Project 2: Transient Analysis

Lab Report

A Requirement in ELEN 100

Submitted by:
Nathan Garvey
Jasper Tan

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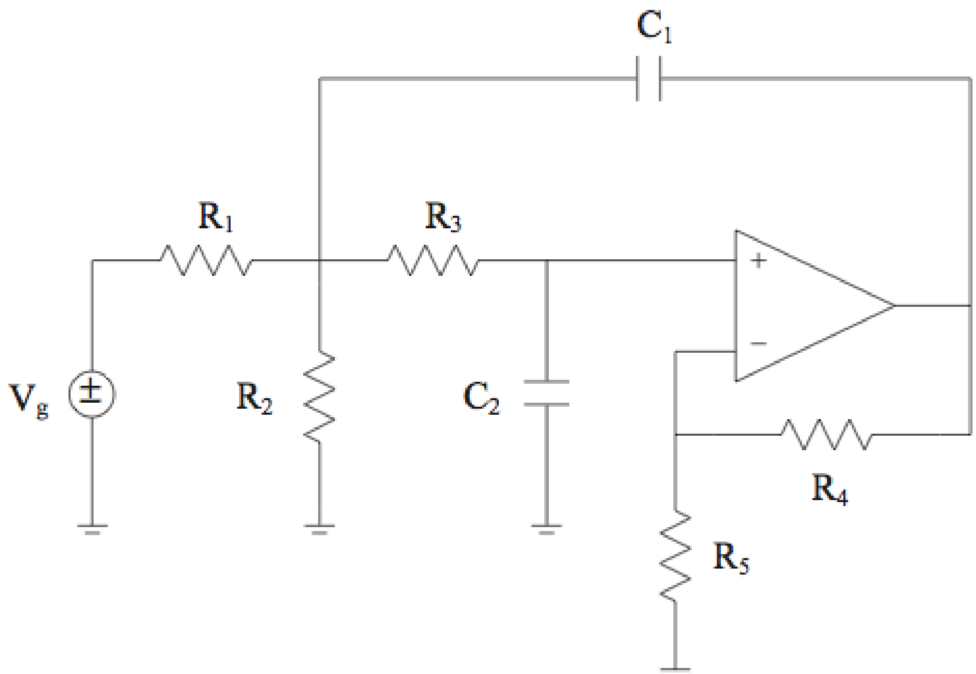
Under Florencia Clemente

Problem 1:

Derive the differential-algebraic equations that describe this circuit (express them in matrix form).



It is first important to know the assumptions that come with ideal op amps.
For this case, the assumptions are that $V_2 = V_3$ and that the currents that enter the + and - input terminals of the op amp are 0.

This problem can be solved by using KCL on nodes V_1, V_2, and V_o (through V_3).
The current through a resistor is V/R (its voltage drop divided by its resistance) and the current through a capacitor is C dV/dt (the product of its capacitance and the time derivative of its voltage drop).

The equations are as follows:

Equation 1:

$$(V_1 - V_g)/R_1 + V_1/R_2 + (V_1 - V_2)/R_3 + C_1 d(V_1 - V_0)/dt = 0$$

Equation 2:

$$(V_2 - V_1)/R_3 + C_2 d(V_2)/dt = 0$$

Equation 4:

$$V_2/R_5 = (V_0 - V_2)/R_4$$

The equations can be easily solved by obtaining the laplace transforms. The voltages would now be in the s-domain instead of the t-domain. Since the voltage source is a step function, its laplace transform is 1/s. The equations are now as follows:

Equation 1:

$$(V_1 - V_g)/R_1 + V_1/R_2 + (V_1 - V_2)/R_3 + C_1 d(V_1 - V_0)/dt = 0$$

$$\Longrightarrow (V_1(s) - 1/s)/R_2 + V_1(s)/R_2 + V_1(s)/R_3 - V_2(s)/R_3 + C_1 s(v_1(s) - V_0(s)$$

$$[1/R_1 + 1/R_2 + 1/R_3 + Cs]V_1(s) + [-1/R_3]V_2(s) + [-C_1 s]V_0(s) = 1/sR_1$$

Equation 2:

$$(V_2 - V_1)/R_3 + C_2 d(V_2)/dt = 0 \Longrightarrow (V_2(s) - V_1(s))/R_3 + C_2 s(V_2(s)) = 0$$

$$[-1/R_3]V_1(s) + [1/R_3 + C_2 s]V_2(s) = 0$$

Equation 4:

$$V_2/R_5 = (V_0 - V_2)/R_4 \Longrightarrow [1/R_1 + 1/R_5]V_2(s) + [-1/R_4]V_0(s) = 0$$

The matrix form of the equations is:

[1/R1+1/R2+1/R3+C1*s, -1/R3, -C1*s; -1/R3, 1/R3+C2*s, 0; 0, 1/R4+1/R5, -1/R4]*[V0; V1; V2] = [1/(s*R1); 0; 0]

On the next page is a handwritten solution.

