EXERCISES
NOTES
IMPORTANT
PROOF

Lecture 2

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We use in this lecture Chapter 2 of Nielsen and Chuang, *Quantum computation and quantum information*, Cambridge University Press (2010) and A. Messiah, *Quantum mechanics*, North Holland Publishing Company (1961). This lecture will cover Heisenberg's equation of motion, Dirac picture, the density operator and von Neumann's equation, and introduce and solve a number of simple Hamiltonians corresponding to a spin$-\frac{1}{2}$, a harmonic oscillator, and the two coupled together.

I. HEISENBERG AND DIRAC PICTURES

Let's revisit the second postulate and go into more detail regarding the time evolution operator. We have

$$|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle , \tag{1}$$

where U is a unitary operator. For Hamiltonian H time-independent, we have $U(t, t_0) = e^{-iH(t-t_0)/\hbar}$. Defining the derivative with respect to time of an operator $O(t)$ as the limit $\lim_{\epsilon\to 0}\frac{O(t+\epsilon)-O(t)}{\epsilon}$, one can show

$$i\hbar\frac{dU(t, t_0)}{dt} = HU(t, t_0). \tag{2}$$

$U(t, t_0)$ solves this first-order ordinary differential equation with initial condition $U(t_0, t_0) = I$. Even if H is time-dependent, and so $U(t, t_0) \neq e^{-iH(t-t_0)/\hbar}$, the equations above can be postulated as the definition of U.

The equations above are equivalent to an integral equation, $U(t, t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} HU(t', t_0)$. Differentiating Eq. (1) with respect to time gives $\frac{d}{dt}|\psi(t)\rangle = \frac{d}{dt}U(t, t_0)|\psi(t_0)\rangle$, and using

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Eq. (2), one finds $i\hbar\frac{d}{dt}|\psi(t)\rangle = HU(t,t_0)|\psi(t_0)\rangle = H|\psi(t)\rangle$. So we could have taken Eq. (1) and Eq. (2) as postulates, and only then derived the Schrödinger equation formulation of the second postulate.

Exercises: a) Show that if $U(t,t_0)$ is differentiable with respect to time t and unitary, then $H(t) = i\hbar\left(\frac{d}{dt}U(t)\right)U^\dagger(t)$ is Hermitian. b) If $U(t)$ satisfies $i\hbar\frac{dU}{dt} = HU$, with H Hermitian and time-dependent, then $U^\dagger U$ is time independent, and $i\hbar\frac{d}{dt}(UU^\dagger) = [H, UU^\dagger]$. In particular, if $U(t=t_0)$ is unitary, then it remains so at all times $t \geq t_0$.

So far, we have formulated the postulates of quantum mechanics in the *Schrödinger picture*. There is an equivalent formulation of the second postulate, in what is called the *Heisenberg picture*. We establish below the relationship between these two pictures. For ease of interpretation, we denote quantities pertaining to Schrödinger picture with a subscript S, and those pertaining to Heisenberg picture by a subscript H. Schrödinger picture states, as discussed so far, are time-dependent $|\psi_S(t)\rangle = U(t,t_0)|\psi_S(t_0)\rangle$. We can turn them into time-independent kets by applying the unitary operator $U^\dagger(t,t_0)$. This gives $|\psi_H(t)\rangle \equiv U^\dagger(t,t_0)|\psi_S(t)\rangle = |\psi_S(t_0)\rangle$. As opposed to their counterparts in the Schrödinger picture, states are time-independent in the Heisenberg picture. On the other hand, time-independent observables become time dependent, namely $O_H(t) \equiv U^\dagger(t,t_0)O_S U(t,t_0)$. The previous equation can be written more generally as $O_H(t) \equiv U^\dagger(t,t_0)O_S(t)U(t,t_0)$ to allow for an observable that has an explicit time-dependence in the Schrödinger picture (such as, for example, terms entering the Hamiltonian of an externally-controlled system).

