

Linear Algebra

Exercises I:

Matrix Operations

Dr Nicholas Sedlmayr

1. Calculate the rank of the following matrices by converting them to echelon form

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 2 \\ 3 & 8 & -6 & 7 & 2 \\ 7 & -6 & 5 & 4 & 4 \\ 1 & 2 & 1 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 & 4 \\ 2 & 1 & 0 & 3 \\ 1 & -1 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

2. Calculate the inverses of the following square matrices

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 8 & 6 & 7 \\ 7 & 6 & 5 & 4 \\ 1 & 2 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 4 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

3. Write down the 5×5 matrices which correspond to the following elementary row operations:

- $\rho_2 := \rho_2 - 3\rho_4$
- $\text{swap}(\rho_2, \rho_5)$

4. Prove that for a square matrix \mathbf{A} with an inverse \mathbf{B} such that $\mathbf{AB} = \mathbb{I}$ that there is a matrix \mathbf{C} satisfying $\mathbf{CA} = \mathbb{I}$ and that the inverse of \mathbf{A} is unique.

- (a) For any $n \times n$ row operation matrix \mathbf{R} show directly that it has an inverse $\mathbf{RS} = \mathbf{SR} = \mathbb{I}$.
- (b) Show for a matrix \mathbf{R} composed of many elementary row operations $\mathbf{R} = \mathbf{R}_1\mathbf{R}_2 \dots \mathbf{R}_n$ that it does not have a row entirely made of zeros. (*Tip: This follows from (a).*)
- (c) Prove that if $\mathbf{AB} = \mathbb{I}$ for square matrices \mathbf{A} and \mathbf{B} then there exists a matrix \mathbf{C} satisfying $\mathbf{CA} = \mathbb{I}$. (*Tip: By considering the matrix \mathbf{RA} which is in echelon form and $\mathbf{AB} = \mathbb{I}$ show that $\text{rk } \mathbf{A} = n$. We note that if an $n \times n$ matrix has rank n it has a left inverse.*)
- (d) Prove that $\mathbf{B} = \mathbf{C}$ and therefore the inverse of \mathbf{A} is unique.