Mathematical methods for quantum engineering
Exercise on dynamical systems and control

pierre.rouchon@minesparis.psl.eu, October 11, 2023

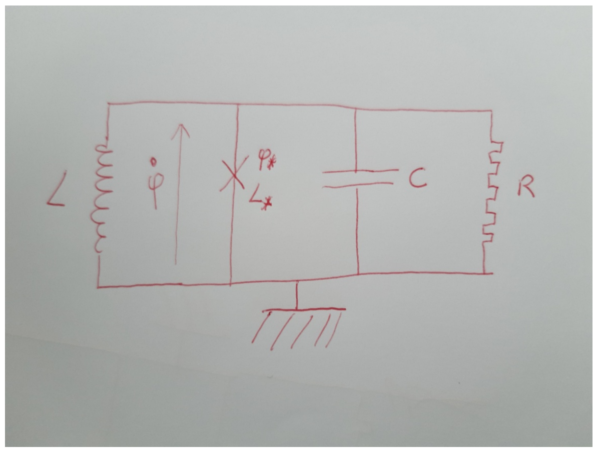

Figure 1: A nonlinear electrical circuit with a capacitor $C > 0$, a resistance $R > 0$, a linear inductance $L > 0$ and a Josephson junction with parameter $L_* > 0$ and $\phi_* \geq 0$.

The dynamical model of the above electrical circuit is

$$C \frac{d^2}{dt^2}\phi(t) = -\frac{\phi(t)}{L} - \frac{\phi_*}{L_*}\sin\left(\frac{\phi(t)}{\phi_*}\right) - \frac{1}{R}\frac{d}{dt}\phi(t)$$

where ϕ is the flux depending on the time t.

1. Assume $R = +\infty$ and $\phi_* = 0$.

   (a) Express the above second-order scalar differential equation as a set of two first-order scalar differential equations with state $x = (\phi, \dot\phi)^T$.

   (b) Show that the solution $\phi(t)$ reads

   $$\phi(t) = \phi(0)\cos(\omega t) + \frac{\dot\phi(0)}{\omega}\sin(\omega t)$$

   and define ω versus L and C.

   (c) Set $z = \phi + i\dot\phi/\omega$, a complex number $i = \sqrt{-1}$. Show that $\frac{d}{dt}z = -i\omega z$. Prove that $z(t) = e^{-i\omega t}z(0)$.

2. Assume $R > 0$ finite and $\phi_* = 0$

   (a) Show that $\frac{d^2}{dt^2}\phi = -\omega^2\phi - 2\xi\omega\frac{d}{dt}\phi$ where ξ and ω have to be defined.

   (b) Express the above second-order scalar differential equation as a set of two first-order scalar differential equations.

(c) Prove that this system is exponentially stable and that $\lim_{t\mapsto+\infty}\phi(t) = 0$.

(d) Discuss versus ξ the position of its eigenvalues in the complex plane.

3. Assume $R = +\infty$ and $\phi_* > 0$

   (a) Express the above second-order scalar differential equation as a set of two first-order scalar differential equations with state $x = (\phi, \dot{\phi})^T$.

   (b) Show that then $L_* \geq L$, exists a unique equilibrium to be define. Compute the first variation around this equilibrium and its eigenvalues. Discuss its stability.

   (c) Assume $L_*/L = 2/(3\pi)$. Show that exist several equilibria to be defined and discuss their stabilities.

4. Assume $R > 0$ finite and $\phi_* > 0$

   (a) Set $E(\phi, \dot{\phi}) = \frac{\phi^2}{2L} - \frac{\phi_*^2}{L_*}\cos\left(\frac{\phi(t)}{\phi_*}\right) + \frac{C}{2}\dot{\phi}^2$. Show that $\frac{d}{dt}E \leq 0$. Conclude that any solution $\phi(t)$ converges to an equilibrium denoted by $\bar{\phi}$ and discuss its stability versus $\bar{\phi}$.

   (b) Denote by $\bar{\theta}$, the unique solution in $]\pi/2, 3\pi/2[$ of $\tan\bar{\theta} = \bar{\theta}$. Assume that $L_*/L = -\cos\bar{\theta}$. Show that $\bar{\phi} = \bar{\theta}\phi_*$ is an equilibrium. Compute its eigenvalues and discuss its stability.

5. Assume $R > 0$ finite, $\phi_* > 0$ and consider the controlled dynamics

$$C\frac{d^2}{dt^2}\phi = -\frac{\phi}{L} - \frac{\phi_*}{L_*}\sin\left(\frac{\phi}{\phi_*}\right) - \frac{1}{R}\frac{d}{dt}\phi + u$$

   with the scalar control input u.

   (a) Propose a feedback law of the form $u = f(\phi)$ that stabilizes the equilibrium ($\phi = 0, u = 0$).

   (b) Assume $R > 0$ and very large. Derive an approximate control model of state dimension one. Propose a feedback law stabilizing the system around any set-point value of the flux ϕ_{sp}.