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Pricing Power Derivatives: Electricity Swing Options

Master's Thesis

in Mathematical Finance

presented by
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on 26.02.2010

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Preisbestimmung bei Elektrizitätsderivaten: Swing Options

Masterarbeit

in Finanzmathematik

vorgelegt von
Nadi Serhan Aydın
am 26.02.2010

Gutachter

Prof. Dr. Rüdiger Kiesel

To my parents,

Für meine Eltern,

Acknowledgements

I am personally grateful to all who had a share in this thesis. I extend my gratitudes to the Institute of Mathematical Finance and, the Faculty of Mathematics and Economics for giving me the permission to commence on this thesis as an exchange student and perform the necessary research work. I also convey my thanks to Prof. Dr. Wolfgang Arendt who confirmed this permission and encouraged me to go ahead with my thesis.

I am deeply indebted to Prof. Dr. Rüdiger Kiesel, the supervisor of this study, who was then the director of the Institute of Mathematical Finance at the Ulm University and has now taken up his new post at the University of Duisburg Essen as the head of the Chair for Energy Trading and Finance. His helpful, insightful and enlightening suggestions were invaluable across the way.

I should also emphasize the full-time support by Richard-Biegler König who is now a PhD candidate in the area of Mathematical Finance at the Ulm University. I owe him a special thank for his contributions in both the construction and the realization of this work.

Last but not the least, I appreciate the significance of the role that the International Office of the Ulm University played by approving my proposal to sign a bilateral exchange agreement between the Middle East Technical University and the Ulm University.

Abstract

The Swing options are the natural outcomes of the increasing uncertainty in the power markets, which came along with the deregulation process triggered by the UK government's action in 1990 to privatize the national electricity supply industry. Since then, the ways of handling the risks in the price generation process have been explored extensively. Producer-consumers of the power market felt confident as they were naturally hedged against the price fluctuations surrounding the large consumers. Companies with high power consumption liabilities on their books demanded tailored financial products that would shelter them from the upside risks while not preventing them from benefiting the low prices.

More effective risk management practices are strongly dependent upon the successful parameterization of the underlying stochastic processes, which is also key to the effective pricing of derivatives traded in the market. In this thesis, we introduce an electricity spot price model, which incorporates jumps and still maintains the analytical tractability. We also derive the forward curve dynamics implied by the spot price model and apply it to generalize Black's 1976 formula so that it incorporates the jump-driven dynamics of the model. As the main discussion of this thesis, the Grid Approach, which is a generalization of the Trinomial Forest Method, is applied to the electricity Swing options. We investigate the effects of spikes on the per right values of the Swing options with various number of exercise rights, as well as the sensitivities of the model-implied prices to several parameters.

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Chapter 1

Preamble

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The deregulation of the electricity prices have brought along the question of how the risk of increasing uncertainty in the price generation process would be handled. Producer-consumers of the power market felt confident as they were naturally hedged against price fluctuations surrounding the large consumers. Companies with high power consumption liabilities on their balance sheet demanded tailored financial products that would protect them against the upside risks while not preventing them from benefiting the low prices. On top of that, due to the non-storability of the power, the deliveries had to be spread over a predetermined period of time, the *delivery period*, causing additional exposures to the risk factors that the energy markets entailed. Thus, the Swing options have come out as the natural outcomes of the increasing uncertainty in the power markets.

On the other side, the distinguishing properties of the electricity markets (see Chapter 2) have substantially led researchers to more complicated methods in their derivative pricing algorithms. Strong multi-seasonality (i.e. intraday, intraweek, intrayear) in power prices implied that the seasonality should be extracted with great care. Their spiky nature has introduced the jump components appearing in the stochastic spot price models and in the forward curve dynamics implied by those, while their non-storability have led to the development of the forwards with delivery periods, along with which the options on forwards have become more challenging to value. Forward contracts with a delivery period, rather than a single point in time for the delivery of the power, has provided the buyer with a constant stream of energy, no matter the internal power use on the buyer's side swung up or down during this delivery period. In the longer delivery periods (e.g. one year), this posed a real problem for the contract buyers as their actual need for energy changed from time to time and the amount purchased had to be adjusted up or down accordingly. The growing need for this flexibility gave rise to what we called 'Swing Options' or 'Swing Rights' today. The pricing of these strongly path-dependent options is based on the optimal exercise rule, where the holder is assumed to maximize profit. However, in reality, they were engineered to serve the need to adjust the purchased volume to match internal energy demands.

This master thesis is motivated mainly by the work done by Hambly-Howison-Kluge (see [12]) and their effort to mathematically generalize the method introduced in Jaillet-Ronn-Tompaidis (see [14]) for pricing the Swing options. They developed a model that incorporates jump structure into the stochastic properties of the spot price process as well as the valuation processes of different type of options. Integration of jumps into the pricing algorithms is an ongoing research for the finance people. The statistical tractabilities of the jump-driven models comes up as a big challenge. Besides, there are clear evidences in the literature in which the jump risk is left unpriced.

1.1 Former Studies

The pricing issues of the options on commodity forwards were first addressed in [3]. The stochastic behavior of the commodity prices was examined by means of a mean-reverting model in [22], however, the model excluded jumps. [19] further extends this model to account for a deterministic seasonality. [7] incorporates both mean-reversion and jumps without a closed-form solution for the forward. A model that captures the most important characteristics of electricity spot prices at the same time is presented in [6]. Both the historical data and the observed forward curve are used for calibration whereas the model assumes a common mean-reversion rate for both the diffusive and the jump part.

The Swing options have recently gained popularity in the energy risk management literature and the valuation issues have been frequently addressed by researchers. A quick survey reveals that potential methods range from (i) Monte Carlo approaches (as in [21]) which are sometimes combined with least-squares regressions (as in [18]) to (ii) PDE-based methods where finite differences tools are employed (as in [17]), (iii) multiple stopping problems (as in [5]), and (iv) multiple tree/grid-based dynamic programming algorithms (as in [14] and [12]). Most of them will be discussed during the next section in a general context.

1.2 Swing Options and the Valuation Issues

The Swing options give the holder a flexibility in both the delivery time and the delivered volume. They, like other options, can be settled by either the physical exchange of goods or cash netting against the spot price. However, since the spot electricity cannot be stored, forward contracts are used as underlying in the hedging transactions. This requires that either the spot price process is modeled effectively and the forward dynamics are derived from it, or the forward curve dynamics are modeled directly. Focusing directly on the forward price dynamics would unsurprisingly be a painful effort for us to extract a high quality curve of forward price dynamics, particularly in cases there are not enough number of liquidly traded forwards in the market. In literature, it is also discussed that focusing directly on the forward curve models can result in a very complex non-Markovian dynamics for the spot price, making it highly challenging to price the path-dependent options.

The intuition behind the valuation of the Swing options is the assumption that the holder exercises rights in a way that maximizes the overall profit. However, one should also keep in mind that the Swing options cannot always be used in a profit-maximizing way since the holder may use them to swing up or down the purchased volume to optimize the flow of internal processes.

For options with an early exercise problem, an analytical solution is yet to be found. In literature, numerical methods are mostly employed to tackle this problem. [9] applies the Longstaff-Schwartz algorithm given in [18] to the Swing options. The Least-Squares Monte Carlo method is used to calculate the conditional continuation values and the exercise decision is made by comparing these values to the exercise payoffs. The pricing algorithm simultaneously uses as an input multiple cash flow matrices for each possible number of remaining exercise rights. [15] treats the valuation problem as an optimal consumption problem and assumes a right continuous stochastic consumption process for the Swing option holder, which satisfies the Markov property. The main result reported in their paper is a lower value bound for the Swing option written on regular electricity derivatives.

1.2.1 Motivating Studies

The methods based on multiple trinomial trees (as mentioned in Jaillet-Ronn-Tompaidis ([14])) seem to maintain their popularity. In this approach, a tree where trinomial refinements are allowed to emanate from each node is built for the forward prices. Then, a number of identical trees are used simultaneously to price the Swing option. However, a tree-way branching is unlikely to suffice for an effective approximation of the distributions, particularly the heavy-tailed ones. As an extension to this method, Hambly-Howison-Kluge ([12]) uses multiple *grid tree*s to capture a wide range of potential movements in the underlying's price. This method enjoys the ability of finer grids to provide better approximations to the state price probabilities than the trinomial trees can do. This approach is also pursued in this thesis.

1.2.2 Extensions to Them

We extend the work of Jaillet-Ronn-Tompaidis by step-by-step generalizing this method to the Grid Approach explained in Hambly-Howison-Kluge ([12]). This generalization requires the integration of the spike process into the valuation algorithm and the use of larger grids with thinner increments to better approximate the probability density functions of the individual sub-processes that mainly drive the spot price process. We also derive a more general approximation to the spike process density than the one given in ([12]). That is, we derive a Gamma density approximation to the stationary distribution of the truncated spike process. Our use of a heterogeneous grid for the truncated spike process in the Grid Approach allow us to more precisely calculate the probabilities for the spike process values which are close to zero. This approach leads to slightly lower values for Swing options than those reported in [12]. Our results are supplemented by the numerical pricing function (*swing.m*) from which they were generated.

1.3 The Aim of This Thesis

In summary, there are four main goals tracked by this thesis:

- Introducing a spot price model for electricity which incorporates jumps while maintaining analytical tractability
- Deriving the forward curve dynamics implied by the spot price model and applying them to generalize Black's 1976 formula so that it incorporates jump-driven dynamics of the model
- Introducing the Grid Approach which is a generalization of the Trinomial Forest Method to price the Swing options
- Numerically pricing standard Swing options and analysing the sensitivity of the Swing option value to changes in the spot price model parameters.

For these purposes, we employ a spot price model which is the exponential of the sum of (i) an Ornstein-Uhlenbeck and (ii) an independent mean-reverting pure jump process, as well as (iii) a deterministic seasonality function, and elaborate on a variety of issues including:

- The moment generating function of the log-spot price process at maturity T
- Approximations to the probability density functions of the log-spot price process and the spike process at maturity T
- Approximations to the distributions of the forward contracts with and without a delivery period by means of the moment-matching method

- Utilization of the Grid Method to price path-dependent options with multiple exercise rights (e.g. Swing contracts) which, in turn, uses approximations to the conditional densities of the individual processes driving the spot price process.

1.4 Organization of the Thesis

As mentioned above, this thesis aims at pricing one of the most complicated options of the energy markets, i.e. the Swing options. Hence, we need to set the preliminary stages first. In this context, next chapter (Chapter 2) will summarize the most widely known characteristics of the electricity markets which distinguish them from the others. Having laid out the main criteria that a spot electricity model should be able to satisfy, this market-implied characteristics will be the base for the mathematical setup of the spot price model, which is given in Chapter 3. This chapter sketches a wide range of stochastic properties implied by the combined spot price model which consists of two independent mean-reverting process and a deterministic seasonality part. Analytical tractability of the model is maintained by means of a set of approximations substituting the original jump process while efficiently retaining its distributional properties. The forward curve dynamics derived from the model will be applied to the options on forwards through a moment-matching interface. Chapter 4 introduces the Trinomial Forest[1] Approach and generalize it to the Grid Method. Chapter 5 summarizes and, discusses future extensions.

[1]The term *forest* has a particular emphasis here since we use a *option value forest* rather than a single tree and it is formed by a number of trees each of which corresponds to different number of remaining exercise rights. Values in each tree has a direct mutual dependence on its neighbour subordinate which results in a forest where the values in each tree at least indirectly affected by those in its subordinates.

Chapter 2

Electricity Markets: Stylized Facts

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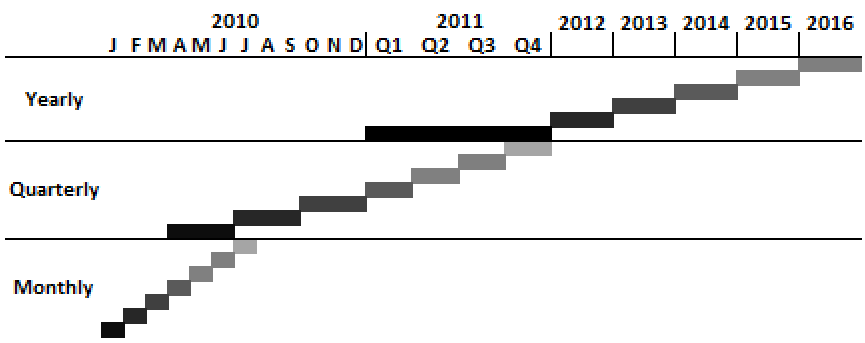
By its nature, the electricity is difficult to store and has to be available on demand. As a consequence, unlike the other energy and non-energy commodities, it is not possible, under normal operating conditions, to keep electricity in stock, ration it or have consumers queue for it. Continuously varying demand and supply further exacerbates the situation. In more technical terms, the *transmission system operators* are the results of the physical requirement for a controlling agency to coordinate the power generator facilities to meet the expected demand in the marketplace. In case there is a mismatch between supply and demand, the generators will speed up or slow down. In spite of all these advanced technologies available on the production side, still, the laws of physics rules the market. Below we discuss the major issues that characterize the electricity markets.

2.1 Non-storability

Unlike oil and liquid gas, which can be stored effectively, electricity is a flow variable. This means that the current technological developments have yet to allow for the physical storage of an amount of electricity which corresponds to the average daily consumption of a big factory. Storage for the daily use of even a small country sounds impossible at this point. Therefore, not only should the power be generated on a continuous basis, but also the supply should match the demand again on a continuous basis. It brings along the need for a very active synchronization between demand and supply together with a very effective reflection of the current production costs in prices (see Section 2.2). This real-time balance between demand and supply results in seasonal patterns in the spot price as the consumption habits exhibit ordinary paths throughout daily, weekly and annual cycles (see Section 2.3), as well as bounded variations around the cost of production in the long-term. Moreover, this *generate-and-transfer* production model requires the power to be delivered on a periodical basis, rather than a lump sum delivery. Thus, the power derivative contracts should always include a delivery period. To illustrate, the delivery periods of all EEX Base Load[1] Futures observed on the first day of the year 2010 for the German Power Market are shown in Figure 2.1. Another characteristic that is usually promoted as being the result of the non-storability of the electricity is the spiky nature of the power markets, which will be discussed in detail in Section 2.4.

Figure 2.1: German Power Market - Baseload Futures



2.2 Mean-reversion

As mentioned in the previous section, the active synchronization between the production and the consumption yields a long-term balance for the underlying's price which generally gravitates around the cost of production. Unlike oil markets, where the long-term prices are affected considerably by the factors such as the convenience yield and the limited reserves, or stock markets, where the prices are allowed to evolve freely, electricity markets trade power contracts whose prices are expected to stay within some certain fair price boundaries throughout the lifetime trajectory. One result of this fact, which should also be reflected in the model-implied volatilities, is that the volatility decreases with increasing time horizon, i.e. there is a long-term equilibrium which is much less volatile than the spot price. The speed of

[1]Base Load contracts characterize the type of load for the delivery of electricity or the procurement of electricity with a constant output over 24 hours of each day of the delivery period. In contrast, Peak Load contracts cover 12 hours from 08:00 am until 08:00 pm on every working day (Monday to Friday) during a delivery period.

the reversion depends heavily on how quickly the imbalances between supply and demand are eliminated.

2.3 Cyclicality

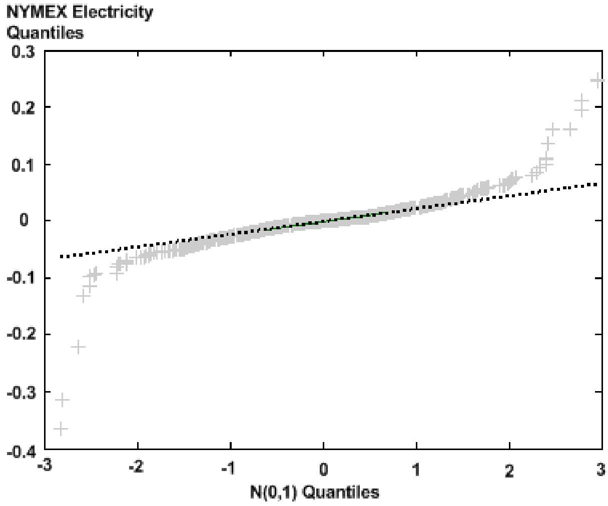
Price cycles occur on different time scales like different times of the day, different days of the week, different seasons of the year and are usually driven by the cyclical changes in demand. The seasonality aspect can be considered as deterministic and therefore easily be extracted from the stochastic part, as claimed in [9]. It is discussed in [17] and [19] that the model-implied forward prices are dominated by the seasonality factor and the seasonality is one of the most important aspects in the shape of the forward curve. Strong deterministic cycles within the daily, weekly and yearly periods are also proved empirically in the power market literature. All these suggest that the cyclicality must be modeled with utmost care, which will lead to a more reliable analysis of the stochastic properties of the deseasonalized data and a better estimation of the future seasonality.

2.4 Spikes

The electricity market consists of supply and demand sides which are both very inelastic in nature. Neither can the power production facilities be turned on immediately at any time to produce power nor the end-users[2] are eager to save energy when the supply is at risk, owing to their long-term fixed price contracts. Hence, the exchange traded electricity prices are highly sensitive to the dynamics between the supply and demand. In case of a power plant disruption, a supply shortage threat can occur causing the prices to jump up suddenly before they normalize again after the issue is resolved. These cases are called *spikes*. Positive spikes can result from extreme political events and weather conditions as well. Not being observed as frequent as positive ones, negative spikes can also occur at times there is a surplus in the electricity supply, which cannot be reduced in a short period of time. As we mentioned previously in Section 2.1, the non-storability of electricity is perhaps the most prominent factor that can easily lead to spikes in the electricity prices. Last but not the least, spikes and the extreme volatility in the market crowds out the normality assumptions when modeling the price dynamics. Figure 2.2 shows a typical curvature of a fat-tailed distribution for the power price logarithmic returns obtained from the NYMEX electricity futures. It also turns out from the figure that the electricity prices are more likely to show positive spikes than negative spikes as the high curvature steepness observed in the upper quartiles is determined by a greater number of outliers.

[2]End-users do not purchase the electricity from exchange, rather from local or national retail power distributors at prices adjusted in longer periods.

Figure 2.2: NYMEX electricity futures log-return quantiles versus Standard Normal quantiles



2.5 Incompleteness

The occurrence of spikes in the electricity price can be translated into not only a fat-tailed distribution for the log-returns but also an incomplete market and a non-hedgeable jump risk. Together with the inability to use the spot commodity in the hedging portfolio, the non-hedgeable jump risk and, as a result, an incomplete market has far reaching implications for risk management and pricing purposes. Section 3.5.1 examines the effects of the market incompleteness on the spot price model and its stochastic properties in the context of risk-neutral pricing.

The next chapter introduces our basis model for the electricity spot price, as well as its stochastic properties which will have direct effects on the values of various electricity-related derivative contracts.

Chapter 3

Properties of the Hambly-Howison-Kluge Model

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Hambly-Howison-Kluge ([12]) employs a spot price model whose stochastic properties are driven by an Ornstein-Uhlenbeck process together with an independent mean reverting pure jump process. The statistical tractability of the model is controlled and maintained via explicitly derived moment generating functions and various approximations to the probability density function of the logarithm of the spot price process at maturity. It is then discussed how to calibrate the model under the risk-neutral measure, i.e. to the observed forward curve, and present approximations to the distributions of the electricity forward contracts with and without a delivery period, which are, in turn, used to value the call and put options written on these forwards.

3.1 The Process at the First Glance

Following [12], we define the spot electricity price model as

$$\begin{aligned} S_t &= e^{f(t)+X_t+Y_t} \\ dX_t &= -\alpha(0-X_t)dt+\sigma dW_t \\ dY_t &= -\beta(0-Y_t)dt+J_t dN_t, \end{aligned} \tag{3.1}$$

where f is a deterministic seasonality function, X_t is a standard OU process with a mean-reversion speed α and Y_t is another mean-reverting process with a Poisson stochastic jump part to incorporate spikes. The Poisson process N_t is characterized by the intensity parameter λ, and J represents a random variable for the jump size. The mean-reversion speed of the spike process is determined by β. Finally, W, N, J are assumed to be mutually independent i.i.d. processes.

One major superiority of this model over the ones given in [7] and [1] is that it allows for separate mean-reversion speeds for both the diffusive and spike parts. This is a plausible assumption particularly for the energy markets since the large price discontinuities result from extraordinary events, hence cannot be sustainable. As a result, the price should revert faster (β) than the rate at which it reverts following gradual deviations spread over longer terms (α). No-arbitrage rule guarantees us this will be the case. In remark below we give the solutions to the individual processes X_t and Y_t.

