Problem Set 3

(M1 Math Methods 2023-2024)

This problem set is due on Monday, October 9th, 2023, at 23:59. The solutions should be sent as a single PDF (scan of handwritten document or LaTeX) to alexandru.petrescu@minesparis.psl.eu by the deadline. If you collaborate with a colleague, please write clearly their names at the top of your solution sheets. If you submit late without a satisfactory reason, the set will be accepted with a 10% penalty in the score.

I. COHERENT AND THERMAL STATES

a) Consider a resonator initially prepared in a coherent state $|\psi(t=0)\rangle = |\alpha\rangle$. Find the time evolution of the system state $|\psi(t)\rangle$ for all subsequent times $t \geq 0$, as governed by the simple harmonic oscillator Hamiltonian $\hat{H}/\hbar = \omega_a \hat{a}^\dagger \hat{a}$ for some real and positive frequency ω_a.

b) In the state $|\psi(t)\rangle$ determined above, what are the time-dependent expectation values of the rescaled position operator $\hat{Q} = (\hat{a} + \hat{a}^\dagger)/\sqrt{2}$ and of the rescaled momentum operator $\hat{P} = (\hat{a} - \hat{a}^\dagger)/(i\sqrt{2})$?

c) Find $\Delta(\hat{P}), \Delta(\hat{Q}), \Delta(\hat{P})\Delta(\hat{Q})$, corresponding to $|\psi(t)\rangle$, at all times. Reminder $\Delta(\hat{A}) = \sqrt{\langle \hat{A}^2\rangle - \langle \hat{A}\rangle^2}$, where $\langle \ldots \rangle$ is shorthand for $\langle \psi | \ldots | \psi \rangle$, is the variance of an observable $\hat{A}$ in state $|\psi\rangle$. How does the variance $\Delta(\hat{Q})$ depend on time? You have shown that coherent states are minimum uncertainty states, as they saturate the inequality in Heisenberg's uncertainty principle.

d) Consider a system that can be found in a mixed state defined by the density operator $\hat{\rho} = \mathcal{N} \exp(-\beta \hat{H})$, where $\hat{H}/\hbar = \omega_a \hat{a}^\dagger \hat{a}$, and $\beta = 1/(k_B T)$ where k_B is Boltzmann's constant and T is a finite temperature. Determine $\mathcal{N}$ such that ρ is correctly defined as a density matrix, and give it a physical interpretation. Find the expectation value and variance of the photon number $\hat{N} = \hat{a}^\dagger \hat{a}$ in this state.

SOLUTION

a)[**5 points**] Considering the expansion of a coherent state over eigenstates of the number operator $|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$, and the time-evolution operator for the simple harmonic oscillator $\hat{U}(t,0) = e^{-i\hat{H}t/\hbar} = \sum_n e^{-in\omega_a t}|n\rangle\langle n|$. Then $\hat{U}(t,0)|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{(\alpha e^{-i\omega_a t})^n}{\sqrt{n!}}|n\rangle = |\alpha e^{-i\omega_a t}\rangle$. Note that this is *absolutely not* the same as $e^{-i\omega_a t}|\alpha\rangle$. Henceforth we let $\alpha(t) = e^{-i\omega_a t}\alpha$.

b)[**5 points**] Note that $\langle\alpha(t)|\,\hat{a}\,|\alpha(t)\rangle = \alpha(t)$, as can be seen by letting the annihilation operator act on its eigenstate (the coherent state) from the left, then using the normalization condition for the eigenstate, i.e. $\langle\alpha(t)|\,\hat{a}\,|\alpha(t)\rangle = \alpha(t)\langle\alpha(t)|\alpha(t)\rangle = \alpha(t)$. Moreover, to handle expectation values involving the creation operator instead, we use the same property of the annihilation operator, and the dual correspondence, i.e. $\hat{a}|\alpha(t)\rangle = \alpha(t)|\alpha(t)\rangle$ implies by dual correspondence $\langle\alpha(t)|\,\hat{a}^\dagger = \langle\alpha(t)|\,\alpha^*(t)$. So $\langle\alpha(t)|\,\hat{Q}\,|\alpha(t)\rangle = \frac{\alpha(t)+\alpha^*(t)}{\sqrt{2}}$, and $\langle\alpha(t)|\,\hat{P}\,|\alpha(t)\rangle = \frac{\alpha(t)-\alpha^*(t)}{\sqrt{2}i}$.

