

1. Consider the following one dimensional Hamiltonian:

$$\hat{H}|\psi\rangle = -J \sum_{j=1}^N \left[ (1 + \delta e^{i\pi j}) \hat{c}_j^\dagger \hat{c}_{j+1} + \text{H.c.} \right] |\psi\rangle = \epsilon |\psi\rangle.$$

This describes a chain of atoms located at sites  $j = 1, 2, \dots, N$ . What does the term  $\hat{c}_j^\dagger \hat{c}_{j+1}$  mean? We assume that the operators are fermionic  $\{\hat{c}_j, \hat{c}_l^\dagger\} = \delta_{j,l}$  etc, and the chain is periodic so that site  $N+1$  is the same as site 1, i.e.  $\hat{c}_1 = \hat{c}_{N+1}$ .

- (a) To solve such a Hamiltonian we want to diagonalise it, i.e. we wish to find a Hamiltonian like  $\hat{H} = \sum_k \epsilon_k \hat{c}_k^\dagger \hat{c}_k$ . Why does this help and what is  $\epsilon_k$ ?  
 (b) For this new Hamiltonian to make sense we need that  $\{\hat{c}_k, \hat{c}_q^\dagger\} = \delta_{k,q}$ . For the transform

$$c_j = \frac{1}{\sqrt{N}} \sum_{k=1}^N e^{ikj} c_k$$

show that this is true.

- (c) Find the Hamiltonian in terms of these new operators and diagonalise it.

2. Consider the following one dimensional Hamiltonian:

$$\hat{H}|\psi\rangle = [\alpha \hat{c}^\dagger \hat{c} + \beta \hat{c}^\dagger \hat{c}^\dagger + \text{H.c.}] |\psi\rangle = \epsilon |\psi\rangle.$$

This is a simplified form of a Hamiltonian we might meet for superconductivity. We can diagonalise this Hamiltonian using a transform like  $\hat{f} = u\hat{c} + v\hat{c}^\dagger$ .

- (a) If  $\hat{f}$  is to obey canonical commutation relations, i.e.  $\{\hat{f}, \hat{f}^\dagger\} = 1$  and  $\{\hat{f}, \hat{f}\} = 0$ , find the condition on  $u$  and  $v$ .  
 (b) Diagonalise the Hamiltonian.