Remark 3.1.1. *The solutions to processes X_t and Y_t in 3.1 are given by*

$$X_t = X_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} dW_s, \quad Y_t = Y_0 e^{-\beta t} + \sum_{i=1}^{N(t)} e^{-\beta(t-\tau_i} J_{\tau_i} \tag{3.2}$$

Proof. By taking

$$Z(t) = X(t)e^{\alpha t}$$

and applying Itô formula to multiplication, we find

$$Z(t) = X(t)e^{\alpha t} = X(0) + \int_0^t e^{-\alpha s} dX(s) - \int_0^t \alpha e^{-\alpha s} X(s) ds.$$

Applying stochastic dynamics of X_t yields

$$\begin{aligned} Z(t) = X(t)e^{\alpha t} &= X(0) + \int_0^t e^{-\alpha s}(-\alpha(0-X(s))ds + \sigma dW(s)) - \int_0^t \alpha e^{-\alpha s} X(s) ds \\ X(t)e^{\alpha t} &= X(0) + \int_0^t e^{-\alpha s} \sigma dW(s) \\ X(t) &= X(0)e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} \sigma dW(s)) \end{aligned}$$

Then, the mean and the variance of the process X_t can be found as

$$E[X(t)|X(0) = x_0] = x_0 e^{-\alpha t}$$

$$\begin{aligned} V[X(t)|X(0) = x_0] &= E[(\int_0^t e^{-\alpha(t-s)}\sigma dW(s) - x_0 e^{-\alpha t})^2] \\ &= \int_0^t e^{-2\alpha(t-s)}\sigma^2 dW(s) \text{ (by isometry)} \\ &= \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}). \end{aligned}$$

Similarly,

$$Q(t) = Y(t)e^{\beta t}$$

yields (again by Itô formula)

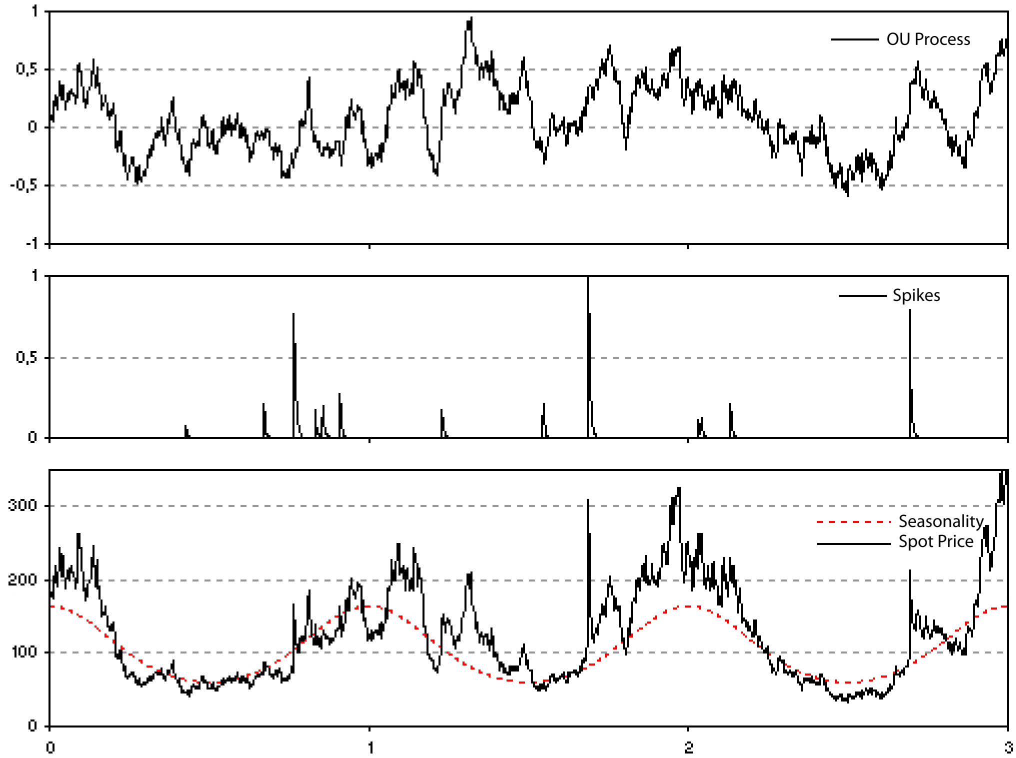
$$Y(t) = Y(0)e^{-\beta t} + \int_0^t e^{-\beta(t-s)} J_s dN(s).$$

Since $N(t)$ is a Poisson process where $dN(t)$ takes the value of one only at jump times $(\tau_1,...,\tau_{N(t)})$ in the interval $[0,t]$, the integral part of the equation can be written as a finite sum. So,

$$Y(t) = Y(0)e^{-\beta t} + \sum_{i=1}^{N(t)} e^{-\beta(t-\tau_i)} J_{\tau_i}. \qquad \square$$

As one might expect, the OU process X_t is conditionally normally distributed, given an initial condition $X(0) = x_0$. However, the distributional properties of the spike process Y_t cannot be concluded at this point. On the other hand, we intend to explore further the statistical properties of Y_t, which is equivalent to saying that we expect that the spikes will have a significant effect on the spot price process S_t, so on the values of various options including the Swing options. Hence, a well-performing approximation to the density of Y_t is needed. Also assumed is that the values $f(t)$, $X(t)$ and $Y(t)$ are individually observable at any time t. Figure 3.1 plots the simulated paths of the individual processes X and Y as well as their composition, S. The calibrated parameters of the Nord Pool market given in [12] are adopted whereas the seasonality function f is chosen arbitrarily. The next section is devoted to the investigation of the stochastic properties of the spike process in detail by means of its moment generating function, $\Phi_Y(\theta, t)$. Section 3.3 introduces a useful truncation of the spike process, which is then used to produce an explicit formula for the probability density function of the spike process, and evaluates its performance in terms of the pointwise convergence of the moment generating functions. This is followed, in Section 3.4, by the derivation of the moment generating function of the logarithm of the combined price process, S_t.

Figure 3.1: Sample paths ($\alpha = 7$, $\beta = 200$, $\lambda = 4$, $\mu_J = 0.4$, $T = 3$, $dt = 1/365$, $f(t) = \ln(100) + 0.5\cos(2\pi t)$, r (risk-free rate) $= 0$ and $X(0) = Y(0) = 0$)



3.2 The Spike Process

Lemma 3.2.1. *(Moment generating function of the spike process, Y_t) Defining by $\Phi_J(\theta) := E[\exp(\theta J)]$ the moment generating function of the jump size process, the spike process in 3.1 with $Y_0 = 0$ has the moment generating function*

$$\Phi_Y(\theta, t) := E[e^{\theta Y_t}] = e^{\lambda \int_0^t (\Phi_J(\theta e^{-\beta s}) - 1) ds}. \tag{3.3}$$

And its mean and variance are given by

$$E[Y(t)] = \frac{\lambda}{\beta} E[J](1 - e^{-\beta t}), \quad V[Y(t)] = \frac{\lambda}{2\beta} E[J^2](1 - e^{-2\beta t}).$$

Proof. Let us first define the conditional moment generating function, given the first jump is occurred at time s, then use the mutual independence of the jump sizes and the jump times (i.e. J_s, J_t and τ_i for all $s \neq t$). Then,

$$\begin{aligned}
E[e^{\theta Y_t}|\tau_1 = s] &= E[e^{\theta e^{-\beta(t-\tau_1)}J_{\tau_1}+\theta\sum_{i=2}^{N(t)} e^{-\beta(t-\tau_i)}J_{\tau_i}}|\tau_1 = s] \\
&= E[e^{\theta e^{-\beta(t-\tau_1)}J_{\tau_1}}|\tau_1 = s]E[e^{\theta\sum_{i=2}^{N(t)} e^{-\beta(t-\tau_i)}J_{\tau_i}}|\tau_1 = s] \text{ (independence)} \\
&= E[e^{\theta e^{-\beta(t-s)}J_s}]E[e^{\theta\sum_{i=2}^{N(t)} e^{-\beta(t-\tau_i)}J_{\tau_i}}|\tau_1 = s] \\
&= E[e^{\theta e^{-\beta(t-s)}J_s}]E[e^{\theta\sum_{i=1}^{N(t-s)} e^{-\beta(t-s-\tau_i)}J_{\tau_i}}|\tau_1 = s] \text{ (stationarity of Poisson)} \\
&= \Phi_J(\theta e^{-\beta(t-s)})\Phi_Y(\theta, t-s).
\end{aligned}$$

Using the property of the conditional expectation, we find

$$\begin{aligned}
\Phi_Y(\theta,t) &= E[E[e^{\theta Y_t}|\tau_1 = s]] \\
&= \int_0^t \Phi_J(\theta e^{-\beta(t-s)})\Phi_Y(\theta, t-s)\lambda e^{-\lambda s} ds \text{ (exponential PDF of } \tau) \\
&= \int_0^t \Phi_J(\theta e^{-\beta(u)})\Phi_Y(\theta, u)\lambda e^{-\lambda(t-u)} du \text{ (change of variable, } u := t-s,\ du := -ds).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\partial\Phi_Y(\theta,t)}{\partial t} &= \Phi_J(\theta e^{-\beta t})\Phi_Y(\theta,t)\lambda - \lambda\int_0^t \Phi_J(\theta e^{-\beta u})\Phi_Y(\theta,u)\lambda e^{-\lambda(t-u)} du \\
\Phi_Y'(\theta,t) &= \Phi_J(\theta e^{-\beta t})\Phi_Y(\theta,t)\lambda - \lambda\Phi_Y(\theta,t) \\
\frac{\Phi_Y'(\theta,t)}{\Phi_Y(\theta,t)} &= \lambda(\Phi_J(\theta e^{-\beta t}) - 1) \\
\ln(\Phi_Y(\theta,t)) &= \int_0^t \lambda(\Phi_J(\theta e^{-\beta s}) - 1) ds \\
\Phi_Y(\theta,t) &= e^{\lambda\int_0^t(\Phi_J(\theta e^{-\beta s})-1)ds}.
\end{aligned}$$

Then, the expectation of the spike process can be found as

$$\begin{aligned}
\frac{\partial\Phi_Y(\theta,t)}{\partial\theta} &= \Phi_Y(\theta,t)\lambda\int_0^t \Phi_J'(\theta e^{-\beta s})e^{-\beta s} ds \\
\frac{\partial\Phi_Y(\theta,t)}{\partial\theta}|_{\theta=0} &= \Phi_Y(0,t)\lambda\int_0^t \Phi_J'(0)e^{-\beta s} ds \\
E[Y(t)] &= \lambda\int_0^t E[J]e^{-\beta s} ds \text{ } (\Phi_Y(0,t) = 1) \\
&= \frac{\lambda}{\beta}E[J](1 - e^{-\beta t}).
\end{aligned}$$

Similarly, the second moment and the variance follow from

$$\begin{aligned}
\frac{\partial^2\Phi_Y(\theta,t)}{\partial\theta^2} &= \frac{\partial\Phi_Y(\theta,t)}{\partial\theta}\lambda\int_0^t \Phi_J'(\theta e^{-\beta s})e^{-\beta s}ds + \Phi_Y(\theta,t)\lambda\int_0^t \Phi_J''(\theta e^{-\beta s})e^{-2\beta s}ds \\
\{\frac{\partial^2\Phi_Y(\theta,t)}{\partial\theta^2}\}_{\theta=0} &= \{\frac{\partial\Phi_Y(\theta,t)}{\partial\theta}\}_{\theta=0}\frac{\lambda}{\beta}E[J](1-e^{-\beta t}) + \Phi_Y(0,t)\frac{\lambda}{2\beta}V[J](1-e^{-2\beta t}) \\
E[Y(t)^2] &= E[Y(t)]^2 + \frac{\lambda}{2\beta}E[J^2](1-e^{-2\beta t}) \\
V[Y(t)] &= \frac{\lambda}{2\beta}E[J^2](1-e^{-2\beta t}).
\end{aligned}$$

□

As one might expect, the long-term mean and variance are inversely proportional to the mean-reversion parameter β. At this point, the length of time that the price process will need to revert to its long-term mean after a shock occurred is expected to be too small. Hence, we also want to investigate the behavior of the moment generating function of the spike process for the asymptotic values of β and illustrate it with a concrete example assuming an exponential distribution for the jump sizes.

For this purpose, recall from Equation 3.3 that

$$\Phi_{Y_t}(\theta) = e^{\lambda\int_0^t(\Phi_J(\theta e^{-\beta s})-1)ds}.$$

Let $u := \theta e^{-\beta s}$ and accordingly $du = -\beta\theta e^{-\beta s}ds$. This yields

$$\begin{aligned}
\Phi_{Y_t}(\theta) &= e^{-\frac{\lambda}{\beta}\int_\theta^{\theta e^{-\beta t}}\frac{(\Phi_J(u)-1)}{u}du} \\
&= e^{\frac{\lambda}{\beta}\int_{\theta e^{-\beta t}}^{\theta}\frac{(\Phi_J(u)-1)}{u}du}.
\end{aligned}$$

Note that our analysis will be based on the fact that the above equation can be written as the combination of two separate integrals, i.e.

$$\Phi_{Y_t}(\theta) = e^{\frac{\lambda}{\beta}(\int_0^\theta\frac{(\Phi_J(u)-1)}{u}du - \int_0^{\theta e^{-\beta t}}\frac{(\Phi_J(u)-1)}{u}du)}. \tag{3.4}$$

Now, let's examine the value of the 2^{nd} integral above as $\beta\to\infty$. As $\beta\to\infty$ and $u\to 0$, using the Taylor expansion gives

$$\begin{aligned}
\Phi_J(u) := E[e^{uJ}] &= E\left[1 + uJ + \frac{u^2J^2}{2}\right] \quad \text{(Taylor expansion)} \\
&= 1 + E[J]u + O(u^2).
\end{aligned}$$

That is to say, for fixed values of θ and t, as $\beta\to\infty$, the aforementioned integral becomes

$$\begin{aligned}
\int_0^{\theta e^{-\beta t}} \frac{\Phi_J(u)-1}{u} du &= \int_0^{\theta e^{-\beta t}} \frac{E[J]u+O(u^2)}{u} du \\
&= \int_0^{\theta e^{-\beta t}} E[J]+O(u)du \\
&= E[J]u+O(u^2)|_0^{\theta e^{-\beta t}} \\
&= E[J]\theta e^{-\beta t}+O(\theta e^{-2\beta t}).
\end{aligned}$$

Plugging this into $\Phi_{Y_t}(\theta)$ in Equation 3.4, we end up with

$$\Phi_{Y_t}(\theta) = \exp\left(\frac{\lambda}{\beta}(\int_0^{\theta} \frac{(\Phi_J(u)-1)}{u} du - E[J]\theta e^{-\beta t} + O(\theta e^{-2\beta t}))\right). \tag{3.5}$$

The following example illustrates the behavior of the moment generating function $\Phi_{Y_t}(\theta)$ for asymptotic values of β.

Example 3.2.2. *We assume $J \sim Exp(\mu_J)$ where μ_J is the mean jump size. Then*

$$\begin{aligned}
\Phi_J(\theta) = E[e^{\theta J}] &= \int_0^{\infty} e^{\theta J}\frac{1}{\mu_J}e^{-\frac{J}{\mu_J}}dJ \\
&= \int_0^{\infty} e^{\theta J}\frac{1}{\mu_J}e^{-\frac{J}{\mu_J}}dJ \\
&= \frac{1}{1-\theta\mu_J} \quad \theta\mu_J<1
\end{aligned}$$

and the integral in Equation 3.3 turns into

$$\begin{aligned}
\Phi_Y(\theta,t) &= \exp\left(\lambda\int_0^t \left[\frac{1}{1-\theta e^{-\beta s}\mu_J}-1\right]ds\right) \\
&= \exp\left(\lambda\int_0^t \frac{e^{\beta s}}{e^{\beta s}-\theta\mu_J}ds\right)\exp\left(-\lambda\int_0^t ds\right) \\
&= \exp\left(\frac{\lambda}{\beta}ln(e^{\beta s}-\theta\mu_J)|_0^t\right)e^{-\lambda t} \\
&= \left(\frac{e^{\beta t}-\theta\mu_J}{1-\theta\mu_J}\right)^{\frac{\lambda}{\beta}}e^{-\lambda t} \\
&= \left(\frac{1-\theta\mu_J e^{-\beta t}}{1-\theta\mu_J}\right)^{\frac{\lambda}{\beta}}e^{\lambda t}e^{-\lambda t} \\
\Phi_Y(\theta,t) &= \left(\frac{1-\theta\mu_J e^{-\beta t}}{1-\theta\mu_J}\right)^{\frac{\lambda}{\beta}}, \quad \theta\mu_J<1.
\end{aligned} \tag{3.6}$$

The mean and variance follow as

$$\frac{\partial \Phi_Y(\theta,t)}{\partial \theta}|_{\theta=0} = E[Y(t)] = \frac{\lambda}{\beta}\mu_J(1-e^{-\beta t}) \text{ and } V[Y(t)] = \frac{\lambda}{2\beta}{\mu_J}^2(1-e^{-2\beta t})$$

Before examining the case where $\beta \to \infty$, we start with the stationary distribution of $Y(t)$ as $t \to \infty$. Obviously, as $t \to \infty$, $\Phi_Y(\theta,t)$ in Equation 3.6 converges to the moment generating function of the Gamma distribution with parameters λ/β and μ_J, i.e.

$$\left(\frac{1}{1-\theta\mu_J}\right)^{\frac{\lambda}{\beta}}.$$

Intuitively, (λ/β)th power of the exponential moment generating function implies that the value of the spike process at time t is determined by the sum of a number of i.i.d. jump sizes -exponential random variables in this case- which is equal to the average number of jumps, λ, adjusted by the jump reversion rate, β. To make this more clear, consider the continuous case where the time t actually denotes the semi-closed interval $(t-1,t]$. λ is the expected number of jump occurrences during a unit of time which should be adjusted by β, the speed of jump decay, to obtain the 'effective' number of jumps, each with an expected size of μ_J in the interval $(t-1,t]$. This is equivalent to saying that the distributional properties of the spike process Y_t is asymptotically determined by the sum of λ/β exponentially distributed jumps. Again, the convergence of the moment generating function of the spike process with exponential jumps to that of a Gamma distribution (i.e. the sum of exponentially distributed i.i.d. random variables) guarantees us that this is the case.

To examine the case where $\beta \to \infty$, we expand the Equation 3.6 around 0 so that it becomes

$$\Phi_Y(\theta,t) \approx 1 + \frac{\lambda}{\beta}\theta\mu_J(1-e^{-\beta t}) + \frac{\lambda}{\beta}\left(\frac{\lambda}{\beta}-1\right)(1-e^{-\beta t})(\theta\mu_J)^2$$

The exponential terms can be left out for the large values of β. Hence,

$$\begin{aligned}\Phi_Y(\theta,t) &\approx 1 + \frac{\lambda}{\beta}\theta\mu_J + \frac{\lambda}{\beta}\left(\frac{\lambda}{\beta}-1\right)(\theta\mu_J)^2 \\ &\approx 1 + \frac{\lambda}{\beta}\theta\mu_J + O\left(\frac{1}{\beta^2}\right).\end{aligned}$$

This is nothing else than the one we can obtain by expanding $(1-\theta\mu_J)^{-\lambda/\beta}$, which is the moment generating function of Gamma distribution with parameters λ/β and μ_J. Hence, Y_t's moment generating function again converges to that of $\Gamma(\lambda/\beta, \mu_J)$ for large values of β as well.

Above example encourages us in our efforts to develop a more tractable process which will replace our original spike process. We will then be able to approximate its statistical characteristics and express the density of the spike process in a more explicit way. In turn, this will lead us to develop a more efficient pricing algorithm for the Swing options as a result of a more explicit pricing formula.

3.3 Approximating the Spike Process

Instead of using inversion methods to derive the probability density function of the spike process from its moment generating function, we introduce a *truncated* spike process, $\tilde{Y}$, from which a more explicit formula for the density can be derived. The fact that for the high mean-reversion rates, β, together with small jump intensities, λ, the value of the spike process, Y_t, is mostly determined by the last jump size (see Figure 3.1) allows us to define $\tilde{Y}$ as

$$\Phi_{\tilde{Y}_t} := \begin{cases} J_{N_t} e^{-\beta(t-\tau_{N_t})} & : N_t > 0 \\ 0 & : N_t = 0 \end{cases} . \tag{3.7}$$

Assuming zero initial values for both Y and $\tilde{Y}$, an explicit formula for the maturity density $\tilde{Y}_T$ will help us form a jump-integrated pricing grid for the combined process S to numerically price a Swing option. Below we start with defining an easier form for the truncated process using the reversibility property of the Poisson process and then derive a density formula for it. Last but not the least, we make it more concrete again by considering an exponential distribution for the jump sizes J. The following definition is very helpful at this point.

Definition 3.3.1. *(Reversible process, see [25]) If the reversed process N_t^* and the original process N_t are statistically indistinguishable, we say that the process X_t is 'time reversible'. More precisely, the reversibility means that*

$$(N_{\tau_1}, N_{\tau_1}, ..., N_{\tau_1}) \sim (N_{t-\tau_1}, N_{t-\tau_2}, ..., N_{t-\tau_n})$$

for all $\tau_1, \tau_2, ..., \tau_n$ and, t, n (i.e. the shown sets of values of random variables have the same joint distributions).

With the aid of this property, one can get more insight on the properties of a process. It often allows for the derivation results simply and elegantly in cases where a direct approach might be quite complicated. Since this property is valid for the Poisson process as well, we can write the truncated process in a simpler form as the Lemma 3.3.2 below suggests.

Lemma 3.3.2. *The distribution of the following process is identical to the distribution of $\tilde{Y}_t$.*

$$V_t := \begin{cases} J_1 e^{-\beta\tau_1} & : \tau_1 \le t \\ 0 & : \tau_1 > t \end{cases} .$$

Proof. Intuitively, the reversibility of the process N_t means that an outside observer cannot tell whether a film is running in the forward or the backward direction. It implies that if $N = \{N_t; t \in R_+\}$ is a Poisson process, then $\hat{N} = \{-N_{-t}; t \in R_+\}$ is also a Poisson process. When there are jumps in the *original process* (i.e. $N_t > 0$), $t - \tau_{N_t}$ corresponds to τ_1, the first jump, in the *reversed process* (i.e. $\tau_1 \le t$). These two are identically distributed. □

The following lemma is a result of the effort to obtain the moment generating function of the new truncated process. After then, it will be shown that $\Phi_{\tilde{Y}}$ and Φ_Y converge rapidly where the difference term depreciates as a function of $1/\beta$ for large values of β.

