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Angular Correlation in the Triple Cascade Transitions*

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The theory of the angular correlation between the radiations involved in three successive nuclear transitions is presented. The formalism is applicable to the case of three successively emitted radiations as well as to the case in which one of the emission processes is replaced by an absorption process. Thus, applications are made to (1) the emission of three cascade gamma-rays and (2) to the double-cascade gamma-emission following capture of particles with nonzero orbital angular momentum. In some cases the angular correlation depends on the order of the transitions. Under process (2), special results are presented which are applicable to $B^{11}(p,\gamma,\gamma)C^{12}$. It is also pointed out that the formalism properly describes the angular correlation in all processes wherein three directions are specified and is therefore applicable when one of these directions refers to a polarization state while the other two refer to propagation vectors. Results are given for the limitation on anisotropy for the triple cascade transition with two radiations parallel or antiparallel and for two radiations in cascade with intervening radiations unobserved.

I. INTRODUCTION

THE angular correlation between two radiations emitted by an excited nucleus in successive cascade has been thoroughly treated in the literature[1-3] and the experimental study thereof has been demonstrated to be an extremely useful procedure for obtaining information concerning nuclear states and for the interpretation of decay schemes.[4] The essential restriction of the theory referred to is the assumption that the sublevels of the initial state are uniformly populated and this may, in fact, be taken as a defining characteristic of the double cascade process.

It is of considerable interest to extend the angular correlation theory to the case in which the above mentioned restriction is removed. This extension is necessary, for example, whenever the initial state is formed by capture of particles with non-vanishing orbital angular momentum (i.e., a propagation vector is defined) and indeed, this capture process followed by the emission of two cascade radiations is a special case of the triple cascade process.[5] In addition, there is considerable interest in the direct triple cascade process in which three radiations are emitted successively. In the following we shall restrict our consideration to two problems: (a) The angular correlation for three gamma-rays emitted in cascade. These results (Sec. II) are readily applied in the case that any one or more of the radiations are spinless particles (α-particles). (b) The angular correlation for the emission of two cascade gamma-rays (or spinless particles) following capture of a particle with spin and orbital angular momentum (Sec. III).[6] For simplicity the detailed applications of the general formalism will be given, for the most part, for those special cases in which two of the radiations are parallel or antiparallel. However, this does not constitute an essential limitation, since such an arrangement can be made to yield the required angular momentum information for the nuclear levels involved just as well as would be the case for the more general and more complicated arrangement. In any case the formalism is presented in such a way as to make clear the procedure to be followed in treating the case of arbitrary directions for the three radiations, and two such applications are given below. In addition the angular correlation between radiations involved in the first and third transitions, with the intervening radiation unobserved, is treated. From the formalism presented, the extension to the angular correlation with any number of unobserved intervening radiations is straightforward.

It should be emphasized that the essential characteristic of the nuclear transitions considered here is the specification of three directions. Thus, so far as the application of the formalism is concerned, we can equally well consider the case that a single radiation is emitted from an initial state which has been formed by capture of polarized particles with a direction of propagation also specified. It is equally true that in this case the substates of the resulting level are not equally populated. Of course, for particle capture this implies the production of a beam of fast polarized particles which is not feasible by present techniques. However, the polarization direction, as one of the three specified directions, need not refer to the initial radiation but may just as well refer to a succeeding radiation observed

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1 D. R. Hamilton, Phys. Rev. **58**, 122 (1940).

2 G. Goertzel, Phys. Rev. **70**, 897 (1946).

3 D. L. Falkoff and G. E. Uhlenbeck, Phys. Rev. **79**, 323 (1950).

4 E. L. Brady and M. Deutsch, Phys. Rev. **78**, 558 (1950); J. R. Beyster and M. L. Wiedenbeck, Phys. Rev. **79**, 411 (1950); W. R. Arnold, Phys. Rev. **80**, 34 (1950) and many others.

5 It follows immediately from the hermitean property of the interaction operator responsible for the capture transition that the distinction between emission and absorption is irrelevant.

6 An example, to which detailed consideration is given below, is the $B^{11}(p,\gamma_1,\gamma_2)C^{12}$ reaction studied by W. M. Good and collaborators in the High Voltage Laboratory, Oak Ridge National Laboratory. The observed anisotropy of γ_1 (12 Mev) relative to the incident proton beam indicates the participation of p (or higher) waves for the incident protons. The authors take this opportunity to thank the members of the High Voltage Group for permission to quote their results before publication.

with a polarization sensitive detector. A case in point would be a reaction resulting in the production of fast polarized particles (by spin-orbit coupling, say) or the emission of a second radiation in cascade with the polarized particle.

II. SUCCESSIVE EMISSION OF THREE GAMMA-RAYS

(A) General Formalism

It is instructive to consider first the problem of triple gamma-correlation because this problem contains all the pertinent features of the general problem with few extraneous complications, and is, moreover, not without experimental interest. Consider, therefore, the successive transitions of a nucleus with initial total angular momentum j_0 going through intermediate states of angular momentum j_1 and j_2 to a final state j_3 emitting gamma-rays of multipolarity 2^{L_0}, 2^{L_1}, and 2^{L_2}, respectively. Let us consider a *maximal* experiment,[7] that is, an experiment in which all measurable magnetic quantum numbers are fixed. Then the desired triple transition is from a state j_0, m_0 by successive emission of the three specified gammas with magnetic quantum numbers p_0, p_1, and p_2, respectively, to a final state j_3, m_3. Here p_i is the magnetic quantum number of the gamma-ray measured with respect to its propagation vector $\mathbf{k}_i$. This restricts p_i to the values ± 1 corresponding to left and right circular polarization. The relative probability for this transition is:

$$P(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2) = \Big| \sum_{m_1 m_2} \langle a_0 j_0 m_0 | H(\mathbf{k}_0 p_0) | a_1 j_1 m_1 \rangle \times \langle a_1 j_1 m_1 | H(\mathbf{k}_1 p_1) | a_2 j_2 m_2 \rangle \times \langle a_2 j_2 m_2 | H(\mathbf{k}_2 p_2) | a_3 j_3 m_3 \rangle \Big|^2. \quad (1)$$