Problem 2:
Write an m-file that simulates the step response of this circuit. Use this file to obtain a plot that corresponds to the following element values: R1 = R2 = 5KΩ, R3 = 400Ω, and C1 = C2 = 0.1 μF . What is the peak overshoot in this case? NOTE: The peak overshoot is defined as the maximal amount by which v0(t) exceeds 1 V .

Our goal in this lab was to analyse the transient response of the circuit. This cannot be done with previous methods using phasor analysis, so conversion to the s domain via laplace transform aided in solving the differential equations. The m-file step.m, took in resistor, capacitor, s and t values as syms:

```
syms R1 R2 R3 R4 R5
syms C1 C2
syms s t
```

which were solved for by the s-domain matrix of problem 1 in these three lines

```
M = [1/R1+1/R2+1/R3+C1*s, -1/R3, -C1*s; -1/R3, 1/R3+C2*s, 0; 0, 1/R4+1/R5,-1/R4];
S = [1/(s*R1); 0; 0];
X = M\S;
```

and the physical values

```
r1 = 5000;
r2 = 5000;
r3 = 400;
r4 =1000;
r5 = 1000;
c1 = .1e-6;
c2 = .1e-6;
```

were substituted back into the voltage expression where the inverse laplace function was applied to obtain physical values of the output voltage as a function of time.

```
Xn = subs (X, {R1 R2 R3 R4 R5 C1 C2} , {r1 r2 r3 r4 r5 c1 c2} )
x = ilaplace(Xn).
```

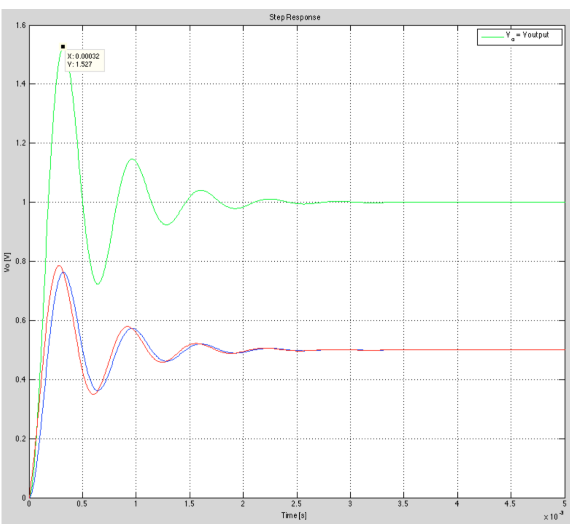
After voltage expressions in the time domain are obtained by matlab, a proper time interval 5 milliseconds is used so that the transient response plot scale is meaningful. A for loop generates the points to be plotted by the plot function.

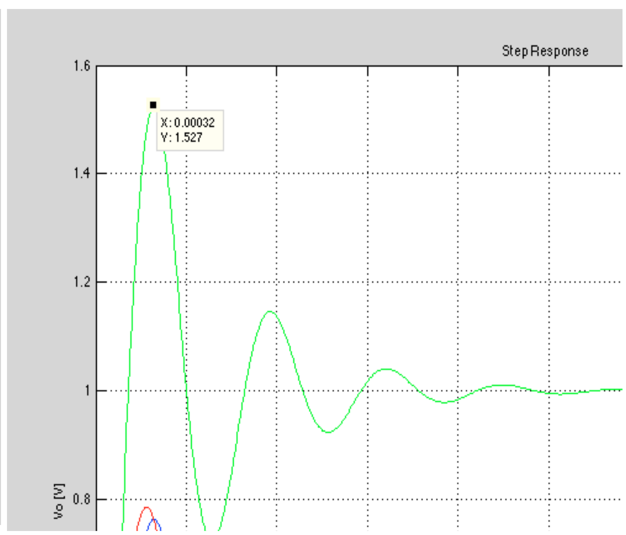
```
tt = [0: 1e-5 : .005];

for i = 1:length(tt)
    v1n (i) = subs ( v2 , {t} , {tt(i)});
     v2n (i) = subs ( v1 , {t} , {tt(i)});
     v0n (i) = subs ( v0 , {t} , {tt(i)});
end;
```

