Theorems and Definitions for *Higher Mathematics in English II*

Topics: Linear Algebra, Multivariate Calculus, and Complex Analysis

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Contents

1 Preamble

1.1 Notation

iff	if and only if
$\Rightarrow$	if then
$\equiv$	defined as
$\therefore$	therefore
$\because$	because
$\square$	end of proof
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{I}$	identity matrix
$\forall$	the universal quantifier, for all
$\exists$	the existential quantifier, there exists
$\in$	is an element of
$\subset$	is a subset of

1.2 Translations

English	po Polsku
Matrix	Macierz
Echelon form	Macierz schodkowa
Transpose	Transponowana
Determinant	Wyznacznik
Trace	Ślad
Rank	Rząd
Identity Matrix	Macierz jednostkowa
Inverse	Odwrotna
Associativity	Łączność
Commutativity	Przemienność
Distributivity	Rozdzielność
Scalar	Skalar
Field	Ciało
Vector	Wektor
Vector Space	Przestrzeń liniowa
Group	Grupa
Linear Independence	Liniowa niezależność
Subspace	Podprzestrzeń
Dimension	Wymiar
Basis	Baza
Inner Product Space	Przestrzeń unitarna
Symmetry	Symetria
Orthogonality	Ortogonalność
Orthonormality	Ortonormalność
Eigenvalues	Wartości własne
Eigenvectors	Wektory własne
Lemma	Lemat
Theorem	Twierdzenie
Proof	Dowód
Axiom	Aksjomat
Equality	Równość
Inequality	Nierówność

2 Linear Algebra

2.1 Matrices

- Definition of an $n \times m$ matrix, $\mathbf{A} = (a_{ij})$ with n row and m columns. Addition of matrices $\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})$.
  - Associativity, commutativity and existence of a zero matrix ($\mathbf{0}$) for addition.
- Definition of multiplication of a matrix by a scalar: $\lambda \mathbf{A} = (\lambda a_{ij})$.
- Definition: Matrix multiplication of an $n \times m$ matrix $\mathbf{A}$ and an $m \times k$ matrix $\mathbf{B}$ is an $n \times k$ matrix $\mathbf{C} = \mathbf{AB} = (c_{ij})$ where $c_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$.
  - Associativity, existence of a zero matrix ($\mathbf{0}$) and an identity matrix $\mathbb{I}$, distributivity.
  - No commutativity!
- A matrix $\mathbf{A}$ can have a right inverse $\mathbf{AB} = \mathbb{I}$ and a left inverse $\mathbf{CA} = \mathbb{I}$.
  - Proposition 1.1: If a square matrix has either a left or right inverse then they have a unique inverse from both the left and right.
  - If a non-square matrix has both a left and right inverse then they are the same and the inverse is unique.
  - Proposition 1.2: If $\mathbf{A}$ and $\mathbf{B}$ are invertible square matrices then $\mathbf{AB}$ is also invertible and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
- The transpose of an $n \times m$ matrix is written as $\mathbf{A}^T$, which is an $m \times n$ matrix found by transposing the rows and columns of $\mathbf{A}$.
  - Proposition 1.3: For two $n \times n$ matrices $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ and $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.
- Elementary row operations perform simple operations on the rows of an $n \times m$ matrix $\mathbf{A}$ and can be written as an $n \times n$ matrix $\mathbf{R}$ with the operation preformed by the multiplication $\mathbf{RA}$. ρ_i is used to refer to row i. There are three of them:
  - $\rho_j := \rho_j + \lambda \rho_i$, add λ copies of row i to row j;
  - $\rho_i := \lambda \rho_i$, multiple row i by λ with $\lambda \neq 0$; and
  - $\mathrm{swap}(\rho_i, \rho_j)$, swap rows i and j.
- Echelon form: A matrix where each row starts with a sequence of zeros, and the number of zeros in this sequences increases from row to row from top to bottom until the final row is reached or all remaining rows are composed entirely of zeros is said to be in echelon form.
  - All matrices can be put into echelon form using elementary row operations.
- Rank: If $\mathbf{A}$ can be converted to the echelon form matrix $\mathbf{B}$ and $\mathbf{B}$ has k non-zero rows then the rank of $\mathbf{A}$ is $\operatorname{rk}\mathbf{A} = k$.
- An $n \times k$ matrix $\mathbf{A}$ with $n \leq k$ and $\operatorname{rk}\mathbf{A} = n$ can be converted to a matrix $\mathbf{B}$ where the left $n \times n$ block is the identity matrix using elementary row operations.
- Augmented matrix: If we have an $n \times m$ matrix $\mathbf{A}$ and an $n \times k$ matrix $\mathbf{A}$ the augmented matrix $(\mathbf{A}|\mathbf{B})$ is the $n \times (m + k)$ matrix created by writing $\mathbf{A}$ and $\mathbf{B}$ next to each other.
- Calculating the inverse: For an $n \times n$ matrix $\mathbf{A}$ of rank n we can calculate the inverse using the augmented matrix $(\mathbf{A}, \mathbb{I})$. Apply row operations, $\mathbf{R}$, such that $\mathbf{R}(\mathbf{A}, \mathbb{I}) = (\mathbb{I}, \mathbf{B})$, clearly $\mathbf{B} = \mathbf{R}$ is the left inverse of $\mathbf{A}$: $\mathbf{BA} = \mathbb{I}$.
- Solving systems of linear equations. A system of linear equations on the variables $x_1, x_2, \dots x_n$ can be written as $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is a $k \times n$ matrix, $\mathbf{x} = (x_1, x_2, \dots x_n)^T$ and $\mathbf{b}$ is a $k \times 1$ column vector. By putting the augmented matrix $(\mathbf{A}|\mathbf{b})$ into echelon form one can read off the solution.

