Information-based pricing with earnings consensus signals*

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Abstract

Information-based framework was first introduced in Brody, Hughston, and Macrina (2007) as an interesting alternative to conventional asset pricing. We present one possible way of harnessing this framework in a real-world setting. In particular, we show that the information process and information-based framework can be practically viable, and a crisp approximation to the signal-based price be recovered with the help of the Kummer's function. Besides, we also verify, through a brief information-theoretic analysis, how signal-based option prices are in agreement with the fact that options are indeed priced on the basis of volatility rather than entropy. By calibrating information flow rate to option prices, we posit that information flow speed can indeed be a more intuitive alternative to implied volatility. Finally, a simple procedure is presented to estimate the signal-to-noise parameter from empirical signals.

Keywords: Information-flow, signal-based pricing, random bridge processes, implied dividend, earnings consensus

Key messages

- Information-based framework remains both intuitive, and applicable to the real-world problem of asset pricing.
- In the present framework, a crisp formula can be worked out for the price of an asset which is also computationally efficient.
- The framework enables the asset price to be adapted to a filtration which is generated by fundamental market factors rather than pre-imposed random processes.
- Changes in option prices can be explained by those in information flow speed in lieu of implied volatility.
- Phenomena such as price discontinuities can be seen as a natural result of sudden changes in information flow.

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1 Introduction

The flow of forward-looking information through signals is essential for the smooth operation of the highly complex financial market engine and it is the most fundamental input to the pricing of any type of asset. The market agents, both human and non-human, on the other hand, are signal processors who continuously mine for and interpret these signals to extract information.

No matter how an information flow pattern is modelled, the aim of financial modelling based on forward-looking information is to ensure that future information (e.g., earnings and/or dividend payments in the case of equity) on an asset's fundamental value is represented −either fully or partially− in price discovery.

Building upon the information-based framework introduced in Brody et al. (2007), this paper aims to put the latter framework into practical use by introducing a slightly modified version of the signal process in Brody et al. (2007) and making a particular choice for real-time signals. To the best of the author's knowledge, this is the first such attempt, with results having significant implications for harnessing the signal-based framework in a real-world setting. We also contribute the literature by presenting a crisp formula for the signal-based price. As a by-product, this paper will have illustrated, in a practical sense, how the problem of predicting the future value of a stochastic random variable can be simplified to interpreting the information concerning its constituents. In particular, we focus on equity market due to the relatively simpler interpretation of the term "fundamental," referring to the value of a cash-generating business.

2 Modelling Information Flow

The information-based framework was first introduced in Brody et al. (2007) as a new way of modelling credit risk and, later on, applied to a broad spectrum of issues in financial mathematics, including the valuation of insurance contracts based on the cumulative gain process in Brody et al. (2008a), modeling of defaultable bonds in Rutkowski and Yu (2007) (as an extension of Brody et al. (2007) to stochastic interest rates), general asset pricing in Brody et al. (2008b), pricing of inflation-linked assets in Hughston and Macrina (2008), and modelling of asymmetric information and insider trading in Brody et al. (2009), before it was generalised to a wider class of Lévy information processes in Hoyle (2010) for valuing credit-risky bonds, vanilla and exotic options, and non-life insurance liabilities. This method was used, in Brody and Hughston (2013), to aggregate individual risk aversion dynamics to form a market pricing kernel, in Macrina and Parbhoo (2010), to price credit-risky assets that may include random recovery upon default, in Mengütürk (2012), to introduce an extension of the theory towards an analysis of information blockages and activations, as well as information-switching dynamics, in Brody et al. (2013a), to introduce a general framework for signal processing with Lévy information, in Yang (2013), to value storable commodities and associated derivatives and, most recently, in Brody and Law (2015), to obtain a stochastic volatility model based on random information flow, and in Bedini et al. (2016), to produce estimates of bankruptcy time.

Accordingly, we introduce the signal process $(\xi_t)_{0\leq t\leq T}$ (or, the information process in the sense of Brody et al. (2007))

$$\xi_t = \sigma t X_T + \beta_t, \tag{2.1}$$

which is a Brownian random bridge (BRB), as defined in Hoyle et al. (2011) for the general class of Lévy processes (i.e., Lévy random bridges or LRBs). Simply put, ξ_t is a Brownian bridge that is conditioned to have a priori law of $\sigma t X_T$. In Eq. (2.1), $\beta_t := \beta_{0T}^{[0,0]}(t)$ is a standard Brownian bridge over the period $[0,T]$ which takes on value 0 at the beginning and end, and σ is a measure of true signal to noise (henceforth, just 'signal-to-noise'). The latter governs the overall speed of revelation of true information about the actual value of the fundamental X_T.

We also remark that Eq. (2.1) is not the only way to represent information flow. Some other forms have also been considered in the literature with slightly different characteristics, such as, $\xi_t = tX_T + \beta_t$, $\xi_t = (t/T)X_T + \sigma\beta_t$ or $\xi_t = (t/T)X_T + \beta_t$.

Proposition 2.1. *The information process $(\xi_t)_{0\leq t\leq T}$, as defined in Eq.* (2.1)*, is conditionally Markovian.*

Proof. (See, e.g., Brody et al. (2007) for alternative proof.) We set $\kappa_t = T/(T-t)$ here and, whenever appropriate, throughout the text. Let ξ_t be intrinsically pinned to an unknown value $X_T = x$. Defining B_t as a Brownian motion, we can indeed express the signal process ξ_t as

$$\sigma t x + \kappa_t^{-\frac{1}{2}} B_t \quad \text{or} \quad \sigma t x + \kappa_t^{-\frac{1}{2}} \int_0^t \mathrm{d}B_s. \tag{2.2}$$

One can verify that these are identical to

$$\xi_t = \sigma t x + (T-t)\int_0^t \frac{\mathrm{d}B_s}{T-s}, \tag{2.3}$$

which, in turn, implies

$$\begin{aligned} \mathrm{d}\xi_t &= \left(\sigma x - \int_0^t \frac{\mathrm{d}B_s}{T-s}\right)\mathrm{d}t + (T-t)\frac{\mathrm{d}B_t}{T-t} \\ &= \left(\sigma x - \frac{\xi_t - \sigma t x}{(T-t)}\right)\mathrm{d}t + \mathrm{d}B_t \\ &= \left(\sigma x - \xi_t/T\right)\kappa_t \mathrm{d}t + \mathrm{d}B_t. \end{aligned} \tag{2.4}$$

Equations (2.3) and (2.4) indeed follow from two other well-known representations of bridges (see, e.g., Øksendal (1998)). Eq. (2.4), on the other hand, directly implies that, given $X_T = x$, ξ_t is a Markov process with respect to its own filtration, i.e.,

$$\mathbb{E}[h(\xi_t)|\sigma\left(\xi_r\right)_{r\leq s}] = \mathbb{E}[h(\xi_t)|\sigma(\xi_s)] \quad (s \leq t), \tag{2.5}$$

for any x, and any measurable, finite-valued function h (cf. Øksendal (1998)). $\square$

More formally, we define a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, on which the filtration $(\mathcal{F}_t^{\xi})_{t\in[0,T]}$ is constructed. Here, $\mathbb{Q}$, i.e., the risk-neutral measure, is assumed to exist. Unless stated otherwise, $\mathbb{Q}$ will be the default measure throughout the paper. The filtration $\mathcal{F}_t^{\xi}$ is generated directly by $(\xi_s)_{0\leq s\leq t}$ and, thus (by Markov property), ξ_t.

We are now in a position to work out, with respect to the available information $\mathcal{F}_t^{\xi}$, the value S_t and dynamics $\mathrm{d}S_t$ of an asset which generates a cashflow $\phi(X_T)$ at time T for some invertible function ϕ. The value S_t, $0 \leq t < T$, is given by

$$S_t = D_t \mathbb{E}\left[\phi(X_T)|\mathcal{F}_t^{\xi}\right] = D_t \mathbb{E}\left[\phi(X_T)|\xi_t\right] = D_t \int_{\mathbb{X}} \phi(x)\pi_t(x)\mathrm{d}x, \tag{2.6}$$

where $D_t := \mathbf{1}_{\{t<T\}} e^{-r(T-t)}$ is the numéraire and $\pi_t(x) := p(x|\xi_t)$ the posterior density of the payoff. The quantities X_T and $\phi(X_T)$ are measurable with respect to $\mathcal{F}_T^{\xi}$, but not necessarily w.r.t. $\mathcal{F}_t^{\xi}$, $t < T$. On an important note, we remark that β_t, i.e., the pure noise, is not measurable w.r.t. $\mathcal{F}_t^{\xi}$, meaning that it is not directly accessible to market agents. Thus, an agent, although he observes ξ_t, cannot separate true signal from noise until time T.

Using Bayesian inference, S_t can be expressed as

$$S_t = D_t \frac{\displaystyle\int_{\mathbb{X}} \phi(x)p(x)e^{\kappa_t\left(\sigma x\xi_t - \frac{1}{2}\sigma^2 x^2 t\right)}\mathrm{d}x}{\displaystyle\int_{\mathbb{X}} p(x)e^{\kappa_t\left(\sigma x\xi_t - \frac{1}{2}\sigma^2 x^2 t\right)}\mathrm{d}x}, \tag{2.7}$$

with $\kappa_t := T/(T-t)$ and $\mathbb{X}$ is the support of the random payoff. Setting $\phi(t, \xi_t) := S_t/D_t$ and applying Itô's Lemma, one can indeed show that the PDE S_t satisfies is

$$\mathrm{d}S_t \quad = \quad rS_t\mathrm{d}t + \Lambda_t\mathrm{d}W_t, \tag{2.8}$$

where

$$\Lambda_t := D_t\sigma\kappa_t\mathbb{C}\mathrm{ov}_t\left(\phi(X_T), X_T\right) \tag{2.9}$$

is the absolute price volatility, and

$$\mathrm{d}W_t := \kappa_t\left(\xi_t/T - \sigma\phi_t(X_T)\right)\mathrm{d}t + \mathrm{d}\xi_t. \tag{2.10}$$

One can indeed show, by referring to Lévy's characterisation (1948), that W_t is a Brownian motion adapted to F_t^ξ (cf. Brody et al. (2007)). Thus, in the present framework which hinges on the reconstruction of the filtration explicitly, we naturally recover a stochastic driver (W_t in this case) rather than pre-imposing it.

