Linear Algebra I
Summary of Lectures:
Orthogonal Bases

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1. Definition 4.1: Two vectors v and w in an inner product space are orthogonal if $\langle v|w\rangle$=0. The set of vectors $\{v_1, v_2, \ldots\}$ is said to be orthogonal, and the vectors $v_1, v_2 \ldots$ in the set are said to be mutually orthogonal if each pair of distinct vectors v_i, v_l with $i \neq l$ are said to be an orthogonal pair, $\langle v_i|v_l\rangle = 0$.

2. Definition 4.2: A set $\{w_1, w_2, \ldots\}$ of vectors in an inner product space is said to be orthonormal if $\langle w_i|w_j\rangle = \delta_{ij}$. If the orthonormal set is a basis then it is called an orthonormal basis.

3. Proposition 4.3: If V is an inner product space over $\mathbb{R}$ or $\mathbb{C}$, $v_1, v_2, \ldots v_n \in V$, $v_i \neq 0\, \forall i = 1 \ldots n$, and the v_i are mutually orthogonal then $\{v_1, v_2, \ldots v_n\}$ is a linearly independent set.

4. Lemma 4.4: If u, v are any two vectors in an inner product space V with $v \neq 0$ then the vector
$$w = u - \frac{\langle v|u\rangle}{\langle v|v\rangle} v$$
is orthogonal to v.

5. Lemma 4.5: If V is an inner product space, $u, v_1, v_2, \ldots v_k \in V$ and $v_1, v_2, \ldots v_k$ are mutually orthogonal non-zero vectors then
$$w = u - \sum_{i=1}^{k} \frac{\langle v_i|u\rangle}{\langle v_i|v_i\rangle} v_i$$
is orthogonal tro $v_1, v_2, \ldots v_k$.

6. Theorem 4.6: (The Gram-Schmidt process) If $\{v_1, \ldots v_n\}$ is a basis of a finite dimensional inner product space V, then $\{w_1, \ldots w_n\}$ obtained by
$$\begin{aligned} w_1 &= v_1 \\ w_2 &= v_2 - \tfrac{\langle w_1|v_2\rangle}{\langle w_!|w_1\rangle} w_1 \\ &\vdots \\ w_k &= v_k - \textstyle\sum_{i=1}^{k-1} \tfrac{\langle w_i|v_k\rangle}{\langle w_i|w_i\rangle} w_i \\ &\vdots \end{aligned}$$
is an orthogonal basis of V.