c)[**5 points**] We have to compute the square of $\hat{Q}, \hat{P}$. The first one is $\hat{Q}^2 = \frac{1}{2}(\hat{a}+\hat{a}^\dagger)^2 = \frac{1}{2}(\hat{a}^2+\hat{a}^{\dagger 2}+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}) = \frac{1}{2}(\hat{a}^2+\hat{a}^{\dagger 2}+2\hat{a}^\dagger\hat{a}+1)$, where in the last step we have used the canonical commutation relation $[\hat{a},\hat{a}^\dagger] = 1$. The last form is called 'normal-ordered' form, where all the creation operators are on the left of all the annihilation operators. The normal-ordered form allows us to calculate the expectation value in any coherent state, since we can use $\hat{a}^p|\alpha\rangle = \alpha^p|\alpha\rangle$ and its dual $\langle\alpha|\,\hat{a}^{\dagger p} = \alpha^{*p}\langle\alpha|$. Using this trick, $\langle\alpha(t)|\,\hat{Q}^2\,|\alpha(t)\rangle = \frac{\alpha^2+\alpha^{*2}+2|\alpha|^2+1}{2}$. By the same kind of algebra, $\hat{P}^2 = -\frac{1}{2}(\hat{a}^2+\hat{a}^{\dagger 2}-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}) = -\frac{1}{2}(\hat{a}^2+\hat{a}^{\dagger 2}-2\hat{a}^\dagger\hat{a}-1)$ so $\langle\alpha(t)|\,\hat{P}^2\,|\alpha(t)\rangle = -\frac{\alpha^2+\alpha^{*2}-2|\alpha|^2-1}{2}$. This finally lets us calculate the two variances $\Delta\hat{Q} = \Delta\hat{P} = \frac{1}{\sqrt{2}}$, and therefore $\Delta\hat{Q}\Delta\hat{P} = \frac{1}{2}$, i.e. , the inequality in the Heisenberg uncertainty principle is saturated.

d)[**5 points**] In the Fock basis of eigenstates of the number operator, the trace of the thermal density matrix is a geometric series with ratio $e^{-\beta\hbar\omega_a} < 1$, which therefore converges and evaluates to $\mathrm{Tr}\hat{\rho} = \mathcal{N}\sum_{n=0}^{\infty}e^{-\beta\hbar\omega_a n} = \mathcal{N}/(1-e^{-\beta\hbar\omega_a}) = 1$ so $\mathcal{N} = 1-e^{-\beta\hbar\omega_a}$. This is the value of the corresponding canonical partition function , $\mathcal{N} = \mathcal{Z}$. Furthermore $\mathrm{Tr}(\hat{n}\hat{\rho}) = \mathcal{N}\sum_{n=0}^{\infty}ne^{-\beta\hbar\omega_a n} = -\mathcal{N}\frac{\partial}{\partial(\beta\hbar\omega_a)}\sum_{n=0}^{\infty}e^{-\beta\hbar\omega_a n} = -\mathcal{N}\frac{\partial}{\partial(\beta\hbar\omega_a)}\frac{1}{\mathcal{N}} = \frac{1}{\mathcal{N}}\frac{\partial\mathcal{N}}{\partial(\beta\hbar\omega_a)} = \frac{e^{-\beta\hbar\omega_a}}{1-e^{-\beta\hbar\omega_a}} = \frac{1}{e^{\beta\hbar\omega_a}-1}$, the Bose-Einstein distribution. Notice how the partition function acts as a generating function and how the first moment was calculated taking a partial derivative. For the second moment, we need to take a second derivative, $\mathrm{Tr}(\hat{n}^2\hat{\rho}) = \mathcal{N}\sum_{n=0}^{\infty}n^2e^{-\beta\hbar\omega_a n} =$

$\mathcal{N}\frac{\partial^2}{\partial^2(\beta\hbar\omega_a)}\frac{1}{\mathcal{N}}$. We need $\frac{\partial^2}{\partial x^2}\frac{1}{1-e^{-x}} = \frac{e^x(1+e^x)}{(-1+e^x)^3}$, so that $\mathrm{Tr}(\hat{n}^2\hat{\rho}) = \frac{1+e^{\beta\hbar\omega_a}}{(-1+e^{\beta\hbar\omega_a})^2}$, and then $\Delta\hat{n} = \frac{e^{\beta\hbar\omega_a/2}}{e^{\beta\hbar\omega_a}-1}$.

