Linear Algebra I
Summary of Lectures:
Orthogonal Bases

Dr Nicholas Sedlmayr

1. Definition 4.1: Two vectors v and w in an inner product space are orthogonal if $\langle v|w\rangle$=0. The set of vectors $\{v_1, v_2, \ldots\}$ is said to be orthogonal, and the vectors $v_1, v_2 \ldots$ in the set are said to be mutually orthogonal if each pair of distinct vectors v_i, v_l with $i \neq l$ are said to be an orthogonal pair, $\langle v_i|v_l\rangle = 0$.

2. Definition 4.2: A set $\{w_1, w_2, \ldots\}$ of vectors in an inner product space is said to be orthonormal if $\langle w_i|w_j\rangle = \delta_{ij}$. If the orthonormal set is a basis then it is called an orthonormal basis.

3. Proposition 4.3: If V is an inner product space over $\mathbb{R}$ or $\mathbb{C}$, $v_1, v_2, \ldots v_n \in V$, $v_i \neq 0\, \forall i = 1 \ldots n$, and the v_i are mutually orthogonal then $\{v_1, v_2, \ldots v_n\}$ is a linearly independent set.

4. Lemma 4.4: If u, v are any two vectors in an inner product space V with $v \neq 0$ then the vector
$$w = u - \frac{\langle v|u\rangle}{\langle v|v\rangle} v$$
is orthogonal to v.

5. Lemma 4.5: If V is an inner product space, $u, v_1, v_2, \ldots v_k \in V$ and $v_1, v_2, \ldots v_k$ are mutually orthogonal non-zero vectors then
$$w = u - \sum_{i=1}^{k} \frac{\langle v_i|u\rangle}{\langle v_i|v_i\rangle} v_i$$
is orthogonal tro $v_1, v_2, \ldots v_k$.

6. Theorem 4.6: (The Gram-Schmidt process) If $\{v_1, \ldots v_n\}$ is a basis of a finite dimensional inner product space V, then $\{w_1, \ldots w_n\}$ obtained by
$$\begin{aligned} w_1 &= v_1 \\ w_2 &= v_2 - \tfrac{\langle w_1|v_2\rangle}{\langle w_!|w_1\rangle} w_1 \\ &\vdots \\ w_k &= v_k - \textstyle\sum_{i=1}^{k-1} \tfrac{\langle w_i|v_k\rangle}{\langle w_i|w_i\rangle} w_i \\ &\vdots \end{aligned}$$
is an orthogonal basis of V.

7. Corollary 4.7: Any finite dimensional inner product space V has an orthonormal basis.

8. Definition 4.8: Two real vector spaces V, W with forms $F : V \times V \to \mathbb{R}$ and $G : W \times W \to \mathbb{R}$ respectively are isomorphic if there is a bijection $f : V \to W$ such that

$$\begin{aligned} f(u+v) &= f(u) + f(v)\,, \\ f(\lambda v) &= \lambda f(v) \text{ and} \\ F(u,v) &= G(f(u), g(v))\,, \end{aligned}$$

$\forall u, v \in V$ and $\forall \lambda \in \mathbb{R}$.
Similarly two complex vector spaces V, W with forms $F : V \times V \to \mathbb{C}$ and $G : W \times W \to \mathbb{C}$ respectively are isomorphic if there is a bijection $f : V \to W$ such that

$$\begin{aligned} f(u+v) &= f(u) + f(v)\,, \\ f(\lambda v) &= \lambda f(v) \text{ and} \\ F(u,v) &= G(f(u), g(v))\,, \end{aligned}$$

$\forall u, v \in V$ and $\forall \lambda \in \mathbb{C}$.

9. Corollary 4.9: Let V be a Euclidean vector space of dimension n. Then V is isomorphic to $\mathbb{R}^n$ with the standard inner product as an inner product space. Similarly each unitary vector space V of dimension n is isomorphic to $\mathbb{C}^n$ with the standard inner product as an inner product space.