It is understood that $P(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2)$ also depends on the polarization parameters p_i and on m_0 and m_3. In Eq. (1) a_i are all the quantum numbers, besides the angular momentum j_i and its projection m_i, which are necessary to characterize the ith level. The interaction operator $H(\mathbf{k}_0 p_0)$ corresponds to the emission of a gamma-ray in the direction $\mathbf{k}_0$ with polarization p_0. Following Goertzel,[2] we may write the matrix elements in the form:

$$\langle a_i j_i m_i | H(\mathbf{k}_i p_i) | a_{i+1} j_{i+1} m_{i+1} \rangle = \sum_{L_i M_i} A_i (j_i L_i m_i M_i | j_{i+1} m_i + M_i) D_{M_i p_i}{}^{(L_i)}(\alpha_i \beta_i \gamma_i). \quad (2)$$

Here A_i is the part of the nuclear matrix element which is independent of magnetic quantum numbers; it is a function of all other quantum numbers including the multipolarity and parity of the radiation. The $(j_i L_i m_i M_i | j_{i+1} m_i + M_i)$ are the Clebsch-Gordan or vector-addition coefficients corresponding to the addition of angular momenta j_i and L_i to form j_{i+1} with magnetic quantum numbers m_i, M_i and $m_i + M_i$, respectively.[8] The $D_{Mp}{}^{(L)}(\alpha\beta\gamma)$ are the Mth pth elements of the rotation matrix of dimension $2L+1$, while α, β, γ are the Euler angles of the direction $\mathbf{k}_i$ with respect to the axis of quantization that is used to define the m_i. We restrict our attention to pure multipoles and thereby discard all but the L_ith term in Eq. (2). Substituting Eq. (2) in Eq. (1) yields the result that:

$$P(\mathbf{k}_0 \mathbf{k}_1 \mathbf{k}_2) = \Big| \sum_{m_1 m_2} (j_0 L_0 m_0 m_1 - m_0 | j_1 m_1) D_{m_1 - m_0 p_0}{}^{(L_0)}(\alpha_0 \beta_0 0) (j_1 L_1 m_1 m_2 - m_1 | j_2 m_2) D_{m_2 - m_1 p_1}{}^{(L_1)}(\alpha_1 \beta_1 0) \times (j_2 L_2 m_2 m_3 - m_2 | j_3 m_3) D_{m_3 - m_2 p_2}{}^{(L_2)}(\alpha_2 \beta_2 0) \Big|^2. \quad (3)$$

We have discarded, in Eq. (3), the "reduced" matrix elements A_i as factors irrelevent for our purposes. In addition the Euler angles, γ_i, have been set equal to zero as is clearly legitimate.

Since the *physical* experiment we consider[9] does not measure the quantum numbers m_0, m_3, p_0, p_1, p_2, we shall sum over these to obtain the desired (relative) probability $\mathcal{W}(\mathbf{k}_0 \mathbf{k}_1 \mathbf{k}_2)$. Upon summing over m_0, m_3 which defined our original axis of quantization, this reference axis becomes arbitrary. It is convenient to take one of the three directions, say $\mathbf{k}_0$, as the z axis of the new reference system since

$$D_{\mu\nu}{}^{(L)}(000) = \delta_{\mu\nu} \quad (4)$$

and this eliminates the summation over m_1 in Eq. (3). This gives the result

$$\mathcal{W}(\mathbf{k}_0, \mathbf{k}_1, \mathbf{k}_2) \equiv w(\vartheta_1 \vartheta_2 \varphi) = \sum_{m_1 m_3 p_0 p_1 p_2} (j_0 L_0 m_1 - p_0 p_0 | j_1 m_1)^2 \times \Big| \sum_{m_2} (j_1 L_1 m_1 m_2 - m_1 | j_2 m_2) D_{m_2 - m_1 p_1}{}^{(L_1)}(0 \vartheta_1 0) \times (j_2 L_2 m_2 m_3 - m_2 | j_3 m_3) D_{m_3 - m_2 p_2}{}^{(L_2)}(\varphi \vartheta_2 0) \Big|^2. \quad (5)$$

Here the angle ϑ_1 is the polar angle of $\mathbf{k}_1$ with respect to $\mathbf{k}_0$ and analogously for ϑ_2; $\varphi(=\alpha_1 - \alpha_2)$ is the dihedral angle between the $(\mathbf{k}_0, \mathbf{k}_1)$ and $(\mathbf{k}_0, \mathbf{k}_2)$ planes.

This result shows a number of significant features: (a) The first factor is effectively the population of the m_1 sublevel of state j_1. It will therefore be possible to remove the restriction that the first radiation is a gamma-ray by substituting the appropriate population function for the desired radiation. This will be discussed in Sec. III. (b) The sum over m_2 in Eq. (5) is coherent, a feature of all correlation problems involving more than two directions. This coherency makes the actual reduction of Eq. (5) to an algebraic form in the three angles, ϑ_1, ϑ_2 and φ a very laborious task. However, its application in special cases is quite straightforward. As an illustration, Eq. (6) gives the general solution for

[7] B. A. Lippmann, Phys. Rev. **81**, 162 (1951).

[8] The notation $(j_i L_i m_i M_i | j_i L_i j_{i+1} m_i + M_i)$ is used in E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, London, 1935), Chapter III.

