Linear Algebra I
Summary of Lectures:
Linear Transformations

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1. Definition 5.1: If V and W are two vectors spaces over the same field F, then a linear transformation from V to W (also called a linear map or homomorphism) is a map $f : V \to W$ satisfying

$$f(\lambda u + \mu v) = \lambda f(u) + \mu f(v)\,, \quad \forall\, u, v \in V \text{ and } \forall\, \lambda, \mu \in F\,.$$

   The space of V is called the domain of F and the space of W is called the co-domain.

2. Lemma 5.2: A linear transformation $f : V \to W$ satisfies

   (a) $f(0) = 0$,
   (b) $f(\lambda u) = \lambda u$,
   (c) $f(-u) = -f(u)$,
   (d) $f(u + v) = f(u) + f(v)$, and
   (e) $f(\sum_{i=1}^{n} \lambda_i u_i) = \sum_{i=1}^{n} \lambda_i f(u_i)$.

3. Definition 5.3: Given $f : V \to W$ as in definition 5.1, the image (or range) of f is $\{f(v) : v \in V\}$. This is written as $f(V)$ or $\mathrm{im}(f)$.
   The kernel (or nullspace) of f is $\{v \in V : f(v) = 0\}$, written $\ker(f)$

4. Proposition 5.4: If $f : V \to W$ is a linear transformation, then $\mathrm{im}(f)$ is a subspace of W and $\ker(f)$ is a subspace of V.

5. Proposition 5.5: A linear transformation $f : V \to W$ is injective iff $\ker(f)$ is the zero subspace $\{0\}$ of V.

6. Definition 5.6: The rank of f is the dimension of $\mathrm{im}(f)$, written $\mathrm{r}(f)$.
   The nullity of f is the dimension of $\ker(f)$, written $\mathrm{n}(f)$.

7. Theorem 5.7: The rank-nullity formula. If $f : V \to W$ is a linear transformation then

$$\mathrm{r}(f) + \mathrm{n}(f) = \dim(V)\,.$$

8. Proposition 5.8: If $f : V \to W$ is a linear transformation of finite dimensional vector spaces V, W over the same field F then

   (a) f is injective iff $n(f) = 0$, and
   (b) f is surjective iff $r(f) = \dim(W)$.

9. Corollary 5.9: If $f : V \to W$ is a linear transformation of finite dimensional vector spaces V, W over the same field F then

   (a) f is injective iff $r(f) = \dim(V)$, and

   (b) f is surjective iff $n(f) = \dim(V) - \dim(W)$.

10. Let $f_{\mathbf{A}} : F^n \to F^m$ be $f_{\mathbf{A}}(\mathbf{v}) = \mathbf{A}\mathbf{v}$ where $\mathbf{v} \in F^n$ and $\mathbf{A}$ is an $m \times n$ matrix over the field F. Then

    (a) $\operatorname{im}(\mathbf{A}) = \{\mathbf{A}\mathbf{v} : \mathbf{v} \in F^n\}$,

    (b) $\ker(\mathbf{A}) = \{\mathbf{v} \in F^n : \mathbf{A}\mathbf{x} = 0\}$, and

    (c) $r(\mathbf{A})$ and $n(\mathbf{A})$ are the rank and nullity of $\mathbf{A}$, i.e. the dimensions of $\operatorname{im}(\mathbf{A})$ and $\ker(\mathbf{A})$ respectively.