Example 2.1. Assume, as a particular case, that ϕ is an identity, i.e., $\phi(x) = x$, and $X_T \sim \mathcal{N}(0,1)$ a priori. Then, Eq. (2.7) implies

$$S_t = D_t \frac{\sigma \kappa_t \xi_t}{\sigma^2 \kappa_t t + 1}. \tag{2.11}$$

As it can easily be verified that $\lim_{t\to T} S_t = \frac{\xi_t}{\sigma T} = X_T$, thus, S_t under $\mathcal{F}_t^\xi$ turns into a Gaussian process conditioned to have the a priori law of X_T.

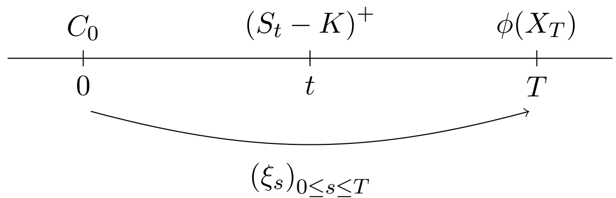

Figure 1: Signal-based option pricing timeline.

2.1 Signal-based Option Pricing

Let us now present a semi-analytical formula for the prices of standard European call and put options that are written at time 0 on an asset which admits S_t. Assume that the options expire at t and have an exercise price K. Let the underlying pay $\phi(X_T) \in (0,\infty)$ at time T where ϕ is not an identity. A simple timeline is given in Figure 1. The signal-based option price (i.e., based on the signal $(\xi_s)_{0\le s\le T}$) is the solution to

$$\begin{aligned} C_0 &= D_0^t \mathbb{E}^{\mathbb{P}} \left[\Phi_t^{-1} \int_{\mathbb{X}} \left(D_t^T \phi(x) - K \right) p_t(x) \mathrm{d}x \right]^+ \\ &= D_0^t \mathbb{E}^{\mathbb{B}} \left[\int_{\mathbb{X}} \left(D_t^T \phi(x) - K \right) p_t(x) \mathrm{d}x \right]^+, \end{aligned} \tag{2.12}$$

where

$$p_t(x) := p(x) e^{\kappa_t \left(\sigma x \xi_t - \frac{1}{2}\sigma^2 x^2 t \right)}, \quad \Phi_t^{-1} := \left(\int_0^\infty p_t(x) \mathrm{d}x \right)^{-1}, \tag{2.13}$$

and $\mathbb{B}$ is the 'bridge measure' which is associated with the Radon−Nikodym derivative (or, exponential martingale)

$$\frac{\mathrm{d}\mathbb{B}}{\mathrm{d}\mathbb{Q}} | \mathcal{F}_t^\xi = \Phi_t^{-1} := \left(\int_0^\infty p_t(x) \mathrm{d}x \right)^{-1} \tag{2.14}$$

under which ξ_t turns into a standard Brownian bridge, i.e., a martingale (see, e.g., Brody et al. (2008b) for details). This yields,

$$C_0 = D_0^T \int_{\mathbb{X}} \phi(x)p(x)\Theta\left(-z^\star + \sqrt{\kappa_t t}\sigma x\right) \mathrm{d}x - D_0^t K \int_{\mathbb{X}} p(x)\Theta\left(-z^\star + \sqrt{\kappa_t t}\sigma x\right) \mathrm{d}x \tag{2.15}$$

and

$$P_0 = D_0^t K \int_{\mathbb{X}} p(x)\Theta\left(z^\star - \sqrt{\kappa_t t}\sigma x\right) \mathrm{d}x - D_0^T \int_{\mathbb{X}} \phi(x)p(x)\Theta\left(z^\star - \sqrt{\kappa_t t}\sigma x\right) \mathrm{d}x \tag{2.16}$$

for call and put prices, respectively, where $\Theta(\cdot)$ is the standard normal CDF. When, as a special case, $\phi(X_T) = S_0 \exp((\mu - \frac{1}{2}\nu^2)T + \nu\sqrt{T}X_T)$ where $\nu > 0$ and $X_T \sim \mathcal{N}(0,1)$, we have an explicit solution for the threshold value, i.e.,

$$z^* = \frac{\left(2\left(\ln\frac{K}{S_0} + \frac{\nu^2}{2}T - rt\right)\right)(\kappa_t\sigma^2 t + 1) - \nu^2 T}{2\kappa_t\sqrt{T}\sigma\nu\sqrt{t/\kappa_t}}. \tag{2.17}$$

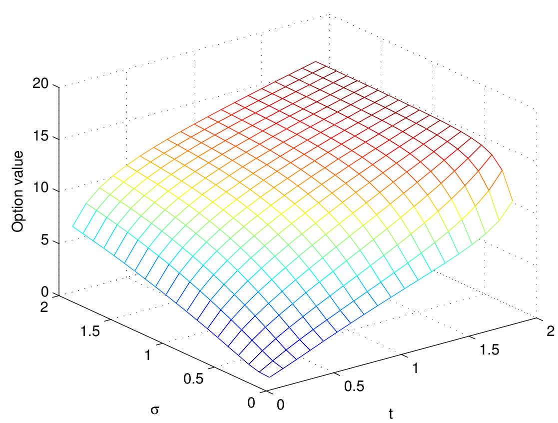

Figure 2: Call option value for ranging pairs (σ, t). Arbitrary parameters: $T = 2$, $\mu = 0.05$, $K = 100$, $\nu = 0.2$, $S_0 = 100$.

Figure 2 depicts the call option value against a range of values for the information flow rate σ and maturity t. As expected, greater time frame to maturity implies greater chances of price exceeding the threshold K. But, it may sound rather contradictory that faster information flow leads to higher option prices. At this point, we would like to distinguish between the pace at which information is being incorporated into prices and the amount of information which has already been incorporated into prices. The former relates to the fact that the overall magnitude of the absolute price volatility Λ_t is determined, as per Eq. (2.8), by σ. The latter, on the other hand, can be explained by measures such as entropy. By using Shannon (1948) entropy (which is a special case of Renyí (1984) entropy) of ξ_t, we can indeed show[1] that, when $\sigma_1 < \sigma_2$,

$$h(\phi(X_T)|\xi_t^1) > h(\phi(X_T)|\xi_t^2) \tag{2.18}$$

holds $\forall t \in (0, T)$ among the conditional entropies of $\phi(X_T)$ w.r.t. ξ_t^1 and ξ_t^2. Hence, the pricing of contingent claims is more about volatility than entropy.

2.2 A Note on Implied Volatility

Information flow, in fact, offers a more intuitive substitute for implied volatility which has no real financial meaning but is an additional degree of freedom to equate market price to model price (not intrinsically sensitive to changes in forward-looking information). To see this, we depict in Figure 3 the relationship between implied volatility ν^{imp} and signal-to-noise ratio σ of call options on two select tickers, namely, `AAPL` and `MSFT`. Each dotted line corresponds to one specific signal maturity T whereas the solid vertical lines are the . Calibrations are carried out as per Eq. (2.15) and (2.17). The interpretation is that one can continue to use a certain measure of volatility (i.e., realised or other) and does not have to explain variations between model and market prices by 'implied' changes in volatility. Rather these variations can now be attributed to a more intuitive factor, i.e., changes in information flow pattern.

3 Multiple Dividends and Market Factors

The risky asset is now characterised at any time t by an infinite number of cashflows which accrue continuously but are announced (or physically distributed) at discrete times intervals. We then call a time-varying subset of these cashflows, i.e., $\{\phi_k\}_{k=1,\dots,n_t}$, which are due $T_1, \dots, T_{n_t}$, the cashflows "within" the horizon. Each payoff ϕ_k can be deemed a function of m_k market factors as a subset of $\{X_1, \dots, X_{\max(m_k)}\}$ for any k, thereby making the price a function of $\max(m_k)$ market factors. This setting indeed allows one to consider a broader spectrum of financial instruments. When each market factor X is associated with an information process $\{\xi_t\}_{0 \le t \le T_k}$, the problem of valuing an equity reduces to identifying a set of potential candidates for X and, therefore, ξ in a real-world setting, and calibrating the signal flow rate.

[1] By using the relation $h(\phi(X_T)) - h(\phi(X_T)|\xi_t) = h(\xi_t) - h(\xi_t|\phi(X_T))$, and the fact that $h(\xi_t^1|\phi(X_T) = h(\xi_t^2|\phi(X_T)$ whereas $h(\xi_t^1) < h(\xi_t^2)$.

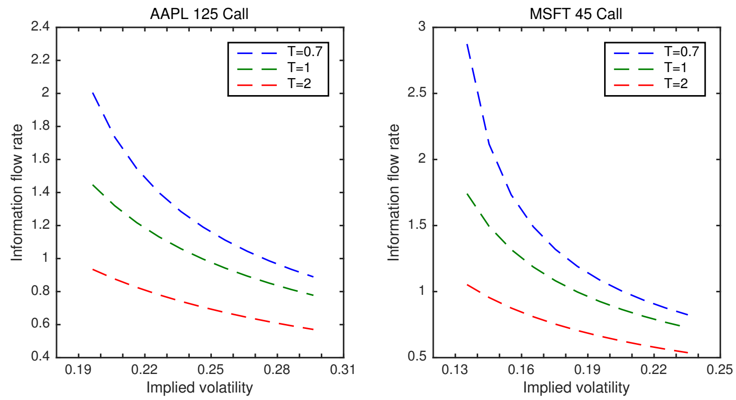

Figure 3: Implied Volatility and information flow rate. Options are valued on May 1, 2015, and mature on August 21, 2015.

