Université PSL M1 Quantum Engineering

Mathematical methods for quantum Engineering
9:30 - 12:00, Thursday 19, January 2023
Exam given by Alexandru Petrescu and Pierre Rouchon

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1 Frequency estimation of a high-Q harmonic oscillator

Consider a classical harmonic oscillator of state $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, unknown pulsation $\overline{\omega} > 0$, unknown viscous coefficient $\overline{\kappa} \geq 0$, scalar control input $u(t)$ and scalar measured output $y = x_1$.

$$\frac{d}{dt}x_1 = \overline{\omega}x_2, \quad \frac{d}{dt}x_2 = -\overline{\omega}x_1 - \overline{\kappa}x_2 + u(t). \tag{1}$$

The goal is to design a simple algorithm based on an output feedback to estimate in real-time $\overline{\omega}$ knowing its order of magnitude $\omega_0 > 0$ ($|\omega_0 - \overline{\omega}| \ll \overline{\omega}$), knowing the order of magnitude κ_0 of $\overline{\kappa}$ ($\kappa_0/\overline{\kappa} \sim 1$) and assuming that $\overline{\kappa} \ll \overline{\omega}$ (high-Q oscillator).

1. For $\overline{\kappa} = 0$, show that this system admits an Hamiltonian formulation and give the expression of the Hamiltonian.

2. Prove that for $\overline{\kappa} > 0$ and $u(t) = 0$, the trajectories of (1) converge exponentially to $(0, 0)$ and give the convergence rate (use $\overline{\omega} \gg \overline{\kappa}$).

3. Consider the following linear and time-varying change of coordinates from $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ to $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ according to

$$x_1 = z_1 \cos\theta + z_2 \sin\theta, \quad x_2 = -z_1 \sin\theta + z_2 \cos\theta$$

with $\frac{d}{dt}\theta = \omega$ and where ω is assumed to be constant and around $\overline{\omega}$ ($|\overline{\omega} - \omega| \ll \overline{\omega}$). Show that

$$\frac{d}{dt}z_1 = (\overline{\omega} - \omega)z_2 - \overline{\kappa}(z_1 \sin^2\theta - z_2 \sin\theta\cos\theta) - u\sin\theta$$
$$\frac{d}{dt}z_2 = -(\overline{\omega} - \omega)z_1 - \overline{\kappa}(-z_1 \sin\theta\cos\theta + z_2 \cos^2\theta) + u\cos\theta$$

4. Consider now the following time-varying control input $u = 2\kappa_0 \cos\theta$. Assume ω constant with $|\overline{\omega} - \omega| \ll \overline{\omega}$. Show that the above dynamics can be approximated by the following averaged dynamics

$$\frac{d}{dt}z_1 = (\overline{\omega} - \omega)z_2 - \kappa z_1$$
$$\frac{d}{dt}z_2 = -(\overline{\omega} - \omega)z_1 - \kappa z_2 + \kappa_0$$

where $\kappa > 0$ is a constant gain (remember that $\kappa_0 \ll \overline{\omega}$). Give the explicit expression of κ versus $\overline{\kappa}$. Show that the above system admits a unique steady-state. Prove that all solutions converge exponentially towards this steady-state.

5. Assume that ω varies according to $\frac{d}{dt}\omega = 2\omega\kappa_0 y(t)\cos\theta(t)$ with $y(t) = x_1(t)$. Thus the output feedback controller has the following dynamics

$$\frac{d}{dt}\theta = \omega, \quad \frac{d}{dt}\omega = 2\omega\kappa_0 y(t)\cos\theta$$

with $u = 2\kappa_0\cos\theta$. The goal is to show the stability of the closed-loop system and to prove that ω converges towards $\overline{\omega}$.

(a) Show that the closed-loop system reads.

$$\begin{aligned}
\frac{d}{dt}x_1 &= \overline{\omega}x_2 \\
\frac{d}{dt}x_2 &= -\overline{\omega}x_1 - \overline{\kappa}x_2 + 2\kappa_0\cos\theta \\
\frac{d}{dt}\omega &= 2\omega\kappa_0 x_1\cos\theta \\
\frac{d}{dt}\theta &= \omega.
\end{aligned}$$

