Hyperbolic PDEs and GENERIC

Nikolas Siccha

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Abstract

Acknowledgements

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For now, we will use this as a dump for formulae and definitions:

Christoffel symbols of the second kind

$$\Gamma^k_{i,j} = \frac{1}{2} g^{k,l} \left(\partial_i g_{j,l} + \partial_j g_{i,l} + -1 \partial_l g_{i,j} \right) \tag{2.1}$$

Riemannian curvature endomorphism

$$R^l_{i,j,k} = \left[\partial_j \Gamma^l_{i,k} + \Gamma^m_{i,k} \Gamma^l_{j,m} \right]_{[i,j]} \tag{2.2}$$

Riemannian curvature tensor

$$Rm_{i,j,k,l} = R^n_{i,j,k} g_{l,n} \tag{2.3}$$

flatness criterion

$$Rm_{i,j,k,l} = 0 \tag{2.4}$$

transformed metric

$$g_{\hat{i},\hat{j}} = u^i_{\hat{i}} g_{i,j} u^j_{\hat{j}} \tag{2.5}$$

transformed hessian

$$h_{\hat{i},\hat{j}} = u^i_{\hat{i}} h_{i,j} u^j_{\hat{j}} + h_i u^i_{\hat{i},\hat{j}} \tag{2.6}$$

Following definitions just a test for back referencing.

Definition 2.1.1. *A metric $g(u)$ is called* Riemannian *if it is everywhere positive definite.*

Definition 2.1.2. *A metric $g(u)$ is called* pseudo-Riemannian *if it is everywhere non-degenerate.*

All following definitions collected from [Lee12].

Definition 2.1.3. *Consider a continuous map $\pi : \mathcal{M}^d \to N$. A* section *of π is a continuous right inverse for π, i.e. a continuous map $\sigma : N \to \mathcal{M}^d$ such that $\pi \circ \sigma = \mathrm{Id}_N$. A* local *section need only be defined on some open subset $U \subseteq \mathcal{M}^d$.*

Definition 2.1.4. *Let $\mathcal{M}^d$ be a topological space. A (real) vector bundle of rank k over $\mathcal{M}^d$ is a topological space E together with a surjective continuous map $\pi : E \to \mathcal{M}^d$ satisying the following conditions:*

- *For each point $p \in \mathcal{M}^d$, the fiber $E_p = \pi^{-1}(p)$ over p is endowed with the structure of a k-dimensional real vector space.*
- *For each point $p \in \mathcal{M}^d$, there exist a neighborhood U of p in $\mathcal{M}^d$ and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ satisfying*
  - *$\pi_U \circ \Phi = \pi$ and*
  - *for each $q \in U$, the restriction of Φ to E_q is a vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k$.*

Definition 2.1.5. *Consider a vector bundle $\pi : E \to \mathcal{M}^d$. A* section *of E is a section of the map π. A* local *section need only be defined on some open subset $U \subseteq \mathcal{M}^d$, while a* global *section is defined on all of $\mathcal{M}^d$.*

Definition 2.1.6. *The* Lie bracket *of two smooth vector fields X and Y is defined by*

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \tag{2.7}$$

for every smooth function $f \in C^\infty(\mathcal{M}^d)$.

Definition 2.1.7. *A smooth distribution D on a smooth manifold $\mathcal{M}^d$ of rank k is a rank-k smooth subbundle of the tangent bundle $T\mathcal{M}^d$.*

Definition 2.1.8. *A smooth distribution D on $\mathcal{M}^d$ is said to be* involutive *if given any pair of smooth local sections of D, their Lie bracket is also a local section of D.*

Definition 2.1.9. *A nonempty immersed submanifold $N \subseteq \mathcal{M}^d$ is called an* integral manifold *of D if $T_pN = D_p$ at each point $p \in N$.*

Definition 2.1.10. *A smooth distribution D on $\mathcal{M}^d$ is said to be* integrable *if each point of $\mathcal{M}^d$ is contained in an integral manifold of D.*

Definition 2.1.11. *Consider a smooth manifold $\mathcal{M}^d$ and let $\mathcal{F}$ be any collection of k-dimensional submanifolds of M. A smooth chart (U, ϕ) is said to be* flat for *$\mathcal{F}$ if $\phi(U)$ is a cube in $\mathbb{R}^d$ and each submanifold in $\mathcal{F}$ intersects U in either the empty set or a countable union of k-dimensional slices of the form $x^i = c^i$ for $i = k+1, \ldots, n$.*

Definition 2.1.12. *A smooth distribution D is said to be* completely integrable *if there exists a flat chart for D in a neighborhood of each point $p \in \mathcal{M}^d$.*

Proposition 2.1.13. *Every involutive distribution is completely integrable and vice versa.*

Proof. See e.g. [Lee12], theorem 19.12. □

Definition 2.1.14. *A collection $\mathcal{F}$ of disjoint, connected, nonempty, immersed k-dimensional submanifolds of $\mathcal{M}^d$, whose union is $\mathcal{M}^d$, such that in a neighborhood of each point $p \in \mathcal{M}^d$ there exists a flat chart for $\mathcal{F}$ is called* foliation of dimension k on $\mathcal{M}^d$.

2.2 Poisson structures

2.2.1 Poisson manifolds

2.2.2 The Poisson bivector

2.2.3 The Schouten bracket

2.2.4 Constant Poisson structures and symplectic structures

2.2.5 Symplectic foliations

2.2.6 Poisson structure reductions

2.3 Lie groups and Lie algebras

2.3.1 Lie-Poisson structures

2.3.2 Extensions of Lie algebras

2.3.3 Groups of diffeomorphisms

2.3.4 Semidirect products

Chapter 3

The geometry of hyperbolic PDEs of metriplectic type

3.1 GENERIC / metriplectic dynamics

The main references are

- [GO97], [OG97] “First” papers
- [DV80] “First” first mention
- [Arn66] French hydrodynamic Lie groups
- [GMR96] SHTC
- [MR98] RET Book
- [Ott05] “Standard” reference
- [PKG14] “Sensible” reversibility
- [PKG18] Better “standard” reference
- [PPRG18] GENERIC <-> SHTC
- [PPK20] Hamiltonian continuum mechanics
- [OST20] XGENERIC

3.1.1 Gradient flows

Give motivation, examples and properties for (physical) gradient flows.

3.1.2 Symplectic flows

Give motivation, examples and properties for (physical) symplectic flows.

3.1.3 Reversibility

Discuss parity, time-reversal transformations etc. See e.g. [PKG14].

3.1.4 Irreversibility

Discuss two options:

- Phase-space area shrinkage
- (Free) energy decrease

3.1.5 Rational extended thermodynamics

Give short discussion of RET.

3.2 Hyperbolic partial differential equations

The main references are

- [BGS06] Standard hyp. PDE Book
- [God62] Ex. solutions
- [Rug] Ex. solutions
- [FL71] Convex entropy extension
- [Tad84] Skew-selfadjoint hyp. PDEs
- [PPRG18] GENERIC <-> SHTC
- [PPK20] Hamiltonian continuum mechanics

Throughout, unless otherwise stated, we will use a d-dimensional spatial manifold $\mathcal{M}^d$, a simply connected n-dimensional target manifold $\mathcal{U}^n$ and assume $u(t) \in C^\infty\left(\mathcal{M}^d;\mathcal{U}^n\right)$ for all t considered. Furthermore, indices in products which appear once above and once below will be summed. Greek indices like μ, ν run from 1 to d and latin indices like i, j, k run from 1 to n, unless otherwise noted.

Definition 3.2.1. *We consider* systems of quasilinear first-order partial differential equations

$$A^{0,i}_j u^j_t(x) + A^{\mu,i}_j(u(x)) u^j_\mu(x) = \sigma^i(u(x)). \tag{3.1}$$

We are interested in the following subclasses:

Definition 3.2.2. *A system of quasilinear first-order PDEs is called* hyperbolic *if the eigenvalue problem*

$$\left[A(\xi) + -1\lambda A^0\right] v = 0 \tag{3.2}$$

has real eigenvalues and n linearly independent eigenvectors for every $\xi \in \mathcal{S}^{[d-1]}$, where

$$A : \mathcal{S}^{[d-1]} \to T^1_1\mathcal{U}^n, A(\xi) = A^\mu \xi_\mu. \tag{3.3}$$

If $A^0 = \mathrm{Id}$ this translates to $A(\xi)$ being diagonalizable over the real numbers for every direction $\xi \in \mathcal{S}^{[d-1]}$.

Note: Unless otherwise noted, we will assume $A^0 = \mathrm{Id}$.

Definition 3.2.3. *A system of quasilinear first-order PDEs is called a system of* balance laws *if there exist smooth flux functions $F^\mu \in \Gamma(T\mathcal{U}^n)$ such that*

$$\frac{\partial F^{\mu,i}}{\partial u^j} = A^{\mu,i}_j \tag{3.4}$$

for $\mu = 0, \ldots, d$. If $\sigma = 0$ it is called a system of conservation laws.

Definition 3.2.4. *A system of quasilinear first-order PDEs is called* symmetric hyperbolic *if A^0 is symmetric positive definite and all A^μ are symmetric. A system that can be brought into this form is called* symmetrizable.