Lemma 3.3.3. *(Moment generating function of the truncated spike process, $\tilde{Y}_t$) With an initial value of $\tilde{Y}_0 = 0$, the moment generating function of the process $\Phi_{\tilde{Y}}$ at time t is given by*

$$\Phi_{\tilde{Y}}(\theta, t) := E[e^{\theta \tilde{Y}_t}] = 1 + \lambda \int_0^t (\Phi_J(\theta e^{-\beta s}) - 1) e^{-\lambda s} ds,$$

and the moments are

$$E[\tilde{Y}(t)] = \frac{\lambda}{\beta + \lambda} E[J](1 - e^{-(\beta+\lambda)t}), \quad E[\tilde{Y}(t)^2] = \frac{\lambda}{2\beta + \lambda} E[J^2](1 - e^{-(2\beta+\lambda)t}).$$

Proof. By Lemma 3.3.2

$$\Phi_{\tilde{Y}}(\theta, t) := E[e^{\theta \tilde{Y}_t}] = E[e^{\theta V_t}]$$

Given that 1^{st} jump in the reversed process occurs at time s, we can write

$$E[e^{\theta V_t}] = E[E[e^{\theta V_t} | \tau_1 = s]].$$

The value of the conditional expectation is

$$\begin{aligned} E[e^{\theta V_t} | \tau_1 = s] &= E[e^{\theta J e^{-\beta s}} I_{\{s \le t\}}] \\ &= \Phi_J(\theta e^{-\beta s} I_{\{s \le t\}}), \end{aligned}$$

where $\tau \sim Exp(1/\lambda)$. Solving the conditional expectation to obtain $E[e^{\theta V_t}]$ yields the following:

$$\begin{aligned} \Phi_{\tilde{Y}}(\theta, t) = E[e^{\theta V_t}] &= \int_0^\infty \Phi_J(\theta e^{-\beta s} I_{\{s \le t\}}) \lambda e^{-\lambda s} ds \\ &= \int_0^t \Phi_J(\theta e^{-\beta s} I_{\{s \le t\}}) \lambda e^{-\lambda s} ds \\ &\quad + \int_t^\infty \Phi_J(\theta e^{-\beta s} I_{\{s \le t\}}) \lambda e^{-\lambda s} ds \quad {\scriptstyle (\Phi_J(\theta e^{-\beta s} I_{\{s \le t\}}) = 1 \text{ in } (t, \infty))} \\ &= \int_0^t \Phi_J(\theta e^{-\beta s}) \lambda e^{-\lambda s} ds + \int_t^\infty \lambda e^{-\lambda s} ds \\ &= \int_0^t \Phi_J(\theta e^{-\beta s}) \lambda e^{-\lambda s} ds + e^{-\lambda t}, \end{aligned}$$

and taking the $e^{-\lambda t}$ term into the first integral results in

$$\begin{aligned}\Phi_{\tilde{Y}}(\theta,t)=E[e^{\theta V_t}] &= 1+\int_0^t \Phi_J(\theta e^{-\beta s})\lambda e^{-\lambda s}-\lambda e^{-\lambda s}ds \\ &= 1+\lambda\int_0^t (\Phi_J(\theta e^{-\beta s})e^{-\lambda s}-1)e^{-\lambda s}ds.\end{aligned}$$

One can calculate the moments as below:

$$\begin{aligned}\frac{\partial\Phi_{\tilde{Y}}(\theta,t)}{\partial\theta} &= \lambda\int_0^t \frac{\partial\Phi_J(\theta e^{-\beta s})}{\partial\theta}e^{-\beta s}e^{-\lambda s}ds \\ \frac{\partial\Phi_{\tilde{Y}}(\theta,t)}{\partial\theta}|_{\theta=0} &= \lambda\int_0^t \frac{\partial\Phi_J(\theta e^{-\beta s})}{\partial\theta}|_{\theta=0}e^{-\beta s}e^{-\lambda s}ds \\ E[\tilde{Y}(t)] &= \lambda\int_0^t E[J]e^{-(\beta+\lambda)s}ds \\ &= \frac{\lambda}{\beta+\lambda}E[J](1-e^{-(\beta+\lambda)t}),\end{aligned}$$

and similarly,

$$\begin{aligned}\frac{\partial^2\Phi_{\tilde{Y}}(\theta,t)}{\partial\theta^2} &= \lambda\int_0^t \frac{\partial^2\Phi_J(\theta e^{-\beta s})}{\partial\theta^2}e^{-2\beta s}e^{-\lambda s}ds \\ \frac{\partial^2\Phi_{\tilde{Y}}(\theta,t)}{\partial\theta^2}|_{\theta=0} &= \lambda\int_0^t \frac{\partial^2\Phi_J(\theta e^{-\beta s})}{\partial\theta^2}|_{\theta=0}e^{-2\beta s}e^{-\lambda s}ds \\ E[\tilde{Y}(t)^2] &= \frac{\lambda}{2\beta+\lambda}E[J^2](1-e^{-(2\beta+\lambda)t}). \qquad \square\end{aligned}$$

Having defined the truncated spike process $\tilde{Y}$ in a simpler (reverse) form and derived its moment generating function, we can now find out whether there is a significant pointwise convergence between $\Phi_Y(\theta)$ and $\Phi_{\tilde{Y}}(\theta)$ to justify our use of the truncated process in approximating the probability density function of the original process (see Lemma 3.3.4 below). Hence, we check the convergence for two different cases, i.e. as $\lambda\to 0$ and as $\beta\to\infty$.

First, for fixed t and θ, consider the case where $\lambda\to 0$. Recall Equation 3.3 which reads

$$\Phi_Y(\theta,t)=\exp\left(\lambda\int_0^t(\Phi_J(\theta e^{-\beta s})-1)ds\right).$$

Using the Maclaurin series expansion, this can be written as

$$\Phi_Y(\theta,t) = 1 + \lambda \int_0^t (\Phi_J(\theta e^{-\beta s}) - 1)ds + O(\lambda^2).$$

Also from Lemma 3.3.3 we have

$$\Phi_{\tilde{Y}}(\theta,t) = 1 + \lambda \int_0^t (\Phi_J(\theta e^{-\beta s}) - 1)e^{-\lambda s}ds.$$

Again applying the expansion rule to the term $e^{-\lambda s}$ around 0, we find

$$\begin{aligned}\Phi_{\tilde{Y}}(\theta,t) &= 1 + \lambda \int_0^t (\Phi_J(\theta e^{-\beta s}) - 1)(1 + O(\lambda))ds \\ &= 1 + \lambda \int_0^t (\Phi_J(\theta e^{-\beta s}) - 1)ds + O(\lambda^2).\end{aligned}$$

Thus, the convergence of the moment generating functions as $\lambda \to 0$ is obvious. Now let us consider the case $\beta \to \infty$ where λ, θ and t are fixed this time. Recall Equation 3.5 this time:

$$\Phi_{Y_t}(\theta) = \exp\left(\frac{\lambda}{\beta}(\int_0^\theta \frac{(\Phi_J(u) - 1)}{u}du - E[J]\theta e^{-\beta t} + O(\theta e^{-2\beta t}))\right).$$

Leaving out exponential terms inside the parentheses and applying the Taylor series expansion yield

$$\Phi_{Y_t}(\theta) = 1 + \frac{\lambda}{\beta}\int_0^\theta \frac{(\Phi_J(u) - 1)}{u}du + O\left(\frac{1}{\beta^2}\right).$$

Now, we define $u := \theta e^{-\beta s}$ in $\Phi_{\tilde{Y}_t}(\theta)$ given in Lemma 3.3.3 with $du = -\beta\theta e^{-\beta s}ds$. Rearranging the equation gives

$$\begin{aligned}\Phi_{\tilde{Y}_t}(\theta) &= 1 + \lambda \int_0^t (\Phi_J(\theta e^{-\beta s}) - 1)e^{-\lambda s}ds \\ &= 1 - \frac{\lambda}{\beta}\int_\theta^{\theta e^{-\beta t}} \frac{(\Phi_J(u) - 1)}{u}(\frac{u}{\theta})^{\frac{\lambda}{\beta}}du \\ &= 1 + \frac{\lambda}{\beta}\int_{\theta e^{-\beta t}}^{\theta} \frac{(\Phi_J(u) - 1)}{u}(\frac{u}{\theta})^{\frac{\lambda}{\beta}}du.\end{aligned}$$

Note that as $\beta \to \infty$, except the ball $u = B(\theta e^{-\frac{\beta}{\lambda}})$, and for the small values of λ, $(\frac{u}{\theta})^{\frac{\lambda}{\beta}} \to 1$. That is to say, its contribution to the integral value is very small when large mean-reversion rates and small jump intensities are the case, hence, negligible. Similarly, as $\beta \to \infty$, the lower limit of the integral can be changed to 0 with a small error. Thus we end up with

$$\Phi_{\tilde{Y}_t}(\theta) = 1 + \frac{\lambda}{\beta} \int_{\theta e^{-\beta t}}^{\theta} \frac{(\Phi_J(\theta e^{-\beta s}) - 1)}{u} du + o(\frac{1}{\beta}) = \Phi_{Y_t}(\theta) + O(\frac{1}{\beta}). \tag{3.8}$$

Equation 3.8 clearly suggests that the process $\tilde{Y}$ is a good approximation to Y. To finalize this section, below we derive an explicit formula for the density of the truncated spike process $\tilde{Y}$ which will substitute the density of the original process in our Swing option pricing algorithm.

Lemma 3.3.4. *(Distribution of the truncated-reversed spike process) Assume that the jump size distribution has the density f_J. Then, the cumulative distribution function of the process $\tilde{Y}$ is*

$$F_{\tilde{Y}_t}(x) = e^{-\lambda t} I_{x \geq 0} + \int_{-\infty}^{x} f_{\tilde{Y}_t}(y) dy, \ \ t \geq 0$$

with the density

$$f_{\tilde{Y}_t}(x) = \frac{\lambda}{\beta} \frac{1}{|x|^{1-\lambda/\beta}} \left| \int_{x}^{x e^{\beta t}} f_J |y|^{-\lambda/\beta} dy \right|, \ \ x \neq 0.$$

Proof. Recall from Lemma 3.3.2 that the distribution of the truncated spike process, $\tilde{Y}_t$, is identical to V_t. Therefore, in our calculations so far, we've taken V_t as our new truncated spike process. Let us first decompose $\tilde{Y}_t$ into two parts, i.e. defining $Q := e^{-\beta\tau}$ the equation

$$\tilde{Y}_t = J e^{-\beta\tau} I_{\tau \leq t}, \ \ \tau \sim \mathrm{Exp}(1/\lambda)$$

reduces to

$$\tilde{Y}_t = JQI_{\tau \leq t}.$$

Now, consider the fact that if the random variable (.) has a standard uniform distribution in the domain $[0, 1]$, then $-\ln(.)/\lambda$ has an exponential distribution with the intensity parameter λ (and the duration parameter $1/\lambda$). We can use this property to show that Q is the (β/λ)th power of a uniformly distributed random variable. In fact, $Q^{\lambda/\beta} = e^{-\lambda\tau}$, and $-ln(e^{-\lambda\tau})/\lambda = \tau$ is, by definition, exponentially distributed. Now, we know that $Q^{\lambda/\beta}$ is uniformly distributed in the domain $[0, 1]$.

Then using this property to determine the distribution function of Q, we can write

$$\begin{aligned}
P(Q \leq q) &= P(e^{-\beta\tau} \leq q) \\
&= P((e^{-\lambda\tau})^{\frac{\beta}{\lambda}} \leq q) \\
&= P((e^{-\lambda\tau}) \leq q^{\frac{\lambda}{\beta}}) \\
&= P((e^{-\lambda\tau}) \leq q^{\frac{\lambda}{\beta}}) = q^{\frac{\lambda}{\beta}} \quad \text{(uniform PDF)} \\
F_Q(q) &= q^{\frac{\lambda}{\beta}} \quad q \in [0,1] \\
f_Q(q) &= \frac{\lambda}{\beta} q^{-(1-\frac{\lambda}{\beta})} I_{q\in[0,1]},
\end{aligned}$$

where $I_{q\in[0,1]}$ is the indicator function which results from the relationship

$$\begin{gathered}
0 \leq e^{-\lambda\tau} \leq 1 \\
0 \leq (e^{-\lambda\tau})^{\frac{\beta}{\lambda}} \leq 1 \\
0 \leq q \leq 1.
\end{gathered}$$

Now, we need to determine $f_{QI_{\tau\leq t}}(q)$. It is obvious that $I_{\tau\leq t}$ is equivalent to $I_{q\in[e^{-\beta t},1]}$ since

$$\tau \leq t \Rightarrow e^{\tau} \leq e^{t} \Rightarrow e^{\beta\tau} \leq e^{\beta t} \Rightarrow e^{-\beta\tau} \geq e^{-\beta t} \Rightarrow q \geq e^{-\beta t}.$$

Then $f_{QI_{\tau\leq t}}(q)$ can be written as

$$f_{QI_{\tau\leq t}}(q) = \frac{\lambda}{\beta} q^{-(1-\frac{\lambda}{\beta})} I_{q\in[e^{-\beta t},1]}.$$

In deriving $F_{QI_{\tau\leq t}}(q)$, one should be aware of the fact that the cumulative distribution function should always sum up to one. That is, even if the probability density function is restricted to a specific interval, i.e. $q \in [e^{-\beta t}, 1]$ in our case -which is a part of the interval $[0,1]$ where q takes values-, the cumulative distribution function should also account for the interval which is out of our interest[1], i.e. $q \in [0, e^{-\beta t}]$. In cumulative distribution function terms, this means that we should separately include the cumulative probability of q being smaller than $e^{-\beta t}$, given $q \geq 0$. Then

$$\begin{aligned}
F_{QI_{\tau\leq t}}(q) &= P(q \leq e^{-\beta t}) I_{q\geq 0} + \int_{-\infty}^{q} f_{QI_{\tau\leq t}}(\xi) d\xi \\
&= P(e^{-\beta\tau} \leq e^{-\beta t}) I_{q\geq 0} + \int_{-\infty}^{q} \frac{\lambda}{\beta} \xi^{-(1-\frac{\lambda}{\beta})} I_{\xi\in[e^{-\beta t},1]} d\xi \\
&= P(\tau \geq t) I_{q\geq 0} + \int_{-\infty}^{q} \frac{\lambda}{\beta} \xi^{-(1-\frac{\lambda}{\beta})} I_{\xi\in[e^{-\beta t},1]} d\xi
\end{aligned}$$

[1]It is out of interest since it implies $t \leq \tau$. In other words, no spikes occurred yet.

Since $P(\tau \geq t) = e^{-\lambda t}$, we conclude that

$$F_{QI_{\tau\leq t}}(q) = e^{-\lambda t} I_{q\geq 0} + \int_{-\infty}^{q} \frac{\lambda}{\beta} \xi^{-(1-\frac{\lambda}{\beta})} I_{\xi\in[e^{-\beta t},1]} d\xi.$$

Below, we refer to the multiplication theorem given in [11] to derive $f_{JQI_{\tau\leq t}}(q)$ and $F_{JQI_{\tau\leq t}}(q)$, namely the probability density function and the cumulative distribution function of the truncated spike process.

Theorem 3.3.5. *(The distribution of the product of two independent random variables) Let us have two random variables, namely X and Y*[2]*. The probability density function of X, $f(x)$ is defined on (a,b) while that of Y, $g(y)$, on (c,d). Given that $0 < a < b < \infty$ and $0 < c < d < \infty$, the probability density function of the product of $V = XY$ is*

$$h(\vartheta) = \begin{cases} \int_{a}^{\vartheta/c} g(\frac{\vartheta}{x}) f(x) \frac{1}{x} dx & : ac < \vartheta < ad \\ \int_{\vartheta/d}^{\vartheta/c} g(\frac{\vartheta}{x}) f(x) \frac{1}{x} dx & : ad < \vartheta < bc \\ \int_{\vartheta/d}^{b} g(\frac{\vartheta}{x}) f(x) \frac{1}{x} dx & : bc < \vartheta < bd \end{cases}$$

if $ad < bc$,

$$h(\vartheta) = \begin{cases} \int_{a}^{\vartheta/c} g(\frac{\vartheta}{x}) f(x) \frac{1}{x} dx & : ac < \vartheta < bc \\ \int_{a}^{b} g(\frac{\vartheta}{x}) f(x) \frac{1}{x} dx & : bc < \vartheta < ad \\ \int_{\vartheta/d}^{b} g(\frac{\vartheta}{x}) f(x) \frac{1}{x} dx & : ad < \vartheta < bd \end{cases}$$

if $ad > bc$,

$$h(\vartheta) = \begin{cases} \int_{a}^{\vartheta/c} g(\frac{\vartheta}{x}) f(x) \frac{1}{x} dx & : ac < \vartheta < ad \\ \int_{\vartheta/d}^{b} g(\frac{\vartheta}{x}) f(x) \frac{1}{x} dx & : bc < \vartheta < bd \end{cases}$$

if $ad = bc$.

Proof. (OUTLINE) Define $K = X$ and $L = XY$. Let u, the forward mapping, and w, the reverse mapping, be the mappings between $\mathcal{A} = \{(x,y) | a < x < b, c < y < d\}$ and $\mathcal{B} = \{(k,l) | a < k < b, ck < l < dk\}$. Then

$$u_1(x,y) = x = k \quad u_2(x,y) = xy = l w_1(k,l) = k = x \quad w_2(k,l) = \frac{l}{k} = y.$$

[2] Note that these variables are completely different than our individual processes included in the spot price model.

The Jacobian matrix and its determinant is given by

$$\mathcal{J} = \begin{vmatrix} \frac{\partial w_1}{\partial k} & \frac{\partial w_1}{\partial l} \\ \frac{\partial w_2}{\partial k} & \frac{\partial w_2}{\partial l} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\frac{l}{k^2} & \frac{1}{k} \end{vmatrix} = \frac{1}{k}$$

Then, the *joint* PDF $f_{K,L}(k,l)$ can be found as

$$\begin{aligned} f_{K,L}(k,l) &= f(w_1(k,l))g(w_2(k,l))|\mathcal{J}| \quad {\scriptstyle |\mathcal{J}| \text{ is the absolute value of the determinant}} \\ &= f(k)g(\frac{l}{k})\frac{1}{|k|}. \end{aligned}$$

Integrating $f_{K,L}(k,l)$ w.r.t. k over appropriate intervals yields the result. □

The first case, $ad < bc$, of the above theorem matches our case where $f_J(j)$ is defined on $(-\infty, \infty)$, whereas $f_{QI_{\tau\leq t}}(q)$ is given to be in the interval $[e^{-\beta t}, 1]$. Defining $K = J$ and $L = QI_{\tau\leq t}$ and the mappings

$$u_1(j, qI_{\tau\leq t}) = j = k \;\; u_2(j, qI_{\tau\leq t}) = qI_{\tau\leq t} = lw_1(k,l) = k = j \;\; w_2(k,l) = \frac{l}{k} = qI_{\tau\leq t},$$

the determinant of the Jacobian matrix can similarly be found as $\mathcal{J} = 1/k$. Then the joint probability density function $f_{K,L}(k,l)$ is

$$\begin{aligned} f_{K,L}(k,l) &= f_J(w_1(k,l))f_{QI_{\tau\leq t}}(w_2(k,l))|\mathcal{J}| \\ &= f_J(k)f_{QI_{\tau\leq t}}(\frac{l}{k})\frac{1}{|k|}. \end{aligned}$$

Since $ac = ad = -\infty$ and $bc = bc = \infty$ in our case, l takes values only in the interval (ad, bc), or $(-\infty, \infty)$, which means that the joint probability density function can be integrated w.r.t. k only over the integral $(-\infty, \infty)$. Thus,

$$f_{JQI_{\tau\leq t}}(x) = (f_L(l)) = \int_{-\infty}^{\infty} f_J(\gamma)f_{QI_{\tau\leq t}}\left(\frac{x}{\gamma}\right)\frac{1}{|\gamma|}d\gamma.$$

Similarly,

$$F_{JQI_{\tau\leq t}}(x) = (F_L(l)) = e^{-\lambda t}I_{x\geq 0} + \int_{-\infty}^{x} f_{JQI_{\tau\leq t}}(\xi)d\xi.$$

Finally, we need to calculate $f_{QI_{\tau\leq t}}(x/\gamma)$ to make the density function more meaningful. Using the density $f_{QI_{\tau\leq t}}$ calculated above, it can be found as

$$\begin{aligned} f_{QI_{\tau\leq t}}(\frac{x}{\gamma}) &= \frac{\lambda}{\beta}(\frac{x}{\gamma})^{-(1-\frac{\lambda}{\beta})}I_{\frac{x}{\gamma}\in[e^{-\beta t},1]} \\ &= \frac{\lambda}{\beta}\frac{1}{x^{1-\frac{\lambda}{\beta}}}I_{\gamma\in[x,xe^{\beta t}]}\gamma^{1-\frac{\lambda}{\beta}}, \quad x > 0 \\ &= \frac{\lambda}{\beta}\frac{1}{|x|^{1-\frac{\lambda}{\beta}}}I_{\gamma\in[x,xe^{\beta t}]}|\gamma|^{1-\frac{\lambda}{\beta}}, \quad x < 0. \end{aligned}$$

Important note. γ and x in the last equation array are the integrators of the random variables J, the jump size, and JQ, the multiplication of the two random variables, respectively. This implies that x can be negative, but only if the jump size is negative since the variable Q is defined in $[0,1]$. This explains why we accordingly took the absolute value of γ, the jump size integrator, as well as x, when $x<0$ is the case. □

In the example below, we illustrate a special case for the density function, where the jump sizes are again exponentially distributed.