(b) Express the above system with (z_1, z_2)-variables instead of (x_1, x_2)-variables according to formulae of question 3. Replace t by θ and show that

$$\begin{aligned}
\frac{d}{d\theta}z_1 &= \frac{\overline{\omega}-\omega}{\omega}z_2 - \tfrac{\overline{\kappa}}{\omega}(z_1\sin^2\theta - z_2\sin\theta\cos\theta) - \frac{\kappa_0\sin 2\theta}{\omega} \\
\frac{d}{d\theta}z_2 &= -\frac{\overline{\omega}-\omega}{\omega}z_1 - \tfrac{\overline{\kappa}}{\omega}(z_1\sin\theta\cos\theta + z_2\cos^2\theta) + \frac{2\kappa_0\cos^2\theta}{\omega} \\
\frac{d}{d\theta}\omega &= 2\kappa_0(z_1\cos^2\theta + z_2\sin\theta\cos\theta).
\end{aligned}$$

(c) Assuming that ω remains around $\overline{\omega}$, show that the averaged system reads

$$\begin{aligned}
\frac{d}{d\theta}z_1 &= \frac{\overline{\omega}-\omega}{\omega}z_2 - \frac{\kappa}{\omega}z_1 \\
\frac{d}{d\theta}z_2 &= -\frac{\overline{\omega}-\omega}{\omega}z_1 - \frac{\kappa}{\omega}z_2 + \frac{\kappa_0}{\omega} \\
\frac{d}{d\theta}\omega &= \kappa_0 z_1.
\end{aligned}$$

(d) Show that the averaged system of question 5c admits a unique steady-state and give its expression. Compute its first variation around this steady state:

$$\frac{d}{d\theta}\begin{pmatrix}\delta z_1\\ \delta z_2\\ \delta\omega\end{pmatrix} = A\begin{pmatrix}\delta z_1\\ \delta z_2\\ \delta\omega\end{pmatrix}$$

and express the 3×3 Jacobian matrix A versus the parameters κ, κ_0 and $\overline{\omega}$.

(e) Show that A is a stable matrix. Assume that $\kappa_0 = \kappa$ and compute the eigen-values of A. Why is it important to check that the size of these eigenvalues are much smaller than one ?

(f) Show that

$$\mathbb{R} \times \mathbb{R} \times]0 + \infty[\ni (z_1, z_2, \omega) \mapsto V(z_1, z_2, \omega) = \tfrac{1}{2} z_1^2 + \tfrac{1}{2}\left(z_2 - \tfrac{\kappa_0}{\kappa}\right)^2 + \frac{\omega - \overline{\omega}\log(\omega/\overline{\omega})}{\kappa}$$

is a Lyapunov function for the averaged system of question 5c. Apply Lasalle invariance principle to conclude that all trajectories converge to the unique steady-state of question 5d?

2 Controlling an approximate model for a transmon qubit

The goal of this problem is to use perturbation theory to find solutions to Schrödinger's equation for an approximate model of a transmon qubit. In this framework, we will then analyze a simple single-qubit gate, the $\pi-$pulse. We consider the following Hamiltonian

$$\begin{aligned} \hat{H} &= \hat{H}_0 + \hat{H}_J, \\ \hat{H}_0 &= \hbar\omega_a \hat{a}^\dagger \hat{a}, \quad \hat{H}_J = \hbar\epsilon \left[\hat{a}^\dagger \hat{a} + \frac{1}{2}\hat{a}^{\dagger 2}\hat{a}^2 + \frac{1}{2}(\hat{a}^2 + \hat{a}^{\dagger 2})\right]. \end{aligned} \tag{2}$$

We have denoted $\hat{H}_0$ the Hamiltonian for a simple harmonic oscillator, as studied in class. Note that $\hat{a}$ and $\hat{a}^\dagger$ are the bosonic annihilation and creation operators obeying the canonical commutation relation $[\hat{a}, \hat{a}^\dagger] = I$, with I being the identity acting on the simple harmonic oscillator Hilbert space.

Furthermore, we let $\hat{H}_J$ be a perturbation proportional to the so-called anharmonicity of the transmon qubit, denoted by ϵ. We assume as usual that $\omega_a > 0$, and let the anharmonicity be negative $\epsilon < 0$. The anharmonicity is considered to be a small perturbation, that is, $\omega_a \gg |\epsilon|$.

1. Write down the eigenvectors and eigenvalues arising from the eigenproblem $\hat{H}_0|\psi\rangle = \hbar\omega_a \hat{a}^\dagger \hat{a}|\psi\rangle = E|\psi\rangle$. Express the eigenvectors in normalized form, in terms of the vacuum vector $|0\rangle$, which you will need to define. You may denote the eigenvectors of H_0 as $|n\rangle$, and the eigenenergies as $E_n^{(0)}$.