Definition 3.2.5. *A pair (η, q) is called an* entropy-entropy flux pair *to the system of quasilinear first-order PDEs if the* additional balance law

$$\frac{\partial \eta}{\partial u^i}\left(A_j^{0,i} u_t^j\,(x) + A_j^{\mu,i}\,(u\,(x)) u_\mu^j\,(x)\right) = \eta_t\,(u\,(x)) + q_\mu^\mu\,(u\,(x)) \tag{3.5}$$

is implied by equation (3.1).

Lemma 3.2.6. *A system of quasilinear first-order PDEs admits an entropy-entropy flux pair (η, q) if the* compatibility condition

$$\frac{\partial q^\mu}{\partial u^j} = \frac{\partial \eta}{\partial u^i} A_j^{\mu,i} \tag{3.6}$$

is satisfied.

Lemma 3.2.7. *Given an entropy density η, a system of quasilinear first-order PDEs admits an entropy-entropy flux pair (η, q) if and only if the* integrability condition

$$\left[\frac{\partial}{\partial u^k}\left(\frac{\partial \eta}{\partial u^i} A_j^{\mu,i}\right)\right]_{[j,k]} = 0 \tag{3.7}$$

is satisfied.

Proof. Follows from lemma 3.2.6 combined with the requirement

$$\left[\frac{\partial^2 q_\mu}{\partial u^j \partial u^k}\right]_{[j,k]} = \left[\frac{\partial}{\partial u^k}\left(\frac{\partial \eta}{\partial u^i} A_j^{\mu,i}\right)\right]_{[j,k]} = 0 \tag{3.8}$$

and from the assertion that the n-dimensional target manifold $\mathcal{U}^n$ is simply connected (see e.g. [Zor16], section 14.3). □

Lemma 3.2.8. *A system of balance laws admitting an entropy-entropy flux pair (η, q) with strictly convex entropy density η is symmetrizable.*

Proof. See e.g. [FL71]. □

Lemma 3.2.9. *A system of quasilinear first-order PDEs can be written as a system of balance laws in original coordinates u if*

$$\left[\frac{\partial A_j^{\mu,i}}{\partial u^k}\right]_{[j,k]} = 0 \tag{3.9}$$

holds. WILL ADD CONTENT FROM [Bog11].

Proof. Follows from the requirement

$$\left[\frac{\partial^2 F^{\mu,i}}{\partial u^j \partial u^k}\right]_{[j,k]} = \left[\frac{\partial A_j^{\mu,i}}{\partial u^k}\right]_{[j,k]} = 0 \tag{3.10}$$

and from the assertion that the n-dimensional target manifold $\mathcal{U}^n$ is simply connected. $\square$

It has been shown that if the entropy and the fluxes are *homogeneous*, symmetrizability, admitting an entropy-entropy flux pair and having a so called *skew- selfadjoint form* are all equivalent (see [Tad84], theorem 2.1 for details). WILL ADD DEFINITION, LEMMA AND DISCUSSION.

A weaker requirement than homogeneity and having a skew-selfadjoint form is the following:

Lemma 3.2.10. *Consider an* entropy functional $H = \int \eta(u(x))\, dx$ *and a* skew-symmetric bilinear form

$$L(v,w) = \left\langle v_i, \left(g^{\mu,i,j}\frac{\partial}{\partial x_\mu} + b_k^{\mu,i,j} u_\mu^k\right) w_j \right\rangle. \tag{3.11}$$

Then the evolution equation

$$u_t^i = \left[L\frac{\partial \eta}{\partial u}\right]^i = \left(g^{\mu,i,j}\eta_{j,k} + b_k^{\mu,i,j}\eta_j\right) u_\mu^k \tag{3.12}$$

admits an entropy-entropy flux pair (η, q)*. If*

$$\left[\frac{\partial}{\partial u^k}\left(b_l^{\mu,j,i}\eta_j\right)\right]_{[k,l]} = 0, \tag{3.13}$$

it is a system of balance laws. If the entropy density η *is strictly convex, the system is symmetrizable.*

Proof. First we note that from the skew-symmetry of L it follows that

$$\begin{aligned}
2L(v,w) &= 2\left\langle v_i, b_k^{\mu,i,j} u_\mu^k w_j + g^{\mu,i,j} w_{j,\mu}\right\rangle & (3.14)\\
&= L(v,w) - L(w,v) & (3.15)\\
&= 2\left\langle v_i, b_k^{\mu,i,j} u_\mu^k w_j\right\rangle_{[v,w]} + \frac{1}{2}\left(\left\langle v_i, g^{\mu,i,j} w_{j,\mu}\right\rangle - \left\langle w_i, g^{\mu,i,j} v_{j,\mu}\right\rangle\right) & (3.16)\\
&= 2\left\langle v_i, \left[b_k^{\mu,i,j}\right]_{[i,j]} u_\mu^k w_j\right\rangle + \left\langle v_j, \frac{\partial}{\partial x_\mu}\left(g^{\mu,i,j} w_i\right)\right\rangle + \left\langle v_i, g^{\mu,i,j} w_{j,\mu}\right\rangle & (3.17)\\
&= 2\left\langle v_i, \left[b_k^{\mu,i,j}\right]_{[i,j]} u_\mu^k w_j\right\rangle + \left\langle v_j, \frac{\partial g^{\mu,i,j}}{\partial u^k} u_\mu^k w_i\right\rangle + 2\left\langle v_i, \left[g^{\mu,i,j}\right]_{(i,j)} w_{j,\mu}\right\rangle & (3.18)
\end{aligned}$$

$$= \left\langle v_i, \underbrace{\left(2\left[b_k^{\mu,i,j}\right]_{[i,j]} + \frac{\partial g^{\mu,j,i}}{\partial u^k}\right)}_{2b_k^{\mu,i,j}} u_\mu^k w_j + \underbrace{2\left[g^{\mu,i,j}\right]_{(i,j)}}_{2g^{\mu,i,j}} w_{j,\mu} \right\rangle. \tag{3.19}$$

Hence, we get

$$2\left[b_k^{\mu,i,j}\right]_{(i,j)} = \frac{\partial g^{\mu,i,j}}{\partial u^k} \text{ and } \left[g^{\mu,i,j}\right]_{[i,j]} = 0. \tag{3.20}$$

To check the existence of an entropy-flux function: From equation (3.12) we get

$$\eta_t\left(u\left(x\right)\right) = \frac{\partial \eta}{\partial u^i}\left(g^{\mu,i,j}\eta_{j,k} + b_k^{\mu,i,j}\eta_j\right)u_\mu^k \tag{3.21}$$

and thus identify the *gradient of the entropy-flux function*

$$\frac{\partial q^\mu}{\partial u^k} = \frac{\partial \eta}{\partial u^i}\left(b_k^{\mu,i,j}\frac{\partial \eta}{\partial u^j} + g^{\mu,i,j}\frac{\partial^2 \eta}{\partial u^j \partial u^k}\right) \tag{3.22}$$

and can check the conditions of lemma 3.2.7:

$$\left[\frac{\partial^2 q^\mu}{\partial u^k \partial u^l}\right]_{[k,l]} = \left[\frac{\partial}{\partial u^l}\left(\frac{\partial \eta}{\partial u^i}\left(b_k^{\mu,i,j}\frac{\partial \eta}{\partial u^j} + g^{\mu,i,j}\frac{\partial^2 \eta}{\partial u^j \partial u^k}\right)\right)\right]_{[k,l]} \tag{3.23}$$

$$= \left[\underbrace{\eta_{i,l}b_k^{\mu,i,j}\eta_j}_{\eta_i b_k^{\mu,j,i}\eta_{j,l}} + \underbrace{\eta_{i,l}g^{\mu,i,j}\eta_{j,k}}_{[()]_{[k,l]}=0} + \dots\right]_{[k,l]} \tag{3.24}$$

$$= \left[\dots + \eta_i\left(\frac{\partial}{\partial u^l}\left(b_k^{\mu,i,j}\eta_j\right) + \underbrace{\frac{\partial}{\partial u^l}\left(g^{\mu,i,j}\eta_{j,k}\right)}_{[()]_{[k,l]}=\left[g_l^{\mu,i,j}\eta_{j,k}\right]_{[k,l]}}\right)\right]_{[k,l]} \tag{3.25}$$

$$= \left[\eta_i\left(b_k^{\mu,j,i}\eta_{j,l} + \frac{\partial}{\partial u^l}\left(b_k^{\mu,i,j}\eta_j\right) + g_l^{\mu,i,j}\eta_{j,k}\right)\right]_{[k,l]} \tag{3.26}$$

$$= \left[\eta_i\left(\underbrace{2\left[b_k^{\mu,j,i}\right]_{(i,j)}}_{g_k^{\mu,i,j}}\eta_{j,l} + \frac{\partial b_k^{\mu,i,j}}{\partial u^l}\eta_j + g_l^{\mu,i,j}\eta_{j,k}\right)\right]_{[k,l]} \tag{3.27}$$

