Averaging and Rotating Wave Approximation

Pierre Rouchon[*]

October 24, 2023

1 Averaging theory

1.1 Introduction

It is assumed here that the rapid effects have an oscillating character. The averaging method has been developed in celestian mechanics for a long time to determine the evolution of the planet orbits under the influence of mutual disturbances between the planets and study the stability of the solar system. Gauss gives the following definition which is most intuitive : *it is advisable to distribute the mass of each planet along its orbit in proportion to time passed in each part of the orbit and replace the attraction of the planets by that of the rings of matter as well as defined.*

In this framework, the unperturbed equations of the motion are those that only take into account the force of attraction due to the sun. The orbit of the earth is then an ellipse of which the sun is one of the focal points. Perturbed Equations are those where we add the forces of attraction between the earth and the other planets assuming that these latter all describe elliptical orbits according to the Kepler's laws. The ε-parameter corresponds to the ratio of the mass of the sun to that of planets : $\varepsilon \approx 1/1000$. The fast time-scale is of the order of a period of revolution, a few years. The slow time-scale is of the order of a few millennia. The question then is whether these small disturbances of order ε can lead on the scale of the millennium to a systematic drift of the lengths of the major axis and the minor axis of the trajectory of the earth. This would have catastrophic consequences for the climate. In fact, the calculations (averaging) show that this is not the case. In contrast, the eccentricity of the orbits oscillates slowly. These oscillations influence the climate.

[*]Laboratoire de Physique de l'Ecole Normale Supérieure, Mines Paris-PSL, Inria, ENS-PSL, Université PSL, CNRS, Paris, France. `pierre.rouchon@minesparis.psl.eu`

1.2 Single-frequency averaging

Consider the following time-periodic dynamical system

$$\frac{dx}{dt} \quad = \quad \varepsilon f(x,t,\varepsilon), \quad 0 \leq \varepsilon \ll 1 \tag{1}$$

where f is regular in $x \in \mathbb{R}^n$ and depends of t periodically (period T).

The averaged system (or slow system) of state denoted by $\bar{x}$ [1] is then

$$\frac{d\bar{x}}{dt} \quad = \quad \varepsilon \tfrac{1}{T}\textstyle\int_0^T f(\bar{x},t,0)\,dt \quad \stackrel{\text{def}}{=} \quad \varepsilon \overline{f}(\bar{x}). \tag{2}$$

Here the notation $\bar{x}$ is just a notation. It approximates the trajectories of (1) by those of (2). If the averaged system admits a hyperbolically stable equilibrium point (the eigenvalues of the linear tangent system are all with strictly negative real part) then the perturbed system oscillates slightly around of this average equilibrium.

More rigorously, the following theorem shows that a *hyperbolic equilibrium* of the average system corresponds to a small periodic orbit of the perturbed system (1) (see [2]).

Theorem 1 (Single frequency averaging)**.** *Consider the perturbed system (1) with f regular. There is a change of variables, $x = z + \varepsilon w(z,t)$ with w of period T to t, such that (1) becomes*

$$\frac{dz}{dt} = \varepsilon \overline{f}(z) + \varepsilon^2 f_1(z,t,\varepsilon)$$

with $\overline{f}$ defined by (2) and f_1 regular versus x and T-periodic versus t. Furthermore,

(i) *if $x(t)$ and $\bar{x}(t)$ are respectively solutions of (1) and (2) with initial conditions x_0 and $\bar{x}_0$ such as $\|x_0 - \bar{x}_0\| = O(\varepsilon)$, then $\|x(t) - \bar{x}(t)\| = O(\varepsilon)$ uniformly for $t \in [0, 1/\varepsilon]$.*

(ii) *If $\overline{a}$ is a stable hyperbolic fixed point of the averaged system (2), then it exists $\overline{\varepsilon} > 0$ such that, for all $\varepsilon \in]0, \overline{\varepsilon}]$, the perturbed system (1) admits a unique periodic-solution $\gamma_\varepsilon(t)$, close to $\overline{a}$ ($\gamma_\varepsilon(t) = \overline{a} + O(\varepsilon)$) and asymptotically stable (trajectories starting near $\gamma_\varepsilon(t)$ tend to wrap around the latter). Then, the approximation of the trajectories of the perturbed system (1) by those of the averaged one (2) becomes valid for $t \in [0, +\infty[$.*

1. $\bar{x}$ is just a notation : $\bar{x}(t)$ does not corresponds exactly to the averaged of x over the interval $[t - T/2, t + T/2]$, even if it is close to such average for small ϵ.

