Linear Algebra
Problem Set 1: Matrix Operations

Dr Nicholas Sedlmayr

1. Calculate the rank of the following matrices by converting them to echelon form (4)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 2 \\ 3 & 8 & -6 & 7 & 2 \\ 7 & -6 & 5 & 4 & 4 \\ 1 & 2 & 1 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 & 4 \\ 2 & 1 & 0 & 3 \\ 1 & -1 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

2. Calculate the inverses of the following square matrices (4)
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 8 & 6 & 7 \\ 7 & 6 & 5 & 4 \\ 1 & 2 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 4 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 \end{pmatrix}$$

3. Write down the 5×5 matrices which correspond to the following elementary row operations: (2)
   - $\rho_2 := \rho_2 - 3\rho_4$
   - $\text{swap}(\rho_2, \rho_5)$

4. Prove that for a square matrix $\mathbf{A}$ with an inverse $\mathbf{B}$ such that $\mathbf{AB} = \mathbb{I}$ that there is a matrix $\mathbf{C}$ satisfying $\mathbf{CA} = \mathbb{I}$ and that the inverse of $\mathbf{A}$ is unique. (10)
   (a) For any $n \times n$ row operation matrix $\mathbf{R}$ show directly that it has an inverse $\mathbf{RS} = \mathbf{SR} = \mathbb{I}$.
   (b) Show for a matrix $\mathbf{R}$ composed of many elementary row operations $\mathbf{R} = \mathbf{R}_1\mathbf{R}_2 \ldots \mathbf{R}_n$ that it does not have a row entirely made of zeros. *(Tip: This follows from (a).)*
   (c) Prove that if $\mathbf{AB} = \mathbb{I}$ for square matrices $\mathbf{A}$ and $\mathbf{B}$ then there exists a matrix $\mathbf{C}$ satisfying $\mathbf{CA} = \mathbb{I}$. *(Tip: By considering the matrix* $\mathbf{RA}$ *which is in echelon form and* $\mathbf{AB} = \mathbb{I}$ *show that* $\operatorname{rk}\mathbf{A} = n$. *We have seen in the class that if an* $n \times n$ *matrix has rank* n *it has a left inverse.)*
   (d) Prove that $\mathbf{B} = \mathbf{C}$ and therefore the inverse of $\mathbf{A}$ is unique.