Linear Algebra I
Summary of Lectures:
Inner Product Spaces

and Bilinear and Sesquilinear Forms

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1. Definition 3.1: If V is a vector space over $\mathbb{R}$, then an inner product on V is a map $(\langle|\rangle)$ from $V \times V$ to $\mathbb{R}$ with the following properties:
   (a) Symmetry: $\langle v|w\rangle = \langle w|v\rangle \ \forall v, w \in V$.
   (b) Linearity: $\langle u|\lambda v + \mu w\rangle = \lambda\langle u|v\rangle + \mu\langle u|w\rangle \ \forall u, v, w \in V$ and $\forall \lambda, \mu \in \mathbb{R}$.
   (c) Positive definiteness:
      (i) $\langle v|v\rangle \geq 0 \ \forall v \in V$, and
      (ii) $\langle v|v\rangle = 0$ iff $v = 0$.

   As the inner product is linear with respect to both variables it is sometimes called bilinear.

2. Definition 3.2: A finite dimensional vector space over $\mathbb{R}$ with an inner product defined is called a Euclidean space.

3. Definition 3.3: The norm (or length) of a vector$v$ is written as $||v|$ and defined by
   $$||v|| = \sqrt{\langle v|v\rangle},$$
   the positive square root of the inner product of v with itself. The distance between two vectors v and w is written as $d(v, w)$ and is $d(v, w) = ||v - w||$.

4. Proposition 3.4: $\forall v \in V$, where V is a Euclidean space, and $\forall \lambda \in \mathbb{R}$ then $||\lambda v|| = |\lambda| \cdot ||v||$.

5. Proposition 3.5: The "Cauchy-Schwarz inequality" says that $\forall v, w \in V$, where V is a Euclidean space, then
   $$|\langle v|w\rangle| \leq ||v|| \cdot ||w|| \, .$$

6. Proposition 3.6: The "triangle inequality" says that $\forall v, w \in V$, where V is a Euclidean space, then
   $$||v + w|| \leq ||v|| + ||w|| \, .$$

7. Definition 3.7: If V is a Euclidean space, and $v, w \in V$, then v and w are said to be orthogonal if $\langle v|w\rangle = 0$. If both v and w are nonzero, then the angle between v and w is defined to be θ, $0 \leq \theta \leq \pi$ and

$$\cos\theta = \frac{\langle v|w\rangle}{||v|| \cdot ||w||}\,.$$

8. Definition 3.8: If V is a vector space over $\mathbb{C}$, then a map $(\langle|\rangle)$ from $V \times V$ to $\mathbb{C}$ is an inner product if the following are true:
   (a) Conjugate-Symmetry: $\langle v|w\rangle = \overline{\langle w|v\rangle}\ \forall v, w \in V$.
   (b) Linearity: $\langle u|\lambda v + \mu w\rangle = \lambda\langle u|v\rangle + \mu\langle u|w\rangle\ \forall u, v, w \in V$ and $\forall \lambda, \mu \in \mathbb{R}$.
   (c) Positive definiteness:
      (i) $\langle v|v\rangle \geq 0\ \forall v \in V$, and
      (ii) $\langle v|v\rangle = 0$ iff $v = 0$.

   This inner product is sometimes called sesquilinear.

9. Definition 3.9: A finite dimensional vector space over $\mathbb{C}$ with an inner product define is called a unitary space.

10. A vector space over $\mathbb{R}$ or $\mathbb{C}$, of any dimension, we will refer to as an inner product space.

11. Definition 3.10: The norm (or length) of a vector $v \in V$, with V a vector space over $\mathbb{C}$,is written as $||v||$ and defined by

$$||v|| = \sqrt{\langle v|v\rangle},$$

   the positive square root of the inner product of v with itself. The distance between two vectors v and w is written as $d(v, w)$ and is $d(v, w) = ||v - w||$.

12. Proposition 3.11: $\forall v \in V$, where V is a unitary space, and $\forall \lambda \in \mathbb{R}$ then $||\lambda v|| = |\lambda| \cdot ||v||$.

13. Proposition 3.12: The "Cauchy-Schwarz inequality" says that $\forall v, w \in V$, where V is a unitary space, then

$$|\langle v|w\rangle| \leq ||v|| \cdot ||w||\,.$$

14. Proposition 3.13: The "triangle inequality" says that $\forall v, w \in V$, where V is a unitary space, then

$$||v + w|| \leq ||v|| + ||w||\,.$$

15. Definition 3.14: A bilinear form on a real vector space V is a map $F : V \times V \to \mathbb{R}$ which $\forall u, v, w \in V$ and $\forall \alpha, \beta \in \mathbb{R}$ satisfies
   (a) $\langle \alpha u + \beta v|w\rangle = \alpha\langle u|w\rangle + \beta\langle v|w\rangle$, and
   (b) $\langle u|\alpha v + \beta w\rangle = \alpha\langle u|v\rangle + \beta\langle u|w\rangle$.