Following [1, 4], it is instructive to see how the change of coordinates $x = z + \varepsilon w(z,t)$ by removing from x some oscillating terms of order ε (w of period T in t). We have, on the one hand,

$$\frac{dx}{dt} = \frac{dz}{dt} + \varepsilon \frac{\partial w}{\partial z}(z,t)\frac{dz}{dt} + \varepsilon \frac{\partial w}{\partial t}(z,t)$$

And on the other hand,

$$\frac{dx}{dt} = \varepsilon f(z + \varepsilon w(z,t), t, \varepsilon)$$

Thereby

$$\begin{aligned} \tfrac{dz}{dt} &= \varepsilon \left(I + \varepsilon \tfrac{\partial w}{\partial z}(z,t)\right)^{-1} \left[f(z + \varepsilon w(z,t), t, \varepsilon) - \tfrac{\partial w}{\partial t}(z,t)\right] \\ &= \varepsilon \left[f(z,t,0) - \tfrac{\partial w}{\partial t}(z,t)\right] + O(\varepsilon^2) \end{aligned}$$

As the t dependency of w is T-periodic, it is not possible to completely cancel first order term versus ε because there is no reason why the function defined by

$$\int_0^t f(z,s,0)\, ds$$

i.e. T-periodic in time. On the other hand, one can eliminate the time dependence of order one in ε :

$$w(z,t) = \int_0^t \left(f(z,s,0) - \overline{f}(z)\right)\, ds$$

(note that w is T-periodic) to get

$$\frac{dz}{dt} = \varepsilon \overline{f}(z) + O(\varepsilon^2)$$

If this approximation is not sufficient, it is necessary to take into account account second order terms and eliminate their time dependence by a change of variable of the type $x = z + \varepsilon w_1(z,t) + \varepsilon^2 w_2(z,t)$ with w_1 and w_2 T-periodic (see, e.g, [4]).

2 Examples

2.1 Resonant control of a high-Q harmonic oscillator

Consider the following harmonic oscillator

$$\frac{d}{dt}x' = \omega p', \quad \frac{d}{dt}p' = -\omega x' - \kappa p' - 2u_1 \sin(\omega t) + 2u_2 \cos(\omega t)$$

where $\omega \gg \max\left(\kappa, \sqrt{u_1^2+u_2^2}\right)$. Consider the following periodic change of variables $(x', p') \mapsto (x, p)$:

$$x' = \cos(\omega t)x + \sin(\omega t)p, \quad p' = -\sin(\omega t)x + \cos(\omega t)p.$$

Then, we have

$$\begin{aligned}
&\cos(\omega t)\frac{d}{dt}x + \sin(\omega t)\frac{d}{dt}p = 0 \\
-&\sin(\omega t)\frac{d}{dt}x + \cos(\omega t)\frac{d}{dt}p = -\kappa(-\sin(\omega t)x + \cos(\omega t)p) - 2u_1\sin(\omega t) + 2u_2\cos(\omega t).
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{d}{dt}x &= -\kappa\sin^2(\omega t)x + 2u_1\sin^2(\omega t) + (\kappa p - 2u_2)\sin(\omega t)\cos(\omega t) \\
\frac{d}{dt}p &= -\kappa\cos^2(\omega t)p + 2u_2\cos^2(\omega t) + (\kappa x - 2u_1)\sin(\omega t)\cos(\omega t).
\end{aligned}$$

Removing highly oscillating terms (rotating wave approximation), we get :

$$\frac{d}{dt}x = -\tfrac{\kappa}{2}x + u_1, \quad \frac{d}{dt}p = -\tfrac{\kappa}{2}p + u_2$$

that reads also with the complex variables $\alpha = x + ip$ and $u = u_1 + iu_2$:

$$\frac{d}{dt}\alpha = -\tfrac{\kappa}{2}\alpha + u. \tag{3}$$

This yields to the following approximate model in the original frame (x', p') :

$$\frac{d}{dt}x' = -\tfrac{\kappa}{2}x' + \omega p + u_1\cos(\omega t) + u_2\sin(\omega t), \quad \frac{d}{dt}p' = -\omega x' - \tfrac{\kappa}{2}p' - u_1\sin(\omega t) + u_2\cos(\omega t)$$

or with complex variable $\alpha' = x' + ip' = e^{-i\omega t}\alpha$:

$$\frac{d}{dt}\alpha' = -(\tfrac{\kappa}{2} + i\omega)\alpha' + ue^{-i\omega t} \tag{4}$$