[9] If linearly polarized gammas are to be detected the changes in the formalism are minor. See, for example, I. Zinnes, Phys. Rev. **80**, 386 (1950), and Biedenharn, Rose, and Arfken, ORNL Report No. 986 (unpublished).

the case $\{1Q2D1D0\}$ where the numbers stand for the values of j_0, j_1, j_2, j_3, respectively, reading from left to right, and the letters give the multipolarity of the radiations (D dipole, Q quadrupole) reading from left to right in the order of emission.

$$w(\vartheta_1\vartheta_2\varphi)=2(13-5\cos^2\vartheta_1-5\cos^2\vartheta_2+9\cos^2\vartheta_1\cos^2\vartheta_2)$$
$$-3\sin 2\vartheta_1\sin 2\vartheta_2\cos\varphi+2\sin^2\vartheta_1\sin^2\vartheta_2\cos 2\varphi. \quad (6)$$

It should be mentioned that the results obtained above apply equally well to any one or more of the radiations being a spinless particle, restricting attention to a single value of the relative angular momentum L_i for the transition. The value of p_i for this case is then zero instead of ± 1 as for the gamma-transition.

(B) Special Cases

Because of the complexity of Eqs. (5) and (6) and the tedious nature of the calculations it is of interest to note that the cross terms of the coherent sum can be made to vanish whenever any two of the gammas are parallel or antiparallel.[10] Consider γ_0 and γ_1 parallel.[11] Then the elements of the rotation matrix referring to γ_1 become Kronecker delta symbols as in Eq. (4). This eliminates the cross terms and gives for the distribution function

$$w(0\vartheta_2)=\sum_{m_1M_2p_0p_1}(j_0L_0m_1-p_0p_0|j_1m_1)^2(j_1L_1m_1p_1|j_2m_1+p_1)^2$$
$$\times(j_2L_2m_1+p_1M_2|j_3m_1+p_1+M_2)^2F_{L_2}{}^{M_2}(\vartheta_2), \quad (7)$$

where

$$F_L{}^M(\theta)=\sum_{p=\pm 1}|D_{Mp}{}^{(L)}(0\vartheta 0)|^2 \quad (8)$$

is the classical angular distribution of the 2^L multipole radiation in the notation of Falkoff and Uhlenbeck.[3] As expected the azimuthal dependence of the angular correlation has disappeared.

If the first and third radiations are parallel, a similar reduction yields

$$w(\vartheta_1 0)=\sum_{m_1M_1p_0p_2}(j_0L_0m_1-p_0p_0|j_1m_1)^2$$
$$\times(j_1L_1m_1M_1|j_2m_1+M_1)^2$$
$$\times(j_2L_2m_1+M_1p_2|j_3m_1+M_1+p_2)^2F_{L_1}{}^{M_1}(\vartheta_1). \quad (9)$$

The case of the second and third radiations parallel or antiparallel may be obtained from Eq. (7) by interchange of subscripts.

Because of the large number of parameters it is not practical to tabulate the results for the above cases in as much detail as has been done for the double correlation. Instead it is suggested that the functions be calculated as needed using the published $F_L{}^M$[12] and Clebsch-Gordan[8] coefficients. As an illustration of the simplicity and directness of the calculation consider the case $\{1Q2D1D0\}$ treated above and let γ_0 and γ_1 be parallel, their common direction defining the polar axis. The third γ will be observed at an angle ϑ_2.

Referring to Eq. (5) the populations of the j_1 magnetic substates m_1 are given by $\sum_{p_0}(j_0L_0m_1-p_0p_0|j_1m_1)^2$ with $p_0=\pm 1$. From a table of Clebsch-Gordan coefficients it is seen that the relative populations a_i of the substates of j_1 are 2, 1, 6, 1, and 2 for $i=2,\cdots,-2$, respectively. (This is equivalent to determining the intensities of the unpolarized Zeeman components.) The populations of the m_2 sublevels are calculated in the same manner except that the transition intensities given by $(j_1L_1m_1p_1|j_2m_2)^2$ are to be weighted by the populations of the m_1 substates. Since $p_1=\pm 1$, two terms at most can contribute to the population of any m_2 sublevel. Renormalizing to avoid fractions the population of the substates are given in terms of

$$b_1=6a_2+a_0=b_{-1}=18,\quad b_0=3(a_1+a_{-1})=6 \quad (10)$$

where a_i and b_j are proportional to the populations of the substates $m_1=i$ and $m_2=j$, respectively. For the final transition $\{1D0\}$ the vector addition coefficients are all equal and

$$w(0\vartheta_2)\sim 3F_1{}^1(\vartheta_2)+F_1{}^0(\vartheta_2)+3F_1{}^{-1}(\vartheta_2). \quad (11)$$

Since

$$F_1{}^1(\vartheta)=F_1{}^{-1}(\vartheta)=\tfrac{1}{2}(1+\cos^2\vartheta) \quad (12a)$$

$$F_1{}^0(\vartheta)=1-\cos^2\vartheta \quad (12b)$$

then

$$w(0\vartheta_2)=1+\tfrac{1}{2}\cos^2\vartheta_2. \quad (13)$$

This result may be verified by setting $\vartheta_1=0$ in Eq. (6). However, it is believed that the calculation by substate populations is the more direct and by far the easier method of obtaining the angular correlation functions for these special cases.

III. PARTICLE CAPTURE FOLLOWED BY TWO GAMMA-RAYS

(A) Formation of First Intermediate State

We consider the capture of a particle with spin and orbital angular momentum forming a compound state which decays by emitting two gamma-rays in cascade. It has already been pointed out that this is actually a special case of Eq. (3). The significant new feature of this process is the possibility of forming the state j_1 in more than one way, corresponding to the "channel spin degeneracy." The incident particle with spin i and relative orbital angular momentum L_0 is unpolarized as is the target nucleus with angular momentum j_0. The channel spin s is the vector sum of i and j_0, and j_1 is the vector sum of L_0 and s. The population of the

[10] This is in agreement with the analysis of S. P. Lloyd, Phys. Rev. **80**, 118 (1950) and J. A. Spiers, Phys. Rev. **80**, 491 (1950) for double correlation.

[11] For unpolarized radiation, the parallel and corresponding antiparallel angular correlation functions are identical. This is a consequence of the relation $D_{Mp}{}^{(L)}(000)=(-1)^L D_{M-p}{}^{(L)}(\pi\pi 0)$.

