Master 1 on Quantum Engineering 2023/24
Mathematical methods for quantum engineering
Dynamical systems and control

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These notes are just an introduction to classical dynamical systems, i.e. study of solutions of Ordinary Differential Equations (ODE), and their feedback control. Much more materials can be found in the books [3, 1] for ODE's and in [2] for the design of Proportional Integral Derivative (PID) regulators widely used in experimental and industrial plants.

1 Dynamical systems

1.1 Initial value problem

Consider the finite dimensional system described by a set of n scalar ordinary differential equations

$$\frac{d}{dt}x_i = f_i(x, u(t)), \quad i = 1, \ldots, n \tag{1}$$

where $x = (x_1, \ldots, x_n)^T$ is the state vector, $u = (u_1, \ldots, u_m)$ the control input vector that are a priori time-dependent, the vector function $f = (f_1, \ldots, f_n)$ is a function of (x, u). System (1) admits the compact expression :

$$\frac{d}{dt}x = f(x, u).$$

The following mathematical result ensures the existence and uniqueness of the following initial value problem.

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Theorem 1 (Initial value Cauchy problem)**.** *Take* (1) *and assume that* f *is continuous and piece-wise differentiable function of* x *and* u*. Then for any initial condition* x^0 *and any piece-wise continuous function of time* $u(t)$*, exists* $T > 0$ *and a unique solution piece-wise time-differentiable function* $[0, T] \ni t \mapsto x(t)$ *of the following Cauchy problem*

$$\frac{d}{dt}x(t) = f(x(t), u(t)), \quad x(0) = x^0.$$

When the norm of $f(x, u)$ *can be bounded by* $c_0(u) + c_1(u)\|x\|$ *with* c_0 *and* c_1 *piece-wise continuous positive functions of* u*,* T *can be set to* $+\infty$ *and the unique solution* $x(t)$ *is defined for any positive time* t*.*

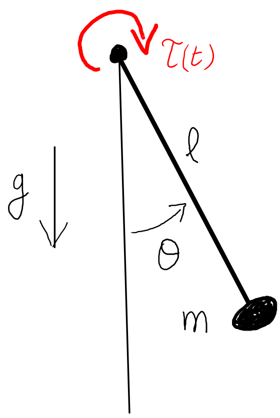

Figure 1 – A punctual pendulum of mass m, length ℓ in a vertical plane, submitted to gravity g, viscous friction with parameter $\lambda \geq 0$, and an external torque $\tau(t)$.

Example 1 (Controlled pendulum with friction)**.** *A punctual pendulum of mass* m*, length* ℓ *in a vertical plane, submitted to gravity* g *and an external torque* $\tau(t) = m\ell^2 u(t)$*, is governed by the following first-order nonlinear differential system*

$$\frac{d}{dt}\theta = \omega, \quad \frac{d}{dt}\omega = -\tfrac{g}{l}\sin\theta - \lambda\omega + u(t), \tag{2}$$

with control input $u(t)$ *and friction coefficient* $\lambda \geq 0$*. The data corresponding to initial value problem are*

$$(\theta(0), \omega(0)), \qquad u(t) \textit{ for } t \geq 0.$$

In this example the state $x = (x_1, x_2)$ *corresponds to* $x_1 = \theta$ *and* $x_2 = \omega$*,*

$$f(x,u) = \left(\begin{array}{c} x_2 \\ -\frac{g}{l}\sin x_1 - \lambda x_2 + u \end{array} \right).$$

Its solutions are defined for all time $t > 0$*, whatever the initial conditions are.*

1.2 Numerical simulation : the Euler scheme

Numerical simulations of initial value problems are based on numerical integration methods of differential equations. The simplest method is the explicit Euler scheme. To compute an approximation of $x(T)$ from x^0 and $u(t)$ for (1) the Euler scheme is the following. Take N a large integer and consider the following recurrence formula on integers k from 0 to $N-1$:

$$x^{(k+1)} = x^{(k)} + T/N f(x^{(k)}, u(kT/N)) \text{ with } x^{(0)} = x^0$$

providing after N iterations an approximation of $x(T)$ as $x^{(N)}$ from x^0 and the control input u on $[0,T]$. This comes from the general convergence result : under the condition of theorem 1, one has, when the unique solution of (1) exists on $[0,T]$,

$$\lim_{N \mapsto +\infty} x^{(N)} = x(T).$$

This means that, once the finite time T is given, one has to use N large enough to obtain an approximation of $x(T)$. An efficient and elementary verification that N is chosen large enough is as follows : check that with $2N$ instead of N, the resulting approximate value of $x(T)$ admits negligible deviation from the one obtained with N.