Now that we have stored enough points on the x-axis (time) and y-axis (output voltage), matlab generates the plot with the following code.

```
plot(tt,v0n,'g',tt,v1n,'b',tt,v2n,'r'); grid on; title('Step Response');
legend('V_o = Voutput');xlabel('Time [s] '); ylabel('Vo [V]');
grid on;
```

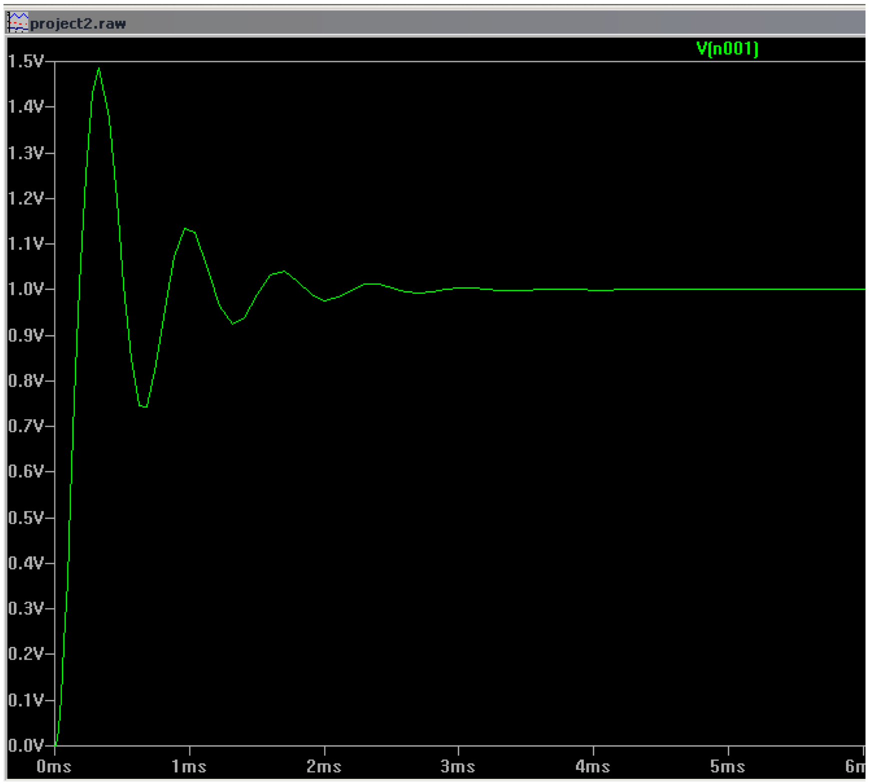
The plot generated with the impedances given is found below. Green is the output voltage. Peak overshoot here is .527 V with the values given.