[12] See reference 1. The $F_L{}^M$ through $L=5$ are tabulated by W. R. Arnold, Phys. Rev. **80**, 34 (1950).

m_1 substate becomes

$$\sum_{m_0}|\sum_s A(s)(j_0 i m_0 m_1-m_0|sm_1)(sL_0 m_1 0|j_1 m_1)|^2 =\sum_s |A(s)|^2(sL_0 m_1 0|j_1 m_1)^2, \quad (14)$$

where $A(s)$ is the part of the matrix element for the capture process exclusive of the Clebsch-Gordan coefficient representing the magnetic quantum number dependence. The cross terms in Eq. (14) have vanished upon summation over m_0 since

$$\sum_{m_0}(j_0 i m_0 m_1-m_0|sm_1)(j_0 i m_0 m_1-m_0|s'm_1)=\delta_{ss'}. \quad (15)$$

This is in agreement with the results of Devons and Hine.[13] The desired angular correlation function may be written by substituting this population function into Eq. (5).

$$w(\vartheta_1\vartheta_2\varphi)=\sum_{m_1 m_3 s p_1 p_2}|A(s)|^2(sL_0 m_1 0|j_1 m_1)^2$$
$$\times|\sum_{m_2}(j_1 L_1 m_1 m_2-m_1|j_2 m_2)D_{m_2-m_1 p_1}{}^{(L_1)}(0\vartheta_1 0)$$
$$\times(j_2 L_2 m_2 m_3-m_2|j_3 m_3)D_{m_3-m_2 p_2}{}^{(L_2)}(\varphi\vartheta_2 0)|^2. \quad (16)$$

(B) Example $B^{11}(p,\gamma,\gamma)C^{12}$ Reaction[6]

As already mentioned, the anisotropy of the first gamma-ray relative to the proton beam ($\sim 1+0.15\cos^2\vartheta$) demonstrates that $L_0=0$ is not alone effective. Since the proton energy, 163 kev, is appreciably smaller than the barrier height it follows that only $L_0=1$ or 2 (depending on the relative parities of the initial and compound states) can make an appreciable contribution. However, it does not appear likely that the observed anisotropy can be accounted for by a mixture of s and d waves and the choice $L_0=1$ is the most probable one for this reaction. With the two possible channel spins, $s=1$ or 2, one has the possibilities $j_1=0$, 1, 2 or 3. The anisotropy eliminates the possibility $j_1=0$ and the break-up of this state into $Be^{8*}+He^4\rightarrow 3He^4$ and $Be^8+He^4\rightarrow 3He^4$ makes the choice $j_1=2$ most plausible. This conclusion follows strictly if the α-particle decay competes with the γ-emission.[14] Of the remaining C^{12} states the angular momentum j_2 of the intermediate state only is in doubt and would presumably be fixed by comparison of experimental results with calculations like the following. Therefore, as an illustration we consider $j_2=2$ and the transition scheme is $\{2D2Q0\}$. The result for the angular correlation function $w(\vartheta_1\vartheta_2\varphi)$ is found, after a somewhat lengthy calculation, to be

$$\begin{aligned}w(\vartheta_1\vartheta_2\varphi)=&47-26x-12(2-3x)\cos^2\vartheta_2\\&-3(1-2x)\cos^2 2\vartheta_2+3\{18x-11+12(1-x)\cos^2\vartheta_2\\&+3(1-2x)\cos^2 2\vartheta_2\}\cos^2\vartheta_1+3\{(3-2x)\sin 2\vartheta_2\\&+(1-2x)\sin 4\vartheta_2\}\sin 2\vartheta_1\cos\varphi+3\{1+2x-4x\cos^2\vartheta_2\\&-(1-2x)\cos^2 2\vartheta_2\}\sin^2\vartheta_1\cos 2\varphi. \quad (17)\end{aligned}$$

The unknown parameter x denotes the fractional contribution of channel spin $s=1$,

$$x=|A(1)|^2/[|A(1)|^2+|A(2)|^2]. \quad (18)$$

As in the case of the triple gamma-cascade, it is highly desirable to restrict attention to the special cases discussed in Sec. II. One might, for example, measure the coincidence rate for γ_1 and γ_2 as a function of angle between them, with γ_1 being detected along the direction of the proton beam. This can also be done with γ_2 being detected along the direction of the beam. In addition the angular distributions of either γ-ray with the beam, the other γ-ray unobserved, may be measured. The corresponding distribution functions are denoted by $w_i(\vartheta_i)$. From such data it will, in many cases, be possible to determine the angular momenta of

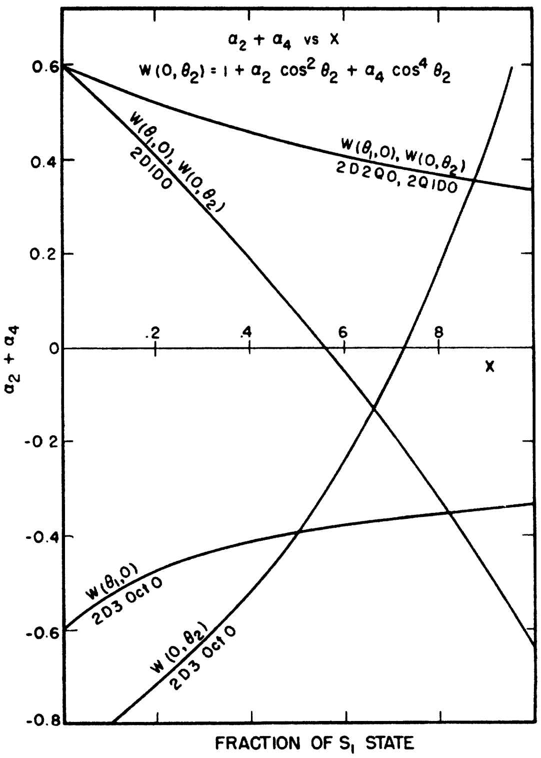

FIG. 1. The degree of anisotropy, $w(\vartheta_i=0)-w(\vartheta_i=\pi/2)$ for the emission of two gamma-rays following p-particle capture by a nucleus of spin $\frac{3}{2}$ as a function of x, the fraction of channel spin $s=1$. The transition schemes considered are designated by the symbols beneath the respective curves. The topmost curve refers to four cases corresponding to either transition scheme with either first or second gamma parallel to the beam.

