Mathematical methods for quantum engineering
Exercise on Lagrangian, Hamiltonian, classical/quantum correspondence
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Take a spherical punctual pendulum of mass m, Cartesian coordinates (x, y, z) and rotating around the origin $(0, 0, 0)$ in the gravity field of acceleration g along the vertical ascending axis z. A possible choice of configuration variables is (x, y) with $z = -\sqrt{\ell^2 - x^2 - y^2}$ since we assume here that pendulum moves on the sphere of radius ℓ and remains under the equator ($z < 0$).

1. Show that its potential energy reads $-mg\sqrt{\ell^2 - x^2 - y^2}$
2. Show that $\dot{z} = -\frac{x\dot{x}+y\dot{y}}{\sqrt{\ell^2-x^2-y^2}}$ and deduce that its kinetic energy reads
$$\frac{m}{2}\left(\ell^2(\dot{x}^2 + \dot{y}^2) + \frac{(x\dot{x} + y\dot{y})^2}{\ell^2 - x^2 - y^2}\right)$$
3. Deduce the Lagrangian.
4. Derive the first-order approximate dynamics around the equilibrium $(x, y) = 0$, compute the eigenvalues and discuss its stability (hint: use the quadratic approximation of the Lagrangian).
5. Take the quadratic Lagrangian of previous question and derive the corresponding quadratic Hamiltonian.
6. From this quadratic Hamiltonian, derive the quantisation via the correspondence principle. Show that the average value $\langle\psi|\hat{x}|\psi\rangle$ (resp. $\langle\psi|\hat{y}|\psi\rangle$) of operator $\hat{x}$ (resp. $\hat{y}$) satisfies a second-order linear differential scalar equation.
7. Assume that the system is rotating around the vertical axis with rotational velocity Ω as a Foucault pendulum on the north pole.
   (a) Show that the kinetic energy reads now
   $$\frac{m}{2}\left(\ell^2(\dot{x}^2 + \dot{y}^2) + \frac{(x\dot{x} + y\dot{y})^2}{\ell^2 - x^2 - y^2} + 2\Omega(y\dot{x} - x\dot{y}) + \Omega^2(x^2 + y^2)\right).$$
   (b) Derive the quadratic approximation of the Lagrangian around the equilibrium $(x, y) = 0$, deduce the corresponding quadratic Hamiltonian.
   (c) From this quadratic Hamiltonian, derive the quantisation via the correspondence principle. Show that the average values of $\hat{x}$ and satisfy second-order linear differential scalar equations.