Differentiating the previous equation term by term, using the differential definition of the time-evolution operator Eq. (2), we find $i\hbar\frac{d}{dt}O_H(t) = i\hbar\dot{U}^\dagger(t,t_0)O_S(t)U(t,t_0) + i\hbar U^\dagger(t,t_0)\dot{O}_S(t)U(t,t_0) + i\hbar U^\dagger(t,t_0)O_S(t)\dot{U}(t,t_0) = U^\dagger(t,t_0)[O_S(t), H(t)]U(t,t_0) + i\hbar U^\dagger(t,t_0)\frac{\partial O_S}{\partial t}U(t,t_0)$. For the last term we write $U^\dagger(t,t_0)\frac{\partial O_S}{\partial t}U(t,t_0) = \frac{\partial O_H}{\partial t}$, with the understanding that this is the Heisenberg-picture operator corresponding to the Schrödinger picture operator $\partial O_S(t)/\partial t$. Moreover, letting $H_H(t) = U^\dagger(t,t_0)H(t)U(t,t_0)$, we arrive at the *Heisenberg equation of motion*

$$i\hbar\frac{dO_H(t)}{dt} = [O_H(t), H_H(t)] + i\hbar\frac{\partial O_H}{\partial t}. \tag{3}$$

Exercises: a) Let $H = \frac{\hbar\omega_q}{2}\sigma^z$ and let $\sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y)$. Find $\sigma_H^\pm(t), \sigma_H^z(t)$. b) If $[O, H] = 0$, and O is explicitly time independent, then O_H is time independent.

There is an intermediate picture, called the *interaction*, or *Dirac*, picture. We present it here since our future discussion of perturbation theory will rely on it. We postulated Eq. (2), the differential definition of the time evolution operator $U(t,t_0)$. Suppose you knew an approximate solution to this equation, $U^{(0)}(t,t_0)$. It is convenient to set $U(t,t_0) = U^{(0)}(t,t_0)U'(t,t_0)$, where $U'(t,t_0)$ is also unitary. To piece together a solution $U(t,t_0)$, we are interested in the dynamics of $U'(t,t_0)$. By using Eq. (2), we have

$$i\hbar\frac{d}{dt}U'(t,t_0) = U^{(0)\dagger}(t,t_0)\left[H(t)U^{(0)}(t,t_0) - i\hbar\frac{dU^{(0)}(t,t_0)}{dt}\right]U'(t,t_0), \tag{4}$$

with initial condition $U'(t_0,t_0) = I$. If $U^{(0)}(t,t_0)$ was a good enough approximate solution to Eq. (2), then the term in the bracket almost vanishes, and hence $U'(t,t_0)$ is almost constant. Let's define a new Hamiltonian $H^{(0)}(t) = i\hbar\left[\frac{d}{dt}U^{(0)}(t,t_0)\right]U^{(0)\dagger}(t,t_0)$ such that $i\hbar\frac{d}{dt}U^{(0)}(t,t_0) = H^{(0)}(t)U^{(0)}(t,t_0)$. Then let $H = H^{(0)}+H'$. In accordance with the discussion above, H' is an operator considered as perturbation, and $H^{(0)}$ is an operator whose time-evolution operator is known. Then

$$i\hbar\frac{d}{dt}U'(t,t_0) = H_I'U'(t,t_0), \tag{5}$$

where $H_I'(t) = U^{(0)\dagger}(t,t_0)H'U^{(0)}(t,t_0)$. The task of perturbation theory, as we will discuss in Lecture 3, is to provide strategies to solve the equation above. As for kets and operators, just as in the Heisenberg picture we may write $|\psi_I(t)\rangle = U^{(0)^\dagger}(t,t_0)\,|\psi_S(t)\rangle$, and $O_I(t) = U^{(0)^\dagger}(t,t_0)O_SU^{(0)}(t,t_0)$. Hence, the Schrödinger equation in the interaction picture is $i\hbar\frac{d}{dt}|\psi_I(t)\rangle = H_I'|\psi_I(t)\rangle$, and the equation analogue to the Heisenberg equation of motion is $i\hbar\frac{d}{dt}O_I = [O_I, H_I^{(0)}] + i\hbar\frac{\partial O_I}{\partial t}$.

II. THE DENSITY OPERATOR

This formulation is equivalent to the formulation presented thus far, which was in terms of state vectors. Its advantage is that it provides a more convenient picture when thinking about some commonly encountered scenarios in quantum mechanics, such as when systems become open.

We first want to define an *ensemble of quantum states.* Suppose a quantum system is in one of the states $\{|\psi_i\rangle\}$ with probabilities $\{p_i\}$. Then we call $\{p_i, |\psi_i\rangle\}$ an ensemble of pure states. The density operator is defined as $\rho = \sum_i p_i\,|\psi_i\rangle\,\langle\psi_i|$. The above is a reformulation of *Postulate 1.* All other postulates have formulations in terms of the density operator.

