Linear Algebra I
Summary of Lectures: Vector Spaces

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1. Definition 2.1: A vector space. A vector space V over a field F (see definition 2.3) is a set containing:

   - a special zero vector $\mathbf{0}$;
   - an operation of addition of two vectors $\mathbf{u}+\mathbf{v} \in V$, for $\mathbf{u}, \mathbf{v} \in V$; and
   - multiplication of a vector V with a number $\lambda \in F$ with $\lambda \mathbf{v} \in V$.

   The vector space must be closed under both of these operations and must satisfy the following laws $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\lambda, \mu \in F$:

   (1) associativity $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$;
   (2) commutativity $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$;
   (3) $\mathbf{u}+\mathbf{0}=\mathbf{u}$;
   (4) $\mathbf{v}+(-1)\mathbf{v}=\mathbf{0}$;
   (5) $\lambda(\mu\mathbf{u})=(\lambda\mu)\mathbf{v}$;
   (6) distributivity $\lambda(\mathbf{u}+\mathbf{v})=\lambda\mathbf{u}+\mu\mathbf{v}$; and
   (7) distributivity $(\lambda+\mu)\mathbf{u}=\lambda\mathbf{u}+\mu\mathbf{u}$.

2. Proposition 2.2: $\forall \mathbf{v} \in V$ and $\forall \lambda \in F$:

   (a) $\mathbf{v}=1\mathbf{v}$;
   (b) $0\mathbf{v}=\mathbf{0}$; and
   (c) $\lambda\mathbf{0}=\mathbf{0}$.

3. Definition 2.3: A field is a set F containing distinct elements 0 and 1 with two binary operations $+$ and $\cdot$ satisfying the axioms $\forall a, b, c \in F$:

   (1) $a+b=b+a$;
   (2) $(a+b)+c=a+(b+c)$;
   (3) $a+0=a$;
   (4) $\forall a\ \exists -a$ such that $a+(-a)=0$;
   (5) $a \cdot b=b \cdot a$;
   (6) $(a \cdot b) \cdot c=a \cdot (b \cdot c)$;
   (7) $a \cdot 1=a$;
   (8) $\forall a \neq 0\ \exists a^{-1}$ such that $a \cdot a^{-1}=1$; and
   (9) $a \cdot (b+c)=a \cdot b+a \cdot c$;

If a field F is finite its order is the number of elements in F.

4. Note that as a field also satisfies all axioms of a vector space a field F is also itself a vector space $V = F$ over the field F and all properties of a vector space apply.

5. Theorem 2.4: For each prime p and each positive integer n, there is a unique field of order p^n. Additionally, every finite field is of this form.

6. Definition 2.5: Given a vector space V over F, a subspace of V is a subset $W \subset V$ which contains the zero vector of V and is closed under the operations of addition and scalar multiplication.

7. Lemma 2.5.1: Let $W \subset V$ be nonempty, where V is a vector space over F. Then W is a subspace of V iff $\mathbf{v} + \lambda\mathbf{u} \in W$ for each $\mathbf{v}, \mathbf{u} \in W$ and each scalar λ.

8. Definition 2.6: Given a vector space V over F, and given a subset of V $A = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots \mathbf{u}_n\}$,

$$W = \{\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 + \lambda_3\mathbf{u}_3 + \ldots \lambda_n\mathbf{u}_n : \lambda_1, \lambda_2, \ldots \lambda_n \in F\}$$

is the subspace of V spanned by A. The elements of W are called linear combinations of vectors from A and the subspace W is denoted as span A.

9. Definition 2.7: If A is an infinite subset of V, where V is a vector space over F, we define span A to be the set of all linear combinations of finite subsets of A.

10. Definition 2.8: A set $A \subset V$ of vectors in a vector space V over F is linearly dependent if there are $n \in \mathbb{N}$ vectors $a_1, a_2, \ldots a_n$ and scalars $\lambda_1, \lambda_2, \ldots \lambda_n$ not all zero such that

$$\lambda_1 a_1 + \lambda_2 a_2 + \ldots \lambda_n a_n = 0\,.$$

Otherwise A is linearly independent.

