Linear Algebra I
Summary of Lectures: Matrices

Dr Nicholas Sedlmayr

1. Definition of an $n \times m$ matrix, $\mathbf{A} = (a_{ij})$ with n row and m columns. Addition of matrices $\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})$.
   - Associativity, commutativity and existence of a zero matrix ($\mathbf{0}$) for addition.
2. Definition of multiplication of a matrix by a scalar: $\lambda\mathbf{A} = (\lambda a_{ij})$.
3. Definition: Matrix multiplication of an $n \times m$ matrix $\mathbf{A}$ and an $m \times k$ matrix $\mathbf{B}$ is an $n \times k$ matrix $\mathbf{C} = \mathbf{AB} = (c_{ij})$ where $c_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$.
   - Associativity, existence of a zero matrix ($\mathbf{0}$) and an identity matrix $\mathbb{I}$, distributivity.
   - No commutativity!
4. A matrix $\mathbf{A}$ can have a right inverse $\mathbf{AB} = \mathbb{I}$ and a left inverse $\mathbf{CA} = \mathbb{I}$.
   - Proposition 1.1: If a square matrix has either a left or right inverse then they have a unique inverse from both the left and right.
   - If a non-square matrix has both a left and right inverse then they are the same and the inverse is unique.
   - Proposition 1.2: If $\mathbf{A}$ and $\mathbf{B}$ are invertible square matrices then $\mathbf{AB}$ is also invertible and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
5. The transpose of an $n \times m$ matrix is written as $\mathbf{A}^T$, which is an $m \times n$ matrix found by transposing the rows and columns of $\mathbf{A}$.
   - Proposition 1.3: For two $n \times n$ matrices $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ and $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.
6. Elementary row operations perform simple operations on the rows of an $n \times m$ matrix $\mathbf{A}$ and can be written as an $n \times n$ matrix $\mathbf{R}$ with the operation preformed by the multiplication $\mathbf{RA}$. ρ_i is used to refer to row i. There are three of them:
   - $\rho_j := \rho_j + \lambda\rho_i$, add λ copies of row i to row j;
   - $\rho_i := \lambda\rho_i$, multiple row i by λ with $\lambda \neq 0$; and
   - swap(ρ_i, ρ_j), swap rows i and j.

7. Echelon form: A matrix where each row starts with a sequence of zeros, and the number of zeros in this sequences increases from row to row from top to bottom until the final row is reached or all remaining rows are composed entirely of zeros is said to be in echelon form.

   - All matrices can be put into echelon form using elementary row operations.

8. Rank: If $\mathbf{A}$ can be converted to the echelon form matrix $\mathbf{B}$ and $\mathbf{B}$ has k non-zero rows then the rank of $\mathbf{A}$ is $\operatorname{rk}\mathbf{A} = k$.

9. An $n \times k$ matrix $\mathbf{A}$ with $n \leq k$ and $\operatorname{rk}\mathbf{A} = n$ can be converted to a matrix $\mathbf{B}$ where the left $n \times n$ block is the identity matrix using elementary row operations.

10. Augmented matrix: If we have an $n \times m$ matrix $\mathbf{A}$ and an $n \times k$ matrix $\mathbf{A}$ the augmented matrix $(\mathbf{A}|\mathbf{B})$ is the $n \times (m+k)$ matrix created by writing $\mathbf{A}$ and $\mathbf{B}$ next to each other.

11. Calculating the inverse: For an $n \times n$ matrix $\mathbf{A}$ of rank n we can calculate the inverse using the augmented matrix $(\mathbf{A}, \mathbb{I})$. Apply row operations, $\mathbf{R}$, such that $\mathbf{R}(\mathbf{A}, \mathbb{I}) = (\mathbb{I}, \mathbf{B})$, clearly $\mathbf{B} = \mathbf{R}$ is the left inverse of $\mathbf{A}$: $\mathbf{BA} = \mathbb{I}$.

12. Solving systems of linear equations. A system of linear equations on the variables $x_1, x_2, \ldots x_n$ can be written as $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is a $k \times n$ matrix, $\mathbf{x} = (x_1, x_2, \ldots x_n)^T$ and $\mathbf{b}$ is a $k \times 1$ column vector. By putting the augmented matrix $(\mathbf{A}|\mathbf{b})$ into echelon form one can read off the solution.

   - Proposition 1.4: An equation $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is a $k \times n$ matrix, $\mathbf{x} = (x_1, x_2, \ldots x_n)^T$ and $\mathbf{b}$ is a $k \times 1$ column vector has at least one solution iff $\operatorname{rk}\mathbf{A} = \operatorname{rk}(\mathbf{A}|\mathbf{b})$. It has exactly one solution if $\operatorname{rk}\mathbf{A} = n$

13. Trace: This operation is defined only for square matrices. A square $n \times n$ matrix $\mathbf{A} = (a_{ij})$ has a trace $\operatorname{tr}\mathbf{A} = \sum_{i=1}^{n} a_{ii}$. I.e. it is the sum over all diagonal entries of the matrix. It has the following properties:

   - For two square $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, $\operatorname{tr}\mathbf{AB} = \operatorname{tr}\mathbf{BA}$.
   - If $\mathbf{P}$ is a square $n \times n$ invertible matrix and $\mathbf{A}$ is a square $n \times n$ matrix then $\operatorname{tr}\mathbf{P}^{-1}\mathbf{AP} = \operatorname{tr}\mathbf{A}$.
   - $\operatorname{tr}(\mathbf{A} + \mathbf{B}) = \operatorname{tr}\mathbf{A} + \operatorname{tr}\mathbf{B}$.
   - $\operatorname{tr}\mathbf{A}^T = \operatorname{tr}\mathbf{A}$.
   - For a scalar λ then $\operatorname{tr}\lambda\mathbf{A} = \lambda \operatorname{tr}\mathbf{A}$.

