Problems marked with **E** are graded on effort, which means that they are graded subjectively on the perceived effort you put into them, rather than on correctness. For bonus questions, we will not provide any insight during office hours or Piazza, and we do not guarantee anything about the difficulty of these questions. We strongly encourage you to typeset your solutions in LaTeX.

If you collaborated with someone, you must state their names. You must write your own solution and may not look at any other student's write-up.

E 1. Use induction to prove that for any integer $n > 1$,

$$\sum_{k=1}^{n-1} \binom{n}{k} = 2^n - 2,$$

where $\binom{n}{k}$ is the binomial coefficient.
Hint: Use the relation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

which is true whenever $0 < k < n$.

Solution: The base case is when $n = 2$. We have

$$\sum_{k=1}^{1} \binom{2}{k} = \binom{2}{1} = 2 = 2^2 - 2.$$

So the statement is true when $n = 2$.
Now, suppose that the statement is true for some positive integer n. We want to show that it must be true for $n + 1$ as well. We compute:

$$\begin{aligned}
\sum_{k=1}^{(n+1)-1} \binom{n+1}{k} &= \sum_{k=1}^{n} \binom{n+1}{k} && (1)\\
&= \sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k} \right] && (2)\\
&= \sum_{k=1}^{n} \binom{n}{k-1} + \sum_{k=1}^{n} \binom{n}{k} && (3)\\
&= \left[\sum_{k=0}^{n-1} \binom{n}{k} \right] + \left[\sum_{k=1}^{n-1} \binom{n}{k} + \binom{n}{n} \right] && (4)\\
&= \left[\binom{n}{0} + \sum_{k=1}^{n-1} \binom{n}{k} \right] + \left[\sum_{k=1}^{n-1} \binom{n}{k} + \binom{n}{n} \right] && (5)\\
&= [1 + 2^n - 2] + [2^n - 2 + 1] && (6)\\
&= 2(2^n) - 2 = 2^{n+1} - 2 && (7)
\end{aligned}$$

To go from step (1) to step (2), we used the relation given in the hint, and to go from step (5) to step (6) we used the inductive hypothesis (twice).

E 2. Prove that $\sqrt{17}$ is irrational. (**Hint:** you may find it easiest to use a proof by contradiction.)

Solution: Suppose for contradiction that $\sqrt{17}$ is rational. Then by definition $\sqrt{17} = \frac{a}{b}$ for some integers a and b. Without loss of generality, assume that this fraction is in lowest terms; that is, a and b have no common divisors.
Squaring both sides, we have $17 = \frac{a^2}{b^2}$, or $17b^2 = a^2$. This means a^2 is a multiple of 17, which means a is a multiple of 17 since 17 is prime. (Note that if a were not a multiple of 17, its prime factorization would be some $p_1^{a_1} \dots p_k^{a_k}$ where $p_i \neq 17$, so the prime factorization of a^2 would be $p_1^{2a_1} \dots p_k^{2a_k}$, which also does not include 17.) Thus, $a = 17m$ for some integer m. We thus have that $17b^2 = 17^2m^2$, or $b^2 = 17m^2$. By the same argument, this implies that b is a multiple of 17. Now we have found that both a and b are multiples of 17. But this contradicts the assumption that they have no common divisors! Thus, we conclude that our original assumption, that $\sqrt{17}$ is rational, is false.

E 3. For the following pairs of $f(n)$ and $g(n)$, is $f(n) = O(g(n))$ true? Justify your answer.

(a) $f(n) = n + \log_2(n^4)$, $g(n) = \frac{1}{9}n - 5$.

Solution: Yes. Setting $M = 90$ and $n_0 = 90$ shows that $f(n) = O(g(n))$ since for all $n > 27$, $n + \log_2(n^4) < 5n \leq 10n - 90 \cdot 5 = 90(\frac{1}{9}n - 5)$. (Note: $\log_2(n^4) = 4\log_2 n < 4n$ for all $n \geq 1$).

(b) $f(n) = (3n)^3$, $g(n) = (n)^3$

Solution: Yes. Note that $f(n) = 27n^3$. Thus, we immediately have that for all $n \geq 1$, $f(n) \leq 27g(n)$ satisfying the definition.

