Higher Mathematics in English II
Exercises II:
Inner Product Spaces

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1. Prove the Cauchy-Schwarz inequality, proposition 3.5, which states that $\forall v, w \in V$, where V is a Euclidean space then $|\langle v|w\rangle| \leq ||v|| \cdot ||w||$.
   Tip: Consider the properties of $||v + \lambda w||^2 \geq 0$ *as a polynomial in* λ.

2. Starting from the triangle inequality, derive the following alternative forms of it:

   $||v-w|| \leq ||v||+||w||\,,\ ||v+w|| \geq ||v||-||w||\,,$ and $||v-w|| \geq ||v||-||w||\,,$

   for $v, w \in V$, where V is a Euclidean space.

3. Prove that in any complex inner product space
   (a) $\langle u|v\rangle = \frac{1}{2}\left(||u+v||^2 + \mathrm{i}||u-\mathrm{i}v||^2 - (1+\mathrm{i})(||u||^2+||v||^2)\right)$, and
   (b) $\langle u|v\rangle = \frac{1}{4}\left(||u+v||^2 - ||u-v||^2 + \mathrm{i}||u-\mathrm{i}v||^2 - \mathrm{i}||u+\mathrm{i}v||^2\right)$.

4. Determine whether

   $$\langle \mathbf{u}|\mathbf{v}\rangle = (u_1 - v_1)^2 + (u_2 - v_2)^2$$

   is an inner product on $\mathbb{R}^2$ with $\mathbf{u} = (u_1, u_2)^T$, and $\mathbf{v} = (v_1, v_2)^T$.

5. Let $V = \mathbb{R}^3$ with the standard inner product. Describe the set of vectors orthogonal to $(1, 0, -3)^T$. Show that this set is a subspace of V, find a basis and hence determine the dimension of the subspace.

6. Let V be the vector space over $\mathbb{R}$ of all polynomials of degree less than 3. I.e. $V = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2, x \in \mathbb{R}\}$. We can define an inner product on this space as

   $$\langle f|g\rangle = \int_0^1 \mathrm{d}x f(x)g(x)\,.$$

   $1, x, x^2$ is an ordered basis of V, find the corresponding matrix of inner products $\mathbf{A}$. Use this matrix to calculate $\langle x+1|x^2-2\rangle$ and $\langle x^2-x+5|x^2+2x\rangle$. Confirm the results by direct integration.

7. Prove that the map $F : V \times V \to \mathbb{C}$ is a sesquilinear form on $V = \mathbb{C}^n$ where $F(\mathbf{x}, \mathbf{y}) = \bar{\mathbf{x}}^T\mathbf{A}\mathbf{y}$, $\mathbf{A}$ is an $n \times n$ matrix, and $\mathbf{x}, \mathbf{y} \in V$. Furthermore prove that if F is conjugate symmetric then $\overline{\mathbf{A}}^T = \mathbf{A}$.

8. A sesquilinear form on $\mathbb{C}^3$ is defined by

$$F((x,y,z)^T,(x',y',z')^T) = \begin{pmatrix}\bar{x} & \bar{y} & \bar{z}\end{pmatrix}\begin{pmatrix}0 & -1-2\mathrm{i} & 0\\ -1+2\mathrm{i} & 0 & \mathrm{i}\\ 0 & -\mathrm{i} & 1\end{pmatrix}\begin{pmatrix}x'\\ y'\\ z'\end{pmatrix}.$$

What is the matrix of F with respect to the bases

(a) $\{(\mathrm{i},1,0)^T,(1,1+\mathrm{i},1)^T,(0,0,\mathrm{i})^T\}$, and

(b) $\{(-1-3\mathrm{i},2+\mathrm{i},-1+2\mathrm{i})^T,(2+\mathrm{i},1-2\mathrm{i},2+\mathrm{i})^T,(0,0,-\mathrm{i})^T\}$?

Is F an inner product?

9. An alternating form F is a bilinear form on a vector space V satisfying $F(v,v)=0\ \forall v\in v$.

(a) Show that if F is an alternating form then $F(u,v) = -F(v,u)$, i.e. that F is skew symmetric.

(b) Give an example of an alternating form (other than the zero function).