Example 3.3.6. *(Exponential jump size distribution) Assume again that* $J \sim \mathrm{Exp}(\mu_J)$. *Then* $f_{\tilde{Y}_t}(x) = f_{JQI_{\tau\leq t}}(x)$ *is*

$$f_{\tilde{Y}_t}(x) = \frac{\lambda}{\beta {\mu_J}^{\lambda/\beta}} \frac{\Gamma(1-\lambda/\beta, y/\mu_J) - \Gamma(1-\lambda/\beta, ye^{\beta t}/\mu_J)}{y^{1-\lambda/\beta}}, \; y > 0. \tag{3.9}$$

Proof. As $J \sim \mathrm{Exp}(\mu_J)$, now we know that the random variables J and JQ take positive values. Referring to the equations $f_{JQI_{\tau\leq t}}(x)$ and $f_{QI_{\tau\leq t}}(\frac{x}{\gamma})$ above, and substituting for the given terms yields

$$\begin{aligned} f_{JQI_{\tau\leq t}}(x) &= \int_{-\infty}^{\infty} \frac{1}{\mu_J} e^{-\frac{\gamma}{\mu_J}} \frac{\lambda}{\beta} \frac{1}{(x)^{1-\frac{\lambda}{\beta}}} I_{\gamma\in[x, xe^{\beta t}]} \gamma^{1-\frac{\lambda}{\beta}} \frac{1}{\gamma} d\gamma \\ &= \int_{x}^{xe^{\beta t}} \gamma^{-\frac{\lambda}{\beta}} e^{-\frac{\gamma}{\mu_J}} \frac{\lambda}{\beta \mu_J (x)^{1-\frac{\lambda}{\beta}}} d\gamma \\ &= \int_{\frac{x}{\mu_J}}^{\frac{xe^{\beta t}}{\mu_J}} \hat{\gamma}^{-\frac{\lambda}{\beta}} {\mu_J}^{-\frac{\lambda}{\beta}} e^{-\hat{\gamma}} \frac{\lambda}{\beta (x)^{1-\frac{\lambda}{\beta}}} d\hat{\gamma} \quad \hat{\gamma} = \tfrac{\gamma}{\mu_J} \text{ and } d\hat{\gamma} = \tfrac{d\gamma}{\mu_J} \\ &= \int_{\frac{x}{\mu_J}}^{\infty} \hat{\gamma}^{-\frac{\lambda}{\beta}} {\mu_J}^{-\frac{\lambda}{\beta}} e^{-\hat{\gamma}} \frac{\lambda}{\beta (x)^{1-\frac{\lambda}{\beta}}} d\hat{\gamma} - \int_{\frac{xe^{\beta t}}{\mu_J}}^{\infty} \hat{\gamma}^{-\frac{\lambda}{\beta}} {\mu_J}^{-\frac{\lambda}{\beta}} e^{-\hat{\gamma}} \frac{\lambda}{\beta (x)^{1-\frac{\lambda}{\beta}}} d\hat{\gamma} \\ &= \frac{\lambda}{\beta (x)^{1-\frac{\lambda}{\beta}}} {\mu_J}^{-\frac{\lambda}{\beta}} \left(\int_{\frac{x}{\mu_J}}^{\infty} \hat{\gamma}^{-\frac{\lambda}{\beta}} e^{-\hat{\gamma}} d\hat{\gamma} - \int_{\frac{xe^{\beta t}}{\mu_J}}^{\infty} \hat{\gamma}^{-\frac{\lambda}{\beta}} e^{-\hat{\gamma}} d\hat{\gamma} \right). \end{aligned}$$

The integrals in the last line are referred to as the *upper incomplete gamma functions* in the literature and this special function is defined as

$$\Gamma(s,x) = \int_{x}^{\infty} \hat{\gamma}^{s-1} e^{-\hat{\gamma}} d\hat{\gamma}.$$

Then, $f_{JQI_{\tau\leq t}}(x)$ can be rewritten as

$$f_{JQI_{\tau\leq t}}(y) = \frac{\lambda}{\beta y^{1-\frac{\lambda}{\beta}}} {\mu_J}^{-\frac{\lambda}{\beta}} \left(\Gamma(1-\lambda/\beta, y/\mu_J) - \Gamma(1-\lambda/\beta, ye^{\beta t}/\mu_J) \right). \qquad \square$$

In pricing the Swing options, this result will be represented by an approximation which fits very well the exact density for typical market parameters. Now we turn to our main focus for this section, the complete spot price process S_t, to conclude our discussion about the stochastic properties of the electricity spot price model. The results so far will intensively be exploited in Section 3.5 as well as Chapter 4 where the valuation issues of various energy options are addressed.

3.4 The Combined Process Revisited

Note that although the processes X_t and Y_t satisfy the Markov property individually, the combined process S_t does not. Therefore, throughout the analysis above, we assumed X_t, Y_t and $f(t)$ components of the spot price process S_t are individually observable. Turning back to the properties of the price process $S_t = \exp(f(t) + X_t + Y_t)$, we give the expectation of S_t as a special case in the proof of the following theorem.

Theorem 3.4.1. *The logarithm of the spot price process S_t defined in Equation 3.1, with X_0 and Y_0 given, has the moment generating function*

$$\Phi_{lnS}(\theta, t) := E[e^{\theta lnS_t}] = e^{\theta f(t) + \theta X_0 e^{-\alpha t} + \theta^2 \frac{\sigma^2}{4\alpha}(1-e^{-2\alpha t}) + \theta Y_0 e^{-\beta t} + \lambda \int_0^t (\Phi_J(\theta e^{-\beta s}) - 1) ds}. \tag{3.10}$$

Proof. The mutual independence between the Ornstein-Uhlenbeck process, X_t, and the spike process, Y_t, allows us to write,

$$\begin{aligned} E[e^{\theta lnS_t}] &= e^{f(t)} E[e^{\theta X_t} | X_0 = X_0] E[e^{\theta Y_t} | Y_0 = Y_0] \\ &= e^{f(t)} e^{\theta X_0 e^{-\alpha t}} e^{\theta Y_0 e^{-\beta t}} E[e^{\theta X_t} | X_0 = 0] E[e^{\theta Y_t} | Y_0 = 0]. \end{aligned}$$

Notice that $E[e^{\theta Y_t}|Y_0 = 0]$ follows from Equation 3.3. Besides, the distributional properties of X_t implies

$$E[e^{\theta X_t} | X_0 = 0, X_t = x] = \int_{-\infty}^{\infty} e^{\theta x} \frac{1}{(\pi \frac{\sigma^2}{\alpha}(1-e^{-2\alpha t}))^{0.5}} \exp\left(-\frac{x^2}{\frac{\sigma^2}{\alpha}(1-e^{-2\alpha t})} \right) dx,$$

which can be solved to find

$$
\begin{aligned}
E[e^{\theta X_t}|X_0=0,X_t=x] &= \int_{-\infty}^{\infty} \frac{1}{(\pi\frac{\sigma^2}{\alpha}(1-e^{-2\alpha t}))^{0.5}} \\
&\quad \exp\left(-\frac{(x^2-2x(\frac{\theta\sigma^2}{2\alpha}(1-e^{-2\alpha t})))}{\frac{\sigma^2}{\alpha}(1-e^{-2\alpha t})}\right)dx \\
&= \frac{(\frac{\theta\sigma^2}{2\alpha}(1-e^{-2\alpha t}))^2}{\frac{\sigma^2}{2\alpha}(1-e^{-2\alpha t})} \int_{-\infty}^{\infty} \frac{1}{(\pi\frac{\sigma^2}{\alpha}(1-e^{-2\alpha t}))^{0.5}} \\
&\quad \exp\left(-\frac{(x-\frac{\theta\sigma^2}{2\alpha}(1-e^{-2\alpha t}))^2}{\frac{\sigma^2}{\alpha}(1-e^{-2\alpha t})}\right)dx \\
&= \theta^2\frac{\sigma^2}{4\alpha}(1-e^{-2\alpha t}).
\end{aligned}
$$

Then, recalling $E[e^{\theta lnS_t}]$ above, we can write

$$
\begin{aligned}
E[e^{\theta lnS_t}] &= e^{f(t)}e^{\theta X_0 e^{-\alpha t}}e^{\theta Y_0 e^{-\beta t}}E[e^{\theta X_t}|X(0)=0]E[e^{\theta Y_t}|Y(0)=0] \\
&= \exp(f(t)+\theta X_0 e^{-\alpha t}+\theta^2\frac{\sigma^2}{4\alpha}(1-e^{-2\alpha t})+\theta Y_0 e^{-\beta t} \\
&\quad +\lambda\int_0^t(\Phi_J(\theta e^{-\beta s})-1)ds)
\end{aligned}
$$

which completes the proof. Now, it is simple to find an expression for $E[S_t]$ by setting θ to 1. This yields

$$
E[S_t]=\exp(f(t)+X_0e^{-\alpha t}+\frac{\sigma^2}{4\alpha}(1-e^{-2\alpha t})+Y_0e^{-\beta t}+\lambda\int_0^t(\Phi_J(e^{-\beta s})-1)ds). \quad \square
$$

3.5 Forward Dynamics and the Options on Forwards with or without a Delivery Period

3.5.1 Risk-Neutral World and Option Pricing

No matter what type of risk attitudes different investors have, the forward prices should perfectly reflect today's common expectations about the future, i.e. a risk-free return on the asset. This implies the necessity of one or more probability measure(s) which is(are) risk-adjusted. The existence, but not necessarily the uniqueness, of the risk-neutral measure makes us believe that the market conforms to no-arbitrage rule. If it is also unique, then the market is complete, meaning that any payoff can be replicated using the traded assets. Assuming a separate spike process Y_t for our spot price model is a result of the existence of a non-hedgeable jump risk in the electricity markets, which cannot be hedged away through traditional hedging methods. Not only are the electricity markets incomplete due to spike risk, but also the electricity itself is characterized by the difficulties in its storage. Hence, investors cannot hold the electricity physically (no pun intended) as a hedging asset against contingent claims written on it. This implies that one cannot simply assume that the discounted price process $\tilde{S}$ is martingale under any risk-neutral measure Q in the set of possible risk-neutral measures. That is, $F(t,T) = E^Q[S_T|\mathcal{F}_t]$ is not necessarily of the simple shape $e^{r(T-t)}$ any more.

However, derivatives traded in the market have to satisfy certain consistency conditions that the spot commodity does necessarily not. Any other scenario could lead to arbitrage opportunities which can be exploited by setting up an appropriate derivatives portfolio. Financial markets cannot continue to exist in the presence of risk-free excess returns. Hence, the arbitrage opportunities should be eliminated and there should exist at least a set of equivalent risk-neutral measures, Q, which may not be determined uniquely but still keep the market arbitrage-free. The existence of Q is equivalent to the existence of a unique price for every single derivative eligible for trade. Given there is a sufficiently large number of derivatives traded in the market, the parameters of the spot price model can be calibrated to the observed prices of these derivatives and they can be used to replicate various financial products tailored to the needs of the energy investors.

The Measure Q

As it is reasonable, we only consider a subset of equivalent measures which leave the structure of the jump process unchanged, i.e. jumps will still be generated by a Poisson process and an independent jump size distribution under the measure Q. This restriction imposed on the set of possible risk-neutral measures might limit the range of the arbitrage free prices we will obtain for certain derivatives. This limitation will be offset by the increased manageability of the stochastic dynamics under the risk-neutral measure.

We define an equivalent measure, Q, by means of the Radon-Nikodyn derivative, L, such that $dQ = L_T dP$. The stochastic dynamics of the state price process, $(L_t)_{t\in[0,T]}$, are given by the equation

$$\frac{dL_t}{L_{t-}} = -\lambda\gamma(t)dt - \theta(X_t,t)dW_t + \gamma(t)dN_t, \quad \text{Ł}_0 = 1.$$

Here θ can be called as the market price of diffusion risk and γ as the market price of jump risk ($\gamma + 1 > 0$). We divide the measure change process into two parts, namely, (1) the change of measure for a Brownian motion with drift and (2) the change of measure for a compound Poisson process. The latter can affect both the intensity and the distribution of the jump sizes, however, we ignore this possibility and prefer to keep the model structure unchanged.

The following theorem is useful to handle both (1) and (2) above.

Theorem 3.5.1. *For the independent Brownian and compound Poisson processes, W_t and C_t, respectively, the process*

$$W_t^Q = W_t + \int_0^t \theta_s ds$$

is a Brownian motion, $C_t^Q = \sum_{i=1}^{N(t)} \hat{J}_i$ is a compound Poisson process under the probability measure Q with the intensity $\hat{\lambda}$ and i.i.d. jump sizes having the density $\hat{f}(j)$, and the processes W_t^Q and C_t^Q are again independent.

Proof. See Theorem 11.6.9 in [23] for the proof of this theorem. □

Then, based on the Girsanov Theorem and the fact that J_t, N_t and W_t are mutually independent, the dynamics under Q measure can be written as

$$\begin{aligned} S_t &= e^{f(t)+X_t+Y_t} \\ dX_t &= (-\alpha X_t - \theta(X_t, t)\sigma)dt + \sigma dW_t^Q \\ dY_t &= -\beta Y_{t-}dt + \hat{J}_t dN_t^Q \end{aligned}$$

It is obvious that the new intensity $\hat{\lambda}$ for the jumps under the measure Q should be of the form $\lambda(1+\gamma(t))$ implying that the market price of the jump risk is included in the new intensity. This assures that the process is adjusted for the jump risk. In order to keep the model in a similar shape, $\gamma(t)$ is defined as the increase in the number of jumps which is proportional to $(\hat{\lambda} - \lambda)/\lambda$. In accordance with our last statement, we set the market price of jump risk, $\gamma(t)$, as

$$\gamma(t) = \frac{\hat{\lambda}}{\lambda} - 1.$$

We also define the market price of diffusion risk, θ, as

$$\theta(x, t) = \frac{\hat{\alpha} - \alpha}{\sigma} x - \frac{\hat{\alpha}}{\sigma} \mu(t)$$

so that the dynamics of X_t and Y_t become

$$\begin{aligned} dX_t &= \hat{\alpha}(\mu(t) - X_t)dt + \sigma dW_t^Q \\ dY_t &= -\beta Y_{t-}dt + \hat{J}_t dN_t^Q, \end{aligned}$$

where N_t^Q has the density $\hat{\lambda}$.

Finally, we arrange X_t above to express $\mu(t)$, the time-varying long-term mean to which the spot price process reverts, as an additional term in the seasonality part. To achieve this, consider the solutions to two different OU processes: one with a long-term mean $\mu(t)$, X_t^μ, and the other with a time-varying long-term mean of zero, X_t^0. The difference between the solutions to these two stochastic differential equations will definitely be the drift term

$$d_t = \hat{\alpha} \int_0^t e^{-\hat{\alpha}(t-s)} \mu(s) ds.$$

Note that this is the solution, $\tilde{f}(t)$, of a first-order ordinary differential equation of the form

$$\tilde{f}^{'}(t) + \hat{\alpha}\tilde{f}(t) = \hat{\alpha}\mu(t).$$

This implies that $\mu(t)$ in the above stochastic process for X_t can be considered as a separate seasonality function $\tilde{f}(t)$ and added to our original seasonality function $f(t)$, turning it into $\hat{f}(t)$, which will be the sum $f(t) + \tilde{f}(t)$. And now the process X_t again has a long-term mean of zero with a shifted seasonality function $\hat{f}(t)$. Expressing the long-term mean separately provides us with a model shape which is similar to the one considered under the P measure. Hence, the fact that adding a deterministic function to a OU process results in another OU process guarantees us that the shape of our original model can still be left unchanged (i.e. a spot price model which is the exponential of the sum of a deterministic seasonality function, a mean-reverting pure OU process and another mean-reverting jump process).

New Dynamics under the Measure Q

The calculations in the previous section implies that the spot electricity price model under the risk-neutral measure Q will look like

$$\begin{aligned} S_t &= e^{\hat{f}(t)+X_t+Y_t} \\ dX_t &= -\hat{\alpha}X_t dt + \sigma dW_t^Q \\ dY_t &= -\beta Y_{t-}dt + \hat{J}_t dN_t^Q. \end{aligned}$$

Here N_t^Q has the intensity $\hat{\lambda}$ under Q. The parameters $\hat{\alpha}$, $\hat{\lambda}$, the function $\hat{f}$ as well as the distribution of jump sizes, $\hat{J}$, are all determined by the particular choice of the measure Q. Only the parameters σ and β remain unchanged under the new risk-neutral measure. Note that the drift of the process Y_t under Q is likely to be different from the one under P. This is simply because the Poisson processes N_t and N_t^Q are not a martingales and they will contribute to the common $-\beta Y_{t-}dt$ term above with two different drift terms, i.e. λdt and $\hat{\lambda} dt$.

For simplicity of notation, the same parameters will be used as the ones in the original model given in Equation 3.1. But, again note that they might differ from the parameters under P, the real-world measure. Therefore, from now on, $f(t)$, α, W_t, J_t, N_t and λ will substitute in notation for their risk-neutral counterparts $\hat{f}(t)$, $\hat{\alpha}$, W_t^Q, $\hat{J}_t$, N_t^Q and $\hat{\lambda}$, and our model of interest will be

$$\begin{aligned} S_t &= e^{f(t)+X_t+Y_t} \\ dX_t &= -\alpha X_t dt + \sigma dW_t \\ dY_t &= -\beta Y_{t-}dt + J_t dN_t. \end{aligned} \tag{3.11}$$

Calibration

It is strongly argued in [19] and [17] that the model-implied forward prices are dominated by the seasonality effect and that the seasonal patterns are one of the most important aspects in the shape of the forward curve. Hence, the modeling of the seasonality function with great care would, in turn, bring two main advantages: (1) the rest of the model parameters could be calibrated more effectively to the deseasonalized historical data and, (2) a more realistic seasonality function which is consistent with and better reflect the forward price dynamics could be extracted from the forward curve. The following lemma can be used to derive a risk-adjusted seasonality curve using the calibrated model parameters and the observed forward curve.

Lemma 3.5.2. *(Seasonal function consistent with the forward curve) The risk-neutral seasonality function is given by the equation*

$$f(T) = F_0^{[T]} - X_0 e^{-\alpha T} - \frac{\sigma^2}{4\alpha}(1 - e^{-2\alpha T}) - Y_0 e^{-\beta T} - \lambda \int_0^T (\Phi_J(e^{-\beta s}) - 1)ds, \quad (3.12)$$

where $F_0^{[T]}$ *is the forward price at time* 0 *which matures at time* T*.*

Proof. Using Theorem 3.4.1 completes the proof, such that

$$\begin{aligned} F_0^{[T]} &= E^Q[S_T|\mathcal{F}_0] = E^Q[S_T] \\ &= E^Q[e^{Z_T}] \\ &= e^{f(T)+X_0 e^{-\alpha T}+\frac{\sigma^2}{4\alpha}(1-e^{-2\alpha T})+Y_0 e^{-\beta T}+\lambda \int_0^T (\Phi_J(e^{-\beta s})-1)ds} \\ \ln F_0^{[T]} &= f(T) + X_0 e^{-\alpha T} + \frac{\sigma^2}{4\alpha}(1 - e^{-2\alpha T}) + Y_0 e^{-\beta T} + \lambda \int_0^T (\Phi_J(e^{-\beta s}) - 1)ds. \square \end{aligned}$$

What we have so far is an incomplete model, which implies the need for a set of liquid derivatives traded in the market. Given that a continuous, observable forward curve, F_0^T for all $T \in [0, T^*]$, exists in the market, all parameters except the seasonality function, $f(t)$, can be calibrated to historical data. Various interpolation methods can be used to obtain an optimal continuous forward curve. The calibration methodology we've just mentioned can be achieved by following the steps below (some of them recursively):

1. $f(t)$ is derived from the real-world measure (i.e. it is calibrated to the historical data)

2. Seasonality is extracted form the historical data

3. The mean-reversion rate α and the volatility σ are calibrated to the seasonality-extracted data set

4. Jumps are detected from this reduced data set
5. Steps (3) and (4) are performed recursively
6. Having determined all parameters, risk-neutral seasonality function, $f(t)$ (i.e. $\hat{f}(t)$ in more precise terms), can be extracted from the observed forward curve, $F_0^{[T]}$, for $T \in [0, T^*]$ using Equation 3.12.

3.5.2 Pricing Options on Forwards

Obviously, we can price an option at time t written on a forward contract using the stochastic dynamics of the forward. The zero-cost[3] forward price is determined by

$$F_t^{[T]} = E^Q[S_T|\mathcal{F}_t].$$

These options are mostly puts or calls of vanilla type and their maturities are usually set equal to those of the forwards on which they are written. Then, the payoff can be given by the equation

$$(F_T^{[T]} - D)^+ = (S_T - D)^+.$$

Analysing the stochastic dynamics of the forward curve implied by the spot price model, we will be able to relate our pricing formula to Black's 1976 formula (see [3]) by approximating forward price distribution by means of the moment matching method.

