Problem Set 1
(M1 Math methods 2023-2024)

This problem set is due on Friday, September 22nd, 2023, at 23:59. The solutions should be emailed as a single PDF (handwritten or typeset) to alexandru.petrescu@minesparis.psl.eu by the deadline. If you collaborate with a colleague, please write their names at the top of your solution. Cite your references (books, websites, chatbots etc.). If you submit late without a satisfactory reason, the set will be accepted with a 10% penalty in the score.

I. PAULI MATRICES (20 POINTS)

We define the set of *Pauli matrices* together with the identity as $\sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 = \sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \sigma_y = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma_3 = \sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

a) Find the eigenvectors, eigenvalues, and diagonal representations of the three Pauli matrices $\sigma_{1,2,3}$.

b) Show that any 2×2 Hermitian matrix, *i.e.* $\begin{pmatrix} a & c+id \\ c-id & b \end{pmatrix}$, with a, b, c, d real, can be expressed as a real linear combination of the Pauli matrices and the identity.

c) Show that, if we define the commutator of two linear operators as $[A, B] = AB - BA$, then $[\sigma_i, \sigma_j] = 2i \sum_k \epsilon_{ijk}\sigma_k$ for $i, j, k \in \{1, 2, 3\}$ and ϵ_{ijk} is the Levi-Civita symbol, *i.e.* $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$, and zero whenever two of its indices are equal. Show that, if we define the anticommutator as $\{A, B\} = AB + BA$, then $\{\sigma_i, \sigma_j\} = 2\delta_{ij} I$. Show that, if the trace of a square matrix is defined by the sum of the elements on its diagonal, $\mathrm{Tr} A = A_{11} + \ldots A_{nn}$, then $\mathrm{Tr}\sigma_i = 0$ for $i = 1, 2, 3$, but $\mathrm{Tr}\{\sigma_i\sigma_j\} = 2\delta_{ij}$. Go back to b) and use what you just proved to extract the coefficient of σ_1 from the linear combination.

d) Let $\vec{a} = a\hat{n}$ with $\hat{n}$ being a unit vector in $\mathbb{R}^3$, i.e. $|\hat{n}| = 1$, and $a > 0$ a real number. Show that $(\hat{n} \cdot \vec{\sigma})^{2p} = I$ for any natural number p. Show that $e^{ia\hat{n}\cdot\vec{\sigma}} = I \cos a + i\hat{n} \cdot \vec{\sigma} \sin a$. $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a vector whose components are the three Pauli matrices, and $\hat{n} \cdot \vec{\sigma} =$

$\hat{n}_1\sigma_1 + \hat{n}_2\sigma_2 + \hat{n}_3\sigma_3$.

e) Show that $R_n(-a)\vec{\sigma}R_n(a) \equiv e^{i\frac{a}{2}\hat{n}\cdot\vec{\sigma}}\vec{\sigma}e^{-i\frac{a}{2}\hat{n}\cdot\vec{\sigma}} = \vec{\sigma}\cos a + \hat{n}\times\vec{\sigma}\sin a + \hat{n}(\hat{n}\cdot\vec{\sigma})(1-\cos a)$. For example, show that if $\hat{n} = (0,1,0)$ is the unit vector on the y axis, then $R_y(-\pi/2)\sigma_1 R_y(\pi/2) = \sigma_3$.

SOLUTION

a) The eigenvectors of Z are $(1,0)^T$, with ^{T} denoting the transpose, with eigenvalue $+1$, and $(0,1)^T$ with eigenvalue -1. The eigenvectors of X are $(1,\pm 1)/\sqrt{2}$ with eigenvalues ± 1. The eigenvectors of Y are $(1,\pm i)/\sqrt{2}$ with eigenvalue ± 1. For brevity, we check the last one explicitly. The eigenvalue equation reads

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}, \tag{1}$$

so that λ is given by the roots of the characteristic polynomial $\begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = \lambda^2 - 1$, which gives the two eigenvalues $\lambda_\pm = \pm 1$.