16. Definition 3.15: A bilinear form on a real vector space V is symmetric if

(c) $F(u, v) = F(v, u)\ \forall u, v \in V$.

17. Definition 3.16: The matrix $\mathbf{A} = (a_{ij}$ with $a_{ij} = F(e_i, e_j)$ is called the 'matrix of the bilinear form F with respect to the ordered basis $e_1, e_2, \ldots e_n$ of V'. If F is symmetric then $\mathbf{B}$ is symmetric.

18. Proposition 3.17: Suppose V is a real vector space with ordered basis $e_1, e_2, \ldots e_n$ and F is a bilinear form defined on V, with matrix $\mathbf{A}$ with respect to this basis. Then for any vectors $v, w \in V$ and their corresponding coordinate forms $\mathbf{v} = (v_1, v_2, \ldots v_n)^T$ and $\mathbf{w} = (w_1, w_2, \ldots w_n)^T$ with respect to the same basis we have

$$F(v, w) = \mathbf{v}^T \mathbf{A} \mathbf{w} \,.$$

19. The base change matrix from a basis $e_1, e_2, \ldots e_n$ to $f_1, f_2, \ldots f_n$ is $\mathbf{P} = (p_{ij})$ where $f_i = \sum_{k=1}^{n} p_{ki} e_k$.

20. Proposition 3.18: (The base change formula) Given two ordered bases of a Euclidean space V, $e_1, e_2, \ldots e_n$ and $f_1, f_2, \ldots f_n$ related by the base change matrix $\mathbf{P}$ from basis $e_1, e_2, \ldots e_n$ to $f_1, f_2, \ldots f_n$, suppose $\mathbf{A}$ and $\mathbf{B}$ are the matrices of the inner product with respect to $e_1, e_2, \ldots e_n$ and $f_1, f_2, \ldots f_n$. Then $\mathbf{B} = \mathbf{P}^T \mathbf{A} \mathbf{P}$.

21. Definition 3.19: A sesquilinear form on a complex vector space V is a map $F : V \times V \to \mathbb{C}$ which $\forall u, v, w \in V$ and $\forall \alpha, \beta \in \mathbb{C}$ satisfies

    (a) $\langle \alpha u + \beta v | w \rangle = \bar{\alpha} \langle u | w \rangle + \bar{\beta} \langle v | w \rangle$, and

    (b) $\langle u | \alpha v + \beta w \rangle = \alpha \langle u | v \rangle + \beta \langle u | w \rangle$.

22. Definition 3.20: A sesquilinear form on a complex vector space V is conjugate-symmetric if

    (c) $F(u, v) = \overline{F(v, u)}\ \forall u, v \in V$.

23. Definition 3.21: The matrix $\mathbf{B} = (a_{ij}$ with $a_{ij} = F(e_i, e_j)$ is called the 'matrix of the bilinear form F with respect to the ordered basis $e_1, e_2, \ldots e_n$ of the complex vector space V'. If F is conjugate-symmetric then $\mathbf{B}$ is conjugate-symmetric, i.e. $\bar{\mathbf{B}}^T = \mathbf{B}$.

24. Proposition 3.22: Suppose V is a complex inner product space with ordered basis $e_1, e_2, \ldots e_n$ and F is a sesquilinear form defined on V, with matrix $\mathbf{A}$ with respect to this basis. Then for any vectors $v, w \in V$ and their corresponding coordinate forms $\mathbf{v} = (v_1, v_2, \ldots v_n)^T$ and $\mathbf{w} = (w_1, w_2, \ldots w_n)^T$ with respect to the same basis we have

$$F(v, w) = \overline{\mathbf{v}}^T \mathbf{A} \mathbf{w} \,.$$

25. Proposition 3.23: (The base change formula) Given two ordered bases of a complex inner product space V, $e_1, e_2, \ldots e_n$ and $f_1, f_2, \ldots f_n$ related by the base change matrix $\mathbf{P}$ from basis $e_1, e_2, \ldots e_n$ to $f_1, f_2, \ldots f_n$, suppose $\mathbf{A}$ and $\mathbf{B}$ are the matrices of the inner product with respect to $e_1, e_2, \ldots e_n$ and $f_1, f_2, \ldots f_n$. Then $\mathbf{B} = \overline{\mathbf{P}}^T \mathbf{A} \mathbf{P}$.