  - Proposition 1.4: An equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{A}$ is a $k \times n$ matrix, $\mathbf{x} = (x_1, x_2, \dots x_n)^T$ and $\mathbf{b}$ is a $k \times 1$ column vector has at least one solution iff $\operatorname{rk}\mathbf{A} = \operatorname{rk}(\mathbf{A}|\mathbf{b})$. It has exactly one solution if $\operatorname{rk}\mathbf{A} = n$

- Trace: This operation is defined only for square matrices. A square $n \times n$ matrix $\mathbf{A} = (a_{ij})$ has a trace $\operatorname{tr}\mathbf{A} = \sum_{i=1}^{n} a_{ii}$. I.e. it is the sum over all diagonal entries of the matrix. It has the following properties:
  - For two square $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, $\operatorname{tr}\mathbf{AB} = \operatorname{tr}\mathbf{BA}$.
  - If $\mathbf{P}$ is a square $n \times n$ invertible matrix and $\mathbf{A}$ is a square $n \times n$ matrix then $\operatorname{tr}\mathbf{P}^{-1}\mathbf{AP} = \operatorname{tr}\mathbf{A}$.
  - $\operatorname{tr}(\mathbf{A} + \mathbf{B}) = \operatorname{tr}\mathbf{A} + \operatorname{tr}\mathbf{B}$.
  - $\operatorname{tr}\mathbf{A}^T = \operatorname{tr}\mathbf{A}$.
  - For a scalar λ then $\operatorname{tr}\lambda\mathbf{A} = \lambda \operatorname{tr}\mathbf{A}$.

- Determinant: This operation is defined only for square matrices and we will define it via "expansion by the first row". For a 1×1 matrix $\mathbf{A} = (a_11)$ we have $\det\mathbf{A} = |\mathbf{A}| = a_{11}$. For an $n \times n$ matrix $\mathbf{A} = (a_{ij})$ it is

$$\begin{aligned}
\det\mathbf{A} &= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \\
&= a_{11} \begin{vmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n3} & \dots & a_{nn} \end{vmatrix} \\
&\quad + \dots + (-1)^{n-1} a_{1n} \begin{vmatrix} a_{21} & a_{22} & \dots & a_{2,n-1} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n,n-1} \end{vmatrix}.
\end{aligned}$$

  - Proposition 1.5: For two $n \times n$ square matrices $\mathbf{A}$ and $\mathbf{B}$, $\det\mathbf{AB} = \det\mathbf{A}\det\mathbf{B}$. It can be proved using the following lemmas.
  - Lemma 1.5.1: Any square matrix $\mathbf{A}$ can be written as the product $\mathbf{S}_1\mathbf{S}_2 \dots \mathbf{S}_k$ of 'generalized' elementary row operations $\rho_i := \lambda\rho_i$ and $\rho_i := \rho_i + \lambda\rho_j$ where λ can be zero.
  - Lemma 1.5.2: The determinant of a matrix whose top two rows are identical is zero.
  - Lemma 1.5.3:

$$\begin{vmatrix} a_{11} + \lambda b_{11} & a_{12} + \lambda b_{12} & \dots & a_{1n} + \lambda b_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + \lambda \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

  - Lemma 1.5.4: The matrix of any generalized elementary row operation can always be written as one of $\mathbf{S}_i\mathbf{T}_j\mathbf{R}\mathbf{T}_j\mathbf{S}_i$, $\mathbf{T}_j\mathbf{R}\mathbf{T}_j$, $\mathbf{S}_i\mathbf{R}\mathbf{S}_i$, or $\mathbf{R}$, where $\mathbf{R}$ is the matrix of a generalized row operation acting only on rows 1 and 2, and $\mathbf{S}_i$, $\mathbf{T}_j$ are the matrices of $\operatorname{swap}(\rho_1, \rho_i)$ and $\operatorname{swap}(\rho_2, \rho_j)$ respectively.
  - Lemma 1.5.5: If $\mathbf{R}$ is the matrix of the swap operation $\operatorname{swap}(\rho_i, \rho_j)$ then $\det\mathbf{RA} = -\det\mathbf{A}$.

