

# Theorems and Definitions for *Higher Mathematics in English II*

## Topics: Linear Algebra, Multivariate Calculus, and Complex Analysis

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# 1 Preamble

## 1.1 Notation

iff	if and only if
$\Rightarrow$	if then
$\equiv$	defined as
$\therefore$	therefore
$\because$	because
$\square$	end of proof
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{I}$	identity matrix
$\forall$	the universal quantifier, for all
$\exists$	the existential quantifier, there exists
$\in$	is an element of
$\subset$	is a subset of

## 1.2 Translations

English	po Polsku
Matrix	Macierz
Echelon form	Macierz schodkowa
Transpose	Transponowana
Determinant	Wyznacznik
Trace	Ślad
Rank	Rząd
Identity Matrix	Macierz jednostkowa
Inverse	Odwrotna
Associativity	Łączność
Commutativity	Przemienność
Distributivity	Rozdzielność
Scalar	Skalar
Field	Ciało
Vector	Wektor
Vector Space	Przestrzeń liniowa
Group	Grupa
Linear Independence	Liniowa niezależność
Subspace	Podprzestrzeń
Dimension	Wymiar
Basis	Baza
Inner Product Space	Przestrzeń unitarna
Symmetry	Symetria
Orthogonality	Ortogonalność
Orthonormality	Ortonormalność
Eigenvalues	Wartości własne
Eigenvectors	Wektory własne
Lemma	Lemat
Theorem	Twierdzenie
Proof	Dowód
Axiom	Aksjomat
Equality	Równość
Inequality	Nierówność

## 2 Linear Algebra

### 2.1 Matrices

- **Definition** of an  $n \times m$  matrix,  $\mathbf{A} = (a_{ij})$  with  $n$  row and  $m$  columns. Addition of matrices  $\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})$ .
  - Associativity, commutativity and existence of a zero matrix ( $\mathbf{0}$ ) for addition.
- **Definition** of multiplication of a matrix by a scalar:  $\lambda \mathbf{A} = (\lambda a_{ij})$ .
- **Definition:** Matrix multiplication of an  $n \times m$  matrix  $\mathbf{A}$  and an  $m \times k$  matrix  $\mathbf{B}$  is an  $n \times k$  matrix  $\mathbf{C} = \mathbf{AB} = (c_{ij})$  where  $c_{ij} = \sum_{r=1}^m a_{ir}b_{rj}$ .
  - Associativity, existence of a zero matrix ( $\mathbf{0}$ ) and an identity matrix  $\mathbb{I}$ , distributivity.
  - No commutativity!
- A matrix  $\mathbf{A}$  can have a right inverse  $\mathbf{AB} = \mathbb{I}$  and a left inverse  $\mathbf{CA} = \mathbb{I}$ .
  - **Proposition 1.1:** If a square matrix has either a left or right inverse then they have a unique inverse from both the left and right.
  - If a non-square matrix has both a left and right inverse then they are the same and the inverse is unique.
  - **Proposition 1.2:** If  $\mathbf{A}$  and  $\mathbf{B}$  are invertible square matrices then  $\mathbf{AB}$  is also invertible and  $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ .
- The **transpose** of an  $n \times m$  matrix is written as  $\mathbf{A}^T$ , which is an  $m \times n$  matrix found by transposing the rows and columns of  $\mathbf{A}$ .
  - **Proposition 1.3:** For two  $n \times n$  matrices  $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$  and  $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$ .
- **Elementary row operations** perform simple operations on the rows of an  $n \times m$  matrix  $\mathbf{A}$  and can be written as an  $n \times n$  matrix  $\mathbf{R}$  with the operation preformed by the multiplication  $\mathbf{RA}$ .  $\rho_i$  is used to refer to row  $i$ . There are three of them:
  - $\rho_j := \rho_j + \lambda\rho_i$ , add  $\lambda$  copies of row  $i$  to row  $j$ ;
  - $\rho_i := \lambda\rho_i$ , multiple row  $i$  by  $\lambda$  with  $\lambda \neq 0$ ; and
  - $\text{swap}(\rho_i, \rho_j)$ , swap rows  $i$  and  $j$ .
- **Echelon form:** A matrix where each row starts with a sequence of zeros, and the number of zeros in this sequences increases from row to row from top to bottom until the final row is reached or all remaining rows are composed entirely of zeros is said to be in echelon form.
  - All matrices can be put into echelon form using elementary row operations.
- **Rank:** If  $\mathbf{A}$  can be converted to the echelon form matrix  $\mathbf{B}$  and  $\mathbf{B}$  has  $k$  non-zero rows then the rank of  $\mathbf{A}$  is  $\text{rk } \mathbf{A} = k$ .
- An  $n \times k$  matrix  $\mathbf{A}$  with  $n \leq k$  and  $\text{rk } \mathbf{A} = n$  can be converted to a matrix  $\mathbf{B}$  where the left  $n \times n$  block is the identity matrix using elementary row operations.
- **Augmented matrix:** If we have an  $n \times m$  matrix  $\mathbf{A}$  and an  $n \times k$  matrix  $\mathbf{A}$  the augmented matrix  $(\mathbf{A}|\mathbf{B})$  is the  $n \times (m + k)$  matrix created by writing  $\mathbf{A}$  and  $\mathbf{B}$  next to each other.
- **Calculating the inverse:** For an  $n \times n$  matrix  $\mathbf{A}$  of rank  $n$  we can calculate the inverse using the augmented matrix  $(\mathbf{A}, \mathbb{I})$ . Apply row operations,  $\mathbf{R}$ , such that  $\mathbf{R}(\mathbf{A}, \mathbb{I}) = (\mathbb{I}, \mathbf{B})$ , clearly  $\mathbf{B} = \mathbf{R}$  is the left inverse of  $\mathbf{A}$ :  $\mathbf{BA} = \mathbb{I}$ .
- **Solving systems of linear equations.** A system of linear equations on the variables  $x_1, x_2, \dots, x_n$  can be written as  $\mathbf{Ax} = \mathbf{b}$  where  $\mathbf{A}$  is a  $k \times n$  matrix,  $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$  and  $\mathbf{b}$  is a  $k \times 1$  column vector. By putting the augmented matrix  $(\mathbf{A}|\mathbf{b})$  into echelon form one can read off the solution.