Forward Dynamics

Using the relationship above and the expectation of S_t given in the proof of Theorem 3.4.1 with initial conditions X_t and Y_t, yields

$$\begin{aligned} F_t^{[T]} &= E[e^{Z_T}|\mathcal{F}_t] = E^Q[S_T|X_t, Y_t] \\ &= \exp\left(f(T) + X_t e^{-\alpha(T-t)} + \frac{\sigma^2}{4\alpha}(1 - e^{-2\alpha(T-t)}) + Y_t e^{-\beta(T-t)}\right) \\ &\quad \exp\left(\lambda \int_0^{T-t} (\Phi_J(e^{-\beta s}) - 1)ds\right). \end{aligned} \tag{3.13}$$

Lemma 3.5.3. *(Dynamics of the forward) The stochastic dynamics of the forward price $F_t^{[T]}$ in Equation 3.13 for a fixed maturity T are given by*

$$\frac{dF_t^{[T]}}{F_t^{[T]}} = -\lambda\left(\Phi_J(e^{-\beta(T-t)}) - 1\right)dt + \sigma e^{-\alpha(T-t)}dW_t + \left(e^{J_t e^{-\beta(T-t)}} - 1\right)dN_t. \tag{3.14}$$

[3] The price of the underlying at time T is expected to be equal to the forward price, thus making the expected cost of the contract 0.

Proof. Let us denote the diffusion and the jump parts of Equation 3.13 separately by

$$\begin{aligned} L_t^{[T]} &= f(X_t,t) = \exp\left(X_s e^{-\alpha(T-t)} + \frac{\sigma^2}{4\alpha}(1-e^{-2\alpha(T-t)})\right) \\ Z_t^{[T]} &= f(\Theta_t^{[T]}) = \exp\left(\Theta_t^{[T]}\right) = \exp\left(Y_t e^{-\beta(T-t)} + \lambda \int_0^{T-t} \phi_J(e^{-\beta s}) - 1 ds\right). \end{aligned}$$

It is clear that $\Theta_t^{[T]}$ has both Riemann and pure jump parts. Then, we can refer to the Itô-Doeblin formula for semimartingales (see Theorem 11.5.1 of [23] for further detail).

Stochastic dynamics of $\Theta_t^{[T]}$ can be written as

$$d\Theta_t^{[T]} = e^{-\beta(T-t)}dY_t + \beta Y_t e^{-\beta(T-t)} - \lambda(\phi_J(e^{-\beta(T-t)}) - 1)dt.$$

Also using the stochastic dynamics of Y_t, this can be further extended to

$$\begin{aligned} d\Theta_t^{[T]} &= e^{-\beta(T-t)}(-\beta Y_t dt + J_s dN_t) + \beta Y_t e^{-\beta(T-t)} - \lambda(\phi_J(e^{-\beta(T-t)}) - 1)dt \\ &= e^{-\beta(T-t)} J_t dN_t - \lambda(\phi_J(e^{-\beta(T-t)}) - 1)dt. \end{aligned}$$

Note that it has a continuous Riemann part

$$d\Theta_{t,c}^{[T]} = -\lambda(\phi_J(e^{-\beta(T-t)}) - 1)dt,$$

whose quadratic variation is simply $d[\Theta,\Theta]_{t,c}^{[T]} = 0$. Next, observe that *only if* the process $Z = \exp(\Theta)$ has a jump at time s, using the explicit solution for Y_t and denoting by $s-$ the moment before the jump occurs allow us to write

$$\begin{aligned} Z_s^{[T]} &= e^{Y_0 e^{-\beta T} + \sum_{i=1}^{N(s)} e^{-\beta(T-\tau_i)} J_{\tau_i} + \lambda \int_0^{T-s} \phi_J(e^{-\beta u}) - 1 du} \\ Z_{s-}^{[T]} &= e^{Y_0 e^{-\beta T} + \sum_{i=1}^{N(s)-1} e^{-\beta(T-\tau_i)} J_{\tau_i} + \lambda \int_0^{T-s} \phi_J(e^{-\beta u}) - 1 du}, \end{aligned}$$

which yields

$$Z_s^{[T]} = Z_{s-}^{[T]} \exp\left(e^{-\beta(T-\tau_{N(s)})} J_{\tau_{N(s)}}\right).$$

Since $\tau_{N(s)} = s$, we can write, $Z_s^{[T]} = Z_{s-}^{[T]} \exp\left(e^{-\beta(T-s)} J_s\right)$. Therefore, the difference can be expressed (i.e. whether there is a jump or not) by the equation

$$Z_s^{[T]} - Z_{s-}^{[T]} = Z_{s-}^{[T]}(\exp\left(e^{-\beta(T-s)} J_s\right) - 1)\Delta N(s).$$

Note that $\Delta N(s)$ is equal to one if a jump occurs and zero otherwise. According to the Itô -Doeblin formula for semimartingales,

$$\begin{aligned} Z_t^{[T]} = f(\Theta_t^{[T]}) &= f(\Theta_0^{[T]}) + \int_0^t -\lambda(\phi_J(e^{-\beta(T-s)}) - 1)Z_s ds + \sum_{0<s\leq t}[Z(s) - Z(s-)] \\ &= f(\Theta_0^{[T]}) + \int_0^t -\lambda(\phi_J(e^{-\beta(T-u)}) - 1)Z_s ds \\ &\quad + \int_0^t (\exp\left(e^{-\beta(T-s)}J_s\right) - 1)Z(s-)dN(s). \end{aligned}$$

Then, the stochastic dynamics of the process can be found as

$$\begin{aligned} dZ_t^{[T]} &= Z_t^{[T]} - \lambda(\phi_J(e^{-\beta(T-t)}) - 1)dt + (\exp\left(e^{-\beta(T-t)}J_t\right) - 1)Z_t^{[T]}dN(t) \\ \frac{dZ_t^{[T]}}{Z_t^{[T]}} &= -\lambda(\phi_J(e^{-\beta(T-t)}) - 1)dt + (\exp\left(e^{-\beta(T-t)}J_t\right) - 1)dN(t). \end{aligned}$$

Similarly, applying the standard Itô formula to $L_t^{[T]} = f(X_t, t)$, we obtain

$$\begin{aligned} L_t^{[T]} = f(X_t, t) &= f(X_0, 0) + \int_0^t L_s^{[T]}(\alpha X_s e^{-\alpha(T-s)} - \frac{\sigma^2}{2}e^{-2\alpha(T-s)})ds \\ &\quad + \int_0^t L_s^{[T]}e^{-\alpha(T-s)}(-\alpha X_s ds + \sigma dW_s) + \frac{1}{2}\int_0^t L_s^{[T]}e^{-\alpha(T-s)}\sigma^2 ds, \end{aligned}$$

which implies

$$\frac{dL_s^{[T]}}{L_s^{[T]}} = \sigma e^{-\alpha(T-t)}dW_t.$$

Finally, writing $dF_t^{[T]}$ in the form

$$\begin{aligned} dF_t^{[T]} &= L_t^{[T]}Z_t^{[T]}\frac{dZ_t^{[T]}}{Z_t^{[T]}} + Z_t^{[T]}L_t^{[T]}\frac{dL_t^{[T]}}{L_t^{[T]}} \\ &= F_t^{[T]}\left(\frac{dZ_t^{[T]}}{Z_t^{[T]}} + \frac{dL_t^{[T]}}{L_t^{[T]}}\right) \end{aligned}$$

yields the result. □

Lemma 3.5.3 introduces us two additional terms:

Compensating Factor. The equality $E^Q[dF_t^{[T]}] = 0$ should hold by definition (forward prices are martingale). This has an implication for the result above: the *invisible* drift of the differential Poisson part in Equation 3.14 should be compensated by the *visible* drift in the equation. This condition results from the fact that the Poisson process is not a martingale under either probability measures, P and Q.

Sensitivity. The effect of a jump in the underlying spot price can be investigated by calculating the ratio of the *percentage* change in the forward price to the *percentage* change in the spot price. This effect obviously remains very limited, particularly for the longer terms to maturity. In mathematical terms, the ratio

$$\frac{\left(F_t^{[T]} - F_{t-}^{[T]}\right)/F_{t-}^{[T]}}{\left(S_t - S_{t-}\right)/S_{t-}} = \frac{\left(F_t^{[T]} - F_{t-}^{[T]}\right)/F_{t-}^{[T]}}{\left(F_t^{[t]} - F_{t-}^{[t]}\right)/F_{t-}^{[t]}} = \frac{e^{J_t e^{-\beta(T-t)}} - 1}{e^{J_t} - 1}$$

is very small for larger values of $T - t$.

Pricing

As we've mentioned at beginning of the previous section, the majority of options written on forwards have the same exercise dates as the forwards on which they are written. Hence, below we find an approximation to the distribution of $F_T^{[T]}$ in terms of the currently observed forward prices $F_t^{[T]}$ in the market. The solutions for both X_t and Y_t imply

$$\begin{aligned} X(T) &= X(t)e^{-\alpha(T-t)} + \int_t^T e^{-\alpha(T-s)}\sigma dW(s) \\ X(T) - X(t)e^{-\alpha(T-t)} &= \int_t^T e^{-\alpha(T-s)}\sigma dW(s) \end{aligned} \tag{3.15}$$

$$\begin{aligned} Y_T &= Y_t e^{-\beta(T-t)} + \sum_{i=1}^{N_T} e^{-\beta(T-\tau_i} J_{\tau_i} \\ Y_T - Y_t e^{-\beta(T-t)} &= \sum_{i=1}^{N_T} e^{-\beta(T-\tau_i} J_{\tau_i}. \end{aligned} \tag{3.16}$$

Also, using Equation 3.13 for $t = T$, we have

$$\ln F_T^{[T]} = f(T) + X_T + Y_T$$

and for time t it is

$$\ln F_t^{[T]} = f(T) + X_t e^{-\alpha(T-t)} + \frac{\sigma^2}{4\alpha}(1 - e^{-2\alpha(T-t)}) + Y_t e^{-\beta(T-t)} + \lambda \int_0^{T-t} (\Phi_J(e^{-\beta s}) - 1) ds.$$

The last two equations yield

$$\begin{aligned}\ln F_T^{[T]} &= \ln F_t^{[T]} + X(T) - X(t)e^{-\alpha(T-t)} + Y_T - Y_t e^{-\beta(T-t)} - \frac{\sigma^2}{4\alpha}(1-e^{-2\alpha(T-t)}) \\ &\quad -\lambda \int_0^{T-t} (\Phi_J(e^{-\beta s}) - 1)ds.\end{aligned}$$

Inserting the Equations 3.15 and 3.16 where necessary results in

$$\begin{aligned}\ln F_T^{[T]} &= \ln F_t^{[T]} + \int_t^T e^{-\alpha(T-s)}\sigma dW(s) + \sum_{i=1}^{N_T} e^{-\beta(T-\tau_i} J_{\tau_i} - \frac{\sigma^2}{4\alpha}(1-e^{-2\alpha(T-t)}) \\ &\quad -\lambda \int_0^{T-t} (\Phi_J(e^{-\beta s}) - 1)ds. \end{aligned} \tag{3.17}$$

In a first approximation, we ignore the effects on the conditional distribution of the jump part $\sum_{i=1}^{N_T} e^{-\beta(T-\tau_i} J_{\tau_i}$ and assume $F_T^{[T]}$ is log-normally distributed. Even in case there are jumps, this still allows us to relate it to Black's 1976 formula. What we trade off against excluding the compound Poisson part is the loss of information due to the ignored heavy-tails in the distribution of $\ln F_t^{[T]}$ that are likely to be caused by the spike risk. Ignoring spike risk will lead to underestimated premiums on the far out-of-the-money calls (call option prices rule out the far probability of $F_T^{[T]}$ exceeding D) and overestimated premiums on the far in-the-money puts (put option prices rule out the far probability of D being exceeded by $F_T^{[T]}$). This situation is conceptually similar to, but just the reverse of, the *volatility smirk* observed in equity markets which is generally known as resulting from the *crashophobia*[4] state of mood in these markets. Still, this approximation is expected to perform good with at-the-money options.

Note that $F_t^{[T]}$ is martingale for the maturity T (by definition) and we can match its first two moments with those of Black's 1976 model where $dF = F\hat{\sigma}dW$ and $F_T = F_t \exp\left(-\frac{1}{2}\hat{\sigma}^2(T-t) + \hat{\sigma}(W_T - W_t)\right)$.

For this purpose, we set

$$\ln F_T^{[T]} \approx \ln F_t^{[T]} + \xi, \quad \xi \sim N(-\frac{1}{2}\tilde{\sigma}^2(T-t), \tilde{\sigma}^2(T-t)),$$

which follows from the martingale property. And 3.17 implies

$$\tilde{\sigma}^2(T-t) := V\left[[\ln F_t^{[T]} | \mathcal{F}_t]\right] = V\left[\int_t^T e^{-\alpha(T-s)}\sigma dW(s) + \sum_{i=1}^{N_T} e^{-\beta(T-\tau_i} J_{\tau_i}\right].$$

[4]The investors put more premium on far out-of-the-money puts than far in-the-money puts against the probability of a crash in the market.

Using the moments derived in Sections 3.1 and 3.2, we find[5]

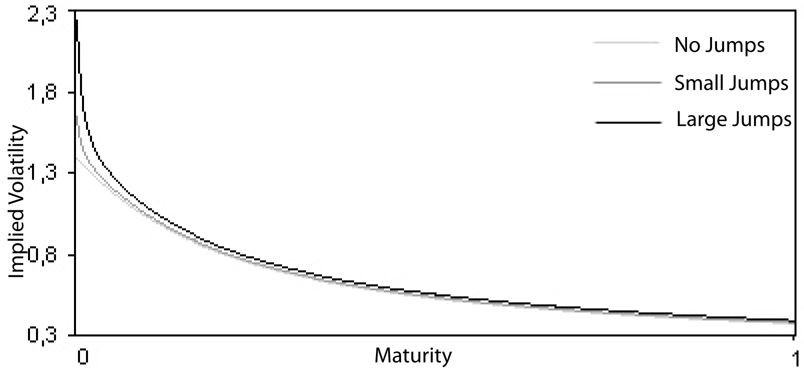
$$\tilde{\sigma}^2(T-t) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha(T-t)}) + \frac{\lambda}{2\beta}E[J^2](1 - e^{-2\beta(T-t)}).$$

Matching $\tilde{\sigma}^2(T-t)$ to implied Black's 1976 volatility $\hat{\sigma}^2(T-t)$, we conclude that $\tilde{\sigma}^2(T-t)$ is the the implied Black's 1976 volatility which can be approximated by

$$\hat{\sigma}^2 \approx \tilde{\sigma}^2 = \frac{\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha(T-t)}) + \frac{\lambda}{2\beta}E[J^2](1 - e^{-2\beta(T-t)})}{T-t}. \tag{3.18}$$

The term structure of the implied volatility is depicted in Figure 3.2 for different jump sizes.

Figure 3.2: Term structure of the implied volatility



As one can easily notice, the model-implied volatility is lower for the forwards with longer maturities. This result is highly consistent with the mean-reverting behavior observed in the electricity markets. The long-term equilibrium price is much less volatile than the prices observed in the shorter term.

In the next part, our calculations will take into account the fact that the electricity is a flow variable and cannot be stored effectively. Accordingly, we will assume a delivery period $[T_1, T_2]$ for the forward contracts on which the options are written.

3.5.3 Pricing Options on Forwards with a Delivery Period

When the power market is concerned, the problem of pricing the options on forward contracts cannot be reduced to one where the option payoff is determined simply by the value of the underlying security on a single day. To illustrate, consider the EEX (European Energy Exchange). The total delivery volume on a baseload futures contract traded on the EEX is specified in one-megawatt-per-hour terms, e.g. the seller of a monthly contract delivers 720 MWs during the specified delivery month,

[5]Note that mutual independence exists between W and J.

given the month has 30 days and the contract covers 24 hours a day. Now, let us denote by $F_t^{[T_1,T_2]}$ a forward power contract which matures any time in between $[T_1, T_2]$. This is similar to an Asian option[6] in the Black-Scholes setting in that we have to deal with the distribution of the integral

$$\int_0^T \varphi(t) S_t dt$$

rather than the underlying's pure price process, S_t. A well known approach to this case is to approximate the distribution of the integral, which is the weighted sum of log-normal variables, by a log-normal distribution using the moment-matching method (see [24], [13] and [10]). This method is similar to the one we utilized in Part 3.5.2 to derive an approximation to Black's 1976 implied volatility.

One method to express the strike price of a forward contract $F_t^{[T_1,T_2]}$ is to weight $F_t^{[T]}$s by a function $\varphi(T)$ where $T \in [T_1, T_2]$, i.e.

$$F_t^{[T_1,T_2]} = \int_{T_1}^{T_2} \varphi(T) F_t^{[T]} dT. \tag{3.19}$$

Since there exists no closed form distribution function or characteristic function for the weighted sum of log-normal variables, the literature in recent years witnessed an extended effort to approximate the distribution of this integral. The term *moment-matching* conceptually means that the mean and variance of the integral above is matched with the mean and variance of a univariate log-normal variable $\tilde{F}$. One important motivation for using the moment-matching method is that it allows us to value the options whose values are determined by the underlying processes that are dependent on the path of a simple process. The valuation of the options based on the approximated one dimensional asset distributions can be achieved by using closed form formulas, e.g. Black and Scholes. Otherwise, pricing options on forwards with a delivery period, as well as other path-dependent derivatives, based on such an integral would require more sophisticated numerical methods and much more time likely. As mentioned above, Asian options are also in this group. Once we determine the parameters of the approximate log-normal distribution, pricing options is nothing else than pricing in the Black-Scholes or Black's 1976 setting. One can use the following remark to relate the mean and variance of a log-normal distribution to those of a normal distribution.

Remark 3.5.4. *If the random variable Y is log-normally distributed with parameters μ_Y, σ_Y, then $X = \ln Y$ is normally distributed where the mean and variance are given by*

$$\mu = \ln\mu_Y - 0.5\ln\left(1 + \frac{\sigma_Y^2}{\mu_Y^2}\right), \quad \sigma^2 = \ln\left(1 + \frac{\sigma_Y^2}{\mu_Y^2}\right).$$

[6] An Asian option (or average value option) is a special type of option contract. For Asian options the payoff is determined by the average underlying price over some preset period of time.

Now, think of a call option on a forward contract with a delivery period $[T_1, T_2]$. Assume that this option expires on the first day of the delivery period, i.e. at time T_1. The payoff at maturity can be defined as $\left(F_{T_1}^{[T_1,T_2]} - D\right)^+$ and the price of the option at time t is then

$$E^Q\left[(F_{T_1}^{[T_1,T_2]} - D)^+|\mathcal{F}_t\right].$$

To be able to calculate this expectation, we need to approximate the distribution of the integral given in Equation 3.19 (i.e. the distribution of $F_{T_1}^{[T_1,T_2]}$) by a log-normal variable $\tilde{F}$. For this purpose, we need to match its mean and variance with those of the approximating variable $\tilde{F}$.

First Moment of $F_{T_1}^{[T_1,T_2]}$

Using the weighted average representation given in Equation 3.19, the expectation $E^Q\left[F_{T_1}^{[T_1,T_2]}|\mathcal{F}_t\right]$ can be written as

$$\begin{aligned} E^Q\left[F_{T_1}^{[T_1,T_2]}|\mathcal{F}_t\right] &= E^Q\left[\int_{T_1}^{T_2} \varphi(T) F_{T_1}^{[T]} dT|\mathcal{F}_t\right] \\ &= \int_{T_1}^{T_2} \varphi(T) E^Q\left[F_{T_1}^{[T]}|\mathcal{F}_t\right] dT. \end{aligned}$$

Now we need to determine $E^Q\left[F_{T_1}^{[T]}|\mathcal{F}_t\right]$. By Equation 3.13, we have

$$\ln F_t^{[T]} = f(T) + X_t e^{-\alpha(T-t)} + \frac{\sigma^2}{4\alpha}(1 - e^{-2\alpha(T-t)}) + Y_t e^{-\beta(T-t)} + \lambda \int_0^{T-t} (\Phi_J(e^{-\beta s}) - 1) ds$$

which can be rewritten for $t = T_1$ as

$$\ln F_{T_1}^{[T]} = f(T) + X_t e^{-\alpha(T-T_1)} + \frac{\sigma^2}{4\alpha}(1 - e^{-2\alpha(T-T_1)}) + Y_t e^{-\beta(T-T_1)} + \lambda \int_0^{T-T_1} (\Phi_J(e^{-\beta s}) - 1) ds.$$

To write the latter in terms of the former, we first calculate $X_t e^{-\alpha(T-T_1)} - X_t e^{-\alpha(T-t)}$ by referring to Equation 3.2, i.e.

$$\begin{aligned} X(T) &= X(T_1) e^{-\alpha(T-T_1)} + \sigma \int_{T_1}^{T} e^{-\alpha(T-s)} dW(s) \\ &= X(t) e^{-\alpha(T-t)} + \sigma \int_t^T e^{-\alpha(T-s)} dW(s) \end{aligned}$$

so that

$$X_{T_1}e^{-\alpha(T-T_1)} - X_t e^{-\alpha(T-t)} = e^{-\alpha(T-T_1)}\sigma \int_t^{T_1} e^{-\alpha(T_1-s)}dW(s).$$

Similarly $Y_{T_1}e^{-\beta(T-T_1)} - Y_t e^{-\beta(T-t)}$ can be calculated by using Equation 3.2 again as

$$Y_{T_1}e^{-\beta(T-T_1)} - Y_t e^{-\beta(T-t)} = \sum_{i=N_t}^{N_{T_1}} J_{\tau_i}e^{-\beta(T-\tau_i)}.$$

Finally, by using these two results, we find

$$\begin{aligned}\ln F_{T_1}^{[T]} &= \ln F_t^{[T]} + e^{-\alpha(T-T_1)}\sigma \int_t^{T_1} e^{-\alpha(T_1-s)}dW(s) + e^{-\beta(T-T_1)} \sum_{i=N_t}^{N_{T_1}} J_{\tau_i}e^{-\beta(T_1-\tau_i)} \\ &\quad -\frac{\sigma^2}{4\alpha}(e^{-2\alpha(T-T_1)} - e^{-2\alpha(T-t)}) - \lambda \int_0^{T_1-t} (\Phi_J(e^{-\beta(T-T_1)}e^{-\beta s}) - 1)ds.\end{aligned}$$

Then, turning back to the desired expectation $E^Q\left[F_{T_1}^{[T]}|\mathcal{F}_t\right]$ and using the mutual independence of W and J, we conclude

$$\begin{aligned}E^Q\left[F_{T_1}^{[T]}|\mathcal{F}_t\right] &= F_t^{[T]}E^Q\left[\exp\left\{e^{-\alpha(T-T_1)}\sigma \int_t^{T_1} e^{-\alpha(T_1-s)}dW(s)\right\}\right] \\ &\quad E^Q\left[\exp\left\{e^{-\beta(T-T_1)} \sum_{i=N_t}^{N_{T_1}} J_{\tau_i}e^{-\beta(T_1-\tau_i)}\right\}\right] \\ &\quad \exp\left\{-(\frac{\sigma^2}{4\alpha}(e^{-2\alpha(T-T_1)} - e^{-2\alpha(T-t)}))\right\} \\ &\quad \exp\left\{-(\lambda \int_0^{T_1-t} (\Phi_J(e^{-\beta(T-T_1)}e^{-\beta s}) - 1)ds)\right\}.\end{aligned}$$

Note that the first expectation in the above expression is a moment generating function of the form $E[e^Z] = e^{\mu_Z + \frac{1}{2}\sigma_Z^2}$ where Z is a normal random variable with zero mean. The second expectation is the moment generating function of the process Y_t with $\theta = e^{-\beta(T-T_1)}$. The fourth term is also equal to $\Phi_{Y_t}(e^{-\beta(T-T_1)})^{-1}$. Arranging the equation accordingly yields

$$\begin{aligned}E^Q\left[F_{T_1}^{[T]}|\mathcal{F}_t\right] &= F_t^{[T]}\exp\left\{\frac{\sigma^2}{4\alpha}(e^{-2\alpha(T-T_1)} - e^{-2\alpha(T-t)})\right\}\Phi_{Y_t}(e^{-\beta(T-T_1)}) \\ &\quad \exp\left\{-(\frac{\sigma^2}{4\alpha}(e^{-2\alpha(T-T_1)} - e^{-2\alpha(T-t)}))\right\}\Phi_{Y_t}(e^{-\beta(T-T_1)})^{-1} \\ &= F_t^{[T]}.\end{aligned}$$

Then, the first moment of $F_{T_1}^{[T_1,T_2]}$ turns out to be

$$\begin{aligned} E^Q\left[F_{T_1}^{[T_1,T_2]}|\mathcal{F}_t\right] &= E^Q\left[\int_{T_1}^{T_2}\varphi(T)F_{T_1}^{[T]}dT|\mathcal{F}_t\right] \\ &= \int_{T_1}^{T_2}\varphi(T)E^Q\left[F_{T_1}^{[T]}|\mathcal{F}_t\right]dT \\ &= \int_{T_1}^{T_2}\varphi(T)F_t^{[T]}dT. \end{aligned}$$

Note that the same result would be reached directly using the martingale property of the forward.

