METHODOLOGY AND POLICY

Asset-backed stable numéraires for sustainable valuation

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ABSTRACT
Interest rates underpin almost every instrument/transaction in conventional financial markets. Valuation of the instruments in relation to interest rates remains meaningful only if *cash* can be attributed a worth of its own (which is generally assumed to accumulate over time). The relevant concepts such as the stochastic short rate and the conventional numéraire (i.e., the money market account) not only become restrictive when one attempts to build more realistic models in quantitative finance, but also −as we demonstrate in this work− have de-stabilizing effects on asset valuations. This paper presents a detailed critique of the conventional numéraire and proposes an asset-backed stable numéraire for sustainable valuation of assets and/or transactions. In particular, we propose some of the key benchmarks of a sustainable footprint as a numéraire currency. Notably, we unveil the subtle assumption behind the practice of straightforward factorization in conventional relative pricing that the asset is perfectly correlated with the numéraire.

KEYWORDS
Asset-backed numéraire; interest-rate benchmarks; fair valuation; conventional pricing; risk-neutral valuation

1. Introduction

In economies based on established sovereign currencies, interest rates seem to be central to the functioning of financial markets. However, there is also growing interest in the possible ways the markets can exist *without* interest rates. A recent example/evidence are the crypto-currencies which −by their nature− offer no direct short-term interest and, hence, does not possess an established forward curve. Thus, holdings of bitcoin as recorded in the ledger do not earn interest. [1] But, of course, this may change as soon as the distributed ledger technologies might eventually find a way of rewarding the holders of positions in virtual currencies with interest on a continuous basis. Although it might seem impossible at the first glance that one can build an interest rate model for which the short rate is identically zero, cyrpto-markets would develop their interest rate models applicable −hence interest rates related− to digital currencies such as bitcoin and etherium. But the meantime will be a chance for a proof of concept for models of

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[1]The ecological cost of crypto-currencies is beyond the scope of this work and assessed in Stoll et al. (2019)

no interest.

The numéraire approach to use a fixed-income instrument as a reference asset is common in the conventional valuation practice. This approach is mainly based on forward rates and the existence of a martingale probability measure, and used primarily to determine the value (i.e., relative price) of financial assets. However, this reference rate is far from duly reflecting the real economic peculiarities of the country where the contract is signed. This is particularly evident at times when the cracks in one financial system are transmitted through interest rate mechanisms into systems even in far geographical regions, resulting in crises at global scale. Furthermore, the practice of using the conventional numéraire for valuation stretches beyond debt instruments and also affects more project-based instruments and/or real assets or transactions that are not necessarily exposed to interest rate risk directly.[2]

In finance, valuation of assets eventually boils down to fairly computing the expected values for various stochastic payoffs using an appropriate probability measure, along with a pricing kernel or numéraire. A 'fair' and 'conscious' valuation, in this sense, would mean judging the value of an asset by constrasting its performance with a real economical/ecological benchmark and/or possible impacts that might stem from the utilization/consumption of fundamental assets (such as potable water, energy and clean weather). In view of this, this study aims to introduce a mathematical framework for the fair and conscious valuation of essential global and local resources as fundamental assets, based on their utilization/consumption rate as compared to their availability. As a future research direction, the authors aim to extend the present framework for setting up a reference currency (as a bundle of critical assets), relative to which the fair/coherent pricing method can be applied to price any good, commodity or service, once the statistical relation of the latter with the fundamental basket is known.

The rest of the study is organized as follows: Section 2 offers a quick overview of the related works. Section 3 establishes the equivalence relation between the two most common conventional numéraires, namely, the money market account and zero-bond, and discusses their major drawbacks. Section 4 introduces the new asset-backed numéraire −along with few practical examples− and demonstrates the possible ways it would yield a more stable and fair valuation. Finally, Section 5 concludes with some forward-looking remarks.

2. Related work

The recent report by Bank of England (see van Steenis (2019)) is a vivid example that, with ultra low interest rates and huge need for adapting to new business models, the interest rate institutions and their contracts will struggle to back up their cost of capital and, therefore, their existence may become at risk. This situation is in fact a late agreement of the market dynamics with the moral principle that 'return should be commensurate to risk.'

On the other hand, in Lipton (2019), it is argued that "by combining the ideas of narrow banking (for stability), digital currencies (for efficiency and transparency), and use of an asset-backed currency for international trade (to reduce trade distortions and inequality), we see the potential for dramatic improvement in the global financial

[2] The latter point also leads to second-round effects.

system" and indeed that 'today, for the first time ever, there is a possibility of designing a digital supranational currency backed by a diverse and widely held assets.'

