

Linear Algebra I

Summary of Lectures:

Eigenvalues and Eigenvectors

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1. **Definition 6.1:** Let \mathbf{A} be an $n \times n$ matrix over a field F . Then a column vector $\mathbf{x} \in F^n$ is called an eigenvector of \mathbf{A} , with eigenvalue $\lambda \in F$, if $\mathbf{x} \neq 0$ and $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$.
2. **Theorem 6.2:** A scalar λ is an eigenvalue of an $n \times n$ matrix \mathbf{A} iff the matrix $\mathbf{A} - \lambda\mathbb{I}_n$ has nullity $n(\mathbf{A} - \lambda\mathbb{I}_n) > 0$.
3. **Theorem 6.3:** Suppose λ is an eigenvalue of an $n \times n$ matrix \mathbf{A} . Then the eigenvectors of \mathbf{A} having eigenvalue λ are the non-zero vectors in $\ker(\mathbf{A} - \lambda\mathbb{I}_n) = \{\mathbf{x} : (\mathbf{A} - \lambda\mathbb{I}_n)\mathbf{x} = 0\}$.
4. **Theorem 6.4:** Every $n \times n$ matrix \mathbf{A} over $F = \mathbb{R}$ or $F = \mathbb{C}$ has an eigenvalue λ in \mathbb{C} and an eigenvector $\mathbf{x} \in \mathbb{C}^n$ with eigenvalue λ .
5. **Definition 6.5:** If $f : V \rightarrow V$ is a linear map, where V is a vector space over a field F , and $0 \neq v \in V$ with $f(v) = \lambda v$ for some $\lambda \in F$, then v is an eigenvector of f , with eigenvalue λ .
6. **Proposition 6.6:** If \mathbf{A} , \mathbf{B} and \mathbf{P} are $n \times n$ matrices related by $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ then \mathbf{B} and \mathbf{A} have the same eigenvalues.
7. **Lemma 6.7:** Suppose that $f : V \rightarrow V$ is a linear transformation of an n -dimensional vector space V over a field F . If f has nullity of at least one then there is a basis v_1, v_2, \dots, v_n such that

$$f(v_j) \in \text{span}(v_1, v_2, \dots, v_{n-1}) \quad \forall j = 1, \dots, n.$$
8. **Proposition 6.8:** Let $V = \mathbb{C}^n$ be the n -dimensional vector space over \mathbb{C} , and suppose f is a linear transformation from V to V . Then there is a basis of V such that, with respect to this basis, the matrix of f is upper triangular.
9. **Proposition 6.9:** If \mathbf{A} is an upper triangular matrix then the diagonal entries in \mathbf{A} are precisely the eigenvalues of \mathbf{A} .
10. **Theorem 6.10:** If \mathbf{A} is any upper triangular $n \times n$ matrix with entries from \mathbb{R} or \mathbb{C} and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the diagonal entries of \mathbf{A} , including repetitions, then the matrix

$$(\mathbf{A} - \lambda_1\mathbb{I})(\mathbf{A} - \lambda_2\mathbb{I}) \dots (\mathbf{A} - \lambda_n\mathbb{I}),$$

is the zero matrix.

11. **Lemma 6.11:** If \mathbf{A} is an upper triangular matrix with eigenvalue λ then $\det[\mathbf{A} - \lambda\mathbb{I}] = 0$.
12. **Proposition 6.12:** If \mathbf{A} is an $n \times n$ matrix over \mathbb{R} or \mathbb{C} with eigenvalue $\lambda \in \mathbb{C}$ then $\det[\mathbf{A} - \lambda\mathbb{I}] = 0$.
13. **Theorem 6.13:** If $\lambda_1, \lambda_2, \dots, \lambda_r$ are distinct eigenvalues of an $n \times n$ matrix \mathbf{A} , with $r \leq n$, with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ are linearly independent.