

Summary of Calculus I

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1 Limits

- **Tangent:** A tangent line is a line that touches a curve, and has the same slope as the curve at the point of contact.
- **Average rate of change:** The average rate of change of a function $f(x)$ between $x = a$ and $x = b$ is

$$\frac{f(b) - f(a)}{b - a}.$$

- **Limit:** Let f be a function defined on some open interval that contains a but not necessarily at a itself. Then we say the limit of $f(x)$ as x approaches a is L and we write

$$\lim_{x \rightarrow a} f(x) = L$$

if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that if

$$0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon.$$

- **Left-hand limit:**

$$\lim_{x \rightarrow a^-} f(x) = L,$$

if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that if

$$a - \delta < x < a \text{ then } |f(x) - L| < \epsilon.$$

- **Right-hand limit:**

$$\lim_{x \rightarrow a^+} f(x) = L,$$

if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that if

$$a < x < a + \delta \text{ then } |f(x) - L| < \epsilon.$$

- **Theorem 1:**

$$\lim_{x \rightarrow a} f(x) = L \text{ if and only if } \lim_{x \rightarrow a^-} f(x) = L \text{ and } \lim_{x \rightarrow a^+} f(x) = L.$$

- **Infinite limits:** Let f be a function defined on some open interval that contains a but not necessarily at a itself. Then

$$\lim_{x \rightarrow a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that if

$$0 < |x - a| < \delta \text{ then } f(x) > M.$$

Let f be a function defined on some open interval that contains a but not necessarily at a itself. Then

$$\lim_{x \rightarrow a} f(x) = -\infty$$

means that for every negative number N there is a positive number δ such that if

$$0 < |x - a| < \delta \text{ then } f(x) < N.$$

- **Vertical asymptote:** The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following is true:

$$\begin{array}{ll} \lim_{x \rightarrow a} f(x) = \infty, & \lim_{x \rightarrow a} f(x) = -\infty, \\ \lim_{x \rightarrow a^-} f(x) = \infty, & \lim_{x \rightarrow a^-} f(x) = -\infty, \\ \lim_{x \rightarrow a^+} f(x) = \infty, \text{ or } & \lim_{x \rightarrow a^+} f(x) = -\infty. \end{array}$$

- **Limit laws:** Suppose that c is a constant and that the limits

$$\lim_{x \rightarrow a} f(x) \text{ and } \lim_{x \rightarrow a} g(x)$$

exist. Then:

1. **Sum law** $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2. **Difference law** $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
3. **Constant multiple law** $\lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$
4. **Product law** $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
5. **Quotient law** $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ if $\lim_{x \rightarrow a} g(x) \neq 0$
6. **Power law** $\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n$ where n is a positive integer
7. $\lim_{x \rightarrow a} c = c$
8. $\lim_{x \rightarrow a} x = a$
9. $\lim_{x \rightarrow a} x^n = a^n$ where n is a positive integer

10. $\lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer and if n is even then we assume that $a > 0$
11. **Root law** $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$ where n is a positive integer and if n is even then we assume that $\lim_{x \rightarrow a} f(x) > 0$

- **Direct substitution property:** If f is a polynomial or a rational function and a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a).$$

Such functions are called continuous at a .

- **Theorem 2:** If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of both f and g exist as x approaches a then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x).$$

- **Theorem 3. The squeeze theorem:** If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L.$$

- **Continuous function:** A function f is said to be continuous at a if $\lim_{x \rightarrow a} f(x) = f(a)$.
 - A function f is continuous from the right at a number a if $\lim_{x \rightarrow a^+} f(x) = f(a)$.
 - A function f is continuous from the left at a number a if $\lim_{x \rightarrow a^-} f(x) = f(a)$.
 - A function f is said to be continuous on an interval if it is continuous at every number in the interval.
- **Theorem 4:** If f and g are continuous at a and c is a constant, then the following functions are also continuous at a :
 - 1.1. $f + g$,
 - 1.2. $f - g$,
 - 1.3. cf ,
 - 1.4. fg , and
 - 1.5. $\frac{f}{g}$ if $g(a) \neq 0$.
- **Theorem 5:**
 - (a) Any polynomial is continuous everywhere, that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

