Preliminaries:

- Inverse temperature: $\beta = 1/k_B T$.
- Canonical ensemble: $\rho(E_n) = e^{-\beta E_n}/Z$.
- Partition function for the canonical ensemble: $Z = \sum_n e^{-\beta E_n}$.

1. Calculate the partition function for a quantum harmonic oscillator with energy levels $E_n = \hbar\omega(n + \frac{1}{2})$.
   (i) Find the average energy E and entropy S as a function of temperature T.
   (ii) A simple model of a solid, due to Einstein, treats the atoms vibrating as a set of N harmonic oscillators. Calculate the heat capacity for $K_B T \gg \hbar\omega$ and $T \to 0$. Compare this to the experimentally known results, for $K_B T \gg \hbar\omega$ one finds $C_V = 3Nk_B$ and for $T \to 0$ one finds $C_V \sim T^3$.

2. We can find a unified way of thinking about various ensembles. Let's start form the Gibbs formula for entropy:
$$S = -k_B \sum_n p(n) \ln p(n)\,.$$
   We can find the microcanonical and canonical ensembles by maximising this with respect to different constraints for $p(n)$.
   (i) Find the microcanonical ensemble for $p(n)$ by adding a constraint that $\sum_n p(n) = 1$ and only states with energy E have non-zero $p(n)$. (To maximise with this constraint use a Lagrange multiplier.)
   (ii) For the canonical ensemble we must add the constraint that the average energy is fixed $\langle E \rangle = \sum_n p(n) E_n$.
   (iii) What happens if we add the constraint that the average energy is fixed $\langle E \rangle = \sum_n p(n) E_n$ and the average particle number is fixed $\langle N \rangle = \sum_n p(n) N_n$?