2. Consider the perturbation $\hat{H}_J$. Show that it can be written as a sum of two operators: the first operator, which we will call $\hat{H}_J^{secular}$, commutes with the number operator $\hat{a}^\dagger \hat{a}$. The second operator, which we shall denote $\hat{H}_J^{nonsecular}$, that does not commute with the number operator $\hat{a}^\dagger \hat{a}$. Write $\hat{H}_J^{secular}$ and $\hat{H}_J^{nonsecular}$ in terms of the ladder operators $\hat{a}$ and $\hat{a}^\dagger$, and write the matrix representations of these two operators over the basis of eigenstates of $\hat{H}_0$, found above.

3. Using first-order nondegenerate time-independent perturbation theory, find the correction to the eigenenergies and eigenvectors of $\hat{H}_0$ due to the perturbation $\hat{H}_J$.

   (a) Explicitly, you have to find the approximation to the eigenenergy of the n^{th} state of the transmon, up to first order in perturbation theory in ϵ, denoted $E_n \approx$

$E_n^{(0)} + E_n^{(1)}$. Here, $\{E_n^{(0)}, |n\rangle | n = 0, 1, 2, \dots\}$ are the solutions to the Schrödinger equation of the simple harmonic oscillator in Question 1 above, and $E_n^{(1)}$ is the correction at first order in ϵ that you have to find.

(b) Moreover, you have to show that the corrected eigenvectors take the form

$$|n^{(1)}\rangle = |n\rangle + \mu_{n+2,n}|n+2\rangle + \mu_{n-2,n}|n-2\rangle, \tag{3}$$

in terms of the eigenstates $|n\rangle$ of the simple harmonic oscillator found above. Find the coefficients $\mu_{i,j}$ in the expression above, according to perturbation theory to first order in ϵ, and such that $|n^{(1)}\rangle$ is properly normalized.

4. Now we will consider the effect of a drive term $\hat{V}(t) = \hbar\mathcal{A}\cos(\omega_d t)(\hat{a} + \hat{a}^\dagger)$, where $\mathcal{A}$ is a small drive amplitude, and ω_d is a drive frequency, both with units of frequency. Assume that the approximate spectrum of the undriven system is correctly described by the results you obtained above in Question 3, that is

$$\hat{H} + \hat{V}(t) \approx \sum_n \left[E_n^{(0)} + E_n^{(1)}\right] |n^{(1)}\rangle\langle n^{(1)}| + \hat{V}(t), \tag{4}$$

with the first term being our approximation of the time-independent unperturbed Hamiltonian, and the second term being the time-dependent perturbation. We assume that the frequency of the perturbation is set to be resonant with our approximation of the $0 \to 1$ transition frequency, that is $\hbar\omega_d = E_1^{(0)} + E_1^{(1)} - E_0^{(0)} + E_0^{(1)}$.

Assume that the system is initialized in the state $|0^{(1)}\rangle$. Find the leading contribution to the probability as a function of time that the system is in the state $|0^{(1)}\rangle$,$|1^{(1)}\rangle$, $|2^{(1)}\rangle$. Give an answer that is correct to first order in ϵ and to first order in $\mathcal{A}$. Argue that, to achieve this order of precision, it is not necessary to normalize the eigenvector in Eq. (3).

5. Project your Hamiltonian Eq. (4) into the lowest two eigenstates $|0^{(1)}\rangle$ and $|1^{(1)}\rangle$. That is, we shall consider $\hat{H}_{qubit}(t) = \hat{P}[\hat{H} + \hat{V}(t)]\hat{P}$, with the operator in the brackets on the right-hand side given by the approximate form of Eq. (4), and with $\hat{P} = |0^{(1)}\rangle\langle 0^{(1)}| + |1^{(1)}\rangle\langle 1^{(1)}|$.

(a) Show that $\hat{P}$, as expressed in terms of the perturbed kets of Eq. (3), is a projector, *i.e.* it obeys $\hat{P}\hat{P} = \hat{P}$.

(b) Express $\hat{H}_{qubit}$ in terms of Pauli matrices. Write first the appropriate Pauli matrices in terms of the perturbed kets and bras of Eq. (3). Hint: $\hat{\sigma}_z = -|0^{(1)}\rangle\langle 0^{(1)}| + |1^{(1)}\rangle\langle 1^{(1)}|$.

