$$\{ q_k \in \mathbb{C} : k \in [-N, N]^2, \; k = (k_1, k_2) \; \underbrace{\begin{matrix} k_2 > 0 \\ \text{or} \\ k_2 = 0, k_1 > 0 \end{matrix}}_{\mathbb{Z}^2_*} \}$$

$$q(x,t) = \sum_{k \in \mathbb{Z}^2} q_k(t) e^{i k \cdot x}$$

$$\overline{q_k} = q_{-k}$$

pick triplets $(k, l, j) \in (\mathbb{Z}^2)^3$

$$k + l + j = 0$$

then evolve those 3 mode's interacting

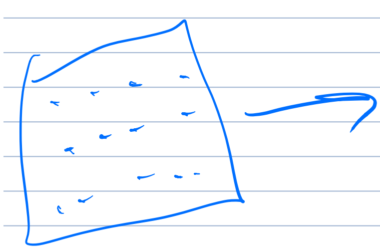
there are different ways

then ① watch & study $q(x,t)$ vorticity

$\downarrow$

$u(x,t)$ velocity

② solve $\dot{\xi}_t = (\nabla^\perp q) = u(\xi_t, t)$



$$\xi_t \in \mathbb{R}^2$$

$$u(x,t) = (u_1(x,t), u_2(x,t))$$

$$\dot{q}(t) = F(q(t)) = \sum_{l+j+k=0} F_{l,j,k}(q(t))$$

$$\shortparallel \qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$$

$$\{\dot{q}_k(t)\} \qquad\qquad\qquad\qquad \varphi(q_l, q_j, q_k)$$

$$\dot{x}_t = (A+B)x_t \qquad e^{hA} e^{hB} e^{hA} \ldots$$

COMPLEX SPLITTING OF NS

JCM, OM, AA

Consider the dynamics on $\mathbf{T} = [0, 2\pi]^2$ with periodic boundary conditions. Let $u(x,t) = (u_1(x,t), u_2(x,t)) \in L^2(\mathbf{T}, \mathbf{R}) \times L^2(\mathbf{T}, \mathbf{R})$ be a velocity field which solves the incompressible two-dimensional Navier-Stokes equation

$$\begin{cases} \partial_t u(x,t) + (u \cdot \nabla) u(x,t) = \cancel{\Delta u(x,t)} - +\nabla P(x,t) \\ \operatorname{div}(u) \overset{def}{=} \nabla \cdot u(x,t) = 0 \end{cases} \tag{1}$$

Here

$$\operatorname{div}(u)(x,t) = \sum_{i=1,2} \frac{\partial u_i}{\partial x_i}(x,t) \qquad \text{and} \qquad (u \cdot \nabla) u(x,t) = (v_1(x,t), v_2(x,t))$$

where

$$v_j(x,t) = \sum_{i=1,2} \frac{\partial u_j}{\partial x_i}(x,t) \quad u_i(x,t)$$

and $P(x,t)$ is a scalar which should be understood as a Lagrange multiplier in forcing the constraint $\operatorname{div}(u) = 0$.

Now we set define the scalar vorticity $q \in nL^2(\mathbf{T}, \mathbf{R})$ by

$$q(x,t) = \operatorname{curl}(u)(x,t) = \nabla \wedge u(x,t) = \frac{\partial u_1}{\partial x_2}(x,t) - \frac{\partial u_2}{\partial x_1}(x,t)$$

Then (1) becomes

$$\partial_t q(x,t) + (\mathcal{K} q \cdot \nabla) q = \cancel{\Delta q}$$

where for the ONB $e_k(x) = \frac{1}{2\pi} e^{ik\dot{x}}$ and the expansion

$$q(x,t) = \sum_{k \in \mathbf{Z}^2} q_k(t) e_k(x)\,,$$

one has

$$\langle \mathcal{K} q, e_k \rangle(t) = -i q_k(t) \frac{k^\perp}{|k|^2} \quad \Rightarrow u(x,t) = (u_1, u_2)$$

with $k^\perp = (-k_2, k_1)$ and $\langle f, g \rangle = \int f(x) \bar{g}(x) dx$. So