II. DRIVEN SIMPLE HARMONIC OSCILLATOR

Recall your results for a *time-dependent change of frame* from the previous problem set. Consider now the following time-dependent Hamiltonian describing a driven simple harmonic oscillator

$$\frac{\hat{H}(t)}{\hbar} = \omega_a \hat{a}^\dagger \hat{a} + \varepsilon_{\mathrm{d}}(t)(\hat{a} + \hat{a}^\dagger), \tag{1}$$

where the time-dependent function in the drive term is given by $\varepsilon_{\mathrm{d}}(t) = \varepsilon_d \sin(\omega_d t)$, determined by an off-resonant drive frequency ω_d, i.e. $\omega_d \neq \omega_a$. Show, by performing a unitary transformation $\hat{U}(t) = e^{-i\phi(t)}\hat{D}[\alpha(t)]$, where $\phi(t)$ is an unknown real function of time, and $\alpha(t)$ is an unknown complex function of time, with $\hat{D}[\alpha] = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, that you can perform a change of frame where the Hamiltonian is independent of terms linear in $\hat{a}, \hat{a}^\dagger$, i.e. $\hat{H}'/\hbar = \omega_a \hat{a}^\dagger\hat{a}$. Find the ordinary differential equations that must be obeyed by $\alpha(t)$ and $\phi(t)$, and give a physical interpretation (*Hint:* This is easiest by reverting back to phase space coordinates via $\alpha(t) = [Q(t) + iP(t)]/\sqrt{2}$, where the rescaled position Q and the rescaled momentum P are real functions of time). Suppose the system is initially prepared in the vacuum state $|0\rangle$ *in the frame of* $\hat{H}'$. What is the subsequent time evolution of this state in the laboratory frame (that of $\hat{H}(t)$)?

SOLUTION

Before beginning, recall the following from the previous problem set about frame changes. Take an arbitrary unitary $\hat{U}(t)$ and define a new ket $|\psi'(t)\rangle = \hat{U}^\dagger(t)|\psi(t)\rangle$. Inserting identity and multiplying from the left by $\hat{U}^\dagger(t)$, the time-dependent Schrödinger equation is

$$\hat{U}^\dagger(t)\left[\frac{\hat{H}(t)}{\hbar} - i\partial_t\right]\hat{U}(t)\hat{U}^\dagger(t)|\psi(t)\rangle = 0, \tag{2}$$

so we have

$$\hat{U}^\dagger(t)\left[\frac{\hat{H}(t)}{\hbar} - i\partial_t\right]\hat{U}(t)|\psi'(t)\rangle = 0. \tag{3}$$

Now the result is obtained by carefully handling the time derivative. It acts on everything to its right, including the unitary $\hat{U}(t)$, so $-i\partial_t\left[\hat{U}(t)\left|\psi'(t)\right\rangle\right] = -i\left[\partial_t\hat{U}(t)\right]\left|\psi'(t)\right\rangle +$ $-i\hat{U}(t)\left[\partial_t\left|\psi'(t)\right\rangle\right]$. So in the new frame we have the following time-dependent Schrödinger equation

$$\left[\frac{\hat{H}'(t)}{\hbar} - i\partial_t\right]\left|\psi'(t)\right\rangle = 0, \tag{4}$$

where

$$\begin{aligned}\hat{H}'(t)/\hbar &= \hat{U}^\dagger(t)\left[\hat{H}(t)/\hbar - i\partial_t\right]\hat{U}(t)\\ &= \hat{U}^\dagger(t)\hat{H}(t)\hat{U}(t)/\hbar - i\hat{U}^\dagger(t)\left[\partial_t\hat{U}(t)\right].\end{aligned} \tag{5}$$

The trickiest step in this problem is calculate the second term of Eq. (5). For the particular unitary chosen here, one has

$$\begin{aligned}-i\hat{U}^\dagger\partial_t\hat{U}(t) &= -ie^{i\phi(t)}\hat{D}^\dagger[\alpha(t)]\partial_t\left(\hat{D}[\alpha(t)]e^{-i\phi(t)}\right)\\ &= -ie^{i\phi(t)}\hat{D}^\dagger[\alpha(t)]\left(\partial_t\hat{D}[\alpha(t)]\right)e^{-i\phi(t)}\\ &\quad - ie^{i\phi(t)}\hat{D}^\dagger[\alpha(t)]\hat{D}[\alpha(t)]\partial_t\left(e^{-i\phi(t)}\right)\\ &= -i\hat{D}^\dagger[\alpha(t)]\partial_t\hat{D}[\alpha(t)] - \dot{\phi}(t),\end{aligned} \tag{6}$$