Postulate 2. Under time evolution, $\rho(t_0) = \sum_i p_i |\psi_i(t_0)\rangle \langle\psi_i(t_0)|$ evolves to $U(t,t_0)\rho(t_0)U^\dagger(t,t_0) = \sum_i p_i U(t,t_0) |\psi_i(t_0)\rangle \langle\psi_i(t_0)| U^\dagger(t,t_0) = \rho(t)$. As an *exercise*, one can prove the *von Neumann equation*. With $\rho(t)$ as above, differentiating with respect to time and using the differential definition of $U(t,t_0)$ Eq. (2), we have

$$i\hbar\frac{d}{dt}\rho = [H, \rho(t)]. \tag{6}$$

Note, however, that $i\hbar\frac{d}{dt}\rho_H = 0$, where ρ_H is the Heisenberg picture density matrix.

For *postulate 3*, applied to state i, the probability to measure outcome m conditioned on the fact that the system was prepared in state i prior to measurement, denoted now $p(m|i)$, is given by $p(m|i) = \langle\psi_i| M_m^\dagger M_m |\psi_i\rangle = \mathrm{Tr}\{M_m^\dagger M_m |\psi_i\rangle\langle\psi_i|\}$. By the law of total probability, the probability to measure outcome m if the system is in a state decribed by density operator ρ is given by $p(m) = \sum_i p_i p(m|i) = \sum_i p_i \mathrm{Tr}\{M_m^\dagger M_m |\psi_i\rangle\langle\psi_i|\} = \mathrm{Tr}\{M_m^\dagger M_m \rho\}$. The state of the system right after measuring outcome m is obtained by applying Postulate 3 for state i. If the system is in state i, the state right after measuring outcome m is $|\psi_i^m\rangle = \frac{M_m|\psi_i\rangle}{\sqrt{p(m|i)}}$. Thus, if one started out in an ensemble $\{p_i, |\psi_i\rangle\}$, immediately after measuring outcome m one would be in an ensemble $\{p(i|m), |\psi_i^m\rangle\}$, where the conditional probability is given by Bayes' rule $p(i|m) = p(m,i)/p(m) = p(m|i)p_i/p(m)$. Thus the density matrix right after measuring m is given by

$$\rho_m = \sum_i p(i|m) |\psi_i^m\rangle\langle\psi_i^m| = \sum_i \frac{p_i M_m |\psi_i\rangle\langle\psi_i| M_m^\dagger}{p(m)} = \frac{M_m \rho M_m^\dagger}{\mathrm{Tr}\{M_m \rho M_m^\dagger\}}. \tag{7}$$

A *pure state* corresponds to an ensemble with a single state, $|\psi\rangle$ with corresponding probability $p = 1$, and $\rho = |\psi\rangle\langle\psi|$. Otherwise, ρ is said to be a *mixed state*. A *mixture of states* ρ_i describes a situation where a system is prepared in state ρ_i with probability p_i. Then one can show that the density operator corresponding to this is $\rho = \sum_i p_i \rho_i$. To see this, simply imagine ρ_i arises from ensemble $\{p_{ij}, |\psi_{ij}\rangle\}$, for each state i. Then the probability of being in the state $|\psi_{ij}\rangle$ is $p_i p_{ij}$, and the resulting ensemble is $\{p_i p_{ij}, |\psi_{ij}\rangle\}$, now sweeping over both i and j, so $\rho = \sum_{ij} p_i p_{ij} |\psi_{ij}\rangle\langle\psi_{ij}| = \sum_i p_i \rho_i$.

As an *example* of the situation described by a mixture of states, suppose you measured a system. According to the analysis above, you would have state ρ_m with probability $p(m)$, for every possible outcome m. Suppose that the measurement outcome was *lost*. Then the state of such a system would be described by $\rho = \sum_m p(m)\rho_m = \sum_m \mathrm{Tr}\{M_m^\dagger M_m \rho\} \frac{M_m \rho M_m^\dagger}{\mathrm{Tr}\{M_m^\dagger M_m \rho\}} = \sum_m M_m \rho M_m^\dagger$.

Theorem. An operator ρ is a density operator associated with some pure state ensemble $\{p_i, |\psi_i\rangle\}$ iff $\text{Tr}\rho = 1$ and ρ is a positive operator. *Proof:* $\rightarrow$ follows from definitions. For $\leftarrow$, ρ is positive implies it is Hermitian, which implies it is normal, so it has a spectral decomposition $\rho = \sum_i \lambda_i |i\rangle \langle i|$ with all $\lambda_i \geq 0$. ρ has trace 1 so $\sum_i \lambda_i = 1$. This defines the appropriate pure state ensemble $\{\lambda_i, |i\rangle\}$.