- Proposition 1.6: For an $n \times n$ matrix $\mathbf{A}$ the following are equivalent:

  (a) $\mathbf{A}^{-1}$ exists,

(b) $\det \mathbf{A} \neq 0$, and
(c) $\operatorname{rk} \mathbf{A} = n$.

- Proposition 1.7: For an $n \times n$ matrix $\mathbf{A}$:
  (a) $\det(\mathbf{A}^T) = \det \mathbf{A}$,
  (b) If $\mathbf{A}$ is invertible $\det \mathbf{A}^{-1} = (\det \mathbf{A})^{-1}$,
  (c) $\det \mathbb{I} = 1$, and
  (d) if $\mathbf{A}$ has a row (or column) entirely composed of zeros $\det \mathbf{A} = 0$.

- Proposition 1.8: The determinant of an upper triangular matrix $\mathbf{A}$ is equal to the product of its diagonal entries.

- Proposition 1.9: The determinants of row operations are:
  (a) If $\mathbf{A}_{ij,\lambda}$ is the matrix for $\rho_i := \rho_i + \lambda\rho_j$ then $\det \mathbf{A}_{ij,\lambda} = 1$,
  (b) If $\mathbf{T}_{i,\lambda}$ is the matrix for $\rho_i := \lambda\rho_i$ then $\det \mathbf{T}_{i,\lambda} = \lambda$, and
  (a) If $\mathbf{S}_{ij}$ is the matrix for $\operatorname{swap}(\rho_i, \rho_j)$ then $\det \mathbf{S}_{ij} = -1$.

- Proposition 1.10: If we swap any two rows in a determinant then the determinant changes sign, see Lemma 1.5.5. It follows that if a matrix has two identical rows then its determinant is zero. (This is equally true for a matrix with two identical columns.)

- Proposition 1.11: A determinant can be expanded along any row (or column). The sign associated with any entry a_{ij} is $(-1)^{i+j}$.

- Minors and cofactors: Let $\mathbf{A} = (a_{ij})$ be an $n \times n$ matrix and let b_{ij} be the determinant of the $(n-1) \times (n-1)$ matrix obtained form $\mathbf{A}$ by deleting row i and column j. Furthermore let $c_{ij} = (-1)^{i+j} b_{ij}$. Then
  - $\mathbf{B} = (b_{ij})$ is the matrix of minors of $\mathbf{A}$,
  - $\mathbf{C} = (c_{ij})$ is the matrix of cofactors of $\mathbf{A}$, and
  - $\operatorname{adj} \mathbf{A} = \mathbf{C}^T$ is the adjugate matrix of $\mathbf{A}$.

2.2 Vector Spaces

- Definition 2.1: A vector space. A vector space V over a field F (see definition 2.3) is a set containing:
  - a special zero vector $\mathbf{0}$;
  - an operation of addition of two vectors $\mathbf{u} + \mathbf{v} \in V$, for $\mathbf{u}, \mathbf{v} \in V$; and
  - multiplication of a vector V with a number $\lambda \in F$ with $\lambda\mathbf{v} \in V$.

  The vector space must be closed under both of these operations and must satisfy the following laws $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\lambda, \mu \in F$:

  (1) associativity $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$;
  (2) commutativity $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$;
  (3) $\mathbf{u} + \mathbf{0} = \mathbf{u}$;
  (4) $\mathbf{v} + (-1)\mathbf{v} = \mathbf{0}$;
  (5) $\lambda(\mu\mathbf{u}) = (\lambda\mu)\mathbf{v}$;
  (6) distributivity $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$; and
  (7) distributivity $(\lambda + \mu)\mathbf{u} = \lambda\mathbf{u} + \mu\mathbf{u}$.

- Proposition 2.2: $\forall \mathbf{v} \in V$ and $\forall \lambda \in F$:

  (a) $\mathbf{v} = 1\mathbf{v}$;
  (b) $0\mathbf{v} = \mathbf{0}$; and
  (c) $\lambda\mathbf{0} = \mathbf{0}$.

