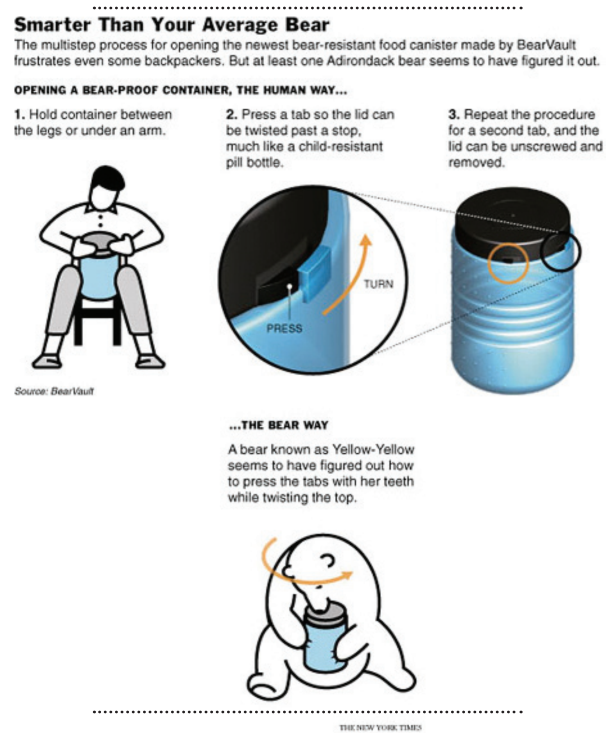
Sequencing is a way of organizing many basic ideas in a purposeful order.

Sequences may be arranged so that their order is a recurrent pattern or a logical progression.

A narrative, for example, is usually a sequence of fictional or non-fictional events.

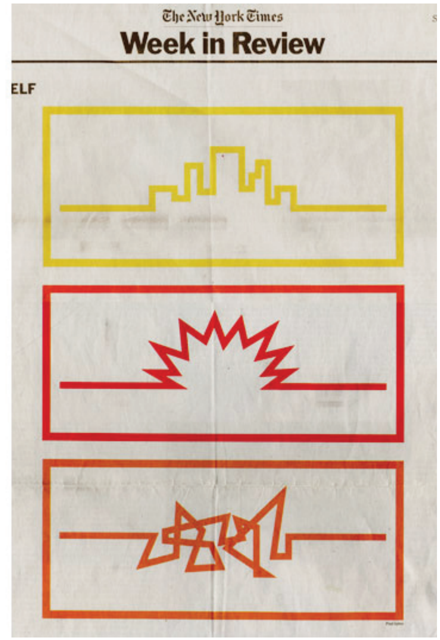
Recipes often feature ordered sequences of steps required for food preparation.

Almost all things we learn are relayed to us by means of some sort of sequence, steps, and so forth.



This New York Times infographic shows how a three step sequence may be approached in a different way. Even the bear method is a two-step process.

Even without words, many sequences are powerful ways of conveying ideas, both big & small.



This three part illustration by Paul Sahre is called "Fear Itself."

7.30 Definition Let $\mathscr{A}$ be a family of functions on a set E. Then $\mathscr{A}$ is said to *separate points* on E if to every pair of distinct points $x_1, x_2 \in E$ there corresponds a function $f \in \mathscr{A}$ such that $f(x_1) \neq f(x_2)$.

If to each $x \in E$ there corresponds a function $g \in \mathscr{A}$ such that $g(x) \neq 0$, we say that *$\mathscr{A}$ vanishes at no point of E.*

The algebra of all polynomials in one variable clearly has these properties on R^1. An example of an algebra which does not separate points is the set of all even polynomials, say on $[-1, 1]$, since $f(-x) = f(x)$ for every even function f.

The following theorem will illustrate these concepts further.

7.31 Theorem *Suppose $\mathscr{A}$ is an algebra of functions on a set E, $\mathscr{A}$ separates points on E, and $\mathscr{A}$ vanishes at no point of E. Suppose x_1, x_2 are distinct points of E, and c_1, c_2 are constants (real if $\mathscr{A}$ is a real algebra). Then $\mathscr{A}$ contains a function f such that*

$$f(x_1) = c_1, \qquad f(x_2) = c_2.$$

Proof The assumptions show that $\mathscr{A}$ contains functions g, h, and k such that

$$g(x_1) \neq g(x_2), \qquad h(x_1) \neq 0, \qquad k(x_2) \neq 0.$$

Put

$$u = gk - g(x_1)k, \qquad v = gh - g(x_2)h.$$

Then $u \in \mathscr{A}$, $v \in \mathscr{A}$, $u(x_1) = v(x_2) = 0$, $u(x_2) \neq 0$, and $v(x_1) \neq 0$. Therefore

$$f = \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}$$

has the desired properties.

