

# Summary of Calculus I

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## Contents

1	Limits	1
2	Differentiation	5

## 1 Limits

- **Tangent:** A tangent line is a line that touches a curve, and has the same slope as the curve at the point of contact.
- **Average rate of change:** The average rate of change of a function  $f(x)$  between  $x = a$  and  $x = b$  is

$$\frac{f(b) - f(a)}{b - a}.$$

- **Limit:** Let  $f$  be a function defined on some open interval that contains  $a$  but not necessarily at  $a$  itself. Then we say the limit of  $f(x)$  as  $x$  approaches  $a$  is  $L$  and we write

$$\lim_{x \rightarrow a} f(x) = L$$

if for every number  $\epsilon > 0$  there is a number  $\delta > 0$  such that if

$$0 < |x - a| < \delta \text{ then } |f(x) - L| < \epsilon.$$

- **Left-hand limit:**

$$\lim_{x \rightarrow a^-} f(x) = L,$$

if for every number  $\epsilon > 0$  there is a number  $\delta > 0$  such that if

$$a - \delta < x < a \text{ then } |f(x) - L| < \epsilon.$$

- **Right-hand limit:**

$$\lim_{x \rightarrow a^+} f(x) = L,$$

if for every number  $\epsilon > 0$  there is a number  $\delta > 0$  such that if

$$a < x < a + \delta \text{ then } |f(x) - L| < \epsilon.$$

- **Theorem 1:**

$$\lim_{x \rightarrow a} f(x) = L \text{ if and only if } \lim_{x \rightarrow a^-} f(x) = L \text{ and } \lim_{x \rightarrow a^+} f(x) = L.$$

- **Infinite limits:** Let  $f$  be a function defined on some open interval that contains  $a$  but not necessarily at  $a$  itself. Then

$$\lim_{x \rightarrow a} f(x) = \infty$$

means that for every positive number  $M$  there is a positive number  $\delta$  such that if

$$0 < |x - a| < \delta \text{ then } f(x) > M.$$

Let  $f$  be a function defined on some open interval that contains  $a$  but not necessarily at  $a$  itself. Then

$$\lim_{x \rightarrow a} f(x) = -\infty$$

means that for every negative number  $N$  there is a positive number  $\delta$  such that if

$$0 < |x - a| < \delta \text{ then } f(x) < N.$$

- **Vertical asymptote:** The line  $x = a$  is called a vertical asymptote of the curve  $y = f(x)$  if at least one of the following is true:

$$\begin{array}{ll} \lim_{x \rightarrow a} f(x) = \infty, & \lim_{x \rightarrow a} f(x) = -\infty, \\ \lim_{x \rightarrow a^-} f(x) = \infty, & \lim_{x \rightarrow a^-} f(x) = -\infty, \\ \lim_{x \rightarrow a^+} f(x) = \infty, \text{ or } & \lim_{x \rightarrow a^+} f(x) = -\infty. \end{array}$$

- **Limit laws:** Suppose that  $c$  is a constant and that the limits

$$\lim_{x \rightarrow a} f(x) \text{ and } \lim_{x \rightarrow a} g(x)$$

exist. Then:

1. **Sum law**  $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
2. **Difference law**  $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
3. **Constant multiple law**  $\lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$
4. **Product law**  $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
5. **Quotient law**  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$  if  $\lim_{x \rightarrow a} g(x) \neq 0$
6. **Power law**  $\lim_{x \rightarrow a} [f(x)]^n = \left[ \lim_{x \rightarrow a} f(x) \right]^n$  where  $n$  is a positive integer
7.  $\lim_{x \rightarrow a} c = c$
8.  $\lim_{x \rightarrow a} x = a$
9.  $\lim_{x \rightarrow a} x^n = a^n$  where  $n$  is a positive integer

10.  $\lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}$  where  $n$  is a positive integer and if  $n$  is even then we assume that  $a > 0$
11. **Root law**  $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$  where  $n$  is a positive integer and if  $n$  is even then we assume that  $\lim_{x \rightarrow a} f(x) > 0$

- **Direct substitution property:** If  $f$  is a polynomial or a rational function and  $a$  is in the domain of  $f$ , then

$$\lim_{x \rightarrow a} f(x) = f(a).$$

Such functions are called continuous at  $a$ .