2.2 Resonant control of a qubit

Let us consider the qubit system [2]

$$i\frac{d}{dt}|\psi\rangle = \left(\tfrac{\omega_{\text{eg}}}{2}\sigma_{\mathbf{z}} + \tfrac{u(t)}{2}\sigma_{\mathbf{x}}\right)|\psi\rangle$$

2. Here : $|\psi\rangle = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix} \in \mathbb{C}$, $\sigma_{\mathbf{x}} = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}$, $\sigma_{\mathbf{y}} = \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}$, $\sigma_{\mathbf{z}} = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}$, $\sigma_- = \begin{pmatrix}0 & 0 \\ 1 & 0\end{pmatrix}$ and $\sigma_+ = \begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}$.

with periodic drive (control input) $u(t) = ve^{i\omega_r t} + v^* e^{-i\omega_r t}$ where the complex amplitude v is chosen such that $|v| \ll \omega_{\text{eg}}$ and the frequency ω_r is close to ω_{eg}, i.e., $|\omega_{\text{eg}} - \omega_r| \ll \omega_{\text{eg}}$.

Denote by $\Delta_r = \omega_{\text{eg}} - \omega_r$ the detuning between the control and the system. The following change of variables $|\psi\rangle = e^{-i\frac{\omega_r t}{2}\sigma_{\mathbf{z}}} |\phi\rangle$, based on the unitary transformation $e^{-i\frac{\omega_r t}{2}\sigma_{\mathbf{z}}}$ yields to the interaction dynamics

$$i\frac{d}{dt}|\phi\rangle = \mathbf{H}_{\text{int}}|\phi\rangle$$

with interaction Hamiltonian

$$\mathbf{H}_{\text{int}} = \frac{\Delta_r}{2}\sigma_{\mathbf{z}} + \tfrac{ve^{i\omega_r t}+v^* e^{-i\omega_r t}}{2} e^{\frac{i\omega_r t}{2}\sigma_{\mathbf{z}}}\sigma_{\mathbf{x}} e^{-\frac{i\omega_r t}{2}\sigma_{\mathbf{z}}}.$$

With the identities $e^{i\theta\sigma_{\mathbf{z}}} = \cos\theta\mathbf{I} + i\sin\theta\sigma_{\mathbf{z}}$ and $\sigma_{\mathbf{z}}\sigma_{\mathbf{x}} = i\sigma_{\mathbf{y}}$ we get the formula

$$e^{i\theta\sigma_{\mathbf{z}}}\sigma_{\mathbf{x}} e^{-i\theta\sigma_{\mathbf{z}}} = e^{2i\theta}\sigma_+ + e^{-2i\theta}\sigma_-.$$

Thus we have

$$\mathbf{H}_{\text{int}} = \tfrac{\Delta_r}{2}\sigma_{\mathbf{z}} + \tfrac{ve^{2i\omega_r t}+v^*}{2}\sigma_+ + \tfrac{v^* e^{-2i\omega_r t}+v}{2}\sigma_-.$$

The following decomposition

$$\mathbf{H}_{\text{int}} = \underbrace{\tfrac{\Delta_r}{2}\sigma_{\mathbf{z}} + \tfrac{v^*}{2}\sigma_+ + \tfrac{v}{2}\sigma_-}_{\text{secular part}} + \underbrace{\tfrac{ve^{2i\omega_r t}}{2}\sigma_+ + \tfrac{v^* e^{-2i\omega_r t}}{2}\sigma_-}_{\text{oscillatory part}}$$

yields to the first order approximation : any solution $|\psi\rangle$ satisfying

$$i\frac{d}{dt}|\psi\rangle = \left(\tfrac{\omega_r+\Delta_r}{2}\sigma_{\mathbf{z}} + \tfrac{ve^{i\omega_r t}+v^* e^{-i\omega_r t}}{2}\sigma_{\mathbf{x}}\right)|\psi\rangle$$

is approximated by $e^{-i\frac{\omega_r t}{2}\sigma_{\mathbf{z}}}|\bar{\phi}\rangle$ where $|\bar{\phi}\rangle$ is solution of the linear time-invariant equation

$$i\frac{d}{dt}|\bar{\phi}\rangle = \left(\tfrac{\Delta_r}{2}\sigma_{\mathbf{z}} + \tfrac{v^*}{2}\sigma_+ + \tfrac{v}{2}\sigma_-\right)|\bar{\phi}\rangle, \quad |\bar{\phi}\rangle_0 = |\psi\rangle_0.$$