Complementing the work above, a rudimentary yet interesting framework for sustainable finance is covered in Schoenmaker (2017). This includes, among others, addressing concentration in carbon-intensive investments, ensuring social foundations (such as access to water, sanitation, etc.) are respected, avoiding short-termism, and refraining from investing in 'sin' sectors. One notable concept mentioned in Schoenmaker (2017) is the 'internalization of externalities' which means that the negative social and environmental *externalities* are incorporated into the decision-making process today to avoid possible future adjustments which may occur abruptly and deem a financial asset and/or transaction unviable. One problem is that even if the externalities related to sustainable development (such as carbon emissions) are internalized at the institutional level, there is no guarantee that micro and macro outcomes will overlap. As the study argues, this is because the institutions use not only false but also distinct benchmarks to value their assets and/or transactions − which most of the time ignore social and ecological impacts on welfare.

One of the key questions in financial mathematics is 'which numéraire should be used in denomination to express the payoff from an asset or transaction to apply an expectation under an appropriate probability measure.' In Hulley et al. (2005), Platen and Heath (2006) and Platen (2006), the concept of 'growth optimal portfolios' (GOPs) is applied to produce benchmarked portfolios/assets that are supermartingale under the real-world martingale measure. In this case, the numéraire portfolio is the one which cannot be outperformed systemically and, thus, it is growth optimal.

In addition to the above, the notion that strict local martingales should play a role in contemporary quantitative finance has been considered by other authors as well. Brody et al. (2019), for example, notably present a term structure for cryptocurrencies in the absence of a short rate and money market account, and they do so by exploiting the strict local martingale property as well as the positivity of the pricing kernel $\{\pi_t\}_{t\geq 0}$.[3]

To authors' knowledge, this paper is one of the earliest works which seeks to link the conceptual discussions about sustainable/responsible finance with some tangible mathematical foundations.

3. The conventional numéraire

The fundamental theorem of asset pricing (FTAP) essentially states that *no free lunch with vanishing risk* rule is equivalent to the existence of a risk-neutral equivalent martingale measure (EMM). But the latter is not more than a mathematical technicality which aims to streamline the asset returns around a corresponding risk-free numéraire, which is −by conventional assumption− the money market account. Indeed, the existence of EMM is more a mathematical convenience than an economic necessity (Platen and Heath 2006). There is no reason a priori that the real *risk-neutral* performance should be around that of a money market account and thus simply be boiled down to *time value*. Furthermore, basing quantitative methods on classical risk neutral pricing is restrictive. One clear example of this arise in the FTAP which ignores trends in asset

[3] The latter two properties imply that $\{\pi_t\}$ is a supermartingale, which ensures that the asset (e.g., bond) price is a decreasing function of time to maturity T.

price. This is also case in alternative methods such as the stochastic discount factor and pricing kernel (Cochrane 2001; Constantinides 1992). That being said, one can choose a different martingale measure for expressing economic *neutrality* or *sustainability*.

To better comprehend the problem, first, the notion of relative/benchmark pricing should be understood. In conventional finance, one typically assumes that there exists the quadruple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{T}}, \mathbb{P})$, i.e., the filtered probability space, with $\mathbb{T} = [0, T]$ with $t, u \in \mathbb{T}$. Let $\{S_t\}_{t\geq 0}$ be the value process of an asset which delivers a single cash flow X_T at time T. The following definition puts the notion of relative pricing a bit more into context.

Definition 3.1 (Numéraire)**.** Let $\{\mathcal{F}_t\}_{t\in\mathbb{R}^+}$ be a filtration (i.e., increasing set of σ-algebras that carry information). Then, a numéraire is any strictly positive $\{\mathcal{F}_t\}_{t\in\mathbb{R}^+}$-adapted process $\{C_t\}_{t\in\mathbb{R}^+}$ that can be taken as a unit of reference when pricing an asset or claim.

Definition 3.2 (Benchmarked value)**.** The benchmarked value $\tilde{S}_t$ is the value of S_t relative to a reference asset C_t (real or financial), i.e., $\tilde{S}_t = S_t/C_t$.