Returning to the system of equations for the components of the eigenvector, plugging in either value of λ determined above we have

$$\begin{aligned} -iy &= \lambda x, \\ ix &= \lambda y, \end{aligned} \tag{2}$$

with the two equations noticeably equivalent (the coefficient matrix, $Y - \lambda I$, has zero determinant by definition, and is rank 1), so by using the first equation we have $\pm ix = y$ for $\lambda = \pm 1$. We are allowed to rescale both x, y by the same constant to normalize the eigenvector, so a solution to the above is $x = 1/\sqrt{2}$ and $y = \pm i/\sqrt{2}$ for $\lambda_\pm = \pm 1$, as advertised above.

b) In any basis, a two-by-two Hermitian matrix H must obey the conditions that H_{00}, H_{11} be real, whereas $H_{01} = H_{10}$, i.e. the matrix in any basis is transpose conjugate. Therefore any Hermitian matrix can be expressed as in the problem statement with a, b, c, d real. Moreover, one can immediately check the validity of the equation below

$$\begin{pmatrix} a & c+id \\ c-id & b \end{pmatrix} = \frac{a+b}{2}I + \frac{a-b}{2}Z + cX - dY. \tag{3}$$

c) Clearly $[\sigma_i, \sigma_i] = 0$. We are left with three commutators to evaluate explicitly

$$
\begin{aligned}
[\sigma_1, \sigma_2] &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 2i\sigma_3, \\
[\sigma_2, \sigma_3] &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = 2i\sigma_1, \\
[\sigma_3, \sigma_1] &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 2i\sigma_2,
\end{aligned}
\tag{4}
$$

This finishes the proof of the commutation relations. Changing the order in the commutators above amounts to inverting the sign, which as indicated by the Levi-Civita tensor.

On to the anticommutators, all Pauli matrices square to the identity, so $\{\sigma_i, \sigma_i\} = I + I = 2I$, for each $i = 1, 2, 3$. For distinct pairs of Pauli matrices, one can simply revisit Eq. (4) and change from commutator to anticommutator to note that $\{\sigma_i, \sigma_j\} = 0$ whenever $i \neq j$.

One sees that the three Pauli matrices are traceless, and moreover, by evaluating the first term on the right-hand side of each of the three equations in Eq. (4), giving $\sigma_1\sigma_2, \sigma_2\sigma_3$ and $\sigma_3\sigma_1$, one finds indeed that $\mathrm{Tr}\{\sigma_i\sigma_j\} = 0$ whenever $i \neq j$. However, for all $i = 1, 2, 3$ $\mathrm{Tr}\{\sigma_i^2\} = \mathrm{Tr}\{I\} = 2$. These facts allow us to extract the coefficient of any one of the Pauli matrices, or of the identity, from the expression of the matrix in b). If we write (note we flipped the sign of d for convenience)

$$
H = \begin{pmatrix} a & c - id \\ c + id & b \end{pmatrix} = \frac{a+b}{2} I + \frac{a-b}{2} \sigma_3 + c\sigma_1 + d\sigma_2 \equiv \sum_{j=0}^{3} h_j \sigma_j, \tag{5}
$$

then

$$
\frac{1}{2} \mathrm{Tr}\{\sigma_i H\} = \frac{1}{2} \sum_{j=0}^{3} h_j \mathrm{Tr}\{\sigma_i \sigma_j\} = \sum_{j=0}^{3} h_j \delta_{ij} = h_i, \tag{6}
$$

where we have used $\mathrm{Tr}\{\sigma_i \sigma_j\} = 2\delta_{ij}$ for $i, j = 0, 1, 2, 3$.