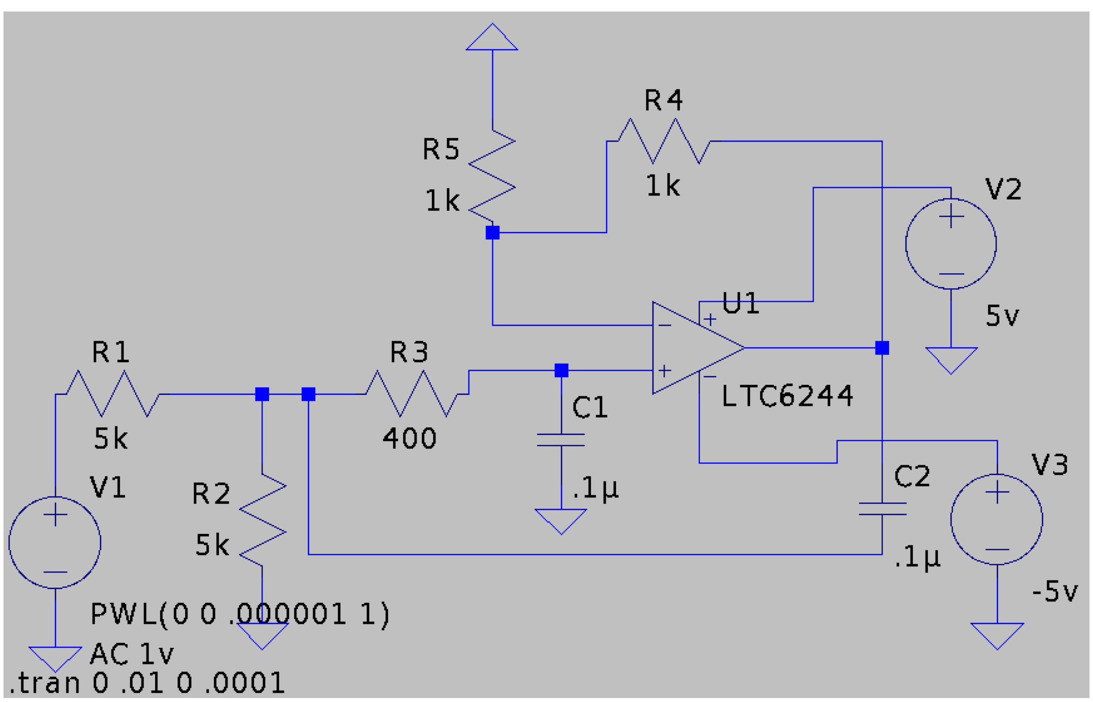




Problem 3. Perform a transient analysis of your circuit in SPICE, and compare with the results obtained using Matlab.

A circuit diagram was modeled in LTSpice using the given impedance values. Results for the output voltage matched the MATLAB calculations very closely, with the peak overshoot occurring at approximately .5 V.





Problem 4:
If the elements of the circuit in Fig. 1 are chosen as $R_4 = R_5 = 1K\Omega$, $C_1 = C_2 = 0.1\mu F$, $R_1 = R_2 = 2/a * 10^7$, $R_3 = a/b * 10^7$, it can be shown that the corresponding transfer function has the form $V_o(s)/V_g(s) = b/(s^2 + as + b)$. Use partial fraction expansion to establish that the resulting step response can be expressed as $v_o(t) = 1 - e^{-\alpha t}\cos\beta t - (\alpha/\beta)e^{-\alpha t}\sin\beta t$ where $\alpha = a/2$ and $\beta = \text{sqrt}(b - (a/2)^2)$.

The function of interest is V_o. It can be obtained through the transfer function by multiplying it with V_g. $V_g(s)$ is simply 1/s because it is a step function. Afterwards, the quadratic equation is used to factorize the denominator of the fraction. This enables it to be expanded through partial fractions. Once broken into partial fractions, it is possible to obtain the inverse laplace transform of $V_o(s)$ to get $v_o(t)$.

Some important information is needed in this solution.

First, from the table of Laplace transforms, we see that the inverse laplace transform of

$$1/(\,(s)(s+x)(s+y)\,) \text{ is } 1/xy + e^{-xt}/x(x-y) + e^{-yt}/y(y-x).$$

In this particular problem, $x = \alpha + j\beta$ and $y = \alpha - j\beta$.

Second, Euler's formula is needed.

$$e^{jx} = \cos xt + j\sin xt$$
$$e^{-jx} = \cos xt - j\sin xt$$

The following pages show the detailed solutions.

Problem 5:

Based on Problem 4, show that the maximal and minimal values of $v_0(t)$ occur at times t_k that satisfy $\sin\beta t_k = 0$. Use this result to show that the first maximum occurs at time

$$t_1 = \frac{\pi}{\beta}$$

and that the peak overshoot is

$$\text{p.o.} = e^{-\alpha\pi/\beta}.$$

Starting with the voltage as a function of time from problem 4,

$$V_0(t) = 1 - e^{-\alpha t}\cos\beta t - \frac{\alpha}{\beta} e^{-\alpha t}\sin\beta t$$

finding the time t_1 at which peak overshoot occurs is a matter of setting its first time derivative equal to zero and verifying that it is a local maximum by way of the second derivative test. The results of the differentiation and second derivative can be found on the next page.

To verify that at time t_1 the peak overshoot occurs for $V_0(t)$, the $\sin\beta t$ term must be equal to zero and $\cos\beta t$ needs to equal - 1. Finding a suitable time t_1 implies that $\theta = \frac{\pi}{\beta}$ and that

$$\cos\theta = \cos\pi = -1.$$

this leaves us with

$$V_1(t) = 1 - e^{-\alpha t}(-1) - \frac{\alpha}{\beta} e^{-\alpha t}(0)$$

where

$$V_{max}(t) = V(t_1) = 1 + e^{-\alpha\pi/\beta}$$

$$\text{so p.o.} = e^{-\alpha\pi/\beta}.$$

Calculations found on the following page.

Problem 6:

Using equations (1)-(4), design R1, R2, and R3 so that the peak overshoot of the circuit in Fig. 1 is no larger than 10%. Use the m-file developed in Problem 2 to verify your design.