Definition 3.3 (Fair price)**.** A price S_t is fair if its value $\tilde{S}_t$ benchmarked under a certain measure $\mathbb{Q}$ forms a martingale, i.e.,

$$\mathbb{E}_t^{\mathbb{Q}}\left[\tilde{S}_u\right] := \mathbb{E}^{\mathbb{Q}}\left[\tilde{S}_u|\mathcal{F}_t\right] = \tilde{S}_t \quad (t \leq u). \tag{1}$$

Having given the basic definitions, it is important to understand that in quantitative finance more meaningful quantities are the relative ones, which facilitate value comparisons, and one always needs a numéraire to express value. To express the value of a banana, one can benchmark bananas to either lemons (e.g., 1 banana = 2 lemons) or a fiat currency (e.g., 1 banana = \$0.5). Thus, lemons and the US\$ are two possible numéraires for expressing value (or, relative price). This performs one of the two functions of a numéraire: to serve as a unit.

The second function, which is to serve as a benchmark, is covered −in conventional sense− by the FTAP which asserts that the *value* of an asset ($\tilde{S}_t$) would remain constant over time if its price (S_t) is evaluated using an appropriate risk-neutral measure and benchmarked against the conventional numéraire: the money market account $\{M_t\}_{t\in\mathbb{R}^+}$. More formally, the $\{\mathcal{F}_t\}$-adapted short rate process $\{r_t\}_{t\geq 0}$ is conventionally assumed to determine the value process for the typical conventional numéraire, the money market account, as

$$M_t = \exp\left(\int_0^t r_s \mathrm{d}s\right) \tag{2}$$

with $M_0 = 1$. Thus, in the sense Eq. (1) also implies $S_t = \mathbb{E}\left[S_T/(M_T/M_t)|\mathcal{F}_t\right]$, any spot price S_t in conventional markets can been seen as the price relative to the unit-initialised money market account denominated in a fiat currency. But, again, there is no reason *a priori* why the risk-neutral performance should correspond to that of $\{M_t\}$.

In the conventional pricing approach, the $\mathcal{F}_t$-adapted kernel process $\{\pi_t\}_{t\geq 0}$ is used to switch from $\mathbb{P} \equiv \mathbb{P}(X_T < x|\mathcal{F}_t)$, i.e. the real-world measure (which induces asset-specific performance), to $\mathbb{Q} \equiv \mathbb{Q}(X_T < x|\mathcal{F}_t)$, i.e. the risk-neutral EMM (which induces a unique performance for all assets).

Conventional price relative to the above numéraire (after a change of numéraire and under the risk-neutral measure) is given by

$$\frac{S_t}{M_t} = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{S_u}{M_u}\right|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{X_T}{M_T}\right|\mathcal{F}_t\right] = \int_{\mathbb{X}} \frac{X_T}{M_T}\mathrm{d}\mathbb{Q} \quad (\forall u \geq t), \tag{3}$$

(with $\mathbb{X}$ being the domain of X_T) whereas before the numéraire change (i.e., under the physical or real-world measure) as

$$S_t\pi_t = \mathbb{E}^{\mathbb{P}}\left[\pi_u S_u|\,\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{P}}\left[\pi_T X_T|\,\mathcal{F}_t\right] = \int_{\mathbb{X}} X_T\pi_T\mathrm{d}\mathbb{P} \quad (\forall u \geq t). \tag{4}$$

Note that $S_{T-} = X_T$. Equation (3) indeed implies that the pricing using a conventional numéraire also derives from relative pricing (as M_u/M_t is the time-u value of a unit-initialized money market account). In (4) the part $\pi_T\mathrm{d}\mathbb{P}$ governs the measure change from $\mathbb{P}$ to $\mathbb{Q}$ (Shreve 2004). Intuitively, the measure change means tuning the drift of the original stochastic process $\{S_t\}_{t\geq 0}$ such that its expected rate of change $\forall t \in \mathbb{T}$ agree with the so-called "risk-free" rate. To express this transformation clearly, we can write

$$\frac{1}{M_T}\mathrm{d}\mathbb{Q} = \pi_T\mathrm{d}\mathbb{P}. \tag{5}$$

Equation (5) yields the well-known Radon−Nikodym derivative

$$\Lambda_t = \left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_t = M_t\pi_t \tag{6}$$

which is a martingale. Here $\mathbb{Q}$ is is equivalent to $\mathbb{P}$ in the sense that they have overlapping null sets.

