Linear Algebra I
Problem Set 4: Inner Products

Dr Nicholas Sedlmayr

Friday February 12th 2016

Due: In class, February 19th 2016

1. Prove the Cauchy-Schwarz inequality, proposition 3.5, which states that $\forall v, w \in V$, where V is a Euclidean space then $|\langle v|w\rangle| \leq ||v|| \cdot ||w||$. (5)
   Tip: Consider the properties of $||v + \lambda w||^2 \geq 0$ as a polynomial in λ.

2. Starting from proposition 3.6, the triangle inequality, derive the following alternative forms of it:

   $$||v - w|| \leq ||v|| + ||w||\,, \ ||v + w|| \geq ||v|| - ||w||\,, \text{ and } ||v - w|| \geq ||v|| - ||w||\,,$$

   for $v, w \in V$, where V is a Euclidean space. (4)

3. Prove that in any complex inner product space (4)

   (a) $\langle u|v\rangle = \frac{1}{2}\left(||u + v||^2 + \mathrm{i}||u - \mathrm{i}v||^2 - (1 + \mathrm{i})(||u||^2 + ||v||^2)\right)$, and

   (b) $\langle u|v\rangle = \frac{1}{4}\left(||u + v||^2 - ||u - v||^2 + \mathrm{i}||u - \mathrm{i}v||^2 - \mathrm{i}||u + \mathrm{i}v||^2\right)$.

4. Determine whether

   $$\langle \mathbf{u}|\mathbf{v}\rangle = (u_1 - v_1)^2 + (u_2 - v_2)^2$$

   is an inner product on $\mathbb{R}^2$ with $\mathbf{u} = (u_1, u_2)^T$, and $\mathbf{v} = (v_1, v_2)^T$. (2)

5. Let $V = \mathbb{R}^3$ with the standard inner product. Describe the set of vectors orthogonal to $(1, 0, -3)^T$. Show that this set is a subspace of V, find a basis and hence determine the dimension of the subspace. (5)

Total available marks: 20