Second Moment of $F_{T_1}^{[T_1,T_2]}$

The second moment of $F_{T_1}^{[T_1,T_2]}$ can be expressed as the product of two individual forwards, namely $F_{T_1}^{[T]}$ and $F_{T_1}^{[T^*]}$, i.e.

$$\begin{aligned} E^Q\left[\left(F_{T_1}^{[T_1,T_2]}\right)^2|\mathcal{F}_t\right] &= E^Q\left[\left(\int_{T_1}^{T_2}\varphi(T)F_{T_1}^{[T]}dT\right)^2|\mathcal{F}_t\right] \\ &= \int_{T_1}^{T_2}\int_{T_1}^{T_2}\varphi(T)\varphi(T^*)E^Q\left[F_{T_1}^{[T]}F_{T_1}^{[T^*]}|\mathcal{F}_t\right]dTdT^*. \end{aligned}$$

Again we need to calculate $E^Q\left[F_{T_1}^{[T]}F_{T_1}^{[T^*]}|\mathcal{F}_t\right]$. Note that, by referring 3.13 we have already calculated $F_{T_1}^{[T]}$ in terms of $F_t^{[T]}$. Hence,

$$\begin{aligned} \ln F_{T_1}^{[T]}+\ln F_{T_1}^{[T^*]} &= \ln F_t^{[T]}+\ln F_t^{[T^*]} \\ &\quad+\left(e^{-\alpha(T-T_1)}+e^{-\alpha(T^*-T_1)}\right)\sigma\int_t^{T_1}e^{-\alpha(T_1-s)}dW(s) \\ &\quad+\left(e^{-\beta(T-T_1)}+e^{-\beta(T^*-T_1)}\right)\sum_{i=N_t}^{N_{T_1}}J_{\tau_i}e^{-\beta(T_1-\tau_i)} \end{aligned}$$

$$-\frac{\sigma^2}{4\alpha}\left(1+e^{-2\alpha(T^*-T)}\right)\left(e^{-2\alpha(T-T_1)}-e^{-2\alpha(T-t)}\right)$$
$$-\lambda\int_0^{T_1-t}(\Phi_J\left(e^{-\beta(T-T_1)}e^{-\beta s}\right)-1)ds$$
$$-\lambda\int_0^{T_1-t}(\Phi_J\left(e^{-\beta(T^*-T_1)}e^{-\beta s}\right)-1)ds.$$

Similarly, using the mutual dependence of W and J, the expectation $E^Q\left[F_{T_1}^{[T]}F_{T_1}^{[T^*]}|\mathcal{F}_t\right]$ can be found as

$$\begin{aligned}
E^Q\left[F_{T_1}^{[T]}F_{T_1}^{[T^*]}|\mathcal{F}_t\right] &= E^Q\left[\exp\left\{\ln\left(F_{T_1}^{[T]}F_{T_1}^{[T^*]}\right)\right\}|\mathcal{F}_t\right]\\
&= E^Q\left[\exp\left\{\ln F_{T_1}^{[T]}+\ln F_{T_1}^{[T^*]}\right\}|\mathcal{F}_t\right]\\
&= F_t^{[T]}F_t^{[T^*]}\\
&\quad \exp\left\{\frac{\sigma^2}{4\alpha}\left(1+e^{-\alpha(T^*-T)}\right)^2\left(e^{-2\alpha(T-T_1)}-e^{-2\alpha(T-t)}\right)\right\}\\
&\quad \exp\left\{-\left(\frac{\sigma^2}{4\alpha}\left(1+e^{-2\alpha(T^*-T)}\right)\left(e^{-2\alpha(T-T_1)}-e^{-2\alpha(T-t)}\right)\right)\right\}\\
&\quad \Phi_{Y_t}\left(e^{-\beta(T-T_1)}+e^{-\beta(T^*-T_1)}\right)\Phi_{Y_t}^{-1}(e^{-\beta(T-T_1)})\Phi_{Y_t}^{-1}(e^{-\beta(T^*-T_1)}).
\end{aligned}$$

Moment-Matching

The weighting factor φ puts different weights on forwards $F_t^{[T]}$ for the different values of $T\in[T_1,T_2]$ depending on the time of the settlement. The parties can either settle the contract after the delivery is complete, i.e. at T_2, or the contract holder can make instantaneous payments whenever the counterparty makes a delivery during the delivery period $[T_1,T_2]$. These two cases differ in that the latter allows for the reinvestment of the proceedings at the risk-free rate. In this case, the zero-cost strike implies

$$E^Q[\int_{T_1}^{T_2}(F_{T_1}^{[T]}-D)e^{r(T_2-T)}dT|\mathcal{F}_t]=0,$$

which results in

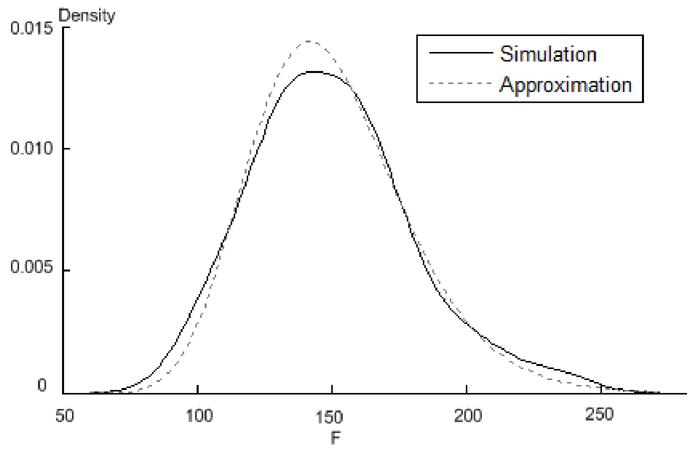
$$D=\int_{T_1}^{T_2}\frac{re^{r(T_2-T)}}{e^{r(T_2-T_1)}-1}E^Q[F_{T_1}^{[T]}|\mathcal{F}_t]dT.$$

In this case, the weighting factor is obviously given by

$$\varphi(T)=\frac{re^{r(T_2-T)}}{e^{r(T_2-T_1)}-1}.$$

Alternatively, if the contract is settled at T_2, the term $e^{r(T_2-T)}$ has a value of 1. This leads to $\varphi(T)=\frac{1}{T_2-T_1}$. Considering these two potential weighting factors which may apply in different settlement cases, we assume that the settlement is made at T_2 and compare the density of the forward $F_{T_1}^{[T_1,T_2]}$ to that of $\tilde{F}_{T_1}^{[T_1,T_2]}$, which is a log-normal approximation to the real density and obtained by matching the two moments calculated above. Figure 3.3 shows that the approximated density exhibit slight differences from the simulated density, particularly around the mean values. However, the option prices based on the approximate distribution are still likely to perform well.

Figure 3.3: Simulated density (10^6 paths) versus approximated density. Parameters: $\alpha = 7$, $\beta = 200$, $\lambda = 4$, $\mu_J = 0.4$, $T_1 = 1$, $T_2 = 1.25$



3.6 Summary

So far, we have introduced a model which satisfies the basic characteristics of the electricity markets. Having obtained a useful approximation to the density of the spike process, we will be able to derive conditional state probabilities in Chapter 4 for the individual process Y_t as well as X_t, which will form the basis of our Swing option pricing algorithm. We also examined the shape of our model in the risk-neutral world setting. It is obvious that the risk-neutral model has a structure which is similar to the one observed under the real-world measure. Since the risk-neutral measure is not uniquely determined in an incomplete electricity market, one can opt for calibrating the parameters in a real-world setting after extracting the real-world seasonality, and convert the model as a whole to a risk-neutral one by obtaining a seasonality function which is consistent with the forward curve and which makes the observed forward curve equal to the forward curve implied by the real-world parameters. The model-implied Black's 1976 volatility indicates that jumps have a considerable effect on the implied volatilities of shorter-maturity options, which can be translated into an extra jump premium on the prices of the forward options. The moment-matching algorithm also performs well with the options on forwards with a delivery period. This type of options are more realistic in real-world settings because large volumes of power cannot be stored and delivered in a few days. Chapter 4 is devoted to the efforts of applying the stochastic properties derived in this chapter to the pricing methodology of the Swing options, i.e. the options with multiple exercise opportunities for certain rights.

Chapter 4

Swing Options and the Grid Approach

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As being path-dependent and exotic in nature, the Swing options are of the particular interest of the energy market participants due to the breath of fresh air they brought to the power risk management practices. Being bundled with the forward contracts, they allow the holders to better match their internal power demands. However, when their valuation is the case, the holder is assumed to follow an optimal strategy which maximizes the expected profit, no matter to what extent the Swing option helped hedge away the risks arising from the fluctuations in the holder's power demand. This is due to the fact that the internal power demand as well as the extent to which the Swing option is utilized is uniquely determined by the holder-specific factors and so, different prices for the same option may unsurprisingly lead to the breach of the no-arbitrage rules. In the following section, we will discuss in detail the reasons for the Swing options' speciality in the marketplace. An intuitional example is provided in the succeeding section to enhance the understanding. This will be followed by the formalization of a standard mathematical setup for the Swing options. To finalize, the two main discussions of this chapter, namely, the Trinomial Forest Approach and its extension to the Grid Approach will be mentioned. As noted earlier at the very beginning of this work, we will pursue the same method as Jaillet-Ronn-Tompaidis ([14]) except that its expansion to the more complicated Grid Approach, which is mentioned in Hambly-Howison-Kluge ([12]), will also be discussed in detail. In an illustrative example, we will price a standard Swing option numerically and further explore its parameter sensitivities.

Our main contribution to the work of Jaillet-Ronn-Tompaidis is a step-by-step generalization of their method to a more comprehensive one. Towards this aim, we integrate the jumps in electricity price into the valuation process and use two-dimensional refinements with a larger set of scenarios for price evaluations. The main advantage of this approach is that it allows for a better approximation of the distributions with heavy tails. We also improve the approximation to the spike process density, which is given in Hambly-Howison-Kluge, by deriving a more general density function. Furthermore, a heterogeneous grid for the truncated spike process better captures the spike process values that are close to zero. This approach leads to slightly lower values for the Swing options than those reported in [12]. Our MATLAB® code for the numerical pricing algorithm is provided in Appendix A.

4.1 What Makes Them Special for Energy Markets

The amount of electricity generated is a flow variable, which means that -unlike a stock variable- it can be measured over an interval of time. Hence, the parties of a forward contract should specify a delivery period, over which the purchased amount of electricity is delivered. Therefore, the pricing results of the options on forwards maturing at a specific point of time, say T, can only be seen as an approximation to those with a short delivery period, like one day. The Swing options are tailored to hedge against uncertainty in the internal demand and other risks arising from the particular attributes in the underlying's price during the delivery period. From this point of view, results given in Section 3.5 can be seen as a ground for the better understanding of what turns the Swing options into very useful tools which complement the energy transactions in the marketplace.

As a result of the complex patterns in the energy consumption and the limited storability of the electricity, the flexibility in delivery in terms of both the *timing* and the *amount* has been an essential part of the electricity contracts. Swing, or *take-or-pay*, contracts provide their owner with the flexibility-of-delivery options and permit the option holder to repeatedly exercise a certain right to *swing* the amount of energy purchased. These rights may be subject to certain conditions which impose some periodical limitations, such as the amount of energy which can be swung up or down during the lifetime of the option.

On the other hand, in an electricity market where a mismatch between supply and demand can suddenly occur, the Swing options match the need for hedging frequent, though not persisting, price spikes which are in general followed by reversion to a long-term level.

A very helpful example in order for the reader to better perceive the raison d'être of these options in the marketplace can be the situation of an agent who has a long position in an energy commodity, say oil. The position will be closed within 3 months. The agent is supposedly aware of the downside risk exposure and is now seeking for ways of eliminating this risk. The probability of considerable OECD production expansions on multiple days during these 3 months is a far tail event. Nevertheless, the agent can marginally buy a strip of 90 European put options maturing in 90 consecutive days. However, this clearly brings an excessive protection. Alternatively, the agent can buy 5 identical American put options whose exercise period covers these 3 months. However, the agent still pays more than he or she will likely to recover. Why? Because, having the same exercise time, these identical American options would benefit most only when they could be exercised at the same optimal exercise time. However, the agent will either not be able to exercise options on the same day due to demand constraints or desire to retain some of them further for potential unfavourable movements in the price of oil. Thus, a Swing contract with 5 exercise rights where each right can be exercised on any day during this 90-day period (and on a *one-right-on-a-day* basis) comes out as a perfect hedging instrument.

4.2 A Quick Review of the Mathematical Setup

A standard Swing option with an *initiation date*, say 0, gives the owner N rights of a *certain type*[1]. These identical rights can be exercised on a number of *potential dates*, $\{\tau_1, \tau_2, ..., \tau_N\} \in [T_1, T_2]$, and on a *one-right-at-a-time* basis. As time passes and the rights are exercised, the owner of the option now has $n \leq N$ right(s) remaining. The supplier of the option may claim a *refraction period*[2], Δt_R, in case it is stated in the initial agreement between parties. The option may also pose some volume constraints at each potential exercise time τ_i, such as,

$$\left[l_i^l, l_i^u\right) \cup \left(u_i^l, u_i^u\right]$$

where $l_i^l \leq l_i^u \leq 0 \leq u_i^l \leq u_i^u$. From left to right, these refer to the lower and upper bounds of the allowed down and up swings at each potential exercise date during the lifetime of the contract. Additionally total volume constraints may apply:

$$L \leq \sum_{i=1}^{N} \omega_{t_i} \leq U.$$

A penalty function, $\rho(V)$, can be set by the option holder against the breach of volume constraints. The swing rights are allowed to be exercised at a predetermined constant strike price, K. Alternatively, the parties can agree on a term structure for strike prices: $K_t,\ t \in \{\tau_1, \tau_2, ..., \tau_N\}$. A typical example of the non-constant

[1]Examples inlude: a number of call/put rights, a mixture of puts and call rights with equal/different strike prices.

[2]Refraction period here refers to the amount of time that the option holder should wait after the last exercise, before exercising another right.

strike price setup is *at-the-money forward strike* where K_t is set to be the forward price, F_0^t.

In the following sections, we address the valuation issues of these complex options using a discrete model. Note that our spot price process S_t is a continuous stochastic process and such a discretization can be made by observing it on discrete points in time. It is also worth noting that, throughout following sections, we simplify the pricing problem by either extracting the seasonality factor f_t from the pricing process (Section 4.3) or simply ignoring seasonality, i.e. $f_t = 0$ (Section 4.4). In the former case, it is equivalent to saying that the trinomial trees in the next session are built based on the deseasonalized forward prices and the expected option payoffs need to be readjusted simply by using the corresponding seasonality factor at time t.

4.3 The Trinomial ***Forest*** Method

The valuation of the Swing options in the trinomial forest[3] setting is basically twofold. First, in the next subsection, we will elaborate on how to construct the tree of nodes that will be used to approximate the distribution of the increments of the individual process X_t. Note that, for the sake of simplicity, we ignore the spike process Y_t for now.

4.3.1 Tree Construction

Recall that the process X_t in the risk-neutral world is written as (see Subsection 3.5.1)

$$dX_t = \hat{\alpha}(\mu(t) - X_t)dt + \sigma dW_t^Q.$$

Given that we observe the term structure of the forward prices, $F_0^{[i]}$, and the seasonality factors, f_i, where $i = 0, 1, ..., I$ and $t = i\Delta t$, a trinomial tree for X_t can be constructed by first considering a tree for the process

$$d\tilde{X}_t = -\hat{\alpha}\tilde{X}_t dt + \tilde{\sigma} dW_t^Q$$

and then incorporating the drift terms for every time step so that the expected deseasonalized spot prices, $E^Q[\exp(X_i)]$, that are calculated numerically, match the deseasonalized forward curve, $F_0^{[i]}/f_i$, which was initially observed.

We first begin with constructing a trinomial tree for $\tilde{X}$, given $\tilde{X}_0 = 0$. Setting $\Delta\tilde{X} = \sigma\sqrt{\kappa\Delta t}$, where κ is a constant, and denoting by (i, j) the node where t and $\tilde{X}$ take the values $i\Delta t$ and $j\Delta\tilde{X}$ (or $j\sigma\sqrt{\kappa\Delta t}$ more explicitly), respectively, the tree will be symmetric around the initial value 0. In a trinomial tree, j will be in the set $\{-1, 0, 1\}$, obviously. Aside from the symmetric branching form (up/straight/down) which will be case for $j \notin \{-1, 1\}$ (i.e. $j = 0$), we also define two additional

[3]A multi-layer tree extension of the traditional trinomial tree dynamic programming approach.

branching scenarios (straight/up/2 x up and straight/down/2 x down) for the values $j=-1$ and $j=1$ to make sure that the process can revert to its long-term mean 0. Now, we need to calculate the probabilities

$$p^i_{j,m} \approx f_{\tilde{X}_{t+\Delta t}|\tilde{X}_t=\tilde{x}_j}(\tilde{x}_m)\Delta\tilde{x}.$$

$p^i_{j,m}$ can be defined as the probability of going from node (i,j) to $(i+1,m)$. For the pairs where these nodes are not connected between consecutive time steps $i\Delta t$ and $(i+1)\Delta t$, this probability is simply set to zero.

