Linear Algebra I
Summary of Lectures:
Quadratic and Hermitian Forms

Dr Nicholas Sedlmayr

1. Definition 7.1: Given a symmetric bilinear form F on a real vector space V, we define a map $Q : V \to \mathbb{R}$ by $Q(v) = F(v,v)$; Q is called the quadratic form associated with the symmetric bilinear form F.

2. Lemma 7.2: Given a symmetric bilinear form F on a real vector space V, and the quadratic form Q associated with F, then

$$F(v,w) = \frac{1}{2}\left(Q(v+w) - Q(v) - Q(w)\right)\,, \quad \forall v,w \in V\,.$$

3. Definition 7.3: Given a conjugate-symmetric sesquilinear form F on a complex vector space V, we define a map $H : V \to \mathbb{R}$ by $H(v) = F(v,v)$; H is called the Hermitian form associated with the conjugate-symmetric sesquilinear form F.

4. Lemma 7.4: Given a conjugate-symmetric sesquilinear form F on a complex vector space V, and the Hermitian form H associated with F, then $\forall v,w \in V$:

$$\begin{aligned} F(v,w) &= \tfrac{1}{2}\left(H(v+w) + \mathrm{i}H(v-\mathrm{i}w) - (1+\mathrm{i})(H(v)+H(w))\right)\,, \\ F(v,w) &= \tfrac{1}{4}\left(H(v+w) - \mathrm{i}H(v-w) + \mathrm{i}H(v-\mathrm{i}w) - \mathrm{i}H(v+\mathrm{i}w)\right)\,. \end{aligned}$$

5. Proposition 7.5: If Q is a quadratic form on a real vector space V, then

$$Q(\lambda x) = \lambda^2 Q(x)\,, \quad \forall \lambda \in \mathbb{R}\,, \text{ and } \forall x \in V\,.$$

Similarly if H is a Hermitian form on a complex vector space V, then

$$H(\lambda x) = |\lambda|^2 H(x)\,, \quad \forall \lambda \in \mathbb{C}\,, \text{ and } \forall x \in V\,.$$

6. Proposition 7.6: Let $V = \mathbb{R}^n$. Then every quadratic form on V is given by a homogeneous function of the coordinates of degree 2. Conversely every homogeneous function of degree 2 of the coordinates is a quadratic form.

7. Theorem 7.7: (Sylvester's law of inertia part 1.) Let V be an n dimensional real vector space, and let F be a symmetric bilinear form on V. Then there are non-negative integers k and m, and a basis $\{w_1, w_2, \ldots w_n$ of V such that:

$$\begin{aligned} F(w_i,w_j) &= 0 \quad \forall i \neq j\,, \\ F(w_i,w_i) &= 1 \quad \text{for } i \leq k\,, \\ F(w_i,w_i) &= -1 \text{ for } k < i \leq k+m\,, \\ F(w_i,w_i) &= 0 \quad \text{for } k+m < i\,. \end{aligned}$$

8. Lemma 7.8: Let F be a symmetric bilinear form on a real vector space V, and suppose that $F(v,v) = 0 \ \forall v \in V$. Then $F(v,w) = 0 \ \forall v, w \in V$.

9. Lemma 7.9: Let F be a bilinear form on a real vector space V and suppose that $w_1, w_2, \ldots w_n$ are vectors from V which are orthogonal with respect to F. For all scalares $\lambda_i \in \mathbb{R}$ if
$$\lambda_1 w_1 + \lambda_2 w_2 + \ldots + \lambda_n w_n = 0$$
then $\lambda_j = 0 \ \forall j$ such that $F(w_j, w_j) = 0$.

10. Lemma 7.10: Let F be a bilinear form on a real vector space V and suppose that $w_1, w_2, \ldots w_k$ are vectors from V which are orthogonal with respect to F, and that $F(w_i, w_i) \neq 0 \ \forall i \leq k$. Then $\forall v \in V \ \exists u \in V$ such that $F(w_i, u) = 0 \ \forall i \leq k$, and v is a linear combination of $w_1, w_2, \ldots w_k, u$. I.e.
$$V = \text{span}(U \cup \{w_1, w_2, \ldots w_k\})$$
where
$$U = \{u \in V : F(w_i, u) = 0, \forall i \leq k\}.$$

11. Theorem 7.11: (Sylvester's law of inertia part 2.)