- Definition 2.3: A field is a set F containing distinct elements 0 and 1 with two binary operations $+$ and $\cdot$ satisfying the axioms $\forall a, b, c \in F$:

  (1) $a + b = b + a$;
  (2) $(a + b) + c = a + (b + c)$;
  (3) $a + 0 = a$;
  (4) $\forall a \; \exists -a$ such that $a + (-a) = 0$;
  (5) $a \cdot b = b \cdot a$;
  (6) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;
  (7) $a \cdot 1 = a$;
  (8) $\forall a \neq 0 \; \exists a^{-1}$ such that $a \cdot a^{-1} = 1$; and
  (9) $a \cdot (b + c) = a \cdot b + a \cdot c$;

  If a field F is finite its order is the number of elements in F.

- Note that as a field also satisfies all axioms of a vector space a field F is also itself a vector space $V = F$ over the field F and all properties of a vector space apply.

- Theorem 2.4: For each prime p and each positive integer n, there is a unique field of order p^n. Additionally, every finite field is of this form.

- Definition 2.5: Given a vector space V over F, a subspace of V is a subset $W \subset V$ which contains the zero vector of V and is closed under the operations of addition and scalar multiplication.

- Lemma 2.5.1: Let $W \subset V$ be nonempty, where V is a vector space over F. Then W is a subspace of V iff $\mathbf{v} + \lambda\mathbf{u} \in W$ for each $\mathbf{v}, \mathbf{u} \in W$ and each scalar λ.

- Definition 2.6: Given a vector space V over F, and given a subset of V $A = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots \mathbf{u}_n\}$,

  $$W = \{\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 + \lambda_3\mathbf{u}_3 + \ldots \lambda_n\mathbf{u}_n : \lambda_1, \lambda_2, \ldots \lambda_n \in F\}$$

  is the subspace of V spanned by A. The elements of W are called linear combinations of vectors from A and the subspace W is denoted as span A.

- Definition 2.7: If A is an infinite subset of V, where V is a vector space over F, we define $\operatorname{span} A$ to be the set of all linear combinations of finite subsets of A.

- Definition 2.8: A set $A \subset V$ of vectors in a vector space V over F is linearly dependent if there are $n \in \mathbb{N}$ vectors $a_1, a_2, \ldots a_n$ and scalars $\lambda_1, \lambda_2, \ldots \lambda_n$ not all zero such that

$$\lambda_1 a_1 + \lambda_2 a_2 + \ldots \lambda_n a_n = 0\,.$$

  Otherwise A is linearly independent.

  - For a finite set $A = \{a_1, a_2, \ldots a_n\}$ it is linearly independent iff $\forall$ scalars $\lambda_1, \lambda_2, \ldots \lambda_n \in F$

$$\lambda_1 a_1 + \lambda_2 a_2 + \ldots \lambda_n a_n = 0 \Rightarrow \lambda_1 = \lambda_2 = \ldots \lambda_n = 0.$$

  - If A is infinite it is linearly independent iff every subset of A is linearly independent.
  - By convention the empty set is linearly independent.

- Proposition 2.9: Suppose $\mathbf{A} = \{a_1, a_2, \ldots a_n\} \subset V$ is linearly independent, where V is a vector space over F. Suppose also that $v \in V$ and there are scalars $\lambda_1, \ldots \lambda_n$ and $\mu_1, \ldots \mu_n$ such that

$$v = \lambda_1 a_1 + \lambda_2 a_2 + \ldots \lambda_n a_n$$

  and

$$v = \mu_1 a_1 + \mu_2 a_2 + \ldots \mu_n a_n$$

  then $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$,$\ldots \lambda_n = \mu_n$.

- Definition 2.10: A basis of a vector space V is a linearly independent set $B \subset V$ which spans V.

- Theorem 2.11: Let V be a vector space over F, and let $B \subset V$ be linearly independent. Then there is a basis B' of V with $B \subset B'$.

- Theorem 2.12: Suppose $\operatorname{span} A = V$ and $B \subset V$ is linearly independent. Then there is a basis B' of V with $B \subset B' \subset A \cup B$.

- Lemma 2.13: The "exchange lemma". Suppose $a_1, a_2, \ldots a_n, b$ are vectors in a vector space V, and suppose that

$$b \in \operatorname{span}(a_1, a_2, \ldots a_{n-1}, a_n)$$

  but

$$b \notin \operatorname{span}(a_1, a_2, \ldots a_{n-1})\,,$$

  then

$$a_n \in \operatorname{span}(a_1, \ldots, a_{n-1}, b)\,.$$

  If in addition $\{a_1, a_2, \ldots a_n\}$ is linearly independent then so is $\{a_1, a_2, \ldots a_{n-1}, b\}$.

- Theorem 2.14: Suppose S and B are both bases of a vector space V over F and. Then A and B have the same number of elements.