Exercise. Show that a density operator ρ is pure iff $\text{Tr}\rho^2 = 1$.

Exercise. Two ensembles can give the same density operator ρ. For example $\rho = \frac{3}{4}|0\rangle \langle 0| + \frac{1}{4}|1\rangle \langle 1|$ corresponds to both $\left\{\left(\frac{3}{4}, |0\rangle\right), \left(\frac{1}{4}, |1\rangle\right)\right\}$, and $\left\{\left(\frac{1}{2}, |a\rangle\right), \left(\frac{1}{2}, |b\rangle\right)\right\}$. Show that $|a/b\rangle = \sqrt{\frac{3}{4}}|0\rangle \pm \sqrt{\frac{1}{4}}|1\rangle$.

Exercise: Bloch sphere representation of a spin$-\frac{1}{2}$. Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. a) Show that the normalization condition $|\alpha|^2 + |\beta|^2 = 1$ means that one can rewrite $|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right)$. b) Show that an arbitrary density matrix for a mixed state or for a pure state of a qubit may be written as $\rho = \frac{I+\vec{r}\cdot\vec{\sigma}}{2}$, where $|\vec{r}| \leq 1$ is a real 3-vector, called the *Bloch vector* for state ρ. c) Find $\vec{r}$ for $\rho = I/2$. d) Show that ρ is pure iff $|\vec{r}| = 1$. e) Show that if ρ is pure, then $\vec{r} = \left(\cos\varphi\sin\frac{\theta}{2}, \sin\varphi\sin\frac{\theta}{2}, \cos\frac{\theta}{2}\right)$, with θ, φ the same as in part a) (the overall phase γ is irrelevant, as discussed previously).

In a tensor product space, the reduced density matrix is defined by a partial trace of the density matrix. If ρ^{AB} defines the state of a composite system formed by subsystems A and B, then $\rho^A = \text{Tr}_B\rho^{AB}$. By definition, we write $\text{Tr}_B\left(|a_1\rangle \langle a_2| \otimes |b_1\rangle \langle b_2|\right) \equiv |a_1\rangle \langle a_2| \,\text{Tr}(|b_1\rangle \langle b_2|) = |a_1\rangle \langle a_2| \langle b_2|b_1\rangle$.

Example. Consider a 2-qubit system prepared in the Bell state $\frac{|00\rangle+|11\rangle}{\sqrt{2}} = |\psi\rangle$. Trace out the second qubit and find the reduced density matrix of the first. What is the corresponding Bloch vector? (Answer: you should find $\vec{r} = 0$. That the reduced density matrix is not pure is an indication of quantum entanglement in the state of the full system.)

III. EXAMPLES OF PHYSICAL SYSTEMS

A. Fock space and the simple harmonic oscillator

The purpose of this section is to introduce bosonic creation and annihilation operators and some useful identities for them. We will do that by refering to the standard treatment of the simple harmonic oscillator.

The simple harmonic oscillator is described by the following Hamiltonian in the position representation

$$\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}. \tag{8}$$

The position q and momentum p are canonically conjugate Hermitian operators, they obey $[q,p] = i\hbar$. Putting $q = \sqrt{\frac{\hbar}{m\omega}}Q$ and $p = \sqrt{m\hbar\omega}P$, we have $H = \mathcal{H}/\hbar\omega = \frac{1}{2}(P^2 + Q^2)$, where $[Q,P] = i$. This would lead to the following differential equation $\frac{1}{2}\left(-d^2/dQ^2 + Q^2\right)\psi(Q) = E\psi(Q)$. This is not a course on wave mechanics, so we'll set this aside and move on to a different representation, amounting to the algebraic solution of the simple harmonic oscillator.

Let $a = \frac{1}{\sqrt{2}}(Q + iP)$ and $a^\dagger = \frac{1}{\sqrt{2}}(Q - iP)$ be the creation and annihilation operator, respectively, which are Hermitian conjugates of each other. They obey the canonical commutator $[a, a^\dagger] = 1$. Then $H = \frac{1}{2}(aa^\dagger + a^\dagger a) = N + \frac{1}{2}$, where $N = a^\dagger a$. Using $[A, BC] = [A, B]C + B[A, C]$, one can show $Na = a(N-1)$ and $Na^\dagger = a^\dagger(N+1)$.

The problem of solving the Schrödinger equation for the simple harmonic oscillator translates then to finding the eigenvectors and eigenvalues of N.