Equivalence of money market and zero-bond numéraires

We can indeed verify that the equivalent martingale measure $\mathbb{Q}$ (corresponding to M_t) and the one that corresponds to zero bond prices $B_t = \{B_t\}_{t\leq T}$) (say $\hat{\mathbb{Q}}$) can be shown to be *different* numéraires although they are strongly coupled substitutes. To see this, set $S_T = 1$ in (3):

$$B_t = M_t\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{1}{M_T}\right|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\left.\exp\left(-\int_t^T r_s\mathrm{d}s\right)\right|\mathcal{F}_t\right], \tag{7}$$

or, equivalently, by (4)

$$B_t = \frac{1}{\pi_t}\mathbb{E}^{\mathbb{P}}\left[\pi_T|\,\mathcal{F}_t\right]. \tag{8}$$

Now consider the relations

$$\frac{S_t}{M_t} = \mathbb{E}^{\mathbb{Q}}\left[\left.\frac{S_T}{M_T}\right|\mathcal{F}_t\right] \implies S_t = M_t\mathbb{E}^{\mathbb{Q}}\left[\left.\frac{S_T}{M_T}\right|\mathcal{F}_t\right] \tag{9}$$

and

$$\frac{S_t}{B_t} = \mathbb{E}^{\hat{\mathbb{Q}}}\left[\left.\frac{S_T}{B_T}\right|\mathcal{F}_t\right] \implies S_t = B_t\mathbb{E}^{\hat{\mathbb{Q}}}\left[\left.\frac{S_T}{B_T}\right|\mathcal{F}_t\right] \tag{10}$$

Then, from (9) and (10) we obtain

$$\mathbb{E}^{\hat{\mathbb{Q}}}\left[S_T|\,\mathcal{F}_t\right] = \frac{1}{B_t/M_t}\mathbb{E}^{\mathbb{Q}}\left[(B_T/M_T)S_T|\,\mathcal{F}_t\right] = \frac{1}{\hat{\pi}_t}\mathbb{E}^{\mathbb{Q}}\left[\hat{\pi}_T S_T|\,\mathcal{F}_t\right], \tag{11}$$

where $\hat{\pi}_t = (B_t/M_t)(\mathrm{d}\hat{\mathbb{Q}}/\mathrm{d}\mathbb{Q})$ is the kernel to shift the measure from $\hat{\mathbb{Q}}$ to $\mathbb{Q}$. Thus, Eq. (11) shows the way the EMM w.r.t. B_t (i.e., $\hat{\mathbb{Q}}$) is related to the EMM w.r.t. M_t ($\mathbb{Q}$).

Referring to (3), we argue that neither $\{M_t\}_{t\geq 0}$ nor $\{B_t\}_{t\geq 0}$ as a benchmark for relative pricing yields a stable relative price over time for the asset being valued. This is because the short-rate process $\{r_t\}_{t\geq 0}$ not only fails to represent real economic circumstances but also leads to unnecessarily high sensitivities for asset prices during expansionary periods (see below). Furthermore, the correlation structure between the conventional benchmark and the asset payoff is generally ignored for the sake of computational simplicity and this further conceals the de-stabilizing effects on relative prices. A clear evidence to this are the sample assertions like "the low Treasury note yields making stock valuations *look* attractive and considerably better investments than bonds", or how "assets look cheaper [or more expensive]" due to fluctuations in interest rates, or the way "[monetary] easing provides a short-term boost to market *psychological* since assets returns are slightly higher than when the interest rates are rising", etc.

De-stabilizing effects of the conventional numéraire

Making the money market account (or zero bond) and real asset class (including even the fundamental ones) two relative markets for pricing (hence, substitutes for investment) leads to cycles of artificial appreciation/depreciation in one market relative to other without any real economic justification. Figure 1 displays one vivid example of how low interest rates illusively boost asset prices. From a conventional point of view, one can argue that the house prices H_t increase because r_t (thus M_u/M_t) slows down, i.e., money becomes *cheaper*. But the money is only a medium of exchange and does not have a value *per se*. House prices should increase only if there is a depreciation in the price of assets to which house prices are benchmarked and this asset bundle should reflect the real economic circumstances (like energy prices). Furthermore, pricing of fundamental assets relative to money market account also falsely assumes that *all* people are financially literate enough (or have access to financial intermediaries) to shift between assets and interest-rate benchmarks in a timely manner to preserve the value of their assets − which is not the case in reality.