We now have all the material needed for Stone's generalization of the Weierstrass theorem.

7.32 Theorem *Let $\mathscr{A}$ be an algebra of real continuous functions on a compact set K. If $\mathscr{A}$ separates points on K and if $\mathscr{A}$ vanishes at no point of K, then the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all real continuous functions on K.*

We shall divide the proof into four steps.

STEP 1 *If $f \in \mathscr{B}$, then $|f| \in \mathscr{B}$.*

Proof Let

(52) $$a = \sup |f(x)| \qquad (x \in K)$$

Proofs are ideas in sequence that follow from basic axioms or other proofs.

and let $\varepsilon > 0$ be given. By Corollary 7.27 there exist real numbers $c_1, \ldots, c_n$ such that

(53) $$\left| \sum_{i=1}^{n} c_i y^i - |y| \right| < \varepsilon \qquad (-a \leq y \leq a).$$

Since $\mathscr{B}$ is an algebra, the function

$$g = \sum_{i=1}^{n} c_i f^i$$

is a member of $\mathscr{B}$. By (52) and (53), we have

$$\big| g(x) - |f(x)| \big| < \varepsilon \qquad (x \in K).$$

Since $\mathscr{B}$ is uniformly closed, this shows that $|f| \in \mathscr{B}$.

STEP 2 *If $f \in \mathscr{B}$ and $g \in \mathscr{B}$, then $\max(f, g) \in \mathscr{B}$ and $\min(f, g) \in \mathscr{B}$.*

By $\max(f, g)$ we mean the function h defined by

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \geq g(x), \\ g(x) & \text{if } f(x) < g(x), \end{cases}$$

and $\min(f, g)$ is defined likewise.

Proof Step 2 follows from step 1 and the identities

$$\max(f, g) = \frac{f+g}{2} + \frac{|f-g|}{2},$$

$$\min(f, g) = \frac{f+g}{2} - \frac{|f-g|}{2}.$$

By iteration, the result can of course be extended to any finite set of functions: If $f_1, \ldots, f_n \in \mathscr{B}$, then $\max(f_1, \ldots, f_n) \in \mathscr{B}$, and

$$\min(f_1, \ldots, f_n) \in \mathscr{B}.$$

STEP 3 *Given a real function f, continuous on K, a point $x \in K$, and $\varepsilon > 0$, there exists a function $g_x \in \mathscr{B}$ such that $g_x(x) = f(x)$ and*

(54) $$g_x(t) > f(t) - \varepsilon \qquad (t \in K).$$

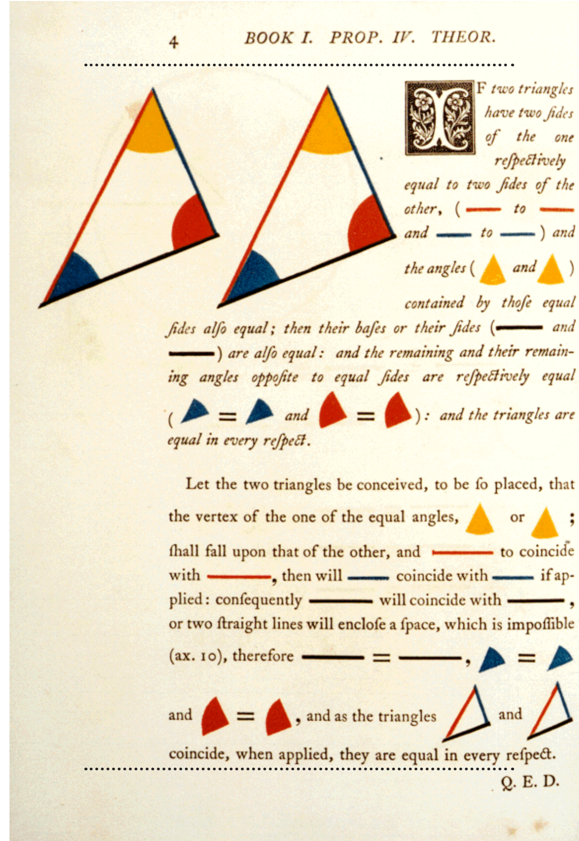
Proof Since $\mathscr{A} \subset \mathscr{B}$ and $\mathscr{A}$ satisfies the hypotheses of Theorem 7.31 so does $\mathscr{B}$. Hence, for every $y \in K$, we can find a function $h_y \in \mathscr{B}$ such that

(55) $$h_y(x) = f(x), \qquad h_y(y) = f(y).$$

But proofs don’t need to “look like proofs” in order to convey a rational sequence of ideas.

