Summary of Calculus I

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Contents

1 Limits

- Tangent: A tangent line is a line that touches a curve, and has the same slope as the curve at the point of contact.
- Average rate of change: The average rate of change of a function $f(x)$ between $x = a$ and $x = b$ is
$$\frac{f(b) - f(a)}{b - a}\,.$$
- Limit: Let f be a function defined on some open interval that contains a but not necessarily at a itself. Then we say the limit of $f(x)$ as x approaches a is L and we write
$$\lim_{x \to a} f(x) = L$$
if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that if
$$0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon\,.$$
- Left-hand limit:
$$\lim_{x \to a^-} f(x) = L\,,$$
if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that if
$$a - \delta < x < a \text{ then } |f(x) - L| < \epsilon\,.$$
- Right-hand limit:
$$\lim_{x \to a^+} f(x) = L\,,$$
if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that if
$$a < x < a + \delta \text{ then } |f(x) - L| < \epsilon\,.$$

- Theorem 1:

$$\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L\,.$$

- Infinit limits: Let f be a function defined on some open interval that contains a but not necessarily at a itself. Then

$$\lim_{x\to a} f(x) = \infty$$

means that for every positive number M there is a positive number δ such that if

$$0 < |x-a| < \delta \text{ then } f(x) > M\,.$$

Let f be a function defined on some open interval that contains a but not necessarily at a itself. Then

$$\lim_{x\to a} f(x) = -\infty$$

means that for every negative number N there is a positive number δ such that if

$$0 < |x-a| < \delta \text{ then } f(x) < N\,.$$

- Vertical asymptote: The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following is true:

$$\begin{array}{ll} \lim_{x\to a} f(x) = \infty\,, & \lim_{x\to a} f(x) = -\infty\,, \\ \lim_{x\to a^-} f(x) = \infty\,, & \lim_{x\to a^-} f(x) = -\infty\,, \\ \lim_{x\to a^+} f(x) = \infty\,, \quad \text{or} & \lim_{x\to a^+} f(x) = -\infty\,. \end{array}$$

- Limit laws: Suppose that c is a constant and that the limits

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x)$$

exist. Then:

1. Sum law $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. Difference law $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. Constant multiple law $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
4. Product law $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$
5. Quotient law $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. Power law $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a}[f(x)]\right]^n$ where n is a positive integer
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ where n is a positive integer

10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer and if n is even then we assume that $a > 0$
11. Root law $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer and if n is even then we assume that $\lim_{x \to a} f(x) > 0$

- Direct substitution property: If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a) \,.$$

Such functions are called continuous at a.

- Theorem 2: If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of both f and g exist as x approaches a then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \,.$$

- Theorem 3. The squeeze theorem: If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L \,.$$

- Continuous function: A function f is said to be continuous at a if $\lim_{x \to a} f(x) = f(a)$.
  - A function f is continuous from the right at a number a if $\lim_{x \to a^+} f(x) = f(a)$.
  - A function f is continuous from the left at a number a if $\lim_{x \to a^-} f(x) = f(a)$.
  - A function f is said to be continuous on an interval if it is continuous at every number in the interval.

- Theorem 4: If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

  1.1. $f + g$,

  1.2. $f - g$,

  1.3. cf,

  1.4. fg, and

  1.5. $\dfrac{f}{g}$ if $g(a) \neq 0$.

- Theorem 5:

  (a) Any polynomial is continuous everywhere, that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

- Lemma 6: $\lim_{\theta \to 0} \cos(\theta) = 1$ and $\lim_{\theta \to 0} \sin(\theta) = 0$.
- Theorem 7: Polynomials, rational functions, root functions, and trigonometric functions are all continuous at every number in their domains.
- Theorem 8: If f is continuous at b and $\lim_{x \to a} g(x) = b$ then $\lim_{x \to a} f(g(x)) = f(b)$. Equivalently $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$.
- Theorem 9: If g is continuous at a and f is continuous ate $g(a)$ then the composite function $f \cdot g$ given by $(f \cdot g)(x) = f((g(x))$ is continuous at a.
- Theorem 10. The intermediate value theorem: Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

2 Differentiation

- Tangent line: The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with the slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists. Equivalently

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

- Derivative: The derivative of a function f at a number a, denoted by $f'(a)$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

- Instantaneous velocity: If $f(t)$ represents position as a function of time then the derivative $f'(a)$ is the instantaneous velocity of $y = f(t)$ with respect to t when $t = a$.

