Linear Algebra I
Problem Set 6: Orthogonal and Orthonormal Bases

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Due: In class, March 4th 2016

1. Prove proposition 4.15, Bessel's inequality. I.e. if $e_1, e_2, \ldots e_k$ is an orthonormal set of vectors in a complex inner product space V, and $v \in V$, then
$$\sum_{i=1}^{k} |\langle e_i|v\rangle|^2 \leq ||v||^2 \,.$$
Tip: Consider $||w||^2$ *with* $w = v - \sum_{i=1}^{k} \langle e_i|v\rangle e_i$. (4)

2. Consider $V = \mathbb{C}^3$, with the standard inner product. Starting from the basis $\{(0, \mathrm{i}, 1)^T, (1+\mathrm{i}, 0, 2)^T, (3, 0, 0)^T\}$ use Gram-Schmidt to construct an orthonormal basis for V. (4)

3. Let V be the vector space over $\mathbb{R}$ of all polynomials of degree less than 3. I.e. $V = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2, x \in \mathbb{R}\}$. We can define an inner product on this space as
$$\langle f|g\rangle = \int_{-1}^{1} \mathrm{d}x f(x) g(x) \,.$$
N.B. this is a different inner product space to that on the last problem sheet! $1, x$ is an orthogonal basis of U, which is a subspace of V. Find the orthogonal complement, $U^\perp$, to U. (4)

4. Prove proposition 4.21, which says that if V is an inner product space, and U is a finite dimensional subspace of V, then
   (a) $U^\perp$ is a subspace of V, (2)
   (b) $U \cap U^\perp = \{0\}$, and (2)
   (c) $U + U^\perp = V$. (4)

   Tips: For (b) consider the properties of a vector which is in both U *and* $U^\perp$. *For (c) it will help to consider an orthonormal basis of* U. *Any* $v \in V$ *can be written as* $v = v_S + v_P$ *where* $v_S \in U$ *and therefore has a representation in terms of the orthonormal basis of* U. *The task is to show that* $v_P \in U^\perp$ *for any* $v \in V$.

Total available marks: 20