Linear Algebra I
Summary of Lectures: Matrices

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1. Definition of an $n \times m$ matrix, $\mathbf{A} = (a_{ij})$ with n row and m columns. Addition of matrices $\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})$.
   - Associativity, commutativity and existence of a zero for addition.
2. Multiplication of a matrix by a scalar: $\lambda\mathbf{A} = (\lambda a_{ij})$.
3. The matrix multiplication of an $n \times m$ matrix $\mathbf{A}$ and an $m \times k$ matrix $\mathbf{B}$ is an $n \times k$ matrix $\mathbf{C} = \mathbf{AB} = (c_{ij})$ where $c_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$.
   - Associativity, existence of a zero matrix ($\mathbf{0}$) and an identity matrix $\mathbb{I}$, distributivity.
   - No commutativity!
4. A matrix $\mathbf{A}$ can have a right inverse $\mathbf{AB} = \mathbb{I}$ and a left inverse $\mathbf{CA} = \mathbb{I}$.
   - Prop. 1.1: If a square matrices has either a left or right inverse then they have a unique inverse from both the left and right.
   - If a non-square matrix has both a left and right inverse then they are the same and the inverse is unique.
   - Prop. 1.2: If $\mathbf{A}$ and $\mathbf{B}$ are invertible square matrices then $\mathbf{AB}$ is also invertible and $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
5. The transpose of an $n \times m$ matrix is written as $\mathbf{A}^T$, which is an $m \times n$ matrix found by transposing the rows and columns of $\mathbf{A}$.
   - Prop. 1.3: For two $n \times n$ matrices $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$ and $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.
6. Elementary row operations perform simple operations on the rows of an $n \times m$ matrix $\mathbf{A}$ and can be written as an $n \times n$ matrix $\mathbf{R}$ with the operation preformed by the multiplication $\mathbf{RA}$. ρ_i is used to refer to row i. There are three of them:
   - $\rho_j := \rho_j + \lambda\rho_i$, add λ copies of row i to row j;
   - $\rho_i := \lambda\rho_i$, multiple row i by λ with $\lambda \neq 0$;
   - swap(ρ_i, ρ_j) swap rows i and j.
7. Echelon form: A matrix where each row starts with a sequence of zeros, and the number of zeros in this sequences increases from row to row from top to bottom until the final row is reached or all remaining rows are composed entirely of zeros is said to be in echelon form.

- All matrices can be put into echelon form using elementary row operations.
- If $\mathbf{A}$ can be converted to the echelon form matrix $\mathbf{B}$ and $\mathbf{B}$ has k non-zero rows then the rank of $\mathbf{A}$ is $\operatorname{rk}\mathbf{A} = k$.