Linear Algebra I
Summary of Lectures:
Linear Transformations

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1. Definition 5.1: If V and W are two vectors spaces over the same field F, then a linear transformation from V to W (also called a linear map or homomorphism) is a map $f : V \to W$ satisfying

   $$f(\lambda u + \mu v) = \lambda f(u) + \mu f(v)\,, \quad \forall\, u, v \in V \text{ and } \forall\, \lambda, \mu \in F\,.$$

   The space of V is called the domain of F and the space of W is called the co-domain.

2. Lemma 5.2: A linear transformation $f : V \to W$ satisfies
   (a) $f(0) = 0$,
   (b) $f(\lambda u) = \lambda u$,
   (c) $f(-u) = -f(u)$,
   (d) $f(u + v) = f(u) + f(v)$, and
   (e) $f(\sum_{i=1}^{n} \lambda_i u_i) = \sum_{i=1}^{n} \lambda_i f(u_i)$.

3. Definition 5.3: Given $f : V \to W$ as in definition 5.1, the image (or range) of f is $\{f(v) : v \in V\}$. This is written as $f(V)$ or $\mathrm{im}(f)$.
   The kernel (or nullspace) of f is $\{v \in V : f(v) = 0\}$, written $\ker(f)$

4. Proposition 5.4: If $f : V \to W$ is a linear transformation, then $\mathrm{im}(f)$ is a subspace of W and $\ker(f)$ is a subspace of V.

5. Proposition 5.5: A linear transformation $f : V \to W$ is injective iff $\ker(f)$ is the zero subspace $\{0\}$ of V.

6. Definition 5.6: The rank of f is the dimension of $\mathrm{im}(f)$, written $\mathrm{r}(f)$.
   The nullity of f is the dimension of $\ker(f)$, written $\mathrm{n}(f)$.

7. Theorem 5.7: The rank-nullity formula. If $f : V \to W$ is a linear transformation then

   $$\mathrm{r}(f) + \mathrm{n}(f) = \dim(V)\,.$$

8. Proposition 5.8: If $f : V \to W$ is a linear transformation of finite dimensional vector spaces V, W over the same field F then
   (a) f is injective iff $n(f) = 0$, and
   (b) f is surjective iff $r(f) = \dim(W)$.

9. Corollary 5.9: If $f : V \to W$ is a linear transformation of finite dimensional vector spaces V, W over the same field F then
   (a) f is injective iff $r(f) = \dim(V)$, and
   (b) f is surjective iff $n(f) = \dim(V) - \dim(W)$.

10. Let $f_{\mathbf{A}} : F^n \to F^m$ be $f_{\mathbf{A}}(\mathbf{v}) = \mathbf{Av}$ where $\mathbf{v} \in F^n$ and $\mathbf{A}$ is an $m \times n$ matrix over the field F. Then
    (a) $\mathrm{im}(\mathbf{A}) = \{\mathbf{Av} : \mathbf{v} \in F^n\}$,
    (b) $\ker(\mathbf{A}) = \{\mathbf{v} \in F^n : \mathbf{Ax} = 0\}$, and
    (c) $r(\mathbf{A})$ and $n(\mathbf{A})$ are the rank and nullity of $\mathbf{A}$, i.e. the dimensions of $\mathrm{im}(\mathbf{A})$ and $\ker(\mathbf{A})$ respectively.

11. For a linear transformation the codomain is a vector space and therefore operations of addition and scalar multiplication must be defined. Any set S of functions from any set X to a vector space W, such that S is closed under addition and scalar multiplication forms a vector space.

12. Let $\mathcal{L}[V, W]$ be the set of all linear transformations from a vector space V to a vector space W over the same field F. The zero element in $\mathcal{L}[V, W]$ is the map that takes every element to the zero in W.

