Linear Algebra I
Summary of Lectures: Vector Spaces

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1. Definition 2.1: A vector space. A vector space V over a field F (see definition 2.3) is a set containing:

   - a special zero vector $\mathbf{0}$;
   - an operation of addition of two vectors $\mathbf{u}+\mathbf{v} \in V$, for $\mathbf{u}, \mathbf{v} \in V$; and
   - multiplication of a vector V with a number $\lambda \in F$ with $\lambda\mathbf{v} \in V$.

   The vector space must be closed under both of these operations and must satisfy the following laws $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\lambda, \mu \in F$:

   (1) associativity $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$;
   (2) commutativity $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$;
   (3) $\mathbf{u}+\mathbf{0} = \mathbf{u}$;
   (4) $\mathbf{v}+(-1)\mathbf{v} = \mathbf{0}$;
   (5) $\lambda(\mu\mathbf{u}) = (\lambda\mu)\mathbf{v}$;
   (6) distributivity $\lambda(\mathbf{u}+\mathbf{v}) = \lambda\mathbf{u}+\mu\mathbf{v}$; and
   (7) distributivity $(\lambda+\mu)\mathbf{u} = \lambda\mathbf{u}+\mu\mathbf{u}$.

2. Proposition 2.2 $\forall \mathbf{v} \in V$ and $\forall \lambda \in F$:

   (a) $\mathbf{v} = 1\mathbf{v}$;
   (b) $0\mathbf{v} = \mathbf{0}$; and
   (c) $\lambda\mathbf{0} = \mathbf{0}$.

3. Definition 2.3 A field is a set F containing distinct elements 0 and 1 with two binary operations $+$ and $\cdot$ satisfying the axioms $\forall a, b, c \in F$:

   (1) $a+b = b+a$;
   (2) $(a+b)+c = a+(b+c)$;
   (3) $a+0 = a$;
   (4) $\forall a \; \exists -a$ such that $a+(-a) = 0$;
   (5) $a \cdot b = b \cdot a$;
   (6) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;
   (7) $a \cdot 1 = a$;
   (8) $\forall a \neq 0 \; \exists a^{-1}$ such that $a \cdot a^{-1} = 1$; and
   (9) $a \cdot (b+c) = a \cdot b + a \cdot c$;

If a field F is finite its order is the number of elements in F.

4. Theorem 2.4: For each prime p and each positive integer n, there is a unique field of order p^n. Additionally, every finite field is of this form.

5. Definition 2.5 Given a vector space V over F, a subspace of V is a subset $W \subset V$ which contains the zero vector of V and is closed under the operations of addition and scalar multiplication.

6. Lemma 2.5.1 Let $W \subset V$ be nonempty, where V is a vector space over F. Then W is a subspace of V iff $\mathbf{v} + \lambda\mathbf{u} \in W$ for each $\mathbf{v}, \mathbf{u} \in W$ and each scalar λ.

7. Definition 2.6 Given a vector space V over F, and given a subset of V $A = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots \mathbf{u}_n\}$,

$$W = \{\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 + \lambda_3\mathbf{u}_3 + \ldots \lambda_n\mathbf{u}_n : \lambda_1, \lambda_2, \ldots \lambda_n \in F\}$$

is the subspace of V spanned by A. The elements of W are called linear combinations of vectors from A and the subspace W is denoted as $\operatorname{span} A$.

8. Definition 2.7 If A is an infinite subset of V, where V is a vector space over F, we define $\operatorname{span} A$ to be the set of all linear combinations of finite subsets of A.

9. Definition 2.8 A set $A \subset V$ of vectors in a vector space V over F is linearly dependent if there are $n \in \mathbb{N}$ vectors $a_1, a_2, \ldots a_n$ and scalars $\lambda_1, \lambda_2, \ldots \lambda_n$ not all zero such that

$$\lambda_1 a_1 + \lambda_2 a_2 + \ldots \lambda_n a_n = 0\,.$$

Otherwise A is linearly independent.

- For a finite set $A = \{a_1, a_2, \ldots a_n\}$ it is linearly independent iff $\forall$ scalars $\lambda_1, \lambda_2, \ldots \lambda_n \in F$

$$\lambda_1 a_1 + \lambda_2 a_2 + \ldots \lambda_n a_n = 0 \Rightarrow \lambda_1 = \lambda_2 = \ldots \lambda_n = 0.$$

- If A is infinite it is linearly independent iff every subset of A is linearly independent.
- By convention the empty set is linearly independent.