$$= \left[\underbrace{\eta_i\left(g_k^{\mu,i,j}\eta_{j,l} + g_l^{\mu,i,j}\eta_{j,k}\right)}_{[()]_{[k,l]}=0} + \underbrace{\eta_i \frac{\partial b_k^{\mu,i,j}}{\partial u^l}\eta_j}_{=\eta_i\left[\frac{\partial b_k^{\mu,i,j}}{\partial u^l}\right]_{(i,j)}\eta_j = \frac{1}{2}\eta_i\frac{\partial g_k^{\mu,i,j}}{\partial u^l}\eta_j} \right]_{[k,l]} \tag{3.28}$$

$$= \left[\frac{1}{2}\eta_i \frac{\partial^2 g^{\mu,i,j}}{\partial u^k \partial u^l}\eta_j\right]_{[k,l]} \tag{3.29}$$

$$= 0. \tag{3.30}$$

To check the existence of flux functions: From the evolution equation we identify $A_k^{\mu,i} = g^{\mu,i,j}\eta_{j,k} + b_k^{\mu,i,j}\eta_j$. We check

$$\left[\frac{\partial A_k^{\mu,i}}{\partial u^l}\right]_{[k,l]} = \left[\frac{\partial}{\partial u^l}\left(g^{\mu,i,j}\eta_{j,k} + b_k^{\mu,i,j}\eta_j\right)\right]_{[k,l]} \tag{3.31}$$

$$= \left[\frac{\partial}{\partial u^l}\left(g^{\mu,i,j}\eta_{j,k}\right) + \frac{\partial}{\partial u^l}\left(b_k^{\mu,i,j}\eta_j\right)\right]_{[k,l]} \tag{3.32}$$

$$= \left[\frac{\partial g^{\mu,i,j}}{\partial u^l}\eta_{j,k} + g^{\mu,i,j}\frac{\partial \eta_{j,k}}{\partial u^l} + \frac{\partial b_k^{\mu,i,j}}{\partial u^l}\eta_j + b_k^{\mu,i,j}\frac{\partial \eta_j}{\partial u^l}\right]_{[k,l]} \tag{3.33}$$

$$= \left[b_l^{\mu,j,i}\eta_{j,k} + \underbrace{b_l^{\mu,i,j}\eta_{j,k} + b_k^{\mu,i,j}\eta_{j,l}}_{[()]_{[k,l]}=0} + \eta_j \frac{\partial b_k^{\mu,i,j}}{\partial u^l} \right]_{[k,l]} \tag{3.34}$$

$$= \left[\frac{\partial}{\partial u^k}\left(b_l^{\mu,j,i}\eta_j\right) + -1\frac{\partial b_l^{\mu,j,i}}{\partial u^k}\eta_j + \eta_j\frac{\partial b_k^{\mu,i,j}}{\partial u^l}\right]_{[k,l]} \tag{3.35}$$

$$= \left[\frac{\partial}{\partial u^k}\left(b_l^{\mu,j,i}\eta_j\right) + \eta_j\frac{\partial^2 g^{\mu,i,j}}{\partial u^k \partial u^l}\right]_{[k,l]} \tag{3.36}$$

$$= \left[\frac{\partial}{\partial u^k}\left(b_l^{\mu,j,i}\eta_j\right)\right]_{[k,l]} \tag{3.37}$$

which completes the proof. □

3.2.1 Symmetric hyperbolic thermodynamically compatible systems

3.2.2 Skew-selfadjoint hyperbolic systems of balance laws

3.3 Poisson brackets of hydrodynamic type

The main references are

- [LGPV13] Standard Poisson structures reference
- [Nov82], [DN83a], [DN83b] (Slightly erroneous) initial papers
- [GD80b], [GD80a], [GD82] preceeding papers
- [Gri85] (Slightly erroneous) treatment of degenerate brackets of **constant rank**
- [Mok89] Correction of [DN83a]
- [Mok92] Correction of [Gri85]
- [Bog07b], [Bog07c] Extension of [Gri85]
- [Sav16b] Extension of [Gri85]
- [Mor80] (Slightly erroneous) treatment of Maxwell-Vlasov
- [MG80] Hamiltonian MHD
- [WM81] Correction of [Mor80]
- [KHO01] Automatic Jacobi-check
- [KH10] Improvement of [KHO01]
- [KVV17] Inacessible, more general improvement of [KH10], [KHO01]
- [Vit19] Like [KVV17]
- [Rey10] Hydrodynamic Poisson structures
- [Sav16a] Hydrodynamic Poisson structures
- [MF90], [Mok92] Nonlocal generalization

Definition 3.3.1. *A bilinear bracket*

$$\{\cdot,\cdot\} : V \times V \to \mathbb{R} \tag{3.38}$$

is called Poisson *if* $\{\cdot,\cdot\}$

- *is antisymmetric:* $\{F, G\} = -\{G, F\}$,
- *is a derivation:* $\{FG, H\} = F\{G, H\} + G\{F, H\}$ *and*
- *satisfies the Jacobi-identity:* $\{F, \{G, H\}\}_{\circlearrowleft F,G,H} = 0$

JUST ADDED THIS TO BE ABLE TO REFERENCE IT, BUT THE BRACKET IS REALLY A BIVECTOR-field, I.E. IT MAY DEPEND ARBITRARILY ON U. WILL MOVE TO POISSON SECTION AND UPDATE.

In [DN83a] and [DN83b] Dubrovin and Novikov introduced Poisson brackets and Hamiltonians of *hydrodynamic type*:

Definition 3.3.2. *A Poisson bracket* $\{\cdot,\cdot\}$ *is called* hydrodynamic *if it satisfies*

$$\left\{u^i(x), u^j(y)\right\} = g^{\mu,i,j}(u(x))[\delta']_\mu (x-y) + u^k_\mu b^{\mu,i,j}_k (u(x))\delta(x-y). \tag{3.39}$$

Definition 3.3.3. *A functional (or Hamiltonian)*

$$H : C^\infty\left(\mathcal{M}^d;\mathcal{U}^n\right) \to \mathbb{R} \tag{3.40}$$

is called hydrodynamic *if it is of the form*

$$H[u] = \int h\left(u\left(x\right)\right)\, dx \tag{3.41}$$

for some energy density $h \in C^{\infty}\left(\mathcal{U}^n\right)$.

By requiring that brackets and Hamiltonians do not depend themselves on spatial derivatives of the unknown field variables $u \in C^{\infty}\left(\mathcal{M}^d; \mathcal{U}^n\right)$, we ensure that the resulting evolution equation will be a system of quasilinear first-order PDEs.

For any two Functionals $F\left(u\right) = \int f\left(u\left(x\right)\right)\, dx$ and $H\left(u\right) = \int h\left(u\left(x\right)\right)\, dx$ we get

$$\{F, H\} = \int \frac{\partial f}{\partial u^i} L^{i,j} \frac{\partial h}{\partial u^j}\, dx \tag{3.42}$$

with *differential operator* $L^{i,j} = g^{\mu,i,j} \frac{\partial}{\partial x_\mu} + b_k^{\mu,i,j} u_\mu^k$. Finally, given a Poisson bracket and a hydrodynamic Hamiltonian functional H we get the *evolution equation*

$$u_t^i = L^{i,j} h_j = \left(g^{\mu,i,j} h_{j,k} + b_k^{\mu,i,j} h_j\right) u_\mu^k, \tag{3.43}$$

where we can identify the $(1,1)$-*tensors*

$$A_k^{\mu,i} = g^{\mu,i,j} h_{j,k} + b_k^{\mu,i,j} h_j. \tag{3.44}$$

For the above bracket to be Poisson, it has to satisfy the *Jacobi identity*

$$\{F, \{G, H\}\}_{\circlearrowleft F,G,H} = 0 \tag{3.45}$$

for all hydrodynamic Functionals F, G, H which, while innocent looking, translates to (see [Mok98], Lemma 2.1):

$$0 = \left[g^{\mu,i,j}\right]_{[i,j]} \tag{3.46}$$

$$0 = \frac{\partial g^{\mu,i,j}}{\partial u^k} + -2\left[b_k^{\mu,i,j}\right]_{(i,j)} \tag{3.47}$$

$$0 = \left[\left[g^{\mu,i,l} b_l^{\nu,j,k}\right]_{[i,j]}\right]_{(\mu,\nu)} \tag{3.48}$$

$$0 = \left[\left[g^{\mu,i,l} b_l^{\nu,j,k}\right]_{[(i,\mu),(j,\nu)]}\right]_{\circlearrowleft i,j,k} \tag{3.49}$$

$$0 = \left[g^{\mu,i,l}\left[\frac{\partial b_l^{\nu,j,k}}{\partial u^m}\right]_{[m,l]} + \left[b_l^{\mu,i,j} b_m^{\nu,l,k}\right]_{[j,k]}\right]_{(\mu,\nu)} \tag{3.50}$$

$$0=\left[g^{\mu,i,l}\frac{\partial b_m^{\nu,j,k}}{\partial u^l}+-1b_l^{\nu,i,j}b_m^{\mu,l,k}+-1b_l^{\nu,i,k}b_m^{\mu,j,l}\right]_{[(i,\mu),(j,\nu)]} \tag{3.51}$$

$$0=\left[\frac{\partial}{\partial u^n}\left(g^{\mu,i,l}\left[\frac{\partial b_l^{\nu,j,k}}{\partial u^m}\right]_{[m,l]}+\left[b_l^{\mu,i,j}b_m^{\nu,l,k}\right]_{[j,k]}\right)-\left[b_m^{\mu,l,i}\left[\frac{\partial b_l^{\nu,j,k}}{\partial u^m}\right]_{[m,l]}\right]_{\circlearrowleft i,j,k}\right]_{((\mu,m),(\nu,n))}. \tag{3.52}$$