14. Determinant: This operation is defined only for square matrices and we will define it via "expansion by the first row". For a 1×1 matrix $\mathbf{A} = (a_1 1)$

we have $\det \mathbf{A} = |\mathbf{A}| = a_{11}$. For an $n \times n$ matrix $\mathbf{A} = (a_{ij})$ it is

$$\begin{aligned}
\det \mathbf{A} &= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \\
&= a_{11} \begin{vmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n3} & \dots & a_{nn} \end{vmatrix} \\
&\quad + \ldots + (-1)^{n-1} a_{1n} \begin{vmatrix} a_{21} & a_{22} & \dots & a_{2,n-1} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n,n-1} \end{vmatrix} .
\end{aligned}$$

- Proposition 1.5: For two $n \times n$ square matrices $\mathbf{A}$ and $\mathbf{B}$, $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$. It can be proved using the following lemmas.
- Lemma 1.5.1: Any square matrix $\mathbf{A}$ can be written as the product $\mathbf{S}_1\mathbf{S}_2 \ldots \mathbf{S}_k$ of 'generalized' elementary row operations $\rho_i := \lambda\rho_i$ and $\rho_i := \rho_i + \lambda\rho_j$ where λ can be zero.
- Lemma 1.5.2: The determinant of a matrix whose top two rows are identical is zero.
- Lemma 1.5.3:

$$\begin{vmatrix} a_{11} + \lambda b_{11} & a_{12} + \lambda b_{12} & \dots & a_{1n} + \lambda b_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + \lambda \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} .$$

- Lemma 1.5.4: The matrix of any generalized elementary row operation can always be written as one of $\mathbf{S}_i\mathbf{T}_j\mathbf{R}\mathbf{T}_j\mathbf{S}_i$, $\mathbf{T}_j\mathbf{R}\mathbf{T}_j$, $\mathbf{S}_i\mathbf{R}\mathbf{S}_i$, or $\mathbf{R}$, where $\mathbf{R}$ is the matrix of a generalized row operation acting only on rows 1 and 2, and $\mathbf{S}_i$, $\mathbf{T}_j$ are the matrices of $\mathrm{swap}(\rho_1, \rho_i)$ and $\mathrm{swap}(\rho_2, \rho_j)$ respectively.
- Lemma 1.5.5: If $\mathbf{R}$ is the matrix of the swap operation $\mathrm{swap}(\rho_i, \rho_j)$ then $\det \mathbf{RA} = -\det \mathbf{A}$.

15. Proposition 1.6: For an $n \times n$ matrix $\mathbf{A}$ the following are equivalent:
    (a) $\mathbf{A}^{-1}$ exists,
    (b) $\det \mathbf{A} \neq 0$, and
    (c) $\mathrm{rk}\, \mathbf{A} = n$.
16. Proposition 1.7: For an $n \times n$ matrix $\mathbf{A}$:

(a) $\det(\mathbf{A}^T) = \det \mathbf{A}$,

(b) If $\mathbf{A}$ is invertible $\det \mathbf{A}^{-1} = (\det \mathbf{A})^{-1}$,

(c) $\det \mathbb{I} = 1$, and

(d) if $\mathbf{A}$ has a row (or column) entirely composed of zeros $\det \mathbf{A} = 0$.

17. Proposition 1.8: The determinant of an upper triangular matrix $\mathbf{A}$ is equal to the product of its diagonal entries.

18. Proposition 1.9: The determinants of row operations are:

    (a) If $\mathbf{A}_{ij,\lambda}$ is the matrix for $\rho_i := \rho_i + \lambda\rho_j$ then $\det \mathbf{A}_{ij,\lambda} = 1$,

    (b) If $\mathbf{T}_{i,\lambda}$ is the matrix for $\rho_i := \lambda\rho_i$ then $\det \mathbf{T}_{i,\lambda} = \lambda$, and

    (a) If $\mathbf{S}_{ij}$ is the matrix for $\text{swap}(\rho_i, \rho_j)$ then $\det \mathbf{S}_{ij} = -1$.

19. Proposition 1.10: If we swap any two rows in a determinant then the determinant changes sign, see Lemma 1.5.5. It follows that if a matrix has two identical rows then its determinant is zero. (This is equally true for a matrix with two identical columns.)

20. Proposition 1.11: A determinant can be expanded along any row (or column). The sign associated with any entry a_{ij} is $(-1)^{i+j}$.

21. Minors and cofactors: Let $\mathbf{A} = (a_{ij})$ be an $n \times n$ matrix and let b_{ij} be the determinant of the $(n-1) \times (n-1)$ matrix obtained form $\mathbf{A}$ by deleting row i and column j. Furthermore let $c_{ij} = (-1)^{i+j} b_{ij}$. Then

    - $\mathbf{B} = (b_{ij})$ is the matrix of minors of $\mathbf{A}$,
    - $\mathbf{C} = (c_{ij})$ is the matrix of cofactors of $\mathbf{A}$, and
    - $\text{adj}\, \mathbf{A} = \mathbf{C}^T$ is the adjugate matrix of $\mathbf{A}$.