Problem Set 2
(M1 Math Methods 2023-2024)

This problem set is due on Friday, September 29th, 2023, at 23:59. The solutions should be sent as a single PDF (scan of handwritten document or LaTeX) to alexandru.petrescu@minesparis.psl.eu by the deadline. If you collaborate with a colleague, please write clearly their names at the top of your solution sheets. If you submit late without a satisfactory reason, the set will be accepted with a 10% penalty in the score.

I. CONTROLLING A QUBIT (20 POINTS)

Consider a qubit under a monochromatic control tone

$$\frac{\hat{H}}{\hbar} = \frac{1}{2}\omega_q \hat{\sigma}_z + \varepsilon_d \cos(\omega_d t)\hat{\sigma}_x. \tag{1}$$

Assume that we choose our frequencies such that $\omega_d, \omega_q \gg \epsilon_d$.

a) *Time-dependent change of frame.* Show that, given the time-dependent Schrödinger equation for an eigenstate of the Hamiltonian $|\psi(t)\rangle$,

$$\left[\frac{\hat{H}(t)}{\hbar} - i\partial_t\right]|\psi(t)\rangle = 0, \tag{2}$$

and an arbitrary time-dependent unitary transformation $\hat{U}(t)$, then the transformed states $\hat{U}^\dagger(t)\,|\psi(t)\rangle$ obey the transformed time-dependent Schrödinger equation

$$\begin{aligned} \left[\frac{\hat{H}'(t)}{\hbar} - i\partial_t\right]\hat{U}^\dagger(t)\,|\psi(t)\rangle = 0, \\ \frac{\hat{H}'}{\hbar}(t) \equiv \hat{U}^\dagger(t)\left[\frac{\hat{H}}{\hbar} - i\partial_t\right]\hat{U}(t). \end{aligned} \tag{3}$$

b) Go to a 'rotating' frame by applying the unitary transformation $\hat{U}(t) = \exp\left(-i\omega_d\hat{\sigma}_z t/2\right)$ and show that in this new rotating frame, up to neglecting terms that contain phase factors that oscillate at large frequencies $\pm(\omega_d + \omega_q)$, the Hamiltonian governing the dynamics is (introducing $\Delta \equiv \omega_q - \omega_d$)

$$\frac{1}{\hbar}\hat{H}' \approx \frac{1}{2}\Delta\hat{\sigma}_z + \frac{1}{2}\varepsilon_d\hat{\sigma}_x. \tag{4}$$

c) Solve the time-dependent Schrödinger equation in this frame, then report the result in the original laboratory frame, assuming the system is prepared in the initial state $|\psi(t=0)\rangle = |0\rangle$.

d) Now fix $\omega_d = \omega_q$ and find the amount of time it will take the system to go to the excited state $|1\rangle$ (this is a π pulse or NOT gate). What about the time to evolve from $|0\rangle$ into the linear combination $|-\rangle = (|0\rangle - i\,|1\rangle)/\sqrt{2}$ (this would amount to a $\pi/2$ pulse)?

e) Numerically integrate the Schrödinger equation corresponding to Hamiltonian Eq. (1) with initial condition $|\psi(t=0)\rangle = |0\rangle$ for the following parameters $\omega_q/(2\pi) = \omega_d/(2\pi) = 5$GHz, $\epsilon_d/(2\pi) = 0.25$GHz, and plot the three components of the Bloch vector $\vec{r}$ corresponding to the density matrix, i.e. $\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)| \equiv \frac{1+\vec{r}(t)\cdot\vec{\sigma}}{2}$. Graphically compare the numerical solution to your result in b). Find a sufficiently small control drive amplitude ϵ_d for which the π-pulse infidelity, defined as the minimum of $1 - \mathrm{Tr}\{\hat{\rho}(t)\hat{\rho}_{\text{target}}\}$ as time t is varied with $\hat{\rho}_{\text{target}} = |1\rangle\langle 1|$, is smaller than 5%.

f) Suppose the system is in state $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Now assume that you measure $\hat{\sigma}_z$, for which the two projectors are $\hat{P}_1 = |1\rangle\langle 1|$ and $\hat{P}_0 = |0\rangle\langle 0|$. What is the probability of measuring 0 (or 1) and what is the state of the system right after the measurement in each case? Now, assume that you performed the same measurement, but *without looking at the outcome*. What is the state of the system? *Hint:* This is represented by a density matrix corresponding to a pure state ensemble defined by the states and probabilities that you determined in the previous question.

g) Suppose that someone prepared the system in either the pure state $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ or the mixed state $\rho = \frac{\hat{P}_0+\hat{P}_1}{2}$. Can you, by performing a combination of π or $\pi/2$ pulses, and $\hat{\sigma}_z$ measurements, figure out whether the system was prepared in one or the other?

II. COHERENT INTERACTION BETWEEN TWO QUBITS (20 POINTS)

Consider the Hamiltonian

$$\frac{\hat{H}}{\hbar} = \frac{\omega_q}{2}\left(\hat{\sigma}_{z1} + \hat{\sigma}_{z2}\right) + g\left(\hat{\sigma}_{+,1}\hat{\sigma}_{-,2} + \hat{\sigma}_{+,2}\hat{\sigma}_{-,1}\right), \tag{5}$$

σ_±,i = (σ_x,i ± i * σ_y,i) / 2

where $\hat{\sigma}_{z1}$ is shorthand for $\hat{\sigma}_{z1} \otimes I_2$, and $\hat{\sigma}_{z2}$ for $I_1 \otimes \hat{\sigma}_{z2}$, and $\hat{\sigma}_{+,1}\hat{\sigma}_{-,2} \equiv \hat{\sigma}_{+,1} \otimes \hat{\sigma}_{-,2}$ etc. Note that $\sigma_{\pm,i} \equiv \frac{\sigma_{xi} \pm \sigma_{yi}}{2}$, I_i is the 2×2 identity matrix, and $\sigma_{xi}, \sigma_{yi}, \sigma_{zi}$ are the three Pauli matrices acting on the qubit $i = 1, 2$.

a) Find the eigenstates and eigenvalues of this Hamiltonian, noting that the observable corresponding to the total number of excitations, $\hat{\sigma}_{z1} + \hat{\sigma}_{z2}$, commutes with the Hamiltonian.

b) Suppose the system is initialized in the one-excitation state $|\psi(t=0)\rangle = |10\rangle$, corresponding to qubit 1 being in the excited state and qubit 2 in the ground state. What is the time evolution of this ket under the Hamiltonian above? How much time will it take for the excitation to get transfered from the first qubit to the second, i.e. find the smallest $t_{swap} > 0$ for which $|\psi(t_{swap})\rangle = |01\rangle$. Similarly, what is $t_{entangle} < t_{swap}$ for which $|\psi(t_{entangle})\rangle = \frac{|10\rangle - i|01\rangle}{\sqrt{2}}$? Find the reduced density matrix corresponding to the first qubit, $\rho_1(t) = \mathrm{Tr}_2\, |\psi(t)\rangle\,\langle\psi(t)|$, and express its purity, defined as $\mathrm{Tr}\left\{\hat{\rho}_1^2(t)\right\}$, as a function of time.

c) Suppose you measured σ_{z1} exactly at $t_{entangle}$ above, and found outcome '1'. What is the probability of this outcome, and what is state of the system right after this measurement?