For simplicity, we shall assume $m_k = 1$, $\forall k$, throughout the text (i.e, a single market factor X determines each cashflow ϕ_k). We further assume that $X_1, \ldots, X_{n_t}$, are i.i.d. At any time t, the σ-algebra $\mathcal{F}_t^{\xi}$ is assumed to be the 'join' of the σ-algebras generated by n_t independent information processes, i.e., $\sigma(\xi_1) \vee \cdots \vee \sigma(\xi_{n_t})$, and the current and past values of the market factors and the risky asset, i.e., $\mathcal{F}_t = \sigma(X_s, S_s : 0 \leq s \leq t)$:

$$\mathcal{F}_t^{\xi} := \sigma(\xi_1) \vee \cdots \vee \sigma(\xi_{n_t}) \vee \mathcal{F}_t. \tag{3.1}$$

Let $n_t = n$. The price of the asset is then simply given by

$$S_t = \sum_{k=1}^{n} \mathbf{1}_{\{t<T_k\}} D_t^k \mathbb{E}\left[\phi_{T_k}(X) | \mathcal{F}_t^{\xi}\right], \tag{3.2}$$

where $\mathbb{E}$ is w.r.t. $\mathbb{Q}$ by setting. Furthermore, the dynamics of asset price process $\{S_t\}_{t\geq 0}$ will be analogous to Eq. (2.8) (cf. Brody et al. (2007)):

$$\begin{aligned}
\mathrm{d}S_t &= r_t S_t \mathrm{d}t \\
&+ \mathbf{1}_{\{t<T_1\}} D_t^{T_1} \sigma_1 \kappa_t^1 \mathbb{C}\mathrm{ov}_t \left[\phi(X_1), X_1\right] \mathrm{d}W_t^1 \\
&\vdots \\
&+ \mathbf{1}_{\{t<T_n\}} D_t^{T_n} \sigma_n \kappa_t^n \mathbb{C}\mathrm{ov}_t \left[\phi(X_n), X_n\right] \mathrm{d}W_t^n \\
&= r_t S_t \mathrm{d}t + \sum_{k=1}^{n} \mathbf{1}_{\{t<T_1\}} D_t^k \sigma_k \kappa_t^k \mathbb{C}\mathrm{ov}_t \left[\phi(X_k), X_k\right] \mathrm{d}W_t^k \\
&\quad - \sum_{k=1}^{n} \phi(X_k) \mathrm{d}\mathbf{1}_{\{t>T_k\}},
\end{aligned} \tag{3.3}$$

where $\kappa_t^k = T_k/(T_k - t)$ and $\mathbb{C}\mathrm{ov}$ is the covariance function. The last term in Eq. (3.3) comes from the price adjustment due to accrual of cashflow (ex-dividend). Eq. (3.3) implies that the asset price dynamics in the multiple cashflow case based on signal-based framework remains fairly tractable. In what follows, we show how the present framework can be applied on real market data with slight modifications.

4 The Case for "Implied" Dividends

In Brody et al. (2013b) and Yang (2013), the present concept is applied to produce a tractable formula for storable commodity prices under the assumption that the asset pays −what authors call− a continuous 'convenience dividend' that is assumed to follow Ornstein−Uhlenbeck (OU) dynamics. The OU process is then associated, through its orthogonal decomposition, to the concept of 'OU bridge,' thereby putting together analytical formulae for commodity spot and derivatives prices. In this paper, we introduce the concept of "implied dividend," which is based on earnings, as the stochastic market factor X that determines the dividend through identity $\phi(X) = X$ and, eventually, the equity price.

There is indeed a large body of literature which argues that it is reasonable to consider earnings data as a proxy for company's expected dividends and measures tied to the former, rather than the latter, will likely provide better information about the actual cash flows generated (see, e.g., Campbell and Shiller (1988), Longstaff (2009), Longstaff and Piazzesi (2004)). Indeed, many growth businesses choose not to pay cash dividends but, instead, to use their earnings to repurchase outstanding shares or to reinvest in future expansion − making earnings a more informative measure of the fundamental value of a business (see Campbell and Shiller (1988), Dong and Hirshleifer (2005)). Investors are also far more interested in the earnings potential of a business rather than its paid dividends (cf. Bakshi and Chen (2005)). Earnings, like dividends, are also generated on a continuous basis, although their true value is revealed at discrete time points (quarterly or annually), justifying their suitability for use in continuous-time setups.

In the sequel, we assume that earnings are the basis for changes in an asset's value as "invisible" dividends and they provide some kind of convenience yield $\phi(X)$ which become known to agents at T_k, $k = 1, \dots, n$. The raw signal process ξ_t in this case conveys noisy information about the true value of the earnings (and, eventually, dividends).

Based on the Bakshi−Chen model introduced in Bakshi and Chen (2005), we relate earnings X to "implied" dividends X' as follows:

$$X'_k = \delta X_k + \epsilon_k, \quad \epsilon_k \sim N(0, \sigma_k^\epsilon). \tag{4.1}$$

where we set ϕ to be identity. Here, $\delta \in [0, 1]$ is the dividend payout ratio. The use of constant payout ratio is common in the equity valuation literature. The classic survey in Lintner (1956) finds that indeed $\delta_t \to \delta$. The rationale for and interpretation behind Eq.

(4.1) is addressed in Bakshi and Chen (2005) and Dong and Hirshleifer (2005). Therefore, independent of whether the firm pays cash dividends or not, we interpret $\delta\phi(X)$ given in Eq. (4.1) as the "implied dividend" which will be the governing factor X in our model behind asset price movements.

Furthermore, we assume that earnings and earnings growth follow a geometric Brownian and Ornstein-Uhlenbeck (OU) dynamics, respectively. That is,

$$\mathrm{d}X_k = \mu_k^X X_k \mathrm{d}T_k + \sigma_X \mathrm{d}W_k^X, \tag{4.2}$$

$$\mathrm{d}\mu_k^X = \alpha(\mu_0 - \mu_k^X)\mathrm{d}T_k + \sigma_\mu \mathrm{d}W_k^\mu, \tag{4.3}$$

where $\mathrm{d}T_k = T_k - T_{k-1}$. Also we set $W_k^X \perp\!\!\!\perp W_k^\mu$ for $\mathcal{F}^\xi$-adapted W_k^X and W_k^μ that are martingale under the pricing measure. In what follows, we will be employing this model to estimate the parameters of our signal-based valuation framework.

4.1 Recovering the Gordon Model in Continuous Time

First we associate the signal-based price S_t to Gordon model Gordon (1962), Gordon and Shapiro (1956) under constant earnings growth assumption, and later on extend this to time-varying growth.

4.1.1 Constant Earnings Growth

Assuming S pays an infinite strip of earnings starting from u, where $u > t$, the continuous time analogous of Eq. (3.2) is

$$S_t = D_t^u \int_u^\infty e^{-r_b(v-u)} \mathbb{E}\left[\delta X_v | \mathcal{F}_t^\xi\right] \mathrm{d}v, \tag{4.4}$$

where $\mathcal{F}_t^\xi$, $\mathcal{F}_t$ are as given in Eq. (3.1), and $D_t^u = e^{-r_t^u(u-t)}$ is again the numéraire. When μ^X is constant, say μ_0, a straightforward calculation yields

$$\begin{aligned}
S_t &= \delta D_t^u \int_u^\infty e^{-r_b(v-u)} \mathbb{E}\left[X_u e^{(\mu_0 - \frac{1}{2}\sigma_X^2)(v-u) + \sigma_X W_{v-u}^X} | \mathcal{F}_t^\xi\right] \mathrm{d}v \\
&= \delta D_t^u \phi_t(X_u) \int_u^\infty e^{-r_b(v-u)} \mathbb{E}\left[e^{(\mu_0 - \frac{1}{2}\sigma_X^2)(v-u) + \sigma_X W_{v-u}^X}\right] \mathrm{d}v \\
&= \delta D_t^u \phi_t(X_u) \int_u^\infty e^{-r_b(v-u)} e^{(\mu_0 - \frac{1}{2}\sigma_X^2)(v-u) + \frac{1}{2}\sigma_X^2(v-u)} \mathrm{d}v \\
&= \delta D_t^u \phi_t(X_u) \int_u^\infty e^{-(r_b - \mu_0)(v-u)} \mathrm{d}v \quad (\eta = v - u) \\
&= \delta D_t^u \phi_t(X_u) \int_0^\infty e^{-(r_b - \mu_0)\eta} \mathrm{d}\eta \\
&= \delta D_t^u \frac{\phi_t(X_u)}{r_b - \mu_0} \quad (r_b > \mu_0),
\end{aligned} \tag{4.5}$$

where r_b is the investment benchmark while r_t^u is the money market rate for maturity u. Eq. (4.5) is nothing but the earnings (or implied dividend) equivalent of the well-known intrinsic value model of Gordon. Note the slight difference in appearance between the discrete and continuous forms of the Gordon model (cf. Kronimus (2003)) which disappears as $(1 + \mu_0 \mathrm{d}\eta)\phi_t(X_u) \to \phi_t(X_u)$ as $\mathrm{d}\eta \to 0$. This model, however, is mostly criticised for assuming that the dividend growth rate as well as the risk-adjusted discount rate remain constant − a point which is confronted in the literature by the well-known St. Petersburg paradox (see, e.g., Eisdorfer and Giaccotto (2014)). In our pricing algorithm, we shall circumvent this issue by considering a constant spread between μ_0 and r_b.

If μ^X admitted the PDE in Eq. (4.3), on the other hand, we would simply have an additional term $\exp(\int_t^u \mu_\nu^X \mathrm{d}\nu)$ substituting for μ_0 from the first line of Eq. (4.5). This, however, is beyond the scope of our analysis in this work. Below, we introduce the earnings signals that will act, among possible others, as our information flow process ξ_t. We also introduce a slightly modified version of the latter.

4.2 Real-world Information Flow

Financial markets, with equity market being a particular example, are forward-looking, i.e., prices are ideally discovered on the basis of expectations pertaining to the future value-generating ability of the underlying business. One vivid example to this is the price adjustments to an equity following unexpected deviations of realised earnings from their consensus values and/or inter-temporal revisions of earnings expectations by brokers.