- **Proposition 1.4:** An equation  $\mathbf{Ax} = \mathbf{b}$  where  $\mathbf{A}$  is a  $k \times n$  matrix,  $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$  and  $\mathbf{b}$  is a  $k \times 1$  column vector has at least one solution iff  $\text{rk } \mathbf{A} = \text{rk}(\mathbf{A}|\mathbf{b})$ . It has exactly one solution if  $\text{rk } \mathbf{A} = n$
- **Trace:** This operation is defined only for square matrices. A square  $n \times n$  matrix  $\mathbf{A} = (a_{ij})$  has a trace  $\text{tr } \mathbf{A} = \sum_{i=1}^n a_{ii}$ . I.e. it is the sum over all diagonal entries of the matrix. It has the following properties:
  - For two square  $n \times n$  matrices  $\mathbf{A}$  and  $\mathbf{B}$ ,  $\text{tr } \mathbf{AB} = \text{tr } \mathbf{BA}$ .
  - If  $\mathbf{P}$  is a square  $n \times n$  invertible matrix and  $\mathbf{A}$  is a square  $n \times n$  matrix then  $\text{tr } \mathbf{P}^{-1}\mathbf{AP} = \text{tr } \mathbf{A}$ .
  - $\text{tr}(\mathbf{A} + \mathbf{B}) = \text{tr } \mathbf{A} + \text{tr } \mathbf{B}$ .
  - $\text{tr } \mathbf{A}^T = \text{tr } \mathbf{A}$ .
  - For a scalar  $\lambda$  then  $\text{tr } \lambda \mathbf{A} = \lambda \text{tr } \mathbf{A}$ .
- **Determinant:** This operation is defined only for square matrices and we will define it via “expansion by the first row”. For a  $1 \times 1$  matrix  $\mathbf{A} = (a_{11})$  we have  $\det \mathbf{A} = |\mathbf{A}| = a_{11}$ . For an  $n \times n$  matrix  $\mathbf{A} = (a_{ij})$  it is

$$\begin{aligned} \det \mathbf{A} &= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n3} & \dots & a_{nn} \end{vmatrix} \\ &\quad + \dots + (-1)^{n-1} a_{1n} \begin{vmatrix} a_{21} & a_{22} & \dots & a_{2,n-1} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n,n-1} \end{vmatrix}. \end{aligned}$$

