

# Summary of Calculus I

Dr Nicholas Sedlmayr

## Contents

<b>1</b>	<b>Limits</b>	<b>1</b>
<b>2</b>	<b>Differentiation</b>	<b>5</b>

## 1 Limits

- **Tangent:** A tangent line is a line that touches a curve, and has the same slope as the curve at the point of contact.
- **Average rate of change:** The average rate of change of a function  $f(x)$  between  $x = a$  and  $x = b$  is

$$\frac{f(b) - f(a)}{b - a}.$$

- **Limit:** Let  $f$  be a function defined on some open interval that contains  $a$  but not necessarily at  $a$  itself. Then we say the limit of  $f(x)$  as  $x$  approaches  $a$  is  $L$  and we write

$$\lim_{x \rightarrow a} f(x) = L$$

if for every number  $\epsilon > 0$  there is a number  $\delta > 0$  such that if

$$0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon.$$

- **Left-hand limit:**

$$\lim_{x \rightarrow a^-} f(x) = L,$$

if for every number  $\epsilon > 0$  there is a number  $\delta > 0$  such that if

$$a - \delta < x < a \text{ then } |f(x) - L| < \epsilon.$$

- **Right-hand limit:**

$$\lim_{x \rightarrow a^+} f(x) = L,$$

if for every number  $\epsilon > 0$  there is a number  $\delta > 0$  such that if

$$a < x < a + \delta \text{ then } |f(x) - L| < \epsilon.$$

- **Theorem 1:**

$$\lim_{x \rightarrow a} f(x) = L \text{ if and only if } \lim_{x \rightarrow a^-} f(x) = L \text{ and } \lim_{x \rightarrow a^+} f(x) = L.$$

- **Infinite limits:** Let  $f$  be a function defined on some open interval that contains  $a$  but not necessarily at  $a$  itself. Then

$$\lim_{x \rightarrow a} f(x) = \infty$$

means that for every positive number  $M$  there is a positive number  $\delta$  such that if

$$0 < |x - a| < \delta \text{ then } f(x) > M.$$

Let  $f$  be a function defined on some open interval that contains  $a$  but not necessarily at  $a$  itself. Then

$$\lim_{x \rightarrow a} f(x) = -\infty$$

means that for every negative number  $N$  there is a positive number  $\delta$  such that if

$$0 < |x - a| < \delta \text{ then } f(x) < N.$$

- **Vertical asymptote:** The line  $x = a$  is called a vertical asymptote of the curve  $y = f(x)$  if at least one of the following is true:

$$\begin{array}{ll} \lim_{x \rightarrow a} f(x) = \infty, & \lim_{x \rightarrow a} f(x) = -\infty, \\ \lim_{x \rightarrow a^-} f(x) = \infty, & \lim_{x \rightarrow a^-} f(x) = -\infty, \\ \lim_{x \rightarrow a^+} f(x) = \infty, \text{ or } & \lim_{x \rightarrow a^+} f(x) = -\infty. \end{array}$$

- **Limit laws:** Suppose that  $c$  is a constant and that the limits

$$\lim_{x \rightarrow a} f(x) \text{ and } \lim_{x \rightarrow a} g(x)$$

exist. Then:

1. **Sum law**  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2. **Difference law**  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
3. **Constant multiple law**  $\lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$
4. **Product law**  $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
5. **Quotient law**  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  if  $\lim_{x \rightarrow a} g(x) \neq 0$
6. **Power law**  $\lim_{x \rightarrow a} [f(x)]^n = \left[ \lim_{x \rightarrow a} f(x) \right]^n$  where  $n$  is a positive integer
7.  $\lim_{x \rightarrow a} c = c$
8.  $\lim_{x \rightarrow a} x = a$
9.  $\lim_{x \rightarrow a} x^n = a^n$  where  $n$  is a positive integer

10.  $\lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}$  where  $n$  is a positive integer and if  $n$  is even then we assume that  $a > 0$
11. **Root law**  $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$  where  $n$  is a positive integer and if  $n$  is even then we assume that  $\lim_{x \rightarrow a} f(x) > 0$

- **Direct substitution property:** If  $f$  is a polynomial or a rational function and  $a$  is in the domain of  $f$ , then

$$\lim_{x \rightarrow a} f(x) = f(a).$$

Such functions are called continuous at  $a$ .

