Linear Algebra I
Summary of Lectures:
Inner Product Spaces

and Bilinear and Sesquilinear Forms

Dr Nicholas Sedlmayr

1. Definition 3.1: If V is a vector space over $\mathbb{R}$, then an inner product on V is a map $(\langle|\rangle)$ from $V \times V$ to $\mathbb{R}$ with the following properties:
   (a) Symmetry: $\langle v|w\rangle = \langle w|v\rangle \ \forall v, w \in V$.
   (b) Linearity: $\langle u|\lambda v+\mu w\rangle = \lambda\langle u|v\rangle+\mu\langle u|w\rangle \ \forall u, v, w \in V$ and $\forall \lambda, \mu \in \mathbb{R}$.
   (c) Positive definiteness:
      (i) $\langle v|v\rangle \geq 0 \ \forall v \in V$, and
      (ii) $\langle v|v\rangle = 0$ iff $v = 0$.

   As the inner product is linear with respect to both variables it is sometimes called bilinear.

2. Definition 3.2: A finite dimensional vector space over $\mathbb{R}$ with an inner product defined is called a Euclidean space.

3. Definition 3.3: The norm (or length) of a vector$v$ is written as $||v|$ and defined by
$$||v|| = \sqrt{\langle v|v\rangle},$$
   the positive square root of the inner product of v with itself. The distance between two vectors v and w is written as $d(v, w)$ and is $d(v, w) = ||v-w||$.

4. Proposition 3.4: $\forall v \in V$, where V is a Euclidean space, and $\forall \lambda \in \mathbb{R}$ then $||\lambda v|| = |\lambda| \cdot ||v||$.

5. Proposition 3.5: The "Cauchy-Schwarz inequality" says that $\forall v, w \in V$, where V is a Euclidean space, then
$$|\langle v|w\rangle| \leq ||v|| \cdot ||w|| \,.$$

6. Proposition 3.6: The "triangle inequality" says that $\forall v, w \in V$, where V is a Euclidean space, then
$$||v + w|| \leq ||v|| + ||w|| \,.$$

7. Definition 3.7: If V is a Euclidean space, and $v, w \in V$, then v and w are said to be orthogonal if $\langle v|w \rangle = 0$. If both v and w are nonzero, then the angle between v and w is defined to be θ, $0 \leq \theta \leq \pi$ and

$$\cos\theta = \frac{\langle v|w \rangle}{||v|| \cdot ||w||} .$$