At time t_k, market experts start disseminating their consensus estimates on the true value of a certain ticker's quarterly earnings value X_k, and therefore its implied dividend, which is due at T_k. These consensus figures are derived from comprehensive assessments of up to 40 brokerage analysts which closely follow a certain ticker and incorporate as much information as available. As for quarterly earnings consensus, $T_k - t_k$ is generally between 2 and 4 years. We exhibit in Figure 4 the quarterly earnings signals extracted from Bloomberg terminal for a large-cap U.S. blue-chip company (ticker: `MSFT`) that are released at an average frequency of four days during 2004Q1−2015Q3. The data are adjusted for corporate issues such as stock splits, exclude non-recurring items, include employee stock options expenses, and incorporate any guidance issued by the company prior to actual earnings announcement. Figure 5, on the other hand, shows the number of active signals and their average length for the time period under consideration.

As we cannot separate X from noise in any observed signal to construct empirically the desired signal ξ_t given in Eq. (2.1)[2], we introduce a slightly modified version of the latter,

[2] For instance, one does not normally observe a noisy signal for $\sigma t X$ in the market but X and, therefore, the desired signal $\sigma t X + \epsilon_t$ cannot be recovered from $X + \epsilon_t$.

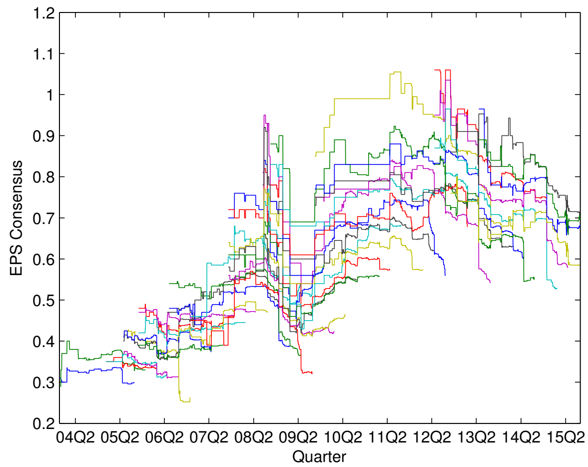

Figure 4: Evolution of quarterly earnings signals. Data source: Bloomberg.

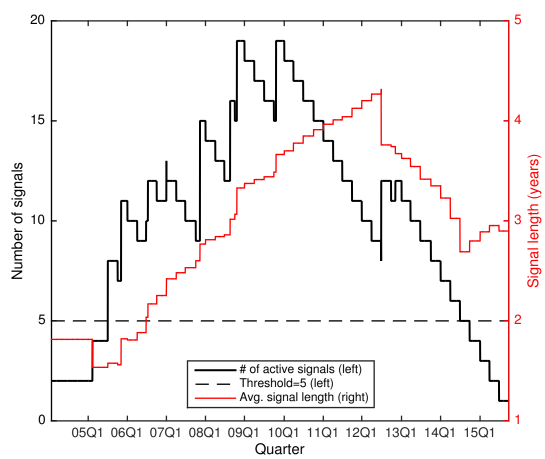

Figure 5: Number n_t (left) and average length $T_k - t_k$ (right) of active signals over time.

while preserving its intuitive properties, as follows:

$$\xi_t^k = \begin{cases} x_k^* + \tau_t^k (X_k - x_k^*) + \frac{\beta_t^k}{\sigma(T_k - t_k)}, & \text{if} \quad t_k \leq t \leq T_k, \\ \emptyset, & \text{otherwise,} \end{cases} \tag{4.6}$$

where x_k^* is the first signal sample received at time t_k about the true value of X_k, σ^{-1} now a measure of noise-to-signal, and

$$\tau_t^k = \mathbf{1}_{t \leq t_k} \frac{t - t_k}{T_k - t_k} \in [0, 1] \tag{4.7}$$

the proportion of signal lifetime elapsed since it started being transmitted. Intuitively, we now allow an increasing σ to suppress noise (thereby, increasing signal-to-noise) rather than to increase the signal content directly, as in Eq. (2.1). The conditional variance of ξ_t^k given $X_k = x$ can be rewritten as

$$\mathbb{V}\left(\xi_t^k | X_k = x\right) = \frac{1}{\sigma^2} \frac{(t - t_k)(T_k - t)}{(T_k - t_k)^3} = \frac{\tau_t^k \left(1 - \tau_t^k\right)}{\sigma^2 (T_k - t_k)}. \tag{4.8}$$

Note that the modified version of the information process ξ_t in Eq. (4.6) is better suited to the real-time signals considered here and, again, do not compromise intuitive and statistical properties of ξ_t. The equivalence of the latter two in the sense of integral in Eq. (2.7) can easily be seen as follows. Let us assume w.l.o.g. that $x^* = 0$ and $t_k = 0$. Then, apparently,

$$\int_A x p(x) e^{-\frac{1}{2}\left(\frac{\xi - ax}{b}\right)^2} \mathrm{d}x = \int_A x p(x) e^{-\frac{1}{2}\left(\frac{\xi' - ax/c}{b/c}\right)^2} \mathrm{d}x \tag{4.9}$$

with $a = \sigma t$, $b = \tau_t (T - t)$, $\xi' = \xi / c$, and $c = \sigma T$.

As indicated by Eq. (4.6), the σ-algebra F_t^ξ constantly enlarges and shrinks whenever the number of available signals increases and decreases, respectively. Once the signal ξ^k is started to be received at time t_k, the market updates its prior information about X_k (i.e., $p(x)$) using $p(\xi_t | x)$. On the other hand, the noise-to-signal measure $1/\sigma$ needs to be determined from the data. Again, for $t > T_k$, i.e., once X_k has been revealed, $(\xi_t)_{t_1 \leq t \leq T}$ becomes degenerate (information-null).

To check the boundary values, apparently, $\xi_{t_k}^k = x_k^*$ and $\xi_{T_k}^k = X_k$, with the latter ensuring that the marginal law of ξ_t^k is the a priori law of X_k (cf. Hoyle (2010), Hoyle et al. (2011)). In Figure 6, we plot the residuals β_t^k from several paths of actual earnings signals, extracted as per Eq. (4.6) whereas their starting and end points are aligned.[3] Sample residuals do indeed exhibit properties that are similar to those of a bridge process. Furthermore, jumps occur occasionally as a result of the significant revisions of consensus data.

[3] We recall that $T_k - t_k$ differs across signals.

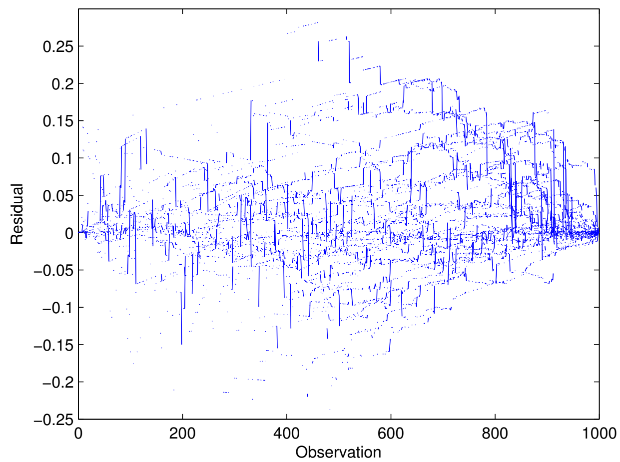

Figure 6: Residuals of empirical information processes depicted in Figure 4.

4.3 Calibrating Information Flow Rate

The information flow parameter σ_k, which is time-homogeneous by our setting, is calibrated based on the modified information process given in Eq. (4.6) as follows. We have an sample history of $N = 41$ quarterly earnings signals (with lengths varying from 1.1 to 4.7 years) for the stock ticker considered. To calibrate σ_k,

1. We first extract the linear part of each signal according to Eq. (4.6) to get various paths of the empirical bridge processes. We refer the reader back to Figure 6 for a visualisation of the residual series β^k, $k = 1, \ldots, N$.

2. For each β^k, we then run the following non-linear regression based on the theoretical variance of β^k_t:

$$\left(\beta^k_t\right)^2 = \frac{1}{\hat{\sigma}_k^2} \frac{\tau^k_t(1-\tau^k_t)}{T_k - t_k} + \epsilon^k_t, \quad t_k \leq t \leq T_k, \tag{4.10}$$

   where, again, $\tau^k_t = (t - t_k)/(T_k - t_k)$ and, presumably, $\epsilon^k_t | \mathcal{F}^\xi_t \sim \mathcal{N}(0, \sigma_\epsilon)$.

   Figure 7 shows the calibration results for σ_k for arbitrary quarterly earnings signals, whereby the Levenberg−Marquardt nonlinear curve-fitting algorithm is used.[4] Parameter estimates for all signals are statistically significant with considerably low p-values.

[4] We also remark that some other statistical learning algorithms, e.g., Expectation-Maximisation algorithm (cf. Hastie et al. (2009)), could also be used to capture possible multi-modal dynamics.

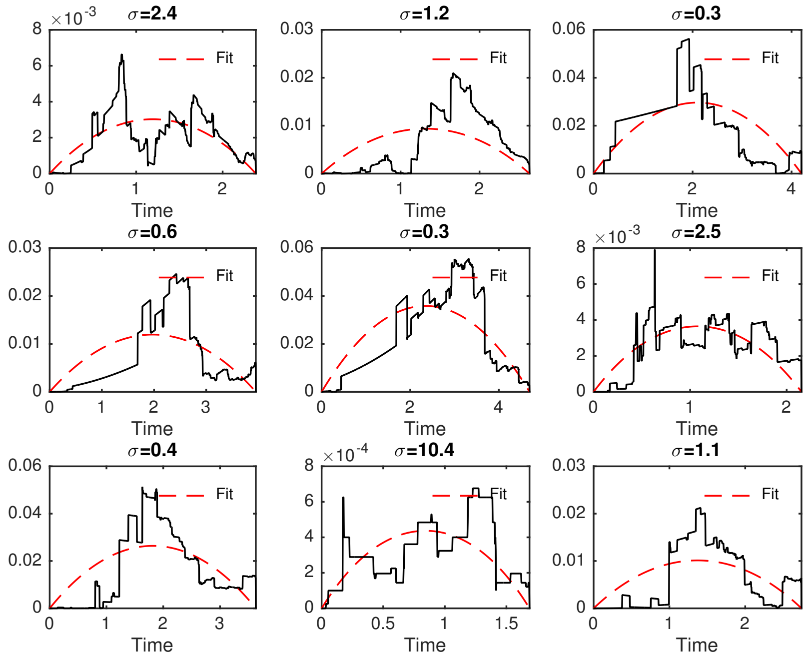

Figure 7: Calibration results of information flow rate, i.e., σ_k, using Levenberg−Marquardt nonlinear curve-fitting algorithm. Arbitrary signals are shown.