- **Theorem 2:** If  $f(x) \leq g(x)$  when  $x$  is near  $a$  (except possibly at  $a$ ) and the limits of both  $f$  and  $g$  exist as  $x$  approaches  $a$  then

$$\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x).$$

- **Theorem 3. The squeeze theorem:** If  $f(x) \leq g(x) \leq h(x)$  when  $x$  is near  $a$  (except possibly at  $a$ ) and

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$$

then

$$\lim_{x \rightarrow a} g(x) = L.$$

- **Continuous function:** A function  $f$  is said to be continuous at  $a$  if  $\lim_{x \rightarrow a} f(x) = f(a)$ .
  - A function  $f$  is continuous from the right at a number  $a$  if  $\lim_{x \rightarrow a^+} f(x) = f(a)$ .
  - A function  $f$  is continuous from the left at a number  $a$  if  $\lim_{x \rightarrow a^-} f(x) = f(a)$ .
  - A function  $f$  is said to be continuous on an interval if it is continuous at every number in the interval.
- **Theorem 4:** If  $f$  and  $g$  are continuous at  $a$  and  $c$  is a constant, then the following functions are also continuous at  $a$ :
  - 1.1.  $f + g$ ,
  - 1.2.  $f - g$ ,
  - 1.3.  $cf$ ,
  - 1.4.  $fg$ , and
  - 1.5.  $\frac{f}{g}$  if  $g(a) \neq 0$ .
- **Theorem 5:**
  - (a) Any polynomial is continuous everywhere, that is, it is continuous on  $\mathbb{R} = (-\infty, \infty)$ .

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

- **Lemma 6:**  $\lim_{\theta \rightarrow 0} \cos(\theta) = 1$  and  $\lim_{\theta \rightarrow 0} \sin(\theta) = 0$ .
- **Theorem 7:** Polynomials, rational functions, root functions, and trigonometric functions are all continuous at every number in their domains.
- **Theorem 8:** If  $f$  is continuous at  $b$  and  $\lim_{x \rightarrow a} g(x) = b$  then  $\lim_{x \rightarrow a} f(g(x)) = f(b)$ . Equivalently  $\lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right)$ .
- **Theorem 9:** If  $g$  is continuous at  $a$  and  $f$  is continuous at  $g(a)$  then the composite function  $f \circ g$  given by  $(f \circ g)(x) = f(g(x))$  is continuous at  $a$ .
- **Theorem 10. The intermediate value theorem:** Suppose that  $f$  is continuous on the closed interval  $[a, b]$  and let  $N$  be any number between  $f(a)$  and  $f(b)$ , where  $f(a) \neq f(b)$ . Then there exists a number  $c$  in  $(a, b)$  such that  $f(c) = N$ .

## 2 Differentiation

- **Tangent line:** The tangent line to the curve  $y = f(x)$  at the point  $P(a, f(a))$  is the line through  $P$  with the slope

$$m = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists. Equivalently

$$m = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}.$$

- **Derivative:** The derivative of a function  $f$  at a number  $a$ , denoted by  $f'(a)$  is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.