Definition 3.3.4. *A collection of metrics g^μ is called* Poisson*, if there exists a collection b^μ such that the pair (g,b) induces a hydrodynamic Poisson bracket.*

3.3.1 with nondegenerate metric

Definition 3.3.5. *A metric $g(u)$ is called* non-degenerate *if* $\det(g(u))\neq 0$ *everywhere.*

Definition 3.3.6. *A Poisson bracket (3.39) is called* non-degenerate *if all metrics $g^\mu(u)$ are non-degenerate.*

3.3.1.1 in one spatial dimension

The main result, due to Dubrovin and Novikov ([DN83b], theorem 1) and most succinctly stated in [Mok00], is the following

Theorem 3.3.7. *Let $\det(g(u))\neq 0$ and $d=1$. The bracket (3.39) is a Poisson bracket if and only if $g(u)$ is an arbitrary flat pseudo-Riemannian metric and $b_k^{i,j}(u)=-1g^{i,l}(u)\Gamma_{l,k}^j(u)$, where $\Gamma_{l,k}^j(u)$ is the Riemannian connection generated by the metric $g(u)$.*

Proof. Stated without explicit proof in [DN83b]. Can be proved by 'direct verification'. □

Due to flatness of the metric, we immediately get existence of (local) coordinates u where $g^{i,j}(u)=e_i\delta_j^i$ with $e_i=\pm1$ and $b(u)=0$. Hence *all* non-degenerate one-dimensional Poisson brackets are classified by the signature of their metric.

3.3.1.2 in multiple spatial dimensions

A generalization of theorem 3.3.7, summarized in [Mok06], uses heavily the notion of *compatible metrics* developed in [Mok00]:

Definition 3.3.8. *Two pseudo-Riemannian contravariant metrics $g^{1,i,j}(u)$ and $g^{2,i,j}(u)$ of constant Riemannian curvature K_1 and K_2, respectively, are said to be* compatible *if an arbitrary linear conbination*

$$g^{i,j(u)}=\lambda_1 g^{1,i,j}(u)+\lambda_2 g^{2,i,j}(u) \tag{3.53}$$

of these metrics, where λ_1 *and* λ_2 *are arbitrary constants such that* $\det\left(g^{i,j}(u)\right) \neq 0$*, is a metric of constant Riemannian curvature* $\lambda_1 K_1 + \lambda_2 K_2$ *and the coefficients of the corresponding Levi-Civita connections are related by the same linear formula*

$$\Gamma^{i,j}_{k}(u) = \lambda_1 \Gamma^{i,j}_{1,k}(u) + \lambda_2 \Gamma^{i,j}_{2,k}(u). \tag{3.54}$$

This definition naturally mirrors the definition of *compatible Poisson brackets*, of which linear combinations are again required to be Poisson brackets (see e.g. [Fer01] or [Mok02c]). Indeed, the following result is immediate:

Theorem 3.3.9. *The metrics* $g^{\mu,i,j}$ *defining a multidimensional Poisson bracket of the form (3.39) are compatible.*

Proof. See [Mok98], chapter II. □

However, this is far from enough. Indeed, again from [Mok98], theorem 2.1:

Theorem 3.3.10. *Flat non-degenerate metrics* $g^{\mu,i,j}(u)$ *and the Riemannian connections* $\Gamma^{\mu,i}_{j,k}$ *defined by these metrics (the Levi-Civita connections) generate a Poisson structure (3.39) if and only if the tensors* $T^{\mu,\nu,i}_{j,k} = \Gamma^{\nu,i}_{j,k} - \Gamma^{\mu,i}_{j,k}$ *satisfy the following relations (no summation over* μ*):*

$$\left[T^{\mu,\nu,i,j,k}\right]_{[i,k]} = 0 \tag{3.55}$$

$$\left[T^{\mu,\nu,i}_{j,k}\right]_{\circlearrowright i,j,k} = 0 \tag{3.56}$$

$$\left[T^{\mu,\nu,i,j,m} T^{\mu,\nu,k}_{m,l}\right]_{[k,l]} = 0 \tag{3.57}$$

$$\nabla^{\mu}_{l} T^{\mu,\nu,i,j,k} = 0 \tag{3.58}$$

where $T^{\mu,\nu,i,j,k} = -1 g^{\nu,k,m} T^{\mu,\nu,i,j}_{m} = -1 g^{\nu,k,m} g^{\mu,i,n} T^{\mu,\nu,j}_{m,n}$ *and* ∇^{μ}_{l} *is the covariant derivative associated with* g^{μ}*.*

Similarly to the case of one spatial dimension, there are multi-dimensional hydrodynamic Poisson brackets that can be reduced to *constant* form. However, contrary to the one-dimensional case, non-degeneracy of the associated metric(s) is not a sufficient condition. Instead, we need additionally that the *obstruction tensors* $T^{\mu,\nu,i}_{j,k}$ are identically zero. We will need the concept of *non-singular* pairs of metrics:

Definition 3.3.11. *A pair of metrics* $g^{1,i,j}(u)$ *and* $g^{2,i,j}(u)$ *is called* non-singular *if the eigenvalues of this pair of metrics, i.e. the roots of the equation* $\det\left(\left(g^{1,i,j}(u) + -1\lambda g^{2,i,j}(u)\right)\right) = 0$*, are distinct.*

Finally we can state

Theorem 3.3.12. *If for a non-degenerate multidimensional Poisson bracket (3.39) one of the metrics* $g^{\mu,i,j}(u)$ *forms non-singular pairs with all the remaining metrics of the bracket, then this Poisson bracket can be reduced to constant form by a local change of coordinates.*

And for all other (non-degenerate) Poisson brackets we have

Theorem 3.3.13. *Consider a Poisson bracket (3.39). If the metric $g^{1,i,j}$ is non-degenerate, then in flat coordinates u where $g^{1,i,j}(u)$ is constant and thus $b_k^{1,i,j} = 0$, every other metric $g^{\mu,i,j}$ is* linear.

Proof. See [DN83a], theorem 1 and [Rey10], proposition 2.15. □

Remark. *Non-degeneracy of* all *metrics is not required.*

3.3.2 with degenerate metric

3.3.2.1 in one spatial dimension

Following the classification of non-degenerate one-dimensional Poisson brackets (3.39) in [DN83a], Grinberg provided answers to the question of classification of *degenerate* one-dimensional Poisson brackets with metric $g(u)$ of constant rank in [Gri85]. However, proofs were lacking in the literature until the publication of [Bog07b]. We will summarize the main statements.

For now, we do not require the *(2,0)-tensor field* g to be associated with a Poisson bracket. First, let us partition the *n-dimensional target manifold* $\mathcal{U}^n$ by defining the *subsets of rank k*

$$\mathcal{O}_k = \{u \in \mathcal{U}^n | \mathrm{rank}\,(g(u)) = k\} \subset \mathcal{U}^n. \tag{3.59}$$

Next, to depart from the non-degenerate case, let us require that at least one $\mathcal{O}_m$ is nonempty and open for some $m < n$. Now, in slight disagreement with [Bog07b], we have:

Proposition 3.3.14. *Consider a* (2,0)-tensor field *g with nonempty, open set $\mathcal{O}_m$ as in 3.59. Then*

- *g being smooth and Poisson does not imply $\overline{\cup_{k=0}^{n-1}\mathcal{O}_k} = \mathcal{U}^n$,*
- *if g is analytic, then $\mathcal{O}_m$ is dense in $\mathcal{U}^n$, i.e. $\overline{\mathcal{O}_m} = \mathcal{U}^n$.*

Proof. For the first point we will construct a counterexample. All functions used in this part will be smooth. Consider a diagonal metric

$$g(u) = \begin{bmatrix} g^1(u) & 0 \\ 0 & g^2(u) \end{bmatrix} \tag{3.60}$$

on the target manifold $\mathcal{U}^n = \mathbb{R}^2$. Together with

$$b(u) = \frac{1}{2}\nabla_u \begin{bmatrix} g^1(u) & b^{1,2}(u) \\ -b^{1,2}(u) & g^2(u) \end{bmatrix} \tag{3.61}$$

this induces a hydrodynamic Poisson bracket if

$$g^2 = b^{1,2} = 0 \tag{3.62}$$

or if

$$-g^2(u) = g^1(u) = b^{1,2}(u) = g^0\left(u^1 + u^2\right) \tag{3.63}$$

for all $u \in \mathcal{U}^n$, see `hydro_smooth_counterexample.ipynb`. We may stitch together the above families of Poisson brackets, as long as they are separated from each other, as for example as follows:

$$g^1(u) = g^0\left(u^1 + u^2\right) \tag{3.64}$$

$$-g^2(u) = \begin{cases} g^1(u) & \text{if } u^1 + u^2 < -\varepsilon, \\ 0 & \text{if } u^1 + u^2 \geq -\varepsilon \end{cases} \tag{3.65}$$

$$g^0(x) = \begin{cases} \neq 0 & \text{if } |x| > \varepsilon, \\ 0 & \text{if } |x| \leq \varepsilon \end{cases}, \tag{3.66}$$

with $\varepsilon > 0$ and g^0 constructed appropriately. We now have

$$\operatorname{rank}(g) = \begin{cases} 2 & \text{if } u^1 + u^2 < -\varepsilon, \\ 0 & \text{if } \left|u^1 + u^2\right| \leq \varepsilon, \\ 1 & \text{if } u^1 + u^2 > \varepsilon \end{cases} \tag{3.67}$$

and thus

$$\overline{\cup_{k=0}^{n-1}\mathcal{O}_k} = \left\{u^1, u^2 \in \mathcal{U}^n \middle| \left(u^1 + u^2\right) \in [-\varepsilon, \infty)\right\} \neq \mathcal{U}^n, \tag{3.68}$$

which completes the proof of the first part.