q_K

$$\langle (\mathcal{K} q \cdot \nabla) q, e_k \rangle(t) = \frac{1}{4\pi} \sum_{\ell + j = k} \langle j^\perp, \ell \rangle \Big(\frac{1}{|\ell|^2} - \frac{1}{|j|^2} \Big) q_\ell(t) q_j(t) = \sum_{\ell + j = k} C_{\ell,j} q_\ell(t) q_j(t)$$

with

$$C_{\ell,j} \overset{def}{=} \frac{\langle j^\perp, \ell \rangle}{4\pi} \Big(\frac{1}{|\ell|^2} - \frac{1}{|j|^2} \Big)$$

First observe that $\langle j^\perp, \ell \rangle = -j_2 \ell_1 + j_1 \ell_2 = -\langle j, \ell^\perp \rangle$ so

$$C_{\ell,j} = C_{j,\ell}$$

$k = \ell + j$

$\langle j^\perp, \ell \rangle = \langle j^\perp, k \rangle$

Next observe that if $\ell + j = k$ then

$$\langle j^\perp, \ell\rangle = \langle j^\perp, k\rangle = \langle k^\perp, \ell\rangle$$

Because $q(x,t)$ is real valued

$$q(x,t) = \bar{q}(x,t) = \sum_{k\in\mathbf{Z}^2} \bar{q}_k(t)\bar{e}_k(x) = \sum_{k\in\mathbf{Z}^2} \bar{q}_k(t)e_{-k}(x)$$
$$= \sum_{k\in\mathbf{Z}^2} \bar{q}_{-k}(t)e_k(x)$$

so we conclude $\bar{q}_{-k} = q_k$ or alternatively $q_{-k} = \bar{q}_k$.

so

$$\partial_t |q_k|^2 = \partial_t \sum |q_k|^2 = \langle (\mathcal{K}q\cdot\nabla)q, q\rangle = \sum_k \sum_{\ell+j=k} C_{\ell,j} q_\ell q_j \bar{q}_k = \sum_k \sum_{\ell+j+k=0} C_{\ell,j} q_\ell q_j q_k = 0$$

The coefficient of $q_\ell q_j q_k$ is

$$\begin{aligned} C_{j,\ell} + C_{\ell,k} + C_{k,j} =& \frac{\langle j, \ell^\perp\rangle}{4\pi}\Big(\frac{1}{|j|^2} - \frac{1}{|\ell|^2}\Big) \\ &+ \frac{\langle \ell, k^\perp\rangle}{4\pi}\Big(\frac{1}{|\ell|^2} - \frac{1}{|k|^2}\Big) \\ &+ \frac{\langle k, j^\perp\rangle}{4\pi}\Big(\frac{1}{|k|^2} - \frac{1}{|j|^2}\Big) \end{aligned}$$

since $\ell + j + k = 0$ we have that

$$\langle j^\perp, \ell\rangle = \langle j, \ell^\perp\rangle = -\langle \ell^\perp, k\rangle$$

so

$$C_{j,\ell} + C_{\ell,k} + C_{k,j} = 0$$

pick $t_n \sim \exp(h)$

$\mathbb{E}\, t_n = h$

evolve for time t_n

Now fixing $\ell + j + k = 0$, consider

$$-\dot{q}_k = -\dot{q}_{-k} = C_{j,\ell} q_j q_\ell$$
$$-\dot{q}_{-j} = C_{\ell,k} q_\ell q_k$$
$$-\dot{q}_{-\ell} = C_{k,j} q_k q_j$$

missing sign

Now define the real basis

$$e_{k,\theta}(x) = e_k(x) f_\theta \quad\text{and}\quad e^\perp_{k,\theta}(x) = e_k(x) f^\perp_\theta$$

pick $\theta \in [0, 2\pi]$ (randomly)

where $f_\theta = e^{i\theta}$ and $f^\perp_\theta = e^{i(\theta+\frac{\pi}{2})}$ with the real inner product