d) Use implicit summation over repeated indices for $\hat{n} \cdot \vec{\sigma} = \hat{n}_i \sigma_i$ where the sum runs over $i = 1, 2, 3$. Just to eliminate any confusion about the notation, we write this matrix down explicitly once, although from now on we will mostly deal with the algebra of Pauli matrices, instead of their matrix representations, to prove identitites

$$\hat{n} \cdot \vec{\sigma} = \hat{n}_i \sigma_i \equiv \sum_{i=1}^{3} \hat{n}_i \sigma_i = \begin{pmatrix} \hat{n}_3 & \hat{n}_2 - i\hat{n}_3 \\ \hat{n}_2 + i\hat{n}_3 & -\hat{n}_3 \end{pmatrix}. \tag{7}$$

To prove that $(\hat{n} \cdot \vec{\sigma})^{2p} = I$ for $p \geq 0$, note that $(\hat{n}_i \sigma_i)^2 = \hat{n}_i \hat{n}_j \sigma_i \sigma_j = \hat{n}_i \hat{n}_j (\delta_{ij} I + i\epsilon_{ijk} \sigma_k) = I$, where we have used $\sigma_i \sigma_j = \delta_{ij} I + i\epsilon_{ijk} \sigma_k$ and the fact that $\hat{n}_i \hat{n}_j \delta_{ij} = 1$ since $\hat{n}$ is a unit vector by definition.

Now we may Taylor expand the exponent on the lhs of the identity to prove

$$e^{ia\hat{n} \cdot \vec{\sigma}} = \sum_{j=0}^{\infty} \frac{(ia\hat{n} \cdot \vec{\sigma})^j}{j!} = I \cos a + i\hat{n} \cdot \vec{\sigma} \sin a, \tag{8}$$

where in the last step we used the identity just proved above for the even powers of $\hat{n} \cdot \vec{\sigma}$, and reassembled the Taylor series of the trigonometric functions.

e) Using Einstein summation makes this derivation least tedious. We first use the result in d) to rewrite

$$\begin{aligned} R_n(-a)\vec{\sigma} R_n(a) \equiv e^{i\frac{a}{2}\hat{n} \cdot \vec{\sigma}} \vec{\sigma} e^{-i\frac{a}{2}\hat{n} \cdot \vec{\sigma}} &= \left(I \cos \frac{a}{2} + i\hat{n} \cdot \vec{\sigma} \sin \frac{a}{2}\right) \vec{\sigma} \left(I \cos \frac{a}{2} - i\hat{n} \cdot \vec{\sigma} \sin \frac{a}{2}\right), \\ R_n(-a)\sigma_\beta R_n(a) &= \left(I \cos \frac{a}{2} + i\hat{n}_\alpha \sigma_\alpha \sin \frac{a}{2}\right) \sigma_\beta \left(I \cos \frac{a}{2} - i\hat{n}_\gamma \sigma_\gamma \sin \frac{a}{2}\right). \end{aligned} \tag{9}$$

In the second row, we traded vector notation for explicit indices $\alpha, \beta, \gamma = 1, 2, 3$, with repeated indices summed over.

We now take the last row of the equation above, and remark that there are four terms to be evaluated