In 1847 Oliver Byrne published the first six books of “The Elements of Euclid” using this mostly helpful method of *visually* relaying the proofs in sequence.

Byrne writes, tellingly, on the title page: “In which coloured diagrams and symbols are used instead of letters for the greater ease of learners.”

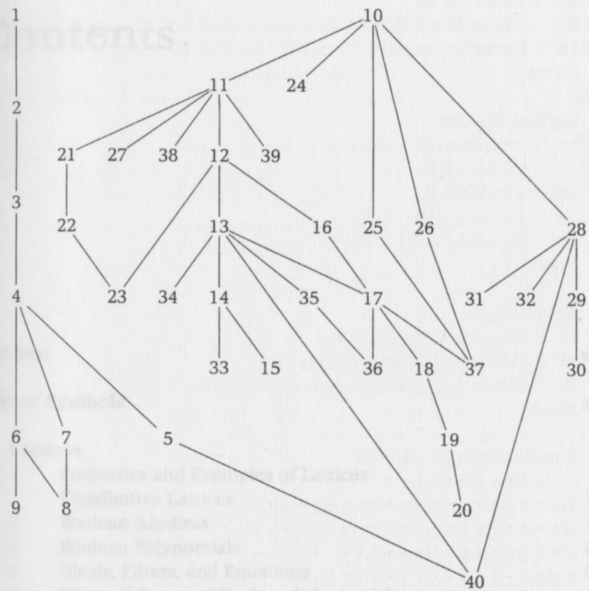

4 BOOK I. PROP. IV. THEOR.

IF two triangles have two ſides of the one reſpectively equal to two ſides of the other, (to and to) and the angles (and) contained by thoſe equal ſides alſo equal; then their baſes or their ſides (and) are alſo equal: and the remaining and their remaining angles oppoſite to equal ſides are reſpectively equal (= and =): and the triangles are equal in every reſpect.

Let the two triangles be conceived, to be ſo placed, that the vertex of the one of the equal angles, or ; ſhall fall upon that of the other, and to coincide with , then will coincide with if applied: conſequently will coincide with , or two ſtraight lines will encloſe a ſpace, which is impoſſible (ax. 10), therefore = , = and = , and as the triangles and coincide, when applied, they are equal in every reſpect.