[13] S. Devons and M. G. N. Hine, Proc. Roy. Soc. (London) A 199, 73 (1949).

[14] Compare G. B. Arfken and L. C. Biedenharn, Phys. Rev. 83, 238(A) (1951). The existence of an *isotropic* 16-Mev γ-ray indicates the presence of two levels excited in the reaction. Despite the energy coincidence ($\gamma_1=12$ Mev, $\gamma_2=4$ Mev) the 16-Mev radiation is apparently not a cross-over. See also W. M. Good *et al.*, Phys. Rev. 83, 211(A), 241(A) (1951).

the levels involved and the multipole order of the gammas. A typical set of predicted distribution functions pertinent to the B^{11} reaction is given below. Using the same transition scheme as for Eq. (17) we obtain

$$w_1(\vartheta_1)=1-21(1-2x)(47-14x)^{-1}\cos^2\vartheta_1 \tag{19a}$$

$$w_2(\vartheta_2)=1-3(1-2x)(9-2x)^{-1}\cos^2\vartheta_2 \tag{19b}$$

$$w(\vartheta_1 0)=1+3(5+4x)^{-1}\cos^2\vartheta_1 \tag{19c}$$

$$w(0\vartheta_2)=1-3(1-4x)(5+4x)^{-1}\cos^2\vartheta_1 +6(1-2x)(5+4x)^{-1}\cos^4\vartheta_2. \tag{19d}$$

These may be seen to follow from Eq. (17) by appropriate reduction. However, direct calculation by populations of magnetic substates is far easier than deriving the general expression. To illustrate the possible use and the limitations of these measurements, the sum of the coefficients of the $\cos^2\vartheta$ and $\cos^4\vartheta$ terms (i.e., the correlation difference between $\vartheta=0$ and $\pi/2$) is plotted as a function of x in Fig. 1 for several transition schemes of interest in the $B^{11}(p,\gamma,\gamma)C^{12}$ reaction. These cases assume the capture of a p-wave proton. For the transition scheme $\{2D3Oct0\}$ the triple angular correlation functions depend upon which gamma-ray (identified by its energy) is emitted first. A possible ambiguity with the transition scheme $\{2Oct1D0\}$ can be eliminated on physical grounds so that in this case it would be possible to make a unique identification of multipolarity, or order of emission, for the two gamma-rays. For the two cases $\{2D2Q0\}$ and $\{2Q1D0\}$ the presence of $\cos^4\vartheta$ terms, which occur when the quadrupole radiation is observed at the variable angle, will be sufficient to remove the ambiguity if the order of the gammas is known or if j_2 is known from independent evidence. The transition $\{2D1D0\}$ will be distinguished from the other cases by the absence of $\cos^4\vartheta$ terms. In this case it is not possible to determine the order of the gammas by angular measurements. It is clear in this reaction that the triple correlation experiment can provide more information than the simple correlation of the first gamma with the proton beam which allows all four of the transition schemes mentioned above.

The presence of $\cos^4\vartheta$ in the angular distribution function of a cascade initiated by a p-wave proton may appear surprising. However, this result is not in contradiction with the theorem of Eisner and Sachs[15] and Yang[16] which applies to the angular distribution of a single (observed) outgoing radiation. The limits of anisotropy for triple correlations are discussed in detail in Sec. IV. The theorem does apply to $w_1(\vartheta_1)$ and $w_2(\vartheta_2)$ and it is seen that no contradiction arises. In the present case when $x=\frac{1}{2}$ the contributions of the two channel spins combine to form the state $j_1=2$ with randomly populated magnetic substates. This means that the triple correlation reduces to the familiar double correlation. It is seen that $w_1(\vartheta_1)$ and $w_2(\vartheta_2)$ do indeed vanish while

$$w(\vartheta_1 0)_{x=\frac{1}{2}}=w(0\vartheta_2)_{x=\frac{1}{2}}=1+(3/7)\cos^2\vartheta, \tag{20}$$

which is just the value given by Hamilton.[1]

IV. LIMITATIONS ON ANISOTROPY

(A) Correlation of Two Radiations

The angular correlation between two radiations can always be expressed as a power series in $\cos^2\vartheta$, under the foregoing assumptions. As emphasized elsewhere it is useful to predict the maximum power of $\cos^2\vartheta$ which may enter the distribution function and we refer to the existence of such a maximum power as a limitation on anisotropy. We also refer to the degree of anisotropy and in what follows this phrase may equally well refer to the maximum power of $\cos^2\vartheta$ or to some measure of the departure from isotropy, such as $|1-w(0)/w(\pi/2)|$.