1.3 First variation around a trajectory

When f depends smoothly on its arguments x and u, it is possible to derive, around a solution of (1), $t \mapsto (x(t), u(t))$, its first variation. Up second order terms, it provides an estimate of the impact of small perturbations on initial state $x^0 + \delta x^0$ and on control input $u(t) + \delta u(t)$ for the perturbed solution $x(t) + \delta x(t)$. This first variation δx is solution of the following linear time-varying initial value problem

$$\frac{d}{dt}\delta x(t) = A(t)\ \delta x(t) + B(t)\ \delta u(t), \quad \delta x(0) = \delta x^0 \tag{3}$$

where A and B are the following $n \times n$ and $n \times m$ Jacobian matrices :

$$A_{i,j}(t) = \frac{\partial f_i}{\partial x_j}(x(t), u(t)) \text{ and } B_{i,j}(t) = \frac{\partial f_i}{\partial u_j}(x(t), u(t)).$$

For steady state trajectories, i.e. solution $x(t) = \bar{x}$ and $u(t) = \bar{u}$ where $f(\bar{x}, \bar{u}) = 0$. Then A and B are constant : up to higher order corrections, the solution around such steady-state (equilibrium) obeys to a linear time-invariant system $\frac{d}{dt}\delta x = A\delta x + B\delta u$.

Example 2 (Controlled pendulum with friction (continued))**.** *The first variation of* (2) *around the trajectory $x(t) = (\theta(t), \omega(t))$ and $u(t)$ is thus*

$$\frac{d}{dt}\delta\theta = \delta\omega, \quad \frac{d}{dt}\delta\omega = -\tfrac{g}{l}\cos\theta(t)\delta\theta - \lambda\delta\omega + \delta u$$

with $A(t) = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l}\cos\theta(t) & -\lambda \end{pmatrix}$ *and* $B(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

For the equilibrium $\bar{x} = (0, 0)$ and $\bar{u} = 0$, this give

$$\frac{d}{dt}\delta\theta = \delta\omega, \quad \frac{d}{dt}\delta\omega = -\tfrac{g}{l}\delta\theta - \lambda\delta\omega + \delta u \tag{4}$$

with $A = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -\lambda \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

For the equilibrium $\bar{x} = (\pi, 0)$ and $\bar{u} = 0$, this give

$$\frac{d}{dt}\delta\theta = \delta\omega, \quad \frac{d}{dt}\delta\omega = +\tfrac{g}{l}\delta\theta - \lambda\delta\omega + \delta u \tag{5}$$

with $A = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & -\lambda \end{pmatrix}$ *and* $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

1.4 Lyapunov function and equilibrium stability

Consider $\frac{d}{dt}x = f(x, u)$ with $u = \bar{u}$ constant. Consider an equilibrium state $\bar{x}$ associated to $\bar{u}$: $f(\bar{x}, \bar{u}) = 0$.

Equilibrium (also called steady-state) $\bar{x}$ is stable (in the Lyapunov sense), if and only if, for any $\epsilon > 0$, exists $\eta_\epsilon \in]0, \epsilon]$ such that, any trajectory starting in ball of center $\bar{x}$ and radius η_ϵ is defined for all time $t > 0$ and remains in the ball of center $\bar{x}$ and of radius ϵ.

An equilibrium is said locally asymptotically stable, if and only if it is stable and moreover exists $\eta > 0$ such that any trajectory $x(t)$ starting in ball of center $\bar{x}$ and radius η is defined for all $t > 0$ and $\lim_{t \mapsto +\infty} x(t) = \bar{x}$.

When $\eta = +\infty$, the equilibrium is said global asymptotically stable. In this case all trajectories converge to $\bar{x}$ when t tends to $+\infty$.

Investigating stability can be done via Lyapunov functions $V(x)$, real values functions of x that decreases along trajectories, i.e. such that $\frac{d}{dt}V(x) = D_x V(x) f(x, \bar{u}) \leq 0$ for any x. Here $D_x V$ is formed by the first partial derivatives of V versus the component of x :

$$\frac{d}{dt}V(x) \equiv \sum_i \frac{\partial V}{\partial x_i}(x) f_i(x, \bar{u}).$$

Lyapunov function have here a key role for dissipative systems, as first integrals for conservative systems governed by Hamilton equations.