$$
\begin{aligned}
&\left(I\cos\frac{a}{2}+i\hat{n}_\alpha\sigma_\alpha\sin\frac{a}{2}\right)\sigma_\beta\left(I\cos\frac{a}{2}-i\hat{n}_\gamma\sigma_\gamma\sin\frac{a}{2}\right)\\
&=\sigma_\beta\cos^2\frac{a}{2}-i\cos\frac{a}{2}\sin\frac{a}{2}\hat{n}_\gamma\sigma_\beta\sigma_\gamma+i\sin\frac{a}{2}\cos\frac{a}{2}\hat{n}_\alpha\sigma_\alpha\sigma_\beta+\hat{n}_\alpha\hat{n}_\gamma\sin^2\frac{a}{2}\sigma_\alpha\sigma_\beta\sigma_\gamma\\
&=\sigma_\beta\cos^2\frac{a}{2}-i\cos\frac{a}{2}\sin\frac{a}{2}\hat{n}_\gamma(\delta_{\beta\gamma}I+i\epsilon_{\beta\gamma\delta}\sigma_\delta)+i\sin\frac{a}{2}\cos\frac{a}{2}\hat{n}_\alpha(\delta_{\alpha\beta}I+i\epsilon_{\alpha\beta\delta}\sigma_\delta)\\
&\quad+\hat{n}_\alpha\hat{n}_\gamma\sin^2\frac{a}{2}(\delta_{\alpha\beta}I+i\epsilon_{\alpha\beta\delta}\sigma_\delta)\sigma_\gamma\\
&=\sigma_\beta\cos^2\frac{a}{2}+\left(\cancel{-i\cos\frac{a}{2}\sin\frac{a}{2}\hat{n}_\beta}+\cancel{i\sin\frac{a}{2}\cos\frac{a}{2}\hat{n}_\beta}\right)I+\sin\frac{a}{2}\cos\frac{a}{2}\epsilon_{\beta\gamma\delta}\hat{n}_\gamma\sigma_\delta-\sin\frac{a}{2}\cos\frac{a}{2}\epsilon_{\alpha\beta\delta}\hat{n}_\alpha\sigma_\delta\\
&\quad+\hat{n}_\alpha\hat{n}_\gamma\sin^2\frac{a}{2}\delta_{\alpha\beta}\sigma_\gamma+i\hat{n}_\alpha\hat{n}_\gamma\sin^2\frac{a}{2}\epsilon_{\alpha\beta\delta}\sigma_\delta\sigma_\gamma\\
&=\cos^2\frac{a}{2}\sigma_\beta+2\sin\frac{a}{2}\cos\frac{a}{2}\epsilon_{\beta\gamma\delta}\hat{n}_\gamma\sigma_\delta+\hat{n}_\beta\hat{n}_\gamma\sin^2\frac{a}{2}\sigma_\gamma+i\hat{n}_\alpha\hat{n}_\gamma\sin^2\frac{a}{2}\epsilon_{\alpha\beta\delta}(\delta_{\delta\gamma}I+i\epsilon_{\delta\gamma\mu}\sigma_\mu)\\
&=\cos^2\frac{a}{2}\sigma_\beta+\sin a\,\epsilon_{\beta\gamma\delta}\hat{n}_\gamma\sigma_\delta+\hat{n}_\beta\hat{n}_\gamma\sin^2\frac{a}{2}\sigma_\gamma+i\hat{n}_\alpha\hat{n}_\gamma\sin^2\frac{a}{2}\epsilon_{\alpha\beta\delta}\delta_{\delta\gamma}I+i\hat{n}_\alpha\hat{n}_\gamma\sin^2\frac{a}{2}\epsilon_{\alpha\beta\delta}i\epsilon_{\delta\gamma\mu}\sigma_\mu\\
&=\cos^2\frac{a}{2}\sigma_\beta+\sin a\,\epsilon_{\beta\gamma\delta}\hat{n}_\gamma\sigma_\delta+\hat{n}_\beta\hat{n}_\gamma\sin^2\frac{a}{2}\sigma_\gamma+\cancel{i\hat{n}_\alpha\hat{n}_\gamma\sin^2\frac{a}{2}\epsilon_{\alpha\beta\gamma}I}-\hat{n}_\alpha\hat{n}_\gamma\sin^2\frac{a}{2}\epsilon_{\alpha\beta\delta}\epsilon_{\delta\gamma\mu}\sigma_\mu\\
&=\cos^2\frac{a}{2}\sigma_\beta+\sin a\,\epsilon_{\beta\gamma\delta}\hat{n}_\gamma\sigma_\delta+\hat{n}_\beta\hat{n}_\gamma\sin^2\frac{a}{2}\sigma_\gamma-\hat{n}_\alpha\hat{n}_\gamma\sin^2\frac{a}{2}(\delta_{\alpha\gamma}\delta_{\beta\mu}-\delta_{\alpha\mu}\delta_{\beta\gamma})\sigma_\mu\\
&=\cos^2\frac{a}{2}\sigma_\beta+\sin a\,\epsilon_{\beta\gamma\delta}\hat{n}_\gamma\sigma_\delta+\hat{n}_\beta\hat{n}_\gamma\sin^2\frac{a}{2}\sigma_\gamma-\hat{n}_\alpha\hat{n}_\gamma\sin^2\frac{a}{2}\delta_{\alpha\gamma}\delta_{\beta\mu}\sigma_\mu+\hat{n}_\alpha\hat{n}_\gamma\sin^2\frac{a}{2}\delta_{\alpha\mu}\delta_{\beta\gamma}\sigma_\mu\\
&=\cos^2\frac{a}{2}\sigma_\beta+\sin a\,\epsilon_{\beta\gamma\delta}\hat{n}_\gamma\sigma_\delta+\hat{n}_\beta\hat{n}_\gamma\sin^2\frac{a}{2}\sigma_\gamma-\sin^2\frac{a}{2}\sigma_\beta+\hat{n}_\alpha\hat{n}_\beta\sin^2\frac{a}{2}\sigma_\alpha\\
&=\cos^2\frac{a}{2}\sigma_\beta+\sin a\,\epsilon_{\beta\gamma\delta}\hat{n}_\gamma\sigma_\delta+2\sin^2\frac{a}{2}\hat{n}_\beta\hat{n}_\gamma\sigma_\gamma-\sin^2\frac{a}{2}\sigma_\beta.
\end{aligned}
\tag{10}
$$

