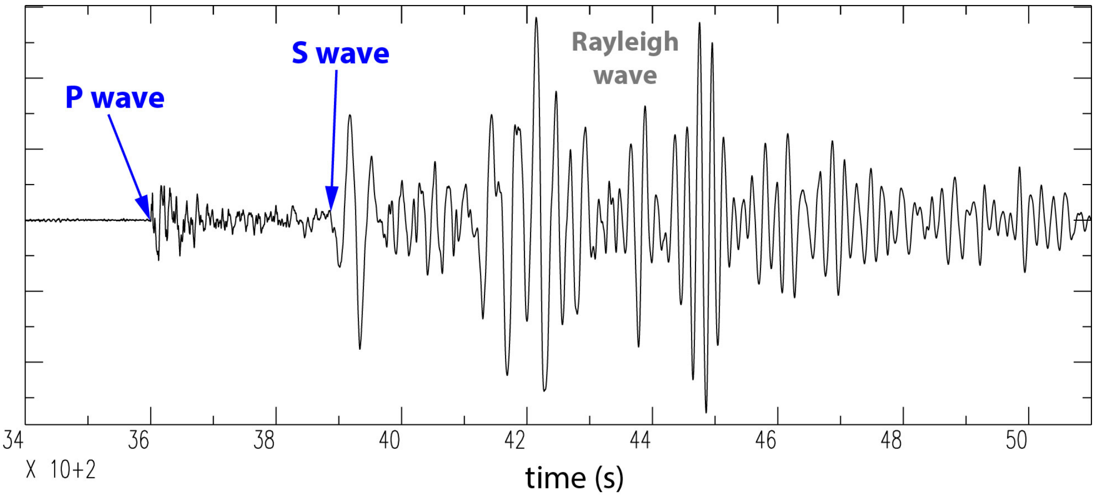
Inverse problem in seismology: location of earthquakes

(location based on the travel time difference between P and S waves)

Measuring and predicting arrival times of P and S waves

On most of seismograms, P and S waves are clearly visible and their arrival times can be measured, as seen in this example:



In a homogeneous media with P-wave speed α and S-wave speed β:

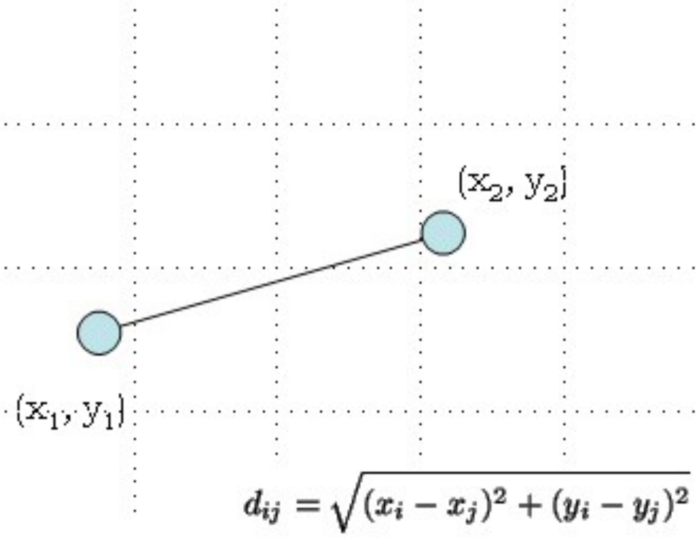
t_0	earthquake origin time
a	distance between the source and the receiver
$t_P = t_0 + d/\alpha$	P-wave arrival time
$t_S = t_0 + d/\beta$	S-wave arrival time

$$t_S - t_P = \left(\frac{1}{\alpha} - \frac{1}{\beta}\right)a \qquad \text{S-P time difference}$$

