

# Linear Algebra

## Problem Set 1:

### Matrix Operations

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1. Calculate the rank of the following matrices by converting them to echelon form

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 2 \\ 3 & 8 & -6 & 7 & 2 \\ 7 & -6 & 5 & 4 & 4 \\ 1 & 2 & 1 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 & 4 \\ 2 & 1 & 0 & 3 \\ 1 & -1 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

2. Calculate the inverses of the following square matrices

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 8 & 6 & 7 \\ 7 & 6 & 5 & 4 \\ 1 & 2 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 4 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

3. Write down the  $5 \times 5$  matrices which correspond to the following elementary row operations:

- $\rho_2 := \rho_2 - 3\rho_4$
- $\text{swap}(\rho_2, \rho_5)$

4. Prove that for a square matrix  $\mathbf{A}$  with an inverse  $\mathbf{B}$  such that  $\mathbf{AB} = \mathbb{I}$  that there is a matrix  $\mathbf{C}$  satisfying  $\mathbf{CA} = \mathbb{I}$  and that the inverse of  $\mathbf{A}$  is unique.

- (a) For any  $n \times n$  row operation matrix  $\mathbf{R}$  show directly that it has an inverse  $\mathbf{RS} = \mathbf{SR} = \mathbb{I}$ .
- (b) Show for a matrix  $\mathbf{R}$  composed of many elementary row operations  $\mathbf{R} = \mathbf{R}_1\mathbf{R}_2 \dots \mathbf{R}_n$  that it does not have a row entirely made of zeros. (*Tip: This follows from (a).*)
- (c) Prove that if  $\mathbf{AB} = \mathbb{I}$  for square matrices  $\mathbf{A}$  and  $\mathbf{B}$  then there exists a matrix  $\mathbf{C}$  satisfying  $\mathbf{CA} = \mathbb{I}$ . (*Tip: By considering the matrix  $\mathbf{RA}$  which is in echelon form and  $\mathbf{AB} = \mathbb{I}$  show that  $\text{rk } \mathbf{A} = n$ . We note that if an  $n \times n$  matrix has rank  $n$  it has a left inverse.*)
- (d) Prove that  $\mathbf{B} = \mathbf{C}$  and therefore the inverse of  $\mathbf{A}$  is unique.