As we discussed above, one notable problem with Eq. (3) is that it is generally assumed that the terminal distributions of $\{M_t\}$ and $\{S_t\}$ are independent, which allows

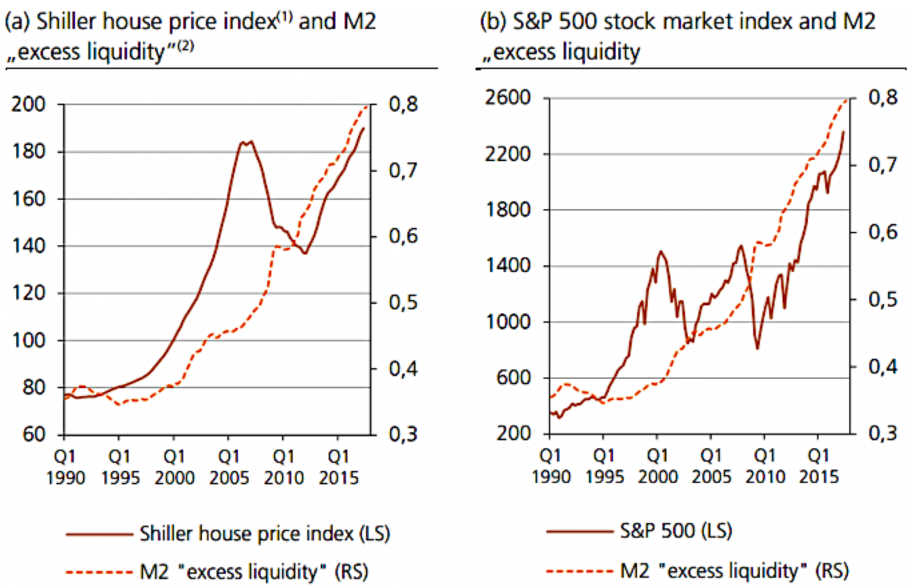

Figure 1. How conventional numéraire causes artificial cycles in asset prices. Adapted from Thomson Financial. (2) Stock of M2 divided by nominal GDP.

the latter expectation to be *illicitly* factorized as

$$\frac{S_t}{M_t} = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{S_u}{M_u}\right] \neq \frac{\mathbb{E}_t^{\mathbb{Q}}\left[S_u\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[M_u\right]} = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(\int_0^u r_s \mathrm{d}s\right)\right]^{-1} \mathbb{E}_t^{\mathbb{Q}}\left[S_u\right] \tag{12}$$

Knowing that an asset's price is determined by the expected value of the sum of its cash flows or convenience yield, the first de-stabilizing impact of pricing relative to an interest-based numéraire comes from the fact that a stream of payments into gratuity is equivalent –in the most simplifictic case– to $D/(r-g)$ where r is the periodical "risk-free" rate bootstrapped from the observed yield curve ($r > g$), whereas D and g are the initial cash inflow and its growth rate, respectively. Although the underlying setup is very simplistic, one can observe that as $r \to g$ the expected value gets extremely sensitive to the shifts in the yield curve. Secondly, the assumption of independence ($S \perp M$) leads to the concealment of additional de-stabilising effects that come from the ignored correlation structure between $\{M_t\}$ and $\{S_t\}$. To see this, consider the random processes $\{S_t\} \in \mathbb{R}^+$ (i.e., the asset price process) and $\{M_t\}$ (i.e., the money market account). A well-known improved approximation to $\mathbb{E}[S_u/M_u]$ for $t \leq u$ around $\boldsymbol{\theta} = (\mu_X, \mu_M)$ is

$$\tilde{\mathbb{E}}_t\left[S_u/M_u\right] = \frac{\mathbb{E}_t\left[S_u\right]}{\mathbb{E}_t\left[M_u\right]} - \frac{\mathbb{C}\mathrm{ov}_t\left[S_u, M_u\right]}{\left(\mathbb{E}_t\left[M_u\right]\right)^2} + \frac{\mathbb{V}_t\left[M_u\right]\mathbb{E}_t\left[S_u\right]}{\left(\mathbb{E}_t\left[M_u\right]\right)^3}. \tag{13}$$

