Linear Algebra I
Summary of Lectures: Quadratic and Hermitian Forms

Dr Nicholas Sedlmayr

1. Definition 7.1: Given a symmetric bilinear form F on a real vector space V, we define a map $Q : V \to \mathbb{R}$ by $Q(v) = F(v, v)$; Q is called the quadratic form associated with the symmetric bilinear form F.

2. Lemma 7.2: Given a symmetric bilinear form F on a real vector space V, and the quadratic form Q associated with F, then

$$F(v, w) = \frac{1}{2}\left(Q(v + w) - Q(v) - Q(w)\right) , \quad \forall v, w \in V .$$

3. Definition 7.3: Given a conjugate-symmetric sesquilinear form F on a complex vector space V, we define a map $H : V \to \mathbb{R}$ by $H(v) = F(v, v)$; H is called the Hermitian form associated with the conjugate-symmetric sesquilinear form F.

4. Lemma 7.4: Given a conjugate-symmetric sesquilinear form F on a complex vector space V, and the Hermitian form H associated with F, then $\forall v, w \in V$:

$$\begin{aligned} F(v, w) &= \tfrac{1}{2}\left(H(v + w) + \mathrm{i}H(v - \mathrm{i}w) - (1 + \mathrm{i})(H(v) + H(w))\right) , \\ F(v, w) &= \tfrac{1}{4}\left(H(v + w) - \mathrm{i}H(v - w) + \mathrm{i}H(v - \mathrm{i}w) - \mathrm{i}H(v + \mathrm{i}w)\right) . \end{aligned}$$

5. Proposition 7.5: If Q is a quadratic form on a real vector space V, then

$$Q(\lambda x) = \lambda^2 Q(x) , \quad \forall \lambda \in \mathbb{R} , \text{ and } \forall x \in V .$$

Similarly if H is a Hermitian form on a complex vector space V, then

$$H(\lambda x) = |\lambda|^2 H(x) , \quad \forall \lambda \in \mathbb{C} , \text{ and } \forall x \in V .$$