13. $\mathcal{L}[V, V]$ allows additional operations, 'composition of functions', $(f \cdot g)(x) = f(g(x))$. $\forall f, g, h \in \mathcal{L}[V, V]$ and $\forall \lambda, \mu \in F$ we have
    (1) $(f + g) + h = f + (g + h)$,
    (2) $f + g = g + f$,
    (3) $0 + f = f + 0$ (there is a zero element),
    (4) $\lambda(\mu f) = (\lambda\mu) f$,
    (5) $(\lambda + \mu) f = \lambda f + \mu f$,
    (6) $0f = 0$,
    (7) $f + (-1)f = 0$,
    (8) $(f \cdot g) \cdot h = f \cdot (g \cdot h)$,
    (9) $\mathbb{I} \cdot f = f \cdot \mathbb{I} = f$ ($\mathbb{I}$ is the identity map $\mathbb{I}(x) = x$),
    (10) $f \cdot (g + h) = f \cdot g + f \cot h$, and
    (11) $((g + h)) \cdot f = g \cdot f + h \cdot f$.

14. If $f : V \to W$ is a linear map, $v_1, v_2, \dots v_n$ is a basis for V and $w_1.w_2, \dots w_m$ is a basis for W then

$$f(v_j) = \sum_{i=1}^{m} a_{ij} w_i$$

with a_{ij} some scalars. The matrix $\mathbf{A} = (a_{ij})$ is called the matrix of f with respect to the ordered bases $v_1, v_2, \dots v_n$ of V and $w_1, w_2, \dots w_m$ of W. In coordinate form with $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots \lambda_n)^T$ the coordinates of $v \in V$ with respect to the basis of V and $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots \mu_m)^T$ the coordinates of $f(v)$ with respect to the basis of W then

$$\boldsymbol{\mu} = \mathbf{A}\boldsymbol{\lambda}.$$

- If we change the basis $v_1, v_2, \dots v_n$ of V to $v'_1, v'_2, \dots v'_n$ related by the base change matrix $\mathbf{P} = (p_{ij})$ so that $v'_j = \sum_{i=1}^{n} p_{ij} v_i$ then the matrix of f with respect to the ordered bases $v'_1, v'_2, \dots v'_n$ of V and $w_1, w_2, \dots w_m$ of W is $\mathbf{AP}$.
- If we change the basis $w_1, w_2, \dots w_m$ of W to $w'_1, w'_2, \dots w'_m$ related by the base change matrix $\mathbf{Q} = (q_{ij})$ so that $w'_j = \sum_{i=1}^{m} q_{ij} w_i$ then the matrix of f with respect to the ordered bases $v_1, v_2, \dots v_n$ of V and $w'_1, w'_2, \dots w'_m$ of W is $\mathbf{Q}^{-1}\mathbf{A}$.
- Together we have $\mathbf{Q}^{-1}\mathbf{AP}$ if we change the basis of both the domain and codomain.

15. Proposition 5.10: Let V be a vector space with ordered basis B given by $v_1, v_2, \dots v_n$ and B' given by $v'_1, v'_2, \dots v'_n$. Let $\mathbf{P} = (p_{ij})$ be the base change matrix so that $v'_j = \sum_{i=1}^{n} p_{ij} v_i$. Suppose $f : V \to V$ is a linear map which has matrix $\mathbf{A}$ with respect to the ordered basis B and matrix $\mathbf{B}$ with respect to the ordered basis B'. Then

$$\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}\,.$$

16. Definition 5.11: If $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$ then $\mathbf{A}$ and $\mathbf{B}$ are called similar matrices. If $\mathbf{B} = \mathbf{P}^{T}\mathbf{AP}$ then $\mathbf{A}$ and $\mathbf{B}$ are called congruent matrices.

17. Proposition 5.12: Let $f, g \in \mathcal{L}[V, W]$ where V, W are finite dimensional vector spaces, and let f, g have matrix representations $\mathbf{A}, \mathbf{B}$ respectively with respect to some ordered bases A, B of V, W respectively. Then
    (a) The matrix representation of λf with respect to A, B is the scalar product $\lambda\mathbf{A}$ of the matrix $\mathbf{A}$.
    (b) The matrix representation of the sum$f + g$ with respect to A, B is the matrix sum $\mathbf{A} + \mathbf{B}$.

