Linear Algebra I
Problem Set 5: Inner Products and Bilinear and Sesquilinear Forms

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Due: In class, February 26th 2016

1. Let V be the vector space over $\mathbb{R}$ of all polynomials of degree less than 3. I.e. $V = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2, x \in \mathbb{R}\}$. We can define an inner product on this space as
$$\langle f|g\rangle = \int_0^1 \mathrm{d}x f(x)g(x)\,.$$
$1, x, x^2$ is an ordered basis of V, find the corresponding matrix of inner products $\mathbf{A}$. Use this matrix to calculate $\langle x+1|x^2-2\rangle$ and $\langle x^2 - x + 5|x^2+2x\rangle$. Confirm the results by direct integration. (6)

2. Prove that the map $F : V \times V \to \mathbb{C}$ is a sesquilinear form on $V = \mathbb{C}^n$ where $F(\mathbf{x}, \mathbf{y}) = \bar{\mathbf{x}}^T \mathbf{A}\mathbf{y}$, $\mathbf{A}$ is an $n \times n$ matrix, and $\mathbf{x}, \mathbf{y} \in V$. Furthermore prove that if F is conjugate symmetric then $\overline{\mathbf{A}}^T = \mathbf{A}$. (4)

3. A sesquilinear form on $\mathbb{C}^3$ is defined by
$$F((x,y,z)^T, (x',y',z')^T) = \begin{pmatrix}\bar{x} & \bar{y} & \bar{z}\end{pmatrix} \begin{pmatrix} 0 & -1-2\mathrm{i} & 0 \\ -1+2\mathrm{i} & 0 & \mathrm{i} \\ 0 & -\mathrm{i} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}\,.$$
What is the matrix of F with respect to the bases
   (a) $\{(\mathrm{i},1,0)^T, (1,1+\mathrm{i},1)^T, (0,0,\mathrm{i})^T\}$, and
   (b) $\{(-1-3\mathrm{i}, 2+\mathrm{i}, -1+2\mathrm{i})^T, (2+\mathrm{i}, 1-2\mathrm{i}, 2+\mathrm{i})^T, (0,0,-\mathrm{i})^T\}$?

   Is F an inner product? (5)

4. Ex. 4.8. An alternating form F is a bilinear form on a vector space V satisfying $F(v,v) = 0\ \forall v \in v$.
   (a) Show that if F is an alternating form then $F(u,v) = -F(v,u)$, i.e. that F is skew symmetric. (3)
   (b) Give an example of an alternating form (other than the zero function). (2)

Total available marks: 20