- Instantaneous rate of change: The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

- Derivative as a function: The derivative of a function $f(x)$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

can be regarded as a new function called the derivative of f. Other notations for the derivative of $y = f(x)$ with respect to x are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x).$$

- Differentiable: A function f is differentiable at a if $f'(a)$ exists. It is differentiable on an open interval (a, b) if it is differentiable at every number in the interval.

- Theorem 1: If f is differentiable at a then it is continuous at a.

- Higher derivatives: The derivative of a derivative is called the second derivative, denoted by $(f')' = f''$ or

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}.$$

In general the nth derivative, where n is a positive integer, is written as $f^n(x)$ or $\dfrac{d^n y}{dx^n}$.

- Differentiation formulas: If c is a constant and n is a real number then:

$$\begin{aligned} \frac{d}{dx}(c) &= 0\,, \text{ and} \\ \frac{d}{dx}(x^n) &= nx^{n-1}\,, \\ \frac{d}{dx}\sin(x) &= \cos(x)\,, \\ \frac{d}{dx}\cos(x) &= -\sin(x)\,. \end{aligned}$$

- Differentiation rules: If c is a constant and f and g are differentiable functions then:

$$\begin{array}{rcl} \frac{d}{dx}[cf(x)] = & cf'(x) & \text{the constant multiple rule,} \\ \frac{d}{dx}[f(x)+g(x)] = & f'(x)+g'(x) & \text{the sum rule,} \\ \frac{d}{dx}[f(x)-g(x)] = & f'(x)-g'(x) & \text{the difference rule,} \\ \frac{d}{dx}[f(x)g(x)] = & g(x)f'(x)+f(x)g'(x) & \text{the product rule,} \\ \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = & \frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)} & \text{the quotient rule, and} \\ \frac{d}{dx}[f(g(x))] = & f'(g(x))\cdot g'(x) & \text{the chain rule.} \end{array}$$

- Implicit differentiation: Using the chain rule, and others as necessary, allows us to also differentiate implicitly. Let $g(x,y)=0$ define y as a function of x. $\frac{dg}{dx}$ will relate $\frac{dy}{dx}$, y, and x.

- Related rates: Let $g(x,y)=0$ relate two functions of a variable t. $\frac{dg}{dt}$ will relate $\frac{dy}{dt}$, $\frac{dx}{dt}$, y, and x.

- Linear approximation: The linear approximation of $f(x)$ near a point a is $L(x)=f(a)+f'(a)(x-a)$.

- Differentials: If $y=f(x)$ where $f(x)$ is a differentiable function, and the differential dx is an independent variable then the differential dy is a dependent variable and is given by

$$dy = f'(x)dx = \frac{dy}{dx}dx\,.$$

- Absolute extrema: Let c be a number in the domain D of a function f. Then $f(c)$ is

  (a) the absolute maximum value of f on D if $f(c)\geq f(x)$ for all x in D, or

  (b) the absolute minimum value of f on D if $f(c)\leq f(x)$ for all x in D.

- Local extrema: Let c be a number in the domain D of a function f. Then $f(c)$ is

(a) a local maximum value of f on D if $f(c) \geq f(x)$ when x is near c, or

(b) a local minimum value of f on D if $f(c) \leq f(x)$ when x is near c.

Near c means on some open interval containing c.

- Theorem 2. The extreme value theorem: If f is continuous on a closed interval $[a, b]$ then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

- Theorem 3. Fermat's theorem: If f has a local extrema at c, and if $f'(c)$ exists, then $f'(c) = 0$.

- Critical number: A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

- The closed interval method: To find the absolute extrema values of a continuous function f on a closed interval $[a, b]$:

  1. Find f at the critical numbers in (a, b).
  2. Find f at the end points a and b.
  3. The largest and smallest of these values are the absolute extrema.

