Linear Algebra I
Exercises IV: Inner Products

Dr Nicholas Sedlmayr

1. Prove the Cauchy-Schwarz inequality which states that $\forall v, w \in V$, where V is a Euclidean space then $|\langle v|w\rangle| \leq ||v|| \cdot ||w||$.
   Tip: Consider the properties of $||v + \lambda w||^2 \geq 0$ *as a polynomial in* λ.

2. Starting from the triangle inequality,
   $$||v + w|| \leq ||v|| + ||w|| \,,$$
   derive the following alternative forms of it:
   $$||v - w|| \leq ||v|| + ||w|| \,, \; ||v + w|| \geq ||v|| - ||w|| \,, \text{ and } ||v - w|| \geq ||v|| - ||w|| \,,$$
   for $v, w \in V$, where V is a Euclidean space.

3. Prove that in any complex inner product space
   (a) $\langle u|v\rangle = \frac{1}{2}\left(||u + v||^2 + \mathrm{i}||u - \mathrm{i}v||^2 - (1 + \mathrm{i})(||u||^2 + ||v||^2)\right)$, and
   (b) $\langle u|v\rangle = \frac{1}{4}\left(||u + v||^2 - ||u - v||^2 + \mathrm{i}||u - \mathrm{i}v||^2 - \mathrm{i}||u + \mathrm{i}v||^2\right)$.

4. Determine whether
   $$\langle \mathbf{u}|\mathbf{v}\rangle = (u_1 - v_1)^2 + (u_2 - v_2)^2$$
   is an inner product on $\mathbb{R}^2$ with $\mathbf{u} = (u_1, u_2)^T$, and $\mathbf{v} = (v_1, v_2)^T$.

5. Let $V = \mathbb{R}^3$ with the standard inner product. Describe the set of vectors orthogonal to $(1, 0, -3)^T$. Show that this set is a subspace of V, find a basis and hence determine the dimension of the subspace.