Name: Nam Huynh

ID: nsh1507

**1.**

**a.** $T(0) = 0$

$T(n+1) = T(n) + 5$

$$
\begin{aligned}
T(n) &= T(n-1) + 5 \\
&= T(n-1) + 5 \\
&= (T(n-2) + 5) + 5 \\
&= ((T(n-3) + 5) + 5) + 5 \\
&= T(n-k) + 5k
\end{aligned}
$$

Let $k = n$:

$T(n) = 5n + T(0) = 5n + 0 = 5n$

$\mathbf{T(n) = 5n}$

**b.** $T(0) = 0$

$T(n+1) = n + T(n)$

$$
\begin{aligned}
T(n) &= (n-1) + T(n-1) \\
&= (n-1) + (n-2) + T(n-2) \\
&= (n-1) + (n-2) + T(n-2) \\
&= (n-1) + (n-2) + \ldots + (n-k+1) + (n-k) + T(n-k)
\end{aligned}
$$

Let $k = n$:

$$T(n) = (n-1) + (n-2) + \ldots + (n-n+1) + (n-n) + T(n-n)$$

$$T(n) = (n-1) + (n-2) + \ldots + 1 + 0 + T(0)$$

$$T(n) = \sum_{i=0}^{n-1} i + T(0)$$

$$T(n) = \frac{(n-1)(n-1+1)}{2}$$

$$\mathbf{T(n)} = \frac{(n-1)(n)}{2}$$

**2.**

**a.** $T(0) = 1$

$T(n+1) = 2 \cdot T(n)$

$$T(n) = 2 \cdot T(n-1)$$

$$= 2 \cdot 2 \cdot T(n-2)$$

$$= 2 \cdot 2 \cdot 2 \cdot T(n-3)$$

$$= 2^k \cdot T(n-k)$$

Let k = n

$$T(n) = 2^n \cdot T(n-n)$$

$$T(n) = 2^n \cdot T(0)$$

$$T(n) = 2^n \cdot 1$$

$$\mathbf{T(n)} = 2^n$$

**b.** $T(0) = 1$

$T(n+1) = 2^{n+1} + T(n)$

$$T(n) = 2^n + T(n-1)$$

$$= 2^n + 2^{n-1} + T(n-2)$$

$$= 2^n + 2^{n-1} + 2^{n-2} + T(n-3)$$

$$= 2^n + 2^{n-1} + 2^{n-2} + \ldots + 2^{n-k+1} + T(n-k)$$

Let $k = n$

$$T(n) = 2^n + 2^{n-1} + 2^{n-2} + \ldots + 2^{n-n+1} + T(n-n)$$

$$T(n) = 2^n + 2^{n-1} + 2^{n-2} + \ldots + 2^1 + T(0)$$

$$T(n) = \sum_{i=0}^{n} 2^i + T(0) - 2^0$$

$$T(n) = (2^{n+1} - 1) + 1 - 1$$

$$\mathbf{T(n)} = 2^{n+1} - 1$$

**3.**

**a.** $T(1) = 1$

$T(n) = n + T(n/2)$ Assume n has the form $n = 2^m$

$$T(2^m) = 2^m + T(\frac{2^m}{2})$$

$$= 2^m + T(2^{m-1})$$

$$= 2^m + 2^{m-1} + T(\frac{2^{m-1}}{2})$$

$$= 2^m + 2^{m-1} + T(2^{m-2})$$

$$= 2^m + 2^{m-1} + 2^{m-2} + T(\frac{2^{m-2}}{2})$$

$$= 2^m + 2^{m-1} + 2^{m-2} + T(2^{m-3})$$

$$= 2^m + 2^{m-1} + 2^{m-2} + \ldots + 2^{m-k+1} + T(2^{m-k})$$

Let k = m:

$$T(2^m) = 2^m + 2^{m-1} + 2^{m-2} + \ldots + 2^{m-m+1} + T(2^{m-m})$$

$$= 2^m + 2^{m-1} + 2^{m-2} + \ldots + 2^1 + T(2^0)$$

$$= (\sum_{i=0}^{m} 2^i) - 2^0 + T(0)$$

$$= (2^{m+1} - 1) - 1 + 1$$

$$= 2^{m+1} - 1$$

$$\Rightarrow \quad n = 2^m \quad \Rightarrow \quad lg(n) = m$$

$$\Rightarrow \quad 2^{m+1} \quad \Rightarrow \quad (2^m)2 \quad \Rightarrow \quad (2^{lg(n)})2 \quad \Rightarrow \quad 2n$$

