Linear Algebra I
Summary of Lectures:
Quadratic and Hermitian Forms

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1. Definition 7.1: Given a symmetric bilinear form F on a real vector space V, we define a map $Q : V \to \mathbb{R}$ by $Q(v) = F(v, v)$; Q is called the quadratic form associated with the symmetric bilinear form F.

2. Lemma 7.2: Given a symmetric bilinear form F on a real vector space V, and the quadratic form Q associated with F, then
$$F(v, w) = \frac{1}{2}\left(Q(v + w) - Q(v) - Q(w)\right) , \quad \forall v, w \in V .$$

3. Definition 7.3: Given a conjugate-symmetric sesquilinear form F on a complex vector space V, we define a map $H : V \to \mathbb{R}$ by $H(v) = F(v, v)$; H is called the Hermitian form associated with the conjugate-symmetric sesquilinear form F.

4. Lemma 7.4: Given a conjugate-symmetric sesquilinear form F on a complex vector space V, and the Hermitian form H associated with F, then $\forall v, w \in V$:
$$\begin{aligned} F(v, w) &= \tfrac{1}{2}\left(H(v + w) + \mathrm{i}H(v - \mathrm{i}w) - (1 + \mathrm{i})(H(v) + H(w))\right) , \\ F(v, w) &= \tfrac{1}{4}\left(H(v + w) - \mathrm{i}H(v - w) + \mathrm{i}H(v - \mathrm{i}w) - \mathrm{i}H(v + \mathrm{i}w)\right) . \end{aligned}$$

5. Proposition 7.5: If Q is a quadratic form on a real vector space V, then
$$Q(\lambda x) = \lambda^2 Q(x) , \quad \forall \lambda \in \mathbb{R} , \text{ and } \forall x \in V .$$
Similarly if H is a Hermitian form on a complex vector space V, then
$$H(\lambda x) = |\lambda|^2 H(x) , \quad \forall \lambda \in \mathbb{C} , \text{ and } \forall x \in V .$$

6. Proposition 7.6: Let $V = \mathbb{R}^n$. Then every quadratic form on V is given by a homogeneous function of the coordinates of degree 2. Conversely every homogeneous function of degree 2 of the coordinates is a quadratic form.

7. Theorem 7.7: (Sylvester's law of inertia part I.) Let V be an n dimensional real vector space, and let F be a symmetric bilinear form on V. Then there are non-negative integers k and m, and a basis $\{w_1, w_2, \dots w_n$ of V such that:
$$\begin{aligned} F(w_i, w_j) &= 0 && \forall i \neq j , \\ F(w_i, w_i) &= 1 && \text{for } i \leq k , \\ F(w_i, w_i) &= -1 && \text{for } k < i \leq k + m , \\ F(w_i, w_i) &= 0 && \text{for } k + m < i . \end{aligned}$$

8. Lemma 7.8: Let F be a symmetric bilinear form on a real vector space V, and suppose that $F(v,v) = 0 \; \forall v \in V$. Then $F(v,w) = 0 \; \forall v, w \in V$.

9. Lemma 7.9: Let F be a bilinear form on a real vector space V and suppose that $w_1, w_2, \ldots w_n$ are vectors from V which are orthogonal with respect to F. For all scalares $\lambda_i \in \mathbb{R}$ if
$$\lambda_1 w_1 + \lambda_2 w_2 + \ldots + \lambda_n w_n = 0$$
then $\lambda_j = 0 \; \forall j$ such that $F(w_j, w_j) = 0$.

10. Lemma 7.10: Let F be a bilinear form on a real vector space V and suppose that $w_1, w_2, \ldots w_k$ are vectors from V which are orthogonal with respect to F, and that $F(w_i, w_i) \neq 0 \; \forall i \leq k$. Then $\forall v \in V \; \exists u \in V$ such that $F(w_i, u) = 0 \; \forall i \leq k$, and v is a linear combination of $w_1, w_2, \ldots w_k, u$. I.e.
$$V = \operatorname{span}(U \cup \{w_1, w_2, \ldots w_k\})$$
where
$$U = \{u \in V : F(w_i, u) = 0, \forall i \leq k\}.$$

11. Theorem 7.11: (Sylvester's law of inertia part II.) Let V be an n dimensional real vector space, and let F be a symmetric bilinear form on V, with two diagonal matrix representations $\mathbf{A}$ and $\mathbf{A}'$, as in theorem 7.7, with respect to the bases $e_1, e_2, \ldots e_n$ and $e'_1, e'_2, \ldots e'_n$ of V. If $\mathbf{A}$ has k positive diagonal entries and m negative diagonal entries, and $\mathbf{A}'$ has k' positive diagonal entries and m' negative diagonal entries, then $k = k'$ and $m = m'$.

12. Definition 7.12: Let V be an n dimensional real vector space, and let F be a symmetric bilinear form on V, with a diagonal matrix representation $\mathbf{A}$, with k and m as in theorem 7.7.

    $k + m$ is the rank of F.

    $k - m$ is the signature of F.

13. Proposition 7.13: With the same notation as definition 7.12, F is positive definite (i.e. is an inner product) iff $k = n$ and $m = 0$.

14. Lemma 7.14: Suppose F is a symmetric bilinear form on a real vector space V, and let $v_1, v_2, \ldots v_n$ be a basis of V. If $F(v_i, v_i) > 0 \; \forall i$ and $F(v_i, v_j) > 0 \; \forall i \neq j$ then F is positive definite.