where we have abbreviated $\dot{\phi}(t) = \partial\phi(t)/\partial(t)$. To calculate that last bit, we use the identity

$$e^{-s\hat{A}(t)}\frac{d}{dt}e^{s\hat{A}(t)} = \int_0^s ds' e^{-s'\hat{A}(t)}\frac{d\hat{A}(t)}{dt}e^{s'\hat{A}(t)} \tag{7}$$

in the particular case of the displacement operator $\hat{D}[\alpha(t)] = e^{\alpha(t)\hat{a}^\dagger - \alpha^*(t)\hat{a}}$.

$$\begin{aligned}-i\hat{D}^\dagger[\alpha(t)]\frac{d}{dt}\hat{D}[\alpha(t)] &= -i\int_0^1 ds\hat{D}^\dagger[s\alpha(t)](-\dot{\alpha}^*\hat{a} + \dot{\alpha}\hat{a}^\dagger)\hat{D}[s\alpha(t)]\\ &= -i\left\{-\dot{\alpha}(t)^*\hat{a} + \dot{\alpha}\hat{a}^\dagger - \frac{1}{2}\left[\dot{\alpha}^*(t)\alpha(t) - \dot{\alpha}(t)\alpha^*(t)\right]\right\}.\end{aligned} \tag{8}$$

We are done with the harder part of the derivation. The first term of the transformed Hamiltonian is easier to express

$$\hat{U}^\dagger(t)\frac{\hat{H}}{\hbar}\hat{U}(t) = \omega_a(\hat{a}^\dagger + \alpha^*(t))(\hat{a} + \alpha(t)) + \epsilon_d(t)(\hat{a}^\dagger + \alpha^*(t) + \hat{a} + \alpha(t)). \tag{9}$$

To arrive at the simple form of $\hat{H}'/\hbar = \omega_a\hat{a}^\dagger\hat{a}$, all one needs to do is set the functions $\alpha(t)$, $\alpha^*(t)$ and $\phi(t)$ such that all terms linear in $\hat{a}$, $\hat{a}^\dagger$, and any eventual terms proportional to the identity operator, are completely canceled. This yields the following three equations

$$\begin{aligned}
i\dot{\alpha}^*(t) + \omega_a \alpha^*(t) + \epsilon_d(t) &= 0, \\
-i\dot{\alpha}(t) + \omega_a \alpha(t) + \epsilon_d(t) &= 0, \\
-\dot{\phi}(t) + \frac{1}{2} i(\dot{\alpha}^*(t)\alpha(t) - \dot{\alpha}(t)\alpha^*(t)) + \omega_a |\alpha(t)|^2 + \epsilon_d(t)(\alpha(t) + \alpha^*(t)) &= 0.
\end{aligned} \tag{10}$$

The first two equations are the classical equations of motion of a driven harmonic oscillator, and the third indicates that $\dot{\phi} \equiv \mathcal{L}(\alpha, \alpha^*, \dot{\alpha}, \dot{\alpha}^*)$ is the corresponding classical Lagrangian. The first two equations are easily seen to be the Euler-Lagrange equations $\frac{\partial \mathcal{L}}{\partial \alpha} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} = 0$ and $\frac{\partial \mathcal{L}}{\partial \alpha^*} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\alpha}^*} = 0$, i.e. α and α^* are taken here as independent degrees of freedom so that $\partial\alpha/\partial\alpha^* = 0$ and $\partial\alpha^*/\partial\alpha = 0$.

If the initial state is $|0\rangle$ in the frame of $\hat{H}'(t)$, then the time-evolved state will be $|\psi'(t)\rangle = e^{-i\hat{H}'t/\hbar}|0\rangle = |0\rangle$ in that frame, and then in the laboratory frame of $\hat{H}(t)$, $|\psi(t)\rangle = \hat{U}(t)|\psi'(t)\rangle = e^{-i\phi(t)}|\alpha(t)\rangle$, where $\alpha(t)$ obeys the differential equation derived above.