- Theorem 4. Rolle's theorem: Let f be a function which satisfies the following three hypotheses:

  1. f is continuous on the closed interval $[a, b]$.
  2. f is differentiable on the open interval (a, b).
  3. $f(a) = f(b)$.

  Then there is a number c in (a, b) such that $f'(c) = 0$.

- Theorem 5. The mean value theorem: Let f be a function which satisfies the following hypotheses:

  1. f is continuous on the closed interval $[a, b]$.
  2. f is differentiable on the open interval (a, b).

  Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} .$$

- Theorem 6: If $f'(x) = 0$ for all x in an interval (a, b) then f is constant on (a, b).

- Corollary 7: If $f'(x) = g'(x)$ for all x in an interval (a, b) then $f - g$ is constant on (a, b). I.e.

$$f(x) = g(x) + c \text{ where } c \text{ is a constant.} .$$

- Increasing/decreasing test:

  (a) If $f'(x) > 0$ on an interval then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval then f is decreasing on that interval.

- First derivative test: Suppose that c is a critical number of a continuous function f.

  (a) If f' changes sign from positive to negative at c, then f has local maximum at c.

  (b) If f' changes sign from negative to positive at c, then f has local minimum at c.

  (c) If f' does not changes sign at c, then f has no local maximum or minimum at c.

- Concavity: If the graph of f lies above all of its tangents on an interval I, then it is concave upward on I. If the graph of f lies below all of its tangents on an interval I, then it is concave downward on I.

- Concavity test:

  (a) If $f''(x) > 0$ for all x in an interval I then f is concave upward on I.

  (b) If $f''(x) < 0$ for all x in I then f is concave downward on I.

- Inflection point: A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

- The second derivative test: Suppose f'' is continuous near c.

  (a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

  (b) If $f'(c) = 0$ and $f''(x) < 0$, then f has a local maximum at c.

- Horizontal asymptotes: Let f be a function defined on a interval (a, ∞). Then

  $$\lim_{x\to\infty} f(x) = L$$

  means that for every $\epsilon > 0$ there is a corresponding number N such that if $x > N$ then $|f(x) - L| < \epsilon$.
  Similarly let f be a function defined on a interval (a, ∞). Then

  $$\lim_{x\to-\infty} f(x) = L$$

  means that for every $\epsilon > 0$ there is a corresponding number N such that if $x < N$ then $|f(x) - L| < \epsilon$.
  The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

  $$\lim_{x\to\infty} f(x) = L \text{ or } \lim_{x\to-\infty} f(x) = L\,.$$

- Limit laws: Suppose that c is a constant and that the limits

  $$\lim_{x\to\infty} f(x) \text{ and } \lim_{x\to\infty} g(x)$$

  exist. Then:

1. Sum law $\lim_{x\to\infty}[f(x)+g(x)] = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$
2. Difference law $\lim_{x\to\infty}[f(x)-g(x)] = \lim_{x\to\infty} f(x) - \lim_{x\to\infty} g(x)$
3. Constant multiple law $\lim_{x\to\infty}[cf(x)] = c \lim_{x\to\infty} f(x)$
4. Product law $\lim_{x\to\infty}[f(x)g(x)] = \lim_{x\to\infty} f(x) \cdot \lim_{x\to\infty} g(x)$
5. Quotient law $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)}$ if $\lim_{x\to\infty} g(x) \neq 0$
6. Power law $\lim_{x\to\infty}[f(x)]^n = \left[\lim_{x\to\infty}[f(x)\right]^n$ where n is a positive integer
7. $\lim_{x\to\infty} c = c$
8. $\lim_{x\to\infty} x = \infty$

11. Root law $\lim_{x\to\infty} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to\infty} f(x)}$ where n is a positive integer and if n is even then we assume that $\lim_{x\to\infty} f(x) > 0$

Similarly for $\lim x \to -\infty$.

- Theorem 8: If $r > 0$ is a rational number then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

.

If $r > 0$ is a rational number such that x^r is defined fir all x then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

.

- Slant asymptotes: The line $y = mx + b$ is a slant asymptote of $f(x)$ if

$$\lim_{x\to\infty}[f(x) - (mx+b)] = 0$$

or

$$\lim_{x\to-\infty}[f(x) - (mx+b)] = 0\,.$$

- The first derivative test for absolute extreme values: Suppose c is a critical number of a continuous function f defined on an interval.
  (a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is an absolute maximum value of f.
  (b) If $f'(x) < 0$ for all $x > c$ and $f'(x) > 0$ for all $x < c$, then $f(c)$ is an absolute minimum value of f.