18. Proposition 5.13: (Composition of functions) Let U, V, W be finite dimensional vector spaces, with ordered bases A, B, C respectively, and let $g : U \to V$ and $f : V \to W$ be linear maps. If f and g are represented by the matrices $\mathbf{A}$ and $\mathbf{B}$ with respect to A, B, C then $f \cdot g$ is represented (with respect to the same bases) by the matrix product $\mathbf{AB}$.

19. In the special case $\mathcal{L}[V, V]$, where V is an n-dimensional vector space over F, then given an ordered basis B of V we have a map from $\mathcal{L}[V, V]$ to the set $M_{n,n}(F)$ of $n \times n$ matrices with entries taken from F, taking f to the matrix representing f (which is unique once B is specified). Additionally:
    - Every matrix is the matrix of some transformation f.
    - The zero transformation $\leftrightarrow$ the zero matrix.
    - The identity transformation $\leftrightarrow$ the identity matrix.
    - Scalar multiplication $\leftrightarrow$ scalar multiplication.
    - Addition of transformations $\leftrightarrow$ addition of matrices.

    I.e. $\mathcal{L}[V, V]$ and $M_{n,n}(F)$ are isomorphic. Moreover composition in $\mathcal{L}[V, V]$ and matrix multiplication in $M_{n,n}(F)$ are isomorphic.

20. Polynomials over a field F are expressions like

$$f(x) = \sum_{r=0}^{n} a_r x^r \text{ with } a_r \in F\,.$$

The degree of the polynomial is the largest r for which $a_r \neq 0$ and is written $\deg(f)$.

21. Proposition 5.14: (Division algorithm) If $f(x)$ and $g(x)$ are two polynomials, and $g(x)$ is not the zero polynomial then there exist polynomials $q(x)$ and $r(x)$ such that

$$f(x) = g(x)q(x) + r(x)$$

and either $r(x)$ is the zero polynomial (i.e. $a_r = 0\ \forall r$) or else $\deg(r) < \deg(g)$.

22. It is generally true for polynomials $p(x), q(x)$ in a single free variable that whenever there is a polynomial identity $p(x) = q(x)$ and an $n \times n$ matrix $\mathbf{A}$ that $p(\mathbf{A}) = q(\mathbf{A})$ holds. For polynomials in more than one variable that is not the case.

23. Proposition 5.15: If $p(x), q(x)$ are polynomials and $\mathbf{A}$ is an $n \times n$ matrix then $p(\mathbf{A})q(\mathbf{A}) = q(\mathbf{A})p(\mathbf{A})$.

24. Proposition 5.16: If $f \in \mathcal{L}[V, V]$ where V is a finite dimensional vector space, and if $p(x), q(x)$ are polynomials, then $p(f)q(f) = q(f)p(f)$.

25. Theorem 5.17: (Remainder theorem) Suppose $p(x)$ is a polynomial of degree at least 1 with coefficients from $\mathbb{R}$ or $\mathbb{C}$ and $\alpha \in \mathbb{C}$. Then α is a root of $p(x)$ iff $(x - \alpha)$ divides $p(x)$ exactly.

26. Corollary 5.18: A polynomial $p(x)$ of degree $d \geq 1$ has at most d roots.

27. Every polynomial in $\mathbb{C}$ has its maximum number of roots, counting multiplicities. For any $p(x)$ of degree $d \geq 1$ $\exists \alpha_1, \alpha_2, \ldots \alpha_d, c \in \mathbb{C}$ such that

$$p(x) = c \prod_{j=1}^{d} (x - \alpha_j)\,.$$

The field of complex numbers is algebraically closed. The field of real numbers is not algebraically closed, but $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$. I.e.:

(1) $\mathbb{C}$ is algebraically closed.

(2) Every element $\xi \in \mathbb{C}$ satisfies a polynomial equation over $\mathbb{R}$.