Above we have contracted two Levi-Civita tensors using $\epsilon_{\alpha\beta\gamma}\epsilon_{ij\gamma}=\delta_{\alpha i}\delta_{\beta j}-\delta_{\alpha j}\delta_{\beta i}$.

Switching to vector notation again, and using trigonometric identities $\cos^2 a/2-\sin^2 a/2=\cos a$, $1-2\sin^2 a/2=\cos a$, we have

$$R_n(-a)\vec{\sigma}R_n(a)\equiv e^{i\frac{a}{2}\hat{n}\cdot\vec{\sigma}}\vec{\sigma}e^{-i\frac{a}{2}\hat{n}\cdot\vec{\sigma}}=\cos a\;\vec{\sigma}+\sin a\;\hat{n}\times\vec{\sigma}+\hat{n}(\hat{n}\cdot\vec{\sigma})(1-\cos a). \tag{11}$$

If $\hat{n}=(0,1,0)$, then

$$R_n(-\pi/2)\sigma_1R_n(\pi/2)\equiv e^{i\frac{\pi}{4}\sigma_2}\sigma_1e^{-i\frac{\pi}{4}\sigma_2}=\cos\pi/2\;\sigma_1+\sin\pi/2\;(\hat{n}\times\vec{\sigma})_1=\sigma_3. \tag{12}$$

II. HERMITIAN AND UNITARY OPERATORS (20 POINTS)

a) Prove that all eigenvalues of a unitary operator have norm 1, and therefore can be written as $e^{i\theta}$ for some real number θ.

b) Show that all Pauli matrices are unitary.

c) Prove that two eigenvectors of a Hermitian operator with different eigenvalues are necessarily orthogonal.

d) Prove that any eigenvalue of a projector P can be either 0 or 1. Can a projector also be unitary?

e) Use the spectral decomposition theorem to find the natural logarithm of the matrix $\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$.