The reader is referred to Stuart and Ord (1998) for a quick verification of this result. Note that $S_u, M_u \in \mathbb{R}^+$ and, hence, $\mathbb{E}_t\left[S_u/M_u\right] \in \mathbb{R}^+ \ \forall t \in \mathbb{T}$. Eq. (13) can be arranged as

$$\begin{aligned}
\tilde{\mathbb{E}}_t\left[S_u/M_u\right] &= \frac{\mathbb{E}_t\left[S_u\right]}{\mathbb{E}_t\left[M_u\right]}\left[1-\left(\frac{\mathbb{C}\mathrm{ov}_t\left[S_u,M_u\right]}{\mathbb{E}_t\left[S_u\right]\mathbb{E}_t\left[M_u\right]}-\frac{\mathbb{V}_t\left[M_u\right]}{\left(\mathbb{E}_t\left[M_u\right]\right)^2}\right)\right] \\
&= \frac{\mathbb{E}_t\left[S_u\right]}{\mathbb{E}_t\left[M_u\right]}\left[1-\left(\frac{\mathbb{E}_t\left[S_uM_u\right]-\mathbb{E}_t\left[S_u\right]\mathbb{E}_t\left[M_u\right]}{\mathbb{E}_t\left[S_u\right]\mathbb{E}_t\left[M_u\right]}-\frac{\mathbb{E}_t\left[M_u^2\right]-\left(\mathbb{E}_t\left[M_u\right]\right)^2}{\left(\mathbb{E}_t\left[M_u\right]\right)^2}\right)\right] \\
&= \frac{\mathbb{E}_t\left[S_u\right]}{\mathbb{E}_t\left[M_u\right]}\left[1-\left(\frac{\mathbb{E}_t\left[S_uM_u\right]}{\mathbb{E}_t\left[S_u\right]\mathbb{E}_t\left[M_u\right]}-\frac{\mathbb{E}_t\left[M_uM_u\right]}{\mathbb{E}_t\left[M_u\right]\mathbb{E}_t\left[M_u\right]}\right)\right] \qquad (14)
\end{aligned}$$

Denoting the term in round brackets by γ_t, one can quickly verify[4] that

$$\gamma_t : \begin{cases} = 0, & \rho = 1 \\ < 0, & \rho < 1. \end{cases} \qquad (15)$$

where ρ is the correlation. The relation (15) also implies −for *all* cases where the asset and numéraire are not perfectly correlated− that $\tilde{\mathbb{E}}_t\left[S_u/M_u\right] \geq \mathbb{E}_t\left[S_u\right]/\mathbb{E}_t\left[M_u\right]$ (meaning that simple factorization will yield consistently biased valuations). This notably points to the fact that the current practice of relative pricing in conventional finance implicitly assumes a perfect correlation between the asset and conventional numéraire, i.e., the money market account, which doesn't agree with the real world. As expected, the de-stabilizing effect is greatest when the asset class is a natural substitute to the interest-based instruments and/or strongly benefits from an adverse move in interest rates (think about government bonds vs. equities).

The weirdness of this subtle assumption is evident as it makes the theoretical variance of the relative price, i.e,

$$\tilde{\mathbb{V}}_t\left[S_u/M_u\right] = \left(\frac{\mathbb{E}_t\left[S_u\right]}{\mathbb{E}_t\left[M_u\right]}\right)^2\left(\frac{\mathbb{V}_t\left[S_u\right]}{\left(\mathbb{E}_t\left[S_u\right]\right)^2}-2\frac{\mathbb{C}\mathrm{ov}_t\left[X_u,M_u\right]}{\mathbb{E}_t\left[X_u\right]\mathbb{E}_t\left[M_u\right]}+\frac{\mathbb{V}_t\left[M_u\right]}{\left(\mathbb{E}_t\left[M_u\right]\right)^2}\right) \qquad (16)$$

equal to 0 (again, simply consider $S_t = kM_t$). For all other cases of imperfect correlation, the above equation would return an non-zero variance that decreases (resp. increases) as ρ converges to (resp. diverges from) +1. From Eq. (14)-(16), we can deduce that the relative price of the asset is most sensitive to the changes in the future performance of the benchmark (thus most unstable) when the asset is most negatively correlated with the latter.

