

Linear Algebra I

Problem Set 5: Inner Products and Bilinear and Sesquilinear Forms

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Due: In class, February 26th 2016

1. Let V be the vector space over \mathbb{R} of all polynomials of degree less than 3. I.e. $V = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2, x \in \mathbb{R}\}$. We can define an inner product on this space as

$$\langle f|g \rangle = \int_0^1 dx f(x)g(x).$$

1. $1, x, x^2$ is an ordered basis of V , find the corresponding matrix of inner products \mathbf{A} . Use this matrix to calculate $\langle x+1|x^2-2 \rangle$ and $\langle x^2-x+5|x^2+2x \rangle$. Confirm the results by direct integration. (6)
2. Prove that the map $F : V \times V \rightarrow \mathbb{C}$ is a sesquilinear form on $V = \mathbb{C}^n$ where $F(\mathbf{x}, \mathbf{y}) = \bar{\mathbf{x}}^T \mathbf{A} \mathbf{y}$, \mathbf{A} is an $n \times n$ matrix, and $\mathbf{x}, \mathbf{y} \in V$. Furthermore prove that if F is conjugate symmetric then $\bar{\mathbf{A}}^T = \mathbf{A}$. (4)
3. A sesquilinear form on \mathbb{C}^3 is defined by

$$F((x, y, z)^T, (x', y', z')^T) = (\bar{x} \ \bar{y} \ \bar{z}) \begin{pmatrix} 0 & -1-2i & 0 \\ -1+2i & 0 & i \\ 0 & -i & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}.$$

What is the matrix of F with respect to the bases

- (a) $\{(i, 1, 0)^T, (1, 1+i, 1)^T, (0, 0, i)^T\}$, and
- (b) $\{(-1-3i, 2+i, -1+2i)^T, (2+i, 1-2i, 2+i)^T, (0, 0, -i)^T\}$?

Is F an inner product? (5)

4. An alternating form F is a bilinear form on a vector space V satisfying $F(v, v) = 0 \ \forall v \in v$.
 - (a) Show that if F is an alternating form then $F(u, v) = -F(v, u)$, i.e. that F is skew symmetric. (3)
 - (b) Give an example of an alternating form (other than the zero function). (2)

Total available marks: 20