- **Proposition 1.5:** For two  $n \times n$  square matrices  $\mathbf{A}$  and  $\mathbf{B}$ ,  $\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}$ . It can be proved using the following lemmas.
- **Lemma 1.5.1:** Any square matrix  $\mathbf{A}$  can be written as the product  $\mathbf{S}_1 \mathbf{S}_2 \dots \mathbf{S}_k$  of ‘generalized’ elementary row operations  $\rho_i := \lambda \rho_i$  and  $\rho_i := \rho_i + \lambda \rho_j$  where  $\lambda$  can be zero.
- **Lemma 1.5.2:** The determinant of a matrix whose top two rows are identical is zero.
- **Lemma 1.5.3:**

$$\begin{vmatrix} a_{11} + \lambda b_{11} & a_{12} + \lambda b_{12} & \dots & a_{1n} + \lambda b_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + \lambda \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$
- **Lemma 1.5.4:** The matrix of any generalized elementary row operation can always be written as one of  $\mathbf{S}_i \mathbf{T}_j \mathbf{R} \mathbf{T}_j \mathbf{S}_i$ ,  $\mathbf{T}_j \mathbf{R} \mathbf{T}_j$ ,  $\mathbf{S}_i \mathbf{R} \mathbf{S}_i$ , or  $\mathbf{R}$ , where  $\mathbf{R}$  is the matrix of a generalized row operation acting only on rows 1 and 2, and  $\mathbf{S}_i$ ,  $\mathbf{T}_j$  are the matrices of  $\text{swap}(\rho_1, \rho_i)$  and  $\text{swap}(\rho_2, \rho_j)$  respectively.
- **Lemma 1.5.5:** If  $\mathbf{R}$  is the matrix of the swap operation  $\text{swap}(\rho_i, \rho_j)$  then  $\det \mathbf{RA} = -\det \mathbf{A}$ .
- **Proposition 1.6:** For an  $n \times n$  matrix  $\mathbf{A}$  the following are equivalent:
  - (a)  $\mathbf{A}^{-1}$  exists,

- (b)  $\det \mathbf{A} \neq 0$ , and
- (c)  $\operatorname{rk} \mathbf{A} = n$ .
- **Proposition 1.7:** For an  $n \times n$  matrix  $\mathbf{A}$ :
  - (a)  $\det(\mathbf{A}^T) = \det \mathbf{A}$ ,
  - (b) If  $\mathbf{A}$  is invertible  $\det \mathbf{A}^{-1} = (\det \mathbf{A})^{-1}$ ,
  - (c)  $\det \mathbb{I} = 1$ , and
  - (d) if  $\mathbf{A}$  has a row (or column) entirely composed of zeros  $\det \mathbf{A} = 0$ .
- **Proposition 1.8:** The determinant of an upper triangular matrix  $\mathbf{A}$  is equal to the product of its diagonal entries.
- **Proposition 1.9:** The determinants of row operations are:
  - (a) If  $\mathbf{A}_{ij,\lambda}$  is the matrix for  $\rho_i := \rho_i + \lambda \rho_j$  then  $\det \mathbf{A}_{ij,\lambda} = 1$ ,
  - (b) If  $\mathbf{T}_{i,\lambda}$  is the matrix for  $\rho_i := \lambda \rho_i$  then  $\det \mathbf{T}_{i,\lambda} = \lambda$ , and
  - (a) If  $\mathbf{S}_{ij}$  is the matrix for  $\operatorname{swap}(\rho_i, \rho_j)$  then  $\det \mathbf{S}_{ij} = -1$ .
- **Proposition 1.10:** If we swap any two rows in a determinant then the determinant changes sign, see Lemma 1.5.5. It follows that if a matrix has two identical rows then its determinant is zero. (This is equally true for a matrix with two identical columns.)
- **Proposition 1.11:** A determinant can be expanded along any row (or column). The sign associated with any entry  $a_{ij}$  is  $(-1)^{i+j}$ .
- **Minors and cofactors:** Let  $\mathbf{A} = (a_{ij})$  be an  $n \times n$  matrix and let  $b_{ij}$  be the determinant of the  $(n-1) \times (n-1)$  matrix obtained from  $\mathbf{A}$  by deleting row  $i$  and column  $j$ . Furthermore let  $c_{ij} = (-1)^{i+j} b_{ij}$ . Then
  - $\mathbf{B} = (b_{ij})$  is the matrix of minors of  $\mathbf{A}$ ,
  - $\mathbf{C} = (c_{ij})$  is the matrix of cofactors of  $\mathbf{A}$ , and
  - $\operatorname{adj} \mathbf{A} = \mathbf{C}^T$  is the adjugate matrix of  $\mathbf{A}$ .