2.3 Phase-Locked Loop (PLL)

We have a noisy input signal $v(t)$ which we wishes to estimate the frequency in real time : this latter is around a reference frequency denoted $\omega_0 > 0$. Thus we have $v(t) = a(t)\cos(\theta(t)) + w(t)$ with $|\dot{\theta} - \omega_0| \ll \omega_0$, $a(t) > 0$ slowly varying, i.e $|\dot{a}| \ll \omega_0 a$, and $w(t)$ a noise. The standard deviation of

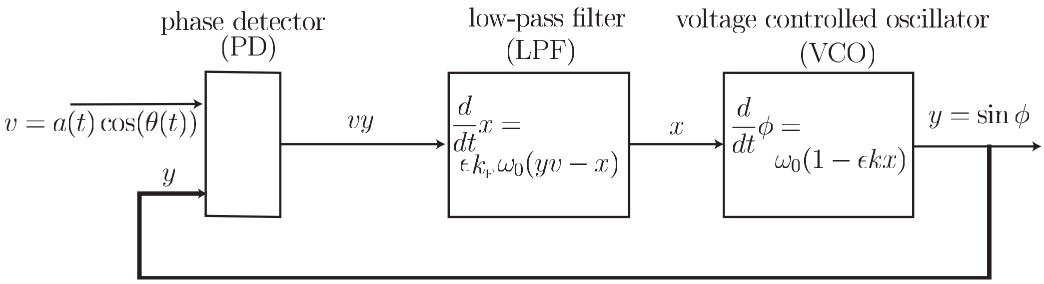

FIGURE 1 – The block diagram of a PLL : the quantity $\omega_0(1-\epsilon kx)$ is a filtered estimate of the frequency $\frac{d}{dt}\theta$ of the input signal v which can be very noisy and whose amplitude a is not known.

w can be much larger than the amplitude a : the signal to noise ratio can be very unfavorable. Practical implementation can be done with analogue electronic circuits or digital systems, known as Phase-Locked Loops (PLL), which give in real-time noise-free estimation of the frequency input $\frac{d}{dt}\theta$: this type of specialized circuits is found in almost all electronic telecommunication systems (mobile phone, WiFi card, fiber optic, laser, atomic clocks, GPS, etc.). PLLs are also used by physicists to measure very precisely the fundamental constants[3]. The operating principle of PLLs is analyzed very well with single-frequency averaging.

Figure 1 gives a simplified block diagram and principle (more elaborated schemes are analyzed in [3]) : each block corresponds either to a specific circuit in the case of an analog PLL, or to one step of the algorithm for a digital PLL. The dynamic is governed by

$$\frac{d}{dt}x = \epsilon k_f \omega_0(v(t)\sin\phi - x), \quad \frac{d}{dt}\phi = \omega_0(1-\epsilon kx)$$

where ϵ is a small positive parameter, k_f and k two positive gains. We set $v(t) = a\cos\theta$ with $\frac{d}{dt}\theta = \omega_0(1+\epsilon p)$ where $a > 0$ and p are unknown but constant parameters. Since $2\cos\theta\sin\phi = \sin(\phi-\theta)+\sin(\phi+\theta)$, with $\Delta = \phi-\theta$ and $\sigma = \phi+\theta$, the system

$$\frac{d}{dt}x = \epsilon k_f \omega_0(a\cos\theta\sin\phi - x), \quad \frac{d}{dt}\phi = \omega_0(1-\epsilon kx), \quad \frac{d}{dt}\theta = \omega_0(1+\epsilon p)$$

becomes in timescale σ ($\frac{d}{dt}\sigma = \omega_0(2+\epsilon(p-kx))$)

$$\frac{d}{d\sigma}x = \epsilon k_f \frac{\frac{a}{2}\sin\Delta + \frac{a}{2}\sin\sigma - x}{2+\epsilon(p-kx)}, \quad \frac{d}{d\sigma}\Delta = -\epsilon\,\frac{p+kx}{2+\epsilon(p-kx)}.$$

3. A.L. Schawlow (physicist, award Nobel Prize in 1981 for his work on laser spectroscopy) said that to have a very precise measurement, it is necessary to measure a frequency.