$$\langle e_{k,\theta}, e^\perp_{k,\theta}\rangle = \Re \int f_{k,\theta}(x) \bar{e}^\perp_{k,\theta}(x)dx = 0$$

$$\langle e^\perp_{k,\theta}, e^\perp_{k,\theta}\rangle = \Re \int f^\perp_{k,\theta}(x) \bar{e}^\perp_{k,\theta}(x)dx = 1 = \Re \int e_{k,\theta}(x) \bar{e}_{k,\theta}(x)dx = \langle f_{k,\theta}, e_{k,\theta}\rangle$$

now fixing a θ, let $q_k = a_k f_\theta + a^\perp_k f^\perp_\theta$. Then

$$q_{-k} = \bar{q}_k = a_k e^{-i\theta} + a^\perp_k e^{-i(\theta+\frac{\pi}{2})} = -a_k f^\perp_{k,\theta} - a^\perp_k f_{k,\theta}$$

and

$$\begin{aligned} q_\ell q_j &= (a_\ell e^{i\theta} + a^\perp_\ell e^{i(\theta+\frac{\pi}{2})})(a_j e^{i\theta} + a^\perp_j e^{i(\theta+\frac{\pi}{2})}) \\ &= a_\ell a_j e^{i2\theta} + (a_\ell a^\perp_j + a_j a^\perp_\ell) e^{i(2\theta+\frac{\pi}{2})} + a^\perp_\ell a^\perp_j e^{i2\theta+i\pi} \end{aligned}$$

now

$$\begin{aligned} e^{i2\theta} &= \cos(\theta) f_\theta + \sin(\theta) f_\theta^\perp \\ e^{i(2\theta+\frac{\pi}{2})} &= -\sin(\theta) f_\theta + \cos(\theta) f_\theta^\perp \\ e^{i2\theta+i\pi} &= -\cos(\theta) f_\theta - \sin(\theta) f_\theta^\perp \end{aligned}$$

$$\begin{aligned} \langle q_\ell q_j, f_\theta\rangle &= \cos(\theta) a_\ell a_j - \sin(\theta)(a_\ell a_j^\perp + a_j a_\ell^\perp) - \cos(\theta) a_\ell^\perp a_j^\perp \\ \langle q_\ell q_j, f_\theta^\perp\rangle &= \sin(\theta) a_\ell a_j + \cos(\theta)(a_\ell a_j^\perp + a_j a_\ell^\perp) - \sin(\theta) a_\ell^\perp a_j^\perp \end{aligned}$$

so

$$\begin{aligned} C_{\ell,j}^{-1} \dot a_k &= -\sin(\theta) a_\ell a_j - \cos(\theta)(a_\ell a_j^\perp + a_j a_\ell^\perp) + \sin(\theta) a_\ell^\perp a_j^\perp \\ C_{\ell,j}^{-1} \dot a_k^\perp &= -\cos(\theta) a_\ell a_j + \sin(\theta)(a_\ell a_j^\perp + a_j a_\ell^\perp) + \cos(\theta) a_\ell^\perp a_j^\perp \end{aligned}$$

so we have

$$\begin{aligned} \dot a_k &= -C_{\ell,j}\sin(\theta) a_\ell a_j & \dot a_k^\perp &= C_{\ell,j}\cos(\theta) a_\ell^\perp a_j^\perp \\ \dot a_j &= -C_{k,\ell}\sin(\theta) a_k a_\ell & \dot a_j^\perp &= C_{k,\ell}\cos(\theta) a_k^\perp a_\ell^\perp \\ \dot a_\ell &= -C_{j,k}\sin(\theta) a_j a_k & \dot a_\ell^\perp &= C_{j,k}\cos(\theta) a_j^\perp a_k^\perp \end{aligned}$$

evolving k & ℓ

$$\begin{aligned} \dot a_k &= -C_{\ell,j}\cos(\theta) a_\ell^\perp a_j & \dot a_k^\perp &= C_{\ell,j}\sin(\theta) a_\ell a_j^\perp \\ \dot a_j &= -C_{k,\ell}\cos(\theta) a_k a_\ell^\perp & \dot a_j^\perp &= C_{k,\ell}\sin(\theta) a_k^\perp a_\ell \\ \dot a_\ell^\perp &= -C_{j,k}\cos(\theta) a_j a_k & \dot a_\ell &= C_{j,k}\sin(\theta) a_j^\perp a_k^\perp \end{aligned}$$