(c) Assume that the system described by $\hat{H}_{qubit}$ is prepared in the initial state $|0^{(1)}\rangle$. What is the wavefunction as a function of time?

(d) Do this result and the one you obtained previously in Question 4 agree in the limit of short evolution time, t, or weak drive amplitude, $\mathcal{A}$? What is the amount of time needed to perform a π pulse?

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3 Frequency estimation of a high-Q harmonic oscillator

1. With $q = x_1$, $p = x_2$ and Hamiltonian $H(q,p,u) = \frac{\overline{\omega}}{2}(q^2+p^2) - uq$, the differential equations read:
$$\frac{d}{dt}q = \frac{\partial H}{\partial p}, \quad \frac{d}{dt}p = -\frac{\partial H}{\partial q}.$$

2. The solutions of the linear time-invariant system $\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with
$$M = \begin{pmatrix} 0 & \overline{\omega} \\ -\overline{\omega} & -\overline{\kappa} \end{pmatrix}$$
converge exponentially towards 0, iff, the eigenvalues of M have a strictly negative real part: it is the case since $\operatorname{Tr}(M) = -\overline{\kappa} < 0$ and $\det(M) = \overline{\omega}^2 > 0$. Since $\overline{\omega} \gg \overline{\kappa}$, the two eigenvalues are complex conjugated with identical real parts $-\overline{\kappa}/2$: the exponential convergence rate is thus $\overline{\kappa}/2$.

3. This comes from the following equations satisfied by $\frac{d}{dt}z_1$ and $\frac{d}{dt}z_2$:
$$\begin{aligned}
\cos\theta\frac{d}{dt}z_1 + \sin\theta\frac{d}{dt}z_2 + \omega(-z_1\sin\theta + z_2\cos\theta) &= \frac{d}{dt}x_1 = \overline{\omega}(-z_1\sin\theta + z_2\cos\theta) \\
-\sin\theta\frac{d}{dt}z_1 + \cos\theta\frac{d}{dt}z_2 - \omega(z_1\cos\theta + z_2\sin\theta) &= \frac{d}{dt}x_2 = -\overline{\omega}(z_1\cos\theta + z_2\sin\theta) \\
&\qquad - \overline{\kappa}(-z_1\sin\theta + z_2\cos\theta) + u.
\end{aligned}$$

4. Since $|\overline{\omega} - \omega|, \overline{\kappa}, \kappa_0 \ll \overline{\omega}$, one can apply the RWA approximation using the fact that
$$\tfrac{1}{2\pi}\int_0^{2\pi} \cos^2\theta d\theta = 1/2 \text{ and } \tfrac{1}{2\pi}\int_0^{2\pi} \sin\theta\cos\theta d\theta = 0.$$
Thus $\kappa = \overline{\kappa}/2$. The unique steady-state is
$$z_1 = \frac{\kappa_0(\overline{\omega} - \omega)}{\kappa^2 + (\overline{\omega}-\omega)^2}, \quad z_2 = \frac{\kappa_0\kappa}{\kappa^2 + (\overline{\omega}-\omega)^2}.$$
The Jacobian matrix $\begin{pmatrix} -\kappa & \overline{\omega} - \omega \\ -\overline{\omega} + \omega & -\kappa \end{pmatrix}$ is stable since its trace -2κ is strictly negative and its determinant $\kappa^2 + (\overline{\omega}-\omega)^2$ is strictly positive.

5. (a) Add to the open-loop dynamics, the dynamics of the controller $\frac{d}{dt}\theta = \omega$ and $\frac{d}{dt}\omega = 2\omega\kappa_0 y(t)\cos\theta$, replace y by x_1 and u by $2\kappa_0\cos\theta$.

(b) Similar computations than the ones done in question 3 yield to

$$
\begin{aligned}
\frac{d}{dt} z_1 &= (\overline{\omega} - \omega) z_2 - \overline{\kappa}(z_1 \sin^2 \theta - z_2 \sin\theta \cos\theta) - \kappa_0 \sin 2\theta \\
\frac{d}{dt} z_2 &= -(\overline{\omega} - \omega) z_1 - \overline{\kappa}(z_1 \sin\theta \cos\theta + z_2 \cos^2 \theta) + 2\kappa_0 \cos^2 \theta \\
\frac{d}{dt} \omega &= 2\omega\kappa_0 (z_1 \cos^2 \theta + z_2 \sin\theta \cos\theta) \\
\frac{d}{dt} \theta &= \omega.
\end{aligned}
$$

One concludes using $d\theta = \omega dt$ and $\frac{d}{d\theta} = \frac{1}{\omega}\frac{d}{dt}$.