3. (Optional) As a final step, assuming $\beta^1 \perp\!\!\!\perp \ldots \perp\!\!\!\perp \beta^k$, we perform a simple variance averaging over all fitted curves resulting from Eq. (4.10) to find $\hat{\sigma}$:

$$\hat{\sigma} = \left(\frac{\sum_{k=1}^{N} \hat{\sigma}_k^{-2}}{N} \right)^{-1/2}. \tag{4.11}$$

The last step yields $\sigma = 0.79$ for the ticker `MSFT` with respect to the period 2005Q3-2015Q3. In what follows, we develop a closed-form approximation to signal-based price and present the pricing results.

5 Analytical Approximation to Signal-based Price

We now turn our attention back to deriving a preferably crisp formula for pricing the risky asset when there is a multiplicity of information processes $\xi_{t_1T_1}, \ldots, \xi_{t_nT_n}$, delivering a continuum of market signals on i.i.d. market factors $X_1, \ldots, X_n$, and, thereby, cashflows $\phi(X_1), \ldots, \phi(X_n)$ where we assumed $\phi(x) = x$ (identity).

For simplicity of exposition and without loss of generality, consider any three signals with

$$(T_{k-1}, T_k, T_{k+1}) = (r, s, u), \quad r < s < u. \tag{5.1}$$

We state the well-known solution to the SDE of X in Eq. (4.2) at time u:

$$\begin{aligned} X_u &= X_s \exp\left(\left[\mu^X_{su} - \sigma^2_X/2\right](u-s) + \sigma_X W^X_{u-s}\right) \\ &= X_s \exp\left(\left[\mu^X_{rs} e^{-\alpha(s-r)} + \mu_0(1 - e^{-\alpha(s-r)}) - \sigma^2_X/2\right](u-s) \right. \\ &\quad \left. + \sigma_X W^X_{u-s} + \sigma_\mu (u-s) \int_r^s e^{-\alpha(s-\nu)} \mathrm{d}W^\mu_\nu \right). \end{aligned} \tag{5.2}$$

where μ_{su} is simply the growth between s to u. Therefore, using the fact that $W^X \perp\!\!\!\perp W^\mu$, we can write X_u/X_s in the conditionally log-normal form

$$Y_{su} = \frac{X_u}{X_s} = \exp\left(\tilde{\mu}_{su}(u-s) + \tilde{\sigma}_{su} Z_{u-s}\right), \tag{5.3}$$

where $Z \sim \mathcal{N}(0, u-s)$, and with

$$\tilde{\mu}_{su} = \mu^X_{rs} e^{-\alpha(s-r)} + \mu_0(1 - e^{-\alpha(s-r)}) - \sigma^2_X/2 \tag{5.4}$$

and

$$\tilde{\sigma}^2_{su} = \sigma^2_X + \frac{\sigma^2_\mu}{2\alpha}(u-s)(1 - e^{-2\alpha(s-r)}). \tag{5.5}$$

5.1 Maximum-likelihood Estimation of Earnings Model

We recall from Section 5 that $Y = \Delta \log X$ is normally distributed with $\tilde{\mu}$ and $\tilde{\sigma}$, given in Eq. (5.4) and (5.5), which are, in turn, functions of the parameters α, μ_0, σ_X and σ_μ. We write the log-likelihood function $\mathcal{L}(\alpha, \mu_0, \sigma_X, \sigma_\mu | \mathbf{y})$, based on the transition density of

$\log X$, to be maximised as follows:

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}(\alpha, \mu_0, \sigma_X, \sigma_\mu | \mathbf{y}) \\
&= \sum_{l=3}^{w+1} \log \left(\frac{1}{\sqrt{2\pi}\tilde{\sigma}_{l-1,l}\sqrt{\Delta T_l}} \exp \left[-\frac{1}{2} \frac{(y_{l-1,l} - \tilde{\mu}_{l-1,l}\Delta T_l)^2}{\tilde{\sigma}^2_{l-1,l}\Delta T_l} \right] \right) \\
&= \sum_{l=3}^{w+1} \log \left(2\pi \left[\sigma_X^2 \Delta T_l + \frac{\sigma_\mu^2}{2\alpha} \Delta T_l^2 (1 - e^{-2\alpha \Delta T_{l-1}}) \right] \right)^{-1/2} \\
&\quad -\frac{1}{2} \sum_{l=3}^{w+1} \left(\frac{\left(y_{l-1,l} - \left[\mu^X_{l-2,l-1} e^{-\alpha \Delta T_{l1}} + \mu_0 (1 - e^{-\alpha \Delta T_{l-1}}) - \frac{\sigma_X^2}{2} \right] \Delta T_l \right)^2}{\sigma_X^2 \Delta T_l + \frac{\sigma_\mu^2}{2\alpha} \Delta T_l^2 (1 - e^{-2\alpha \Delta T_{l-1}})} \right) \\
&= -\frac{w-1}{2} \log(2\pi) - \frac{1}{2} \log \left(\prod_{l=3}^{w+1} \sigma_X^2 \Delta T_l + \frac{\sigma_\mu^2}{2\alpha} \Delta T_l^2 (1 - e^{-2\alpha \Delta T_{l-1}}) \right) \\
&\quad -\frac{1}{2} \sum_{l=3}^{w+1} \left(\frac{\left(y_{l-1,l} - \left[\mu^X_{l-2,l-1} e^{-\alpha \Delta T_{l-1}} + \mu_0 (1 - e^{-\alpha \Delta T_{l1}}) - \frac{\sigma_X^2}{2} \right] \Delta T_l \right)^2}{\sigma_X^2 \Delta T_l + \frac{\sigma_\mu^2}{2\alpha} \Delta T_l^2 (1 - e^{-2\alpha \Delta T_{l-1}})} \right),
\end{aligned} \tag{5.6}$$

where $\Delta T_l := T_l - T_{l-1}$ and w is the estimation window size (i.e., number of Y samples at each iteration). Indeed, one can easily verify that the function $\mathcal{L}$ is concave.

Log-likelihood calibration procedures for a two-layer stochastic asset pricing model with latent growth parameter (or volatility factor) are not very explicit in the literature, at least to the author's knowledge, and possesses some challenges. In Aït-Sahalia and Kimmel (2007), for instance, authors develop a maximum-likelihood calibration method for a two-layer stochastic volatility model where option prices are inverted to produce an estimate of the unobservable volatility state variable. Our GBM model with OU drift, as given in Eq. (4.2) and (4.3), can also be considered within this difficulty category. The issue with estimating the parameters of our earnings model is that a mean-reverting drift is not directly observable, which can lead to a distortion of parameter estimations, particularly of α.

We therefore replace the unobservable $\mu^X_{l-2,l-1}$, $l \leq n$, which goes into Eq. (5.6), with its signed empirical proxies $\hat{\mu}^{X+}_{l-2,l-1}$ and $\hat{\mu}^{X+}_{l-2,l-1}$, as per Eq. (5.2), depending on whether $\mathrm{sgn}(\log X_{l-1}/X_{l-2}) = +1$ or $\mathrm{sgn}(\log X_{l-1}/X_{l-2}) = -1$. Then, the solving the problem

$$\arg \max_{\alpha, \mu_0, \sigma_X, \sigma_\mu} \mathcal{L}(\alpha, \mu_0, \sigma_X, \sigma_\mu | \mathbf{y}) \tag{5.7}$$

referring to its necessary first-order optimality conditions

$$\frac{\partial \mathcal{L}}{\partial \alpha} = \frac{\partial \mathcal{L}}{\partial \mu_0} = \frac{\partial \mathcal{L}}{\partial \sigma_X} = \frac{\partial \mathcal{L}}{\partial \sigma_\mu} = 0 \tag{5.8}$$

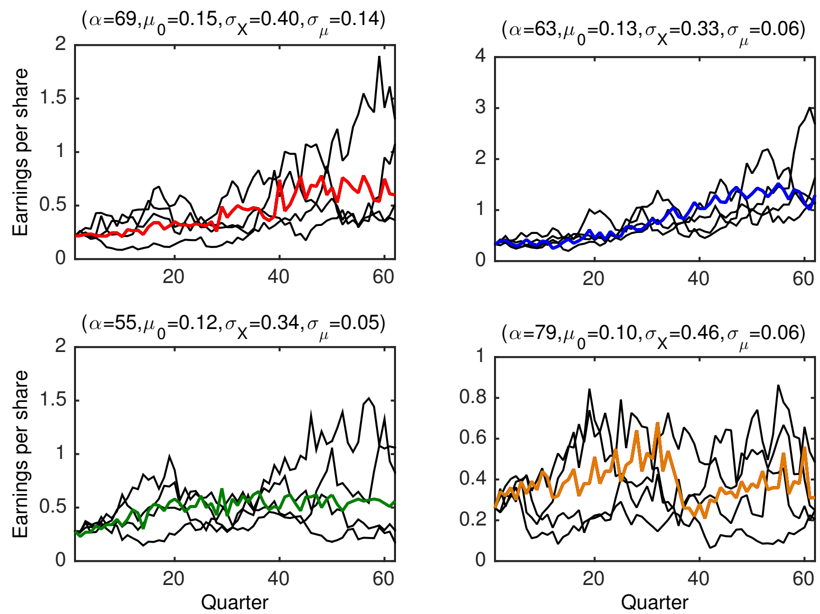

Figure 8: Sample paths of actual earnings (solid lines) compared to the calibrated earnings model output (with parameters in headers).