- Definition 2.15: The number of elements of a basis of a vector space V over F is called the dimension of V and is written as $\dim V$.

- Corollary 2.16: If V is a vector space over F and $U \subset V$ is a subspace of V then $\dim U \leq \dim V$. If, additionally, $\dim V$ is finite and $U \neq V$ then $\dim U < \dim V$.

- Corollary 2.17: Suppose that V is a vector space over F, $\dim V$ is finite, and $U \subset V$ is a subspace of V with $\dim U = \dim V$, then $U = V$.

- The coordinates of a vector $v \in V$, with V a vector space over F, with respect to an *ordered* basis $B = \{v_1, v_2, \ldots v_n\}$ are $(\lambda_1, \lambda_2, \ldots, v_n)^T$ where

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \ldots \lambda_n v_n\,.$$

- Definition 2.18: Two vector spaces V and W, both over the same field F, are isomorphic if there is a bijection $f : V \to W$ such that
$$f(u+v) = f(u) + f(v)$$
and
$$f(\lambda v) = \lambda f(v)\,,$$
$\forall u, v \in V$ and $\forall \lambda \in F$. The bijection is said to be an isomorphism from V to W and we write $V \cong W$ or $f : V \xrightarrow{\sim} W$.

- Theorem 2.19: Suppose V is a vector space over $\mathbb{R}$ with finite dimension $n \geq 0$. Then $V \cong \mathbb{R}^n$ as real vector spaces. Similarly if V is a vector space over $\mathbb{C}$ with finite dimension $n \geq 0$. Then $V \cong \mathbb{C}^n$ as complex vector spaces.

3 Inner Product Spaces

- Definition 3.1: If V is a vector space over $\mathbb{R}$, then an inner product on V is a map $(\langle | \rangle)$ from $V \times V$ to $\mathbb{R}$ with the following properties:

  (a) Symmetry: $\langle v|w\rangle = \langle w|v\rangle \; \forall v, w \in V$.

  (b) Linearity: $\langle u|\lambda v + \mu w\rangle = \lambda\langle u|v\rangle + \mu\langle u|w\rangle \; \forall u, v, w \in V$ and $\forall \lambda, \mu \in \mathbb{R}$.

  (c) Positive definiteness:

    (i) $\langle v|v\rangle \geq 0 \; \forall v \in V$, and

    (ii) $\langle v|v\rangle = 0$ iff $v = 0$.

  As the inner product is linear with respect to both variables it is sometimes called bilinear.

- Definition 3.2: A finite dimensional vector space over $\mathbb{R}$ with an inner product defined is called a Euclidean space.

- Definition 3.3: The norm (or length) of a vector$v$ is written as $||v|$ and defined by

$$||v|| = \sqrt{\langle v|v\rangle},$$

  the positive square root of the inner product of v with itself. The distance between two vectors v and w is written as $d(v, w)$ and is $d(v, w) = ||v - w||$.

- Proposition 3.4: $\forall v \in V$, where V is a Euclidean space, and $\forall \lambda \in \mathbb{R}$ then $||\lambda v|| = |\lambda| \cdot ||v||$.

- Proposition 3.5: The "Cauchy-Schwarz inequality" says that $\forall v, w \in V$, where V is a Euclidean space, then

$$|\langle v|w\rangle| \leq ||v|| \cdot ||w|| \,.$$

- Proposition 3.6: The "triangle inequality" says that $\forall v, w \in V$, where V is a Euclidean space, then

$$||v + w|| \leq ||v|| + ||w|| \,.$$

- Definition 3.7: If V is a Euclidean space, and $v, w \in V$, then v and w are said to be orthogonal if $\langle v|w\rangle = 0$. If both v and w are nonzero, then the angle between v and w is defined to be θ, $0 \leq \theta \leq \pi$ and

$$\cos\theta = \frac{\langle v|w\rangle}{||v|| \cdot ||w||} \,.$$

- Definition 3.8: If V is a vector space over $\mathbb{C}$, then a map $(\langle | \rangle)$ from $V \times V$ to $\mathbb{C}$ is an inner product if the following are true:

  (a) Conjugate-Symmetry: $\langle v|w\rangle = \overline{\langle w|v\rangle} \; \forall v, w \in V$.

  (b) Linearity: $\langle u|\lambda v + \mu w\rangle = \lambda\langle u|v\rangle + \mu\langle u|w\rangle \; \forall u, v, w \in V$ and $\forall \lambda, \mu \in \mathbb{R}$.

  (c) Positive definiteness:

    (i) $\langle v|v\rangle \geq 0 \; \forall v \in V$, and

    (ii) $\langle v|v\rangle = 0$ iff $v = 0$.