One has the following global result

Theorem 2 (Lasalle's invariance principe)**.** *Take $\frac{d}{dt}x = f(x, \bar{u})$ with f smooth function of x, $V(x)$ a smooth real-value functions of x, non-negative and tending to $+\infty$ when the length of x tends to $+\infty$. Assume that for all x, $\frac{d}{dt}V(x) \leq 0$. Then all trajectories are defined for $t > 0$ and they converge to the largest invariant set satisfying $\frac{d}{dt}V = 0$. This largest set is formed by the sub-set of trajectories $x(t)$ satisfying the set of $\dim x + 1$ scalar equations :*

$$\frac{d}{dt}x = f(x, \bar{u}), \quad \sum_i \frac{\partial V}{\partial x_i}(x) f_i(x, \bar{u}) = 0.$$

Example 3 (Controlled pendulum with friction (continued))**.** *Consider (2) with $\bar{u} = 0$ and the mechanical energy $V = -\frac{g}{l}\cos\theta + \omega^2/2$. Then V is a Lyapunov since*

$$\frac{d}{dt}V = -\lambda\omega^2 \leq 0.$$

Since θ is periodic and V tends to infinity when $|\omega|$ tends to infinity, all trajectories of (2) with $u = 0$ converge to the solution of the over-determined system

$$\frac{d}{dt}\theta = \omega, \quad \frac{d}{dt}\omega = -\tfrac{g}{l}\sin\theta - \lambda\omega, \quad \frac{d}{dt}V = -\lambda\omega^2 = 0.$$

For $\lambda > 0$, the above system admits two solutions corresponding to the equilibria given by $\sin\bar{\theta} = 0$, i.e. $\bar{\theta} = 0$ and $\bar{\theta} = \pi$

One has the following local result.

Theorem 3 (Lyapunov functions around an equilibrium)**.** *Take $\frac{d}{dt}x = f(x, \bar{u})$ with f smooth function of x, an equilibrium $\bar{x}$ ($f(\bar{x}, \bar{u}) = 0$) and $V(x)$ a* *smooth real-value functions of x defined on a ball of radius $\epsilon > 0$ around $\bar{x}$. Assume that for all x in this ball, $\frac{d}{dt}V(x) \leq 0$ and that $\bar{x}$ is a strictly minima of V on this ball. Then $\bar{x}$ is a stable equilibrium. If, moreover, $\frac{d}{dt}V(x) < 0$ for all x in this ball excepted at $\bar{x}$, then $\bar{x}$ is locally asymptotically stable.*

Example 4 (Controlled pendulum with friction (continued)). *For (2), its energy $V = -\frac{g}{l}\cos\theta + \omega^2/2$ is minimal at $\theta = 0$ and $\omega = 0$. This minimum is strict, thus the equilibrium $\bar{\theta} = 0$ is stable. One cannot conclude with the above theorem since $\frac{d}{dt}V < 0$ is not strictly negative for $x \neq 0$ around $(0, 0)$.*

Theorem 4 (Stability of equilibrium based on linear tangent approximation). *Take $\frac{d}{dt}x = f(x, \bar{u})$ with f smooth function of x and an equilibrium $\bar{x}$ ($f(\bar{x}, \bar{u}) = 0$). Consider its first variation around $\bar{x}$: $\frac{d}{dt}\delta x = A\delta x$ where A is the Jacobian matrix of f at $\bar{x}$:*

$$A = \left(\frac{\partial f_i}{\partial x_j}(\bar{x}, \bar{u})\right)_{i,j=1,\dots,n}.$$

Then the following implications are satisfied

- *if the real parts of the eigenvalues of A are strictly negative, then $\bar{x}$ is locally asymptotically stable. One says in this case that it is exponentially stable, since locally, the convergence is exponential in t.*
- *if the real part of one or several eigen-values of A is strictly positive , then $\bar{x}$ is unstable (exponentially unstable).*

One says that A is stable or Hurwitz when all its eigenvalues have a strictly negative real-part.