While the limitations in the case of two successive radiations have been completely discussed,[1,2] the limitation in the case of correlating two radiations, with intervening transitions unobserved, has not been treated in detail and the previously determined limits[15,16] for such transitions can be more narrowly restricted. For the case of particle capture followed by a cascade of gamma-rays it has been suggested that the degree of anisotropy, correlating individual gamma-transitions with the initial beam, decreases as one proceeds down a cascade. It is easily seen from the formalism and from investigation of special cases that this is not necessarily so. In particular, it is not difficult to see that the coefficients of the various powers of $\cos^2\vartheta$ may increase or decrease and the highest power of $\cos^2\vartheta$ may also increase or decrease. The treatment of this problem is a special case of triple correlation in which one averages over the directions of intervening radiations. Using the orthogonality property of the rotation matrices

$$\int D_{\lambda\mu}^{(L)}(\alpha\beta 0)D_{\nu\mu}^{(L)*}(\alpha\beta 0)d\Omega_{\alpha\beta}=4\pi(2L+1)^{-1}\delta_{\lambda\nu} \tag{21}$$

cross terms in Eq. (5) vanish and

$$w_2(\vartheta_2)=\sum_{m_1m_2sM_2}|A(s)|^2(sL_0m_10|j_1m_1)^2 \times(j_1L_1m_1m_2-m_1|j_2m_2)^2 \times(j_2L_2m_2M_2|j_3m_2+M_2)^2F_{L_2}^{M_2}(\vartheta_2). \tag{22}$$

The limitations on anisotropy may be shown explicitly by using the methods developed by Racah.[17] Using the properties of the rotation matrices, Eq. (8) may be transformed to give

$$F_L^M(\vartheta)=(-)^{L+1}4\pi^{\frac{1}{2}}(2L+1)^{-\frac{1}{2}}\sum_{\nu=0}^{L}(L2\nu M0|LM) \times(LL1-1|2\nu 0)Y_{2\nu}^0(\cos\vartheta) \tag{23}$$

[15] E. Eisner and R. G. Sachs, Phys. Rev. **72**, 680 (1947).
[16] C. N. Yang, Phys. Rev. **74**, 764 (1948).
[17] G. Racah, Phys. Rev. **62**, 438 (1942); Phys. Rev. **63**, 367 (1943).

where the $Y_{2\nu}^0(\cos\vartheta)$ are the usual normalized spherical harmonics. Consider the case in which the second γ-ray (γ_2) is not observed and the distribution of γ_1 relative to the incident particle beam is investigated. Substituting the form of F_L^M given by Eq. (23) into Eq. (22) and using the Racah coefficients, $W(abcd;ef)$, to change the coupling schemes, it is possible to carry out the summation over magnetic quantum numbers. The derivation is given in Appendix I and the result is

$$\begin{aligned} w_2(\vartheta_2)=\sum_{s\nu}|A(s)|^2&(L_2L_21-1|2\nu0)(2\nu L_000|L_00)\\ &\times W(j_12\nu sL_0;j_1L_0)W(j_2L_12\nu j_1;j_1j_2)\\ &\times W(L_22\nu j_3j_2;L_2j_2)Y_{2\nu}^0(\cos\vartheta). \quad (24)\end{aligned}$$

For γ_2 unobserved the angular distribution of γ_1 relative to the beam is given by

$$\begin{aligned} w_1(\vartheta_1)=\sum_{s\nu}|A(s)|^2&(L_1L_11-1|2\nu0)(L_02\nu00|L_00)\\ &\times W(L_12\nu j_2j_1;L_1j_1)\\ &\times W(sL_0j_12\nu;j_1L_0)Y_{2\nu}^0(\cos\vartheta) \quad (25)\end{aligned}$$

as is easily verified by the methods of Appendix I.

It is a property of the Racah coefficients that $W(abcd;ef)$ vanishes unless the four triads (abe), (cde), (acf) and (bdf) satisfy triangular inequalities.[17] It follows that the maximum power of $\cos^2\vartheta$ in Eq. (25) is $\mathfrak{M}(L_0, j_1, L_1)$ as has been shown previously.[1,2] Here $\mathfrak{M}(a, b, \cdots)$ denotes the largest integer equal to or less than the smallest member of the set $(a, b, \cdots)$.[18] In a similar manner the maximum power of $\cos^2\vartheta$ in Eq. (24) is seen to be $\mathfrak{M}(L_0, j_1, j_2, L_2)$. This result extends previous statements[15,16] to show explicitly the additional restrictions which exist when intervening levels are present. It will be noted that the multipolarity of the unobserved radiations does not enter this limit.

As an illustration of the foregoing, consider the case of capture of a spin $\frac{1}{2}$ particle with $L_0=2$ (d wave) by a nucleus with spin $\frac{3}{2}$ (as in the proton capture by B^{11}) and, for $x=0$ and the transition scheme $\{3D2Q0\}$, the distribution of γ_1 with γ_2 unobserved is

$$w_1(\vartheta_1)=1-(9/73)\cos^2\vartheta_1 \quad (26a)$$

while the distribution of γ_2 with γ_1 unobserved is given by

$$w_2(\vartheta_2)=1-(42/31)\cos^2\vartheta_2+(55/31)\cos^4\vartheta_2. \quad (26b)$$

Here the degree of anisotropy of the second gamma, by either of the above definitions, is clearly greater than that of the first.

[18] In rare cases the maximum power of $\cos^2\vartheta$ may not be realized due to the possible vanishing of one of the Racah coefficients. Tables of these coefficients are now in process of preparation and will be circulated at a later date. Of the cases investigated thus far there occurs only one case, of physical interest, in which a Racah coefficient vanishes. This corresponds, in Eq. (25) to $L_0=1$, $L_2=2$ and $j_1=j_2=\frac{3}{2}$ which gives an isotropic $w_1(\vartheta_1)$ and for Eq. (24) to $L_2=2$, $j_2=j_3=\frac{3}{2}$, and $\mathfrak{M}(L_0, j_1)\geqslant 1$ which gives an isotropic $w_2(\vartheta_2)$.