## 2.2 Vector Spaces

- **Definition 2.1:** A vector space. A vector space  $V$  over a field  $F$  (see definition 2.3) is a set containing:
  - a special zero vector  $\mathbf{0}$ ;
  - an operation of addition of two vectors  $\mathbf{u} + \mathbf{v} \in V$ , for  $\mathbf{u}, \mathbf{v} \in V$ ; and
  - multiplication of a vector  $V$  with a number  $\lambda \in F$  with  $\lambda \mathbf{v} \in V$ .

The vector space must be closed under both of these operations and must satisfy the following laws  $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$  and  $\lambda, \mu \in F$ :

- (1) associativity  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ ;
  - (2) commutativity  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ ;
  - (3)  $\mathbf{u} + \mathbf{0} = \mathbf{u}$ ;
  - (4)  $\mathbf{v} + (-1)\mathbf{v} = \mathbf{0}$ ;
  - (5)  $\lambda(\mu\mathbf{u}) = (\lambda\mu)\mathbf{v}$ ;
  - (6) distributivity  $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$ ; and
  - (7) distributivity  $(\lambda + \mu)\mathbf{u} = \lambda\mathbf{u} + \mu\mathbf{u}$ .
- **Proposition 2.2:**  $\forall \mathbf{v} \in V$  and  $\forall \lambda \in F$ :
    - (a)  $\mathbf{v} = 1\mathbf{v}$ ;
    - (b)  $0\mathbf{v} = \mathbf{0}$ ; and
    - (c)  $\lambda\mathbf{0} = \mathbf{0}$ .
  - **Definition 2.3:** A field is a set  $F$  containing distinct elements 0 and 1 with two binary operations  $+$  and  $\cdot$  satisfying the axioms  $\forall a, b, c \in F$ :
    - (1)  $a + b = b + a$ ;
    - (2)  $(a + b) + c = a + (b + c)$ ;
    - (3)  $a + 0 = a$ ;
    - (4)  $\forall a \exists -a$  such that  $a + (-a) = 0$ ;
    - (5)  $a \cdot b = b \cdot a$ ;
    - (6)  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ ;
    - (7)  $a \cdot 1 = a$ ;
    - (8)  $\forall a \neq 0 \exists a^{-1}$  such that  $a \cdot a^{-1} = 1$ ; and
    - (9)  $a \cdot (b + c) = a \cdot b + a \cdot c$ ;

If a field  $F$  is finite its order is the number of elements in  $F$ .

- Note that as a field also satisfies all axioms of a vector space a field  $F$  is also itself a vector space  $V = F$  over the field  $F$  and all properties of a vector space apply.
- **Theorem 2.4:** For each prime  $p$  and each positive integer  $n$ , there is a unique field of order  $p^n$ . Additionally, every finite field is of this form.
- **Definition 2.5:** Given a vector space  $V$  over  $F$ , a subspace of  $V$  is a subset  $W \subset V$  which contains the zero vector of  $V$  and is closed under the operations of addition and scalar multiplication.
- **Lemma 2.5.1:** Let  $W \subset V$  be nonempty, where  $V$  is a vector space over  $F$ . Then  $W$  is a subspace of  $V$  iff  $\mathbf{v} + \lambda\mathbf{u} \in W$  for each  $\mathbf{v}, \mathbf{u} \in W$  and each scalar  $\lambda$ .
- **Definition 2.6:** Given a vector space  $V$  over  $F$ , and given a subset of  $V$   $A = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ ,

$$W = \{\lambda_1\mathbf{u}_1 + \lambda_2\mathbf{u}_2 + \lambda_3\mathbf{u}_3 + \dots \lambda_n\mathbf{u}_n : \lambda_1, \lambda_2, \dots, \lambda_n \in F\}$$

is the subspace of  $V$  spanned by  $A$ . The elements of  $W$  are called linear combinations of vectors from  $A$  and the subspace  $W$  is denoted as  $\text{span } A$ .

- **Definition 2.7:** If  $A$  is an infinite subset of  $V$ , where  $V$  is a vector space over  $F$ , we define  $\text{span } A$  to be the set of all linear combinations of finite subsets of  $A$ .
- **Definition 2.8:** A set  $A \subset V$  of vectors in a vector space  $V$  over  $F$  is linearly dependent if there are  $n \in \mathbb{N}$  vectors  $a_1, a_2, \dots, a_n$  and scalars  $\lambda_1, \lambda_2, \dots, \lambda_n$  not all zero such that

$$\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n = 0.$$

Otherwise  $A$  is linearly independent.