(c) With $\eta = \omega/\overline{\omega}$ instead of ω one gets

$$
\begin{aligned}
\frac{d}{d\theta} z_1 &= \frac{1-\eta}{\eta} z_2 - \tfrac{\overline{\kappa}}{\overline{\omega}\eta}(z_1 \sin^2 \theta - z_2 \sin\theta \cos\theta) - \frac{\kappa_0 \sin 2\theta}{\overline{\omega}\eta} \\
\frac{d}{d\theta} z_2 &= -\frac{1-\eta}{\eta} z_1 - \tfrac{\overline{\kappa}}{\overline{\omega}\eta}(z_1 \sin\theta \cos\theta + z_2 \cos^2 \theta) + \frac{2\kappa_0 \cos^2 \theta}{\overline{\omega}\eta} \\
\frac{d}{d\theta} \eta &= 2\frac{\kappa_0}{\overline{\omega}}(z_1 \cos^2 \theta + z_2 \sin\theta \cos\theta).
\end{aligned}
$$

Since $\left|\frac{\overline{\omega}-\omega}{\omega}\right| \ll 1$, η is close to one, and thus $\frac{1-\eta}{\eta}$, $\frac{\overline{\kappa}}{\overline{\omega}\eta}$, $\frac{\kappa_0}{\overline{\omega}\eta}$ and $\frac{\kappa_0}{\overline{\omega}}$ are much smaller than 1. Thus the RWA provides a good approximation:

$$
\begin{aligned}
\frac{d}{d\theta} z_1 &= \frac{1-\eta}{\eta} z_2 - \tfrac{\overline{\kappa}}{2\overline{\omega}\eta} z_1 \\
\frac{d}{d\theta} z_2 &= -\frac{1-\eta}{\eta} z_1 - \tfrac{\overline{\kappa}}{2\overline{\omega}\eta} z_2 + \frac{\kappa_0}{\overline{\omega}\eta} \\
\frac{d}{d\theta} \eta &= \frac{\kappa_0}{\overline{\omega}} z_1.
\end{aligned}
$$

One concludes with $\omega = \overline{\omega}\eta$.

(d) The algebraic system characterizing the possible steady-states

$$
\begin{aligned}
0 &= \frac{\overline{\omega} - \omega}{\omega} z_2 - \frac{\kappa}{\omega} z_1 \\
0 &= -\frac{\overline{\omega} - \omega}{\omega} z_1 - \frac{\kappa}{\omega} z_2 + \frac{\kappa_0}{\omega} \\
0 &= \kappa_0 z_1
\end{aligned}
$$

admits an unique solution $z_1 = 0$, $z_2 = \kappa_0/\kappa > 0$ and $\omega = \overline{\omega}$. A direct computation of the Jacobian matrix at this stead-state gives

$$
A = \begin{pmatrix} -\frac{\kappa}{\overline{\omega}} & 0 & -\frac{\kappa_0}{\kappa\overline{\omega}} \\ 0 & -\frac{\kappa}{\overline{\omega}} & 0 \\ \kappa_0 & 0 & 0 \end{pmatrix}
$$

(e) The linearized system separates in two blocs:

- the 1D bloc $\frac{d}{d\theta}\delta z_2 = -\frac{\kappa}{\overline{\omega}}\delta z_2$ that converge exponentially to 0 with eigenvalue $-\frac{\kappa}{\overline{\omega}}$

- the 2D bloc
$$\frac{d}{d\theta}\delta z_1 = -\tfrac{\kappa}{\overline{\omega}}\delta z_1 - \tfrac{\kappa_0}{\kappa\overline{\omega}}\delta\omega, \quad \frac{d}{d\theta}\delta\omega = \kappa_0 \delta z_1$$
corresponding to $\frac{d^2}{d\theta^2}\delta\omega = -\frac{\kappa}{\overline{\omega}}\frac{d}{d\theta}\delta\omega - \frac{\kappa_0^2}{\kappa\overline{\omega}}\delta\omega$ with is also exponentially stable (damped harmonic oscillator).

When $\kappa = \kappa_0$, one eigenvalue is real $-\frac{\kappa}{\overline{\omega}}$ and the two other ones are complex conjugated: $-\frac{\kappa}{2\overline{\omega}} \pm i\sqrt{\frac{\kappa}{\overline{\omega}} - \frac{\kappa^2}{4\overline{\omega}^2}}$. Their magnitude are much smaller that 1 since less that $\sqrt{\kappa/\overline{\omega}} \ll 1$. This is crutial to ensure the validity of the RWA.

