Linear Algebra I
Exercises V: Orthogonal and Orthonormal Bases

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1. Prove Bessel's inequality. I.e. if $e_1, e_2, \ldots e_k$ is an orthonormal set of vectors in a complex inner product space V, and $v \in V$, then

$$\sum_{i=1}^{k} |\langle e_i|v\rangle|^2 \leq ||v||^2 \,.$$

Tip: Consider $||w||^2$ *with* $w = v - \sum_{i=1}^{k} \langle e_i|v\rangle e_i$.

2. Consider $V = \mathbb{C}^3$, with the standard inner product. Starting from the basis $\{(0,\mathrm{i},1)^T, (1+\mathrm{i},0,2)^T, (3,0,0)^T\}$ use Gram-Schmidt to construct an orthonormal basis for V.

3. Let V be the vector space over $\mathbb{R}$ of all polynomials of degree less than 3. I.e. $V = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2, x \in \mathbb{R}\}$. We can define an inner product on this space as

$$\langle f|g\rangle = \int_{-1}^{1} \mathrm{d}x f(x)g(x) \,.$$

$1, x$ is an orthogonal basis of U, which is a subspace of V. Find the orthogonal complement, $U^\perp$, to U.

4. Prove that if V is an inner product space, and U is a finite dimensional subspace of V, then

   (a) $U^\perp$ is a subspace of V,

   (b) $U \cap U^\perp = \{0\}$, and

   (c) $U + U^\perp = V$.

Tips: For (b) consider the properties of a vector which is in both U *and* $U^\perp$. *For (c) it will help to consider an orthonormal basis of* U. *Any* $v \in V$ *can be written as* $v = v_S + v_P$ *where* $v_S \in U$ *and therefore has a representation in terms of the orthonormal basis of* U. *The task is to show that* $v_P \in U^\perp$ *for any* $v \in V$.