- For a finite set  $A = \{a_1, a_2, \dots, a_n\}$  it is linearly independent iff  $\forall$  scalars  $\lambda_1, \lambda_2, \dots, \lambda_n \in F$

$$\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n = 0 \Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_n = 0.$$

- If  $A$  is infinite it is linearly independent iff every subset of  $A$  is linearly independent.
- By convention the empty set is linearly independent.

- **Proposition 2.9:** Suppose  $\mathbf{A} = \{a_1, a_2, \dots, a_n\} \subset V$  is linearly independent, where  $V$  is a vector space over  $F$ . Suppose also that  $v \in V$  and there are scalars  $\lambda_1, \dots, \lambda_n$  and  $\mu_1, \dots, \mu_n$  such that

$$v = \lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n$$

and

$$v = \mu_1 a_1 + \mu_2 a_2 + \dots + \mu_n a_n$$

then  $\lambda_1 = \mu_1, \lambda_2 = \mu_2, \dots, \lambda_n = \mu_n$ .

- **Definition 2.10:** A basis of a vector space  $V$  is a linearly independent set  $B \subset V$  which spans  $V$ .
- **Theorem 2.11:** Let  $V$  be a vector space over  $F$ , and let  $B \subset V$  be linearly independent. Then there is a basis  $B'$  of  $V$  with  $B \subset B'$ .
- **Theorem 2.12:** Suppose  $\text{span } A = V$  and  $B \subset V$  is linearly independent. Then there is a basis  $B'$  of  $V$  with  $B \subset B' \subset A \cup B$ .
- **Lemma 2.13:** The “exchange lemma”. Suppose  $a_1, a_2, \dots, a_n, b$  are vectors in a vector space  $V$ , and suppose that

$$b \in \text{span}(a_1, a_2, \dots, a_{n-1}, a_n)$$

but

$$b \notin \text{span}(a_1, a_2, \dots, a_{n-1}),$$

then

$$a_n \in \text{span}(a_1, \dots, a_{n-1}, b).$$

If in addition  $\{a_1, a_2, \dots, a_n\}$  is linearly independent then so is  $\{a_1, a_2, \dots, a_{n-1}, b\}$ .

- **Theorem 2.14:** Suppose  $S$  and  $B$  are both bases of a vector space  $V$  over  $F$  and. Then  $A$  and  $B$  have the same number of elements.
- **Definition 2.15:** The number of elements of a basis of a vector space  $V$  over  $F$  is called the dimension of  $V$  and is written as  $\dim V$ .
- **Corollary 2.16:** If  $V$  is a vector space over  $F$  and  $U \subset V$  is a subspace of  $V$  then  $\dim U \leq \dim V$ . If, additionally,  $\dim V$  is finite and  $U \neq V$  then  $\dim U < \dim V$ .
- **Corollary 2.17:** Suppose that  $V$  is a vector space over  $F$ ,  $\dim V$  is finite, and  $U \subset V$  is a subspace of  $V$  with  $\dim U = \dim V$ , then  $U = V$ .
- The coordinates of a vector  $v \in V$ , with  $V$  a vector space over  $F$ , with respect to an *ordered* basis  $B = \{v_1, v_2, \dots, v_n\}$  are  $(\lambda_1, \lambda_2, \dots, \lambda_n)^T$  where

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n.$$



- **Definition 2.18:** Two vector spaces  $V$  and  $W$ , both over the same field  $F$ , are isomorphic if there is a bijection  $f : V \rightarrow W$  such that

$$f(u + v) = f(u) + f(v)$$

and

$$f(\lambda v) = \lambda f(v),$$

$\forall u, v \in V$  and  $\forall \lambda \in F$ . The bijection is said to be an isomorphism from  $V$  to  $W$  and we write  $V \cong W$  or  $f : V \xrightarrow{\sim} W$ .

- **Theorem 2.19:** Suppose  $V$  is a vector space over  $\mathbb{R}$  with finite dimension  $n \geq 0$ . Then  $V \cong \mathbb{R}^n$  as real vector spaces. Similarly if  $V$  is a vector space over  $\mathbb{C}$  with finite dimension  $n \geq 0$ . Then  $V \cong \mathbb{C}^n$  as complex vector spaces.

### 3 Multivariate Calculus

## 4 Complex Analysis