(B) Special Cases of Triple Correlation

As mentioned in Sec. III the maximum power of $\cos^2\vartheta$ that enters in the distribution for the case of two of the three observed radiations parallel or antiparallel may be greater than that allowed for a simple double correlation. To determine the new limit for the first two radiations parallel it is only necessary to transform Eq. (7) by the use of Racah coefficients (see Appendix II). The summation over magnetic quantum numbers leads to

$$\begin{aligned} w(0\vartheta_2)=\sum_{ll'\nu}&(2l+1)^{\frac{1}{2}}(2l'+1)^{\frac{1}{2}}(L_22\nu10|L_21)\\ &\times[(L_0L_111|l2)(L_0L_111|l'2)(ll'2-2|2\nu0)\\ &+(L_0L_11-1|l0)(L_0L_11-1|l'0)(ll'00|2\nu0)]\\ &\times W(j_0L_0j_2L_1;j_1l)W(j_0L_0j_2L_1;j_1l')W(ll'j_2j_2;2\nu j_0)\\ &\times W(L_2L_2j_2j_2;2\nu j_3)Y_{2\nu}^0(\cos\vartheta_2). \quad (27)\end{aligned}$$

Again the properties of the Racah coefficients, W, enable one to set an upper limit to the powers of $\cos^2\vartheta$. In this case it may be written as

$$\mathfrak{M}[\mathfrak{M}(L_0, j_1)+L_1, j_2, L_2]. \quad (28)$$

It is quite possible to use Eq. (27) for calculating angular correlation functions. However, for this purpose, the expressions given in Sec. II are more direct and, in the absence of tables of the Racah coefficients, somewhat simpler.

The upper limit on the power of $\cos^2\vartheta$ may be determined in a similar way for the case of the first and third radiations parallel but the analysis is rather lengthy. The result is

$$\mathfrak{M}[\mathfrak{M}(L_0, j_1)+\mathfrak{M}(L_2, j_2), L_1]. \quad (29)$$

It is clear that these limits hold also when the first transition is the capture of an unpolarized particle, L_0 referring then to the relative orbital angular momentum.

There is a very simple physical interpretation of all of these results which makes them somewhat more transparent. It is clear that a 2^L order multipole has an angular distribution containing no powers of $\cos^2\vartheta$ greater than L. This sets a limit to what may be called the "angular information" of a given radiation. It is also clear that a level $j=0$ or $j=\frac{1}{2}$ can contain or transmit no angular information in the sense that all emitted radiation from such a state is isotropic. This is a special case of the result that the highest power of $\cos^2\vartheta$ that can appear in radiation from a level with angular momentum j is just $\mathfrak{M}(j)$ as can be shown quite readily. Then it is possible to schematize the above results as follows: Consider the case of the first and second radiations parallel. The observation of each of these transitions may be interpreted as putting angular information into the system. The first observation limits the powers of $\cos^2\vartheta$ to L_0 and of this no more than $\cos^{2j_1}\vartheta$ can be emitted by the state j_1. The observation of the second transition then adds angular information limited by $\cos^{2L_1}\vartheta$. Of this angular information no higher power than $\cos^{2j_2}\vartheta$ can be passed by the level j_2 and no higher power than $\cos^{2L_2}\vartheta$ can appear in the final radiation. Hence, the maximum power of $\cos^2\vartheta$ that can appear in the angular distribution is just that given by the foregoing result (28). For the case of the first and third radiations parallel one considers angular information as being introduced into the system by these two transitions. The information transmitted is limited by the j values of the levels as in the previous case. The result is that the maximum power of $\cos^2\vartheta$ that can appear in the angular distribution is that given by (29). It is to be emphasized that this schematization is presented primarily as a mnemonic. The formal justification of the results is found through the application of Racah functions as illustrated in Appendix II.

APPENDIX I

We consider a special case of the triple correlation problem where a particle is captured and two successive gamma-rays are emitted, but only the second gamma-ray is observed. The desired angular distribution of the observed gamma-ray with respect to the propagation direction of the incident particle, $w_2(\vartheta_2)$, is given by Eq. (22). As mentioned earlier, it is possible to perform explicitly the summations over all magnetic quantum numbers that appear in Eq. (22). To do this, we substitute for $F_{L_2}{}^{M_2}(\vartheta_2)$ the result given in Eq. (23) and obtain

$$w(\vartheta_2)\sim\sum_{sm_1M_1M_2\nu}|A(s)|^2(sL_0m_10|j_1m_1)^2 \times(j_1L_1m_1M_1|j_2m_1+M_1)^2 \times(j_2L_2m_1+M_1M_2|j_3m_1+M_1+M_2)^2(L_2L_21-1|2\nu0) \times(L_22\nu M_20|L_2M_2)Y_{2\nu}{}^0(\cos\vartheta_2). \quad (A1)$$

Now we perform the summation over M_2 using the result that

$$\sum_{M_2}(j_2L_2m_1+M_1M_2|j_3m_1+M_1+M_2)^2(L_22\nu M_20|L_2M_2) =(2j_3+1)(2L_2+1)^{\frac{1}{2}}(2j_2+1)^{-\frac{1}{2}} \times(2\nu j_20m_1+M_1|j_2m_1+M_1) \times W(L_22\nu j_3j_2;L_2j_2). \quad (A2)$$

The summation over M_1 can now be performed, namely,

$$\sum_{M_1}(j_1L_1m_1M_1|j_2m_1+M_1)^2(2\nu j_20m_1+M_1|j_2m_1+M_1) =(2j_2+1)^{\frac{3}{2}}(2j_1+1)^{-\frac{1}{2}}(j_12\nu m_10|j_1m_1) \times W(j_2L_12\nu j_1;j_1j_2). \quad (A3)$$

This leaves only the summation over m_1 which is done in the same way.

$$\sum_{m_1}(sL_0m_10|j_1m_1)^2(j_12\nu m_10|j_1m_1) =(2j_1+1)^{\frac{3}{2}}(2L_0+1)^{-\frac{1}{2}}(2\nu L_000|L_00) \times W(j_12\nu sL_0;j_1L_0). \quad (A4)$$

The final result for $w_2(\vartheta_2)$ is given in Eq. (24), discarding irrelevant factors. It should be mentioned that the specialization to particle capture for the first transition is not essential. Had we desired to calculate a similar problem for a triple gamma-cascade we merely omit the channel spin sum, replacing s by j_0 (the angular momentum of the initial state) and introduce the polarization index p_0 for the magnetic quantum number of the first radiation. For unpolarized gammas this index is summed.