For the second part we use the following factoid (see e.g. [Hog13], chapter 2.4, fact 25):

> A square matrix A has rank k if and only if A has a nonsingular $k \times k$ submatrix, and every $(k+1) \times (k+1)$ submatrix of A is singular.

As the determinant of a matrix with analytic coefficients is itself analytic, square submatrices of g that are singular on some open set $\mathcal{O}_m \subset \mathcal{U}^n$ stay singular on the whole of the *n-dimensional target manifold $\mathcal{U}^n$*. For the same reason, square submatrices of g that are nonsingular on some open set $\mathcal{O}_m \subset \mathcal{U}^n$ can not be singular on all of any open set. Given any $x \in \mathcal{U}^n$, this includes *any* open set $\mathcal{O}' \subset \mathcal{U}^n$ containing x. Thus, every neighborhood of every point $x \in \mathcal{U}^n$ contains at least one point from $\mathcal{O}_m$, which is thus dense in $\mathcal{U}^n$, i.e. its closure equals $\mathcal{U}^n$ (see e.g. [Bou95], chapter 1, definition 10). □

Remark. *The phrasing in [Bog07b] is slightly ambiguous. On page 540 he claims that "if the functions $g^{i,j}(u)$ are smooth, [...] the [target manifold $\mathcal{U}^n$] is the closure of the union $\cup_{k=0}^{n-1}\mathcal{O}_k$."*

Two sentences before, he talks about "any degenerate (2,0)-tensor $g^{i,j}(u)$", suggesting that the requirement for g to be Poisson can be dropped. The following theorem hower relies on g being Poisson, so we must interpret the whole *section to carry this requirement implicitly. If indeed g were not required to be Poisson, an even simpler counterexample could have been constructed, as we are not constrained by the considerable requirements induced by the Jacobi identity. Next, the* exact *meaning of g being degenerate has not been stated in [Bog07b]. If it were to mean that $\mathcal{O}_n$ is empty, the above statement would be trivially true, but then there would be no need of closing the union. The statement closest to a definition of degeneracy can be found on page 539:*

> *In Section 2, we disclose the invariant meaning of the Poisson brackets for any values of k with degenerate (2,0)-tensor $g^{i,j}(u)$ that has a constant rank $g^{i,j} = m < n$ in an open domain $\mathcal{O}_m$.*

Note that the phrasing does not imply that $\mathcal{O}_m$ is nonempty, that $\mathcal{O}_m$ is defined as in 3.59, or that $\mathcal{O}_n$ is empty. Again, if the requirement were not that at least one $\mathcal{O}_m$ as defined in 3.59 with $m < n$ is open (and nonempty), a simpler counterexample with a metric g that is Poisson could have been constructed.

Finally, at the end of Section 1, page 540 in [Bog07b] a further constraint is mentioned, namely $1 < m < n$. This constraint only seems to apply to Corollary 1 in [Bog07b], where it is explicitly invoked. Even if it were implied in the contested statement, we could simply construct a $\hat{g}$ which is block-diagonal, with the g from the proof as one block and another matrix with arbitrary rank as the other block.

Cf. figure 3.1 for all constructions. For $k < m$ we have

$$\overline{\mathcal{O}_k} \cap \overline{\mathcal{O}_m} \subset \cup_{l=0}^{k-1} \mathcal{O}_l \tag{3.69}$$

and thus

- *for the left figure $\mathcal{O}_1$ is open but g is not Poisson,*
- *for the middle figure g is Poisson but $\mathcal{O}_1$ is not open as it shares a boundary with $\mathcal{O}_2$,*
- *for the right figure, g is Poisson and $\mathcal{O}_1$ is open*
- *for all figures, $\mathcal{O}_0$ is not open and $\mathcal{O}_2$ is open.*

(3.69) SEEMS *RIGHT, BUT I SHOULD FIND A REFERENCE.*

Theorem 3.3.15. *Let L_u and $\mathcal{O}_k$ be as in* **??** *and 3.59. Then*

- *An invariant non-degenerate metric is defined on the distribution L_u,*
- *The distribution L_u is invariant under any $(1,1)$-tensor $A^i_j(u)$ (3.44) and operator $b^{i,j}_k(u)$ for $j = \text{const}$,*
- *In the open domain $\mathcal{O}_m$, the distribution L_u is involutive and defines an m-dimensional foliation $\mathcal{F}_m$,*
- *The metric on the leaves of $\mathcal{F}_m$ is flat.*

Proof. See [Bog07b], theorem 1. □

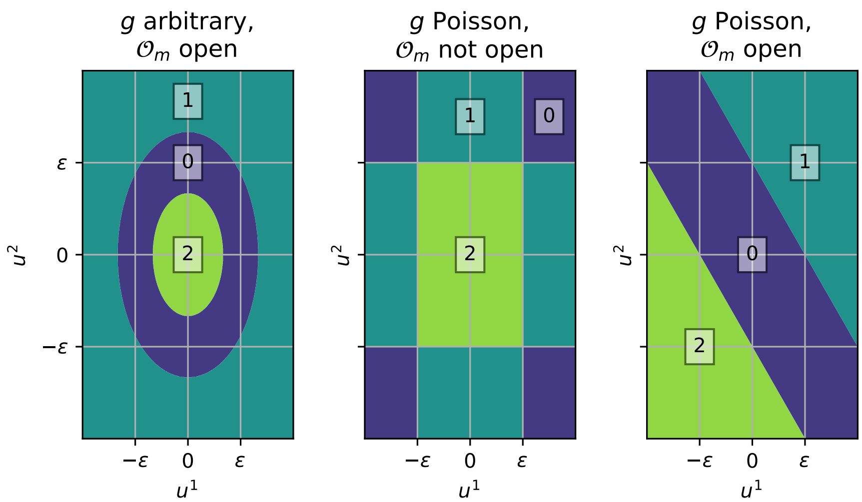

Figure 3.1: Constructions for PROPOSITION. Colors and numbers indicate Rank(g(u)).

3.3.2.2 in multiple spatial dimensions

3.3.3 of Lie-Poisson type

3.3.3.1 in one spatial dimension

[BN85]

3.3.3.2 in multiple spatial dimensions

3.4 Hyperbolic Poisson brackets

Throughout, we will assume *vanishing of boundary terms*

$$\int \frac{\partial}{\partial x} \left(f \left(u \left(x \right) \right) \right) \, dx = 0. \tag{3.70}$$

3.4.1 of constant type

For constant Poisson brackets, just like for linear hyperbolic systems of balance laws, many things simplify. Nevertheless, they have attracted some interest over the years, mainly in the context of multi-Hamiltonian systems, see e.g. [GN90]. Olver and Nutku examined one particular class of one-dimensional, 2×2 constant hydrodynamic Poisson brackets in [ON88], following [Nut85] and [Nut87]. The same class has been connected to the shallow water equations in [CM82], to Poisson's equation in [Nut87], to the Born-Infeld equation in [ANN$^+$89] and to Riemann invariants in [Ney89]. This class has been further investigated in [Car95]. Benjamin and Bowman considered discontinuous solutions of general one-dimensional constant Poisson brackets in [Ben87].

Here, we will only state some immediate results connecting constant hydrodynamic Poisson brackets to (possibly hyperbolic) systems of quasilinear first-order PDEs with an additional conservation law.

We adapt the customary definition of constant Poisson brackets on finite dimensional vector spaces as given e.g. in [LGPV13], definition 6.1:

Definition 3.4.1. *A hydrodynamic Poisson bracket π on the infinite-dimensional function space $C^\infty\left(\mathcal{M}^d;\mathcal{U}^n\right)$ is called* constant*, if for each pair of linear functionals F and G of the form $I = \int f_i\left(x\right) u^i\left(x\right) dx$, their Poisson bracket $\pi\left(F, G\right)$ is a constant function on $C^\infty\left(\mathcal{M}^d;\mathcal{U}^n\right)$.*

This translates to the equivalent conditions $g^\mu = \text{symmconst}$ and $b^\mu = 0$, hence we have as evolution equation

$$u_t = g^\mu \nabla^2 h u_\mu. \tag{3.71}$$

As can be immediately verified from relations **??**, any constant skew-symmetric bracket is a Poisson bracket. Furthermore, from lemma 3.2.10 we get:

Corollary 3.4.2. *Any constant hydrodynamic Poisson bracket induces a system of conservation laws with an additional balance law for the energy density. If the energy density is strictly convex, the system is symmetrizable.*

Proof. Follows from lemma 3.2.10 and $b^\mu = 0$. □

Corollary 3.4.3. *In one spatial dimension ($d = 1$),* all *constant hydrodynamic Poisson brackets with positive definite matrix g induce a symmetrizable system of conservation laws for* any *hydrodynamic Hamiltonian.*

Remark.