(f) It is clear that V tends to $+\infty$ when (z_1, z_2, ω) tends to the boundary of $\mathbb{R}\times\mathbb{R}\times]0+\infty[$. Moreover one has:
$$\frac{d}{d\theta}V = -\frac{\kappa}{\omega}\left(z_1^2 + \left(z_2 - \tfrac{\kappa_0}{\kappa}\right)^2\right) \leq 0.$$
Thus V is a Lyapunov function. Lasalle invariance principle gives then the convergence to the set of solutions of
$$\begin{aligned}
\frac{d}{d\theta}z_1 &= \frac{\overline{\omega}-\omega}{\omega}z_2 - \frac{\kappa}{\omega}z_1\\
\frac{d}{d\theta}z_2 &= -\frac{\overline{\omega}-\omega}{\omega}z_1 - \frac{\kappa}{\omega}z_2 + \frac{\kappa_0}{\omega}\\
\frac{d}{d\theta}\omega &= \kappa_0 z_1\\
\frac{d}{d\theta}V &= -\frac{\kappa}{\omega}\left(z_1^2 + \left(z_2 - \tfrac{\kappa_0}{\kappa}\right)^2\right) = 0.
\end{aligned}$$
This over-determined system admits a unique solution, the steady-state $z_1 = 0$, $z_2 = \kappa_0/\kappa$ and $\omega = \overline{\omega}$.

4 Controlling an approximate model for a transmon qubit

1. Defining the vacuum to be the unique state $|0\rangle$ that obeys $\hat{a}|0\rangle = 0$, we let $|n\rangle = \frac{\hat{a}^{\dagger n}}{\sqrt{n!}}|0\rangle$. Then $\hat{H}_0|n\rangle = E_n^{(0)}|n\rangle = n\hbar\omega_a|n\rangle$.

2. The perturbation can be split into two terms,
$$\begin{aligned}
\hat{H}_J^{secular} &= \hbar\epsilon\hat{a}^\dagger\hat{a} + \frac{\hbar\epsilon}{2}\hat{a}^{\dagger 2}\hat{a}^2 = \hbar\epsilon\hat{a}^\dagger\hat{a} + \frac{\hbar\epsilon}{2}\hat{a}^\dagger\hat{a}(\hat{a}^\dagger\hat{a} - 1),\\
\hat{H}_J^{nonsecular} &= \hat{H}_J - \hat{H}_J^{secular} = \frac{\hbar\epsilon}{2}(\hat{a}^2 + \hat{a}^{\dagger 2}).
\end{aligned} \tag{5}$$
The first one is diagonal in the Fock basis $|n\rangle$ defined above, whereas the second one is not. The nonzero matrix elements of $\hat{H}_J^{secular}$ are
$$\langle n|\hat{H}_J^{secular}|n\rangle = n\hbar\epsilon + \frac{1}{2}n(n-1)\hbar\epsilon, \text{ for } n = 0, 1, 2, \ldots. \tag{6}$$
Meanwhile, the nonzero matrix elements of $\hat{H}_J^{nonsecular}$ are
$$\langle n|\hat{H}_J^{secular}|n+2\rangle = \langle n+2|\hat{H}_J^{secular}|n\rangle = \frac{1}{2}\sqrt{n+2}\sqrt{n+1}\hbar\epsilon, \text{ for } n = 0, 1, 2, \ldots. \tag{7}$$

These matrix representations allow us to see that $\hat{H}_J^{secular}$ commutes with the number operator $\hat{a}^\dagger \hat{a}$, whereas $\hat{H}_J^{nonsecular}$ does not, without explicitly evaluating the commutator.

3. (a) By using the formula in the course notes [Eq. (23)] the eigenenergies of the transmon up to corrections that are first order in ϵ are

$$\begin{aligned} E_n^{(0)} + E_n^{(1)} &= n\hbar\omega_a + \langle n|\hat{H}_J|n\rangle = n\hbar\omega_a + \langle n|\hat{H}_J^{secular}|n\rangle \\ &= n\hbar\omega_a + n\hbar\epsilon + \frac{n(n-1)}{2}\hbar\epsilon. \end{aligned} \tag{8}$$

Only the part of the perturbation that is diagonal in the eigenbasis of H_0, $\hat{H}_J^{secular}$, contributes to the eigenenergy correction at linear order in ϵ.