Theorem. Let $|v\rangle$ be an eigenvector of N with eigenvalue v. Then (a) $v \geq 0$; (b) if $v = 0$, then $a|v\rangle = 0$, else $a|v\rangle$ is a nonzero vector whose norm squared is $v\langle v|v\rangle$, and it is an eigenvector of N, whose eigenvalue is $v-1$; (c) $a^\dagger|v\rangle \neq 0$. Its norm is $\langle v|v\rangle(v+1)$, and it is an eigenvector of N with eigenvalue $v+1$.

Proof. $N|v\rangle = v|v\rangle$ and $\langle v|v\rangle > 0$ by hypothesis. The main step of the proof is to calculate the norms of $a|v\rangle$ and $a^\dagger|v\rangle$. These are

$$\begin{aligned} \langle v|\, a^\dagger a\, |v\rangle &= \langle v|\, N\, |v\rangle = v\langle v|v\rangle, \\ \langle v|\, aa^\dagger\, |v\rangle &= \langle v|\, (N+1)\, |v\rangle = (v+1)\langle v|v\rangle. \end{aligned} \tag{9}$$

Since $|v\rangle$ belongs to the state space, $\langle v|v\rangle \geq 0$, and only $|v\rangle = 0$ would satisfy $\langle v|v\rangle = 0$. Then the first Eq. (9) is equivalent to $v \geq 0$, which proves (a).

If $v = 0$, then the norm of $a|v\rangle$ is 0 so $a|v\rangle = 0$. Moreover, for $v \neq 0$, $Na|v\rangle = a(N-1)|v\rangle = (v-1)a|v\rangle$, which, together with the first Eq. (9), completes the proof of (b). Finally, $Na^\dagger|v\rangle = a^\dagger(N+1)|v\rangle = (v+1)a^\dagger|v\rangle$, which completes the proof of (c).

Now, to build the spectrum, take an arbitrary eigenvector $|v\rangle$ (at least one exists by the fundamental theorem of algebra), and construct the sequence $|v\rangle, a|v\rangle, \ldots, a^p|v\rangle, \ldots$, with corresponding eigenvalues $v, v-1, \dots, v-p, \dots$. For some integer n, $v-n \geq 0$, but $v-n-1 < 0$. Since the eigenvalues of N are ≥ 0, $aa^n |v\rangle = 0$. Then it must be that $v-n=0$, so v is a nonnegative integer. Therefore the spectrrum of eigenvalues of N is formed by the set of non-negative integers. The whole spectrum can be reconstructed by acting repeatedly with a or $a^\dagger$ on one of the eigenvectors.

To normalize the vectors, form the set of orthonormal vectors $|0\rangle, |1\rangle, \dots$ corresponding to the eigenvalues of N, i.e. $0, 1, \dots$. The vectors are related to each other via the recursion relations

$$\begin{aligned} a^\dagger |n\rangle &= \sqrt{n+1}\, |n+1\rangle, \\ a |n\rangle &= \sqrt{n}\, |n-1\rangle, \\ a|0\rangle &= 0, \end{aligned} \tag{10}$$

such that the normalized n^{th} eigenvector takes the form

$$|n\rangle = \frac{1}{\sqrt{n!}} a^{\dagger n} |0\rangle. \tag{11}$$

From these definitions, $N|n\rangle = n|n\rangle$, and $\langle n | n' \rangle = \langle 0 | \frac{1}{\sqrt{n!}} \frac{1}{\sqrt{n'!}} a^n a^{\dagger n'} |0\rangle = \delta_{n,n'}$.

Exercises. a) Show that $[N, a^p] = -pa^p$, and $[N, a^{\dagger p}] = pa^p$, for $p > 0$ integer, and that the only algevraic functions of a and $a^\dagger$ that commute with N are the functions of N only. b) Show that a and $a^\dagger$ have no inverse.