- *The symmetrizer is g^{-1}*

- *For $d > 1$ the system is generally not symmetrizable.*

Corollary 3.4.4. Any *linear system of first-order PDEs*

$$u_t + A^\mu u_\mu = 0 \tag{3.72}$$

that admits an additional conservation law for $\eta(u)$ with non-singular and constant Hessian can be written using a constant hydrodynamic Poisson bracket with $g^\mu = A^\mu \nabla^{-2}\eta$.

Proof. From the existence of a conservation law for $\eta(u)$ we get from the integrability condition (cf. equation (3.7))

$$\nabla^2\eta A^\mu = \left(\nabla^2\eta A^\mu\right)^T = (A^\mu)^T \nabla^2\eta \tag{3.73}$$

and hence after multiplication with $\nabla^{-2}\eta$ from the left and from the right

$$g^\mu = A^\mu\nabla^{-2}\eta = \nabla^{-2}\eta(A^\mu)^T = \left(A^\mu\nabla^{-2}\eta\right)^T = (g^\mu)^T = \text{symm.const}, \tag{3.74}$$

thus the g^μ induce a Poisson bracket. □

Remark.
- *Generally we can assume $\nabla^2\eta = \text{const}$ if the system of first-order PDEs is linear.*
- *If $\nabla^2\eta$ is singular, the A^μ are 2×2 block-diagonal in coordinates in which $\nabla^2\eta$ is diagonal. The blocks associated with the non-zero eigenvalues of $\nabla^2\eta$ can be written using a constant hydrodynamic Poisson bracket, while we can say nothing about the other blocks.*

We note that, while all linear hyperbolic systems of conservation laws with an additional conservation law admit a constant hydrodynamic Poisson bracket, not all linear evolution equations induced by constant hydrodynamic Poisson brackets are hyperbolic. Consider for example

$$g = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, h\left(u^1, u^2\right) = u^1 u^2 \implies A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3.75}$$

Similarly, only in one spatial dimension is positive definiteness of the matrices g^μ sufficient for hyperbolicity. Consider for example

$$g^1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, g^2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \implies g^1 - g^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{3.76}$$

i.e. we may choose h as above such that in direction $\xi = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ the system matrix $A(\xi)$ is not real diagonalizable.

In fact, we may show the following:

Corollary 3.4.5. *In one spatial dimension ($d = 1$) a constant hydrodynamic Poisson bracket induces hyperbolic PDEs for all hydrodynamic functionals if and only if its metric g is definite or identically zero. In multiple spatial dimensions,* in addition *all metrics g^μ have to be scalar multiples of each other.*

Proof. For the one-dimensional case, we may simply construct a Hamiltonian as above in coordinates in which the metric is diagonal and contains only $1, 0$ or -1. For the multi-dimensional case we note that from the 1D result it follows that for every *direction* $\xi \in \mathcal{S}^{[d-1]}$ the matrix $A(\xi)$ has to be either definite or identically zero. From continuity of eigenvalues of $A(\xi)$ (see e.g. [Kat95], theorem 6.8) it follows that for all i, j, every curve connecting e_i and $\pm e_j$ on $\mathcal{S}^{[d-1]}$ has to pass a point ξ where $A(\xi) = 0$. This however can only happen, if $A(e_i)$ is a scalar multiple of $A(e_j)$. □

Finally we want to note the following simple result, which will help us relate non-constant hydrodynamic Poisson brackets to constant ones:

Corollary 3.4.6. *Let g be the constant metric associated with a constant hydrodynamic Poisson bracket π. After an invertible change of variables*

$$\hat{u} = \hat{u}(u), \tag{3.77}$$

we have as transformed expressions

$$\hat{g}^{\mu,\hat{\imath},\hat{\jmath}} = \hat{u}_i^{\hat{\imath}} g^{\mu,i,j} \hat{u}_j^{\hat{\jmath}} \tag{3.78}$$

$$\hat{b}_{\hat{k}}^{\mu,\hat{\imath},\hat{\jmath}} = \hat{u}_i^{\hat{\imath}} g^{\mu,i,j} \hat{u}_{j,k}^{\hat{\jmath}} u_{\hat{k}}^k \tag{3.79}$$

where

$$\hat{u}_i^{\hat{\imath}} = [\nabla \hat{u}]_i^{\hat{\imath}} \tag{3.80}$$

$$\hat{u}_{j,k}^{\hat{\jmath}} = \left[\nabla^2 \hat{u}\right]_{j,k}^{\hat{\jmath}} \tag{3.81}$$

$$u_{\hat{k}}^k = \left[\hat{\nabla} u\right]_{\hat{k}}^k \tag{3.82}$$

Proof. A straightforward computation yields

$$\pi(F, H) = \left\langle F_i, g^{\mu,i,j} \partial_\mu (H_j) + \overbrace{b_k^{\mu,i,j}}^{0} u_\mu^k H_j \right\rangle \tag{3.83}$$

$$= \left\langle \hat{F}_{\hat{\imath}} \hat{u}_i^{\hat{\imath}}, g^{\mu,i,j} \partial_\mu \left(\hat{u}_j^{\hat{\jmath}} \hat{H}_{\hat{\jmath}} \right) \right\rangle \tag{3.84}$$

$$= \left\langle \hat{F}_{\hat{\imath}}, \underbrace{\hat{u}_i^{\hat{\imath}} g^{\mu,i,j} \hat{u}_j^{\hat{\jmath}}}_{\hat{g}^{\mu,\hat{\imath},\hat{\jmath}}} \partial_\mu \left(\hat{H}_{\hat{\jmath}} \right) + \underbrace{\hat{u}_i^{\hat{\imath}} g^{\mu,i,j} \hat{u}_{j,k}^{\hat{\jmath}} u_{\hat{k}}^k}_{\hat{b}_{\hat{k}}^{\mu,\hat{\imath},\hat{\jmath}}} \hat{u}_\mu^{\hat{k}} \hat{H}_{\hat{\jmath}} \right\rangle \tag{3.85}$$

$$= \hat{\pi}\left(\hat{F}, \hat{H}\right). \tag{3.86}$$

□

3.4.2 with non-degenerate metric

3.4.2.1 in one spatial dimension

In [PPK20], the following result is stated (Theorem 2.10):

Theorem 3.4.7. *Consider a one-dimensional Hamiltonian system of hydrodynamic type with non-degenerate metric. Assuming that the energy of the system is of hydrodynamic type, convex and a proper scalar, it follows that the evolution equations can be regarded as a first-order quasilinear symmetric hyperbolic PDE system.*

This appears to be not correct. Consider the following diagonal metric g with *matrix representation*

$$g\left(u\right) = \begin{bmatrix} g^1\left(u^1\right) & 0 \\ 0 & g^2\left(u^2\right) \end{bmatrix}. \tag{3.87}$$

We get the corresponding *differential operator*

$$\begin{aligned}
Lv &= g\left(u\right)\frac{\partial v}{\partial x} + \frac{1}{2}\frac{\partial}{\partial u^k}\left(g\left(u\right)\right)u_x^k v &(3.88)\\
&= \begin{bmatrix} g^1 & 0 \\ 0 & g^2 \end{bmatrix}\begin{bmatrix} \frac{\partial v_1}{\partial u^1} & \frac{\partial v_1}{\partial u^2} \\ \frac{\partial v_2}{\partial u^1} & \frac{\partial v_2}{\partial u^2} \end{bmatrix}\begin{bmatrix} u_x^1 \\ u_x^2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \frac{\partial g^1}{\partial u^1}u_x^1 & 0 \\ 0 & \frac{\partial g^2}{\partial u^2}u_x^2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &(3.89)\\
&= \begin{bmatrix} g^1\frac{\partial v_1}{\partial u^1}u_x^1 + g^1\frac{\partial v_1}{\partial u^2}u_x^2 + \frac{\partial g^1}{\partial u^1}u_x^1\frac{1}{2}v_1 \\ g^2\frac{\partial v_2}{\partial u^1}u_x^1 + g^2\frac{\partial v_2}{\partial u^2}u_x^2 + \frac{\partial g^2}{\partial u^2}u_x^2\frac{1}{2}v_2 \end{bmatrix} &(3.90)\\
&= \begin{bmatrix} g^1\frac{\partial v_1}{\partial u^1} + \frac{1}{2}\frac{\partial g^1}{\partial u^1}v_1 & g^1\frac{\partial v_1}{\partial u^2} \\ g^2\frac{\partial v_2}{\partial u^1} & g^2\frac{\partial v_2}{\partial u^2} + \frac{1}{2}\frac{\partial g^2}{\partial u^2}v_2 \end{bmatrix}\begin{bmatrix} u_x^1 \\ u_x^2 \end{bmatrix}, &(3.91)
\end{aligned}$$

defined by its action on a generic field $v = \hat{v}\circ u \in C^\infty\left(\mathbb{R};\mathbb{R}^2\right)$. The induced bracket is Poisson, as can be verified by inserting

$$\begin{aligned}
g^{i,j} &= \delta^{i,j}g^i &(3.92)\\
b_k^{i,j} &= \frac{1}{2}\frac{\partial g^{i,j}}{\partial u^k} = \frac{1}{2}\delta_k^{i,j}\frac{\partial g^i}{\partial u^k} &(3.93)
\end{aligned}$$

into relations **??**. Here $\delta_k^{i,j}$ is a generalization of the Kronecker delta, which is 1 exactly if all indices are equal and 0 otherwise. Inner products of g and b retain this structure, hence are symmetric in all free indices, and thus all terms in the Jacobi identity vanish.