Example 5 (Controlled pendulum with friction (continued)). *For the equilibrium $\bar{\theta} = 0$ one has $A = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -\lambda \end{pmatrix}$. For $\lambda > 0$, this 2×2 real matrix is stable since Trace$(A) = -\lambda < 0$ and $\det(A) = \frac{g}{l} > 0$. Thus this equilibrium is exponentially stable.*

For the equilibrium $\bar{\theta} = \pi$ one has $A = \begin{pmatrix} 0 & 1 \\ \frac{g}{l} & -\lambda \end{pmatrix}$. For $\lambda > 0$, this 2×2 real matrix is unstable since Trace$(A) = -\lambda < 0$ and $\det(A) = -\frac{g}{l} < 0$. Thus this equilibrium is exponentially unstable. It is a saddle since one eigenvalue is strictly positive (exponential divergence along the corresponding eigenvector) and the other one is strictly negative (exponential convergence along the corresponding eigenvector).

1.5 Slow/fast systems, adiabatic elimination and model reduction

Assume now that the state x is made of two sets of components $x = (x_s, x_f)$ where x_s corresponds to the slow sub-state of dimension n_s and x_f to the fast one of dimension n_f with $n = n_s + n_f$. This decomposition is based on the following slow/fast dynamics

$$\frac{d}{dt}x_s = \epsilon f_s(x_s, x_f, \bar{u}) \tag{6}$$

$$\frac{d}{dt}x_f = f_f(x_s, x_f, \bar{u}) \tag{7}$$

$$\tag{8}$$

where $f = (\epsilon f_s, f_f)$ with $0 < \epsilon \ll 1$. Following the so-called quasi-static approximation, one can eliminate adiabatically x_f : it converges after a transient of length 1 to a steady $\bar{x}_f$ depending on x_s through $f_f(x_s, \bar{x}_f, \bar{u}) = 0$. Roughly speaking such approximation is valid as soon as, for x_s constant $\bar{x}_f$ is an exponentially stable equilibrium of the fast sub-system $\frac{d}{dt}x_f = f_f(x_s, x_f, \bar{u})$, i.e., the eigenvalues of the $n_f \times n_f$ fast Jacobian matrix $\frac{\partial f_f}{\partial x_f}$ evaluated at $(x_s, \bar{x}_f, \bar{u})$ are with strictly negative real parts. To get then the reduced slow dynamics of state x_s, one has to express $\bar{x}_f$ as a function of x_s and $\bar{u}$, $\bar{x}_f = g_f(x_s, \bar{u})$ and plug this into the dynamics of x_s :

$$\frac{d}{dt}x_s = \epsilon f_s(x_s, g(x_s, \bar{u}), \bar{u}) \triangleq \epsilon g_s(x_s, \bar{u}). \tag{9}$$

This means that the trajectories of the full slow/fast system (6) starting from (x_s^0, x_f^0) at $t = 0$ can be approximated by the trajectories of the reduced slow system (9) starting from x_s^0 after a possible fast initial transient on a time-scale of length 1 when the initial value x_f^0 of x_f is not equal to $g_f(x_s, \bar{u})$, i.e., doesn't satisfy the boundary layer condition $f_f(x_f^0, x_s^0, \bar{u}) = 0$.

The theory behind such adiabatic approximation for slow/fast system model reduction relies on the so-called Tikhonov theorem and singular perturbation theory of ODE's.

Example 6 (Controlled pendulum with friction (continued))**.** *In practice the dynamics (2) of the pendulum with control input u neglect the dynamics of the actuator, an electrical motor generating a torque from a voltage input. By design, such dynamics are fast and stable. Typically, they can be described by the simple first-order filter with input voltage proportional to $\bar{u}$ and*

$$\frac{d}{dt}u_f = (\bar{u} - u_f)/\epsilon$$

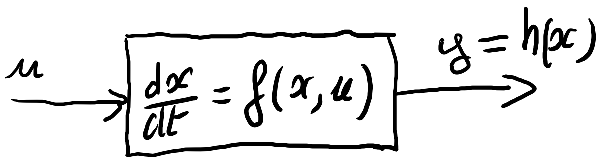

FIGURE 2 – Open-loop control system with control input u, state x and measured output y, related by the causal dynamics $\frac{d}{dt}x = f(x,u)$ with $y = h(x)$.

where ϵ is a small positive time constant corresponding to the motor bandwidth (RL electrical circuit) and u_f proportional to the effective electromagnetic torque applied to the pendulum. This means that (2) is the slow reduced model derived from a more accurate slow/fast system including the actuator dynamics

$$\frac{d}{dt}\theta = \omega, \quad \frac{d}{dt}\omega = -\tfrac{g}{l}\sin\theta - \lambda\omega + u_f, \quad \frac{d}{dt}u_f = (\bar{u} - u_f)/\epsilon.$$

Setting the slow time $\tau = t/\epsilon$, one recovers the above (6) with $x_s = (\theta, \omega)$, $x_f = u_f$, $dx_s/d\tau = \epsilon\frac{d}{dt}x_s$ and $dx_f/d\tau = \bar{u} - x_f$.