Mathematically, the problem basically requires that $e^{-\alpha\pi/\beta} \leq 0.10$

$$\alpha = a/2$$
$$\beta = \mathrm{sqrt}(b - (a/2)^2)$$

We can rewrite these two equations to obtain

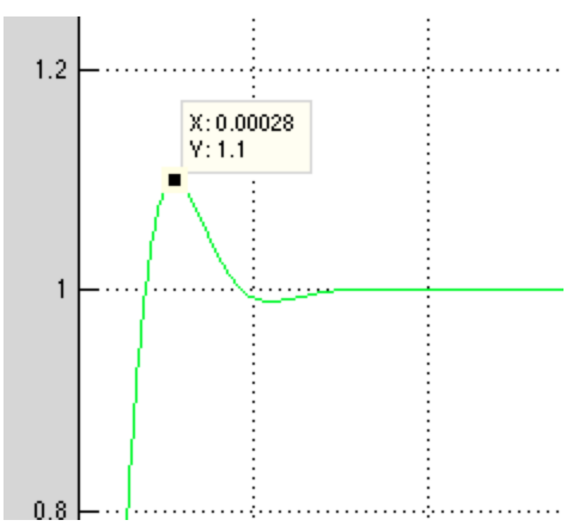
$$a = 2\alpha$$
$$b = \alpha^2 + \beta^2$$

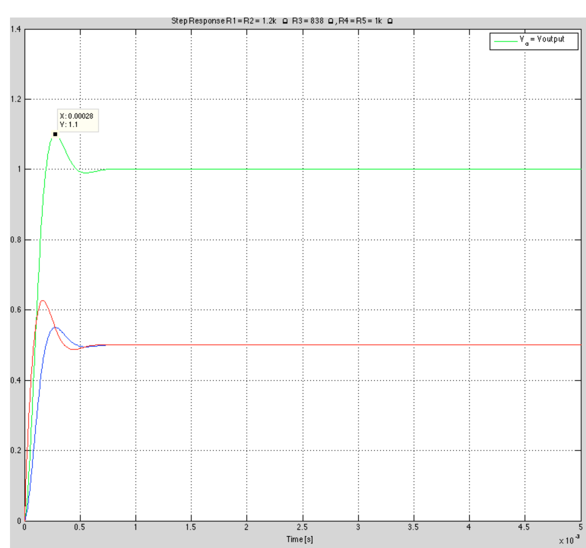
First, we noticed that $e^{-\pi} = 0.0432$, which is less than 0.1. If we then set $\alpha = \beta$, then we get a peak overshoot of 0.0432.

Therefore, $a = 2\beta$ and $b = 2\beta^2$.

Equation 3 says $R_1 = R_2 = 2/a * 10^7$. Therefore, $R_1 = R_2 = 10^7/\beta$.
Equation 4 says $R_3 = a/b * 10^7$. $R_3 = 2\beta/2\beta^2 * 10^7$. $R_3 = 10^7/\beta$.
$R_1 = R_2 = R_3$, and each depend on β. We can choose any value of β, as long as they are all equal.
We chose
$R_1 = R_2 = R_3 = 1.2\text{k}\Omega$, as these are easily found.

We also obtained another set of values for the resistors that gives a peak overshoot of exactly 0.010. This requires more solutions.
$e^{-\alpha\pi/\beta} = 0.10$
The solutions are found on the next page.