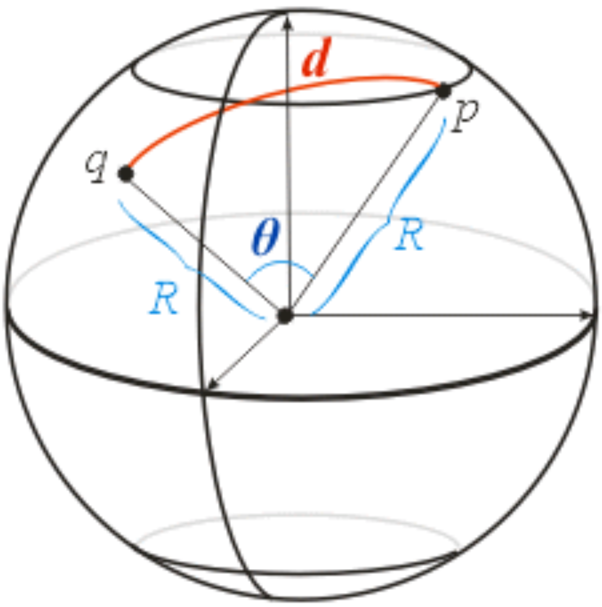
$$S_{diff} = \left(\frac{1}{\alpha} - \frac{1}{\beta}\right) \qquad \text{S-P slowness difference}$$

$$t_S - t_P = S_{diff}d$$

Distance between two points on the surface



Distance between two points on the spherical surface (Earth)



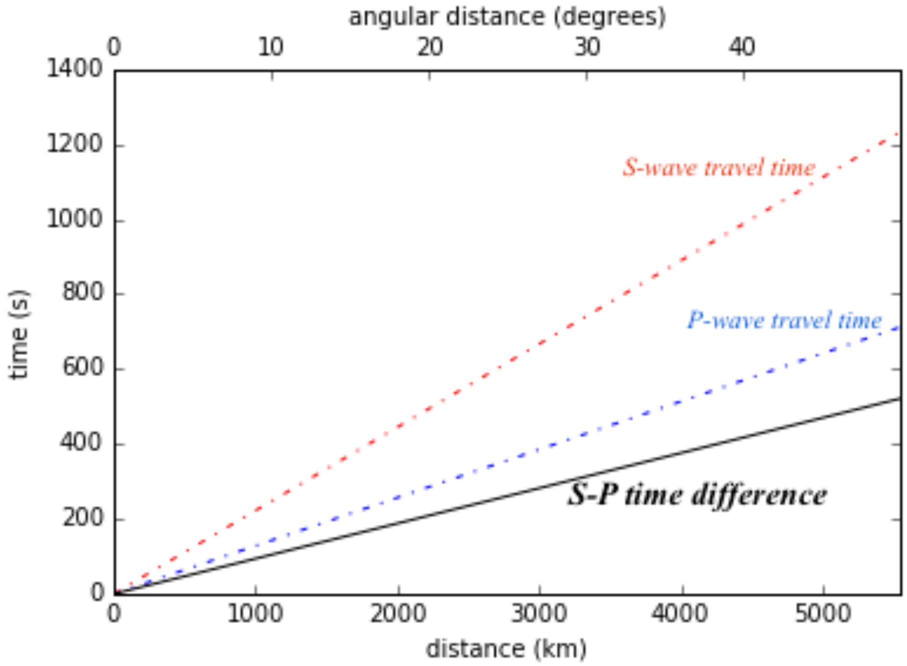
Relationship between the great-circle distance along the Earth's surface in km (***d***) and the angular distance in degrees (***θ***):