- **Theorem 2:** If  $f(x) \leq g(x)$  when  $x$  is near  $a$  (except possibly at  $a$ ) and the limits of both  $f$  and  $g$  exist as  $x$  approaches  $a$  then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x).$$

- **Theorem 3:** The squeeze theorem. If  $f(x) \leq g(x) \leq h(x)$  when  $x$  is near  $a$  (except possibly at  $a$ ) and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L.$$

- **Continuous function:** A function  $f$  is said to be continuous at  $a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$ .
  - A function  $f$  is continuous from the right at a number  $a$  if  $\lim_{x \rightarrow a^+} f(x) = f(a)$ .
  - A function  $f$  is continuous from the left at a number  $a$  if  $\lim_{x \rightarrow a^-} f(x) = f(a)$ .
  - A function  $f$  is said to be continuous on an interval if it is continuous at every number in the interval.
- **Theorem 4:** If  $f$  and  $g$  are continuous at  $a$  and  $c$  is a constant, then the following functions are also continuous at  $a$ :
  - 1.1.  $f + g$ ,
  - 1.2.  $f - g$ ,
  - 1.3.  $cf$ ,
  - 1.4.  $fg$ , and
  - 1.5.  $\frac{f}{g}$  if  $g(a) \neq 0$ .
- **Theorem 5:**
  - (a) Any polynomial is continuous everywhere, that is, it is continuous on  $\mathbb{R} = (-\infty, \infty)$ .

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

- **Lemma 6:**  $\lim_{\theta \rightarrow 0} \cos(\theta) = 1$  and  $\lim_{\theta \rightarrow 0} \sin(\theta) = 0$ .
- **Theorem 7:** Polynomials, rational functions, root functions, and trigonometric functions are all continuous at every number in their domains.
- **Theorem 8:** If  $f$  is continuous at  $b$  and  $\lim_{x \rightarrow a} g(x) = b$  then  $\lim_{x \rightarrow a} f(g(x)) = f(b)$ . Equivalently  $\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right)$ .
- **Theorem 9:** If  $g$  is continuous at  $a$  and  $f$  is continuous at  $g(a)$  then the composite function  $f \circ g$  given by  $(f \circ g)(x) = f(g(x))$  is continuous at  $a$ .
- **Theorem 10:** The intermediate value theorem. Suppose that  $f$  is continuous on the closed interval  $[a, b]$  and let  $N$  be any number between  $f(a)$  and  $f(b)$ , where  $f(a) \neq f(b)$ . Then there exists a number  $c$  in  $(a, b)$  such that  $f(c) = N$ .

## 2 Differentiation

- **Tangent line:** The tangent line to the curve  $y = f(x)$  at the point  $P(a, f(a))$  is the line through  $P$  with the slope

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists. Equivalently

$$m = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}.$$

- **Derivative:** The derivative of a function  $f$  at a number  $a$ , denoted by  $f'(a)$  is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.

- **Instantaneous velocity:** If  $f(t)$  represents position as a function of time then the derivative  $f'(a)$  is the instantaneous velocity of  $y = f(t)$  with respect to  $t$  when  $t = a$ .
- **Instantaneous rate of change:** The derivative  $f'(a)$  is the instantaneous rate of change of  $y = f(x)$  with respect to  $x$  when  $x = a$ .
- **Derivative as a function:** The derivative of a function  $f(x)$ ,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h},$$

can be regarded as a new function called the derivative of  $f$ . Other notations for the derivative of  $y = f(x)$  with respect to  $x$  are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x).$$

- **Differentiable:** A function  $f$  is differentiable at  $a$  if  $f'(a)$  exists. It is differentiable on an open interval  $(a, b)$  if it is differentiable at every number in the interval.
- **Theorem 1:** If  $f$  is differentiable at  $a$  then it is continuous at  $a$ .
- **Higher derivatives:** The derivative of a derivative is called the second derivative, denoted by  $(f')' = f''$  or

$$\frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2 y}{dx^2}.$$

In general the  $n$ th derivative, where  $n$  is a positive integer, is written as  $f^n(x)$  or  $\frac{d^n y}{dx^n}$ .

- **Differentiation formulas:** If  $c$  is a constant and  $n$  is a real number then:

$$\frac{d}{dx}(c) = 0, \text{ and}$$

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

- **Differentiation rules:** If  $c$  is a constant and  $f$  and  $g$  are differentiable functions then:

$$\frac{d}{dx}[cf(x)] = cf'(x) \quad \text{the constant multiple rule,}$$

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) \quad \text{the sum rule,}$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x) \quad \text{the difference rule,}$$

$$\frac{d}{dx}[f(x)g(x)] = g(x)f'(x) + f(x)g'(x) \quad \text{the product rule, and}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} \quad \text{the quotient rule.}$$