- **Instantaneous velocity:** If  $f(t)$  represents position as a function of time then the derivative  $f'(a)$  is the instantaneous velocity of  $y = f(t)$  with respect to  $t$  when  $t = a$ .
- **Instantaneous rate of change:** The derivative  $f'(a)$  is the instantaneous rate of change of  $y = f(x)$  with respect to  $x$  when  $x = a$ .
- **Derivative as a function:** The derivative of a function  $f(x)$ ,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h},$$

can be regarded as a new function called the derivative of  $f$ . Other notations for the derivative of  $y = f(x)$  with respect to  $x$  are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x).$$

- **Differentiable:** A function  $f$  is differentiable at  $a$  if  $f'(a)$  exists. It is differentiable on an open interval  $(a, b)$  if it is differentiable at every number in the interval.
- **Theorem 1:** If  $f$  is differentiable at  $a$  then it is continuous at  $a$ .
- **Higher derivatives:** The derivative of a derivative is called the second derivative, denoted by  $(f')' = f''$  or

$$\frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2 y}{dx^2}.$$

In general the  $n$ th derivative, where  $n$  is a positive integer, is written as  $f^n(x)$  or  $\frac{d^n y}{dx^n}$ .

- **Differentiation formulas:** If  $c$  is a constant and  $n$  is a real number then:

$$\begin{aligned}\frac{d}{dx}(c) &= 0, \text{ and} \\ \frac{d}{dx}(x^n) &= nx^{n-1}, \\ \frac{d}{dx}\sin(x) &= \cos(x), \\ \frac{d}{dx}\cos(x) &= -\sin(x).\end{aligned}$$

- **Differentiation rules:** If  $c$  is a constant and  $f$  and  $g$  are differentiable functions then:

$$\begin{aligned}\frac{d}{dx}[cf(x)] &= cf'(x) && \text{the constant multiple rule,} \\ \frac{d}{dx}[f(x) + g(x)] &= f'(x) + g'(x) && \text{the sum rule,} \\ \frac{d}{dx}[f(x) - g(x)] &= f'(x) - g'(x) && \text{the difference rule,} \\ \frac{d}{dx}[f(x)g(x)] &= g(x)f'(x) + f(x)g'(x) && \text{the product rule,} \\ \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] &= \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)} && \text{the quotient rule, and} \\ \frac{d}{dx}[f(g(x))] &= f'(g(x)) \cdot g'(x) && \text{the chain rule.}\end{aligned}$$

- **Implicit differentiation:** Using the chain rule, and others as necessary, allows us to also differentiate implicitly. Let  $g(x, y) = 0$  define  $y$  as a function of  $x$ .  $\frac{dy}{dx}$  will relate  $\frac{dy}{dx}$ ,  $y$ , and  $x$ .
- **Related rates:** Let  $g(x, y) = 0$  relate two functions of a variable  $t$ .  $\frac{dg}{dt}$  will relate  $\frac{dy}{dt}$ ,  $\frac{dx}{dt}$ ,  $y$ , and  $x$ .
- **Linear approximation:** The linear approximation of  $f(x)$  near a point  $a$  is  $L(x) = f(a) + f'(a)(x - a)$ .
- **Differentials:** If  $y = f(x)$  where  $f(x)$  is a differentiable function, and the differential  $dx$  is an independent variable then the differential  $dy$  is a dependent variable and is given by

$$dy = f'(x)dx = \frac{dy}{dx}dx.$$

- **Absolute extrema:** Let  $c$  be a number in the domain  $D$  of a function  $f$ . Then  $f(c)$  is
  - the absolute maximum value of  $f$  on  $D$  if  $f(c) \geq f(x)$  for all  $x$  in  $D$ ,  
or
  - the absolute minimum value of  $f$  on  $D$  if  $f(c) \leq f(x)$  for all  $x$  in  $D$ .
- **Local extrema:** Let  $c$  be a number in the domain  $D$  of a function  $f$ . Then  $f(c)$  is

- (a) a local maximum value of  $f$  on  $D$  if  $f(c) \geq f(x)$  when  $x$  is near  $c$ , or
- (b) a local minimum value of  $f$  on  $D$  if  $f(c) \leq f(x)$  when  $x$  is near  $c$ .

Near  $c$  means on some open interval containing  $c$ .