$$d = \frac{2\pi\theta}{360} R$$

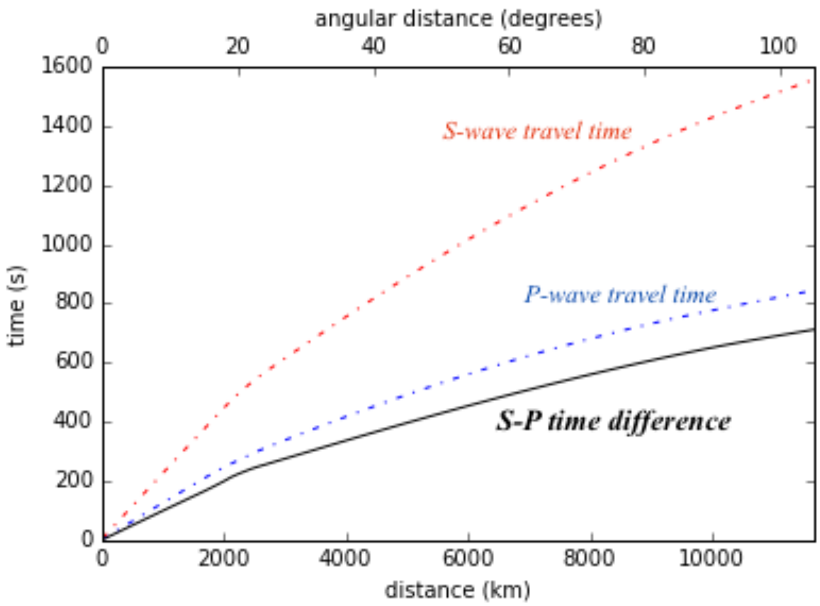
1 degree is equivalent to ***~111.11 km***

<u>Synthetic example of a media with</u> ***<u>β=4.5 km/s</u>*** **<u>and</u>** ***<u>α=7.7942 km/s</u>*** **:**

$\boldsymbol{S_{diff}}$ = 0.0939 s/km   (10.44 s/degree)



Travel times in a spherical Earth's model (IASP91):



Testing the prediction against observations ("misfit" or "cost" function)

$t_i^{S-P\,obs}$ observed S-P time difference at station ***i***

$t_i^{S-P\,pred}$ predicted S-P time difference at station ***i***

We search to minimize the differences:

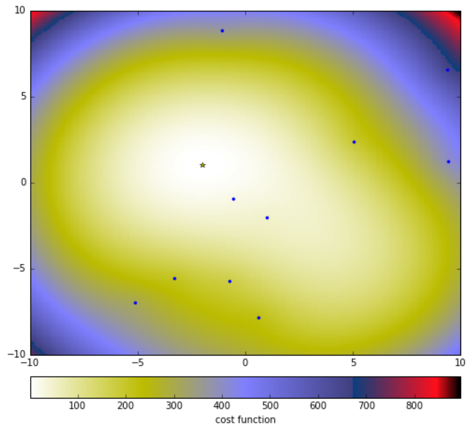
$$t_i^{S-P\,obs} - t_i^{S-P\,pred}$$

A simple form of misfit function, sum of squared residuals : $\sum_i \left(t_i^{S-P\,obs} - t_i^{S-P\,pred} \right)^2$.

If we normalize it with measurement errors $\boldsymbol{\sigma}$ and number of stations $\boldsymbol{N}$:

$$MF_1 = \frac{1}{N} \sum_i \frac{\left(t_i^{S-P\,obs} - t_i^{S-P\,pred} \right)^2}{\sigma_i^2}$$

This function minimizes when we predict correctly the travel times and is expected to have a minimal value close to 1 (if the values of $\boldsymbol{\sigma_i}$ represent correctly the measurement errors):

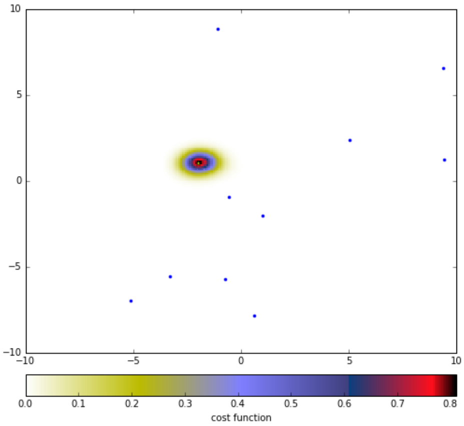


*Synthetic illustration of a misfit function **MF₁** (stations - blue dots; source - star)*

As is seen in the previous figure, the minimum of $\boldsymbol{MF_1}$ is too broad. Therefore, we can replace it with an exponential form:

$$MF_2 = \exp\left(-\frac{1}{2} MF_1 \right) = \exp\left(-\frac{1}{2N} \sum_i \frac{\left(t_i^{S-P\,obs} - t_i^{S-P\,pred} \right)^2}{\sigma_i^2} \right)$$