Recall that (global) hyperbolicity means that the matrix in front of the spatial gradient of the unknown fields is diagonalizable over the real numbers (everywhere). A necessary condition for a *real-valued matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{3.94}$$

to be real diagonalizable is for the discriminant of its characteristic polynomial to be non-negative:

$$(a-d)^2 + 4bc \geq 0. \tag{3.95}$$

A sufficient condition for the above real-valued matrix A to be symmetric positive definite is

$$b = c, |b| < \min(a, d).$$

Let us now apply the above differential operator to the field associated with the variational derivative of some (strictly) convex hydrodynamic functional $H(u) = \int h(u(x))\, dx$. We get

$$L^{i,j}\frac{\partial h}{\partial u^j} = \begin{bmatrix} g^1 \frac{\partial^2 h}{\partial u^1 \partial u^1} + \frac{1}{2}\frac{\partial g^1}{\partial u^1}\frac{\partial h}{\partial u^1} & g^1 \frac{\partial^2 h}{\partial u^1 \partial u^2} \\ g^2 \frac{\partial^2 h}{\partial u^2 \partial u^1} & g^2 \frac{\partial^2 h}{\partial u^2 \partial u^2} + \frac{1}{2}\frac{\partial g^2}{\partial u^2}\frac{\partial h}{\partial u^2} \end{bmatrix} \begin{bmatrix} u^1_x \\ u^2_x \end{bmatrix} \tag{3.96}$$

$$= A \begin{bmatrix} u^1_x \\ u^2_x \end{bmatrix} \tag{3.97}$$

and $\nabla_i \nabla_j h = \left[g^{-1}A\right]_{i,j}$ where ∇_i represents the covariant derivative associated with the Levi-Civita connection of the metric g.

We observe:

- the metric g is flat and the above differential operator induces a Poisson bracket
- the metric g is non-degenerate (everywhere), if $g^1 \neq 0 \neq g^2$ (everywhere)
- the energy functional H is of hydrodynamic type
- the energy functional/density is (strictly) convex if $\left|\frac{\partial^2 h}{\partial u^1 \partial u^2}\right| < \min(\frac{\partial^2 h}{\partial u^1 \partial u^1}, \frac{\partial^2 h}{\partial u^2 \partial u^2})$
- the energy density h is a proper scalar, if u^1 and u^2 are proper scalars.

However

- the signs $\sigma\left(g^1\right)$ and $\sigma\left(g^2\right)$ do not matter for non-degeneracy
- the first order partial derivatives of the energy density $\frac{\partial h}{\partial u^1}$ and $\frac{\partial h}{\partial u^2}$ do not matter for convexity
- the matrix defined by the entries $\nabla_i \nabla_j h$, while symmetric, is **not** necessarily positive definite

and we can thus choose our non-degenerate metric g and construct a (strictly) convex energy density h such that for some point u^*

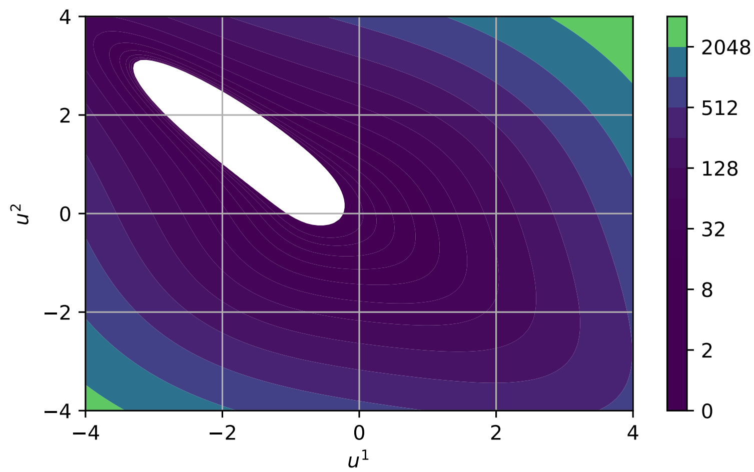

Figure 3.2: "Excess hyperbolicity" $\left((a-d)^2+4bc\right)$ for $g^i = (-1)^i\left(1+{u^i}^2\right)$ and $h(u) = \left\langle u, \frac{1}{2}\begin{bmatrix} 1 & 1-\varepsilon \\ 1-\varepsilon & 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$, $\varepsilon = 0.1$. White corresponds to non-hyperbolic regions.

- $g^1 < 0 < g^2$
- $0 < \left| \frac{\partial^2 h}{\partial u^1 \partial u^2} \right| < \min\left(\frac{\partial^2 h}{\partial u^1 \partial u^1}, \frac{\partial^2 h}{\partial u^2 \partial u^2}\right)$
- $\frac{1}{2} \frac{\partial g^i}{\partial u^i} \frac{\partial h}{\partial u^i} = -1 g^i \frac{\partial^2 h}{\partial u^i \partial u^i}$ for $i = 1, 2$.

The first two points ensure that the product of the off-diagonal entries is (strictly) negative, whilst the last point ensures that the diagonal entries vanish, hence yielding non-hyperbolicity at u^* and in a neighbourhood around u^*.

See also figure 3.2 for a slightly different example.

We may wonder whether there is something peculiar about the above constructions. It is obvious that the above construction would not have worked, if the metric were definite or if $\frac{\partial^2 h}{\partial u^1 \partial u^2} = 0$. Are there special Poisson brackets or Hamiltonians which guarantee hyperbolicity? The following Lemma gives the answer:

Lemma 3.4.8. *Consider a Poisson bracket of hydrodynamic type with non-degenerate metric g in one space dimension and with arbitrary number of fields $n \geq 2$.*

- *If the metric is* definite, every *Hamiltonian of hydrodynamic type induces a system of hyperbolic PDEs.*
- *On the other hand, for* every *(non-constant) Hamiltonian of hydrodynamic type, we can construct a Poisson bracket with non-degenerate metric such*

that the induced system of PDEs is non-hyperbolic on a set of non-zero measure.

Proof. We know that the metric g is flat. If it is definite, for every point $p \in \mathcal{U}^n$ we may find local coordinates $\hat{u}$ such that

$$g_{i,j} = \pm \delta_{i,j} \tag{3.98}$$

on a neighborhood of p. On this neighborhood g is constant, hence the Christoffel symbols of the second kind (cf. equation (2.1)) vanish and thus $b_k^{i,j}$ also does.

In these coordinates, the evolution equation reads

$$\hat{u}_t = \hat{L}\hat{\nabla}h = g\hat{\nabla}^2\hat{h}\hat{u}_x \tag{3.99}$$
$$= \pm \hat{\nabla}^2\hat{h}\hat{u}_x. \tag{3.100}$$

Due to symmetry of the Hessian, the system is hyperbolic. In fact, if we could ensure that the Hessian of the energy density were positive definite in those flat coordinates $\hat{u}$, we would also immediately get hyperbolicity for indefinite metrics. However, given a Hamiltonian of hydrodynamic type, we can in general guarantee neither this sufficient condition nor a weaker one.

Let us first look at the 2×2 case. Let

$$\hat{h}_{\hat{i},\hat{j}} = \left[\hat{\nabla}^2\hat{h}\right]_{\hat{i},\hat{j}} \tag{3.101}$$

denote the entries of the Hessian in flat coordinates $\hat{u}$ where the metric is constantly

$$\hat{g} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.102}$$

Then

$$A = \hat{g}\hat{\nabla}^2\hat{h} \tag{3.103}$$

is real diagonalizable if and only if

$$\left|\hat{h}_{1,1} + \hat{h}_{2,2}\right| \geq 2\left|\hat{h}_{1,2}\right|, \tag{3.104}$$

which is weaker than convexity of h.