Given the process $\tilde{X}$ is at node $(i,0)$, $(i,-1)$ or $(i,1)$, we need to calculate the probabilities $p^i_{j,0}$, $p^i_{j,-1}$ and $p^i_{j,1}$ explicitly for each of the three possible branching scenarios defined above. Having derived the stochastic dynamics of $\tilde{X}$, this can be done by exploiting three equations, namely,

$$\begin{aligned}
\sum_{m=-1}^{1} p^i_{j,m} &= 1 \\
\sum_{m=-1}^{1} p^i_{j,m}(\tilde{X}_{i+1,m}-\tilde{X}_{i,j}) &= E^Q[\tilde{X}_{i+1}-\tilde{X}_i] \\
\sum_{m=-1}^{1} p^i_{j,m}(\tilde{X}_{i+1,m}-\tilde{X}_{i,j})^2 - (E^Q[\tilde{X}_{i+1}-\tilde{X}_i])^2 &= V^Q[\tilde{X}_{i+1}-\tilde{X}_i].
\end{aligned}$$

It is important to note that depending on the different values of j at time $t = i\Delta t$, $\tilde{X}_{i+1,m}-\tilde{X}_{i,j}$ can have three different magnitudes for every single j value in the set $\{-1,0,1\}$, i.e. $\{\Delta\tilde{X},0,-\Delta\tilde{X}\}$ for $j=0$, $\{0,-\Delta\tilde{X},-2\Delta\tilde{X}\}$ for $j=1$ or $\{0,\Delta\tilde{X},2\Delta\tilde{X}\}$ for $j=-1$. To proceed, we will assume $j=0$ at time $i\Delta t$. Solving the above equations by using the mean and variance of $\tilde{X}_{i+1}-\tilde{X}_i$, which are given by the equations

$$\begin{aligned}
E^Q[\tilde{X}_{i+1}-\tilde{X}_i] &= -\hat{\alpha}\tilde{X}_t\Delta t = -\hat{\alpha} j\sigma\sqrt{\kappa\Delta t}\Delta t \\
V^Q[\tilde{X}_{i+1}-\tilde{X}_i] &= \tilde{\sigma}^2\Delta t,
\end{aligned}$$

the probabilities of going up one $\Delta\tilde{X}$ unit, horizontal move and going down one $\Delta\tilde{X}$ unit are found respectively as

$$\begin{aligned}
p^i_{0,1} &= \frac{1}{6} + \frac{\alpha j\Delta t(\alpha j\Delta t - 1)}{2} \\
p^i_{0,0} &= \frac{2}{3} - (\alpha j\Delta t)^2 \\
p^i_{0,-1} &= \frac{1}{6} + \frac{\alpha j\Delta t(\alpha j\Delta t + 1)}{2}.
\end{aligned}$$

$p^i_{j,m}$'s for other values of j can be found similarly. To be able to incorporate drift term, $d(i)$, we define the probability that the node $(i+1,m)$ will be reached, P^{i+1}_m, simply as

$$P^{i+1}_m = \sum_{j=-1}^{1} P^i_j p^i_{j,m}$$

where $p^i_{j,m}$ is again the probability of reaching from node (i,j) to node $(i+1,m)$. Obviously, $P^0_0 = 0$ and P^i_j's can simply be calculated by using $p^i_{j,m}$'s as inputs to above equation.

In theoretical terms, the drift term $d(i)$, which was given by d_t in Section 3.5.1, corresponds to the difference term between the solutions to two OU processes, one with a long-term mean of zero and the other with a time-varying long-term mean of $\mu(t)$. This means that $d(i)$ in the solution can be treated as a time-varying long-term mean, $\mu(t)$, in an OU stochastic differential equation. Hence, in practice, these shifts, d_i, should be determined so that the following two coincide: (i) the expectation of the deasonalized spot price, $E^Q[\exp(\tilde{X}_{i,j}+d_i)] = E^Q[\exp(X_{i,j})]$, and (ii) the deseasonalized forward curve, F^i_0/f_i. Again, F^i_0 denotes the forward price at time 0 which matures at time $i\Delta t$. Solving for d_i, we reach at

$$d_i = \ln(\frac{F_i}{f_i}) - \ln(\sum_j P^i_j e^{\tilde{X}_{i,j}}).$$

Thus, adjusting the node values at each time step using the drifts d_i, we end up with a trinomial tree for the deseasonalized price process $\exp(X_t)$.

4.3.2 Pricing

Defining $S_{i,j} = \exp(f_i + X_{i,j})$ one can use the relation

$$V(n,s,i,j) = \max \left\{ \begin{array}{l} e^{-r\Delta t} E^Q[V(n,S_{i+1},i+1)|S_{i,j}=s], \\ e^{-r\Delta t} E^Q[V(n-1,S_{i+1},i+1)|S_{i,j}=s] + (s-D)^+ \end{array} \right\}$$

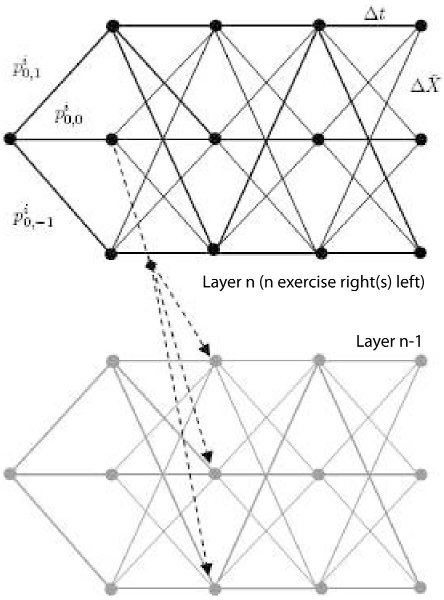
to derive the value of a Swing option at time $t = i\Delta t$ at any node (i,j) where $V(n,S,I,J) = (S_{I,J} - D)^+$, $0 < n \leq N$ and $V(0,s,i,j) = 0$. To avoid a potential abuse of notation, we denote by D the exercise price of the option. The conditional expectations can be approximated in a discrete form by

$$E^Q[V(n,S_{i+1},i+1)|S_{i,j}=s] \approx \sum_m V(n,s_{i+1,m},i+1)p^i_{j,m}.$$

The last two equations imply that the pricing algorithm should simultaneously deal with at least two different trees at any time point, one with n and the other is $(n-1)$ exercise rights remaining (See Figure 4.1 for an illustration). At this point, the following remark is useful in that we should know the total number of trees that a candidate pricing algorithm will need to handle. With slight modifications, the same approach can be applied in calculating the total number of grid trees in the Grid Approach, which will be mentioned in Section 4.4.

Remark 4.3.1. *Given that the Swing option holder can exercise a prespecified 'basis amount' multiplied by a set of n_V consecutive volume multipliers at each exercise date (e.g. 1 x* basis amount *or 2 x* basis amount *for $n_V = 2$ and $V = \{1,2\}$), the*

Figure 4.1: A trinomial tree for the process $\tilde{X}_t$



total number of trees necessary to price a Swing option with N exercise rights is given by

$$\sum_{n=0}^{N}(N-n)(n_V-1)+1,$$

where N and n again denote the quantities of the initial exercise rights and the remaining exercise rights, respectively.

Proof. Assume that the holder is only allowed to *swing up* at each potential exercise date τ_i according to the rule

$$\left(u_i^l, u_i^u\right], \quad 0 \leq u_i^l \leq u_i^u.$$

Further assume that the writer specifies a certain set of multipliers (v_1 ,v_1+1,...,v_1+n_V-1) so that, at each exercise time, the holder can choose one from the set of possible exercise volumes (V_1 x *basis amount* ,(V_1+1) x *basis amount*,...,(V_1+n_V-1) x *basis amount*). This implies a number of cumulative usage amounts after the holder exercises $(N-n)$ rights (i.e. n rights remaining), which can be expressed by the set

$$((N-n)V_1 \text{ x basis amount}, ..., (N-n)(V_1+n_V-1) \text{ x basis amount}).$$

This set has a total number of $(N-n)(n_V-1)+1$ usage amounts, for each of which an independent tree should be constructed. Given that the number of remaining rights range between 0 and N, the required number of trees to value a Swing option is found as

$$\sum_{n=0}^{N}(N-n)(n_V-1)+1. \qquad \square$$

However, the Trinomial Forest Method mentioned so far cannot be used with our spot price model without any modifications, since there are some drawbacks embedded in it which can limit accuracy of the method. First, the refinement of the tree nodes in the spot direction is not likely to have an improving effect on the results. Second, heavy-tailed conditional distribution imposed by the jump structure cannot be captured using a three-point approximation.

4.4 Generalization to the Grid Approach

Hence, we now generalize the method discussed in Section 4.3 to a grid tree approach where the tips of the price tree branches emanating from the node (i,j) now has a two-dimensional grid shape when viewed from above, rather than three points on a line, as in the Trinomial Forest Method. The grid tree has nodes (i,j,k) where each grid in this tree has nodes (j,k) at time $t=i\Delta t$. We denote the spike process by the third dimension k and it adds an additional y direction over which the jump values are spread, whereas the branching method in the trinomial trees allowed only for the potential refinements in the x direction. A useful visualization can be found in Figure 4.2.

Again assuming that the processes X and Y are individually observable, the valuation function given in Equation 4.1 can be rewritten as

$$V(n,x,y,i,j,k)=\max\left\{\begin{array}{l} e^{-r\Delta t}E^Q[V(n,X_{i+1},Y_{i+1},i+1)|X_{i,j}=x,Y_{i,k}=y], \\ e^{-r\Delta t}E^Q[V(n-1,X_{i+1},Y_{i+1},i+1)|X_{i,j}=x,Y_{i,k}=y] \\ +(e^{f(t)+x+y}-D)^+ \end{array}\right\}$$

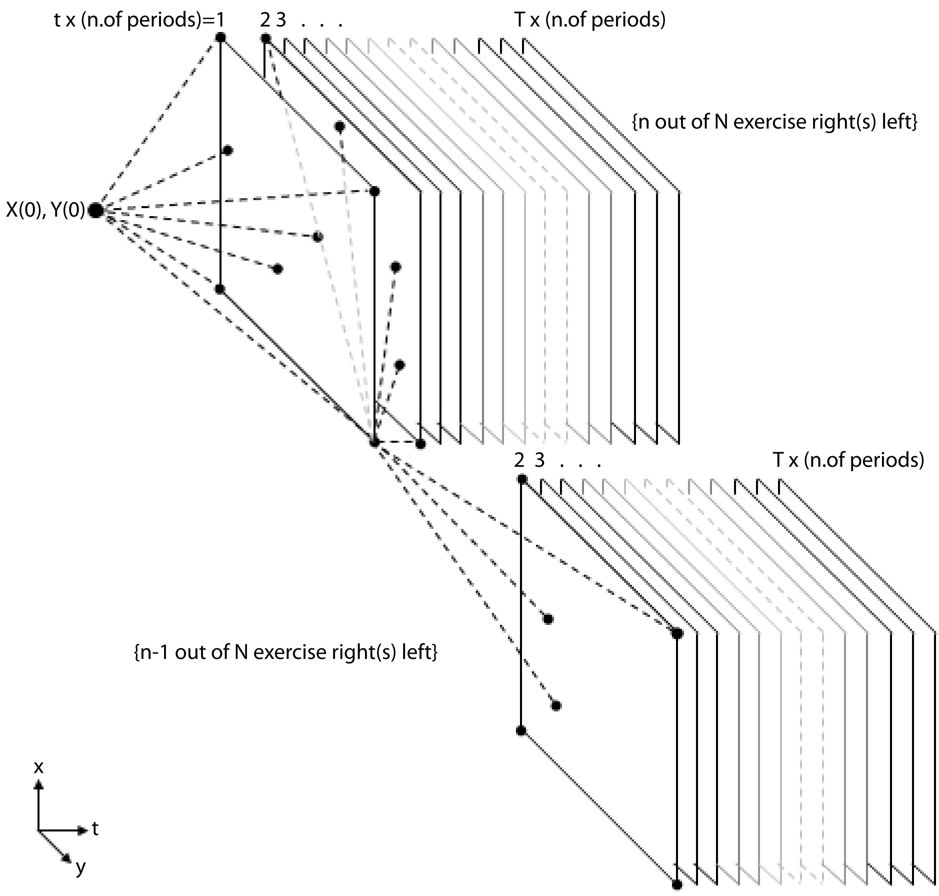
to derive the value of a Swing option at time $t=i\Delta t$ at any node (i,j,k) where $V(n,S,I,J,K)=(S_{I,J,K}-D)^+$, $0<n\le N$ and $V(0,s,i,j,k)=0$. Again, the conditional expectations can be approximated in a discrete form by

$$E^Q[V(n,X_{i+1},Y_{i+1},i+1)|X_{i,j}=x,Y_{i,k}=y]\approx\sum_{m,n}V(n,s_{i+1,m,n},i+1)p^i_{j,k,m,n}.$$

$p^i_{j,k,m,n}$ is the probability of reaching the node $(i+1,m,n)$ at time $t+\Delta t=(i+1)\Delta t$ given that the process is at node (i,j,k) at time $t=i\Delta t$ and this probability can be approximated by

$$p^i_{j,k,m,n}\approx f_{X_{t+\Delta t}|X_t=x_j}(x_m)\Delta x f_{Y_{t+\Delta t}|Y_t=y_k}(y_n)\Delta y$$

Figure 4.2: Grid Approach



since X_t and Y_t are independent. Below we discuss the conditional densities of these Markov processes.

Conditional density $f_{X_{t+\Delta t}|X_t=x_j}(x_m)$. Based on the first two moments of the OU process derived in Section 3.1, $X_{t+\Delta t}$, given $X_t = x_j$, is conditionally normally distributed with the mean and variance

$$N\left(x_j e^{-\alpha\Delta t}, \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})\right).$$

Conditional density $f_{Y_{t+\Delta t}|Y_t=y_k}(y_n)$. In order to derive an explicit approximation to the conditional density of Y_t we should first make a distribution assumption for the jump sizes, J. Here we assume exponentially distributed jump sizes and recall the density we derived for the truncated process $\tilde{Y}_t$ in Section 3.3, which was given by

$$f_{\tilde{Y}_t}(y) = \frac{\lambda}{\beta {\mu_J}^{\lambda/\beta}} \frac{\Gamma(1-\lambda/\beta, y/\mu_J) - \Gamma(1-\lambda/\beta, ye^{\beta t}/\mu_J)}{y^{1-\lambda/\beta}}, \ \ y > 0.$$

Note that if the value of λ/β is too small (i.e. the jumps appear very rarely but their effects vanish very quickly), an upper incomplete gamma function in the form

$$\Gamma(1-\lambda/\beta, y) = \int_{y}^{\infty} \hat{\gamma}^{-\frac{\lambda}{\beta}} e^{-\hat{\gamma}} d\hat{\gamma}$$

can be approximated by

$$\Gamma(1-\lambda/\beta, y) \approx \int_{y}^{\infty} \hat{\gamma}^{0} \beta e^{-\hat{\gamma}} d\hat{\gamma} \approx e^{-y}.$$

Then, we have

$$g_{\tilde{Y}_t}(y) = \frac{\lambda}{\beta {\mu_J}^{\lambda/\beta}} \frac{e^{-y/\mu_J} - e^{-ye^{\beta t}/\mu_J}}{y^{1-\lambda/\beta}}, \ \ y > 0.$$

The reason why we denoted the approximation by g rather than f is to distinguish it from a real density function which is normalized. To normalize g, we have to find a normalizing constant and make sure that our density function sums up to $1 - e^{-\lambda t}$, i.e. the cumulative density on the set $\tau \in [0, t]$ (refer to Lemma 3.3.4). The value of g in the interval $(0, \infty)$ is calculated as

$$\begin{aligned}
\int_0^\infty g_{\tilde{Y}_t}(y) dy &= \int_0^\infty \frac{\lambda}{\beta {\mu_J}^{\lambda/\beta}} \frac{e^{-y/\mu_J}}{x^{1-\lambda/\beta}} - \int_0^\infty \frac{\lambda}{\beta {\mu_J}^{\lambda/\beta}} \frac{e^{-ye^{\beta t}/\mu_J}}{y^{1-\lambda/\beta}} \\
&= \frac{\lambda}{\beta} \Gamma\left(\frac{\lambda}{\beta}\right) \int_0^\infty \frac{1}{{\mu_J}^{\lambda/\beta} \Gamma\left(\frac{\lambda}{\beta}\right)} e^{-y/\mu_J} y^{\lambda/\beta - 1} \\
&\quad - \frac{\lambda}{\beta} \Gamma\left(\frac{\lambda}{\beta}\right) e^{-\lambda t} \int_0^\infty \frac{1}{\frac{\mu_J}{e^{-\beta t}}^{\lambda/\beta} \Gamma\left(\frac{\lambda}{\beta}\right)} \frac{e^{-ye^{\beta t}/\mu_J}}{y^{1-\lambda/\beta}} \\
&= \frac{\lambda}{\beta} \Gamma\left(\frac{\lambda}{\beta}\right) \left(1 - e^{-\lambda t}\right).
\end{aligned}$$

It turns out that the approximation g should be normalized by using a factor $\frac{\lambda}{\beta}\Gamma\left(\frac{\lambda}{\beta}\right)$ to obtain an approximation for $f_{\tilde{Y}_t}(y)$ which satisfies

$$P(\tau \geq t) + \int_0^\infty f_{\tilde{Y}_t}(y) dy = e^{-\lambda t} + (1 - e^{-\lambda t}) = 1.$$

From that we have

$$f_{\tilde{Y}_t}(y) \approx \frac{1}{{\mu_J}^{\lambda/\beta} \Gamma\left(\frac{\lambda}{\beta}\right)} \frac{e^{-y/\mu_J} - e^{-ye^{\beta t}/\mu_J}}{y^{1-\lambda/\beta}}, \ \ y > 0. \tag{4.1}$$

As a result, it is obvious that the stationary distribution as $t \to \infty$ is similar to Gamma distribution, i.e

$$f_{\tilde{Y}_t}(y) \approx \frac{1}{{\mu_J}^{\lambda/\beta}\Gamma\left(\frac{\lambda}{\beta}\right)} e^{-y/\mu_J} y^{\lambda/\beta-1}, \;\; y > 0. \tag{4.2}$$

Finally, one can either use the unconditional stationary distribution given in Equation 4.2 or simply approximate the conditional densities at each time step from Equation 4.1 by the single equation

$$f_{\tilde{Y}_{\Delta t}}\left(y_n - y_k e^{-\beta \Delta t}\right).$$

That means, we approximate the conditional density of the value y_n at any time $t + \Delta t$, given the process has the value y_k at time t, by the unconditional density of the difference $y_n - y_k e^{-\beta \Delta t}$. Numerically, preferring one to the other has ignorable effects on the results.

4.5 Numerical Algorithm

Throughout this section, we adopt the same parameter values as given in [12] to value the Swing options. These are the calibrated parameters of the Nord Pool market. It is also assumed that the number of the consecutive volume multipliers (recall Remark 4.3.1), n_V, is equal to 1 with $v > 0$, i.e. the Swing holder has an option with N up-swing (call) rights. Also, the basis amount is taken to be one unit of underlying. We first define a two-dimensional grid whose increments in the both x and y directions are determined by the functions

$$\begin{aligned} dX &= 2\sigma_X \sqrt{T}\frac{1}{n_x}, \quad X_0 = 0 \\ dY_i &= \epsilon(1+\kappa)^{i-1}, \quad i = 1, ..., n_y, \quad Y_0 = 0 \end{aligned} \tag{4.3}$$

where $T = T_2 - T_1$ and $\epsilon > 0$. n_x and n_y denote the number of desired refinements in the x and y directions. κ is the rate at which the length of the intervals in the y direction grows. Note that we use a heterogeneous grid for the spike process whereas we prefer a homogeneous grid for the diffusive part. This is simply because approximating the transition probabilities from the spike density at maturity given in Equation 4.2 is highly challenging. For low values of λ and high values of β, the density approaches to infinity for the spike values close to zero while it depreciates very quickly and approaches to zero for the spike values even slightly greater than zero. As a result, we use an adaptive grid which has small increments (in the y direction) for the small values of the spike process and large increments (in the y direction) for the larger values of the spike process. This allows us to adjust the sensitivity of the approximation and cover a larger proportion of the cumulative density. On the other hand, such an approach does not improve the results for the OU part, X_t, and an approximation to the cumulative density of X_t with the homogeneous intervals in the x direction sums up to a value greater than 0.99 in almost all cases.

At this point, it is worth noting that an optimization of the algorithm complexity due to increasing number of refinements versus the goodness of the conditional density approximation is required. However, it is beyond the range of this work. Still, one can check the goodness of the approximation by simply computing the cumulative conditional densities along each of the initial values included in the grid and make the grid increments finer to reach a cumulative density value as close as possible to 1 without spoiling the algorithm's effectiveness.

Depending on the fact that the value of a Swing option with N exercise rights and a set of possible swing volumes allowed for each right will depend on the valuation of a large number of identical grids, where each layer is dependent upon the subordinate layers[4], Remark 4.3.1 implies that a computer should apply a huge number of decision rules, which is equal to

$$\left(\sum_{n=0}^{N}(N-n)(n_V-1)+1\right)(n_T n_x n_y n_V+1)$$

where n_T denotes the number of time periods in the Swing option's lifetime. However, assuming $n_V = 1$, we enjoy a considerable reduction in the memory requirements since the number of the decision nodes collapses to

$$(N+1)\,(n_T n_x n_y+1)\,. \tag{4.4}$$

To illustrate, for a Swing option with parameters $N = 100$, $n_T = 365$, $n_x = 100$ and $n_y = 50$, the last two equations corresponds to total number of decision nodes equal to $1.88\,x\,10^{10}$ (5151 grid trees) for $n_V = 2$ and $0.02\,x\,10^{10}$ (101 grid trees) , $n_V = 1$, respectively. We also keep only two (instead of $(N+1)$) grid trees with n and $(n-1)$ number of remaining exercise rights in memory at any time and recursively replace the lower grid tree with the upper one until the grid tree with N exercise rights is reached. At each of $n_T n_x n_y$ nodes of a particular grid tree, the node value is determined according to the decision rule given in Equation 4.1 ($0.02\,x\,10^{10}$ times in total for the parameter values given above). Each additional refinement in spike process direction, y, incorporated into our algorithm increases the complexity of the algorithm (measured by the number of decision rules to be applied) by a multiplier

$$(N+1)\,(n_T n_x)\,.$$

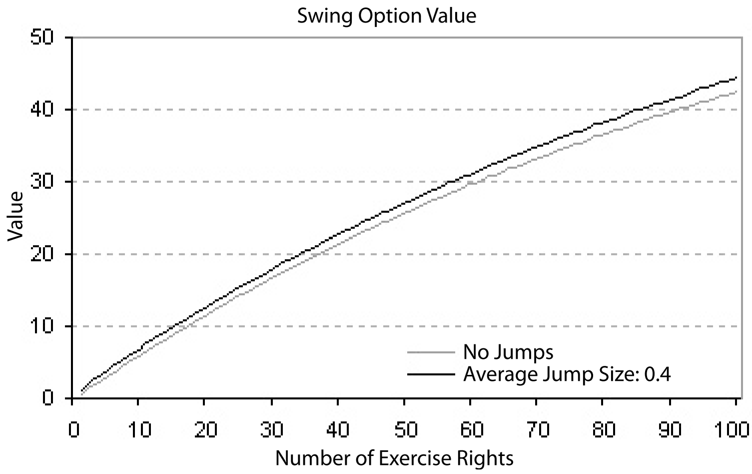
Our pricing function *swing.m*[5] performs well[6] and values 100 Swing Options with consecutive number of exercise rights between 1 to 100 approximately in 40 minutes. Figure 4.3 depicts the cumulative (not per right) values of one-year Swing options

[4] Subordinate layers can be seen as a set of individual Swing options with consecutive number of exercise rights ranging between $N-1$ and 0.