**T(n) = 2n-1**

**b.** $T(1) = 1$

$T(n) = 1 + T(n/3)$ Assume n has the form $n = 3^m$

$$T(3^m) = 1 + T(\frac{3^m}{3})$$

$$= 1 + T(3^{m-1})$$

$$= 1 + 1 + T(\frac{3^{m-1}}{3})$$

$$= 1 + 1 + T(3^{m-2})$$

$$= k + T(3^{m-k})$$

Let k = m:

$$T(3^m) = m + T(3^{m-m})$$

$$= m + T(3^0)$$

$$= m + T(1)$$

$$= m + 1$$

$\Rightarrow$ $n = 3^m$

$\Rightarrow$ $log_3(n) = m$

$$\mathbf{T(n) = log_3(n) + 1}$$

**6.**

**f.**

**i.**

$Theorem\text{: } For\ any\ \ n \in \mathbb{N},\ T_F(n) = F_{n+1} - 1$

$Proof\text{: } By\ the\ strong\ form\ of\ induction$

$Observe\ that\ when\ n = 0\ we\ have\text{:}$

$$T_F(0) = 0 = F_{0+1} - 1$$

$$= 1 - 1$$

$$= 0$$

$Observe\ that\ when\ n = 1\ we\ have\text{:}$

$$T_F(1) = 0 = F_2 - 1$$

$$= 1 - 1$$

$$= 0$$

Assume $T_F(k) = F_{k+1} - 1 \quad if\ 0 \leq k < n$

$$T_F(k + 1) = 1 + T_F(k) + T_F(k - 1)$$

$$= 1 + F_{k+1} - 1 + F_k - 1$$

$$= F_{k+1} + F_k - 1$$

$$T_F(k + 1) = F_{k+2} - 1$$

**ii.**

$$= \lim_{n \to \infty} \frac{\frac{1}{\sqrt{5}}(\varphi^{n+1} - \widehat{\varphi}^{n+1}) - 1}{\varphi^n}$$

$$= \lim_{n \to \infty} \frac{\frac{1}{\sqrt{5}}(\varphi^{n+1} - \widehat{\varphi}^{n+1} - \sqrt{5})}{\varphi^n}$$

$$= \frac{1}{\sqrt{5}} \lim_{n \to \infty} \frac{(\varphi^{n+1} - \widehat{\varphi}^{n+1} - \sqrt{5})}{\varphi^n}$$

$$= \frac{1}{\sqrt{5}} \left[\left( \lim_{n \to \infty} \frac{\varphi^{n+1}}{\varphi^n}\right) - \left( \lim_{n \to \infty} \frac{\widehat{\varphi}^{n+1}}{\varphi^n}\right) - \left( \lim_{n \to \infty} \frac{\sqrt{5}}{\varphi^n}\right)\right]$$

$$= \frac{1}{\sqrt{5}} (\varphi - 0 - 0)$$

$$= \frac{\varphi}{\sqrt{5}}$$

**Since** $\lim_{n \to \infty} \frac{T_F(n)}{\varphi^n} = c \text{ and } c \neq 0, \text{ therefore } T_F(n) \in \Theta(\varphi^n)$

**g.**

$$T_f(0) = 0$$

$$T_f(1) = 0$$

$$T_f(n+1) = 1 + T_f(n)$$

$$\begin{aligned} T_f(n) &= 1 + T_f(n-1) \\ &= 1 + 1 + T_f(n-2) \\ &= 1 + 1 + 1 + T_f(n-3) \\ &= k + T_f(n-k) \end{aligned}$$

Let k = n:

$T_f(n) \quad = \quad n + T_f(n - n) \; = \; n + T_f(0) \; = n + 0 \; = \; n$

$T_f(n) \quad = \quad \mathbf{n}$

**h.**

Theorem: For any n $\in \mathbb{N}$, $L^n(a, b)$ = (f(n;a,b), f(n+1;a,b)).