(b) To obtain the corrections for the eigenvectors, we apply Eq. (24) of the course notes, to obtain

$$|n^{(1)}\rangle = |n\rangle + \sum_{k\neq n} |k\rangle \frac{\langle k|\hat{H}_J^{nonsecular}|n\rangle}{E_n^{(0)} - E_k^{(0)}} + O(\epsilon^2). \tag{9}$$

Note that all kets, bras, and energies on the right-hand side of Eq. (9) stand for the unperturbed kets, bras, and eigenenergies that you determined at Question 1. To ease notations, let

$$\hat{v} = \frac{\hat{H}_J^{nonsecular}}{\hbar\epsilon} = \frac{1}{2}(\hat{a}^2 + \hat{a}^{\dagger 2}). \tag{10}$$

The following are the only nonzero elements of the matrix representation of the operator $\hat{v}$ over Fock states

$$\langle n+2|\hat{v}|n\rangle = \langle n|\hat{v}|n+2\rangle = \frac{1}{2}\sqrt{(n+1)(n+2)}. \tag{11}$$

So $\hat{v}$ only couples Fock states that differ by 2 excitations. Using then Eq. (9) we arrive at the corrected eigenvector to first order in α

$$|n^{(1)}\rangle = |n\rangle + \eta_{n+2,n}|n+2\rangle + \eta_{n-2,n}|n-2\rangle, \tag{12}$$

where the coefficients are

$$\begin{aligned} \eta_{ij} &= \frac{\hbar\epsilon}{E_j^{(0)} - E_i^{(0)}} \langle i|\hat{v}|j\rangle \text{ for all } i, j \geq 0, \\ \eta_{ij} &= 0 \text{ if } i \text{ or } j < 0. \end{aligned} \tag{13}$$

The norm of the corrected eigenstate Eq. (3) is

$$\mathcal{N}_n = \sqrt{\langle n^{(1)}|n^{(1)}\rangle} = \sqrt{1 + \eta_{n+2,n}^2 + \eta_{n-2,n}^2} = 1 + O(\epsilon^2). \tag{14}$$

Therefore,

$$\mu_{n\pm 2,n} = \mathcal{N}_n^{-1}\eta_{n\pm 2,n} = \eta_{n\pm 2,n} + O(\epsilon^3). \tag{15}$$

If we aim to report all results to order ϵ, we do not need to worry about evaluating this norm and normalizing the eigenvector $|n^{(1)}\rangle$.

4. If our goal is to express all quantities to linear order in ϵ or $\mathcal{A}$, it is sufficient to work with the Hamiltonian in the form

$$\hat{H}(t) \approx \sum_n \left[E_n^{(0)} + E_n^{(1)}\right] |n^{(1)}\rangle\langle n^{(1)}| + \hat{V}(t). \tag{16}$$

By the prescription of time-dependent perturbation theory, what we need is the interaction-picture form of the time-dependent perturbation. This is

$$\hat{V}_I(t) = \sum_{m,n\geq 0}^{\infty} \hbar\mathcal{A}\cos(\omega_d t)\langle n^{(1)}|\hat{a}+\hat{a}^\dagger|m^{(1)}\rangle e^{-i\left[E_m^{(1)}-E_n^{(1)}+E_m^{(0)}-E_n^{(0)}\right]t/\hbar}|n^{(1)}\rangle\langle m^{(1)}|. \tag{17}$$

The relevant matrix elements to calculate are

$$\langle n^{(1)}|\hat{V}_I(t)|0^{(1)}\rangle, \text{ for } n = 0, 1, 2. \tag{18}$$

This reduces to evaluating the following three matrix elements of the operator $\hat{a} + \hat{a}^\dagger$. If we only keep terms linear in ϵ,

$$\begin{aligned} \langle 0^{(1)}|\hat{a}+\hat{a}^\dagger|0^{(1)}\rangle &\approx 0, \\ \langle 1^{(1)}|\hat{a}+\hat{a}^\dagger|0^{(1)}\rangle &\approx 1, \\ \langle 2^{(1)}|\hat{a}+\hat{a}^\dagger|0^{(1)}\rangle &\approx 0. \end{aligned} \tag{19}$$

We then find that the only relevant nonzero matrix element of the interaction-picture perturbation Hamiltonian is

$$\langle 1^{(1)}|\hat{V}_I(t)|0^{(1)}\rangle = \hbar\mathcal{A}\cos(\omega_d t)e^{i\left[E_1^{(1)}-E_0^{(1)}+E_1^{(0)}-E_0^{(0)}\right]t/\hbar}. \tag{20}$$

