Linear Algebra
Exercises II:
Matrices and Vector Spaces

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1. Are the following matrices invertible?

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 9 & 6 & 2 \\ 3 & 0 & 0 & 6 \\ 5 & 8 & 9 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 9 & 6 & 2 \\ 3 & 0 & 0 & 6 \\ 5 & 8 & 9 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 9 & 2 \\ 3 & 0 & 6 \end{pmatrix}.$$

2. Prove proposition 1.8 from the lectures, that the determinant of an upper triangular square matrix is equal to the product of its diagonal entries.

3. Prove that if a square matrix $\mathbf{A}$ is invertible then $\det \mathbf{A}^{-1} = (\det \mathbf{A})^{-1}$.

4. Prove that the set of vectors $\mathbb{R}^2$ forms a vector space when the addition of two vectors $\mathbf{u} = (x_1, y_1)^T$ and $\mathbf{v} = (x_2, y_2)^T$ is defined as $\mathbf{u} + \mathbf{v} = (x_1 x_2, y_1 y_2)^T$ and scalar multiplication as $\lambda \mathbf{u} = (x_1^\lambda, y_1^\lambda)^T$. What is the zero vector in this vector space?

5. Let V be the set of all functions $f, g, \ldots$ from the natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ to $\mathbb{R}$, with addition defined by $(f + g)(n) = f(n) + g(n)$ $\forall n \in \mathbb{N}$ and scalar multiplication $(\lambda f)(n) = \lambda \cdot f(n)$ $\forall n \in \mathbb{N}$. (This can also be written as the set of functions $f : \mathbb{N} \to \mathbb{R}$.) Prove that V is a vector space. What is the zero vector in this vector space?