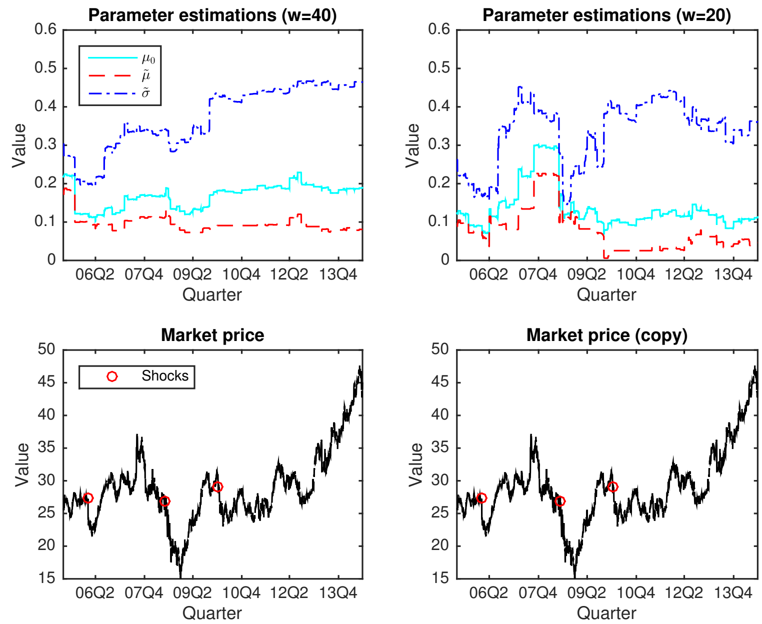

Figure 9: Maximum likelihood parameter estimation of stochastic drift model for implied dividends (top panel) and market price (bottom panel, two copies to ease vertical comparison). Note: The first red bullet in the bottom charts corresponds to a quarterly earnings announcement in 2006 which was significantly below expectations. Second bullet marks the collapse a systemically important financial institution in 2008. The third bullet is the 'Flash Crash' in 2010.

yields the desired results. To illustrate, we estimate the earnings model on selected tickers for the period 2000Q1-2015Q1 using more than 60 quarters of earnings data for each. The model output for each ticker is depicted against the actual earnings data in Figure 8, where the calibrated parameters are reported as figure titles. It can be inferred from the figure that earnings growth is generally characterised by large diversions from, as well as extremely fast reversions to, a long-term growth trajectory.

For pricing purposes, the parameters of $\mathcal{L}$ shall be estimated recursively using an appropriate rolling window length w that incorporates both past ($w - n_t$ samples) and future information (n_t samples). Figure 9 depicts the values over time of log-likelihood calibrated parameters, namely, $\tilde{\mu}$, $\tilde{\sigma}$ and μ_0, for the stock ticker `MSFT` (top panels), along with (two copies of) the observed market price for the same period (bottom panels) where major financial events (both at firm and system levels) are also indicated. Estimated values for α, on the other hand, lie in the band $[54.8, 191.0]$. One notable observation from Figure 9 could be that the estimated model parameters are able to capture major idiosyncratic and systemic incidents of financial stress.

5.2 Approximation via Kummer's Function

The pricing relation at time t, $s \le t \le u$, will then be based on

$$S_t = D_t^u \mathbb{E}\left[\delta X_u + \epsilon_u | \mathcal{F}_t^\xi\right] = D_t^u \mathbb{E}\left[\delta X_u | \mathcal{F}_t^\xi\right], \tag{5.9}$$

where, again, $\mathcal{F}_t^\xi$ is defined as in Eq. (3.1). The second equality, in fact, follows from the fact that ξ_t carries information only about X_u (i.e., $\epsilon_u \perp\!\!\!\perp \xi_t$) and the assumption that $\mathbb{E}\left[\epsilon_u | \{X_t\}_{t<u}\right] = 0$.

We now know from Eq. (5.2), (5.5) and (5.4) that, conditionally,

$$X_u | X_s \sim \log \mathcal{N}(\tilde{\mu}'_{su}, \tilde{\sigma}_{su}) \tag{5.10}$$

with $\tilde{\mu}'_{su} := \ln X_s/(u-s) + \tilde{\mu}_{su}$. Then, the pricing relation in Eq. (5.9) implies,

$$\begin{aligned} S_t &= D_t^u \mathbb{E}\left[\delta X | \xi_t\right] \\ &= \delta D_t^u \frac{\displaystyle\int_{\mathbb{X}} \exp\left(-\frac{1}{2}\frac{(\ln x - \tilde{\mu}'_{su}(u-s))^2}{\tilde{\sigma}^2_{su}(u-s)} - \frac{1}{2}\frac{(\xi_t - [x^* + \tau_t(x - x^*)])^2}{\tau_t(1-\tau_t)/(\sigma^2(u-t))}\right) \mathrm{d}x}{\displaystyle\int_{\mathbb{X}} x^{-1} \exp\left(-\frac{1}{2}\frac{(\ln x - \tilde{\mu}'_{su}(u-s))^2}{\tilde{\sigma}^2_{su}(u-s)} - \frac{1}{2}\frac{(\xi_t - [x^* + \tau_t(x - x^*)])^2}{\tau_t(1-\tau_t)/(\sigma^2(u-t))}\right) \mathrm{d}x} \end{aligned} \tag{5.11}$$

where $\mathbb{X} = (0, \infty)$. Equation (5.11) is apparently not very handy without recourse to numerical methods. We try to circumvent this issue by using two possible analytical approximations, namely, through gamma and log-gamma distributions.

More specifically, this uses either $X \sim \Gamma(a, b)$ approximately, or $Z \sim \log\Gamma(a, b)$, again, approximately, where $Z = \log X$, and Γ and $\log\Gamma$ are gamma and log-gamma[5] probability laws with densities

$$f_X(x|a,b) = \frac{x^{a-1}}{b^a \Gamma(a)} e^{-x/b}, \quad x \in (0, \infty), \tag{5.12}$$

and

$$f_Z(z|a,b) = \frac{1}{b^a \Gamma(a)} e^{az - e^z/b}, \quad z \in (-\infty, \infty), \tag{5.13}$$

respectively.[6]. For approximation, we choose to minimise the Kullback−Leibler (1951) divergence between the theoretical and approximating density functions, i.e.,

$$\begin{aligned} \arg\min_{(a,b)} \int_{\mathbb{X}} \; & f_X(\tilde{\mu}'_{su}(u-s), \tilde{\sigma}_{su}\sqrt{u-s}) \\ & \cdot \log_2\left(\frac{f_X(\tilde{\mu}'_{su}(u-s), \tilde{\sigma}\sqrt{u-s})}{f'_X(a_{su}, b_{su})}\right) \mathrm{d}x, \end{aligned} \tag{5.14}$$

[5] "Log-gamma" in the sense that it is logarithm of a gamma random variable (not that its logarithm is a gamma distribution as in the case of, e.g., lognormal distribution).

[6] We remark that gamma distribution is conjugate prior to log-normal distribution with a known mean.

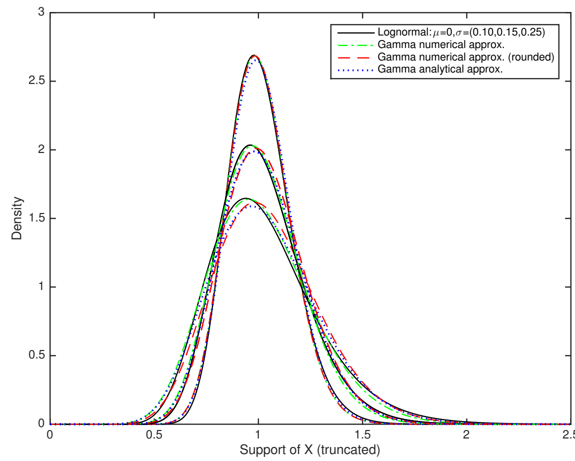

Figure 10: Approximation of conditional log-normal by its conjugate prior gamma.

or

$$\begin{aligned} \arg\min_{(a,b)} \quad \int_{\mathbb{Z}} \quad & f_Z(\tilde{\mu}'_{su}(u-s), \tilde{\sigma}_{su}\sqrt{u-s}) \\ & \cdot \log_2 \left(\frac{f_Z(\tilde{\mu}'_{su}(u-s), \tilde{\sigma}\sqrt{u-s})}{f'_Z(a_{su}, b_{su})} \right) \mathrm{d}z, \end{aligned} \tag{5.15}$$

which ensures the expected entropic (or, informational) distance between the latter two is minimised. Approximate analytical solution to problems given in Eq. (5.14) and (5.15), on the other hand, is given by

$$a_{su} \approx \frac{1}{\tilde{\sigma}^2_{su}(u-s)}, \quad b_{su} \approx \tilde{\sigma}^2_{su}(u-s) \exp\left(\left(\tilde{\mu}'_{su} + \frac{\tilde{\sigma}^2_{su}}{2} \right) (u-s) \right) \tag{5.16}$$

(see Appendix A for a sketch of proof). Figure 10 shows the results of gamma approximation to the log-normal density for different parameter values, using both numerical and analytical solutions to the Kullback−Leibler minimisation problem. The approximation works extremely good, particularly for small variances values, and this is why it will work particularly good in our context.