SOLUTION

a) Suppose we have one eigenvector of unitary U, i.e. $U|i\rangle = \lambda_i |i\rangle$. Conjugation gives $\langle i| U^\dagger = \lambda_i^* \langle i|$. Multiplying the second equation by the first gives $\langle i| U^\dagger U |i\rangle = \lambda_i^* \lambda_i \langle i|i\rangle$. On the lhs we use the fact that $U^\dagger U = I$ by unitarity, and the fact that the eigenvector is normalized $\langle i|i\rangle = 1$. So $|\lambda_i|^2 = 1$, which proves the statement.

b) All Pauli matrices are Hermitian $\sigma_i = \sigma_i^\dagger$, i=0,1,2,3 (identity included). Moreover, they all square to the identity I, so they are unitary.

c) Consider two nondegenerate eigenvectors of a Hermitian operator $H|i\rangle = \lambda_i |i\rangle$ and $H|j\rangle = \lambda_j |j\rangle$. We first multiply the first equation by the dual $\langle i|$ to get $\langle i| H |i\rangle = \lambda_i \langle i|i\rangle = \lambda_i$. Since $H = H^\dagger$, we can show that the diagonal matrix element is real $\langle i| H |i\rangle = \langle i| H^\dagger |i\rangle^* = \langle i| H |i\rangle^*$, so λ_i is real. Thus all eigenvalues of a Hermitian operator are real.

We now multiply the first equation by the dual $\langle j|$ to get $\langle j| H |i\rangle = \lambda_i \langle j|i\rangle$, and the second equation by the dual $\langle i|$ to get $\langle i| H |j\rangle = \lambda_j \langle i|j\rangle$. We subtract from the former the conjugate of the latter relation $\langle j| H |i\rangle = \lambda_j \langle j|i\rangle$ to get $0 = (\lambda_i - \lambda_j)\langle j|i\rangle$. Since $\lambda_i \neq \lambda_j$ by hypothesis, $\langle i|j\rangle = 0$.

d) A projector satisfies $P^2 = P$. Let $P|i\rangle = \lambda_i |i\rangle$ for some eigenvector $|i\rangle$. Then a second application of P to the eigenvalue equation gives $PP|i\rangle = \lambda_i P|i\rangle$. On the lhs, we use $PP = P$, and on the rhs, we use the eigenvalue equation itself, to get $P|i\rangle = \lambda_i^2 |i\rangle$. So $\lambda_i^2 = \lambda_i$, or $\lambda_i(\lambda_i - 1) = 0$, so the eigenvalue λ_i can be 0 or 1.

Projectors are Hermitian. If a projector is unitary, then $PP^\dagger = P^\dagger P = I$. Replacing the Hermitian conjugate by the projector itself, $P^2 = I$. But by the projector property, $P^2 = P$. So $P = I$. The only projector that is also unitary is the identity operator, all of whose eigenvalues are 1 (there are no zero eigenvalues). Indeed, if one of the eigenvalues of a projector is 0, then it is not invertible, so it cannot be unitary.

e) The eigenvalues of this matrix derive from the characteristic equation

$$\begin{vmatrix} 4-\lambda & 3 \\ 3 & 4-\lambda \end{vmatrix} = 0, \tag{13}$$

i.e. $(4-\lambda)^2 - 3^2 = 0$, so $4-\lambda = \pm 3$, with two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 7$. The corresponding eigenvectors must be normalized and obey the following equation

$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}, \tag{14}$$

equivalently one has to solve the equation

$$(4-\lambda_{1,2})u_{1,2} + 3v_{1,2} = 0. \tag{15}$$

For λ_1, we have $u_1 = -v_1$, whereas for λ_2, we have $u_2 = v_2$, leading to eigenvectors $(-1/\sqrt{2}, 1/\sqrt{2})^T$ for $\lambda_1 = 1$ and $(1/\sqrt{2}, 1/\sqrt{2})^T$ for $\lambda_2 = 7$. We may evaluate their respective projectors

$$\begin{aligned} P_1 = |1\rangle\langle 1| &= \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}, \\ P_2 = |2\rangle\langle 2| &= \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}. \end{aligned} \tag{16}$$