This function maximizes close to the source with a maximal value close to 1

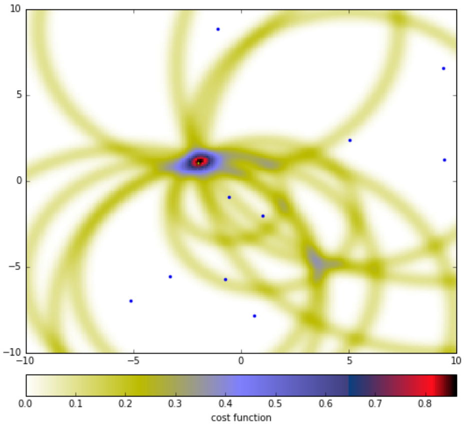


Synthetic illustration of a misfit function ***MF₂*** *(stations - blue dots; source - star)*

Another possible form of the misfit function:

$$MF_3 = \frac{1}{N}\sum_i \exp\left(-\frac{1}{2}\frac{\left(t_i^{obs\,S-P} - t_i^{pred\,S-P}\right)^2}{\sigma_i^2}\right)$$

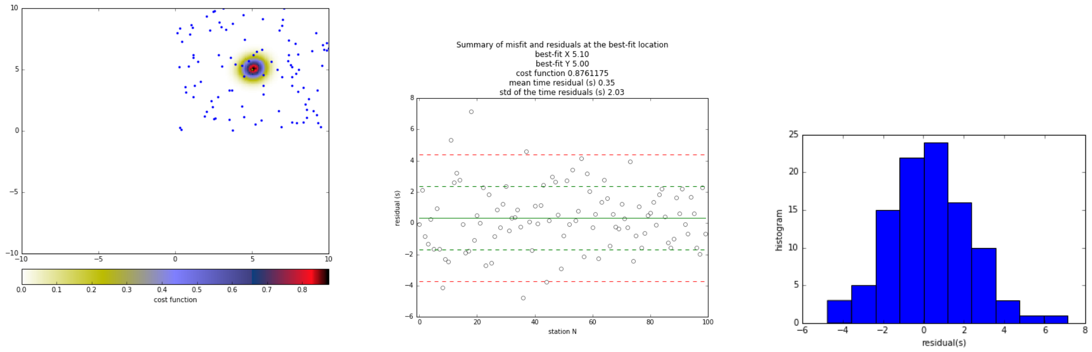
This function maximizes close to the source with a maximal value close to 1



Synthetic illustration of a misfit function ***MF₃*** *(stations - blue dots; source - star)*

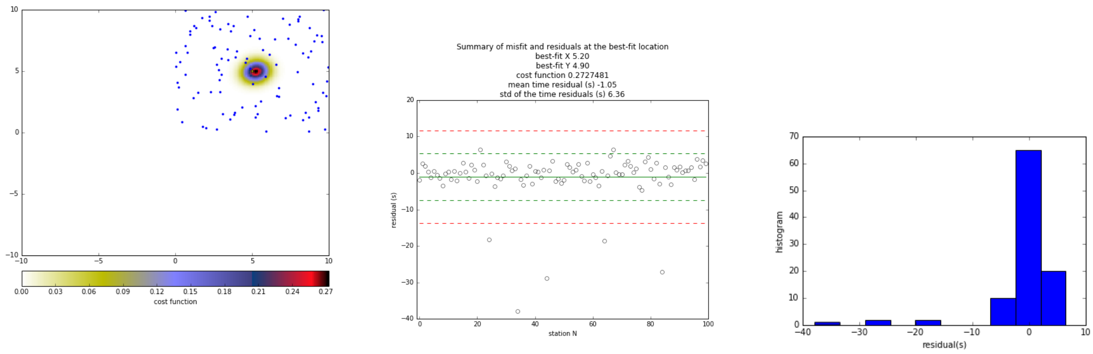
Analyzing residuals: quality of the measurements and the inversion

