Problem Set 3
(M1 Math Methods 2023-2024)

This problem set is due on Monday, October 9th, 2023, at 23:59. The solutions should be sent as a single PDF (scan of handwritten document or LaTeX) to alexandru.petrescu@minesparis.psl.eu by the deadline. If you collaborate with a colleague, please write clearly their names at the top of your solution sheets. If you submit late without a satisfactory reason, the set will be accepted with a 10% penalty in the score.

I. COHERENT AND THERMAL STATES

a) Consider a resonator initially prepared in a coherent state $|\psi(t=0)\rangle = |\alpha\rangle$. Find the time evolution of the system state $|\psi(t)\rangle$ for all subsequent times $t \geq 0$, as governed by the simple harmonic oscillator Hamiltonian $\hat{H}/\hbar = \omega_a \hat{a}^\dagger \hat{a}$ for some real and positive frequency ω_a.

b) In the state $|\psi(t)\rangle$ determined above, what are the time-dependent expectation values of the rescaled position operator $\hat{Q} = (\hat{a} + \hat{a}^\dagger)/\sqrt{2}$ and of the rescaled momentum operator $\hat{P} = (\hat{a} - \hat{a}^\dagger)/(i\sqrt{2})$?

c) Find $\Delta(\hat{P}), \Delta(\hat{Q}), \Delta(\hat{P})\Delta(\hat{Q})$, corresponding to $|\psi(t)\rangle$, at all times. Reminder $\Delta(\hat{A}) = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$, where $\langle \ldots \rangle$ is shorthand for $\langle \psi | \ldots | \psi \rangle$, is the variance of an observable $\hat{A}$ in state $|\psi\rangle$. How does the variance $\Delta(\hat{Q})$ depend on time? You have shown that coherent states are minimum uncertainty states, as they saturate the inequality in Heisenberg's uncertainty principle.

d) Consider a system that can be found in a mixed state defined by the density operator $\hat{\rho} = \mathcal{N} \exp(-\beta \hat{H})$, where $\hat{H}/\hbar = \omega_a \hat{a}^\dagger \hat{a}$, and $\beta = 1/(k_B T)$ where k_B is Boltzmann's constant and T is a finite temperature. Determine $\mathcal{N}$ such that ρ is correctly defined as a density matrix, and give it a physical interpretation. Find the expectation value and variance of the photon number $\hat{N} = \hat{a}^\dagger \hat{a}$ in this state.

II. DRIVEN SIMPLE HARMONIC OSCILLATOR

Recall your results for a *time-dependent change of frame* from the previous problem set. Consider now the following time-dependent Hamiltonian describing a driven simple harmonic oscillator

$$\frac{\hat{H}(t)}{\hbar} = \omega_a \hat{a}^\dagger \hat{a} + \varepsilon_{\mathrm{d}}(t)(\hat{a} + \hat{a}^\dagger), \tag{1}$$

where the time-dependent function in the drive term is given by $\varepsilon_{\mathrm{d}}(t) = \varepsilon_d \sin(\omega_d t)$, determined by an off-resonant drive frequency ω_d, i.e. $\omega_d \neq \omega_a$. Show, by performing a unitary transformation $\hat{U}(t) = e^{-i\phi(t)} \hat{D}[\alpha(t)]$, where $\phi(t)$ is an unknown real function of time, and $\alpha(t)$ is an unknown complex function of time, with $\hat{D}[\alpha] = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a})$, that you can perform a change of frame where the Hamiltonian is independent of terms linear in $\hat{a}, \hat{a}^\dagger$, i.e. $\hat{H}'/\hbar = \omega_a \hat{a}^\dagger \hat{a}$. Find the ordinary differential equations that must be obeyed by $\alpha(t)$ and $\phi(t)$, and give a physical interpretation (*Hint:* This is easiest by reverting back to phase space coordinates via $\alpha(t) = [Q(t) + iP(t)]/\sqrt{2}$, where the rescaled position Q and the rescaled momentum P are real functions of time). Suppose the system is initially prepared in the vacuum state $|0\rangle$ *in the frame of* $\hat{H}'$. What is the subsequent time evolution of this state in the laboratory frame (that of $\hat{H}(t)$)?