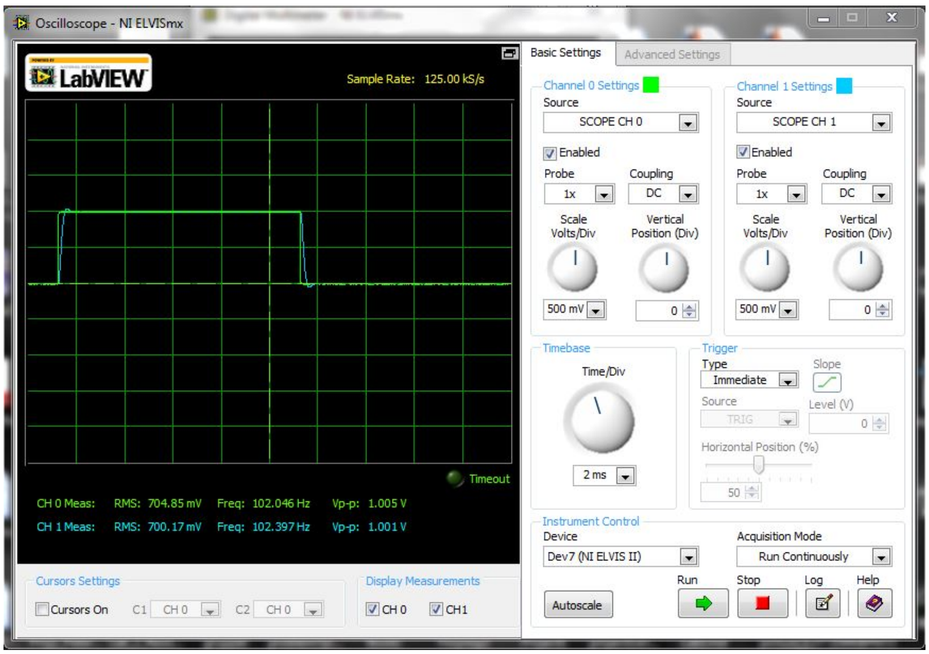


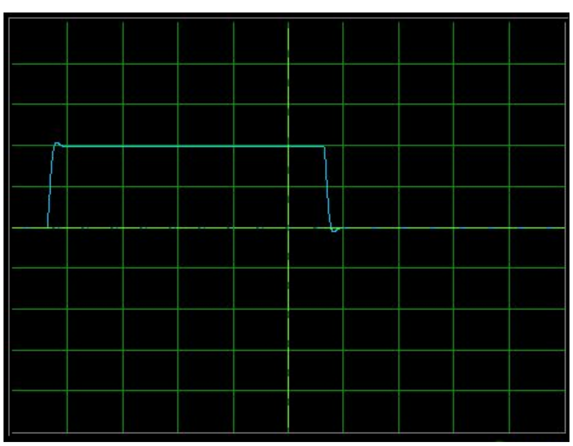
Problem 7: Assemble the circuit with the element values given in Problem 2, and measure the peak overshoot. Repeat this for the values obtained in Problem 6. In both cases, compare your measurements to the simulation results.

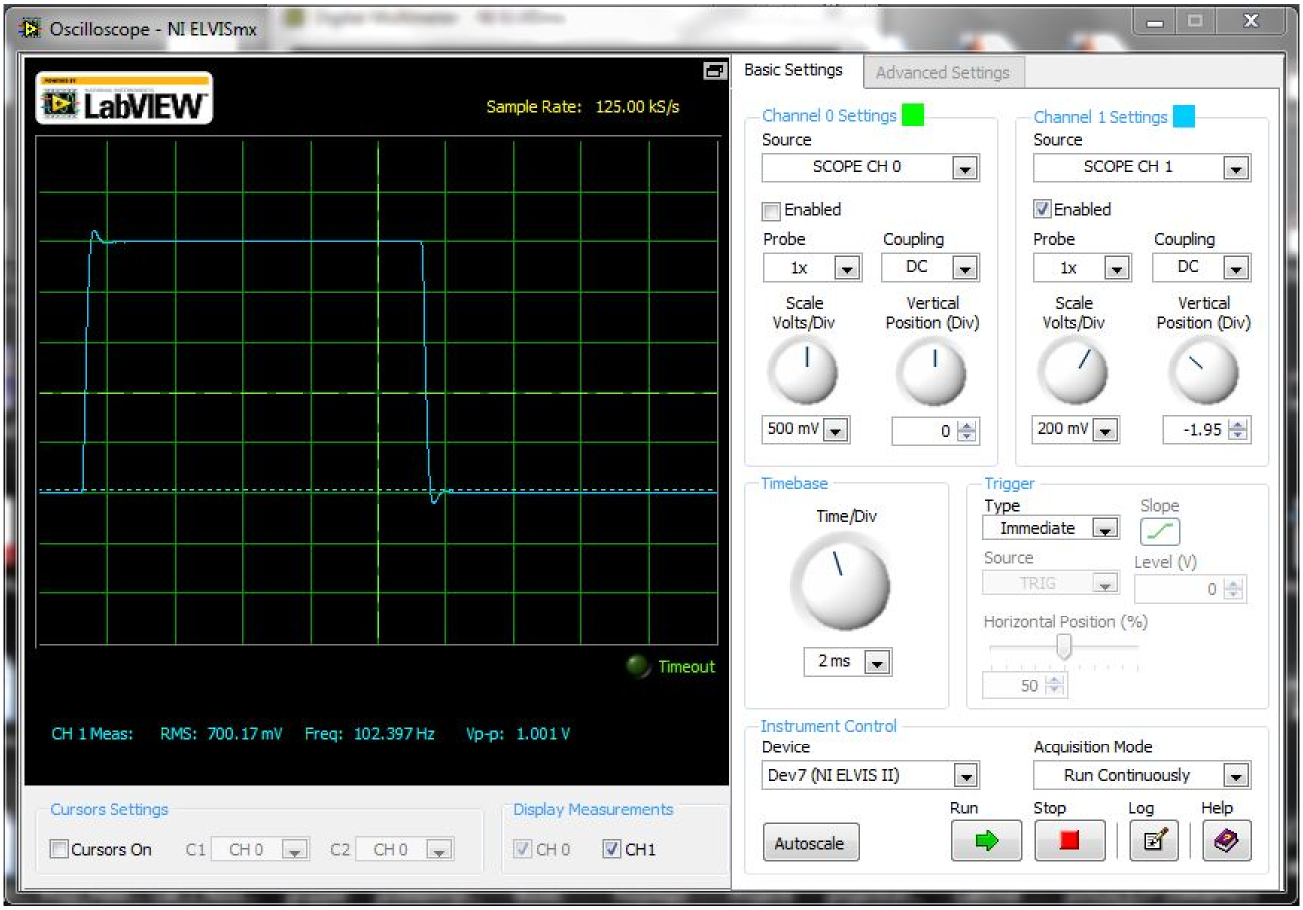
A 1V square wave function with a frequency of 50Hz was used as the voltage source. This serves as a step function. Channel 0 (green) is the voltage source and Channel 1 (blue) is the output voltage.