We use this property to replace the log-normal density (normal density) with its gamma (log-gamma) conjugate prior with shape and scale parameters $a_{su} = a(\tilde{\mu}'_{su}, \tilde{\sigma}_{su})$ and $b_{su} =$

$b(\tilde{\mu}'_{su}, \tilde{\sigma}_{su})$, which yields

$$\tilde{S}_t = \delta D_t^u \frac{\displaystyle\int_{\mathbb{X}} x^{a_{su}} \exp\left(-\frac{x}{b_{su}} - \frac{1}{2}\frac{(\xi_t - [x^* + \tau_t(x - x^*)])^2}{\tau_t(1-\tau_t)/(\sigma^2(u-t))}\right) \mathrm{d}x}{\displaystyle\int_{\mathbb{X}} x^{a_{su}-1} \exp\left(-\frac{x}{b_{su}} - \frac{1}{2}\frac{(\xi_t - [x^* + \tau_t(x - x^*)])^2}{\tau_t(1-\tau_t)/(\sigma^2(u-t))}\right) \mathrm{d}x}, \tag{5.17}$$

where $\tilde{S}_t$ is the approximation to S_t with a gamma prior and, again, $\mathbb{X} = (0, \infty)$. We can further simplify Eq. (5.17) as follows:

$$\begin{aligned}
\tilde{S}_t &= \delta D_t^u \frac{\displaystyle\int_{\mathbb{X}} x^{a_{su}} \exp\left(-\frac{x}{b_{su}} - \frac{1}{2}\frac{(\xi_t - (1-\tau_t)x^* - \tau x)^2}{\tau_t(1-\tau_t)/(\sigma^2(u-t))}\right) \mathrm{d}x}{\displaystyle\int_{\mathbb{X}} x^{a_{su}-1} \exp\left(-\frac{x}{b_{su}} - \frac{1}{2}\frac{(\xi_t - (1-\tau)x^* - \tau x)^2}{\tau_t(1-\tau_t)/(\sigma^2(u-t))}\right) \mathrm{d}x} \\
&= \delta D_t^u \frac{\displaystyle\int_{\mathbb{X}} x^{a_{su}} \exp\left(-\frac{1}{2}\frac{(\tau_t x)^2 - 2\tau_t x\left(\xi_t - (1-\tau_t)x^* - \frac{1-\tau_t}{b_{su}\sigma^2(u-t)}\right)}{\tau_t(1-\tau_t)/(\sigma^2(u-t))}\right) \mathrm{d}x}{\displaystyle\int_{\mathbb{X}} x^{a_{su}-1} \exp\left(-\frac{1}{2}\frac{(\tau_t x)^2 - 2\tau_t x\left(\xi_t - (1-\tau_t)x^* - \frac{1-\tau_t}{b_{su}\sigma^2(u-t)}\right)}{\tau_t(1-\tau_t)/(\sigma^2(u-t))}\right) \mathrm{d}x} \\
&= \delta D_t^u \frac{\displaystyle\int_{\mathbb{X}} x^{a_{su}} \exp\left(-\frac{1}{2}\left(\frac{\tau_t x - \psi_t}{\gamma_t}\right)^2\right) \mathrm{d}x}{\displaystyle\int_{\mathbb{X}} x^{a_{su}-1} \exp\left(-\frac{1}{2}\left(\frac{\tau_t x - \psi_t}{\gamma_t}\right)^2\right) \mathrm{d}x},
\end{aligned} \tag{5.18}$$

where $\psi_t := \xi_t - (1-\tau_t)x^* - \frac{1-\tau_t}{b_{su}\sigma^2(u-t)}$ and $\gamma_t := \frac{\sqrt{\tau_t(1-\tau_t)}}{\sigma\sqrt{(u-t)}}$. A double change of variable, i.e., first

$$\tilde{S}_t = \delta D_t^u \frac{\displaystyle\int_{\mathbb{X}'} \left(\frac{\gamma_t x' + \psi_t}{\tau_t}\right)^{a_{su}} \exp\left(-\frac{1}{2}(x')^2\right) \frac{\gamma_t}{\tau_t} \mathrm{d}x'}{\displaystyle\int_{\mathbb{X}'} \left(\frac{\gamma_t x' + \psi_t}{\tau_t}\right)^{a_{su}-1} \exp\left(-\frac{1}{2}(x')^2\right) \frac{\gamma_t}{\tau_t} \mathrm{d}x'}, \tag{5.19}$$

with $x' := (\tau_t x - \psi_t)/\gamma_t$ and $\mathbb{X}' = (-\psi_t/\gamma_t, \infty)$, and, second,

$$
\begin{aligned}
\tilde{S}_t &= \delta D_t^u \frac{\displaystyle\int_{\mathbb{X}} \left(\frac{x}{\tau_t}\right)^{a_{su}} \exp\left(-\frac{1}{2}\left(\frac{x-\psi_t}{\gamma_t}\right)^2\right) \frac{\gamma_t}{\tau_t}\frac{1}{\gamma_t} \mathrm{d}x}{\displaystyle\int_{\mathbb{X}} \left(\frac{x}{\tau_t}\right)^{a_{su}-1} \exp\left(-\frac{1}{2}\left(\frac{x-\psi_t}{\gamma_t}\right)^2\right) \frac{\gamma_t}{\tau_t}\frac{1}{\gamma_t} \mathrm{d}x} \\
&= \delta D_t^u \frac{1}{\tau_t} \frac{\displaystyle\int_{\mathbb{X}} x^{a_{su}} \exp\left(-\frac{1}{2}\left(\frac{x-\psi_t}{\gamma_t}\right)^2\right) \mathrm{d}x}{\displaystyle\int_{\mathbb{X}} x^{a_{su}-1} \exp\left(-\frac{1}{2}\left(\frac{x-\psi_t}{\gamma_t}\right)^2\right) \mathrm{d}x},
\end{aligned}
\tag{5.20}
$$

with $x := \gamma_t x' + \psi_t$ and $\mathbb{X} = (0, \infty)$, reveals that the signal-based price S_t can crisply be expressed as the ratio of two consecutive raw (uncentered) absolute (left-truncated at 0) moments of the normal random variable

$$
X \sim \mathcal{N}\left(\psi_t, \gamma_t^2\right), \tag{5.21}
$$

where, again,

$$
\psi_t := \xi_t - (1-\tau_t)x^* - \frac{1-\tau_t}{b_{su}\sigma^2(u-t)} \quad \text{and} \quad \gamma_t := \frac{\sqrt{\tau_t(1-\tau_t)}}{\sigma\sqrt{(u-t)}}. \tag{5.22}
$$

With reference to, e.g., Winkelbauer (2012), Eq. (5.20) can be rephrased even more neatly as

$$
\begin{aligned}
\tilde{S}_t &= \delta D_t^u \frac{1}{\tau_t} \frac{\gamma_t^{a_{su}} 2^{\frac{a_{su}}{2}} \frac{\Gamma\left(\frac{a_{su}+1}{2}\right)}{\sqrt{\pi}}}{\gamma_t^{a_{su}-1} 2^{\frac{a_{su}-1}{2}} \frac{\Gamma\left(\frac{a_{su}}{2}\right)}{\sqrt{\pi}}} \frac{{}_1F_1\left(-\frac{a_{su}}{2}, \frac{1}{2}; -\frac{1}{2}\left(\frac{\psi_t}{\gamma_t}\right)^2\right)}{{}_1F_1\left(-\frac{a_{su}-1}{2}, \frac{1}{2}; -\frac{1}{2}\left(\frac{\psi_t}{\gamma_t}\right)^2\right)} \\
&= \delta D_t^u \tau_t^{-1} \gamma_t \sqrt{2} \frac{\Gamma\left(\frac{1+a_{su}}{2}\right)}{\Gamma\left(\frac{a_{su}}{2}\right)} \frac{{}_1F_1\left(-\frac{a_{su}}{2}, \frac{1}{2}; -\frac{1}{2}\left(\frac{\psi_t}{\gamma_t}\right)^2\right)}{{}_1F_1\left(-\frac{a_{su}-1}{2}, \frac{1}{2}; -\frac{1}{2}\left(\frac{\psi_t}{\gamma_t}\right)^2\right)},
\end{aligned}
\tag{5.23}
$$

where ${}_1F_1(\kappa, \varphi; z)$ corresponds to the confluent hypergeometric function of the first kind (or Kummer's function) that is given by

$$
{}_1F_1(\lambda, \varphi; z) \equiv \sum_{m=0}^{\infty} \frac{(\lambda)_m}{(\varphi)_m} \frac{z^m}{m!} \tag{5.24}
$$

with $(\lambda)_m$ being the Pochhammer symbol defined by

$$
(\lambda)_m \equiv \begin{cases} 1, & \text{if} \quad m = 0, \\ \lambda(\lambda+1)\ldots(\lambda+m-1), & \text{if} \quad m > 0. \end{cases} \tag{5.25}
$$

The confluent hypergeometric functions appear rarely in the financial mathematics literature and are generally used as a tool to derive the characteristic function of an average F-distribution as part of the general theory of asset pricing (see, e.g., Hwang and Satchell (2012)). In Boyle and Potapchik (2006), a confluent hypergeometric function appears in the computation of the Laplace transform of the normalised price for arithmetic Asian options. Computation of the confluent hypergeometric functions can pose, however, significant challenges, particularly, when $|z| \gg 0$ (see, e.g., Boyle and Potapchik (2006), Pearson (2009)).

Eq. (5.24) is known to converge for any $z \in \mathbb{C}$ and is defined for any $\lambda \in \mathbb{C}$, $\varphi \in \mathbb{C}\backslash\{\mathbb{Z}^{-} \cup \{0\}\}$, with $\mathbb{Z}^{-}$ being the set of negative integers. We also note that ${}_1F_1(\lambda, \varphi; 0) = 1$ for all feasible λ, φ. Further details on this type of functions are provided in Pearson (2009). Furthermore, in Brody et al. (2007), the authors reach a closed-form result in terms of a finite sum of Legendre-type polynomials that is somewhat analogous to Equation (5.23).