[5] See Appendix A for MATLAB® routines. A more comprehensive version of the function is available upon request from *nadi.aydin@uni-ulm.de*

[6] Combined with a Core2Duo 2.33GHz processor.

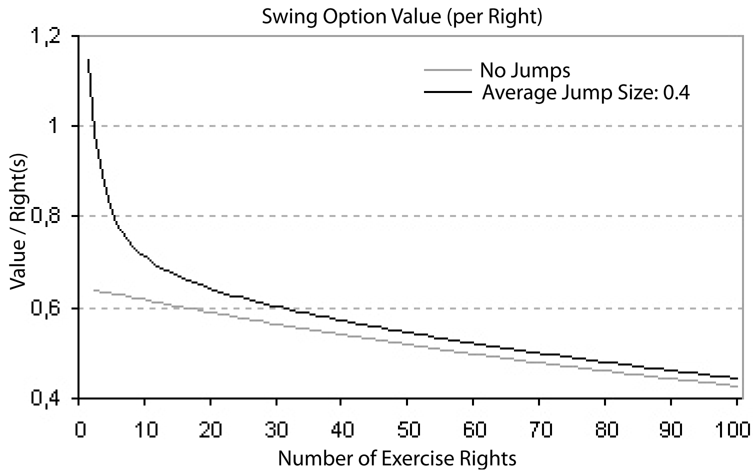
Figure 4.3: Values of Swing options with different number of exercise rights ($\alpha = 7$, $\beta = 200$, $\lambda = 4$, $\mu_J = 0.4$, $T = T_2 - T_1 = 1$, $dT = 1/365$, $f(t) = 0$ and r (risk-free rate) $= 0$)
[h]



with up to 100 exercise rights for the average jump sizes 0.0 and 0.4. We observe that there is a considerable jump risk premium on the cumulative option prices, which increases with the number of the exercise rights. Furthermore, in either cases, the marginal cost of the exercise rights to the option holder decreases as the number of rights increases. This is consistent with the nature of spikes since the probability of witnessing a large number of extraordinary events in the market during a single year is almost 0. On the other hand, the additional rights still add some value to the option.

Figure 4.4 plots the per right values of the options given in Figure 4.3. Although, the total jump risk premium on the cumulative Swing option value (Figure 4.3) increases as the number of call rights increases, the largest contribution comes from the first exercise right. Also, it is once again clear in the figure that the value of additional rights decreases with the cumulative number of exercise rights. Again, this can be explained by the decreasing marginal utility of exercise rights which is measured by the likelihood that an exercise right will be used against large spikes rather than moderate jumps driven by the Brownian part. Conceptually, this is similar to a standard basket CDS (Credit Default Swap) in that the credit swap buyers are protected against first n defaults. Given the correlation between defaults is close to zero, investors always pay a larger *spread-per-default* value as n gets smaller. Also, recalling our example in Section 4.1, it is now more clear that why the hedger will rationally prefer an option with n exercise rights to a bunch of n identical options of American or European type, each with a single right. If a buyer goes for the latter option, she/he will be overpaying for a number of reasons that we've already mentioned again in Section 4.1.

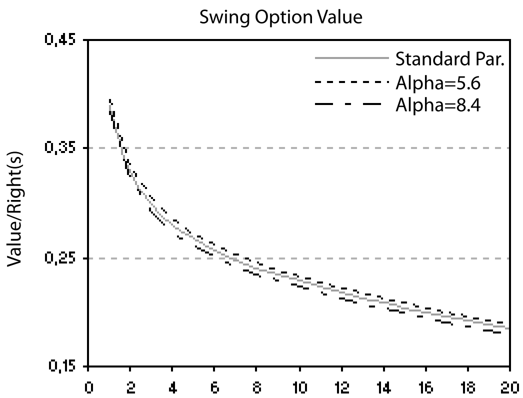
Figure 4.4: (per Right) Values of Swing options with different number of exercise rights ($\alpha = 7$, $\beta = 200$, $\lambda = 4$, $\mu_J = 0.4$, $T = T_2 - T_1 = 1$, $dT = 1/365$, $f(t) = 0$ and r (risk-free rate) $= 0$)



The effect of the correlation between extraordinary events should be examined separately. By nature, our spike process given in 3.1 driven by Poisson increments assumes that the number of jumps occurred in disjoint intervals are independent from each other. This is equivalent to saying, the likelihood witnessing an extraordinary event on one day is constant no matter what happened in the market on the day before. A detailed examination of how to generate spike trains with specified correlation coefficients is given in [20] and [4].

The spot price process S_t given in Equation 3.1 is obviously dominated by the process X_t whereas the process Y_t has only a limited effect when the occasional jumps with high mean-reversion rates are considered (see also Figure 3.1). Hence, the long-term variance of the process X_t will mainly determine the value of the model-implied long-term variance of S_t and is expected to have a direct relationship with the Swing option value. Unlike the Geometric Brownian Motion, where the variance of the process grows proportionally to the maturity, the OU process has a long-term variance which is bounded from above by $\sigma^2/2\alpha$, which is inversely proportional to the parameter α. The sensitivity results for a 60-day delivery period including up to 20 exercise rights, in Figure 4.5, confirms this reverse relationship between α and the Swing option value. The faster the price anomalies disappear, the less likely that the price will exhibit large deviations in the long-term and the lower the value of such an option will be. A similar relationship can be observed also between β and the Swing option value, however, the effect on price converges to zero as the number of rights increases. This can be explained by the same intuition behind the decreasing effect of the spike process itself on the per right Swing option values given in Figure 4.4. As the magnitude of the expected oscillations in the spike process decreases by an increase in the value of β (recall proof of Remark

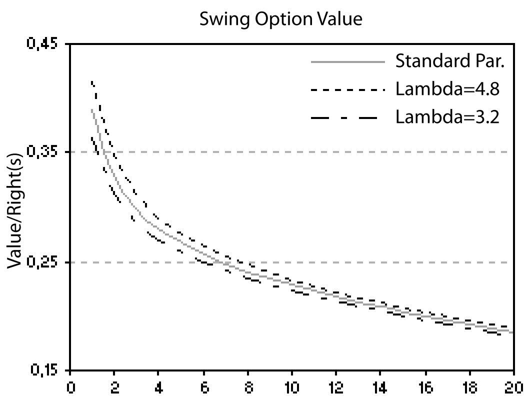
Figure 4.5: Sensitivities: (per Right) Values of Swing options with the standard parameters given in Figures 4.3 and 4.4 versus a $+/-20\%$ change in the mean-reversion parameters α and β.



3.1.1), so does the contribution of the spike process, which is characterized by only occasionally observed jumps, on the Swing option value.

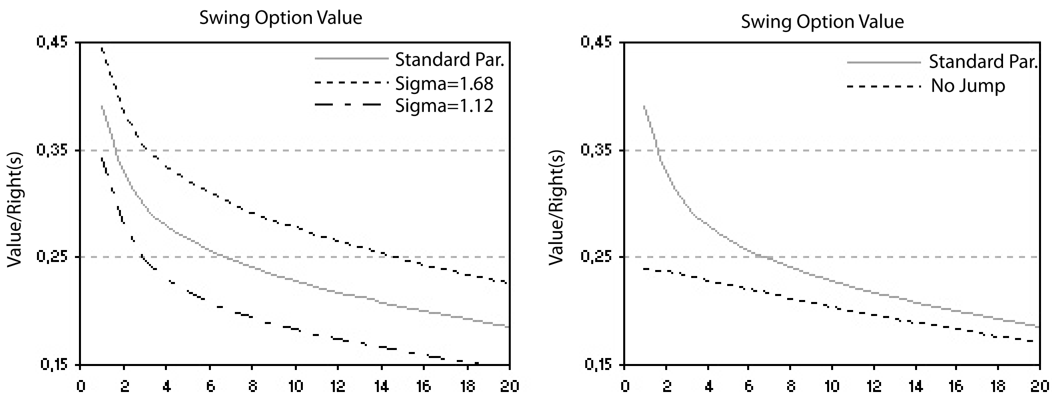
Results given in Figure 4.6 indicate that, given a 60-day delivery period and up to 20 exercise rights, a 20% increase (decrease) in the value of the jump frequency parameter λ results in an average of 6% increase (decrease) in the per right values up to first 3 rights. This turns into a 10% increase (decrease) on average when the average jump size parameter μ_J is considered. It is worth noting that the average jump size parameter has the largest effect among the other parameters on the value of a single-right Swing option. However, the most consistent and significant change

Figure 4.6: Sensitivities: (per Right) Values of Swing options with the standard parameters given in Figures 4.3 and 4.4 versus a $+/-20\%$ change in the jump parameters λ and μ_J.



is caused by the volatility paramater σ (see Figure 4.7). Although, a 20% increase (decrease) in the value of the volatility parameter σ causes an average of 15% increase (decrease) in the per right values up to first 3 rights, this ratio is strongly downward-biased due to the high per right values of the options with a few exercise rights, owing to the spike risk premium. In greater number of exercise rights, a 20% increase (decrease) in the value of the volatility parameter σ implies almost a constant 20% increase (decrease) in the per right values.

Figure 4.7: Sensitivities: (per Right) Values of Swing options with the standard parameters given in Figures 4.3 and 4.4 versus a $+/-20\%$ change in the volatility parameter σ. The left graph plots the value of the Swing option assuming a spot process both with and without jumps.



Chapter 5

Summary and Outlook

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5.1 Summary

In the present thesis, the valuation issues of a variety of power market options have been investigated with a special emphasis to the Swing options. We started with sketching the most prominent attributes of the electricity markets that distinguish them from the other commodity and non-commodity markets. Then, a spot price model which takes into account the most fundamental characteristics of the electricity markets, such as mean-reversion, seasonality and spikes, is introduced and its analytical properties are explored. This is followed by the approximation of the distributions of the forwards with and without a delivery period by means of the moment-matching method. Before valuing the standard type Swing options with various exercise rights numerically, an in-depth analysis of the Trinomial Forest Method has been made in the light of a thorough examination of Jaillet-Ronn-Tompaidis ([14]). Next, we've extended it by generalizing this method to the Grid Approach explained in Hambly-Howison-Kluge ([12]). This generalization has brought along two main challenges: (i) integration of the spike process as a second dimension into the valuation algoritm, (ii) use of larger grids with thinner increments to be able to efficiently approximate the cumulative density functions of the individual processes X_t and Y_t. The former challenge has also introduced two further challenges: (i.1) derivation of an explicit density function for the spike process, (i.2) dealing with the increasing complexity of the pricing algorithm by a factor $(N+1)\,(n_T n_x)$, a problem which was exacerbated by (ii) as well.

Towards this aim, the first approach was realized by analysing (a) both processes (i.e. the OU process and the spike process) with respect to the solutions to their stochastic differential equations, (b) the distributional properties of these solutions (i.e. forward prices) and (c) the values of the options written on forwards. After then, (d) an in-depth analysis of the Trinomial Forest Method is made and (e) the extended Jaillet-Ronn-Tompaidis ([14]) algorithm was applied to Swing options with multiple up-swings (call option rights). We also contribute to Hambly-Howison-Kluge ([12]) by deriving a Gamma density approximation to the stationary distribution of the truncated spike process and using a heterogeneous grid to calculate transition probabilities in the Grid Approach. The results indicate that the price of a Swing option with N upswing rights is always lower than the total value of N identical American options, whether the spot price process foresees spikes or not. However, taking the account of spikes makes the Swing option almost a unique tool for the energy risk managers. A Swing option with 100 exercise rights is as much as 70% cheaper than a bunch of 100 identical American options. This ratio implies only 33% saving in a market model without jumps. Thus, the raison d'être of the Swing options in the marketplace is economically clear. They serve the energy spike risk management purposes in a highly efficient way. The success of our numerical algorithm has been investigated with respect to the dependence of the valuation results on process parameters like the mean-reversion speeds for both processes, the jump frequency and magnitude parameters, and the volatility of the OU process. Our algorithm successfully captures the sensitivities of the Swing option value to different model parameters in terms of both the directions and the expected magnitudes. The decay in per right values (Figure 4.4) with the increasing number of rights is successfully reflected in spike parameters sensitivities. That is, the effect of a shift in either the jump frequency or average jump size on the per right option values converges to zero as the number of up-swing rights increases.

5.2 Outlook

Based on the present work, three main issues could be addressed:

1. Implementation of a correlated jump structure
2. Integration of the penalty functions into the pricing algorithm
3. Finding a semi-analytic approximation formula for the Swing option value.

The first issue is likely to have the most significant effect on the Swing option values and should be examined separately. Our Poisson-driven spike process assumes that the number of jumps occurred in the disjoint intervals are independent from each other. Hence, whether a spike occurred on a certain day or not, our *memoryless* Poisson process treats the likelihoods of witnessing or not witnessing an extraordinary event on the next day as equal. A detailed examination of how to generate spike trains with specified correlation coefficients is given in [20] and [4].

The second issue has already been mentioned shortly in the context of a standard Swing option in Section 4.2. A penalty function can be specified at the initialization of the Swing contract either by setting a per-unit penalty if the holder exercises a volume which corresponds to a multiple of the *basis amount* which is not included in the set of multiples, V, (recall Remark 4.3.1) or by compromising on a one-time penalty to be paid at maturity if the total swung volume exceeds a certain level.

The last issue should be addressed by finding an approximation that expresses the Swing option value in terms of a convex combination of upper and lower bounds, i.e. a set of Bermudan[1] and European options, respectively. Obviously, the solutions to options with an early exercise problem (e.g. Bermudan, American) could not be fully analytic, so would a possible approximate solution to the Swing option value.

[1] Or American options, as in our case where the Swing option can be exercised at any date throughout the delivery period.

Appendix A

MATLAB® Routines

swing.m

Inputs: length of delivery period (**T**); number of exercise rights (**N**); exercise price (**D**); risk-free rate (**r**); number of x, y refinements and number of periods (**nofx**, **nofy**, **nofp**); mean-reversion rates (**alpha**, **beta**); volatility (**sigma**); average jump size (**muJ**), jump frequency (**lambda**).

Outputs: Swing option value (**value**); transition probability matrix (**trans**)

The following function is designed to value a Swing option which gives the holder N *call* rights. It can be run directly by transferring the code into a MATLAB® editor, saving the function as *swing.m* and setting its directory as default. Please note that high degrees of refinement selected (i.e. *nofx* x *nofy*) can turn minutes into long hours. The function *swing.m* can price both a single Swing option with N exercise rights (array='off') and an array of N Swing options with up to N exercise rights (array='on').

```
function [value,trans]=swing(T,N,D,r,nofx,nofy,nofp,alpha,...
sigma,beta,mu_J,lambda,array)
if mu_J==0 nofy=1; else end
dt=1/nofp;
X0=0; Y0=0;
dX=2*sigma*sqrt(T)*(1/nofx);
gridY(1)=10^(-10);
for i=2:49
    dY(i-1)=(10^(-10))*(1.6)^(i-1);
    gridY(i)=gridY(i-1)+dY(i-1);
end
dY=dY';
gridY=gridY';
pdfY=((1/(gamma(lambda/beta)*mu_J^(lambda/beta)))*(exp(-(gridY)/mu_J)...
-exp(-(gridY)*exp(beta*T)/mu_J)).*((gridY).^(lambda/beta-1)));
```

```
prob=pdfY.*[gridY(1);dY];
gridX=[nofx/2:-1:-nofx/2+1]'*dX+X0;
gridY=[gridY;0];
for i=1:nofy
    grid((i-1)*nofx+1:(i-1)*nofx+nofx,:)=[repmat(gridY(i),nofx,1),gridX,...
     repmat(gridY(i),nofx,1)+gridX];
end

%Transition probabilities
gridXA=repmat(gridX,1,nofx); gridXB=gridXA';
piX=normpdf(gridXB,gridXA*exp(-alpha*dt),sqrt((sigma^2/(2*alpha))*...
(1-exp(-2*alpha*dt))))*dX;
if (mu_J==0) piY=1;
else piY=prob; piY(end+1,1)=1-sum(piY);
end
piXYA=repmat(piX,nofy,nofy);
for i=1:nofy
    piXYB(1:nofx*nofy,(i-1)*nofx+1:(i-1)*nofx+nofx)=piY(i);
end
trans=piXYA.*piXYB;
clc;
fprintf('Transition matrix (%g x %g) calculated...\n',nofx*nofy,nofx*nofy);

%Swing Valuation
fprintf('Swing valuation started...\n');
layerdown=repmat(0,nofx*nofy,T*nofp);
pricedown=layerdown;
priceup(:,T*nofp)=exp(grid(:,3))-D;
for j=T*nofp-1:-1:1
priceup(:,j)=max(trans*priceup(:,j+1)*exp(-r*dt),(max(exp(grid(:,3))-D,0)+...
trans*pricedown(:,j+1)*exp(-r*dt)));
end
priceup_initial=priceup;
switch array
    case 'off'
        clc;
        fprintf('nadi.aydin@uni-ulm.de');
        fprintf('\n\nSwing is being valued for %g rights. Progress: ',N);
        if N==1 price=priceup;
            if mod(1*100/N,10)==0;
                fprintf('%g %% ',1*100/N);
            end
        else
        for i=2:N
            clear pricedown;
```

```
            pricedown=priceup;
            clear priceup;
            priceup(:,T*nofp)=max(exp(grid(:,3))-D,0);
            for j=T*nofp-1:-1:1
                priceup(:,j)=max(trans*priceup(:,j+1)*exp(-r*dt),...
                 (max(exp(grid(:,3))-D,0)+trans*pricedown(:,j+1)*exp(-r*dt)));
            end
            if mod(i*100/N,10)==0;
                fprintf('%g %% ',i*100/N);
            end
            if i==N
                fprintf('\n');
            end
        end
            price=priceup;
            end

        % Value Swing Option
        piX0=normpdf(gridX,0,sqrt((sigma^2/(2*alpha))*(1-exp(-2*alpha*dt...
         ))))*dX;
        piXYA0=repmat(piX0,nofy,1);
        for i=1:nofy piXYB0((i-1)*nofx+1:(i-1)*nofx+nofx,1)=piY(i); end
        piXY0=piXYA0'.*piXYB0';
        value=piXY0*price(:,1);
    case 'on'
        instep=input('Which step to initialize the algoritm?: ')
        for k=instep:N
            if k==instep clc;
                fprintf('nadi.aydin@uni-ulm.de\n\nProgress: %g %%\n',0);
                fprintf('Swing is being valued for %g right(s)',k);
                if k==1
                    price(k).p=priceup;
                else
                    for i=1:k-1
                        clear pricedown;
                        pricedown=priceup;
                        clear priceup;
                        priceup(:,T*nofp)=max(exp(grid(:,3))-D,0);
                        for j=T*nofp-1:-1:1
                        priceup(:,j)=max(piXY*priceup(:,j+1)*exp(-r*dt),...
                         (max(exp(grid(:,3))-D,0)+piXY*pricedown(:,j+1)...
                         *exp(-r*dt)));
                        end
                        if mod(i*100/k,10)==0
                            fprintf('%g %% ',i*100/k);
```

```
                        end
                        if i==k
                            fprintf('\n');
                        end
                    end
                    price(k).p=priceup;
                end
            else
                fprintf('Swing is being valued for %g exercise right(s)',k);
                clear pricedown;
                pricedown=priceup;
                clear priceup;
                priceup(:,T*nofp)=max(exp(grid(:,3))-D,0);
                for j=T*nofp-1:-1:1
                    priceup(:,j)=max(piXY*priceup(:,j+1)*exp(-r*dt),...
                     (max(exp(grid(:,3))-D,0)+piXY*pricedown(:,j+1)...
                     *exp(-r*dt)));
                end
            end
            price(k).p=priceup;
            % Value Swing Option
            piX0=normpdf(gridX,0,sqrt((sigma^2/(2*alpha))*(1-exp...
             (-2*alpha*dt))))*dX;
            piXYA0=repmat(piX0,nofy,1);
            for i=1:nofy
                piXYB0((i-1)*nofx+1:(i-1)*nofx+nofx,1)=piY(i);
            end
            piXY0=piXYA0'.*piXYB0';
            value(k)=piXY0*price(k).p(:,1);
            if k<=N-1
                clear price(k).p; clc;
                fprintf('nadi.aydin@uni-ulm.de\n\nProgress: %g %%\n',k*100/N);
            elseif k==N price=price(k).p; clc; fprintf('100 %% Finished.\n');
            end
        end
        hold on
        xlabel('Number of Exercise Right(s)');
        ylabel('Value/Right(s)');
        title('PerRight Value of the Swing Option');
        plot([instep:1:N],value(instep:1:N)./[instep:1:N]);
        hold off
end
```

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