The question now is, given some energy density h, which is possibly (strictly) convex in non-flat, "original" coordinates u, can we somehow guarantee hyperbolicity based on properties of the energy density alone, independently of the Poisson bracket? The answer, as stated in the lemma, is no:

Consider coordinates

$$\begin{bmatrix} u^1 \\ u^2 \end{bmatrix} = \begin{bmatrix} \hat{u}^1 \\ f\left(\hat{u}^1, \hat{u}^2\right) \end{bmatrix} \tag{3.105}$$

for some smooth real-valued function f with $f' \neq 0$. Denoting as entries of the Jacobian $u^i_{\hat{i}} = \hat{\nabla}_{\hat{i}} u^i$ we get

$$\hat{h}_{\hat{i},\hat{j}} = h_{i,j} u^i_{\hat{i}} u^j_{\hat{j}} + h_i u^i_{\hat{i},\hat{j}} \tag{3.106}$$

$$= \begin{bmatrix} h_{i,j} u^i_1 u^j_1 + h_i u^i_{1,1} & h_{i,j} u^i_1 u^j_2 + h_i u^i_{1,2} \\ h_{i,j} u^i_2 u^j_1 + h_i u^i_{2,1} & h_{i,j} u^i_2 u^j_2 + h_i u^i_{2,2} \end{bmatrix} \tag{3.107}$$

$$= \begin{bmatrix} h_2 f_{1,1} + f_1^2 h_{2,2} + 2 f_1 h_{1,2} + h_{1,1} & h_2 f_{1,2} + f_1 f_2 h_{2,2} + f_2 h_{1,2} \\ h_2 f_{2,1} + f_1 f_2 h_{2,2} + f_2 h_{1,2} & h_2 f_{2,2} + f_2^2 h_{2,2} \end{bmatrix}. \tag{3.108}$$

We get

$$\left| h_2 f_{1,1} + h_2 f_{2,2} + f_1^2 h_{2,2} + 2 f_1 h_{1,2} + f_2^2 h_{2,2} + h_{1,1} \right| \geq 2 |h_2 f_{1,2} + f_1 f_2 h_{2,2} + f_2 h_{1,2}| \tag{3.109}$$

as a necessary condition for hyperbolicity. Without loss of generality we will assume $h_2(0) \neq 0$. Note that given some function f, we may construct a function $\hat{f}(x) = \frac{f(cx)}{c}$ such that $\hat{f}_i(0) = f_i(0)$ but $\hat{f}_{i,j}(0) = c f_{i,j}(0)$ for an arbitrary constant $c \neq 0$. We achieve violation of the above inequality by choosing an f such that

$$2 |f_{1,2}| \geq |f_{1,1} + f_{2,2}| \tag{3.110}$$

at the origin and then constructing an appropriate $\hat{f}$ with a large enough c. One possible choice of f is

$$f(x,y) = \frac{1}{2}\left(1 + (1+x)^2\right)\left(y + y^3\right). \tag{3.111}$$

Let us now look at the case $n > 2$. Given a non-constant energy density h, we will construct a Poisson bracket with non-degenerate indefinite metric g such that the induced PDE is non-hyperbolic on a set of non-zero measure. As above we will assume $h_2(0) \neq 0$. We will extend the constructions above. Let the metric in flat coordinates be

$$\hat{g} = \begin{bmatrix} 1 & 0 & 0_{1,[n-2]} \\ 0 & -1 & 0_{1,[n-2]} \\ 0_{[n-2],1} & 0_{[n-2],1} & I_{[n-2],[n-2]} \end{bmatrix} \tag{3.112}$$

and consider coordinates

$$\begin{bmatrix} u^1 \\ u^2 \\ u^3 \\ \vdots \end{bmatrix} = \begin{bmatrix} \hat{u}^1 \\ f\left(\hat{u}^1, \hat{u}^2\right) \\ \hat{u}^3 \\ \vdots \end{bmatrix}. \tag{3.113}$$

Then we get

$$\hat{h}_{\hat{\imath},\hat{\jmath}} = \begin{cases} h_2 f_{1,1} + f_1^2 h_{2,2} + 2 f_1 h_{1,2} + h_{1,1} & \text{if } \hat{\imath} = \hat{\jmath} = 1, \\ h_2 f_{1,2} + f_1 f_2 h_{2,2} + f_2 h_{1,2} & \text{if sorted}(\hat{\imath},\hat{\jmath}) = 1,2, \\ h_2 f_{2,2} + f_2^2 h_{2,2} & \text{if } \hat{\imath} = \hat{\jmath} = 2, \\ h_{1,\hat{\jmath}} + h_{2,\hat{\jmath}} f_1 & \text{if } \min(\hat{\imath},\hat{\jmath}) = 1 \text{ and } \max(\hat{\imath},\hat{\jmath}) \geq 3, \\ h_{2,\hat{\jmath}} f_2 & \text{if } \min(\hat{\imath},\hat{\jmath}) = 2 \text{ and } \max(\hat{\imath},\hat{\jmath}) \geq 3, \\ h_{\hat{\imath},\hat{\jmath}} & \text{else} \end{cases}. \tag{3.114}$$

Note that for the f given above, at the origin we have $f_{1,1} = f_{2,2} = f_1 = 0$ and $f_2 = 1$, hence

$$\hat{h}_{\hat{\imath},\hat{\jmath}}(0) = \begin{cases} h_{1,2}(0) + f_{1,2}(0) h_2(0) & \text{if sorted}(\hat{\imath},\hat{\jmath}) = 1,2, \\ h_{\hat{\imath},\hat{\jmath}}(0) & \text{else} \end{cases}. \tag{3.115}$$

For arbitrary $n > 2$, we of course do not have the luxury of an explicit expression for the eigenvalues of a general matrix. However, we may still construct a matrix such that we can ensure that it has at least one eigenvalue pair with non-zero imaginary part. We will use Gerschgorin's theorem (cf. e.g. [Ste90], chapter IV, theorem 2.1), ensuring that at least two Gerschgorin disks are each separated from all others while not touching the real line.

Consider

$$\hat{g}\hat{\nabla}^2\hat{h} = \underbrace{\begin{bmatrix} h_{1,1} & 0 & h_{1,3} & \cdots \\ 0 & -h_{2,2} & 0 & \cdots \\ h_{1,3} & 0 & h_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{A_{\text{Symm}}} + \underbrace{\begin{bmatrix} 0 & \hat{h}_{1,2} & 0 & \cdots \\ -\hat{h}_{1,2} & 0 & -h_{2,3} & \cdots \\ 0 & h_{2,3} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{A_{\text{Skew}}} \tag{3.116}$$

at the origin. The matrix A_{Skew} has non-zero eigenvalues

$$\lambda_{1,2} = \pm i \left| \hat{h}_{1,2} + \sum_{j=3}^{n} h_{2,j} \right| \tag{3.117}$$

and, by virtue of being antisymmetric, is unitarily diagonalizable (over the complex numbers) with matrix Q. After again replacing f by $\hat{f}$, we see that the only entry of $\hat{g}\hat{\nabla}^2\hat{h}$ that depends on c is $\hat{h}_{1,2}$, which only appears in the antisymmetric second part. We now apply Gerschgorin's theorem to $Q\hat{g}\hat{\nabla}^2\hat{h}Q^*$.

The absolute values of all entries of $QA_{\text{Symm}}Q^*$ can be bounded by $n^2\|A_{\text{Symm}}\|_\infty$, independently of c. $QA_{\text{Skew}}Q^*$ on the other hand will be diagonal and have as only non-zero entries two purely imaginary diagonal entries, whose absolute values will *grow* asymptotically linearly with c. Hence, the radii of all Gerschgorin disks can be bounded independently of c, as can the absolute values of all but two centers of the Gerschgorin disks. For the centers whose absolute values can not be bounded independently of c, we can bound the distance to $\lambda_{1,2}$ independently of c. However, as we can make $|\lambda_{1,2}|$ arbitrarily large, this means that we can choose c such that two Gerschgorin disks are isolated from all others and do not touch the real line. This is sufficient to conclude that, at the origin, $\hat{g}\hat{\nabla}^2\hat{h}$ has at least one eigenvalue pair with non-zero imaginary part, and hence the induced PDE is non-hyperbolic at the origin and in a neighborhood around the origin. This concludes the proof.

$\square$

Remark.

- *For $n = 1$ every first order quasilinear PDE is hyperbolic and Poisson.*
- *For constant Hamiltonians the evolution equation is trivially zero.*
- *For $n > 2$, the constructed metric need not be Lorentzian, but it simplifies the presentation.*
- *If $h_2(0) = 0$ we simply permute and shift the coordinate system.*

3.4.2.2 in multiple spatial dimensions

3.4.3 with degenerate metric

3.4.3.1 in one spatial dimension

3.4.3.2 in multiple spatial dimensions

3.4.4 of Lie-Poisson type

3.4.4.1 in one spatial dimension

3.4.4.2 in multiple spatial dimensions

3.5 Hyperbolic gradient flows

Chapter 4

Structure preserving integrators

4.1 Symplectic integrators

4.2 Energy preserving integrators

4.3 Poisson integrators

4.4 Multi-symplectic integrators

4.5 Dissipation preserving integrators

4.6 Conformal symplectic integrators

4.7 Metriplectic integrators

4.8 Finite volume methods

4.9 Symmetric integrators

4.9.1 The discontinuous Galerkin time stepping method

4.9.2 The adjoint of the discontinuous Galerkin time stepping method

4.9.3 The symmetric discontinuous Galerkin time stepping method

4.9.4 The symmetric-dissipative discontinuous Galerkin time stepping method

Chapter 5

Conclusion

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