Summary of Calculus I

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Contents

1 Limits

- Tangent: A tangent line is a line that touches a curve, and has the same slope as the curve at the point of contact.
- Average rate of change: The average rate of change of a function $f(x)$ between $x = a$ and $x = b$ is
$$\frac{f(b) - f(a)}{b - a} .$$
- Limit: Let f be a function defined on some open interval that contains a but not necessarily at a itself. Then we say the limit of $f(x)$ as x approaches a is L and we write
$$\lim_{x \to a} f(x) = L$$
if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that if
$$0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon .$$
- Left-hand limit:
$$\lim_{x \to a^-} f(x) = L ,$$
if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that if
$$a - \delta < x < a \text{ then } |f(x) - L| < \epsilon .$$
- Right-hand limit:
$$\lim_{x \to a^+} f(x) = L ,$$
if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that if
$$a < x < a + \delta \text{ then } |f(x) - L| < \epsilon .$$

- Theorem 1:

$$\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L\,.$$

- Infinit limits: Let f be a function defined on some open interval that contains a but not necessarily at a itself. Then

$$\lim_{x\to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that if

$$0 < |x-a| < \delta \text{ then } f(x) > M\,.$$

Let f be a function defined on some open interval that contains a but not necessarily at a itself. Then

$$\lim_{x\to a} f(x) = -\infty$$

means that for every negative number N there is a positive number δ such that if

$$0 < |x-a| < \delta \text{ then } f(x) < N\,.$$

- Vertical asymptote: The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following is true:

$$\begin{array}{ll} \lim_{x\to a} f(x) = \infty\,, & \lim_{x\to a} f(x) = -\infty\,, \\ \lim_{x\to a^-} f(x) = \infty\,, & \lim_{x\to a^-} f(x) = -\infty\,, \\ \lim_{x\to a^+} f(x) = \infty\,, \quad \text{or} & \lim_{x\to a^+} f(x) = -\infty\,. \end{array}$$

- Limit laws: Suppose that c is a constant and that the limits

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x)$$

exist. Then:

1. Sum law $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. Difference law $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. Constant multiple law $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
4. Product law $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$
5. Quotient law $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. Power law $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a}[f(x)]\right]^n$ where n is a positive integer
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ where n is a positive integer

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer and if n is even then we assume that $a > 0$
11. Root law $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer and if n is even then we assume that $\lim_{x \to a} f(x) > 0$

- Direct substitution property: If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)\,.$$

Such functions are called continuous at a.

- Theorem 2: If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of both f and g exist as x approaches a then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)\,.$$

- Theorem 3: The squeeze theorem. If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L\,.$$

- Continuous function: A function f is said to be continuous at a if $\lim_{x \to a} f(x) = f(a)$.
  - A function f is continuous from the right at a number a if $\lim_{x \to a^+} f(x) = f(a)$.
  - A function f is continuous from the left at a number a if $\lim_{x \to a^-} f(x) = f(a)$.
  - A function f is said to be continuous on an interval if it is continuous at every number in the interval.

- Theorem 4: If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

  1.1. $f + g$,

  1.2. $f - g$,

  1.3. cf,

  1.4. fg, and

  1.5. $\dfrac{f}{g}$ if $g(a) \neq 0$.

- Theorem 5:

  (a) Any polynomial is continuous everywhere, that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

- Lemma 6: $\lim_{\theta \to 0} \cos(\theta) = 1$ and $\lim_{\theta \to 0} \sin(\theta) = 0$.
- Theorem 7: Polynomials, rational functions, root functions, and trigonometric functions are all continuous at every number in their domains.
- Theorem 8: If f is continuous at b and $\lim_{x \to a} g(x) = b$ then $\lim_{x \to a} f(g(x)) = f(b)$. Equivalently $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$.
- Theorem 9: If g is continuous at a and f is continuous ate $g(a)$ then the composite function $f \cdot g$ given by $(f \cdot g)(x) = f((g(x))$ is continuous at a.
- Theorem 10: The intermediate value theorem. Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

2 Differentiation

- Tangent line: The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with the slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists. Equivalently

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

- Derivative: The derivative of a function f at a number a, denoted by $f'(a)$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

- Instantaneous velocity: If $f(t)$ represents position as a function of time then the derivative $f'(a)$ is the instantaneous velocity of $y = f(t)$ with respect to t when $t = a$.

- Instantaneous rate of change: The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

- Derivative as a function: The derivative of a function $f(x)$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

can be regarded as a new function called the derivative of f. Other notations for the derivative of $y = f(x)$ with respect to x are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x).$$

- Differentiable: A function f is differentiable at a if $f'(a)$ exists. It is differentiable on an open interval (a, b) if it is differentiable at every number in the interval.

- Theorem 1: If f is differentiable at a then it is continuous at a.

- Higher derivatives: The derivative of a derivative is called the second derivative, denoted by $(f')' = f''$ or

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}.$$

In general the nth derivative, where n is a positive integer, is written as $f^n(x)$ or $\dfrac{d^ny}{dx^n}$.