Now we can use Eq. (60) of the lecture notes

$$c_1^{(1)}(t) = -\frac{i}{\hbar}\int_0^t dt'\langle 1^{(1)}|V_I(t')|0^{(1)}\rangle. \tag{21}$$

Assuming that $\omega_d = [E_1^{(1)} - E_0^{(1)} + E_1^{(0)} - E_0^{(0)}]/\hbar$, the first integral evaluates to

$$c_1^{(1)}(t) = -\frac{i\hbar\mathcal{A}}{2\hbar}t + \text{osc.} = -i\frac{\mathcal{A}}{2}t + \text{osc.}, \tag{22}$$

where by osc. we have denoted terms that oscillate as a function of time, which we shall drop. Thus the probability to find the system in state 1 after time t is $|\mathcal{A}|^2t^2/4$. Obviously this result is only valid for time scales $t \ll |\mathcal{A}|^{-1}$.

5. (a) We can evaluate $\hat{P}\hat{P}$ in the eigenbasis of $\hat{H}_0$ by using the expression for $\hat{P}$ in the text of the exam, and the expressions of the perturbed kets of Eq. (3). We get

$$\begin{aligned} \hat{P}\hat{P} =& |0^{(1)}\rangle\langle 0^{(1)}|0^{(1)}\rangle\langle 0^{(1)}| + |1^{(1)}\rangle\langle 1^{(1)}|1^{(1)}\rangle\langle 1^{(1)}| \\ &+ |0^{(1)}\rangle\langle 0^{(1)}|1^{(1)}\rangle\langle 1^{(1)}| + |1^{(1)}\rangle\langle 1^{(1)}|0^{(1)}\rangle\langle 0^{(1)}| \\ =& \hat{P}. \end{aligned} \tag{23}$$

The terms on the second row give zero, upon using Eq. (3), whereas the first row gives $\hat{P}$ if the expressions are properly normalized.

(b) The resulting two-by-two Hamiltonian is then

$$
\begin{aligned}
H_{qubit} = & \left(E_0^{(0)} + E_0^{(1)}\right) |0^{(1)}\rangle\langle 0^{(1)}| + \left(E_1^{(0)} + E_1^{(1)}\right) |1^{(1)}\rangle\langle 1^{(1)}| \\
& + \mathcal{A}\sin(\omega_d t)\langle 1^{(1)}|\hat{a} + \hat{a}^\dagger|0^{(1)}\rangle|1^{(1)}\rangle\langle 0^{(1)}| \\
& + \mathcal{A}\sin(\omega_d t)\langle 0^{(1)}|\hat{a} + \hat{a}^\dagger|1^{(1)}\rangle|0^{(1)}\rangle\langle 1^{(1)}| \\
= & \begin{pmatrix} E_0^{(0)} + E_0^{(1)} & \mathcal{A}\sin(\omega_d t) \\ \mathcal{A}\sin(\omega_d t) & E_1^{(0)} + E_1^{(1)} \end{pmatrix} = \frac{1}{2}\Delta_{10}\sigma^z + \mathcal{A}\sin(\omega_d t)\sigma^x + E_{\mathit{offset}} I_2,
\end{aligned}
\tag{24}
$$

where $\Delta_{10} = E_1^{(0)} + E_1^{(1)} - E_0^{(0)} - E_0^{(1)}$, and E_{offset} is an irrelevant energy offset multiplying the identity matrix acting on this subspace.

(c) Revisiting our solution for the driven qubit, we find that, for resonant drive, the probability to find the qubit in the excited state is $\sin^2\left(\frac{1}{2}\mathcal{A}t\right)$, cf. Eq. (26) of the problem set 1 solution. The time of a π-pulse is therefore the shortest time needed for this function to go from 0 to 1, namely $\frac{1}{2}\pi = \frac{1}{2}\mathcal{A}T_\pi$, so $T_\pi = \pi/\mathcal{A}$.

(d) The fact that the probability of transition from 0 to 1 is $\sin^2(\mathcal{A}t/2) \approx \mathcal{A}^2t^2/4$, agrees in the short time limit $\mathcal{A}t \ll 1$ with the perturbative result obtained above. We can only expect the perturbative result to hold in the limit $\mathcal{A}t \ll 1$, so either for very short times, or very small drive amplitude.