

Linear Algebra I

Summary of Lectures: Vector Spaces

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1. **Definition 2.1:** A vector space. A vector space V over a field F (see definition 2.3) is a set containing:

- a special zero vector $\mathbf{0}$;
- an operation of addition of two vectors $\mathbf{u} + \mathbf{v} \in V$, for $\mathbf{u}, \mathbf{v} \in V$; and
- multiplication of a vector V with a number $\lambda \in F$ with $\lambda \mathbf{v} \in V$.

The vector space must be closed under both of these operations and must satisfy the following laws $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\lambda, \mu \in F$:

- (1) associativity $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$;
- (2) commutativity $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$;
- (3) $\mathbf{u} + \mathbf{0} = \mathbf{u}$;
- (4) $\mathbf{v} + (-1)\mathbf{v} = \mathbf{0}$;
- (5) $\lambda(\mu\mathbf{u}) = (\lambda\mu)\mathbf{v}$;
- (6) distributivity $\lambda(\mathbf{u} + \mathbf{v}) = \lambda\mathbf{u} + \lambda\mathbf{v}$; and
- (7) distributivity $(\lambda + \mu)\mathbf{u} = \lambda\mathbf{u} + \mu\mathbf{u}$.

2. **Proposition 2.2** $\forall \mathbf{v} \in V$ and $\forall \lambda \in F$:

- (a) $\mathbf{v} = 1\mathbf{v}$;
- (b) $0\mathbf{v} = \mathbf{0}$; and
- (c) $\lambda\mathbf{0} = \mathbf{0}$.

3. **Definition 2.3** A field is a set F containing distinct elements 0 and 1 with two binary operations $+$ and \cdot satisfying the axioms $\forall a, b, c \in F$:

- (1) $a + b = b + a$;
- (2) $(a + b) + c = a + (b + c)$;
- (3) $a + 0 = a$;
- (4) $\forall a \exists -a$ such that $a + (-a) = 0$;
- (5) $a \cdot b = b \cdot a$;
- (6) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;
- (7) $a \cdot 1 = a$;
- (8) $\forall a \neq 0 \exists a^{-1}$ such that $a \cdot a^{-1} = 1$; and
- (9) $a \cdot (b + c) = a \cdot b + a \cdot c$;

If a field F is finite its order is the number of elements in F .

4. Note that as a field also satisfies all axioms of a vector space a field F is also itself a vector space $V = F$ over the field F and all properties of a vector space apply.
5. **Theorem 2.4:** For each prime p and each positive integer n , there is a unique field of order p^n . Additionally, every finite field is of this form.
6. **Definition 2.5** Given a vector space V over F , a subspace of V is a subset $W \subset V$ which contains the zero vector of V and is closed under the operations of addition and scalar multiplication.
7. **Lemma 2.5.1** Let $W \subset V$ be nonempty, where V is a vector space over F . Then W is a subspace of V iff $\mathbf{v} + \lambda \mathbf{u} \in W$ for each $\mathbf{v}, \mathbf{u} \in W$ and each scalar λ .
8. **Definition 2.6** Given a vector space V over F , and given a subset of V $A = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$,

$$W = \{\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \lambda_3 \mathbf{u}_3 + \dots \lambda_n \mathbf{u}_n : \lambda_1, \lambda_2, \dots, \lambda_n \in F\}$$

is the subspace of V spanned by A . The elements of W are called linear combinations of vectors from A and the subspace W is denoted as $\text{span } A$.

9. **Definition 2.7** If A is an infinite subset of V , where V is a vector space over F , we define $\text{span } A$ to be the set of all linear combinations of finite subsets of A .
10. **Definition 2.8** A set $A \subset V$ of vectors in a vector space V over F is linearly dependent if there are $n \in \mathbb{N}$ vectors a_1, a_2, \dots, a_n and scalars $\lambda_1, \lambda_2, \dots, \lambda_n$ not all zero such that

$$\lambda_1 a_1 + \lambda_2 a_2 + \dots \lambda_n a_n = 0.$$

Otherwise A is linearly independent.

- For a finite set $A = \{a_1, a_2, \dots, a_n\}$ it is linearly independent iff \forall scalars $\lambda_1, \lambda_2, \dots, \lambda_n \in F$

$$\lambda_1 a_1 + \lambda_2 a_2 + \dots \lambda_n a_n = 0 \Rightarrow \lambda_1 = \lambda_2 = \dots \lambda_n = 0.$$

- If A is infinite it is linearly independent iff every subset of A is linearly independent.
- By convention the empty set is linearly independent.

11. **Proposition 2.9** Suppose $\mathbf{A} = \{a_1, a_2, \dots, a_n\} \subset V$ is linearly independent, where V is a vector space over F . Suppose also that $v \in V$ and there are scalars $\lambda_1, \dots, \lambda_n$ and μ_1, \dots, μ_n such that

$$v = \lambda_1 a_1 + \lambda_2 a_2 + \dots \lambda_n a_n$$

and

$$v = \mu_1 a_1 + \mu_2 a_2 + \dots \mu_n a_n$$

then $\lambda_1 = \mu_1, \lambda_2 = \mu_2, \dots, \lambda_n = \mu_n$.

12. **Definition 2.10** A basis of a vector space V is a linearly independent set $B \subset V$ which spans V .
13. **Theorem 2.11** Let V be a vector space over F , and let $B \subset V$ be linearly independent. Then there is a basis B' of V with $B \subset B'$.
14. **Theorem 2.12** Suppose $\text{span } A = V$ and $B \subset V$ is linearly independent. Then there is a basis B' of V with $B \subset B' \subset A \cup B$.
15. **Lemma 2.13** The “exchange lemma”. Suppose a_1, a_2, \dots, a_n, b are vectors in a vector space V , and suppose that

$$b \in \text{span}(a_1, a_2, \dots, a_{n-1}, a_n)$$

but

$$b \notin \text{span}(a_1, a_2, \dots, a_{n-1}),$$

then

$$a_n \in \text{span}(a_1, \dots, a_{n-1}, b).$$

If in addition $\{a_1, a_2, \dots, a_n\}$ is linearly independent then so is $\{a_1, a_2, \dots, a_{n-1}, b\}$.

16. **Theorem 2.14** Suppose S and B are both bases of a vector space V over F and. Then A and B have the same number of elements.
17. **Definition 2.15** The number of elements of a basis of a vector space V over F is called the dimension of V and is written as $\dim V$.
18. **Corollary 2.16** If V is a vector space over F and $U \subset V$ is a subspace of V then $\dim U \leq \dim V$. If, additionally, $\dim V$ is finite and $U \neq V$ then $\dim U < \dim V$.
19. **Corollary 2.17** Suppose that V is a vector space over F , $\dim V$ is finite, and $U \subset V$ is a subspace of V with $\dim U = \dim V$, then $U = V$.
20. The coordinates of a vector $v \in V$, with V a vector space over F , with respect to an *ordered* basis $B = \{v_1, v_2, \dots, v_n\}$ are $(\lambda_1, \lambda_2, \dots, \lambda_n)^T$ where

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n.$$