B. Displacement operator. BCH Lemma. Coherent states

Consider the *displacement operator* $D(\alpha) \equiv e^{\alpha a^\dagger - \alpha^* a}$, for α complex. Note that $D(\alpha)^\dagger D(\alpha) = D(\alpha)D(\alpha)^\dagger = I$, so it is unitary, and that $D(\alpha)^\dagger = D(\alpha)^{-1} = D(-\alpha)$. We want to find the action of this operator on the creation and annihilation operators, and show the following

$$D^\dagger(\alpha) a D(\alpha) = a + \alpha. \tag{12}$$

To prove this identity, we use the *Baker-Campbell-Haussdorff lemma*. It states that, for X and Y linear operators

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \dots \tag{13}$$

Proof. Let $f(s, Y) = e^{sX} Y e^{-sX}$, and differentiate with respect to s. Then

$$\frac{d}{ds} f(s, Y) = X e^{sX} Y e^{-sX} - e^{sX} Y e^{-sX} X = [X, f(s, Y)], \tag{14}$$

together with the boundary condition $f(0, Y) = Y$. This differential equation is solved by $f(s, Y) = e^{s[X,\cdot]}Y \equiv Y + s[X, Y] + \frac{s^2}{2!}[X, [X, Y]] + \ldots$, as can be checked by direct differentiation of the right hand side. Setting $s = 1$ proves the lemma.

In particular, note that if the commutator $[X, Y]$ commutes with X, then the series terminates and we get $e^X Y e^{-X} = Y + [X, Y]$. Identity Eq. (12) is a direct consequence of the BCH Lemma.

Now one can prove that if $[A, [A, B]] = 0$ and $[B, [A, B]] = 0$, then $e^{A+B} = e^A e^B e^{+\frac{1}{2}[A,B]}$. *Proof.* Let $f(s) = e^{sA}e^{sB}$. Then $df/ds = Ae^{sA}e^{sB} + e^{sA}e^{sB}B = (A + e^{sA}Be^{-sA})f(s) = (A + B + s[A, B])f(s)$, with the last equality following from the BCH lemma. This differential equation is solved with initial condition $f(s) = I$. It is solved by $f(s) = e^{sA+sB+\frac{s^2}{2!}[A,B]} = e^{sA+sB}e^{\frac{s^2}{2!}[A,B]}$. Setting $s = 1$, we find $e^A e^B e^{-\frac{1}{2}[A,B]} = e^{A+B}$. In particular, for the displacement operator,

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} = e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-\frac{|\alpha|^2}{2}}. \tag{15}$$

The composition of two displacement operators is also a displacement operator, up to a pure phase factor

$$\begin{aligned} D(\alpha)D(\beta) &= e^{\alpha a^\dagger - \alpha^* a} e^{\beta a^\dagger - \beta^* a} = e^{(\alpha+\beta)a^\dagger - (\alpha^*+\beta^*)a} e^{\frac{1}{2}[\alpha a^\dagger - \alpha^* a, \beta a^\dagger - \beta^* a]} \\ &= e^{(\alpha+\beta)a^\dagger - (\alpha^*+\beta^*)a} e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)}. \end{aligned} \tag{16}$$

An analogous calculation leads to the following braiding relation that shows that displacement operators do not, in general, commute

$$D(\alpha)D(\beta) = e^{\alpha\beta^* - \alpha^*\beta} D(\beta)D(\alpha). \tag{17}$$

Note that the prefactor on the right-hand-side above is a phase factor. The phase equals twice the signed area of the parallelogram subtended by the phase factors α and β. If $\alpha = re^{i\alpha}, \beta = pe^{i\beta}$, with $r, p > 0$ and a, b phases in $[0, 2\pi)$, then $\alpha\beta^* - \alpha^*\beta = rpe^{i(a-b)} - e^{i(b-a)} = 2irp\sin(a - b)$. The operators commute if this phase factor is a multiple of 2π.

Exercise. Consider $H/\hbar = \Delta a^\dagger a + \epsilon(a + a^\dagger)$. Show that its eigenstates are displaced Fock states. What are its eigenvalues?

Coherent states: Let $D(\alpha)|0\rangle \equiv |\alpha\rangle$. This state is an eigenstate of the annihilation operator, with $a|\alpha\rangle = \alpha|\alpha\rangle$. This is proven as follows, using Eq. (12), $aD(\alpha)|0\rangle = [D(\alpha)a + \alpha D(\alpha)]|0\rangle = \alpha D(\alpha)|0\rangle$.

One can easily find the expansion of the coherent states over the Fock basis, as follows

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\,. \tag{18}$$

That is, if you performed a projective measurement of photon number $N = \sum_{n=0}^{\infty} n\,|n\rangle\langle n|$, you would obtain outcome n with probability given by the Poisson distribution $\langle\alpha|n\rangle\langle n|\alpha\rangle = \frac{e^{-|\alpha|^2}}{n!}|\alpha|^{2n}$. The average photon number in the coherent state is $\langle\alpha|N|\alpha\rangle = |\alpha|^2$, and the variance is $\Delta(N)^2 = |\alpha|^2$. The standard deviation scales like the square root of the mean.