  This inner product is sometimes called sesquilinear.

- Definition 3.9: A finite dimensional vector space over $\mathbb{C}$ with an inner product define is called a unitary space.

- A vector space over $\mathbb{R}$ or $\mathbb{C}$, of any dimension, we will refer to as an inner product space.

- Definition 3.10: The norm (or length) of a vector $v \in V$, with V a vector space over $\mathbb{C}$,is written as $||v||$ and defined by

$$||v|| = \sqrt{\langle v|v\rangle},$$

  the positive square root of the inner product of v with itself. The distance between two vectors v and w is written as $d(v, w)$ and is $d(v, w) = ||v - w||$.

- Proposition 3.11: $\forall v \in V$, where V is a unitary space, and $\forall \lambda \in \mathbb{R}$ then $||\lambda v|| = |\lambda| \cdot ||v||$.
- Proposition 3.12: The "Cauchy-Schwarz inequality" says that $\forall v, w \in V$, where V is a unitary space, then
$$|\langle v|w \rangle| \leq ||v|| \cdot ||w|| \,.$$
- Proposition 3.13: The "triangle inequality" says that $\forall v, w \in V$, where V is a unitary space, then
$$||v + w|| \leq ||v|| + ||w|| \,.$$
- Definition 3.14: A bilinear form on a real vector space V is a map $F : V \times V \to \mathbb{R}$ which $\forall u, v, w \in V$ and $\forall \alpha, \beta \in \mathbb{R}$ satisfies
  (a) $\langle \alpha u + \beta v | w \rangle = \alpha \langle u|w \rangle + \beta \langle v|w \rangle$, and
  (b) $\langle u | \alpha v + \beta w \rangle = \alpha \langle u|v \rangle + \beta \langle u|w \rangle$.
- Definition 3.15: A bilinear form on a real vector space V is symmetric if
  (c) $F(u, v) = F(v, u)$ $\forall u, v \in V$.
- Definition 3.16: The matrix $\mathbf{A} = (a_{ij})$ with $a_{ij} = F(e_i, e_j)$ is called the 'matrix of the bilinear form F with respect to the ordered basis $e_1, e_2, \ldots e_n$ of V'. If F is symmetric then $\mathbf{B}$ is symmetric.
- Proposition 3.17: Suppose V is a real vector space with ordered basis $e_1, e_2, \ldots e_n$ and F is a bilinear form defined on V, with matrix $\mathbf{A}$ with respect to this basis. Then for any vectors $v, w \in V$ and their corresponding coordinate forms $\mathbf{v} = (v_1, v_2, \ldots v_n)^T$ and $\mathbf{w} = (w_1, w_2, \ldots w_n)^T$ with respect to the same basis we have
$$F(v, w) = \mathbf{v}^T \mathbf{A} \mathbf{w} \,.$$
- The base change matrix from a basis $e_1, e_2, \ldots e_n$ to $f_1, f_2, \ldots f_n$ is $\mathbf{P} = (p_{ij})$ where $f_i = \sum_{k=1}^{n} p_{ki} e_k$.
- Proposition 3.18: (The base change formula) Given two ordered bases of a Euclidean space V, $e_1, e_2, \ldots e_n$ and $f_1, f_2, \ldots f_n$ related by the base change matrix $\mathbf{P}$ from basis $e_1, e_2, \ldots e_n$ to $f_1, f_2, \ldots f_n$, suppose $\mathbf{A}$ and $\mathbf{B}$ are the matrices of the inner product with respect to $e_1, e_2, \ldots e_n$ and $f_1, f_2, \ldots f_n$. Then $\mathbf{B} = \mathbf{P}^T \mathbf{A} \mathbf{P}$.
- Definition 3.19: A sesquilinear form on a complex vector space V is a map $F : V \times V \to \mathbb{C}$ which $\forall u, v, w \in V$ and $\forall \alpha, \beta \in \mathbb{C}$ satisfies
  (a) $\langle \alpha u + \beta v | w \rangle = \bar{\alpha} \langle u|w \rangle + \bar{\beta} \langle v|w \rangle$, and
  (b) $\langle u | \alpha v + \beta w \rangle = \alpha \langle u|v \rangle + \beta \langle u|w \rangle$.
- Definition 3.20: A sesquilinear form on a complex vector space V is conjugate-symmetric if
  (c) $F(u, v) = \overline{F(v, u)}$ $\forall u, v \in V$.
- Definition 3.21: The matrix $\mathbf{B} = (a_{ij}$ with $a_{ij} = F(e_i, e_j)$ is called the 'matrix of the bilinear form F with respect to the ordered basis $e_1, e_2, \ldots e_n$ of the complex vector space V'. If F is conjugate-symmetric then $\mathbf{B}$ is conjugate-symmetric, i.e. $\bar{\mathbf{B}}^T = \mathbf{B}$.
- Proposition 3.22: Suppose V is a complex inner product space with ordered basis $e_1, e_2, \ldots e_n$ and F is a sesquilinear form defined on V, with matrix $\mathbf{A}$ with respect to this basis. Then for any vectors $v, w \in V$ and their corresponding coordinate forms $\mathbf{v} = (v_1, v_2, \ldots v_n)^T$ and $\mathbf{w} = (w_1, w_2, \ldots w_n)^T$ with respect to the same basis we have
$$F(v, w) = \overline{\mathbf{v}}^T \mathbf{A} \mathbf{w} \,.$$
- Proposition 3.23: (The base change formula) Given two ordered bases of a complex inner product space V, $e_1, e_2, \ldots e_n$ and $f_1, f_2, \ldots f_n$ related by the base change matrix $\mathbf{P}$ from basis $e_1, e_2, \ldots e_n$ to $f_1, f_2, \ldots f_n$, suppose $\mathbf{A}$ and $\mathbf{B}$ are the matrices of the inner product with respect to $e_1, e_2, \ldots e_n$ and $f_1, f_2, \ldots f_n$. Then $\mathbf{B} = \overline{\mathbf{P}}^T \mathbf{A} \mathbf{P}$.