III. SIMPLIFIED MODEL FOR A TRANSMON QUBIT

The goal of this problem is to find the approximate solutions to Schrödinger's equation for a transmon qubit using perturbation theory, then analyze a simple single-qubit gate, the $\pi-$pulse. We consider the following Hamiltonian

$$\begin{aligned}
\hat{H} &= \hat{H}_0 + \hat{H}_J, \\
\hat{H}_0 &= \hbar\omega_a \hat{a}^\dagger \hat{a}, \\
\hat{H}_J &= \frac{\hbar\alpha}{12}(\hat{a} + \hat{a}^\dagger)^4 = \frac{\hbar\alpha}{4} I + \hbar\alpha\left[\hat{a}^\dagger\hat{a} + \frac{1}{2}(\hat{a}^2 + \hat{a}^{\dagger 2})\right] + \frac{\hbar\alpha}{12}(\hat{a}^4 + \hat{a}^{\dagger 4} + 4\hat{a}^{\dagger 3}\hat{a} + 4\hat{a}^\dagger\hat{a}^3 + 6\hat{a}^{\dagger 2}\hat{a}^2).
\end{aligned} \tag{11}$$

We assume the second equality in the last equation above. You do *not* have to derive it. We have denoted $\hat{H}_0$ the Hamiltonian corresponding to a simple harmonic oscillator, and $\hat{H}_J$ a perturbation proportional to the so-called anharmonicity of the transmon qubit, denoted by α. We assume $\omega_a > 0$ and $\alpha < 0$, and $\omega_a \gg |\alpha|$. Note that $\hat{a}$ and $\hat{a}^\dagger$ are the bosonic annihilation and creation operators obeying the canonical commutation relation $[\hat{a}, \hat{a}^\dagger] = I$, with I being the identity acting on the simple harmonic oscillator Hilbert space.

a) As a warm-up, recall your knowledge of the simple harmonic oscillator, and write down the spectrum of $\hat{H}_0$, i.e. the eigenvectors and eigenvalues arising from the eigenproblem $\hat{H}_0 |\psi\rangle = \hbar\omega_a \hat{a}^\dagger \hat{a} |\psi\rangle = E |\psi\rangle$. Express the eigenvectors in normalized form, in terms of the vacuum vector $|0\rangle$, which you will need to define.

b) Consider the perturbation $\hat{H}_J$. Show that it can be written as a sum of two operators: the first operator, which we will call $\hat{H}_J^{secular}$, commutes with the number operator $\hat{a}^\dagger \hat{a}$. The second operator, which we shall denote $\hat{H}_J^{nonsecular}$, that does not commute with the number operator. Read off the expressions of these two operators from the expanded form of $\hat{H}_J$. To save time, try to make the arguments above *without* explicitly evaluating the commutator.

c) Using the nondegenerate time-independent perturbation theory, find the lowest-order nontrivial correction to the eigenenergies of $\hat{H}_0$, and the correction to the eigenvectors of $\hat{H}_0$, due to $\hat{H}_J$. Show that the corrected eigenvectors are

$$\left|n^{(1)}\right\rangle = |n\rangle + \mu_{n+2,n} |n+2\rangle + \mu_{n-2,n} |n-2\rangle + \mu_{n+4,n} |n+4\rangle + \mu_{n-4,n} |n-4\rangle\,, \tag{12}$$

in terms of the eigenstates $|n\rangle$ of the simple harmonic oscillator found at a). Find the coefficients μ_{ij} in the expression above, according to perturbation theory, without normalizing the corrected eigenvectors $\left|n^{(1)}\right\rangle$.

SOLUTION

a) Defining the vacuum to be the unique state $|0\rangle$ that obeys $\hat{a}|0\rangle = 0$, we let $|n\rangle = \frac{\hat{a}^{\dagger n}}{\sqrt{n!}} |0\rangle$. Then $\hat{H}_0 |n\rangle = E_n |n\rangle = n\hbar\omega_a |n\rangle$.

b) The perturbation can be split into two terms,

$$\begin{aligned} \hat{H}_J^{secular} &= \frac{\hbar\alpha}{4} I + \hbar\alpha \hat{a}^\dagger \hat{a} + \frac{\hbar\alpha}{2} \hat{a}^{\dagger 2} \hat{a}^2 = \frac{\hbar\alpha}{4} I + \hbar\alpha \hat{a}^\dagger \hat{a} + \frac{\hbar\alpha}{2} \hat{a}^\dagger \hat{a} (\hat{a}^\dagger \hat{a} - 1) \\ \hat{H}_J^{nonsecular} &= \hat{H}_J - \hat{H}_J^{secular} = \frac{\hbar\alpha}{2} (\hat{a}^2 + \hat{a}^{\dagger 2}) + \frac{\hbar\alpha}{12} (\hat{a}^4 + \hat{a}^{\dagger 4} + 4\hat{a}^{\dagger 3}\hat{a} + 4\hat{a}^\dagger \hat{a}^3). \end{aligned} \tag{13}$$