There is in fact a range of fast and effective algorithms available in the literature (see, e.g., Pearson (2009)) to compute ${}_1F_1(\lambda, \varphi; z)$, such as Taylor series, single fraction, Buchholz polynomials, asymptotic series expansion, quadrature methods, or via solving the confluent hypergeometric differential equation (CHDE):

$$z\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + (\varphi - z)\frac{\mathrm{d}f}{\mathrm{d}z} - \lambda f = 0. \tag{5.26}$$

A thorough survey of algorithms that deal with confluent hypergeometric functions is beyond the scope of this paper, but Taylor series expansion seems to stand out as the most simple and least costly method to compute Eq. (5.24). Picking an appropriate tolerance level, say $e = 10^{-15}$, and introducing, based on Eq. (5.24), the series

$$A_m := \frac{(\lambda)_m}{(\varphi)_m}\frac{z^m}{m!}, \quad \hat{F}_m := \sum_{m=0}^{\infty} A_m, \tag{5.27}$$

with $A_0 = 1$, $\hat{F}_0 = A_0$, and

$$A_{m+1} = A_m \left(\frac{\lambda + m}{\varphi + m}\right)\left(\frac{z}{m+1}\right), \quad \hat{F}_{m+1} = \hat{F}_m + A_m, \quad \hat{F}_\infty = {}_1F_1, \tag{5.28}$$

the desired function ${}_1F_1$ can easily be computed to a high precision using the following truncation procedure:

$$\hat{F}_M = \sum_{m=0}^{M} A_m, \quad \text{such that} \quad \frac{|A_{M+1}|}{|\hat{F}_M|} < e. \tag{5.29}$$

This method indeed yields the desired values of ${}_1F_1$ in a small fraction of a second, rendering the pricing method computationally efficient. Figure 11 shows the ratio of two

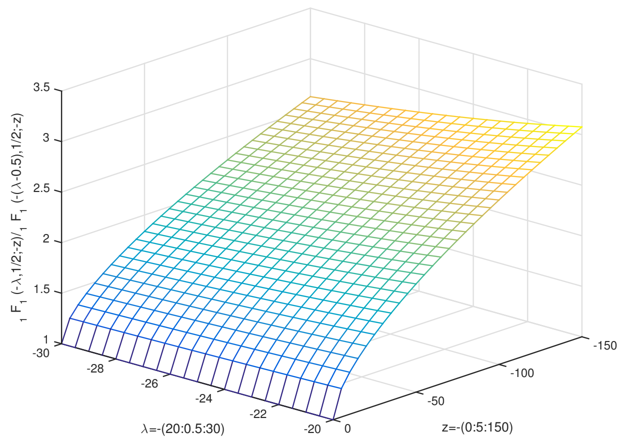

Figure 11: Ratio of two confluent hypergeometric functions whereas values are calculated using Taylor expansion with a tolerance level $e = 10^{-15}$.

confluent hypergeometric functions for several values of λ and z, calculated based on the above method.

Thus, all in all, we are able to recover a crisp tractable approximation formula for the signal-based price of a risky asset at time t which will pay an implied dividend of $\phi(X_u)$ at time u. For computational purposes, we finally note from Eq. (5.23) that, when $s < t \leq u$ (i.e., with a_s^u and b_s^u having already been inferred from the data) only the last argument of ${}_1F_1$ needs to be updated with the arrival of new information ξ_t − which is expected to improve the algorithm's speed.

5.3 Signal-based Asset Price

At any time t, there will be a total of $k = 1, \ldots, n(t)$, earnings signals, with each of them being τ_k into their lifetime. Thus, the approximate price $\tilde{S}_t$ in the multiple cashflow case is the sum of information-based net present values $\tilde{S}_t^1, \ldots, \tilde{S}_t^{n(t)}$, of a strip of $n(t)$ cashflows, and a Gordon continuation value in the sense of Section 4.1 above, i.e.,

$$\begin{aligned} \tilde{S}_t &= \delta \left(\sum_{k=1}^{n} D_t^k \left(\phi_t(X_{T_k}) + \mathbf{1}_{\{k=n\}} \frac{\phi_t(X_{T_{k+1}})}{r_b - \mu_0} \right) \right) \\ &= \delta \left(\sum_{k=1}^{n} D_t^k \left(\phi_t(X_{T_k}) + \mathbf{1}_{\{k=n\}} \frac{\phi_t(\phi_{T_k}(X_{T_{k+1}}))}{r - \mu_0} \right) \right) \\ &= \delta \left(\sum_{k=1}^{n} \tilde{S}_t^k \left(1 + \mathbf{1}_{\{k=n\}} \frac{e^{\mu_0 \mathrm{d}T_k}}{r_b - \mu_0} \right) \right), \end{aligned} \tag{5.30}$$

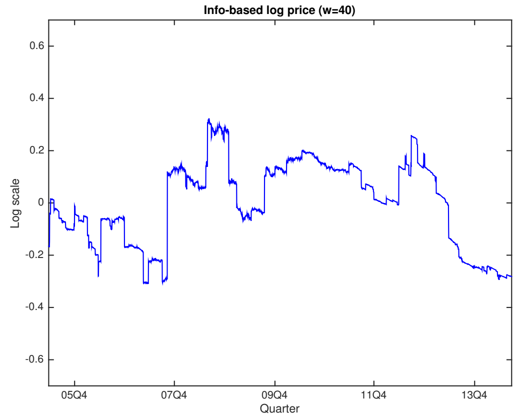

Figure 12: Signal-based price based on multiple signals on quarterly earnings.

where $D_t^k := e^{-r_t^k(T_k-t)}$ with $r_t^k \neq r_b$, $r_b > \mu_0$, and each $\tilde{S}_t^k$ is as given in Eq. (5.23) above, and with

$$\phi_t(X_v) = \phi_t(\phi_u(X_v)) \quad (t \leq u \leq v) \tag{5.31}$$

following from the tower property given the definition $\phi_t(X_u) := \mathbb{E}_t[\phi(X_u)]$, or $\mathbb{E}[\phi(X_u)|\xi_t]$. Figure 12 depicts the evolution of $\tilde{S}_t$, based on Eq. (5.30), over the sample period.

6 Summary and Conclusion

Thus, in this paper, we have presented one possible way of harnessing the signal-based framework of Brody et al. (2007) in a real-world setting by adapting it to a certain choice for the signal process. In particular, we have shown that the information process and information-based framework can be both practically and computationally viable, and a crisp approximation to the signal-based price be recovered via the confluent hypergeometric function of the first kind (or, Kummer's function). Introducing a slightly modified version of ξ_t and using quarterly earnings consensus data as the basis for constructing the required signals empirically, we have presented some concrete pricing results. Besides, we have also verified, through a brief information-theoretic analysis, how signal-based option prices are in agreement with the fact that options are priced on the basis of volatility (i.e., how rapidly the information is incorporated into prices) rather than entropy (i.e., how much uncertainty remains about the true value of the fundamental). By calibrating signal-to-noise parameter to option prices, we have posited that information flow speed can indeed be a more intuitive alternative to implied volatility. Finally, a simple procedure is presented to calibrate the signal-to-noise parameter to empirical signals.

Appendix

A Kullback−Leibler Minimisation (Log-normal to Gamma)

We recall the objective function related to Kullback−Leibler distance minimisation problem (5.14):

$$D(a_t, b_t) = \int_{\mathbb{X}} f_X(\tilde{\mu}_t, \tilde{\sigma}_t) \log\left(\frac{f_X(\tilde{\mu}_t, \tilde{\sigma}_t)}{g_X(a_t, b_t)}\right) \mathrm{d}x, \tag{A.1}$$

where $\mathbb{X} = (0, \infty)$. Let $h(\tilde{\mu}_t, \tilde{\sigma}_t)$ denote the terms which don't depend on a_t and b_t. We have

$$D(a_t, b_t) = h(\tilde{\mu}_t, \tilde{\sigma}_t) + \log\left(\Gamma\left(a_t\right)\right) + a_t \log\left(b_t\right) + \frac{1}{b_t}\mathbb{E}_f\left[X\right] - (a_t - 1)\mathbb{E}_f\left[\log\left(X\right)\right]. \tag{A.2}$$

Taking derivatives of D with respect to its arguments, each set to zero, we get

$$\frac{\partial D(a_t, b_t)}{\partial a_t} = \Psi^{(0)}\left(a_t\right) + \log\left(b_t\right) - \mathbb{E}_f\left[\log\left(X\right)\right] = 0 \tag{A.3}$$

$$\begin{aligned} \frac{\partial D(a_t, b_t)}{\partial b_t} &= \frac{a_t}{b_t} - \frac{1}{b_t^2}\mathbb{E}_f\left[X\right] = 0 \\ &= a_t b_t - \mathbb{E}_f\left[X\right] = 0. \end{aligned} \tag{A.4}$$

where

$$\Psi^{(m)}(a_t) \equiv \mathrm{d}^{m+1} \log\Gamma(a_t)/\mathrm{d}a_t^{m+1} \tag{A.5}$$

is the polygamma function. Knowing that $\mathbb{E}_f[\log(X)] = \tilde{\mu}_t$ and $\mathbb{E}_f[X] = \exp(\tilde{\mu}_t + \tilde{\sigma}_t^2/2)$, we obtain the following system of equations to solve:

$$\begin{aligned} \Psi^{(0)}\left(a_t\right) + \log\left(b_t\right) &= \tilde{\mu}_t \\ a_t b_t &= \exp\left(\tilde{\mu}_t + \frac{\tilde{\sigma}_t^2}{2}\right). \end{aligned} \tag{A.6}$$

Next, we eliminate b_t by inserting first equation into the latter

$$a_t = \exp\left(\Psi^{(0)}\left(a_t\right) + \frac{\tilde{\sigma}_t^2}{2}\right). \tag{A.7}$$

A first-degree approximation to $\Psi^{(0)}\left(a_t\right)$ is given by

$$\Psi^{(0)}\left(a_t\right) \approx \log\left(a_t\right) - \frac{1}{2a_t} \tag{A.8}$$

which yields

$$a_t \approx \frac{1}{\tilde{\sigma}_t^2}, \quad b_t \approx \tilde{\sigma}_t^2 \exp\left(\tilde{\mu}_t + \frac{\tilde{\sigma}_t^2}{2}\right). \tag{A.9}$$

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The author reports no conflicts of interest. The author alone is responsible for the content and writing of the paper.

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