- **Theorem 2. The extreme value theorem:** If  $f$  is continuous on a closed interval  $[a, b]$  then  $f$  attains an absolute maximum value  $f(c)$  and an absolute minimum value  $f(d)$  at some numbers  $c$  and  $d$  in  $[a, b]$ .
- **Theorem 3. Fermat's theorem:** If  $f$  has a local extrema at  $c$ , and if  $f'(c)$  exists, then  $f'(c) = 0$ .
- **Critical number:** A critical number of a function  $f$  is a number  $c$  in the domain of  $f$  such that either  $f'(c) = 0$  or  $f'(c)$  does not exist.
- **The closed interval method:** To find the absolute extrema values of a continuous function  $f$  on a closed interval  $[a, b]$ :
  1. Find  $f$  at the critical numbers in  $(a, b)$ .
  2. Find  $f$  at the end points  $a$  and  $b$ .
  3. The largest and smallest of these values are the absolute extrema.
- **Theorem 4. Rolle's theorem:** Let  $f$  be a function which satisfies the following three hypotheses:
  1.  $f$  is continuous on the closed interval  $[a, b]$ .
  2.  $f$  is differentiable on the open interval  $(a, b)$ .
  3.  $f(a) = f(b)$ .

Then there is a number  $c$  in  $(a, b)$  such that  $f'(c) = 0$ .

- **Theorem 5. The mean value theorem:** Let  $f$  be a function which satisfies the following hypotheses:
  1.  $f$  is continuous on the closed interval  $[a, b]$ .
  2.  $f$  is differentiable on the open interval  $(a, b)$ .

Then there is a number  $c$  in  $(a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

- **Theorem 6:** If  $f'(x) = 0$  for all  $x$  in an interval  $(a, b)$  then  $f$  is constant on  $(a, b)$ .
- **Corollary 7:** If  $f'(x) = g'(x)$  for all  $x$  in an interval  $(a, b)$  then  $f - g$  is constant on  $(a, b)$ . I.e.

$$f(x) = g(x) + c \text{ where } c \text{ is a constant.}$$

- **Increasing/decreasing test:**
  - (a) If  $f'(x) > 0$  on an interval then  $f$  is increasing on that interval.

- (b) If  $f'(x) < 0$  on an interval then  $f$  is decreasing on that interval.
- **First derivative test:** Suppose that  $c$  is a critical number of a continuous function  $f$ .
  - (a) If  $f'$  changes sign from positive to negative at  $c$ , then  $f$  has local maximum at  $c$ .
  - (b) If  $f'$  changes sign from negative to positive at  $c$ , then  $f$  has local minimum at  $c$ .
  - (c) If  $f'$  does not change sign at  $c$ , then  $f$  has no local maximum or minimum at  $c$ .
- **Concavity:** If the graph of  $f$  lies above all of its tangents on an interval  $I$ , then it is concave upward on  $I$ . If the graph of  $f$  lies below all of its tangents on an interval  $I$ , then it is concave downward on  $I$ .
- **Concavity test:**
  - (a) If  $f''(x) > 0$  for all  $x$  in an interval  $I$  then  $f$  is concave upward on  $I$ .
  - (b) If  $f''(x) < 0$  for all  $x$  in  $I$  then  $f$  is concave downward on  $I$ .
- **Inflection point:** A point  $P$  on a curve  $y = f(x)$  is called an inflection point if  $f$  is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at  $P$ .
- **The second derivative test:** Suppose  $f''$  is continuous near  $c$ .
  - (a) If  $f'(c) = 0$  and  $f''(c) > 0$ , then  $f$  has a local minimum at  $c$ .
  - (b) If  $f'(c) = 0$  and  $f''(c) < 0$ , then  $f$  has a local maximum at  $c$ .
- **Horizontal asymptotes:** Let  $f$  be a function defined on a interval  $(a, \infty)$ . Then

$$\lim_{x \rightarrow \infty} f(x) = L$$

means that for every  $\epsilon > 0$  there is a corresponding number  $N$  such that if  $x > N$  then  $|f(x) - L| < \epsilon$ .

