Problem Set 1
(M1 Math methods 2023-2024)

This problem set is due on Friday, September 22nd, 2023, at 23:59. The solutions should be emailed as a single PDF (handwritten or typeset) to alexandru.petrescu@minesparis.psl.eu by the deadline. If you collaborate with a colleague, please write their names at the top of your solution. Cite your references (books, websites, chatbots etc.). If you submit late without a satisfactory reason, the set will be accepted with a 10% penalty in the score.

I. PAULI MATRICES (20 POINTS)

We define the set of *Pauli matrices* together with the identity as $\sigma_0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 = \sigma_x = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \sigma_y = Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma_3 = \sigma_z = Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

a) Find the eigenvectors, eigenvalues, and diagonal representations of the three Pauli matrices $\sigma_{1,2,3}$.

b) Show that any 2×2 Hermitian matrix, *i.e.* $\begin{pmatrix} a & c+id \\ c-id & b \end{pmatrix}$, with a, b, c, d real, can be expressed as a real linear combination of the Pauli matrices and the identity.

c) Show that, if we define the commutator of two linear operators as $[A, B] = AB - BA$, then $[\sigma_i, \sigma_j] = 2i \sum_k \epsilon_{ijk}\sigma_k$ for $i, j, k \in \{1, 2, 3\}$ and ϵ_{ijk} is the Levi-Civita symbol, *i.e.* $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{213} = \epsilon_{321} = \epsilon_{132} = -1$, and zero whenever two of its indices are equal. Show that, if we define the anticommutator as $\{A, B\} = AB + BA$, then $\{\sigma_i, \sigma_j\} = 2\delta_{ij} I$. Show that, if the trace of a square matrix is defined by the sum of the elements on its diagonal, $\mathrm{Tr}A = A_{11} + \ldots A_{nn}$, then $\mathrm{Tr}\sigma_i = 0$ for $i = 1, 2, 3$, but $\mathrm{Tr}\{\sigma_i \sigma_j\} = 2\delta_{ij}$. Go back to b) and use what you just proved to extract the coefficient of σ_1 from the linear combination.

d) Let $\vec{a} = a\hat{n}$ with $\hat{n}$ being a unit vector in $\mathbb{R}^3$, i.e. $|\hat{n}| = 1$, and $a > 0$ a real number. Show that $(\hat{n} \cdot \vec{\sigma})^{2p} = I$ for any natural number p. Show that $e^{ia\hat{n}\cdot\vec{\sigma}} = I \cos a + i\hat{n} \cdot \vec{\sigma} \sin a$. $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a vector whose components are the three Pauli matrices, and $\hat{n} \cdot \vec{\sigma} = \hat{n}_1\sigma_1 + \hat{n}_2\sigma_2 + \hat{n}_3\sigma_3$.

e) Show that $R_n(-a)\vec{\sigma}R_n(a) \equiv e^{i\frac{a}{2}\hat{n}\cdot\vec{\sigma}}\vec{\sigma}e^{-i\frac{a}{2}\hat{n}\cdot\vec{\sigma}} = \vec{\sigma}\cos a + \hat{n} \times \vec{\sigma} \sin a + \hat{n}(\hat{n} \cdot \vec{\sigma})(1 - \cos a)$. For example, show that if $\hat{n} = (0, 1, 0)$ is the unit vector on the y axis, then $R_y(-\pi/2)\sigma_1 R_y(\pi/2) = \sigma_3$.

II. HERMITIAN AND UNITARY OPERATORS (20 POINTS)

a) Prove that all eigenvalues of a unitary operator have norm 1, and therefore can be written as $e^{i\theta}$ for some real number θ.

b) Show that all Pauli matrices are unitary.

c) Prove that two eigenvectors of a Hermitian operator with different eigenvalues are necessarily orthogonal.

d) Prove that any eigenvalue of a projector P can be either 0 or 1. Can a projector also be unitary?

e) Use the spectral decomposition theorem to find the natural logarithm of the matrix $\begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}$.

III. TENSOR PRODUCTS (20 POINTS)

a) Show that transpose, complex conjugation, and adjoint distribute over the tensor product, that is $(A \otimes B)^* = A^* \otimes B^*, (A \otimes B)^T = A^T \otimes B^T$, and $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.

b) Show that $\exp(A_1 \otimes I_2 + I_1 \otimes A_2) = \exp(A_1) \otimes \exp(A_2)$ for some linear operators $A_{1,2}$ acting on Hilbert spaces $V_{1,2}$, and $I_{1,2}$ the respective identity operators.

c) Find the eigenvalues and eigenvectors of $X \otimes Z$, using the definitions of Pauli operators from a previous exercise.

d) Prove that $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$ cannot be expressed as a tensor product $|a\rangle \otimes |b\rangle$ for some pair of vectors $|a\rangle$ and $|b\rangle$ [we take the first (second) entry of the tensor products above to be a vector in some vector space V_1 (V_2)].