2 Control systems

2.1 Open-loop/closed-loop dynamics

Open-loop dynamics illustrated on figure 2 correspond in fact to an initial value problem when the given data corresponds to the initial state of the system $x^0 = x(0)$ and the open-loop control $u(t)$ for $t \geq 0$:

$$\frac{d}{dt}x = f(x,u), \quad y = h(x)$$

where x is the state vector of dimension n (n could be infinite for partial differential systems), u the control-input vector of dimension m and y the measured-output vector of dimension p. Here f and h are arbitrary functions of x and u ($\dim f = n$ and $\dim h = p$).

Under mild conditions when n is finite (see theorem 1), these data with the dynamical model, a set of ordinary/partial differential equations derived from physical conservation laws, defines then a unique open-loop state trajectory $x(t)$ for $t > 0$.

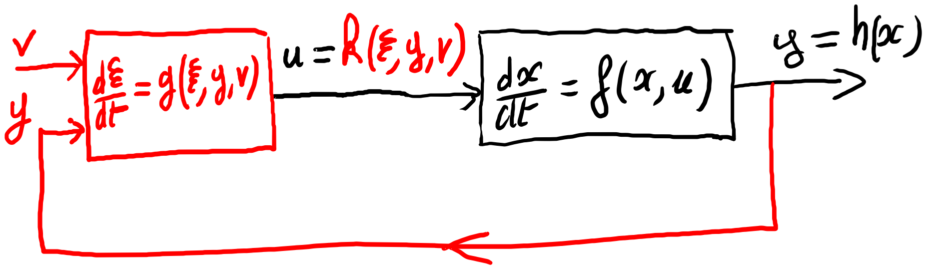

FIGURE 3 – Closed-loop control system with the new control input v, the system/controller state (x, ξ) and measured output y, related by the causal dynamics $\frac{d}{dt}x = f(x, k(\xi, y, v))$ and $\frac{d}{dt}\xi = g(\xi, y, v)$ with $y = h(x)$.

Usually, the sensors and measurement signals $y(t)$ do not give the full state x, but to some partial knowledge of x through an output function $h(x)$ where the dimension of h is less than the dimension of x. Closing the loop means here that the input u at time t will depends on the measurement signal y in a causal way, i.e. on its values at time τ with $\tau \leq t$. As illustrated on figure 3, the output signal is fed back to define the input u in a causal manner via another system, called the controller system. The controller input is then $y(t)$ and usually another exogenous signal $v(t)$. Its dynamics is then governed by another differential system

$$\frac{d}{dt}\xi = g(\xi, y, v), \quad u = k(\xi, y, v)$$

with ξ of dimension r with the functions g and k coding the feedback law.

The controller output defines then in causal manner the value of $u(t)$. Thus in closed-loop we have changed the system to a new one, with closed-loop state being formed by x and ξ, the new input v instead of u, but with the same output y :

$$\frac{d}{dt}x = f(x, k(\xi, y, v)), \quad \frac{d}{dt}\xi = g(\xi, h(x), v), \quad y = h(x)$$

In practice, the goal of such feedback scheme is to stabilize or to track some reference trajectory for x or set-point for the measured output y. In this introductory notes, we will only consider regulation issues with u and y of dimension one, i.e. Single Input / Single Output (SISO) systems and $v = \bar{y}$ corresponding to the set-point $\bar{y}$ that y has to reach and follows. Moreover we will consider only time-invariant linear systems, i.e., first order approximation around a steady-state of the dynamical equations relying on physical conservation laws. From

$$\frac{d}{dt}x = f(x,u), \quad y = h(x)$$

and a steady state $\bar{x}, \bar{u}$ such that $f(\bar{x}, \bar{u}) = 0$, one derives the linearized model

$$\frac{d}{dt}\delta x = A\delta x + B\delta u, \quad \delta y = C\delta x$$

where $A = \frac{\partial f}{\partial x}(\bar{x}, \bar{u})$, $B = \frac{\partial f}{\partial u}(\bar{x}, \bar{u})$ and $C = \frac{\partial h}{\partial x}(\bar{x})$