Thus, we are in the standard form of the theorem 1 with σ instead of t. The average system is

$$\frac{d}{d\sigma}x = \epsilon k_f \; \frac{\frac{a}{2}\sin\Delta - x}{2+\epsilon(p-kx)}, \quad \frac{d}{d\sigma}\Delta = -\epsilon \; \frac{p+kx}{2+\epsilon(p-kx)}.$$

We neglect terms of order 2 in ϵ and we take as average system :

$$\frac{d}{d\sigma}x = \frac{\epsilon k_f}{2} \; \left(\frac{a}{2}\sin\Delta - x\right), \quad \frac{d}{d\sigma}\Delta = -\frac{\epsilon}{2} \; (p+kx).$$

This system can also be written in the form of a single equation of the second order with $\sigma/\varsigma = \epsilon\sqrt{k_f ka/8}$

$$\frac{d^2}{d\varsigma^2}\Delta = -\sqrt{\frac{2k_f}{ka}} \; \frac{d}{d\varsigma}\Delta - \left(\sin\Delta - \frac{2p}{ak}\right).$$

We choose the gain k large enough so that $\left|\frac{2p}{ak}\right|$ <1. So we put $\sin\bar{\Delta} = \frac{2p}{ak}$ with $\bar{\Delta} \in]-\frac{\pi}{2}, \frac{\pi}{2}[$. This system therefore admits two equilibria :

- $\Delta = \pi - \bar{\Delta}$ is a saddle (two real eigenvalues of opposite signs)
- $\Delta = \bar{\Delta}$ is locally asymptotically stable (two eigenvalues with a strictly negative real part).

For k and k_f large enough $\Delta = \phi - \theta$ therefore converges to a constant. This is phase locking : the difference Δ between the phase ϕ of the PLL and the phase θ of the input signal to be analyzed converges to a constant : $\phi = \theta + \bar{\Delta} \mod (2\pi)$.

Thus $\frac{d}{dt}\phi$ converges to $\frac{d}{dt}\theta$, as $\frac{d}{dt}\phi = \omega_0(1-\epsilon kx)$, we see that we have a low-noise estimate of the input frequency $\frac{d}{dt}\theta$ with $\omega_0(1-\epsilon kx)$. For this we did not need to know precisely the amplitude a. Note that the same PLL gives, when the frequency remains fixed, the phase modulations.

2.4 Kapitsa pendulum

Dynamics of a pendulum with a fixed suspension point :

$$\frac{d^2}{dt^2}\theta = \frac{g}{l}\sin\theta$$

g : free fall acceleration, l : pendulum's length, θ : angle to the vertical ; $\theta = \pi$ stable and $\theta = 0$ unstable equilibrium.

Dynamics when the suspension point oscillates vertically

Motion of the suspension point : $z = \frac{v}{\Omega}\cos(\Omega t)$ ($a = v/\Omega > 0$ amplitude and Ω frequency).

Pendulum's dynamics : replace acceleration g by $g + \ddot{z} = g - v\Omega\cos(\Omega t)$,

$$\frac{d}{dt}\theta = \omega, \quad \frac{d}{dt}\omega = \frac{g - v\Omega\cos(\Omega t)}{l}\sin\theta.$$

Replacing the velocity ω by the momentum $p_\theta = \omega + \frac{v\sin(\Omega t)}{l}\sin\theta$:

$$\begin{aligned}\frac{d}{dt}\theta &= p_\theta - \tfrac{v\sin(\Omega t)}{l}\sin\theta,\\ \frac{d}{dt}p_\theta &= \left(\tfrac{g}{l} - \tfrac{v^2\sin^2(\Omega t)}{l^2}\cos\theta\right)\sin\theta + \tfrac{v\sin(\Omega t)}{l}p_\theta\cos\theta.\end{aligned}$$

For large enough Ω, we can average these time-periodic dynamics over $[t - \pi/\Omega, t + \pi/\Omega]$:

$$\frac{d}{dt}\theta = p_\theta, \quad \frac{d}{dt}p_\theta = \left(\tfrac{g}{l} - \tfrac{v^2}{2l^2}\cos\theta\right)\sin\theta.$$

Around $\theta = 0$ the approximation of small angles gives

$$\frac{d^2}{dt^2}\theta = \frac{g - v^2/2l}{l}\theta = \frac{g - a^2\Omega^2/2l}{l}\theta$$

If $a^2\Omega^2/2l > g$ then the averaged system becomes stable around $\theta = 0$.

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