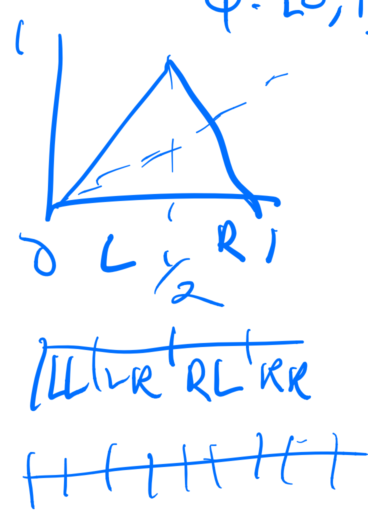
$$\boxed{\begin{aligned} \dot x &= -cy \\ \dot y &= cx \end{aligned}}$$

Maybe we should view this as an angle θ and

$$\begin{aligned} \dot a_k &= -C_{\ell,j}\sin(\theta) a_\ell a_j & \dot a_k^\perp &= C_{\ell,j}\sin(\theta) a_\ell a_j^\perp \\ \dot a_j &= -C_{k,\ell}\sin(\theta) a_k a_\ell & \dot a_j^\perp &= C_{k,\ell}\sin(\theta) a_k^\perp a_\ell \\ \dot a_\ell &= -C_{j,k}\sin(\theta) a_j a_k & \dot a_\ell &= C_{j,k}\sin(\theta) a_j^\perp a_k^\perp \end{aligned}$$

The other sets of equations come from picking the angle $-\theta - \frac{\pi}{2}$

$\phi : [0,1] \circlearrowleft$



1. Conserved Quantities

First observe that if $j + \ell + k = 0$ then

$$C_{j,\ell} + C_{\ell,k} + C_{k,j} = 0 = \frac{C_{j,\ell}}{|k^2|} + \frac{C_{\ell,k}}{|j|^2} + \frac{C_{k,j}}{|\ell|^2}$$

This implies that

$$|q_\ell|^2 + |q_j|^2 + |q_k|^2 \quad \text{and} \quad \frac{1}{|\ell|^2}|q_\ell|^2 + \frac{1}{|j|^2}|q_j|^2 + \frac{1}{|k|^2}|q_k|^2$$

are conserved by the dynamics

$$\begin{aligned} \dot q_{-k} &= C_{j,\ell} q_j q_\ell \\ \dot q_{-j} &= C_{\ell,k} q_\ell q_k \\ \dot q_{-\ell} &= C_{k,j} q_k q_j \end{aligned}$$

Similarly if $C_{j,\ell}(\theta) = C_{j,\ell}\sin\theta$ then the following sets of three equations also conserve the analogous norms.

$$\begin{aligned} \dot a_k &= -C_{\ell,j}(\theta)a_\ell a_j & \qquad \dot a_k^\perp &= C_{\ell,j}(\theta)a_\ell a_j^\perp \\ \dot a_j &= -C_{k,\ell}(\theta)a_k a_\ell & \dot a_j^\perp &= C_{k,\ell}(\theta)a_k^\perp a_\ell \\ \dot a_\ell &= -C_{j,k}(\theta)a_j a_k & \dot a_\ell &= C_{j,k}(\theta)a_j^\perp a_k^\perp \end{aligned}$$

1.1. **Rotations.** If we want to have dynamice on only two variables pick

$$A^k_{j,\ell} + A^k_{\ell,j} = 0 \qquad \text{and} \qquad C_{j,k} = A^k_{\ell,j} + A^j_{\ell,k}$$

$$\begin{aligned} \dot q_{-k} &= A^j_{k,\ell} q_j q_\ell \\ \dot q_{-\ell} &= A^j_{\ell,k} q_j q_k \end{aligned}$$

then we can split of the real parts as before. `Not sure I got this right`