2.2 Linear time-invariant state-space form

We consider now a linear SISO system with $\dim x = n$ and u and y of dimension 1 :

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx$$

where A is an $n \times n$ matrix, B an $n \times 1$ and C an $1 \times n$. The open-loop solution when u is a prescribed time function admits the following expression

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau)d\tau.$$

where $e^{tA} = \sum_{k=0}^{+\infty} \frac{t^k A^k}{k!}$ is the solution W of the matrix differential equation

$$\frac{d}{dt}W = AW, \quad W(0) = I.$$

Two usefull formulae :
— if $M = PAP^{-1}$ with P invertible, then $e^{tM} = Pe^{tA}P^{-1}$.
— $\det(e^A) = e^{\mathrm{Trace}(A)}$.

2.3 Transfer function

Replacing $\frac{d}{dt}$ by the Laplace variable s in $\frac{d}{dt}x = Ax + Bu$ and $y = Cx$ yields to

$$sx = Ax + Bu, \quad y = Cx.$$

One can eliminate x to get the transfer function between u and y

$$y = C(sI - A)^{-1}Bu = H(s)u$$

where $H(s) = P(s)/Q(s)$ is the quotient of two polynomials in s, $P(s)$ and $Q(s)$, i.e. a rational function of s. The roots of $Q(s)$ are the poles, i.e. corresponding to the eigenvalues of A. The roots of $P(s)$ are the transmission zeros. Moreover the degree of Q always exceeds the degree of P by at least one. This corresponds in fact to a strictly causal transfer rational function. With $s = i\omega$, one recovers the Fourier domain description of the input/output relations.

Conversely, any strictly causal rational transfer function $H(s) = P(s)/Q(s)$ between u and y is derived from a so-called Kalman state space formulation in the time domain : this means that exist matrices (A, B, C) (not unique) such that $H(s) = C(sI - A)^{-1}B$ and thus associated to the time-domain description $\frac{d}{dt}x = Ax + Bu$ with $y = Cx$ where the dimension of x is given by the degree of $Q(s)$.

Example 7 (First order system)**.** *The transfer function of a first-order low-pass filter,*

$$H(s) = \frac{g}{\tau s + 1}$$

where g is the static gain and $\tau > 0$ the time-constant, derives from the following state-space description

$$\frac{d}{dt}x = (gu - x)/\tau, \quad y = x.$$

Example 8 (Second order system)**.** *The second-order transfer function*

$$H(s) = \frac{g}{(s/\omega_0)^2 + 2\xi(s/\omega_0) + 1}$$

with $\omega_0 > 0$ and $\xi > 0$ corresponds to the following state-space description

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\xi\omega_0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \omega_0^2 g \end{pmatrix} u, \quad y = (1, 0)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{10}$$

since

$$\frac{d^2}{dt^2}y + 2\xi\omega_0\frac{d}{dt}y + \omega_0^2(y - gu) = 0.$$

It corresponds to a damped harmonic oscillator of frequency ω_0 and quality factor $1/(2\xi)$:

- *$\xi < 1$ corresponds to the under-damped regime with complex conjugated poles having an identical negative real part $-\xi\omega_0$.*
- *For $0 < \xi \ll 1$ the two poles are well approximated by $(-\xi \pm i)\omega_0$.*
- *$\xi > 1$ corresponds to the over-damped regime with two real, negative and distinct poles.*
- *For $\xi \gg 1$, the pole close to zero is approximated by $-\omega_0/(2\xi)$ and the other one is approximated by $-2\xi\omega_0$. This is a slow/fast system*

whose slow dynamics is can be approximated by the following first-order system

$$2\xi\omega_0 \frac{d}{dt} y + \omega_0^2 (y - gu) = 0.$$

with the first-order transfer function $\frac{g}{\tau s+1}$ *with* $\tau = 2\xi/\omega_0$.