(c) $f(n) = (\log_2 n)^3$, $g(n) = 2^{\log_2 n}$

Solution: Yes. By L'Hopital's Rule,

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{m \equiv \log n \to\infty} \frac{m^3}{2^m} = \lim_{m\to\infty} \frac{3m^2}{2^m \cdot \ln(2)} = \lim_{m\to\infty} \frac{6m}{2^m \cdot (\ln(2))^2} =$$
$$\lim_{m\to\infty} \frac{6}{2^m \cdot (\ln(2))^3} = 0.$$

Therefore, $f(n) = O(g(n))$; in fact this shows that $f(n) = o(g(n))$.

(d) $f(n) = 2^{1.4n}$, $g(n) = 3e^n$

Solution: Yes.

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{2^{1.4n}}{3e^n} = \lim_{n\to\infty} \frac{2.639^n}{3 \cdot 2.718^n} = 0.$$

Therefore, $f(n) = O(g(n))$; in fact this shows that $f(n) = o(g(n))$.

(e) $f(n) = \log_2(2n)$, $g(n) = \log_2(n)$

Solution: Yes. Setting $M = 2$ and $n_0 = 3$, we have that for all $n \geq 3$ $\log_2(2n) = \log_2(2) + \log_2(n) \leq 2\log_2(n)$.

E 4. Give an asymptotic complexity bound of the running time of the following homework solver

```
function SOLVE-HW(X):
    for i in range(15):
        print 'Keep it up grader! :)'
    return X
```

as a function of the size of the input X. Make this bound as tight as possible.
Note: our providing this program does not constitute a guarantee of any particular lower bound on the number of points you should expect to receive if you choose to implement it and run it to solve your own homeworks.

> **Solution:** Notice that the program ignores the input, and only returns it at the end. So, the program runs in a fixed amount of time regardless of the input. Note that we don't know how long it may take to print the statement (that depends on the implementation, hardware, etc.), but as a function of the input it is a constant. So the program runs in $O(1)$, or "constant," time.

E 5. Suppose you want to unambiguously represent all of the elements of $\{0, 1, \ldots, n-1\}$ using ternary strings of a certain length k, i.e., strings with k characters from the set $\{a, b, c\}$. What's the smallest possible choice for k?

> **Solution:** We want to determine the smallest possible k that allows us to represent n elements using distinct strings of length k. First observe that for any integer $k \geq 0$, the number of distinct strings of length k is exactly 3^k. To see this, note that such a string has three possibilities for each of the k positions, so the total number of possibilities is $\underbrace{3 \cdot 3 \cdot 3}_{k \text{ times}} = 3^k$. (For length $k = 0$, we have just the single "empty" string ε.)
> So to be able to represent each of the n elements by a distinct string of length k, we need to choose k so that
> $$3^k \geq n \iff k \geq \log_3 n$$
> By rounding, the smallest integer k we can use is therefore $k = \lceil \log_3 n \rceil$.

6. **Extra credit:** *You do not have to do this question to receive full credit on this assignment.* To receive the bonus points, you must typeset this *entire* assignment in LaTeX and draw a table with two columns that includes the *name* (i.e., "fraction") and an *example* of each of the following:

   - fraction (using `\frac`)
   - less than or equal to
   - union of two sets
   - summation using sigma (Σ) notation
   - the set of real numbers is denoted $\mathbb{R}$; write a mathematically correct statement that applies to *all* real numbers $x \in \mathbb{R}$.

Solution: The following tex

```
\renewcommand{\arraystretch}{2}
\begin{tabular}[t]{|c|c|}
  \hline
  superscript and subscript & $\E^{\I \theta_0} =
      \cos \theta_0 + \I \sin \theta_0$ \\
  \hline
  fraction & $\displaystyle \frac{1}{\I} = - \I$. \\
  \hline
  less than or equal to & $\abs{\vec{x} \cdot \vec{y}}^2 \leq
      \length{\vec{x}}^2 \length{\vec{y}}^2$\\
  \hline
  union of two sets & $\Pr[A \cup B] \leq \Pr[A] + \Pr[B]$ \\
  \hline
  sum using Sigma notation & $\displaystyle
      \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$\\
  \hline
\end{tabular}
```

which uses macros from the provided `header.tex`, was used to create the following table:

fraction	$\dfrac{1}{\mathrm{i}} = -\mathrm{i}.$
less than or equal to	$\lvert\vec{x} \cdot \vec{y}\rvert^2 \leq \lVert\vec{x}\rVert^2 \lVert\vec{y}\rVert^2$
union of two sets	$\Pr[A \cup B] \leq \Pr[A] + \Pr[B]$
sum using Sigma notation	$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$
true statement $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Z}$	$\mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}$