

Linear Algebra I

Exercises III:

Vector Spaces and Fields

Dr Nicholas Sedlmayr

1. Prove the following: Suppose $\mathbf{A} = \{a_1, a_2, \dots, a_n\} \subset V$ is linearly independent, where V is a vector space over F . Suppose also that $v \in V$ and there are scalars $\lambda_1, \dots, \lambda_n$ and μ_1, \dots, μ_n such that

$$v = \lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n$$

and

$$v = \mu_1 a_1 + \mu_2 a_2 + \dots + \mu_n a_n$$

then $\lambda_1 = \mu_1, \lambda_2 = \mu_2, \dots, \lambda_n = \mu_n$. (*Tip: consider the definition of linear independence.*)

2. Prove that the set of Pauli matrices and the 2×2 identity matrix, $A = \{\mathbb{I}_2, \sigma^x, \sigma^y, \sigma^z\}$, is a basis of the vector space V , the set of all 2×2 matrices over \mathbb{C} , with addition and scalar multiplication defined in the usual way. The Pauli matrices are

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

3. Find a basis for the following vector spaces:
- (a) The set of all 2×2 matrices over \mathbb{R} .
 - (b) \mathbb{C}^4 , i.e. the set of 4×1 column vectors with complex entries.
4. Which of the following are bases over \mathbb{R}^3 ? Give reasons!
- (a) $A = \{(1, 1, 0)^T, (0, 1, 0)^T, (1, 0, 1)^T, (0, 0, 1)^T\}$.
 - (b) $B = \{(1, 1, 0)^T, (0, 1, 0)^T, (1, 0, 1)^T\}$.
 - (c) $C = \{(1, 0, 0)^T, (0, i, 0)^T, (1, 0, i)^T\}$.
 - (d) $D = \{(1, 1, 1)^T, (2, 2, 1)^T, (1, 1, 0)^T\}$.
5. Find bases over the following subspaces of \mathbb{R}^3 .
- (a) $A = \{(x, y, z)^T : 2x + y - z = 0\}$.
 - (b) $B = \{(x, y, z)^T : x + y - 2z = 0, x - y = 0\}$.
6. Prove that the set of vectors $U = \{(x, y, 0)^T : x, y \in \mathbb{R}\}$ forms a subspace of the vector space \mathbb{R}^3 .

7. Prove that one can not define a field of order 4 with the integers modulo 4, $\{0, 1, 2, 3\}$.
8. Prove that one can define a field of order 4 from $\{0, 1, a, a+1\}$. *Tip: Start from normal addition and multiplication and determine what needs to be changed to satisfy the definition of a field. Marks will be given if a clear outline of what needs to be done can be given, even without the complete solution.*