- **Lemma 6:** $\lim_{\theta \rightarrow 0} \cos(\theta) = 1$ and $\lim_{\theta \rightarrow 0} \sin(\theta) = 0$.
- **Theorem 7:** Polynomials, rational functions, root functions, and trigonometric functions are all continuous at every number in their domains.
- **Theorem 8:** If f is continuous at b and $\lim_{x \rightarrow a} g(x) = b$ then $\lim_{x \rightarrow a} f(g(x)) = f(b)$. Equivalently $\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right)$.
- **Theorem 9:** If g is continuous at a and f is continuous at $g(a)$ then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a .
- **Theorem 10. The intermediate value theorem:** Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

2 Differentiation

- **Tangent line:** The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with the slope

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists. Equivalently

$$m = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}.$$

- **Derivative:** The derivative of a function f at a number a , denoted by $f'(a)$ is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.

- **Instantaneous velocity:** If $f(t)$ represents position as a function of time then the derivative $f'(a)$ is the instantaneous velocity of $y = f(t)$ with respect to t when $t = a$.
- **Instantaneous rate of change:** The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.
- **Derivative as a function:** The derivative of a function $f(x)$,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h},$$

can be regarded as a new function called the derivative of f . Other notations for the derivative of $y = f(x)$ with respect to x are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x).$$

- **Differentiable:** A function f is differentiable at a if $f'(a)$ exists. It is differentiable on an open interval (a, b) if it is differentiable at every number in the interval.
- **Theorem 1:** If f is differentiable at a then it is continuous at a .
- **Higher derivatives:** The derivative of a derivative is called the second derivative, denoted by $(f')' = f''$ or

$$\frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d^2 y}{dx^2}.$$

In general the n th derivative, where n is a positive integer, is written as $f^n(x)$ or $\frac{d^n y}{dx^n}$.

- **Differentiation formulas:** If c is a constant and n is a real number then:

$$\begin{aligned}\frac{d}{dx}(c) &= 0, \text{ and} \\ \frac{d}{dx}(x^n) &= nx^{n-1}, \\ \frac{d}{dx}\sin(x) &= \cos(x), \\ \frac{d}{dx}\cos(x) &= -\sin(x).\end{aligned}$$

- **Differentiation rules:** If c is a constant and f and g are differentiable functions then:

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= cf'(x) && \text{the constant multiple rule,} \\ \frac{d}{dx}[f(x) + g(x)] &= f'(x) + g'(x) && \text{the sum rule,} \\ \frac{d}{dx}[f(x) - g(x)] &= f'(x) - g'(x) && \text{the difference rule,} \\ \frac{d}{dx}[f(x)g(x)] &= g(x)f'(x) + f(x)g'(x) && \text{the product rule,} \\ \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} && \text{the quotient rule, and} \\ \frac{d}{dx}[f(g(x))] &= f'(g(x)) \cdot g'(x) && \text{the chain rule.}\end{aligned}$$

- **Implicit differentiation:** Using the chain rule, and others as necessary, allows us to also differentiate implicitly. Let $g(x, y) = 0$ define y as a function of x . $\frac{dy}{dx}$ will relate $\frac{dy}{dx}$, y , and x .
- **Related rates:** Let $g(x, y) = 0$ relate two functions of a variable t . $\frac{dg}{dt}$ will relate $\frac{dy}{dt}$, $\frac{dx}{dt}$, y , and x .
- **Linear approximation:** The linear approximation of $f(x)$ near a point a is $L(x) = f(a) + f'(a)(x - a)$.
- **Differentials:** If $y = f(x)$ where $f(x)$ is a differentiable function, and the differential dx is an independent variable then the differential dy is a dependent variable and is given by

$$dy = f'(x)dx = \frac{dy}{dx}dx.$$

- **Absolute extrema:** Let c be a number in the domain D of a function f . Then $f(c)$ is
 - (a) the absolute maximum value of f on D if $f(c) \geq f(x)$ for all x in D ,
or
 - (b) the absolute minimum value of f on D if $f(c) \leq f(x)$ for all x in D .
- **Local extrema:** Let c be a number in the domain D of a function f . Then $f(c)$ is

- (a) a local maximum value of f on D if $f(c) \geq f(x)$ when x is near c , or
- (b) a local minimum value of f on D if $f(c) \leq f(x)$ when x is near c .

Near c means on some open interval containing c .