4 Orthogonal Bases

- Definition 4.1: Two vectors v and w in an inner product space are orthogonal if $\langle v|w\rangle=0$. The set of vectors $\{v_1, v_2, \ldots\}$ is said to be orthogonal, and the vectors $v_1, v_2 \ldots$ in the set are said to be mutually orthogonal if each pair of distinct vectors v_i, v_l with $i \neq l$ are said to be an orthogonal pair, $\langle v_i|v_l\rangle = 0$.

- Definition 4.2: A set $\{w_1, w_2, \ldots\}$ of vectors in an inner product space is said to be orthonormal if $\langle w_i|w_j\rangle = \delta_{ij}$. If the orthonormal set is a basis then it is called an orthonormal basis.

- Proposition 4.3: If V is an inner product space over $\mathbb{R}$ or $\mathbb{C}$, $v_1, v_2, \ldots v_n \in V$, $v_i \neq 0$ $\forall i = 1 \ldots n$, and the v_i are mutually orthogonal then $\{v_1, v_2, \ldots v_n\}$ is a linearly independent set.

- Lemma 4.4: If u, v are any two vectors in an inner product space V with $v \neq 0$ then the vector

$$w = u - \frac{\langle v|u\rangle}{\langle v|v\rangle} v$$

  is orthogonal to v.

- Lemma 4.5: If V is an inner product space, $u, v_1, v_2, \ldots v_k \in V$ and $v_1, v_2, \ldots v_k$ are mutually orthogonal non-zero vectors then

$$w = u - \sum_{i=1}^{k} \frac{\langle v_i|u\rangle}{\langle v_i|v_i\rangle} v_i$$

  is orthogonal tro $v_1, v_2, \ldots v_k$.

- Theorem 4.6: (The Gram-Schmidt process) If $\{v_1, \ldots v_n\}$ is a basis of a finite dimensional inner product space V, then $\{w_1, \ldots w_n\}$ obtained by

$$\begin{aligned} w_1 &= v_1 \\ w_2 &= v_2 - \tfrac{\langle w_1|v_2\rangle}{\langle w_1|w_1\rangle} w_1 \\ &\vdots \\ w_k &= v_k - \textstyle\sum_{i=1}^{k-1} \tfrac{\langle w_i|v_k\rangle}{\langle w_i|w_i\rangle} w_i \\ &\vdots \end{aligned}$$

  is an orthogonal basis of V.

- Corollary 4.7: Any finite dimensional inner product space V has an orthonormal basis.

- Definition 4.8: Two real vector spaces V, W with forms $F : V \times V \to \mathbb{R}$ and $G : W \times W \to \mathbb{R}$ respectively are isomorphic if there is a bijection $f : V \to W$ such that

$$\begin{aligned} f(u+v) &= f(u) + f(v)\,, \\ f(\lambda v) &= \lambda f(v) \text{ and} \\ F(u,v) &= G(f(u), g(v))\,, \end{aligned}$$

  $\forall u, v \in V$ and $\forall \lambda \in \mathbb{R}$.
  Similarly two complex vector spaces V, W with forms $F : V \times V \to \mathbb{C}$ and $G : W \times W \to \mathbb{C}$ respectively are isomorphic if there is a bijection $f : V \to W$ such that

$$\begin{aligned} f(u+v) &= f(u) + f(v)\,, \\ f(\lambda v) &= \lambda f(v) \text{ and} \\ F(u,v) &= G(f(u), g(v))\,, \end{aligned}$$

  $\forall u, v \in V$ and $\forall \lambda \in \mathbb{C}$.