- For a finite set $A = \{a_1, a_2, \ldots a_n\}$ it is linearly independent iff $\forall$ scalars $\lambda_1, \lambda_2, \ldots \lambda_n \in F$

$$\lambda_1 a_1 + \lambda_2 a_2 + \ldots \lambda_n a_n = 0 \Rightarrow \lambda_1 = \lambda_2 = \ldots \lambda_n = 0.$$

- If A is infinite it is linearly independent iff every subset of A is linearly independent.
- By convention the empty set is linearly independent.

11. Proposition 2.9: Suppose $\mathbf{A} = \{a_1, a_2, \ldots a_n\} \subset V$ is linearly independent, where V is a vector space over F. Suppose also that $v \in V$ and there are scalars $\lambda_1, \ldots \lambda_n$ and $\mu_1, \ldots \mu_n$ such that

$$v = \lambda_1 a_1 + \lambda_2 a_2 + \ldots \lambda_n a_n$$

and

$$v = \mu_1 a_1 + \mu_2 a_2 + \ldots \mu_n a_n$$

then $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$,$\ldots \lambda_n = \mu_n$.

12. Definition 2.10: A basis of a vector space V is a linearly independent set $B \subset V$ which spans V.

13. Theorem 2.11: Let V be a vector space over F, and let $B \subset V$ be linearly independent. Then there is a basis B' of V with $B \subset B'$.

14. Theorem 2.12: Suppose span $A = V$ and $B \subset V$ is linearly independent. Then there is a basis B' of V with $B \subset B' \subset A \cup B$.

15. Lemma 2.13: The "exchange lemma". Suppose $a_1, a_2, \ldots a_n, b$ are vectors in a vector space V, and suppose that

$$b \in \operatorname{span}(a_1, a_2, \ldots a_{n-1}, a_n)$$

but

$$b \notin \operatorname{span}(a_1, a_2, \ldots a_{n-1}),$$

then

$$a_n \in \operatorname{span}(a_1, \ldots, a_{n-1}, b).$$

If in addition $\{a_1, a_2, \ldots a_n\}$ is linearly independent then so is $\{a_1, a_2, \ldots a_{n-1}, b\}$.

16. Theorem 2.14: Suppose S and B are both bases of a vector space V over F and. Then A and B have the same number of elements.

17. Definition 2.15: The number of elements of a basis of a vector space V over F is called the dimension of V and is written as $\dim V$.

18. Corollary 2.16: If V is a vector space over F and $U \subset V$ is a subspace of V then $\dim U \leq \dim V$. If, additionally, $\dim V$ is finite and $U \neq V$ then $\dim U < \dim V$.

19. Corollary 2.17: Suppose that V is a vector space over F, $\dim V$ is finite, and $U \subset V$ is a subspace of V with $\dim U = \dim V$, then $U = V$.

20. The coordinates of a vector $v \in V$, with V a vector space over F, with respect to an *ordered* basis $B = \{v_1, v_2, \ldots v_n\}$ are $(\lambda_1, \lambda_2, \ldots, v_n)^T$ where

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \ldots \lambda_n v_n .$$

21. Definition 2.18: Two vector spaces V and W, both over the same field F, are isomorphic if there is a bijection $f : V \to W$ such that

$$f(u + v) = f(u) + f(v)$$

and

$$f(\lambda v) = \lambda f(v),$$

$\forall u, v \in V$ and $\forall \lambda \in F$. The bijection is said to be an isomorphism from V to W and we write $V \cong W$ or $f : V \xrightarrow{\sim} W$.

22. Theorem 2.19: Suppose V is a vector space over $\mathbb{R}$ with finite dimension $n \geq 0$. Then $V \cong \mathbb{R}^n$ as real vector spaces. Similarly if V is a vector space over $\mathbb{C}$ with finite dimension $n \geq 0$. Then $V \cong \mathbb{C}^n$ as complex vector spaces.