Similarly let  $f$  be a function defined on a interval  $(a, \infty)$ . Then

$$\lim_{x \rightarrow -\infty} f(x) = L$$

means that for every  $\epsilon > 0$  there is a corresponding number  $N$  such that if  $x < N$  then  $|f(x) - L| < \epsilon$ .

The line  $y = L$  is called a horizontal asymptote of the curve  $y = f(x)$  if either

$$\lim_{x \rightarrow \infty} f(x) = L \text{ or } \lim_{x \rightarrow -\infty} f(x) = L.$$

- **Limit laws:** Suppose that  $c$  is a constant and that the limits

$$\lim_{x \rightarrow \infty} f(x) \text{ and } \lim_{x \rightarrow \infty} g(x)$$

exist. Then:



1. **Sum law**  $\lim_{x \rightarrow \infty} [f(x) + g(x)] = \lim_{x \rightarrow \infty} f(x) + \lim_{x \rightarrow \infty} g(x)$
2. **Difference law**  $\lim_{x \rightarrow \infty} [f(x) - g(x)] = \lim_{x \rightarrow \infty} f(x) - \lim_{x \rightarrow \infty} g(x)$
3. **Constant multiple law**  $\lim_{x \rightarrow \infty} [cf(x)] = c \lim_{x \rightarrow \infty} f(x)$
4. **Product law**  $\lim_{x \rightarrow \infty} [f(x)g(x)] = \lim_{x \rightarrow \infty} f(x) \cdot \lim_{x \rightarrow \infty} g(x)$
5. **Quotient law**  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow \infty} f(x)}{\lim_{x \rightarrow \infty} g(x)}$  if  $\lim_{x \rightarrow \infty} g(x) \neq 0$
6. **Power law**  $\lim_{x \rightarrow \infty} [f(x)]^n = [\lim_{x \rightarrow \infty} f(x)]^n$  where  $n$  is a positive integer
7.  $\lim_{x \rightarrow \infty} c = c$
8.  $\lim_{x \rightarrow \infty} x = \infty$
11. **Root law**  $\lim_{x \rightarrow \infty} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow \infty} f(x)}$  where  $n$  is a positive integer and if  $n$  is even then we assume that  $\lim_{x \rightarrow \infty} f(x) > 0$

Similarly for  $\lim x \rightarrow -\infty$ .

- **Theorem 8:** If  $r > 0$  is a rational number then

$$\lim_{x \rightarrow \infty} \frac{1}{x^r} = 0$$

If  $r > 0$  is a rational number such that  $x^r$  is defined for all  $x$  then

$$\lim_{x \rightarrow -\infty} \frac{1}{x^r} = 0$$

- **Slant asymptotes:** The line  $y = mx + b$  is a slant asymptote of  $f(x)$  if

$$\lim_{x \rightarrow \infty} [f(x) - (mx + b)] = 0$$

or

$$\lim_{x \rightarrow -\infty} [f(x) - (mx + b)] = 0.$$

- **The first derivative test for absolute extreme values:** Suppose  $c$  is a critical number of a continuous function  $f$  defined on an interval.
  - (a) If  $f'(x) > 0$  for all  $x < c$  and  $f'(x) < 0$  for all  $x > c$ , then  $f(c)$  is an absolute maximum value of  $f$ .
  - (b) If  $f'(x) < 0$  for all  $x > c$  and  $f'(x) > 0$  for all  $x < c$ , then  $f(c)$  is an absolute minimum value of  $f$ .
- **Antiderivative:** A function  $F$  is called the antiderivative of  $f$  on an interval  $I$  if  $F'(x) = f(x)$  for all  $x$  in  $I$ .
- **Theorem 9:** If  $F$  is an antiderivative of  $f$  on an interval  $I$ , then the most general antiderivative of  $f$  on  $I$  is  $F(x) + C$  where  $C$  is an arbitrary constant.