- Corollary 4.9: Let V be a Euclidean vector space of dimension n. Then V is isomorphic to $\mathbb{R}^n$ with the standard inner product as an inner product space. Similarly each unitary vector space V of dimension n is isomorphic to $\mathbb{C}^n$ with the standard inner product as an inner product space.

- Proposition 4.10: Suppose that $\{e_1, e_2, \dots e_n\}$ is an orthonormal basis of a Euclidean space V. Then for any $v \in V$:
$$v = \sum_{i=1}^{n} \langle e_i|v\rangle e_i \, .$$

- Proposition 4.11: (Pythagoras' theorem) Suppose $e_1, e_2, \dots e_n$ is an orthonormal basis of a Euclidean space V. Then for all $v \in V$
$$||v||^2 = \sum_{i=1}^{n} \langle e_i|v\rangle^2 \, .$$

- Corollary 4.12: (Parseval's identity) If $e_1, e_2, \dots e_n$ is an orthonormal basis of a Euclidean space V, and $v, w \in V$, then
$$\langle v|w\rangle = \sum_{i=1}^{n} \langle v|\,\mathrm{e}_i\,\rangle\langle e_i|w\rangle \, . .$$

- Proposition 4.13: (Bessel's inequality) If $e_1, e_2, \dots e_k$ is an orthonormal set of vectors in a real inner product space V, and $v \in V$, then
$$\sum_{i=1}^{k} \langle e_i|v\rangle^2 \leq ||v||^2 \, .$$

- Proposition 4.14: If $e_1, e_2, \dots e_n$ is an orthonormal basis of a complex inner prodoct space V, and $v, w \in V$, then:
  (a) $v = \sum_{i=1}^{n} \langle e_i|v\rangle e_i$,
  (b) $||v||^2 = \sum_{i=1}^{n} \langle e_i|v\rangle^2$, (Pythagoras' theorem) and
  (c) $\langle v|w\rangle = \sum_{i=1}^{n} \langle v|\,\mathrm{e}_i\,\rangle\langle e_i|w\rangle = \sum_{i=1}^{n} \overline{\langle e_i|v\rangle}\langle e_i|w\rangle$ (Parseval's identity).

- Proposition 4.15: (Bessel's inequality) If $e_1, e_2, \dots e_k$ is an orthonormal set of vectors in a complex inner product space V, and $v \in V$, then
$$\sum_{i=1}^{k} |\langle e_i|v\rangle|^2 \leq ||v||^2 \, .$$

- Definition 4.16: If U and W are subspaces of a vector space V then the sum of U and W is defined as
$$U + W = \{u + w : u \in U, w \in W\} \, .$$

- Proposition 4.17: $U + W$ is a subspace of a vector space V if U and W are subspaces of V.

- The union of two sets is $A \cup B = \{x : x \in A \vee x \in B\}$. I.e. the elements in either A or B. The intersection of two sets is $A \cap B = \{x : x \in A \wedge x \in B\}$. I.e. the elements in both A or B.

- Definition 4.18: If V is a vector space and U is a subspace of V, then W is called a complement to U in V if
  (a) W is a subspace of V,
  (b) $V = U + W$, and
  (c) $U \cap W = \{0\}$.

  When these conditions are met we write $V = U \oplus W$, and say that V is the direct sum of U and W.

- Definition 4.19: If V is an inner product space and U is a subspace of V we define
$$U^\perp = \{v \in V : \langle u|v\rangle = 0 \, \forall u \in U\} \, .$$
  This is called the orthogonal complement of U in V, or "U perp" for short.

- Lemma 4.20: If V is an inner product space, U is a subspace of V, and U has a basis $\{u_1, \dots u_k\}$, then
$$U^\perp = \{v \in V : \langle u_i | v \rangle = 0 \, \forall i = 1, \dots k\} \, .$$
- Proposition 4.21: If V is an inner product space, and U is a finite dimensional subspace of V, then
  (a) $U^\perp$ is a subspace of V,
  (b) $U \cap U^\perp = \{0\}$, and
  (c) $U + U^\perp = V$.
- Proposition 4.22: If $V = U \oplus W$ then $\dim(V) = \dim(U) + \dim(W)$.
- Corollary 4.23: If V is a finite dimensional inner product space, and U is a subspace of V, then
  (a) $\dim(U) + \dim(U^\perp) = \dim(V)$, and
  (b) $(U^\perp)^\perp = U$.

5 Multivariate Calculus

6 Complex Analysis