The first one is diagonal in the Fock basis $|n\rangle$ defined above, whereas the second one is not. This implies that $\hat{H}_J^{secular}$ commutes with the number operator $\hat{a}^\dagger \hat{a}$, whereas $\hat{H}_J^{nonsecular}$ does not.

c) By using the formula in the course notes [Eq. (23)] the first-order correction to the eigenenergies is

$$E_n^{(0)} + E_n^{(1)} = n\hbar\omega_a + \langle n|\,\hat{H}_J\,|n\rangle = n\hbar\omega_a + \langle n|\,\hat{H}_J^{secular}\,|n\rangle = n\hbar\omega_a + \frac{\hbar\alpha}{4} + n\hbar\alpha + \frac{n(n-1)}{2}\hbar\alpha. \tag{14}$$

Only the part of the perturbation that is diagonal in the eigenbasis of H_0, $\hat{H}_J^{secular}$, contributes to the eigenenergy correction at linear order in α.

To obtain the corrections for the eigenvectors, we apply Eq. (24) of nondegenerate time-independent perturbation theory, to obtain

$$\left|n^{(1)}\right\rangle = |n\rangle + \sum_{k\neq n} |k\rangle \frac{\langle k|\,\hat{H}_J^{nonsecular}\,|n\rangle}{E_n - E_k} + O(\alpha^2). \tag{15}$$

Note that all kets, bras, and energies on the right-hand side above stand for the unperturbed kets, bras, and eigenenergies that you determined at a).

To ease notations, let

$$\hat{v} = \frac{\hat{H}_J^{nonsecular}}{\hbar\alpha} = \frac{1}{2}(\hat{a}^2 + \hat{a}^{\dagger 2}) + \frac{1}{12}(\hat{a}^4 + \hat{a}^{\dagger 4} + 4\hat{a}^{\dagger 3}\hat{a} + 4\hat{a}^{\dagger}\hat{a}^3). \tag{16}$$

The following are the only nonzero elements of the matrix representation of the operator $\hat{v}$ over Fock states

$$\begin{aligned} \langle n+2|\,\hat{v}\,|n\rangle &= \langle n|\,\hat{v}\,|n+2\rangle = \frac{1}{2}\sqrt{(n+1)(n+2)} + \frac{n}{3}\sqrt{(n+1)(n+2)}, \\ \langle n+4|\,\hat{v}\,|n\rangle &= \langle n|\,\hat{v}\,|n+4\rangle = \frac{1}{12}\sqrt{(n+1)(n+2)(n+3)(n+4)}. \end{aligned} \tag{17}$$

Intuitively, $\hat{v}$ only couples Fock states that differ by 2 or 4 excitations.

Using then Eq. (15) we arrive at the corrected eigenvector to first order in α

$$\left|n^{(1)}\right\rangle = |n\rangle + \mu_{n+2,n}\,|n+2\rangle + \mu_{n-2,n}\,|n-2\rangle + \mu_{n+4,n}\,|n+4\rangle + \mu_{n-4,n}\,|n-4\rangle\,, \tag{18}$$

where the coefficients are

$$\begin{aligned} \mu_{ij} &= \frac{\hbar\alpha}{E_j - E_i}\,\langle i|\,\hat{v}\,|j\rangle \text{ for all } i, j \geq 0, \\ \mu_{ij} &= 0 \text{ if } i \text{ or } j < 0. \end{aligned} \tag{19}$$

The matrix element of the dimensionless perturbation operator $\hat{v}$ is only nonzero between Fock states whose indices differ by 2 or 4 as in Eq. (17). The norm of the corrected eigenstate Eq. (18) is $\sqrt{\left\langle n^{(1)}\middle|n^{(1)}\right\rangle} = \sqrt{1 + O(\alpha^2)} = 1 + O(\alpha^2)$. If we aim to report all results to order α, we do not need to worry about evaluating this norm and normalizing the eigenvector $\left|n^{(1)}\right\rangle$.