we have

$$H_i(u_i) = E(f_i y_i + f_{-i} y_{-i}) = f_{-i} \int_{-\infty}^{u_i} (u_i - \beta_i)\, dF_i(\beta_i) + f_i \int_{u_i}^{\infty} (\beta_i - u_i)\, dF_i(\beta_i),$$

which reduces to $f_i(\mu_i - u_i) + g_i u_i F_i(u_i) - g_i \int_{-\infty}^{u_i} \beta_i \, dF_i(\beta_i)$, where μ_i denotes the mean value of β_i.

Hence

$$\frac{d}{du_i} H_i(u_i) = -f_i + g_i F_i(u_i). \qquad (6.3)$$

Unfortunately, even in this special case, there seems to be no simple general algorithm providing a complete solution of the problem. However it is usually quite easy to improve upon the approximation obtained by replacing each β_i in (6.1) by its mean value μ_i. We define new free variables v_i by $u_i = \mu_i + v_i$.

Keeping these variables as nonbasic and equal to zero, we use ordinary linear programming methods to minimize $\Sigma\, c_j x_j$ for non-negative x_j satisfying

$$\sum_{j=1}^{n} A_{ij} x_j - v_i = \mu_i.$$

When this is done, we have the approximate solution obtained by putting $\beta_i = \mu_i$, expressed in the form

$$x_h = a_{h0} + \sum_{l=1}^{n-m} a_{hl} x_{j_{m+l}} + \sum_{i=1}^{m} a_{h,-i} v_i, \text{ for } h = j_1, \ldots, j_m,$$

where

$$E(C) = c_0 + \sum_{l=1}^{n-m} c_l x_{j_{m+l}} + \sum_{i=1}^{m} \{c_{-i} v_i + H_i(\mu_i + v_i)\},$$

and each $c_l \geqslant 0$ for $l = 1, \ldots, n - m$.

If we temporarily ignore the restriction of the basic variables to non-negative values, we see that $E(C)$ is minimized for non-negative nonbasic restricted variables when these all vanish and each $v_i = F^{-1}\left(\frac{f_i - c_{-i}}{g_i}\right) - \mu_i$, where $F^{-1}(x)$ is conventionally interpreted as $-\infty$ for $x < 0$ and $+\infty$ for $x > 1$.

Sometimes, particularly if the β_i all have small variances, this will provide the complete solution to the original problem. It will nearly always be possible to go some way towards giving the v_i these values. However we may be unable to give some v_i, say v_k, its indicated value without making some basic variable negative. If we make this variable nonbasic in place of v_k, the expression for $E(C)$ in terms of the new nonbasic variables may be very complicated, and there seems to be no really satisfactory way of completing the process. Perhaps the best procedure is to make the indicated change in the nonbasic set, and to use the Newton-Raphsŏn method for minimizing a general convex function as suggested in section 1.

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