If we observe the first gamma and not the second, the angular distribution of the observed gamma with respect to the beam direction, $w_1(\vartheta_1)$, is given by the usual results for double correlation as

$$w_1(\vartheta_1)=\sum_{sm_1M_1}|A(s)|^2(sL_0m_10|j_1m_1)^2 \times(j_1L_1m_1M_1|j_2m_1+M_1)^2F_{L_1}{}^{M_1}(\vartheta_1). \quad (A5)$$

Using Eq. (23), $w_1(\vartheta_1)$ takes the form,

$$w_1(\vartheta_1)\sim\sum_{s\nu m_1M_1}|A(s)|^2(sL_0m_10|j_1m_1)^2 \times(j_1L_1m_1M_1|j_2m_1+M_1)^2(L_12\nu M_10|L_1M_1) \times(L_1L_11-1|2\nu0)Y_{2\nu}{}^0(\cos\vartheta). \quad (A6)$$

We can perform the summation over M_1 as previously using the equation,

$$\sum_{M_1}(j_1L_1m_1M_1|j_2m_1+M_1)^2(L2\nu M_10|L_1M_1) =(2j_2+1)(2L_1+1)^{\frac{1}{2}}(2j_1+1)^{-\frac{1}{2}}(j_12\nu m_10|j_1m_1) \times W(L_12\nu j_2j_1;L_1j_1). \quad (A7)$$

The sum over m_1 can now be done similarly to Eq. (A4).

$$\sum_{m_1}(sL_0m_10|j_1m_1)^2(j_12\nu m_10|j_1m_1) =(2j_1+1)^{\frac{3}{2}}(2L_0+1)^{-\frac{1}{2}}(L_02\nu00|L_00) \times W(sL_0j_12\nu;j_1L_0). \quad (A8)$$

The final result, aside from irrelevant multiplicative constants, is given by Eq. (25).

APPENDIX II

It is desired to find the limits on the anisotropy in the case where the first two gamma-rays in a triple cascade are parallel. For this case we find from Eqs. (7) and (23) that the angular correlation for the third gamma has the form,

$$
\begin{aligned}
w(0\vartheta_2)\sim \sum_{m_0m_3p_0p_1p_2\nu} & (j_0L_0m_0p_0|j_1m_0+p_0)^2 \\
& \times(j_1L_1m_0+p_0p_1|j_2m_0+p_0+p_1)^2 \\
& \times(j_2L_2m_0+p_0+p_1m_3-m_0-p_0-p_1|j_3m_3)^2 \\
& \times(L_22\nu m_3-m_0-p_0-p_10|L_2m_3-m_0-p_0-p_1) \\
& \times(L_2L_2p_2-p_2|2\nu 0)Y_{2\nu}{}^0(\cos\vartheta_2). \quad \text{(A9)}
\end{aligned}
$$

The summation over m_3 can be carried out in Eq. (A9), exactly as in Appendix I, leading to

$$
\begin{aligned}
w(0\vartheta_2)\sim \sum_{m_0p_0p_1p_2\nu} & (j_0L_0m_0p_0|j_1m_0+p_0)^2 \\
& \times(j_1L_1m_0+p_0p_1|j_2m_0+p_0+p_1)^2 \\
& \times(j_22\nu m_0+p_0+p_10|j_2m_0+p_0+p_1)(L_2L_2p_2-p_2|2\nu 0) \\
& \times W(L_22\nu j_3j_2;L_2j_2)Y_{2\nu}{}^0(\cos\vartheta_2). \quad \text{(A10)}
\end{aligned}
$$

To perform the sum over m_0 we use the result that

$$
\begin{aligned}
& (j_0L_0m_0p_0|j_1m_0+p_0)^2(j_1L_1m_0+p_0p_1|j_2m_0+p_0+p_1)^2 \\
& \quad =(-)^{j_0-j_2}(2j_1+1)(2j_2+1)^{\frac{1}{2}}\sum_{ll'l''}(2l+1)^{\frac{1}{2}}(2l'+1)^{\frac{1}{2}} \\
& \quad \times(2l''+1)^{\frac{1}{2}}W(j_0L_0j_2L_1;j_1l)W(j_0L_0j_2L_1;j_1l) \\
& \quad \times W(j_2lj_2l';j_0l'')(L_0L_1p_0p_1|lp_0+p_1) \\
& \quad \times(L_0L_1p_0p_1|l'p_0+p_1)(ll'-p_0-p_1p_0+p_1|l''0) \\
& \quad\quad \times(j_2l''m_0+p_0+p_10|j_2m_0+p_0+p_1). \quad \text{(A11)}
\end{aligned}
$$

The sum over m_0 can now be performed and the result is just Eq. (27).

Using the property that the Racah coefficient $W(abcd;ef)$ vanishes unless a triangular inequality is satisfied for each of the four triads (abe) (cde) (acf) (bdf), we can conclude that value of ν in Eq. (27) is limited by the triangular inequalities on the triads $(2\nu j_2j_2)$ $(2\nu L_2L_2)$, $(2\nu JJ')$, while J (and J') are limited by similar conditions on the triads (j_0j_2J), (L_0L_1J). We can further conclude from this that ν is limited by the condition: $\nu\leq\mathfrak{M}(j_2;L_2;L_1+\mathfrak{M}(L_0,j_1))$.

The case where the first and third radiations are parallel is treated in a similar fashion and results in a product of six Racah functions. The condition on the maximum power (ν) of $\cos^2\vartheta_1$, is then found to be Eq. (29).

In order to see that exactly the same results apply for antiparallel gammas, we note that $D_{\nu\mu}{}^{(L)}(\pi,\pi-\beta,0)=(-1)^LD_{\nu-\mu}{}^{(L)}(0,\beta,0)$, and hence (since the $(-1)^L$ is immaterial here) we merely reverse the polarization of the antiparallel gamma. Since we sum over the polarizations the results are clearly unchanged.