- **Theorem 2. The extreme value theorem:** If f is continuous on a closed interval $[a, b]$ then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.
- **Theorem 3. Fermat's theorem:** If f has a local extrema at c , and if $f'(c)$ exists, then $f'(c) = 0$.
- **Critical number:** A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.
- **The closed interval method:** To find the absolute extrema values of a continuous function f on a closed interval $[a, b]$:
 1. Find f at the critical numbers in (a, b) .
 2. Find f at the end points a and b .
 3. The largest and smallest of these values are the absolute extrema.
- **Theorem 4. Rolle's theorem:** Let f be a function which satisfies the following three hypotheses:
 1. f is continuous on the closed interval $[a, b]$.
 2. f is differentiable on the open interval (a, b) .
 3. $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.

- **Theorem 5. The mean value theorem:** Let f be a function which satisfies the following hypotheses:
 1. f is continuous on the closed interval $[a, b]$.
 2. f is differentiable on the open interval (a, b) .

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

- **Theorem 6:** If $f'(x) = 0$ for all x in an interval (a, b) then f is constant on (a, b) .
- **Corollary 7:** If $f'(x) = g'(x)$ for all x in an interval (a, b) then $f - g$ is constant on (a, b) . I.e.

$$f(x) = g(x) + c \text{ where } c \text{ is a constant.}$$

- **Increasing/decreasing test:**
 - (a) If $f'(x) > 0$ on an interval then f is increasing on that interval.

- (b) If $f'(x) < 0$ on an interval then f is decreasing on that interval.
- **First derivative test:** Suppose that c is a critical number of a continuous function f .
 - (a) If f' changes sign from positive to negative at c , then f has local maximum at c .
 - (b) If f' changes sign from negative to positive at c , then f has local minimum at c .
 - (c) If f' does not change sign at c , then f has no local maximum or minimum at c .
- **Concavity:** If the graph of f lies above all of its tangents on an interval I , then it is concave upward on I . If the graph of f lies below all of its tangents on an interval I , then it is concave downward on I .
- **Concavity test:**
 - (a) If $f''(x) > 0$ for all x in an interval I then f is concave upward on I .
 - (b) If $f''(x) < 0$ for all x in I then f is concave downward on I .
- **Inflection point:** A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P .
- **The second derivative test:** Suppose f'' is continuous near c .
 - (a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c .
 - (b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c .
- **Horizontal asymptotes:** Let f be a function defined on a interval (a, ∞) . Then

$$\lim_{x \rightarrow \infty} f(x) = L$$

means that for every $\epsilon > 0$ there is a corresponding number N such that if $x > N$ then $|f(x) - L| < \epsilon$.

Similarly let f be a function defined on a interval (a, ∞) . Then

$$\lim_{x \rightarrow -\infty} f(x) = L$$

means that for every $\epsilon > 0$ there is a corresponding number N such that if $x < N$ then $|f(x) - L| < \epsilon$.

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x \rightarrow \infty} f(x) = L \text{ or } \lim_{x \rightarrow -\infty} f(x) = L.$$

- **Limit laws:** Suppose that c is a constant and that the limits

$$\lim_{x \rightarrow \infty} f(x) \text{ and } \lim_{x \rightarrow \infty} g(x)$$

exist. Then:

1. **Sum law** $\lim_{x \rightarrow \infty} [f(x) + g(x)] = \lim_{x \rightarrow \infty} f(x) + \lim_{x \rightarrow \infty} g(x)$
2. **Difference law** $\lim_{x \rightarrow \infty} [f(x) - g(x)] = \lim_{x \rightarrow \infty} f(x) - \lim_{x \rightarrow \infty} g(x)$
3. **Constant multiple law** $\lim_{x \rightarrow \infty} [cf(x)] = c \lim_{x \rightarrow \infty} f(x)$
4. **Product law** $\lim_{x \rightarrow \infty} [f(x)g(x)] = \lim_{x \rightarrow \infty} f(x) \cdot \lim_{x \rightarrow \infty} g(x)$
5. **Quotient law** $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow \infty} f(x)}{\lim_{x \rightarrow \infty} g(x)}$ if $\lim_{x \rightarrow \infty} g(x) \neq 0$
6. **Power law** $\lim_{x \rightarrow \infty} [f(x)]^n = [\lim_{x \rightarrow \infty} f(x)]^n$ where n is a positive integer
7. $\lim_{x \rightarrow \infty} c = c$
8. $\lim_{x \rightarrow \infty} x = \infty$
11. **Root law** $\lim_{x \rightarrow \infty} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow \infty} f(x)}$ where n is a positive integer and if n is even then we assume that $\lim_{x \rightarrow \infty} f(x) > 0$

Similarly for $\lim x \rightarrow -\infty$.