Q. E. D.

Interdependence Chart

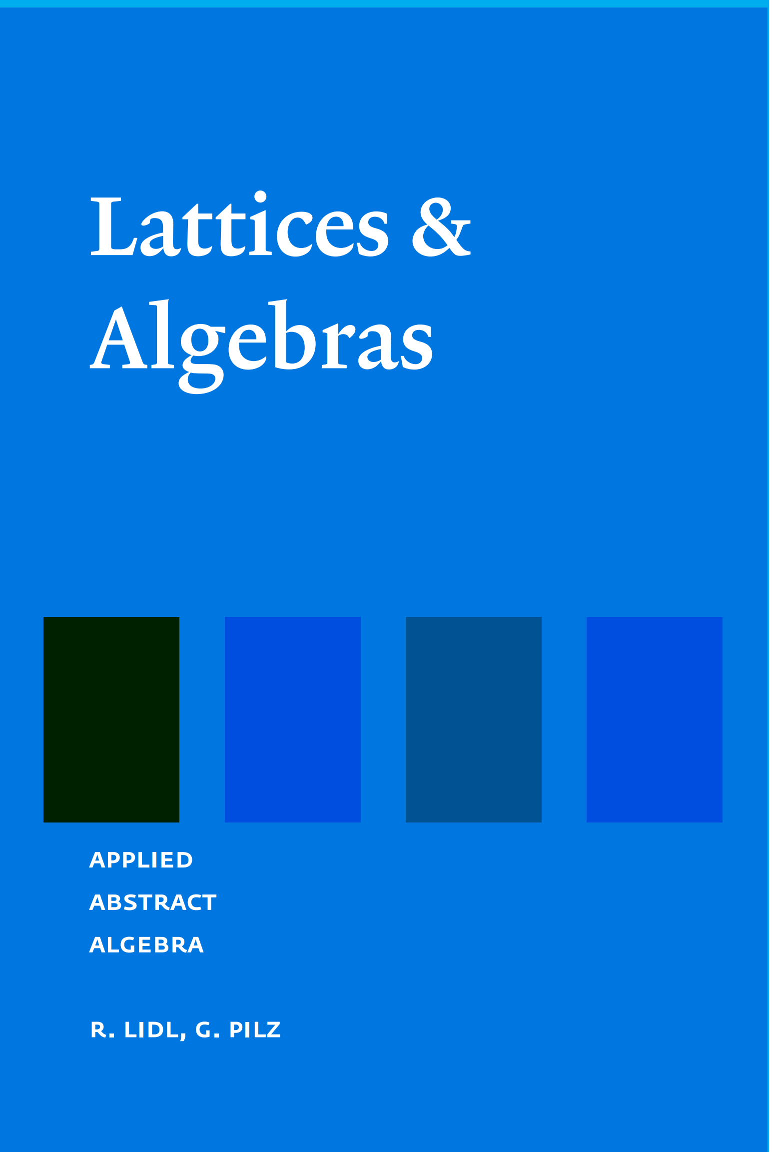


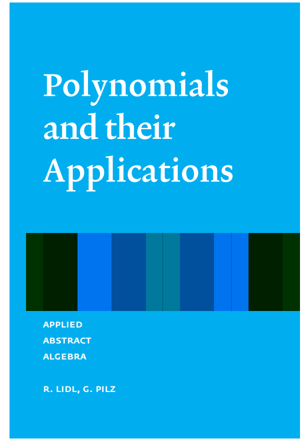
Among the numerous general texts on algebra, we mention Birkhoff & MacLane (1977), Childs (1995), Herstein (1975), Jacobson (1985), and Lang (1984). Application-oriented books include Biggs (1985), Birkhoff & Bartee (1970), Bobrow & Arbib (1974), Cohen, Giusti & Mora (1996), Dorninger & Müller (1984), Fisher (1977), Gilbert (1976), Prather (1976), Preparata & Yeh (1973), Spindler (1994), and Stone (1973). A survey of the present "state of the art" in algebra is Hazewinkel (1996) (with several more volumes to follow). Historic notes on algebra can be found in Birkhoff (1976). Applications of linear algebra (which are not covered in this book) can be found in Goult (1978), Noble & Daniel (1977), Rorres & Anton (1984), and Usmani (1987). Lipschutz (1976) contains a large collection of Exercises. Good books on computational aspects ("Computer Algebra") include Geddes, Czapor & Labahn (1993), Knuth (1981), Lipson (1981), Sims (1984), Sims (1994), and Winkler (1996).

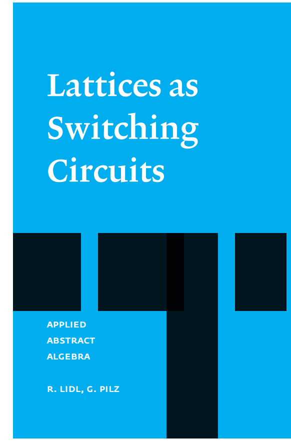
Complicated series often appear to be made up of many rational sub-sequences.

In these cases, the subsequences could be potentially abstracted in order to create a more legible high level path.

For example, sections 1-4 stand on their own as a non-divergent linear sequence, whereas 6,9 and 7,8 also appear to be separate sequences.

