Linear Algebra I
Summary of Lectures: Inner Product Spaces

and Bilinear and Sesquilinear Forms

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1. Definition 3.1: If V is a vector space over $\mathbb{R}$, then an inner product on V is a map $(\langle|\rangle)$ from $V \times V$ to $\mathbb{R}$ with the following properties:
    (a) Symmetry: $\langle v|w\rangle = \langle w|v\rangle \ \forall v, w \in V$.
    (b) Linearity: $\langle u|\lambda v+\mu w\rangle = \lambda\langle u|v\rangle+\mu\langle u|w\rangle \ \forall u, v, w \in V$ and $\forall \lambda, \mu \in \mathbb{R}$.
    (c) Positive definiteness:
        (i) $\langle v|v\rangle \geq 0 \ \forall v \in V$, and
        (ii) $\langle v|v\rangle = 0$ iff $v = 0$.

    As the inner product is linear with respect to both variables it is sometimes called bilinear.

2. Definition 3.2: A finite dimensional vector space over $\mathbb{R}$ with an inner product defined is called a Euclidean space.

3. Definition 3.3: The norm (or length) of a vector$v$ is written as $||v|$ and defined by
$$||v|| = \sqrt{\langle v|v\rangle},$$
the positive square root of the inner product of v with itself. The distance between two vectors v and w is written as $d(v, w)$ and is $d(v, w) = ||v-w||$.

4. Proposition 3.4: $\forall v \in V$, where V is a Euclidean space, and $\forall \lambda \in \mathbb{R}$ then $||\lambda v|| = |\lambda| \cdot ||v||$.

5. Proposition 3.5: The "Cauchy-Schwarz inequality" says that $\forall v, w \in V$, where V is a Euclidean space, then
$$|\langle v|w\rangle| \leq ||v|| \cdot ||w|| \,.$$

6. Proposition 3.6: The "triangle inequality" says that $\forall v, w \in V$, where V is a Euclidean space, then
$$||v + w|| \leq ||v|| + ||w|| \,.$$

7. Definition 3.7: If V is a Euclidean space, and $v, w \in V$, then v and w are said to be orthogonal if $\langle v|w\rangle = 0$. If both v and w are nonzero, then the angle between v and w is defined to be θ, $0 \le \theta \le \pi$ and

$$\cos\theta = \frac{\langle v|w\rangle}{||v|| \cdot ||w||}.$$

8. Definition 3.8: If V is a vector space over $\mathbb{C}$, then a map $(\langle|\rangle)$ from $V \times V$ to $\mathbb{C}$ is an inner product if the following are true:

   (a) Conjugate-Symmetry: $\langle v|w\rangle = \overline{\langle w|v\rangle}$ $\forall v, w \in V$.

   (b) Linearity: $\langle u|\lambda v + \mu w\rangle = \lambda\langle u|v\rangle + \mu\langle u|w\rangle$ $\forall u, v, w \in V$ and $\forall \lambda, \mu \in \mathbb{R}$.

   (c) Positive definiteness:

      (i) $\langle v|v\rangle \ge 0$ $\forall v \in V$, and

      (ii) $\langle v|v\rangle = 0$ iff $v = 0$.

   This inner product is sometimes called sesquilinear.

9. Definition 3.9: A finite dimensional vector space over $\mathbb{C}$ with an inner product define is called a unitary space.

10. A vector space over $\mathbb{R}$ or $\mathbb{C}$, of any dimension, we will refer to as an inner product space.

11. Definition 3.10: The norm (or length) of a vector $v \in V$, with V a vector space over $\mathbb{C}$,is written as $||v|$ and defined by

$$||v|| = \sqrt{\langle v|v\rangle},$$

   the positive square root of the inner product of v with itself. The distance between two vectors v and w is written as $d(v, w)$ and is $d(v, w) = ||v - w||$.

12. Proposition 3.11: $\forall v \in V$, where V is a unitary space, and $\forall \lambda \in \mathbb{R}$ then $||\lambda v|| = |\lambda| \cdot ||v||$.

13. Proposition 3.12: The "Cauchy-Schwarz inequality" says that $\forall v, w \in V$, where V is a unitary space, then

$$|\langle v|w\rangle| \le ||v|| \cdot ||w||.$$

14. Proposition 3.13: The "triangle inequality" says that $\forall v, w \in V$, where V is a unitary space, then

$$||v + w|| \le ||v|| + ||w||.$$

15. Definition 3.14: A bilinear form on a real vector space V is a map $F : V \times V \to \mathbb{R}$ which $\forall u, v, w \in V$ and $\forall \alpha, \beta \in \mathbb{R}$ satisfies

   (a) $\langle \alpha u + \beta v|w\rangle = \alpha\langle u|w\rangle + \beta\langle v|w\rangle$, and

   (b) $\langle u|\alpha v + \beta w\rangle = \alpha\langle u|v\rangle + \beta\langle u|w\rangle$.

16. Definition 3.14: A bilinear form on a real vector space V is symmetric if

   (c) $F(u, v) = F(v, u)$ $\forall u, v \in V$.