Thus, we demonstrated on important flaw behind the current "shallow" relative pricing (i.e., using straightforward factorization). In the next section, we try to resolve a second major drawback with the conventional pricing approach by proposing −instead of M_t or B_t− a new numéraire C_t, which based on some of the key benchmarks for a sustainable ecological/economical footprint. The main benefit of this asset-backed numéraire will be its *stabilizing* effect on the relative asset prices as it more realistically represents real developmental dynamics. We argue that an asset or transaction should be more (resp. less) valuable over time only if it outperforms (resp. underperform) a sustainable footprint benchmark. This is somewhat similar to the viewpoint presented

[4] For this, consider cases $\rho = 1$ ($S_u = kM_u$) and $\rho = 0$. For the first case, it is apparent that $\gamma_t = 0$ whereas the second case implies $\mathbb{E}_t\left[S_uM_u\right] = \mathbb{E}_t\left[S_u\right]\mathbb{E}_t\left[M_u\right]$ and therefore $\gamma_t < 0$.

in Schoenmaker (2017) that the public discount factor should be very small or zero[5] and, hence, the current and future be valued equally. The conventional numéraire approach is in fact built on the very assumption that it will be more expensive to consume/utilize a fundamental good *in future* than it is *today*.

4. Asset-backed numéraire

Although the technicalities of building a reference basket/currency falls beyond the scope of this work, we would like to touch on at least one possible way of doing this. We illustrate this by proposing a number of carefully selected fundamental assets (or a currency based on them) as an asset-backed numéraire. In the first example below, we demonstrate how a currency numéraire can be backed by an environmentally critical asset (i.e., drinkable water) but still reflect regional differences. In the second, we associate the numéraire with another environmentally critical asset (CO_2 allowance) which is of regulated supply and this time let the numéraire be dependent on the utilised quantity.

Example 1: Potable water and regional dependence

It is particularly problematic if a currency is fixed for a region which is too large to display uniform economic conditions − like the EU.[6] Let $C_\ell(t)$ denote the time-t price of per cubic meter drinkable water at Location ℓ. This is an example to a natural currency which is globally available, yet at different prices at different locations $\mathcal{L} = \{1, 2, 3, \dots\}$. Taking currency C_ℓ as numéraire, any other asset $S(t)$ yields $\tilde{S}(t) \equiv S(t)/C_\ell(t)$ as a relative price, benchmarked to the price of potable water. Thus, at any particular location ℓ, there would exist an EMM $\mathbb{Q}_\ell$ under which $\tilde{S}(t)$ is a martingale, i.e.,

$$\tilde{S}(t) = \mathbb{E}_t^{\mathbb{Q}_\ell}\left[\tilde{S}(u)\right] \quad (t \le u), \tag{17}$$

with S and C_ℓ being modelled using stochastic processes for their uncertainty. For instance, availability of potable water at location ℓ (hence, its local price process $C_\ell(t)$) can depend, in addition to a deterministic trend factor μ_ℓ, on some stochastic climate factor L_ℓ (note that the latter is also allowed vary locally), such that

$$C_\ell = f(X(t)) \quad \text{with} \quad \mathrm{d}X(t) = \mu_\ell(t)\mathrm{d}t + \sigma_\ell(t)\mathrm{d}L_\ell. \tag{18}$$

In particular, L_ℓ can be some stable, variance-normalized Lévy process and $f(X)$ would admit various forms, such as exponential.[7]

Now, how would cross-regional dynamics work? To answer this, assume the availability of potable water in location 1 is expected to be scarce. Thus, the quantity of interest is $\mathbb{E}_t^{\mathbb{Q}_\ell}\left[X_T/C_\ell(T)\right]$, which can be approximated using Eq. (13). Now consider two extreme cases: the correlation (ρ) between $\{X_t\}$ and $\{C(t)\}$ is close to either 0 or 1. The latter equation tells us if drinkable water resources get scarce at $\ell = 1$, in

[5] It should be very small or zero because C_t in $S_t = \mathbb{E}_t^{\mathbb{Q}}\left[S_T/(C_T/C_t)\right]$ should reflect all possible information about future value of sustainable footprint C_T, thus making C_T/C_t as close as possible to 1.

[6] See Aydin and Rainer (2015) for a more detailed discussion.

[7] From $\mathrm{d}e^X = e^X\mathrm{d}X$, this would imply $\mathrm{d}C_\ell = C_\ell\left[\mu_\ell(t)\mathrm{d}t + \sigma_\ell(t)\mathrm{d}L_\ell\right]$.

the first scenario ($\rho = 0$), this will lead to a depreciation in the particular class of assets/transactions relative to the critical asset C_1. Note that this is more than just an artificial depreciation/appreciation of assets due to a "politically-motivated" shift in the yield curve and it reflects the ecological/economical footprint. Furthermore, comparing different locations, if potable water turns out to be scarcer at $\ell = 1$ than $\ell = 2$ at some future time $u > t$, then $C_1(u) > C_2(u)$ would hold and assets/transactions that are little or not correlated with at $\ell = 1$ would depreciate more relative to $\ell = 2$. In the second case ($\rho = 1$), however, the price denominated in terms of the critical asset would remain largely constant as $\{X_t\}$ and $\{C(t)\}$ would move together. We learn from this example that using a critical good as a numéraire in fact have a stabilizing impact on prices.