III. SIMPLIFIED MODEL FOR A TRANSMON QUBIT

The goal of this problem is to find the approximate solutions to Schrödinger's equation for a transmon qubit using perturbation theory, then analyze a simple single-qubit gate, the $\pi-$pulse. We consider the following Hamiltonian

$$\begin{aligned}
\hat{H} &= \hat{H}_0 + \hat{H}_J, \\
\hat{H}_0 &= \hbar \omega_a \hat{a}^\dagger \hat{a}, \\
\hat{H}_J &= \frac{\hbar\alpha}{12}(\hat{a} + \hat{a}^\dagger)^4 = \frac{\hbar\alpha}{4} I + \hbar\alpha \left[\hat{a}^\dagger \hat{a} + \frac{1}{2}(\hat{a}^2 + \hat{a}^{\dagger 2}) \right] + \frac{\hbar\alpha}{12}(\hat{a}^4 + \hat{a}^{\dagger 4} + 4\hat{a}^{\dagger 3}\hat{a} + 4\hat{a}^\dagger \hat{a}^3 + 6\hat{a}^{\dagger 2}\hat{a}^2).
\end{aligned} \tag{2}$$

We assume the second equality in the last equation above. You do *not* have to derive it. We have denoted $\hat{H}_0$ the Hamiltonian corresponding to a simple harmonic oscillator, and $\hat{H}_J$ a perturbation proportional to the so-called anharmonicity of the transmon qubit, denoted by α. We assume $\omega_a > 0$ and $\alpha < 0$, and $\omega_a \gg |\alpha|$. Note that $\hat{a}$ and $\hat{a}^\dagger$ are the bosonic annihilation and creation operators obeying the canonical commutation relation $[\hat{a}, \hat{a}^\dagger] = I$, with I being the identity acting on the simple harmonic oscillator Hilbert space.

a) As a warm-up, recall your knowledge of the simple harmonic oscillator, and write down the spectrum of $\hat{H}_0$, i.e. the eigenvectors and eigenvalues arising from the eigenproblem $\hat{H}_0 |\psi\rangle = \hbar\omega_a \hat{a}^\dagger \hat{a} |\psi\rangle = E |\psi\rangle$. Express the eigenvectors in normalized form, in terms of the vacuum vector $|0\rangle$, which you will need to define.

b) Consider the perturbation $\hat{H}_J$. Show that it can be written as a sum of two operators: the first operator, which we will call $\hat{H}_J^{secular}$, commutes with the number operator $\hat{a}^\dagger \hat{a}$. The second operator, which we shall denote $\hat{H}_J^{nonsecular}$, that does not commute with the number operator. Read off the expressions of these two operators from the expanded form of $\hat{H}_J$. To save time, try to make the arguments above *without* explicitly evaluating the commutator.

c) Using the nondegenerate time-independent perturbation theory, find the lowest-order nontrivial correction to the eigenenergies of $\hat{H}_0$, and the correction to the eigenvectors of $\hat{H}_0$, due to $\hat{H}_J$. Show that the corrected eigenvectors are

$$\left|n^{(1)}\right\rangle = |n\rangle + \mu_{n+2,n} |n+2\rangle + \mu_{n-2,n} |n-2\rangle + \mu_{n+4,n} |n+4\rangle + \mu_{n-4,n} |n-4\rangle \,, \tag{3}$$

in terms of the eigenstates $|n\rangle$ of the simple harmonic oscillator found at a). Find the coefficients μ_{ij} in the expression above, according to perturbation theory, without normalizing the corrected eigenvectors $\left|n^{(1)}\right\rangle$.