- Antiderivative: A function F is called the antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

- Theorem 9: If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$ where C is an arbitrary constant.

3 Summation

- Sigma notation: We use $\sum_{i=m}^{n} a_i = a_1 + a_2 + \ldots + a_n$ as notation for sum over the terms a_i from $i = m$ to $i = n$. is is the index of summation.

- Theorem 1: If c is any constant then:

  (a) $\sum_{i=m}^{n} ca_i = c \sum_{i=m}^{n} a_i$,

  (b) $\sum_{i=m}^{n} (f_i + g_i) = \sum_{i=m}^{n} f_i + \sum_{i=m}^{n} g_i$, and

  (c) $\sum_{i=m}^{n} (f_i - g_i) = \sum_{i=m}^{n} f_i - \sum_{i=m}^{n} g_i$.

- Some useful sums:

  (a) $\sum_{i=1}^{n} 1 = n$,

  (b) $\sum_{i=1}^{n} c = cn$,

  (c) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$,

  (d) $\sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}$, and

  (e) $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$.

4 Integration

- Area under a curve: The area A of the region S that lies under the graph of the continuous function f curve is the limit of the sum of the area of approximating rectangles:

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i)\,.$$

  For rectangles of equal width which meet the curve on their top right corner then $\Delta x = (b-a)/n$ where the region S lies between $x = a$ and $x = b$, and $x_i = a + i\Delta x$.

- The definite integral: If f is a function defined for $a \le x \le b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. We let $x_0 = a, x_1, x_2, \ldots x_n = b$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots x_n^*$ be any smapel points in these subintervals, so x_i^* lies in the ith subinterval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x\,,$$

  provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is integrable on $[a, b]$.

- Theorem 1: If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$.

- Properties of definite integrals: Let c be any constant.

  1. $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$.
  2. $\int_a^a f(x)\,\mathrm{d}x = 0$.
  3. $\int_a^b c\,\mathrm{d}x = c(b-a)$.
  4. $\int_a^b [f(x) + g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$.
  5. $\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$.
  6. $\int_a^b [f(x) - g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x$.
  7. If $f(x) \ge 0$ for $a \le x \le b$, then $\int_a^b f(x)\,\mathrm{d}x \ge 0$.
  8. If $f(x) \ge g(x)$ for $a \le x \le b$, then $\int_a^b f(x)\,\mathrm{d}x \ge \int_a^b g(x)\,\mathrm{d}x$.

9. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then
   $m(b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M(b-a)$.

- Theorem 2. Fundamental theorem of calculus: If f is continuous on $[a, b]$ then:

  1. The function g defined by $g(x) = \int_a^x f(t)\mathrm{d}t$ for $a \leq x \leq b$ is continuous on $[a, b]$, differentiable on (a, b), and $g'(x) = f(x)$.
  2. $\int_a^b f(t)\mathrm{d}t = F(a) - F(b)$ where F is any antiderivative of f, i.e. we have $F'(x) = f(x)$.

- Indefinite integrals: We write

$$\int f(x)\mathrm{d}x = F(x) + C$$

  where $F'(x) = f(x)$.

- Theorem 3. Net change theorem: The integral of a rate of change is the net change:

$$\int_a^b F'(x)\mathrm{d}x = F(b) - F(a)\,.$$

- The substitution rule: If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I then

$$\int f(g(x))g'(x)\mathrm{d}x = \int f(u)\mathrm{d}u\,.$$

- The substitution rule for definite integrals: If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$ then

$$\int_a^b f(g(x))g'(x)\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\mathrm{d}u\,.$$

- Theorem 4. Integrals of symmetric functions: Suppose f is continuous on $[-a, a]$.

  (a) If f is even then $\int_{-a}^a f(x)\mathrm{d}x = 2\int_0^a f(x)\mathrm{d}x$.

  (b) If f is odd then $\int_{-a}^a f(x)\mathrm{d}x = 0$.