Exercise. Show that in any coherent state $\Delta(Q)\Delta(P) = \sqrt{\frac{1}{2}}$, such that coherent states saturate the Heisenberg uncertainty principle inequality (i.e. they are minimum uncertainty states).

Finally, the overlap of two coherent states is always nonzero, that is one can easily show $\langle\beta|\alpha\rangle = e^{-\frac{1}{2}(|\beta^2|+|\alpha|^2-2\beta^*\alpha)} \neq \delta(\alpha-\beta)$, but $\langle\alpha|\alpha\rangle = 1$. One can show that (*exercise*) $\frac{1}{\pi}\int d^2\alpha\,|\alpha\rangle\langle\alpha| = I$, where $d^2\alpha = d\mathrm{Re}(\alpha)d\mathrm{Im}(\alpha)$. Together with the nonorthogonality condition above, this shows that the set of coherent states is overcomplete.

C. Spin-$\frac{1}{2}$. The Bloch sphere

The following Hamiltonian describes a spin$-\frac{1}{2}$ in a magnetic field

$$H = \vec{r}\cdot\vec{\sigma}, \tag{19}$$

where $\vec{\sigma} = (\sigma^x,\sigma^y,\sigma^z)$ and $\vec{r} = (r_x,r_y,r_z)$. The eigenvalues of this Hamiltonian are $\pm|\vec{r}| = \pm r$, and the projectors onto the two eigenstates are $P_\pm = \frac{I\pm\hat{r}\cdot\vec{\sigma}}{2}$. Equivalently, $P_\pm = \rho_\pm$ the pure density matrices corresponding to the eigenvectors.

Let's express the field in terms of Euler angles $\vec{r} = r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Consider a ket $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$. The associated density matrix is $|\psi\rangle\langle\psi| = \cos^2\frac{\theta}{2}|0\rangle\langle 0| + \sin^2\frac{\theta}{2}|1\rangle\langle 1| + e^{-i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}|0\rangle\langle 1| + e^{i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}|1\rangle\langle 0| = \frac{1+\cos\theta}{2}|0\rangle\langle 0| + \frac{1-\cos\theta}{2}|1\rangle\langle 1| + (\cos\varphi - i\sin\varphi)\frac{\sin\theta}{2}|0\rangle\langle 1| + (\cos\varphi + i\sin\varphi)\frac{\sin\theta}{2}|1\rangle\langle 0| = \frac{I+\hat{r}\cdot\vec{\sigma}}{2}$. $|\psi\rangle$ is the eigenvector of eigenvalue $+r$. To see this, do $H|\psi\rangle\langle\psi| = r\hat{r}\cdot\vec{\sigma}\frac{I+\hat{r}\cdot\vec{\sigma}}{2} = r|\psi\rangle\langle\psi|$, since $(\hat{r}\cdot\vec{\sigma})^2 = I$. The density matrix corresponding to $|\psi\rangle$ has Bloch vector $\hat{r}$, i.e. aligned with the external field.

The eigenvector of eigenvalue $-r$ is oriented exactly opposite on the Bloch sphere, i.e. we set $\theta\to\pi-\theta$ and $\varphi\to\varphi+\pi$, so that $|\psi'\rangle = \cos\frac{\pi-\theta}{2}|0\rangle + e^{i\varphi+i\pi}\sin\frac{\pi-\theta}{2}|1\rangle = \sin\frac{\theta}{2}|0\rangle - e^{i\varphi}\cos\frac{\theta}{2}|1\rangle$. The projector onto $|\psi'\rangle$ is $|\psi'\rangle\langle\psi'| = \sin^2\frac{\theta}{2}|0\rangle\langle 0| + \cos^2\frac{\theta}{2}|1\rangle\langle 1| -$

$e^{-i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}|0\rangle\langle 1| - e^{i\varphi}\cos\frac{\theta}{2}\sin\frac{\theta}{2}|1\rangle\langle 0| = \frac{1-\cos\theta}{2}|0\rangle\langle 0| + \frac{1+\cos\theta}{2}|1\rangle\langle 1| - (\cos\varphi - i\sin\varphi)\frac{\sin\theta}{2}|0\rangle\langle 1| - (\cos\varphi + i\sin\varphi)\frac{\sin\theta}{2}|1\rangle\langle 0| = \frac{I-\hat{r}\cdot\vec{\sigma}}{2}$. Analogously to the above, one finds $H|\psi'\rangle\langle\psi'| = -r|\psi'\rangle\langle\psi'|$. The density matrix corresponding to $|\psi'\rangle$ has Bloch vector $-\hat{r}$, i.e. anti-aligned with the external field.