This allows us to write the spectral decomposition of the matrix as

$$\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix} = 1P_1 + 7P_2, \tag{17}$$

and then evaluate any function of the matrix

$$f\left(\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}\right) = f(1)P_1 + f(7)P_2. \tag{18}$$

III. TENSOR PRODUCTS (20 POINTS)

a) Show that transpose, complex conjugation, and adjoint distribute over the tensor product, that is $(A \otimes B)^* = A^* \otimes B^*, (A \otimes B)^T = A^T \otimes B^T$, and $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.

b) Show that $\exp(A_1 \otimes I_2 + I_1 \otimes A_2) = \exp(A_1) \otimes \exp(A_2)$ for some linear operators $A_{1,2}$ acting on Hilbert spaces $V_{1,2}$, and $I_{1,2}$ the respective identity operators.

c) Find the eigenvalues and eigenvectors of $X \otimes Z$, using the definitions of Pauli operators from a previous exercise.

d) Prove that $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$ cannot be expressed as a tensor product $|a\rangle \otimes |b\rangle$ for some pair of vectors $|a\rangle$ and $|b\rangle$ [we take the first (second) entry of the tensor products above to be a vector in some vector space V_1 (V_2)].

SOLUTION

These properties are tedious to prove but straightforward in the Kronecker product form of the tensor product

$$A \otimes B = \begin{pmatrix} A_{11}B & \dots & A_{1n}B \\ \vdots & & \vdots \\ A_{m1}B & \dots & A_{mn}B \end{pmatrix}. \tag{19}$$

For conjugation,

$$(A \otimes B)^* = \begin{pmatrix} A_{11}B & \dots & A_{1n}B \\ \vdots & & \vdots \\ A_{m1}B & \dots & A_{mn}B \end{pmatrix}^* = \begin{pmatrix} A_{11}^*B^* & \dots & A_{1n}^*B^* \\ \vdots & & \vdots \\ A_{m1}^*B^* & \dots & A_{mn}^*B^* \end{pmatrix} = A^* \otimes B^*. \tag{20}$$

For transposition,

$$(A \otimes B)^T = \begin{pmatrix} A_{11}B & \dots & A_{1n}B \\ \vdots & & \vdots \\ A_{m1}B & \dots & A_{mn}B \end{pmatrix}^T = \begin{pmatrix} A_{11}B^T & \dots & A_{n1}B^T \\ \vdots & & \vdots \\ A_{1m}B^T & \dots & A_{nm}B^T \end{pmatrix} = A^T \otimes B^T. \tag{21}$$

For transpose-conjugation, the property to prove is a consequence of the previous two.

b) $\exp(A_1 \otimes I_2 + I_1 \otimes A_2) = \exp(A_1 \otimes I_2)\exp(I_1 \otimes A_2)$ since $[A_1 \otimes I_2, I_1 \otimes A_2] = (A_1 \otimes I_2)(I_1 \otimes A_2) - (I_1 \otimes A_2)(A_1 \otimes I_2) = A_1 \otimes A_2 - A_1 \otimes A_2 = 0$. Now $\exp(A_1 \otimes I_2) = \exp(A_1) \otimes I_2$, since $\exp(A_1 \otimes I_2) = \sum_{k=0}^{\infty} \frac{(A_1 \otimes I_2)^k}{k!} = \sum_{k=0}^{\infty} \frac{A_1^k \otimes I_2^k}{k!} = \sum_{k=0}^{\infty} \frac{A_1^k}{k!} \otimes I_2$. Similarly, $\exp(I_1 \otimes A_2) = I_1 \otimes \exp(A_2)$. We can then write our original expression as $\exp(A_1 \otimes I_2 + I_1 \otimes A_2) = \exp(A_1 \otimes I_2)\exp(I_1 \otimes A_2) = [\exp(A_1) \otimes I_2][I_1 \otimes \exp(A_2)] = \exp(A_1) \otimes \exp(A_2)$, which concludes the proof.