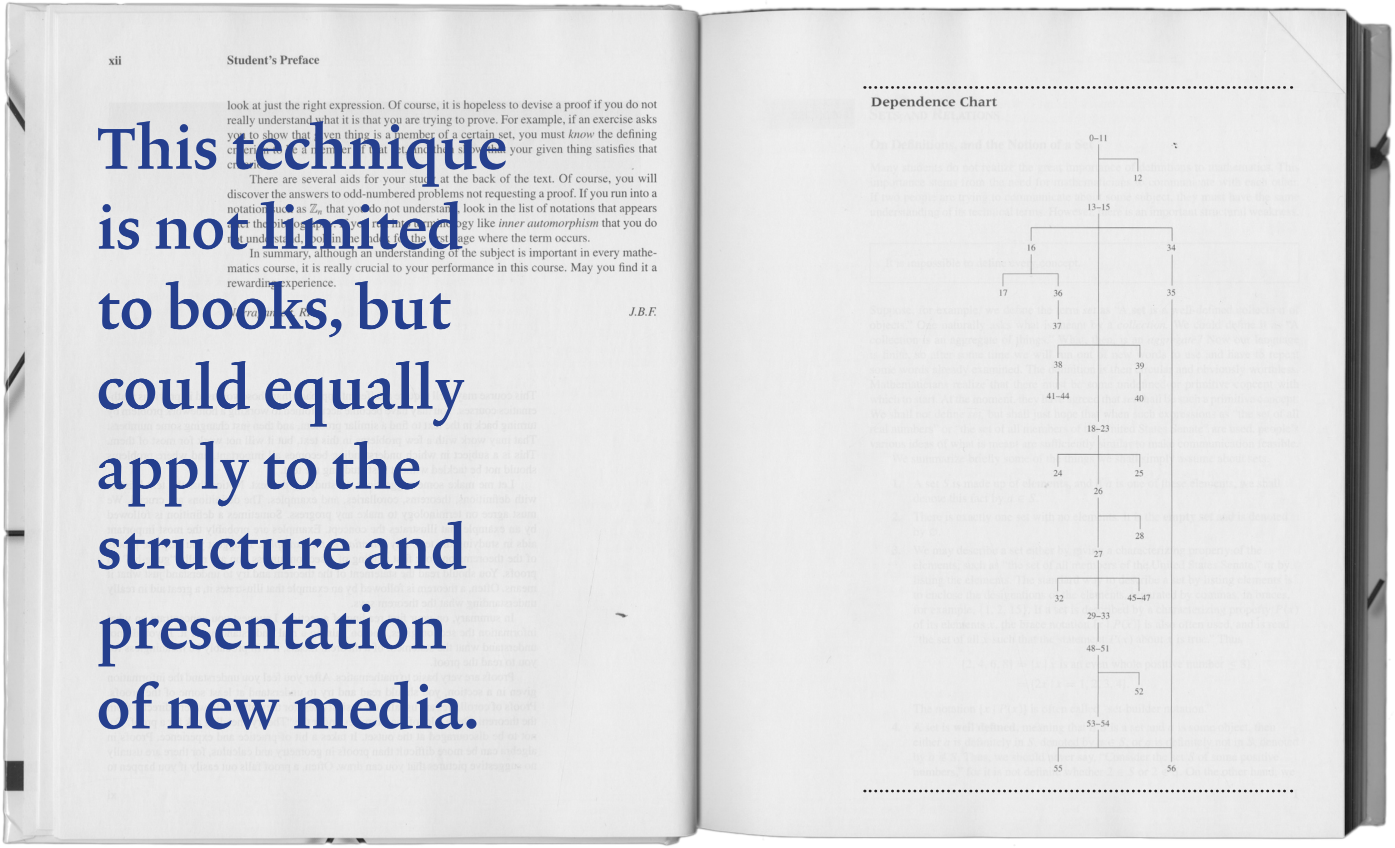






For example, breaking the previous graph into three distinct topics where it diverges may reveal a more compelling narrative.

I believe that treating the book (both inside and out) as part of a visual system that follows the organization and content of its high level topics is an integral way of making its content accessible.


xii Student's Preface

look at just the right expression. Of course, it is hopeless to devise a proof if you do not really understand what it is that you are trying to prove. For example, if an exercise asks you to show that a given thing is a member of a certain set, you must *know* the defining criterion to be a member of that set, and then show that your given thing satisfies that criterion.

There are several aids for your study at the back of the text. Of course, you will discover the answers to odd-numbered problems not requesting a proof. If you run into a notation such as $\mathbb{Z}_n$ that you do not understand, look in the list of notations that appears after the bibliography. If you run into terminology like *inner automorphism* that you do not understand, look in the index for the first page where the term occurs.

In summary, although an understanding of the subject is important in every mathematics course, it is really crucial to your performance in this course. May you find it a rewarding experience.

Narragansett, RI J.B.F.

This technique is not limited to books, but could equally apply to the structure and presentation of new media.

Marcos Ojeda
30 Sept, 2009
http://generic.cx/thesis