D. Jaynes-Cummings Hamiltonian

The following Hamiltonian describes a spin$-\frac{1}{2}$ interacting with a harmonic oscillator

$$H = \frac{\omega_q}{2}\sigma^z \otimes I_r + \omega_r I_q \otimes a^\dagger a + g(\sigma^+ \otimes a + \sigma^- \otimes a^\dagger) = \frac{\omega_q}{2}\sigma^z + \omega_r a^\dagger a + g(\sigma^+ a + \sigma^- a^\dagger). \tag{20}$$

The first expression contains the tensor product structure. The second uses shorthand notation, which will is often implicitly used. When we write a we mean $I_q \otimes a$ and when we write σ^+ what we mean is $\sigma^+ \otimes I_r$ etc. The spin is described by the Pauli matrices σ^i introduced before, together with the identity I_q, whereas for the harmonic oscillator we have the bosonic commutation relation $[a, a^\dagger] = 1$ as before.

To diagonalize this Hamiltonian, it is simplest to find a conserved quantity, i.e. an operator that commutes with it. This is the excitation number $N = a^\dagger a + \frac{1+\sigma^z}{2}$. We leave the proof that $[N, H] = 0$ as an *exercise*. Then N and H will be diagonal in the same basis.

The eigenspaces of N are $V_0 = \{|0,0\rangle\}$, $V_1 = \{|0,1\rangle, |1,0\rangle\}, \ldots, V_n = \{|n-1,1\rangle, |n,0\rangle\}, \ldots,$ where the subscript of V denotes the eigenvalue of N, and the two labels of the kets count the number of excitations in the simple harmonic oscillator and in the spin, respectively. For the one-dimensional eigenspace V_0, the eigenenergy is $E_{0,0} = -\omega_q/2$. Over V_n for $n \geq 1$, the Hamiltonian is represented by the two-dimensional block

$$\begin{aligned} H_n &= \begin{pmatrix} \langle n-1,1| H |n-1,1\rangle & \langle n-1,1| H |n,0\rangle \\ \langle n,0| H |n-1,1\rangle & \langle n,0| H |n,0\rangle \end{pmatrix} = \begin{pmatrix} (n-1)\omega_r + \frac{\omega_q}{2} & g\sqrt{n} \\ g\sqrt{n} & n\omega_r - \frac{\omega_q}{2} \end{pmatrix} \\ &= \left(n - \frac{1}{2}\right)\omega_r I_2 + \frac{\omega_q - \omega_r}{2}\tau^z + g\sqrt{n}\tau^x. \end{aligned} \tag{21}$$

We have introduced Pauli matrices τ^i, along with the identity operator, that act on the two-dimensional subspace V_n. The full Hamiltonian is block-diagonal, i.e. we write $H = H_0 \oplus H_1 \oplus H_2 \oplus \ldots$ acting on $V = V_0 \oplus V_1 \oplus V_2 \oplus \ldots$.

We may further write

$$
\begin{aligned}
H_n &= \vec{r}_n \cdot \vec{\tau} + \left(n - \frac{1}{2}\right) \omega_r I_2 \\
\vec{r}_n &= (g\sqrt{n}, 0, \Delta/2) \equiv r_n(\sin\theta_n, 0, \cos\theta_n), \\
r_n &= |\vec{r}_n| = \sqrt{ng^2 + \Delta^2/4},\ \sin\theta_n = g\sqrt{n}/r_n,\ \cos\theta_n = \Delta/(2r_n).
\end{aligned} \tag{22}
$$

From this form, we can calculate using the previous subsection the eigenenergies and eigenvectors in the subspace V_n for $n \geq 1$

$$
\begin{aligned}
E_{\pm,n} &= \pm r_n, \\
|\psi_{+,n}\rangle &= \cos\left(\frac{\theta_n}{2}\right)|n,0\rangle + \sin\left(\frac{\theta_n}{2}\right)|n-1,1\rangle, \\
|\psi_{-,n}\rangle &= \sin\left(\frac{\theta_n}{2}\right)|n,0\rangle - \cos\left(\frac{\theta_n}{2}\right)|n-1,1\rangle.
\end{aligned} \tag{23}
$$

$\theta_n/2$ can be interpreted as a 'mixing angle'.