c) The quicker way to do this is to notice that the problem is separable (we have to calculate the eigenstates of a tensor product). We know from the first problem here that $X = |+\rangle\langle+| - |-\rangle\langle-|$, with $|\pm\rangle = \frac{|0\rangle\pm|1\rangle}{\sqrt{2}}$, and $Z = |1\rangle\langle 1| - |0\rangle\langle 0|$ so $X \otimes Z = (|+\rangle\langle+| - |-\rangle\langle-|) \otimes (|1\rangle\langle 1| - |0\rangle\langle 0|) = |+,1\rangle\langle+,1| + |-,0\rangle\langle-,0| - |-,1\rangle\langle-,1| - |+,0\rangle\langle+,0|$, where we have denoted $|\pm,0\rangle \equiv |\pm\rangle \otimes |0\rangle$ and $|\pm,1\rangle \equiv |\pm\rangle \otimes |1\rangle$. Then the eigenvalues and vectors can be read off from the spectral decomposition.

The brute force way to solve this is using the Kronecker product Eq. (19), we have

$$X \otimes Z = \begin{pmatrix} 0 & Z \\ Z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{22}$$

The characteristic polynomial is biquadratic

$$\begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & -1 \\ 1 & 0 & -\lambda & 0 \\ 0 & -1 & 0 & -\lambda \end{vmatrix} = \lambda^4 - 2\lambda^2 + 1 = 0, \tag{23}$$

with solutions $\lambda_{\alpha\beta} = (-1)^\alpha(-1)^\beta$ for $\alpha, \beta = 0, 1$. One would then solve for the eigenvectors. This is too much work. In quantum mechanics, exploiting separability goes a long way in simplifying a problem. Above, we only exploited the diagonalization of the 2×2 Pauli matrix to build the solution. Here we have avoided the explicit diagonalization of a four by four matrix. We would not be able to do this, if, for example, we were asked for the eigenvalues and eigenvectors of, say, $Z \otimes X + X \otimes Y$.

d) Write $|a\rangle = a_0|0\rangle + a_1|1\rangle + \ldots$ and $|b\rangle = b_0|0\rangle + b_1|1\rangle + \ldots$ in their respective spaces V_1 and V_2. Then $|a\rangle \otimes |b\rangle = \sum_{ij} a_i b_j |i\rangle \otimes |j\rangle$. If $|\psi\rangle = |a\rangle \otimes |b\rangle$, then the following must hold for the coefficients a_i, b_j

$$\begin{aligned} a_0 b_0 &= 1/\sqrt{2}, \\ a_1 b_1 &= 1/\sqrt{2}, \\ a_0 b_1 &= 0, \\ a_1 b_0 &= 0, \\ a_i, b_i &= 0, i \geq 2. \end{aligned} \tag{24}$$

One clearly sees that the first four conditions cannot hold simultaneously. If the third condition holds, then either a_0 or b_1 must vanish, so either the first condition Eq. (24) is false, or the second condition. We have reached a contradiction. So $|\psi\rangle$ is not a tensor product state.

Alternative proof. Assume that $|\psi\rangle = |a\rangle \otimes |b\rangle$ in some basis. Then clearly taking a partial trace with respect to V_1 would give the reduced density matrix $\rho_1 = |a\rangle\langle a|$. This density matrix has the property that it is a projector, i.e. $\rho_1^2 = \rho_1$. Looking at the partial trace with respect to V_2 of $|\psi\rangle\langle\psi|$ we get $\mathrm{Tr}_2\{|\psi\rangle\langle\psi|\} = \rho_1' = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$. This matrix is *not* a projector, for $\rho_1'^2 = \frac{1}{2}\rho_1'$. So it cannot be that $|\psi\rangle = |a\rangle \otimes |b\rangle$ in some basis.