1. Near the boundary of the first Brillouin zone, for a one-dimensional chain of atoms, at $k = \pi/a$, the nearly free electron theory predicts that the most important term in the lattice potential is $V(x) \approx V_1 \cos[2\pi x/a]$. The wavefunction is then approximately

$$\psi(x) \approx \alpha \mathrm{e}^{\mathrm{i}kx} + \beta \mathrm{e}^{\mathrm{i}(k-2\pi/a)x} \,.$$

Substitute this wavefunction into Schrödinger's equation:

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = \epsilon\psi(x) \,.$$

To find equations for α and β we can use that $\mathrm{e}^{\mathrm{i}kx}$ and $\mathrm{e}^{\mathrm{i}(k-2\pi/a)x}$ are *orthogonal*. Let us do this explicitly. Multiply the result of your substitution of the $\psi(x)$ into Schroödinger's equation by

(a) $\mathrm{e}^{\mathrm{i}kx}$, and

(b) $\mathrm{e}^{\mathrm{i}(k-2\pi/a)x}$.

Then integrate over all space to find two equations for α and β. Solve these equations and show that

$$\epsilon = \frac{\hbar^2 k^2}{2m} + \frac{\pi\hbar^2}{ma}\left[\frac{\pi}{a} - k \pm \sqrt{\left(\frac{\pi}{a} - k\right)^2 + \left(\frac{amV_1}{2\pi\hbar^2}\right)^2}\,\right]$$

is the energy. Show that this agree with our solution near $k = \pi/a$ and $k = 0$.

2. Consider a monovalent metal with a simple cubic lattice of spacing a. Calculate the radius of the Fermi sphere using the free electron theory. Is this sphere contained completely within the first Brillouin zone?