Linear Algebra I
Problem Set III: Vector Spaces

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1. Prove the following: Suppose $\mathbf{A} = \{a_1, a_2, \dots a_n\} \subset V$ is linearly independent, where V is a vector space over F. Suppose also that $v \in V$ and there are scalars $\lambda_1, \dots \lambda_n$ and $\mu_1, \dots \mu_n$ such that
$$v = \lambda_1 a_1 + \lambda_2 a_2 + \dots \lambda_n a_n$$
and
$$v = \mu_1 a_1 + \mu_2 a_2 + \dots \mu_n a_n$$
then $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$,$\dots \lambda_n = \mu_n$. *(Tip: consider the definition of linear independence.)*

2. Prove that the set of Pauli matrices and the 2×2 identity matrix, $A = \{\mathbb{I}_2, \boldsymbol{\sigma}^x, \boldsymbol{\sigma}^y, \boldsymbol{\sigma}^z\}$, is a basis of the vector space V, the set of all 2×2 matrices over $\mathbb{C}$, with addition and scalar multiplication defined in the usual way. The Pauli matrices are
$$\boldsymbol{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

3. Find a basis for the following vector spaces:
   (a) The set of all 2×2 matrices over $\mathbb{R}$.
   (b) $\mathbb{C}^4$, i.e. the set of 4×1 column vectors with complex entries.

4. Which of the following are bases over $\mathbb{R}^3$? Give reasons!
   (a) $A = \{(1,1,0)^T, (0,1,0)^T, (1,0,1)^T, (0,0,1)^T\}$.
   (b) $B = \{(1,1,0)^T, (0,1,0)^T, (1,0,1)^T\}$.
   (c) $C = \{(1,0,0)^T, (0,\mathrm{i},0)^T, (1,0,\mathrm{i})^T\}$.
   (d) $D = \{(1,1,1)^T, (2,2,1)^T, (1,1,0)^T, \}$.

5. Find bases over the following subspaces of $\mathbb{R}^3$.
   (a) $A = \{(x,y,z)^T : 2x + y - z = 0\}$.
   (b) $B = \{(x,y,z)^T : x + y - 2z = 0,\ x - y = 0, \}$.