2.4 PI(D) regulator

A Proportional-Integral-Derivative controller (PID regulator) is an output feedback law generating the control input u from the measured output y and its reference y_r via the following formula :

$$u(t) = K_P(y(t) - y_r(t)) + K_I \int_0^t (y(t) - y_r(t)) + K_D \left(\frac{d}{dt} y(t) - \frac{d}{dt} y_r \right)$$

with proportional gain K_P, integral gain K_I and derivative gain K_D. Denoting by $e = y - y_r$ the regulation error, it just means that the control is linear combination of the error, its time integral and derivative. It corresponds to the transfer function

$$u = \left(K_P + \frac{K_I}{s} + K_D s \right) (y - y_r).$$

Such controller is widely used. Most of the time the derivative gain K_D is set to 0 [1] and then one gets a PI controller. This subsection shows just how to use PI controller for controlling a first order system subject to unknown perturbation.

Example 9 (PI control of a first order system with unknown perturbation input w)**.** *Take the first order system of figure 4 corresponding to a tank containing liquid of measured volume y with liquid inflow w seen here as an unknown input (un-controlled and un-measured perturbation) and the liquid outflow u seen here as a control input since its can be set to any time function via a valve. The balance equation is trivial :*

$$\frac{d}{dt} y = w - u.$$

The control goal is just to maintain the volume y around a prescribed set-point y_r through u, thus to limit the impact on y of the time-varying perturbation

1. $K_D = 0$ usually since, in practice, taking the derivative of y amplifies the noise that always corrupts the measurement signal given by sensors.

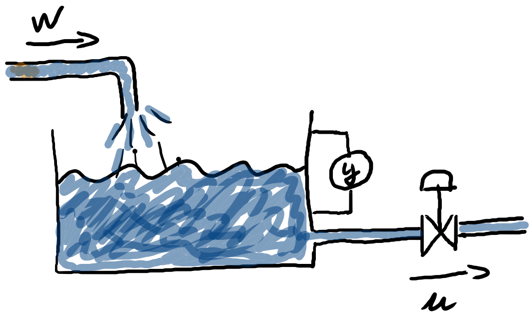

FIGURE 4 – A tank as prototype of first order system with two inputs : a perturbation input w that is an unknown time-function and the control u use to maintain via a PI controller the liquid volume y around its reference y_r.

$w(t)$ (perturbation rejection). This can be simply done via a PI feedback law where u is given by

$$\frac{d}{dt}\xi = y - y_r, \quad u = K_P(y - y_r) + K_I z.$$

The closed loop system obeys then to a second-order dynamics

$$\frac{d}{dt}y = w - K_P(y - y_r) - K_I z, \quad \frac{d}{dt}z = y - y_r.$$

Assume that y_r is constant, then z follows a second order equation, mimicking a damped harmonic oscillator of position z and subject to an unknown external force w :

$$\frac{d^2}{dt^2}z = -K_I z - K_P \frac{d}{dt}z + w.$$

For any strictly positive gains K_I and K_P, z converge exponentially to w/K_I as soon as w is constant. Thus $\frac{d}{dt}z = y - y_r$ converges to zero, whatever the constant value of w. A possible tuning of the gains K_P and K_I is the following :

$$K_P = 1/\tau_p, \quad K_I = \epsilon K_P^2$$

with τ_p giving a stabilization time constant and $0 < \epsilon \ll 1$ fixing the resetting time-constant τ/ϵ when the averages of w over intervals of length τ admit significant changes.

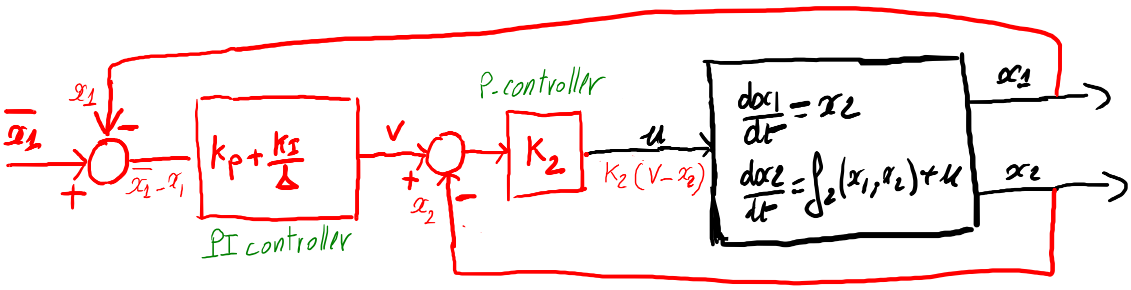

FIGURE 5 – Control of the position x_1 of a mechanical system with unknown deterministic force via a fast velocity P-regulator those reference is given by a slow position PI-regulator.