If the measurements and the “model” used for their predictions are consistent with each other and if the measurement errors are distributed normally, the final residuals are expected to be distributed normally around zero and the misfit function is expected to be close to its “optimal value” (1 for $\boldsymbol{MF_2}$) :



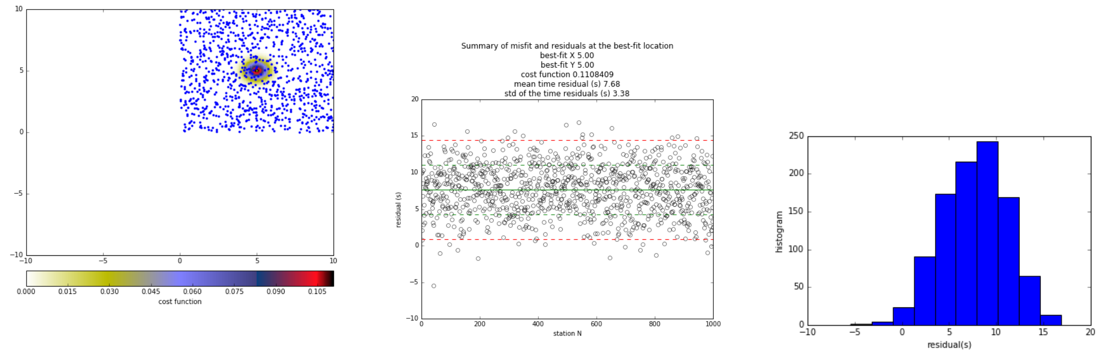
<u>Synthetic example of inversion with 100 stations</u>. Left frame: station source positions and the misfit function. Central frames: residuals. Right frame: residuals histogram.

Inversion results can be deteriorated because of a few very large errors in the measurements (outliers). Sometimes, they can be seen in posteriori residuals and, in this case, one possibility is to eliminate the corresponding measurements from the dataset.



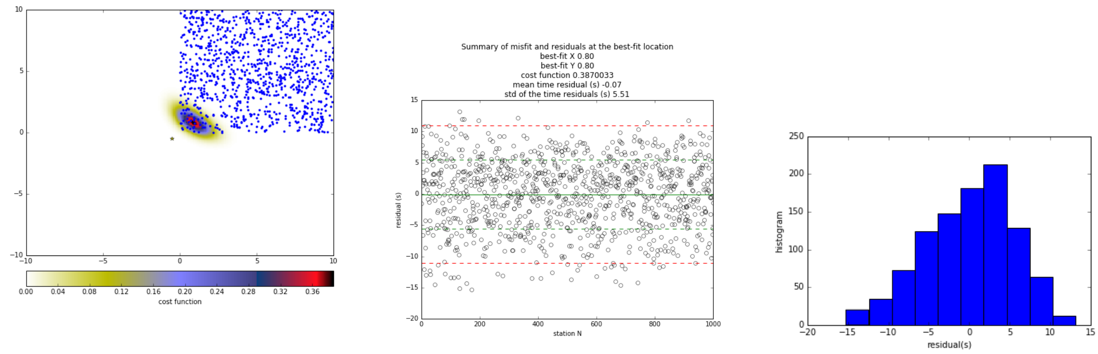
<u>Synthetic example of inversion with 100 stations and a few outliers</u>. Left frame: station source positions and the misfit function. Central frames: residuals. Right frame: residuals histogram.

Inversion results can be deteriorated because of a “wrong” model used for predicting the observations (bias). This bias can strongly reduce the misfit function and sometimes results in a strongly shifted distribution of residuals (to compensate for the systematic errors in “predictions”).



Synthetic example of inversion with 1000 stations and a biased travel time model ($\boldsymbol{S}_{diff}$ 10 instead of 8). Left frame: station source positions and the misfit function. Central frames: residuals. Right frame: residuals histogram.

In the case of the earthquake source location, the biased travel-time predictions can strongly shift the determined epicenter when the source is located at the edge of the network.



Synthetic example of inversion with 1000 stations and a biased travel time model ($\boldsymbol{S}_{diff}$ 10 instead of 8). Left frame: station source positions and the misfit function. Central frames: residuals. Right frame: residuals histogram.