- **Theorem 8:** If $r > 0$ is a rational number then

$$\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x then

$$\lim_{x \rightarrow -\infty} \frac{1}{x^r} = 0$$

- **Slant asymptotes:** The line $y = mx + b$ is a slant asymptote of $f(x)$ if

$$\lim_{x \rightarrow \infty} [f(x) - (mx + b)] = 0$$

or

$$\lim_{x \rightarrow -\infty} [f(x) - (mx + b)] = 0.$$

- **The first derivative test for absolute extreme values:** Suppose c is a critical number of a continuous function f defined on an interval.
 - (a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is an absolute maximum value of f .
 - (b) If $f'(x) < 0$ for all $x > c$ and $f'(x) > 0$ for all $x < c$, then $f(c)$ is an absolute minimum value of f .
- **Antiderivative:** A function F is called the antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I .
- **Theorem 9:** If F is an antiderivative of f on an interval I , then the most general antiderivative of f on I is $F(x) + C$ where C is an arbitrary constant.

3 Summation

- **Sigma notation:** We use $\sum_{i=m}^n a_i = a_1 + a_2 + \dots + a_n$ as notation for sum over the terms a_i from $i = m$ to $i = n$. i is the index of summation.

- **Theorem 1:** If c is any constant then:

$$(a) \sum_{i=m}^n ca_i = c \sum_{i=m}^n a_i,$$

$$(b) \sum_{i=m}^n (f_i + g_i) = \sum_{i=m}^n f_i + \sum_{i=m}^n g_i, \text{ and}$$

$$(c) \sum_{i=m}^n (f_i - g_i) = \sum_{i=m}^n f_i - \sum_{i=m}^n g_i.$$

- **Some useful sums:**

$$(a) \sum_{i=1}^n 1 = n,$$

$$(b) \sum_{i=1}^n c = cn,$$

$$(c) \sum_{i=1}^n i = \frac{n(n+1)}{2},$$

$$(d) \sum_{i=1}^n i^2 = \frac{n(2n+1)(n+1)}{6}, \text{ and}$$

$$(e) \sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2} \right]^2.$$

4 Integration

- **Area under a curve:** The area A of the region S that lies under the graph of the continuous function f curve is the limit of the sum of the area of approximating rectangles:

$$A = \lim_{n \rightarrow \infty} A_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta x f(x_i).$$

For rectangles of equal width which meet the curve on their top right corner then $\Delta x = (b - a)/n$ where the region S lies between $x = a$ and $x = b$, and $x_i = a + i\Delta x$.

- **The definite integral:** If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$. We let $x_0 = a, x_1, x_2, \dots, x_n = b$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \dots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the i th subinterval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x,$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is integrable on $[a, b]$.

- **Theorem 1:** If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$.
- **Properties of definite integrals:** Let c be any constant.

1. $\int_a^b f(x) dx = - \int_b^a f(x) dx.$
2. $\int_a^a f(x) dx = 0.$
3. $\int_a^b c dx = c(b - a).$
4. $\int_a^b [f(x) + g(x)] dx = \int_a^b f(x) dx + \int_a^b g(x) dx.$
5. $\int_a^b cf(x) dx = c \int_a^b f(x) dx.$
6. $\int_a^b [f(x) - g(x)] dx = \int_a^b f(x) dx - \int_a^b g(x) dx.$
7. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x) dx \geq 0.$
8. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x) dx \geq \int_a^b g(x) dx.$

9. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x) \, dx \leq M(b-a).$$

• **Fundamental theorem of calculus:** If f is continuous on $[a, b]$ then:

1. The function g defined by $g(x) = \int_a^x f(t) \, dt$ for $a \leq x \leq b$ is continuous on $[a, b]$, differentiable on (a, b) , and $g'(x) = f(x)$.
2. $\int_a^b f(t) \, dt = F(b) - F(a)$ where F is any antiderivative of f , i.e. we have $F'(x) = f(x)$.