To elaborate a bit more, think about scarcer water resources (higher C) leading to lower crop yields (thus, higher X), or vice versa. Since $\mathbb{C}\mathrm{ov}\,[X, C] > 0$, we can argue that the relative prices for crops would be more stable than that of any other asset which is weakly or not correlated with water resources.

Example 2: Carbon emissions[8] *and quantity dependence*

Another critical asset we can related to the present framework is the climate (or clean weather). Consider the price of allowance for q metric tonnes of carbon emission: $C_q : [0, T] \to \mathbb{R}^+$. One of the advantages of the emissions trading schemes (such as the EU-ETS) is that the regulator simply caps the total amount of carbon emissions by $\bar{q}$ (the sustainable limit) over the given period $[0, T]$ and the market discovers the price per tonne, i.e., $C_1(t)$ $(t \leq T)$, depending on the market demand and supply for certificates. However, in an efficient market, one can expect the price of allowance to hike as $q \to \bar{q}$, i.e.,

$$\lim_{q \to \bar{q}} C_q(t) = \infty. \tag{19}$$

On the other hand, when $q \ll \bar{q}$, the allowance price would scale linearly with q. These aspects can be combined under the pricing formula

$$C_q(t) = \frac{1}{1 - q/\bar{q}} q C_1(t), \tag{20}$$

where again $C_1(t)$ is quoted in the market. Now, taking $C_q(t)$ as the numéraire, any other asset or transaction would yield $\tilde{S}_t = S(t)/C_q(t)$ as a relative price for which, again, there would exist an EMM such that the benchmarked price is a martingale, that is,

$$\frac{S(t)}{C_q(t)} = \mathbb{E}_t^{\mathbb{Q}_q}\left[\frac{S(u)}{C_q(u)}\right] \quad (t \leq u) \tag{21}$$

Quantity dependence ensures that as the expected carbon emissions related to the asset or transaction over the period $\mathbb{T} = [0, T]$ increases (resp. decreases), the $C_q(t)$ will increase (resp. decrease) and the relative price $\{\tilde{S}_t\}_{t \in \mathbb{T}}$ would depreciate (resp.

[8]We use carbon emissions as shorthand for greenhouse gas emissions such as CO_2, CH_4 and N_2O.

appreciate) against per metric tonnes of carbon emissions. In this way, relative pricing includes the critical carbon emissions in a way that it makes the benchmarked price more fair by reflecting the environmental impacts of economic activities. We argue that it will also be more stable because most of the assets (even the digital ones like cryptocurrencies) and transactions will have an environmental component and, hence, their values be coupled (to varying degrees) to that of emission allowances.

Motivated by this example, and to better see why the expected risk-neutral performance is not necessarily M_t (which is a non-negative and, in general, increasing function of t), consider the stock price of a publicly traded polluter and two possible scenarios for time T. The expected emission amount q could either

1) decrease and remain far from $\bar{q}$ (high energy efficiency), or
2) increase and get close to $\bar{q}$ (low energy efficiency).

Apparently, in the first scenario, S_t would go up whereas, in the second, it would go down. So, if a particular country is strict on transforming to a carbon-free economy, then it might be that $C_q(t)$ (not M_t) is the right benchmark for valuing the companies operating therein.

5. Conclusion

Using conventional money market account or zero-bond as a pricing numéraire poses two main problems. First, straightforward factorization of the expected relative price into relative expected price is problematic because this is only possible if the asset and numéraire are perfectly correlated − which is not the case in reality. Ignore trends in the assets to be priced. Does not yield a stable price. Second, it does not fit well the purpose of relative pricing assets in a way that it fairly represent real economic/environmental developments and concerns such as sustainability. That being said, there is a pertinent need for change of paradigm from debt-driven to asset-driven benchmarks, from nominal to asset-linked currencies representing sustainable value, and from nominal valuation to valuation relative to real economic performance. In view of this, this paper has suggested asset-linked benchmarks as more stable alternatives to the conventional numéraires. They can be used both as a benchmark for investment and/or a numéraire for relative pricing.

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