Linear Algebra I
Problem Set 7: Linear Transformations

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Friday March 4th 2016

Due: In class, March 18th 2016

1. Let $f(\mathbf{u}) = \mathbf{A}\mathbf{u}$ be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ with $\mathbf{u} \in \mathbb{R}^4$ and $\mathbf{A}$ the 3×4 matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 2 & -1 & 1 \\ 1 & 3 & -4 & -1 \end{pmatrix} .$$

By converting $\mathbf{A}$ into echelon form, $\mathbf{B}$, calculate the rank and nullity of $\mathbf{A}$. If $\mathbf{R}\mathbf{A} = \mathbf{B}$ what is the matrix $\mathbf{R}$? By constructing a basis which spans $\mathrm{im}(\mathbf{B})$ find a basis which spans $\mathrm{im}(\mathbf{A})$. Note that if $\mathbf{u}_1, \mathbf{u}_2, \ldots$ is a basis for $\mathrm{im}(\mathbf{A})$ then $\mathbf{R}^{-1}\mathbf{u}_1, \mathbf{R}^{-1}\mathbf{u}_2, \ldots$ is a basis for $\mathrm{im}(\mathbf{A})$. (8)

2. For the set of vectors $S = \{(1,2,0)^T, (1,0,-1)^T\}$ in $\mathbb{R}^3$ find a linear map $f : \mathbb{R}^n \to \mathbb{R}^3$ whose kernel is spanned by S. (3)

3. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$f((x,y,z)^T) = (-x + y - z, x + 2y, -y - 2z)^T .$$

Calculate the images of the vectors (*i.e.* $f(\mathbf{v}_1)$, *etc.*)

$$\mathbf{v}_1 = (0,1,1)^T , \quad \mathbf{v}_2 = (1,-1,1)^T , \text{ and } \mathbf{v}_3 = (2,1,0)^T .$$

Verify that $f(\mathbf{v}_1) = -\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3$, and derive similar expressions for $f(\mathbf{v}_2)$ and $f(\mathbf{v}_3)$. Hence write down the matrix of f with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of $\mathbb{R}^3$. (7)

4. Let $f(u) = \frac{\mathrm{d}u}{\mathrm{d}x}$ be a linear transformation $f : V \to V$ where V is the space of polynomials of order 3 or less over $\mathbb{R}$. What is the kernel of f? What is the nullity of f? (2)

Total available marks: 20