2.5 Cascade of PI regulators

We consider here a second order system, not necessarily linear, of the following form

$$\frac{d}{dt}x_1 = x_2, \quad \frac{d}{dt}x_2 = f_2(x_1, x_2) + u$$

where $f_2(x_1, x_2)$ is a smooth function of (x_1, x_2) with bounded partial derivatives : $\left|\frac{\partial f_2}{\partial x_1}\right|, \left|\frac{\partial f_2}{\partial x_2}\right| \leq M$ for all (x_1, x_2). The goal is to stabilize the position x_1 to a reference $\bar{x}_1$ without knowing precisely f_2. We assume here that we measure $y_1 = x_1$ and also $y_2 = x_2$, the position and the velocity. The question is thus : how to define u from the real-time knowledge of y_1 and y_2 and the bound M on f_2 derivatives. This can be done via a cascade of two simple regulators as illustrated on figure 5 :

— a fast velocity P regulator with proportional gain K_2 and velocity reference v :

$$u = K_2(v - x_2).$$

— a slow PI position regulator with proportional gain K_p, integral gain K_I, reference $\bar{x}_1$ and defining the velocity reference v

$$v = K_P(\bar{x}_1 - x_1) + K_I \int (\bar{x}_1 - x_1).$$

The time scale separation between the fast P-controller (the slave controller) those reference is set by the slow PI-controller (the master controller), means that one has the following gain scaling :

$$K_2 = \omega_0/\epsilon, \quad K_P = 2\xi\omega_0, \quad , K_I = \omega_0^2$$

with $0 < \epsilon \ll 1$ and the slow time-scale associated to the frequency $\omega_0 > 0$ and $\xi > 0$ not too large.

The closed-loop system is governed by three coupled scalar differential equations :

$$\frac{d}{dt}x_1 = x_2, \quad \epsilon\frac{d}{dt}x_2 = \epsilon f_2(x_1, x_2)+\omega_0(2\xi\omega_0(\bar{x}_1-x_1)+\omega_0^2 z-x_2), \quad \frac{d}{dt}z = \bar{x}_1-x_1.$$

In the time scale $\tau = t/\epsilon$, one gets :

$$\frac{d}{d\tau}x_1 = \epsilon x_2, , \quad \frac{d}{d\tau}z = \epsilon(\bar{x}_1-x_1), \quad \frac{d}{d\tau}x_2 = \epsilon f_2(x_1, x_2)+\omega_0(2\xi\omega_0(\bar{x}_1-x_1)+\omega_0^2 z-x_2).$$

The slow state is (x_1, z) and the fast state is x_2. The quasi-static approximation

$$\epsilon f_2(x_1, x_2) + \omega_0(2\xi\omega_0(\bar{x}_1 - x_1) + \omega_0^2 z - \bar{x}_2) = 0$$

is valid since, up to first-order corrections versus ϵ, $\bar{x}_2 = 2\xi\omega_0(\bar{x}_1-x_1)+\omega_0^2 z+O(\epsilon)$ is an exponentially steady state of the fast dynamics with convergence rate ω_0 in the τ-scale.

Consequently, the slow sub-system reads

$$\frac{d}{d\tau}x_1 = \epsilon\left(2\xi\omega_0(\bar{x}_1 - x_1) + \omega_0^2 z\right) + O(\epsilon^2), \quad \frac{d}{d\tau}z = \epsilon(\bar{x}_1 - x_1)..$$

It is an exponentially stable system converging towards a steady state with position $x_1 = \bar{x}_1$.

Such elementary cascade of PI-regulators ensures, for ϵ small enough, the convergence towards steady-state position $\bar{x}_1$ without knowing precisely f_2, the deterministic external forces f_2 applied on the system.

Références

[1] V.I. Arnold. *Ordinary Differential Equations.* MIT Press, 1973.

[2] K.J. Aström and T. Hägglund. *Advanced PID control.* ISA-The Instrumentation, Systems and Automation Society, 2006.

[3] M.W. Hirsch and S. Smale. *Differential Equations, Dynamical Systems and an Introduction to Chaos.* Elsevier Acamedic Press, 2004.