10. Proposition 4.10: Suppose that $\{e_1, e_2, \ldots e_n\}$ is an orthonormal basis of a Euclidean space V. Then for any $v \in V$:

$$v = \sum_{i=1}^{n} \langle e_i | v \rangle e_i\,.$$

11. Proposition 4.11: (Pythagoras' theorem) Suppose $e_1, e_2, \ldots e_n$ is an orthonormal basis of a Euclidean space V. Then for all $v \in V$

$$||v||^2 = \sum_{i=1}^{n} \langle e_i | v \rangle^2\,.$$

12. Corollary 4.12: (Parseval's identity) If $e_1, e_2, \ldots e_n$ is an orthonormal basis of a Euclidean space V, and $v, w \in V$, then

$$\langle v | w \rangle = \sum_{i=1}^{n} \langle v | \, \mathrm{e}_i \, \rangle \langle e_i | w \rangle\,..$$

13. Proposition 4.13: (Bessel's inequality) If $e_1, e_2, \ldots e_k$ is an orthonormal set of vectors in a real inner product space V, and $v \in V$, then

$$\sum_{i=1}^{k} \langle e_i | v \rangle^2 \leq ||v||^2\,.$$

14. Proposition 4.14: If $e_1, e_2, \ldots e_n$ is an orthonormal basis of a complex inner prodoct space V, and $v, w \in V$, then:

    (a) $v = \sum_{i=1}^{n} \langle e_i | v \rangle e_i$,

    (b) $||v||^2 = \sum_{i=1}^{n} \langle e_i | v \rangle^2$, (Pythagoras' theorem) and

    (c) $\langle v | w \rangle = \sum_{i=1}^{n} \langle v | \mathrm{e}_i \rangle \langle e_i | w \rangle = \sum_{i=1}^{n} \overline{\langle e_i | v \rangle} \langle e_i | w \rangle$ (Parseval's identity).

15. Proposition 4.15: (Bessel's inequality) If $e_1, e_2, \ldots e_k$ is an orthonormal set of vectors in a complex inner product space V, and $v \in V$, then

$$\sum_{i=1}^{k} |\langle e_i | v \rangle|^2 \leq ||v||^2 .$$

16. Definition 4.16: If U and W are subspaces of a vector space V then the sum of U and W is defined as

$$U + W = \{u + w : u \in U, w \in W\} .$$

17. Proposition 4.17: $U + W$ is a subspace of a vector space V if U and W are subspaces of V.

18. The union of two sets is $A \cup B = \{x : x \in A \vee x \in B\}$. I.e. the elements in either A or B. The intersection of two sets is $A \cap B = \{x : x \in A \wedge x \in B\}$. I.e. the elements in both A or B.

19. Definition 4.18: If V is a vector space and U is a subspace of V, then W is called a complement to U in V if

    (a) W is a subspace of V,

    (b) $V = U + W$, and

    (c) $U \cap W = \{0\}$.

    When these conditions are met we write $V = U \oplus W$, and say that V is the direct sum of U and W.

20. Definition 4.19: If V is an inner product space and U is a subspace of V we define

$$U^{\perp} = \{v \in V : \langle u | v \rangle = 0 \, \forall u \in U\} .$$

    This is called the orthogonal complement of U in V, or "U perp" for short.

21. Lemma 4.20: If V is an inner product space, U is a subspace of V, and U has a basis $\{u_1, \ldots u_k\}$, then

$$U^{\perp} = \{v \in V : \langle u_i | v \rangle = 0 \, \forall i = 1, \ldots k\} .$$

22. Proposition 4.21: If V is an inner product space, and U is a finite dimensional subspace of V, then

    (a) $U^{\perp}$ is a subspace of V,

    (b) $U \cap U^{\perp} = \{0\}$, and

(c) $U + U^{\perp} = V$.

23. Proposition 4.22: If $V = U \oplus W$ then $\dim(V) = \dim(U) + \dim(W)$.

24. Corollary 4.23: If V is a finite dimensional inner product space, and U is a subspace of V, then

   (a) $\dim(U) + \dim(U^{\perp}) = \dim(V)$, and

   (b) $(U^{\perp})^{\perp} = U$.