# Higher Mathematics in English II
# Exercises I:
# Matrices and Vector Spaces

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1. Calculate the rank of the following matrices by converting them to echelon form
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 2 \\ 3 & 8 & -6 & 7 & 2 \\ 7 & -6 & 5 & 4 & 4 \\ 1 & 2 & 1 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 & 4 \\ 2 & 1 & 0 & 3 \\ 1 & -1 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

2. Calculate the inverses of the following square matrices
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 8 & 6 & 7 \\ 7 & 6 & 5 & 4 \\ 1 & 2 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 4 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

3. Write down the $5 \times 5$ matrices which correspond to the following elementary row operations:
   - $\rho_2 := \rho_2 - 3\rho_4$
   - $\text{swap}(\rho_2, \rho_5)$

4. Prove that for a square matrix $\mathbf{A}$ with an inverse $\mathbf{B}$ such that $\mathbf{AB} = \mathbb{I}$ that there is a matrix $\mathbf{C}$ satisfying $\mathbf{CA} = \mathbb{I}$ and that the inverse of $\mathbf{A}$ is unique.
   (a) For any $n \times n$ row operation matrix $\mathbf{R}$ show directly that it has an inverse $\mathbf{RS} = \mathbf{SR} = \mathbb{I}$.
   (b) Show for a matrix $\mathbf{R}$ composed of many elementary row operations $\mathbf{R} = \mathbf{R}_1\mathbf{R}_2 \ldots \mathbf{R}_n$ that it does not have a row entirely made of zeros. *(Tip: This follows from (a).)*
   (c) Prove that if $\mathbf{AB} = \mathbb{I}$ for square matrices $\mathbf{A}$ and $\mathbf{B}$ then there exists a matrix $\mathbf{C}$ satisfying $\mathbf{CA} = \mathbb{I}$. *(Tip: By considering the matrix* $\mathbf{RA}$ *which is in echelon form and* $\mathbf{AB} = \mathbb{I}$ *show that* $\operatorname{rk} \mathbf{A} = n$. *We note that if an* $n \times n$ *matrix has rank* $n$ *it has a left inverse.)*
   (d) Prove that $\mathbf{B} = \mathbf{C}$ and therefore the inverse of $\mathbf{A}$ is unique.

5. Are the following matrices invertible?

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 1 & 9 & 6 & 2 \\ 3 & 0 & 0 & 6 \\ 5 & 8 & 9 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 9 & 6 & 2 \\ 3 & 0 & 0 & 6 \\ 5 & 8 & 9 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 9 & 2 \\ 3 & 0 & 6 \end{pmatrix}.$$

6. Prove that the determinant of an upper triangular square matrix is equal to the product of its diagonal entries.

7. Prove that if a square matrix $\mathbf{A}$ is invertible then $\det \mathbf{A}^{-1} = (\det \mathbf{A})^{-1}$.

8. Prove that the set of vectors $\mathbb{R}^2$ forms a vector space when the addition of two vectors $\mathbf{u} = (x_1, y_1)^T$ and $\mathbf{v} = (x_2, y_2)^T$ is defined as $\mathbf{u} + \mathbf{v} = (x_1 x_2, y_1 y_2)^T$ and scalar multiplication as $\lambda \mathbf{u} = (x_1^\lambda, y_1^\lambda)^T$. What is the zero vector in this vector space?

9. Let $V$ be the set of all functions $f, g, \ldots$ from the natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ to $\mathbb{R}$, with addition defined by $(f + g)(n) = f(n) + g(n)$ $\forall n \in \mathbb{N}$ and scalar multiplication $(\lambda f)(n) = \lambda \cdot f(n)$ $\forall n \in \mathbb{N}$. (This can also be written as the set of functions $f : \mathbb{N} \to \mathbb{R}$.) Prove that $V$ is a vector space. What is the zero vector in this vector space?

10. Prove the following: Suppose $\mathbf{A} = \{a_1, a_2, \ldots a_n\} \subset V$ is linearly independent, where $V$ is a vector space over $F$. Suppose also that $v \in V$ and there are scalars $\lambda_1, \ldots \lambda_n$ and $\mu_1, \ldots \mu_n$ such that

$$v = \lambda_1 a_1 + \lambda_2 a_2 + \ldots \lambda_n a_n$$

and

$$v = \mu_1 a_1 + \mu_2 a_2 + \ldots \mu_n a_n$$

then $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$,$\ldots$ $\lambda_n = \mu_n$. *(Tip: consider the definition of linear independence.)*

11. Prove that the set of Pauli matrices and the $2 \times 2$ identity matrix, $A = \{\mathbb{I}_2, \boldsymbol{\sigma}^x, \boldsymbol{\sigma}^y, \boldsymbol{\sigma}^z\}$, is a basis of the vector space $V$, the set of all $2 \times 2$ matrices over $\mathbb{C}$, with addition and scalar multiplication defined in the usual way. The Pauli matrices are

$$\boldsymbol{\sigma}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \boldsymbol{\sigma}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

12. Find a basis for the following vector spaces:

    (a) The set of all $2 \times 2$ matrices over $\mathbb{R}$.

    (b) $\mathbb{C}^4$, i.e. the set of $4 \times 1$ column vectors with complex entries.

13. Which of the following are bases over $\mathbb{R}^3$? Give reasons!

    (a) $A = \{(1, 1, 0)^T, (0, 1, 0)^T, (1, 0, 1)^T, (0, 0, 1)^T\}$.
    (b) $B = \{(1, 1, 0)^T, (0, 1, 0)^T, (1, 0, 1)^T\}$.
    (c) $C = \{(1, 0, 0)^T, (0, \mathrm{i}, 0)^T, (1, 0, \mathrm{i})^T\}$.
    (d) $D = \{(1, 1, 1)^T, (2, 2, 1)^T, (1, 1, 0)^T, \}$.

14. Find bases over the following subspaces of $\mathbb{R}^3$.
    (a) $A = \{(x, y, z)^T : 2x + y - z = 0\}$.
    (b) $B = \{(x, y, z)^T : x + y - 2z = 0,\ x - y = 0, \}$.
15. Prove that the set of vectors $U = \{(x, y, 0)^T : x, y \in \mathbb{R}\}$ forms a subspace of the vector space $\mathbb{R}^3$.
16. Prove that one can not define a field of order 4 with the integers modulo 4, $\{0, 1, 2, 3\}$.
17. Prove that one can define a field of order 4 from $\{0, 1, a, a+1\}$. *Tip: Start from normal addition and multiplication and determine what needs to be changed to satisfy the definition of a field.*