For $R_1 = R_2 = R_3 = 1.2\ \text{k}\Omega$

Expected peak overshoot: 0.0432





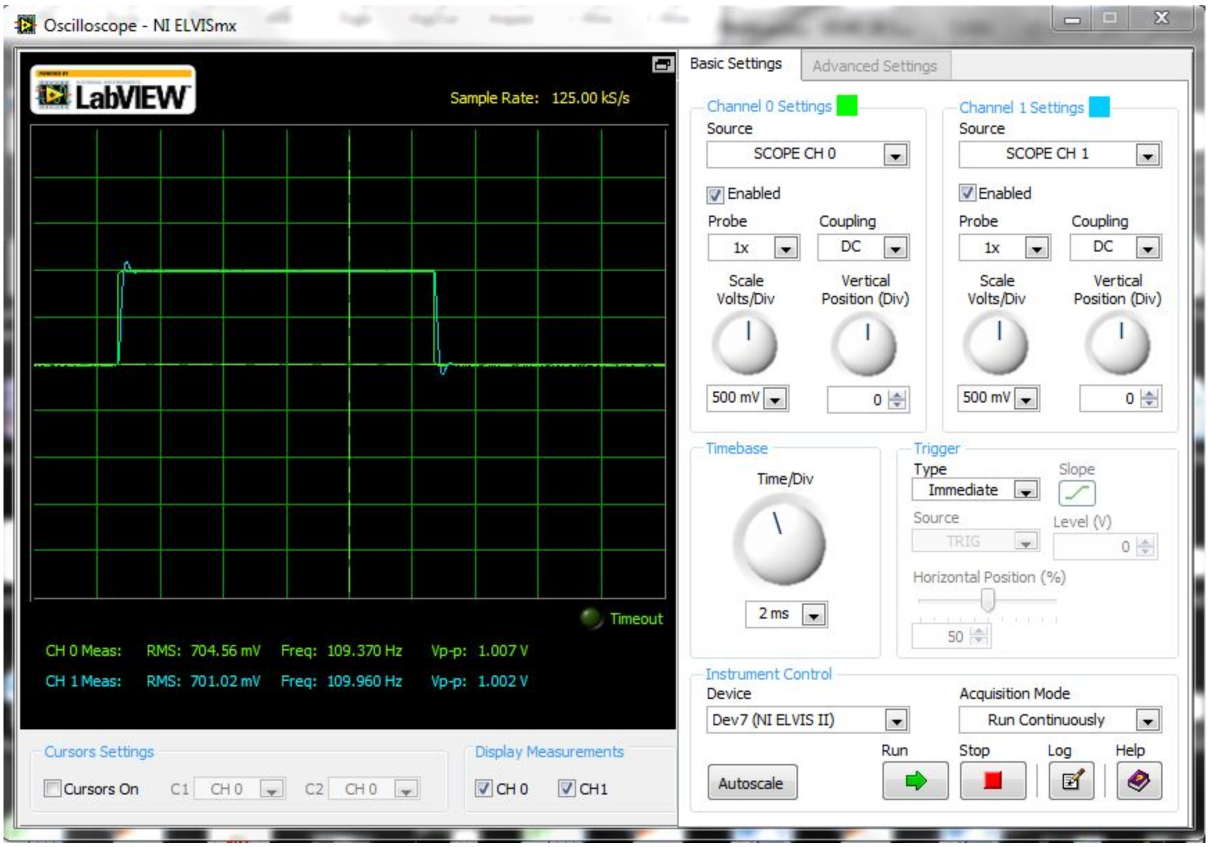


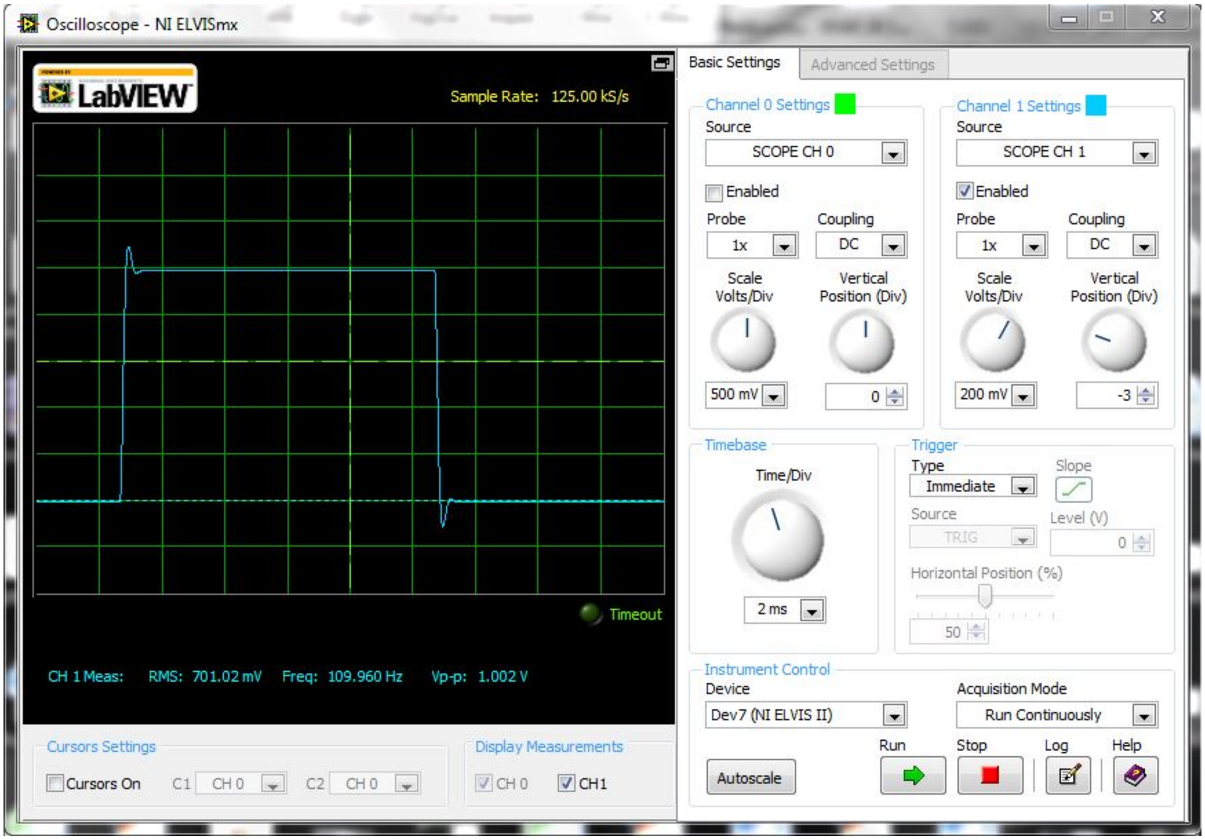
Here, five boxes represent 1V. One box represents 0.2V. 0.0432 is about a fifth of 0.2V, so the peak overshoot must be about one fifth of a box above the steady voltage, and that is the case here in our measurements.

For $R_1 = R_2 = 1.2k\Omega$

$R_3 = 838\Omega$

Expected peak overshoot = 0.1





Here, five boxes represents 1V. One box represents 0.2V. 0.1 is half of a box. We expect the peak overshoot to be half a box over the steady voltage, and that is the case here.