$Proof$: *By weak induction*

Observe when n = 0 we have

$L^0(a, b) \quad = \quad$ (a,b)

$\quad = \quad$ (f(0;a,b), f(1;a,b))

Assume $L^k(a, b) \quad = \quad$ (f(k;a,b), f(k+1;a,b))

$$
\begin{aligned}
L^{k+1}(a, b) &= L(L^k(a, b)) \\
&= L(f(k; a, b),\ f(k + 1; a, b)) \\
&= (\ f(k + 1; a, b), f(k; a, b) + \ f(k + 1; a, b)) \\
&= (\ f(k + 1; a, b), f(k + 2; a, b))
\end{aligned}
$$

**i.**

Consider the vector $\vec{x} = (x_1, x_2)$ *and* $\vec{y} = (y_1, y_2)$

**Prove** $\mathbf{L}(\vec{x} + \vec{y}) = \mathbf{L}(\vec{x}) + \mathbf{L}(\vec{y})$ **:**

$L(\vec{x} + \vec{y}) = L(x_1 + y_1,\ x_2 + y_2) = (x_2 + y_2, x_1 + y_1 + x_2 + y_2)$

$L(\vec{x}) + L(\vec{y}) = L(x_1, x_2) + L(y_1, y_2) = (x_2, x_1 + x_2) + (y_2, y_1 + y_2)$

$= (x_2 + y_2, x_1 + y_1 + x_2 + y_2)$

Since **L**($\vec{x} + \vec{y}$) $= (x_2 + y_2, x_1 + y_1 + x_2 + y_2)$ and **L**($\vec{x}$) **+ L** ($\vec{y}$) $=$

$(x_2 + y_2, x_1 + y_1 + x_2 + y_2)$, therefore, **L**($\vec{x} + \vec{y}$) $=$ **L**($\vec{x}$) **+ L** ($\vec{y}$) **.**

**Prove L(c·$\vec{x}$) = c · L($\vec{x}$):**

L(c.$\vec{x}$) = L(c.$(x_1, x_2)$) = L$(cx_1, cx_2)$ = $(cx_2, cx_1 + cx_2)$

c . L($\vec{x}$) = c . L$(x_1, x_2)$ = c . $(x_2, x_1 + x_2)$ = $(cx_2, cx_1 + cx_2)$

Since L(c.$\vec{x}$) = $(cx_2, cx_1 + cx_2)$ and c.L($\vec{x}$) = $(cx_2, cx_1 + cx_2)$, therefore,

**L(c.$\vec{x}$) = c . L($\vec{x}$).**

**Since L($\vec{x} + \vec{y}$) = L($\vec{x}$) + L ($\vec{y}$) and L(c.$\vec{x}$) = c . L($\vec{x}$), the function L is linear.**

**j.**

$$
\begin{aligned}
L(a,b) &= L(a.\vec{e_1} + b.\vec{e_2}) \\
&= L(a.\vec{e_1}) + L(b.\vec{e_2}) \\
&= a \,.\, L(\vec{e_1}) + b.\, L(\vec{e_2}) \\
&= a \,.\, (0, 1) + b \,.\, (1,1) \\
&= (0,a) + (b,b)
\end{aligned}
$$

= (b, a+b)

= M . (a,b) for some 2x2 matrix M

Since M . (a,b) = (b, a+b), we can represent the matrix M in the form of system of equations:

$$aM_{11} + bM_{12} = b$$

$$aM_{21} + bM_{22} = a + b$$

Only $m_{21} = m_{22} = 1$ *would satisfy the second equation.*

Only $m_{11} = 0$ *and* $m_{12} = 1$ *would satisfy* the first equation.

Therefore the matrix representation of L would be the matrix:

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

**n.** fibPow ∈ Θ(log(n)), since the function called upon matPow() which requires in the worst case Θ(log(n)) recursive calls to compute a result.

**o.**

**a.** fib is not a pseudo polynomial time because $\Theta(\varphi^n)$ can not be bounded by a polynomial function since it is exponential.

**b.** fibIt is a pseudo polynomial time since $\Theta(n)$ can be bounded by another polynomial function.

**c.** fibPow is a pseudo polynomial time since $\Theta(\log(n))$ can be bounded by another polynomial function.